minimax inequality

Canonical name	MinimaxInequality
Date of creation	2013-03-22 16:57:16
Last modified on	2013-03-22 16:57:16
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Last modified by	bchui (10427)
Numerical id	32
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Entry type	Theorem
Classification	msc 91A99
Classification	msc 91A05

The minimax inequality was first proved by John von Neumann. It is the starting point to discuss two-players zero-sum static games theory.

Theorem 1 : <u>minimax inequality, simple strategies</u>

For any $m \times n$ matrix $A_{i,j}$, we have

(1) $\max\limits_{1\le i\le m}\min\limits_{1\le j\le n} A_{i,j} \le \min\limits_{1\le j\le n}\max\limits_{1\le i\le m} A_{i,j}$

(2) $\max\limits_{1\le i\le m}\min\limits_{1\le j\le n} A_{i,j} = \min\limits_{1\le j\le n}\max\limits_{1\le i\le m} A_{i,j}$ if and only if $A_{i,\tilde{j}} \le A_{\tilde{i},\tilde{j}} \le A_{\tilde{i},j} \quad \forall i,j$ is valid for some $(\tilde{i},\tilde{j})$

For a 2 players zero-sum game, the entries of $A_{i,j}$ is interpreted as the payoff when player 1 has chosen the i^{th} strategy while player 2 has chosen the j^{th} strategy. The value $A_{\tilde{i},\tilde{j}}$ is known as the *value* of the game.

Proof Since $\min\limits_{1\le j\le n} A_{i,j} \le \max\limits_{1\le i\le m} A_{i,j} \quad \forall i,j$. The LHS is independent of j while the RHS is independent of i, therefore we obtain $\max\limits_{1\le i\le m}\min\limits_{1\le j\le n} A_{i,j} \le \min\limits_{1\le j\le n}\max\limits_{1\le i\le m} A_{i,j}$

Theorem 2 : <u>minimax inequality, mixed strategies</u>

Let $S_m = \{x \in \mathbb{R}^m \mid x_i \ge 0 \ \forall i \ , \ \sum\limits_{i=1}^{m} x_i = 1\} \subseteq \mathbb{R}^m$. For any $m\times n$ matrix $A_{i,j}$, we have

$$\max_{x\in S_m}\min_{y\in S_n}\sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}x_iy_j = \min_{y\in S_n}\max_{x\in S_m}\sum_{i=}^{m}\sum_{j=1}^{n} A_{i,j}x_iy_j$$

Here $0 \le x_i \le 1$ is interpreted as the probability that Player 1 will choose strategy i while $0 \le y_j \le 1$ is the probability that Player 2 will choose strategy j.

Proof For any $x \in S_m$ and any $y \in S_n$ we have $\max\limits_{x\in S_m}\min\limits_{y\in S_n}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} A_{i,j}x_iy_j \le \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} A_{i,j}x_iy_j$

Taking maximum for $x \in S_m$ on both sides, we have $\max\limits_{x\in S_m}\min\limits_{x\in S_n}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} A_{i,j}x_iy_j \le \max\limits_{s\in S_m}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} A_{i,j}x$

Taking minimum for $y \in S_n$ on both sides, we have $v_1 = \max\limits_{x\in S_m}\min\limits_{y\in S_n}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} A_{i,j}x_iy_j \le v_2 = \min\limits_{y\in S_n}\max\limits_{x\in S}$

The prove of other half of the inequality takes two steps:

Step 1 Suppose there is a $y \in S_n$ such that $\sum\limits_{j=1}^{n} A_{i,j}y_j \le 0 \quad \Rightarrow \quad$ There is some $\tilde{x} \in S_m$ such that $\sum\limits_{i=1}^{m}\left(\sum\limits_{j=1}^{n} A_{i,j}y_j\right)\tilde{x}_i \le 0$

$$\Rightarrow \max_{x \in S_n} \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} x_i y_j \leq 0 \quad \Rightarrow \quad v_2 = \min_{y \in S_n} \max_{x \in S_m} \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} x_i y_j \leq 0 \quad (*1)$$

Step 2 Suppose there is a $x \in S_m$ such that $\sum_{i=1}^{m} A_{i,j} x_i y_j > 0 \quad \Rightarrow$ There is some $\tilde{y} \in S_n$ such that $\sum_{j=1}^{n} \left(\sum_{i=1}^{m} A_{i,j} x_i \right) \tilde{y}_j \geq 0$

$$\Rightarrow \min_{y \in S_n} \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} x_i y_j \geq 0 \quad \Rightarrow \quad v_1 = \max_{x \in S_m} \min_{y \in S_n} \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} x_i y_j \geq 0 \quad (*2)$$

Combining (*1) and (*2) we see that either $0 \leq v_1$ or $v_2 \leq 0$ is the case and $v_1 < 0 < v_2$ cannot be valid. Repeat the same procedure to the matrix $\tilde{A}_{i,j} = A_{i,j} - \lambda$ and we see that $v_1 - \lambda < 0 < v_2 - \lambda$ is invalid, i.e. $v_1 < \lambda < v_2$ is not valid for any λ. Since λ is arbitrary, we conclude that $v_2 \leq v_1$.

An entire theory on minimax has already been developed and is one of the major research area in optimization theory. The following contains some good sources for further reference:

References

[1] V.F.Demyanov and V.N.Malozemov, *Introduction to Minimax*, Keter Publishing House Jerusalem Ltd, 1974.