Black-Scholes PDE

Canonical name	BlackScholesPDE
Date of creation	2013-03-22 16:31:08
Last modified on	2013-03-22 16:31:08
Owner	stevecheng (10074)
Last modified by	stevecheng (10074)
Numerical id	6
Author	stevecheng (10074)
Entry type	Result
Classification	msc 91B28
Classification	msc 60H10
Synonym	Black-Scholes partial differential equation
Synonym	Black-Scholes equation
Related topic	AnalyticSolutionOfBlackScholesPDE
Related topic	ExampleOfSolvingTheHeatEquation

Contents

The *Black-Scholes partial differential equation* is the partial differentiation equation:

$$rf = \frac{\partial f}{\partial t} + rx\,\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2\,\frac{\partial^2 f}{\partial x^2}\,, \quad f = f(t,x)\,, \tag{1}$$

on the domain $0 \le x < \infty$, $0 \le t \le T$. Its solution gives the price function of a stock option (or any other contingent claim on a tradable asset) under the assumptions of the Black-Scholes model for prices.

The parameters r, $\sigma > 0$ and $T > 0$ are, respectively, the prevailing continuously-compounded risk-free rate of interest, the volatility of the stock, and the time to maturity of the stock option.

More generally, the term *Black-Scholes partial differential equation* can refer to other partial differential equations similar in form to equation (**??**), derived in similar ways under more general modelling assumptions than the most basic Black-Scholes model.

0.1   Derivation from martingale form

The *Black-Scholes formula*, in its martingale form, gives the theoretical value of the stock option

$$V(t) = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}[V(T)\mid \mathcal{F}_t]\,, \quad 0 \le t \le T\,, \tag{2}$$

where $V(T)$ is a given pay-off function or terminal condition, assumed to be a $\mathbf{L}^1(\mathbb{Q})$ random variable; $\mathcal{F}_t$ is the filtration generated by the the standard Wiener process $\widetilde{W}(t)$ under the probability measure $\mathbb{Q}$ (the "risk-neutral measure").

Equation (**??**) gives the solution to the stochastic differential equations:

$$V(t) = \Delta(t)X(t) + \Theta(t)M(t)\,, \quad M(t) = e^{rt}\,, \tag{3}$$
$$dV(t) = \Delta(t)\,dX(t) + \Theta(t)\,dM(t)\,, \quad dM(t) = rM(t)\,dt\,, \tag{4}$$
$$dX(t) = rX(t)\,dt + \sigma X(t)\,d\widetilde{W}(t)\,. \tag{5}$$

From the martingale form of the result, there are two ways to derive the PDE form (**??**).

0.1.1 Using the Feynman-Kac formula

One way is to appeal to the Feynman-Kac formula, which states in this case, that under certain regularity conditions[1] on the function $\psi \colon (0, \infty) \to \mathbb{R}$, the function $f \colon [0, T] \times (0, \infty) \to \mathbb{R}$ defined by

$$f(t, x) = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[\, \psi(X(T)) \mid X(t) = x \,], \tag{6}$$

satisfies the partial differential equation (**??**) with the terminal condition

$$f(T, x) = \psi(x) \,.$$

(The stochastic process $X(t)$ is to satisfy equation (**??**).)

Now, equations (**??**) and (**??**) look quite similar, but with an important difference: the conditional expectation on equation (**??**) is with respect to the event $\{X(t) = x\}$, while equation (**??**) is conditioning on the filtration $\mathcal{F}_t$. We claim that these are actually the same thing — that is, for fixed t, the random variable $\mathbb{E}^{\mathbb{Q}}[V(T) \mid \mathcal{F}_t]$ can be written as a function of $X(t)$ — *provided that the terminal condition is itself a function of $X(T)$*:

$$V(T) = \psi(X(T)) \,.$$

We will demonstrate this claim later. Assuming that the claim holds, we may thus set $V(t) = f(t, X(t))$, and the Black-Scholes partial differential equation follows from the Feynman-Kac formula as explained earlier.

0.1.2 Proof of Markov property

Formally, any $\mathcal{F}_t$-measurable stochastic process $X(t)$ that ensures $\mathbb{E}[\psi(X(T)) \mid \mathcal{F}_t]$ is always a measurable function of $X(t)$, for any Borel-measurable ψ, is called a *Markov process*, or is said to have the *Markov property*. Thus we want to show that the stock process $X(t)$ has the Markov property. For this, we will need the following explicit solution for it at time T in terms of an initial condition at time t:

$$X(T) = X(t) \exp\Big((r - \tfrac{1}{2}\sigma^2)(T - t) + \sigma\big(\widetilde{W}(T) - \widetilde{W}(t)\big) \Big)$$
$$:= X(t) \, g\big(\widetilde{W}(T) - \widetilde{W}(t)\big) \,.$$

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[1] The standard assumption is that ψ is twice continuously-differentiable and $\mathbb{E}^{\mathbb{Q}}[|\psi(X(T))|] < \infty$. However, for the particular PDE we consider, these assumptions can be relaxed.

