stability of transfer functions in the Laplace domain

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The following describes SISO (single input single output) system descriptions. More complex systems require a more sophisticated analysis.

For a general transfer function, $H(s)$, in Laplace domain, we have

$$H(s) = \frac{\prod_{i=0}^{Z}(s - z_i)}{\prod_{i=0}^{P}(s - p_i)} \tag{1}$$

The conclusions below can all be derived and understood by expansion of $H(s)$ in terms of partial fractions, and then doing a inverse Laplace transform fraction by fraction.

z_i denotes the zeros and p_i denotes the poles of the linear time invariant system (LTI). Stability of the system $H(s)$ is characterized by the location of the poles in the complex s-plane. There are many definitions of stability in the control system literature, the most common one used (for transfer functions) is the bounded-input-bounded-output stability (BIBO), which states that for a BIBO stable system, for any bounded input, or finite amplitude input, the output of the system will also be bounded.

For example, a typical second order system such as the mass-spring-dashpot system has the following transfer function,

$$H(p) = \frac{\omega^2}{p^2 + 2\zeta\omega p + \omega^2} \tag{2}$$

with a pair of complex poles at $p = -\zeta\omega \pm \omega\sqrt{\zeta^2 - 1}$. In common control system literature, ζ is usually denoted as the *damping ratio* and ω is denoted as the *natural frequency* of the system. In the case of the mass-spring-dashpot system, we can tell that the oscillation of the mass attached to a spring should be a function of the weight of the mass and the stiffness of the spring, hence in the literature we have $\omega = \sqrt{\frac{K}{M}}$, where K is the spring constant and M is the weight of the mass. From the same logic, the amount time for the mass to stop oscillating should be a function of the spring stiffness and the characteristics of the dashpot, so we have $\zeta = \frac{D\omega}{2K}$, where D is the dashpot constant.

To determine is the system $H(p)$ BIBO stable, the simplest solution is to scrutinize the time domain solution of **??** via inverse Laplace transform, and such discussion can easily be found in common control system related text books and online lecture notes. However, such approach tells nothing about the *physical nature* of the dynamic system, so here I will try to establish relationship between the location of the poles and the stability of the system. Here I first state that the system in **??**, specifically the mass-spring-dashpot system, is stable, since it is physically impossible for the system to produce an output increasing in amplitude forever as time goes to infinity.

First let's consider an ideal mass-spring system. If we push the mass once (impulse response), the mass will oscillate forever with the same amplitude and frequency, since there are no dashpot to dampen the motion. In this case, $D = 0 \rightarrow \zeta = 0$, which the pair of complex poles of the system will be located on the imaginary axis of the complex s-plane, and the stronger the spring (K is large) the further away the poles from the origin. So the imaginary part of the poles $img(p_i)$ dominates the oscillatory nature of the system. Now let's assume that we have a relatively weak spring compare to the dashpot that we are going to add to the system, so the real part of the poles $real(p_i)$ will be dominating. If we push the mass the same way as before, now we can expect the mass to oscillate for sometime then come to a rest, depending on the strength of the dashpot. The stronger the dashpot, the further away the poles from the origin along the real axis on the s-plane. We have almost covered the whole left-hand side of the s-plane, and so far we have the following observations.

- The imaginary part of complex poles correspond to oscillatory energy storage mechanisms. Note if the original differential equation has real coefficients, the complex poles are always in complex conjugate pairs . As a result these can always be represented by sine and cos terms in the time domain.

- The real parts of poles govern the decay rate of responses to stimulations; with negative values corresponding to decaying terms, and positive parts to growth terms (which are not stable).

- The ratio between the real and imaginary parts of the poles govern the shape of the response. In particular ratios less than one hide the visible part of the oscillatory terms in the time domain.

- Poles that have neither real or imaginary parts, i.e. at the origin, correspond to successive integrations in the time domain. One integration for each pole. Since these have an unbounded output for a step input, systems like these are not BIBO; but are common.

For a system consisted of poles with imaginary parts only, $p_i = \pm j\omega$, it is usually referred to as a *marginally stable system. Notice that the real part of the poles for system* **??** *are always negative, since both* ζ *and* ω *are always positive. If a stable system has negative poles (real or complex), unstable system must has positives poles, in the right-hand side of the s-plane. This can easily be visualize if we place a amplifier (with* $-\zeta$*) instead of a dashpot in the system.*