fundamental theorem of demography, proof of

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- First we will prove that there exist $m, M > 0$ such that

$$m \leq \frac{\|x_{k+1}\|}{\|x_k\|} \leq M \tag{1}$$

for all k, with m and M of the sequence. In to show this we use the primitivity of the matrices A_k and A_∞. Primitivity of A_∞ implies that there exists $l \in \mathbb{N}$ such that

$$A_\infty^l \gg 0$$

By continuity, this implies that there exists k_0 such that, for all $k \geq k_0$, we have

$$A_{k+l}A_{k+l-1} \cdots A_k \gg 0$$

Let us then write x_{k+l+1} as a function of x_k:

$$x_{k+l+1} = A_{k+l} \cdots A_k x_k$$

We thus have

$$\|x_{k+l+1}\| \leq C^{l+1} \|x_k\| \tag{2}$$

But since the matrices $A_{k+l}, \ldots, A_k$ are strictly positive for $k \geq k_0$, there exists a $\varepsilon > 0$ such that each of these matrices is superior or equal to ε. From this we deduce that

$$\|x_{k+l+1}\| \geq \varepsilon \|x_k\|$$

for all $k \geq k_0$. Applying (**??**), we then have that

$$C^l \|x_{k+1}\| \geq \varepsilon \|x_k\|$$

which yields

$$\|x_{k+1}\| \geq \frac{\varepsilon}{C^l} \|x_k\|$$

for all $k \geq 0$, and so we indeed have (**??**).

- Let us denote by e_k the (normalised) Perron eigenvector of A_k. Thus

$$A_k e_k = \lambda_k e_k \quad \|e_k\| = 1$$

Let us denote by π_k the projection on the supplementary space of $\{e_k\}$ invariant by A_k. Choosing a proper norm, we can find $\varepsilon > 0$ such that

$$|A_k \pi_k| \leq (\lambda_k - \varepsilon)$$

for all k. • We shall now prove that

$$\frac{\left\langle e^*_{k+1}, x_{k+1}\right\rangle}{\langle e^*_k, x_k\rangle} \to \lambda_\infty \text{ when } k \to \infty$$

In order to do this, we compute the inner product of the sequence $x_{k+1} = A_k x_k$ with the e_k's:

$$\begin{aligned}\left\langle e^*_{k+1}, x_{k+1}\right\rangle &= \left\langle e^*_{k+1} - e^*_k, A_k x_k\right\rangle + \lambda_k \langle e^*_k, x_k\rangle \\ &= o\left(\langle e^*_k, x_k\rangle\right) + \lambda_k \langle e^*_k, x_k\rangle\end{aligned}$$

Therefore we have

$$\frac{\left\langle e^*_{k+1}, x_{k+1}\right\rangle}{\langle e^*_k, x_k\rangle} = o(1) + \lambda_k$$

• Now assume

$$u_k = \frac{\pi_k x_k}{\langle e^*_k, x_k\rangle}$$

We will verify that $u_k \to 0$ when $k \to \infty$. We have

$$u_{k+1} = (\pi_{k+1} - \pi_k) A_k \frac{x_k}{\left\langle e^*_{k+1}, x_{k+1}\right\rangle} + \frac{\langle e^*_k, x_k\rangle}{\langle e^*_k, x_{k+1}\rangle} A_k \pi_k \frac{x_k}{\langle e^*_k, x_k\rangle}$$

and so

$$|u_{k+1}| \le |\pi_{k+1} - \pi_k| C' + \frac{\langle e^*_k, x_k\rangle}{\left\langle e^*_{k+1}, x_{k+1}\right\rangle}(\lambda_k - \varepsilon)|u_k|$$

We deduce that there exists $k_1 \ge k_0$ such that, for all $k \ge k_1$

$$|u_{k+1}| \le \delta_k + (\lambda_\infty - \frac{\varepsilon}{2})|u_k|$$

where we have noted

$$\delta_k = (\pi_{k+1} - \pi_k) C'$$

We have $\delta_k \to 0$ when $t \to \infty$, we thus finally deduce that

$$|u_k| \to 0 \text{ when } k \to \infty$$

Remark that this also implies that

$$z_k = \frac{\pi_k x_k}{\|x_k\|} \to 0 \text{ when } k \to \infty$$

- We have $z_k \to 0$ when $k \to \infty$, and $x_k/\|x_k\|$ can be written

$$\frac{x_k}{\|x_k\|} = \alpha_k e_k + z_k$$

Therefore, we have $\alpha_k e_k \to 1$ when $k \to \infty$, which implies that α_k tends to 1, since we have chosen e_k to be normalised ($i.e.$,$\|e_k\| = 1$).

We then can conclude that

$$\frac{x_k}{\|x_k\|} \to e_\infty \text{ when } k \to \infty$$

and the proof is done.