corner point theorem

Canonical name	CornerPointTheorem
Date of creation	2013-03-22 15:39:16
Last modified on	2013-03-22 15:39:16
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	12
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 90C05
Synonym	fundamental theorem of linear programming
Defines	corner point

Let P be a primal linear programming problem with f the objective function and polyhedron R as its feasible region. A *corner point*, or extreme point of P is a of R.

Corner Point Theorem. If P has an optimal solution $a < \infty$, then there is a corner point p of P such that $f(p) = a$. If another corner point q also satisfies $f(x) = a$, then $f(\overline{pq}) = \{a\}$. If r is a third corner point such that $f(r) = a$, then $f(\triangle pqr) = \{a\}$.

Note that the line segment and triangle mentioned above are necessarily subsets of R.

A cousin of the above theorem is the following:

Theorem. If P has an optimal solution $a < \infty$ occurring at an interior point of an edge E (or a face F) on the boundary of the feasible region R, then $f(E) = \{a\}$ (or $f(F) = \{a\}$). In fact, if the solution occurs at an interior point of R, then the solution is satisfied by all points of R: $f(R) = \{a\}$. In other words, the objective function f is constant on R.

On way to visualize the above theorems is to simplify them into the case when the objective function is a line $f(x) = mx + b$ on the "$x - y$ plane" and the feasible region is a line segment $I = [x_0, x_1]$ on the x-axis. It is easy to see now that the maximum (or minimum) of f occurs at either x_0 or x_1. If the optimal value occurs either at both end points, or at an interior point $x_2 \in (x_0, x_1)$, then f is a horizontal line segment on I.

An application of the above theorems can be demonstrated by the following example: If the feasible region R is a unit square and if corner points $(0, 0), (1, 1)$ satisfy the optimal solution of P, then all points on $\{(x, y) \mid x = y\} \cap R$ satisfy the solution. In particular, $(\frac{1}{2}, \frac{1}{2})$, an interior point of R, satisfies the solution. As a result, all points of R satisfy the solution.