spin groups

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Defines	spin symmetry
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Defines	$SO(n)$
Defines	$Spin(3)$
Defines	$Spin(4)$
Defines	$Sp(1)$
Defines	short exact sequence of Lie groups

1 Spin groups

1.1 Description

Spins and spin group mathematics are important subjects both in theoretical physics and mathematics. In physics, the term *spin 'groups'* is often used with the broad meaning of a collection of coupled, or interacting spins, and thus covers the broad 'spectrum' of spin clusters ranging from gravitons (as in spin networks and spin foams, for example) to 'up' (u) and 'down' (d) quark spins (fermions) coupled by gluons in nuclei (as treated in quantum chromodynamics or theoretical nuclear physics), and electron spin Cooper pairs (regarded as bosons) in low-temperature superconductivity. On the other hand, in relation to *quantum symmetry*, *spin groups* are defined in quantum mechanics and quantum field theories (QFT) in a precise, mathematical (algebraic) sense as properly defined groups, as introduced next. (In a semi-classical approach, the related concept of a *spinor* has been introduced and studied in depth by É. Cartan, who found that with his definition of spinors the (special) relativistic Lorentz covariance properties were not recovered, or applicable.)

Definition 1.1. In the mathematical, precise sense of the term, a *spin group* –as for example the Lie group $Spin(n)$– is defined as a *double cover of the special orthogonal (Lie) group* $SO(n)$ satisfying the additional condition that there exists the *short exact sequence of Lie groups*:

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$$

Alternatively one can say that the above exact sequence of Lie groups defines the spin group $Spin(n)$. Furthermore, $Spin(n)$ can also be defined as the *proper subgroup (or groupoid) of the invertible elements in the* ***http://planetmath.org/CliffordAlgebra2****Clifford algebra* $\mathbb{C}l(n)$; (when defined as a double cover this should be $Cl_{p,q}(R)$, a `http://planetmath.org/CliffordAl` algebra built up from an orthonormal basis of $n = p + q$ mutually orthogonal vectors under addition and multiplication, p of which have norm $+1$ and q of which have norm -1, as further explained in the `http://planetmath.org/Spinor`spinor definition).

Note also that other spin groups such as $Spin\ d$ (ref. [?]) are mathematically defined, and also important, in `http://planetmath.org/QFTOrQuantumFieldTheories`QFT.

Important examples of $Spin(n)$ and quantum symmetries: there exist the following isomorphisms:

1. $Spin(1) \cong O(1)$
2. $Spin(2) \cong U(1) \cong SO(2)$
3. $Spin(3) \cong Sp(1) \cong SU(2)$
4. $Spin(4) \cong Sp(1) \times Sp(1)$
5. $Spin(5) \cong Sp(2)$
6. $Spin(6) \cong SU(4)$

Thus, the symmetry groups in the Standard Model (SUSY) of current Physics can also be written as : $Spin(2) \times Spin(3) \times SU(3)$.

Remarks

- In modern Physics, non-Abelian spin groups are also defined, as for example, spin quantum groups and spin quantum groupoids.
- An extension of the concepts of spin group and spinor, is the notion of a 'twistor', a mathematical concept introduced by Sir Roger Penrose, generally with distinct symmetry/mathematical properties from those of spin groups, such as those defined above.

1.2 The Fundamental Groups of $Spin(p, q)$

With the usual notation, the fundamental groups $\pi_1(Spin(p, q))$ are as follows:

1. $\{0\}$, for $(p, q) = (1, 1)$ and $(p, q) = (1, 0)$;
2. $\{0\}$, if $p > 2$ and $q = 0, 1$;
3. $\mathbb{Z}$ for $(p, q) = (2, 0)$ and $(p, q) = (2, 1)$;
4. $\mathbb{Z} \times \mathbb{Z}$ for $(p, q) = (2, 2)$;
5. $\mathbb{Z}$ for $p > 2, q = 2$
6. $\mathbb{Z}_2$ for $p > 2, q > 2$

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