

New Eigenvector-type Calculation

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Abstract

These are a continuation of some notes on the eigenvector calculation for priorities from a pairwise comparison matrix. In this version I explain the new calculation as a tweak to the eigenvector calculation.

1 Introduction

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1/3 & 3 & 1 \\ 3 & 1 & 1/3 & 3 \\ 1 & 3 & 1 & 1/9 \\ 1 & 1/3 & 9 & 1 \end{bmatrix}$$

The matrix of the inverse votes is the same as the transpose A^T :

$$A^T = \begin{bmatrix} 1 & 3 & 1/3 & 1 \\ 1/3 & 1 & 3 & 1/3 \\ 1 & 1/3 & 1 & 9 \\ 1 & 3 & 1/9 & 1 \end{bmatrix}$$

If we let λ be the function that returns the largest eigenvector of a matrix, we have the following from the standard calculations.

$$\begin{aligned} \lambda(A) &= (0.1788, 0.2967, 0.1797, 0.3448) \\ \lambda(A^T) &= (0.1868, 0.2401, 0.4000, 0.1731) \end{aligned}$$

Notice that the rankings are:

$$\begin{aligned} \text{Rank}(A) &= (4, 2, 3, 1) \\ \text{Rank}(A^T) &= (3, 2, 1, 4) \end{aligned}$$

In other words, the rankings do not reverse as expected!

2 Matrix Multiplication Review

This new calculation is based upon the eigenvector calculation, but by replacing the summation that comes from matrix multiplication by a product. Recall the definition of matrix multiplication:

Definition 1 (Standard Matrix Multiplication). *Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Then the result of multiplying A by B is an $m \times k$ matrix, denote by AB whose row i column j entry is given by:*

$$(AB)_{i,j} := \sum_{l=1}^n a_{i,l} b_{l,j} \quad (1)$$

Note 1. *This is really based on the vector dot product. Let $A_{i,-}$ be the i^{th} row of A and $B_{-,j}$ be the j^{th} column of B . Then the right hand side of Equation 1 is really $A_{i,-} \cdot B_{-,j}$, i.e. the vector dot product of row i of A with column j of B .*

This leads us naturally to another formula for matrix multiplication.

Definition 2 (Matrix Multiplication Dot Product Form). *Let A and B be as in Definition 1, then:*

$$(AB)_{i,j} := A_{i,-} \cdot B_{-,j}$$

3 Standard Largest Eigenvector Method

There is a standard trick for calculating the largest eigenvector, namely we raise the matrix to larger powers and act on the unit column vector (the column vector of all 1's), and normalize. That calculation converges to the largest eigenvector.

However, we can also simply raise the matrix to larger and larger powers and analyze the columns. Multiplying by the unit column vector is simply averaging the columns of those power matrices. So, to a certain extent the largest eigenvector simply falls out of the limit of the powers of the matrix (after dividing those matrices by a suitable scalar, for instance the appropriate power of the determinant).

There is a good reason to use this calculation to calculate the priority vector for a pairwise comparison matrix A . That is because A^k represents the sum of the weights of all length k paths. What I mean here is that $(A^k)_{i,j}$ is the sum of all of the weights derived from length k paths from i to j .

Note 2. *That is the key point, and our key area of difficulty. A^k is the sum of all of the weights, not the product!*

Note 3. *A starts off as a reciprocal matrix, but none of the A^k for $k > 1$ will be reciprocal because of the summation occurring. It would seem natural to want that reciprocity to be true for those powers.*

Note 4 (Spoiler). *The new calculation has the A^k -like calculated matrices as all being reciprocal if A was reciprocal.*

4 New Calculation

I propose we change the A^k terms by a new type of matrix multiplication, for the new definition.

Definition 3 (New Multiplication). *Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Then the result of bmultiplying A by B is an $m \times k$ matrix, denote by $A \circledast B$ whose row i column j entry is given by:*

$$(A \circledast B)_{i,j} := \prod_{l=1}^n a_{i,l} b_{l,j} \quad (2)$$

Note 5 (New dot product). *The above definition screams to be rewritten in terms of a modified dot product, the bdot, which we define here.*

Definition 4 (BDot Product). *If \mathbf{v} and \mathbf{w} are vectors of length n , then we denote the bdot product of \mathbf{v} and \mathbf{w} by $\mathbf{v} \odot \mathbf{w}$ and define it as follows:*

$$\mathbf{v} \odot \mathbf{w} := \prod_{i=1}^n v_i w_i$$

Using the above definition of the bdot product of vectors, we can rewrite Definition 3 similar to the way we rewrote Definition 1 using the standard dot product to arrive at Definition 2.

Definition 5 (New Multiplication BDot Form). *Let A and B be as in Definition 3, then:*

$$(A \circledast B)_{i,j} := A_{i,-} \odot B_{-,j}$$