Ricci tensor

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Defines	scalar curvature
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Defines	ricci scalar
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Defines	Weyl curvature tensor

Definition. The *Ricci curvature tensor* is a rank 2, symmetric tensor that arises naturally in pseudo-Riemannian geometry. Let (M, g_{ij}) be a smooth, n-dimensional pseudo-Riemannian manifold, and let $R^i{}_{jkl}$ denote the corresponding Riemann curvature tensor. The Ricci tensor R_{ij} is commonly defined as the following contraction of the full curvature tensor:

$$R_{ij} = R^k{}_{ikj}.$$

The index symmetry of R_{ij}, so defined, follows from the symmetry properties of the Riemann curvature. To wit,

$$R_{ij} = R^k{}_{ikj} = R_{ki}{}^k{}_j = R^k{}_{jki} = R_{ji}.$$

It is also convenient to regard the Ricci tensor as a symmetric bilinear form. To that end for vector-fields X, Y we will write

$$\mathrm{Ric}(X, Y) = X^i Y^j R_{ij}.$$

Related objects. Contracting the Ricci tensor, we obtain an important scalar invariant

$$R = R^i{}_i,$$

called the scalar curvature, and sometimes also called the Ricci scalar. Closely related to the Ricci tensor is the tensor

$$G_{ij} = R_{ij} - \frac{1}{2} R\, g_{ij},$$

called the *Einstein tensor*. The Einstein tensor is also known as the trace-reversed Ricci tensor owing to the fact that

$$G^i{}_i = -R.$$

Another related tensor is

$$S_{ij} = R_{ij} - \frac{1}{n} R\, g_{ij}.$$

This is called the trace-free Ricci tensor, owing to the fact that the above definition implies that

$$S^i{}_i = 0.$$

Geometric interpretation. In Riemannian geometry, the Ricci tensor represents the average value of the sectional curvature along a particular direction. Let

$$K_x(u,v) = \frac{R_x(u,v,v,u)}{g_x(u,u)g_x(v,v) - g_x(u,v)^2}$$

denote the sectional curvature of M along the plane spanned by vectors $u, v \in T_xM$. Fix a point $x \in M$ and a tangent vector $v \in T_xM$, and let

$$S_x(v) = \{u \in T_xM \colon g_x(u,u) = 1,\ g_x(u,v) = 0\}$$

denote the $n-2$ dimensional sphere of those unit vectors at x that are perpendicular to v. Let μ_x denote the natural $(n-2)$-dimensional volume measure on T_xM, normalized so that

$$\int_{S_x(v)} \mu_x = 1.$$

In this way, the quantity

$$\int_{S_x(v)} K_x(\cdot, v)\mu_x,$$

describes the average value of the sectional curvature for all planes in T_xM that contain v. It is possible to show that

$$\mathrm{Ric}_x(v,v) = (1-n)\int_{S_x(v)} K_x(\cdot, v)\mu_x,$$

thereby giving us the desired geometric interpretation.

Decomposition of the curvature tensor. For $n \geq 3$, the Ricci tensor can be characterized in terms of the decomposition of the full curvature tensor into three covariantly defined summands, namely

$$\begin{aligned}
F_{ijkl} &= \tfrac{1}{n-2}\left(S_{jl}\, g_{ik} + S_{ik}\, g_{jl} - S_{il}\, g_{jk} - S_{jk}\, g_{il}\right), \\
E_{ijkl} &= \tfrac{1}{n(n-1)} R\left(g_{jl}\, g_{ik} - g_{il}\, g_{jk}\right), \\
W_{ijkl} &= R_{ijkl} - F_{ijkl} - E_{ijkl}.
\end{aligned}$$

The W_{ijkl} is called the *Weyl curvature tensor*. It is the conformally invariant, trace-free part of the curvature tensor. Indeed, with the above definitions, we have

$$W^k{}_{ikj} = 0.$$

The E_{ijkl} and F_{ijkl} correspond to the trace-free part of the Ricci curvature tensor, and to the Ricci scalar. Indeed, we can recover S_{ij} and R from E_{ijkl} and F_{ijkl} as follows:

$$S_{ij} = F^{k}{}_{ikj},$$
$$E^{ij}{}_{ij} = R.$$

Relativity. The Ricci tensor also plays an important role in the theory of general relativity. In this keystone application, M is a 4-dimensional pseudo-Riemannian manifold with signature $(3, 1)$. The Einstein field equations assert that the energy-momentum tensor is proportional to the Einstein tensor. In particular, the equation

$$R_{ij} = 0$$

is the field equation for a vacuum space-time. In geometry, a pseudo-Riemannian manifold that satisfies this equation is called Ricci-flat. It is possible to prove that a manifold is Ricci flat if and only if locally, the manifold, is conformally equivalent to flat space.