Then

$$\mathbb{E}^{\mathbb{Q}}[V(T) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\psi(X(T)) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}\left[\psi\big(X(t)\,g(\widetilde{W}(T) - \widetilde{W}(t))\big) \mid \mathcal{F}_t\right].$$

Since $X(t)$ is $\mathcal{F}_t$-measurable, it may be treated as a constant x while taking conditional expectations with respect to $\mathcal{F}_t$. And $\widetilde{W}(T) - \widetilde{W}(t)$ is a random variable independent of $\mathcal{F}_t$ (and hence of $X(t)$). Therefore the expression that appears on the right-hand side above is a function of $x = X(t)$, and would be unchanged if the conditioning is changed from $\mathcal{F}_t$ to $X(t)$.

0.1.3 Using Itō's formula

There is also a more direct method of deriving equation (**??**) using Itō's formula for Itō processes.

Set $V(t) = f(t, X(t))$ as before. Make the assumption that f is twice continuously differentiable, to be checked later; then we can apply Itō's formula to expand $dV(t)$: (partial derivatives evaluated at $x = X(t)$)

$$\begin{aligned}
dV(t) &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,dX(t)dX(t) \\
&= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 X(t)^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x}\,dX(t)\,.
\end{aligned} \tag{7}$$

By theorems on the uniqueness of solutions to stochastic differential equations, the coefficients of the correponding dt and $dX(t)$ terms in equations (**??**) and (**??**) must be equal almost surely. Equating the coefficients, we find:

$$\Delta(t) = \frac{\partial f}{\partial x}\,, \tag{8}$$

$$r\big(V(t) - \Delta(t)X(t)\big) = \Theta(t)rM(t) = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 X(t)^2 \frac{\partial^2 f}{\partial x^2}\,. \tag{9}$$

Equation (**??**) is essentially equation (**??**), except that x has been replaced by $X(t)$ in some places. However, because $X(t)$ is a random variable that takes on all values on $(0, \infty)$, and f and its derivatives are assumed to be continuous, equation (**??**) must hold for arbitrary values x substituted for $X(t)$. Hence we obtain equation (**??**).

To verify the initial assumption that f is twice continuously differentiable, we write formula (**??**) in a more explicit form (using the Markov property

for $X(t)$ as before):

$$f(t, x) = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[\psi(X(T)) \mid X_t = x]$$
$$= \int_{-\infty}^{\infty} e^{-r(T-t)} \psi(y) \, p(x, y, T - t) \, dy \, ,$$

where $p(x, y, \tau)$ is the transition density of the stock process from x to y over a time interval of length τ. In fact, from the solution for $X(t)$, the density $p(x, \cdot, \tau)$ is the density for the log-normal random variable whose logarithm has mean $\log x + (r - \frac{1}{2}\sigma^2)\tau$ and variance $\sigma^2 \tau$. By differentiation under the integral sign, we see that $f(t, x)$ must be twice continously differentiable for $0 \leq t < T$ and $0 < x < \infty$.

0.2 Comparison of the martingale and PDE forms

The martingale formulation of the result, in equation (**??**) is more general than the PDE formulation (**??**): the latter imposes extra regularity conditions, and more importantly, it can only describe prices $V(t) = f(t, X(t))$ that are *functions only of the time t and the current stock price $X(t)$*. In the case of the call option ($\psi(x) = x - K$ for some $K > 0$), this assumption is true; but there are other sorts of contingent claims whose pay-off depend on the history of the stock process $X(t)$ — for example, a common kind of contingent claim has a final pay-off that depends on an average of the stock price $X(t)$ over time $t \in [0, T]$.

However, the PDE form of the solution is often more amenable to a numerical solution. While the expectation in equation (**??**) can be approximated numerically with Monte-Carlo simulation, for low-dimensional problems, solutions based on finite-difference approximations of PDEs are often quicker to compute than those based on Monte-Carlo.

0.3 Analytic solution

Though the solution for the Black-Scholes PDE is already known using the martingale representation (**??**), we can also http://planetmath.org/AnalyticSolutionOfBlackS the PDE directly using classical analytical methods.

Actually, the classical solution can be quite instructive in that it shows more or less the physical meaning behind the PDE, and also the behavior of solutions as the terminal condition is varied. (For example, the PDE is

a transformation of a diffusion equation, and consequently its solutions are always infinitely smooth, as long as the terminal condition satisfies some local-integrability properties.)

0.4   Quantity of stock needed to replicate option

Another important result that must not be left unmentioned is equation (**??**), derived during the course of showing the Black-Scholes PDE.

In the article on the Black-Scholes pricing formula, a solution for $V(t)$ was produced that satisfied (**??**), but the $\mathcal{F}_t$-adapted process $\delta(t)$ that appears in that stochastic differential equation was obtained by appealing to an existence theorem. However, under the assumption that $V(t) = f(t, X(t))$, equation (**??**) gives an actual computable formula:

$$\Delta(t) = \left.\frac{\partial f(t, x)}{\partial x}\right|_{x=X(t)}.$$

which is crucial in practice, for the theoretical "price" of a stock option would be useless if one cannot produce in reality that option for the stated price.

0.5   Generalizations

(To be written.)