where $t_1 = \tan(\theta_1/2)$, Eq. (5.33) yields a quartic polynomial equation in t_1. With t_1 computed, a value of θ_1 is obtained and θ_3 can be computed uniquely from Eqs. (5.29) and (5.32). The remaining angles (θ_2 and θ_4) can be computed as indicated earlier.

We propose a method that allows better insight without the complexity of a quartic polynomial equation. For simplicity, the sine and cosine of a sum of angles will be represented according to $C_{ijk}=\cos(\theta_i+\theta_j)$ and $S_{ij}=\sin(\theta_i+\theta_j)$.

As described in chapter 4, a set of inverse kinematic equations can be obtained by expressing the components of vectors **t** and **p** and the inner products **t.p** and **p.p** in terms of the joint variables θ_i, i=1, . . . , 3. The equations obtained are

$$t_x = \sigma_3 S_{123} \tag{5.34}$$

$$t_y = -\sigma_3 C_{123} \tag{5.35}$$

$$p_x = a_3 C_{123} + a_2 C_{12} + a_1 C_1 \tag{5.36}$$

$$p_y = a_3 S_{123} + a_2 S_{12} + a_1 S_1 \tag{5.37}$$

$$\mathbf{t.p} = a_1 C_{23} + a_2\sigma_3 S_3 + \tau_3(d_1+d_2) + d_3 \tag{5.38}$$

$$\mathbf{p.p} = 2(a_1a_3 C_{23} + a_2a_3 C_3 + a_1a_2 C_2) + ct \tag{5.39}$$

where

$$ct = a_1^2 + a_2^2 + a_3^2 + d_1^2 + d_2^2 + d_3^2 + 2(d_1d_2 + d_1d_3 + d_2d_3).$$

Equations (5.34) and (5.35) yield S_{123} and C_{123} directly, so a unique value of $\theta_{123} = \theta_1+\theta_2+\theta_3$ is obtained.

With θ_{123} known, Eqs. (5.36) and (5.37) become (elbow equations)

$$p_x - a_3 C_{123} = a_2 C_{12} + a_1 C_1 \qquad (5.36')$$

$$p_y - a_3 S_{123} = a_2 S_{12} + a_1 S_1 \qquad (5.37')$$

and can be solved for C_2 by

$$C_2 = [(p_x - a_3 C_{123})^2 + (p_y - a_3 S_{123})^2 - a_2^2 - a_1^2] / (2 a_1 a_2)$$

which is obtained by applying the cosine law to the triangle having links 1 and 2 as its sides. Two values of θ_2 follow from $\theta_2 = \text{atan2}(\pm \sqrt{(1-C_2^2)}, C_2)$

and a unique value of θ_1 can then be computed from Eqs. (5.36') and (5.37') which yield a linear system in S_1 and C_1 when S_{12} and C_{12} are expanded using sum of angles trigonometric identities. Joint variable 3, θ_3 is given by

$$\theta_3 = \theta_{123} - \theta_1 - \theta_2$$

and the solution set is completed when the last angle θ_4 is computed as shown earlier. This development proves that there can be at most 2 solution sets for a 4-DOF arm with this particular geometry.

Case 2: $\sigma_1 = \sigma_3 = 0$. The first two joint axes are parallel and the last two joint axes are parallel. The reduced system is

$$a_1t_y\ S_1 + a_1t_x\ C_1 = r_1 \tag{5.40}$$

$$0 = r_2 \tag{5.41}$$

$$-\sigma_2a_3\ S_3 = r_3 \tag{5.42}$$

$$a_1p_y\ S_1 + a_1p_x\ C_1 + \sigma_2a_3d_2\ S_3 + a_2a_3\ C_3 = r_4. \tag{5.43}$$

Equation (5.41) imposes a constraint on pose parameter t_z, $t_z = \tau_2\tau_3/\tau_1$. When this constraint is satisfied, Eq. (5.40) can be solved and yields two distinct values for θ_1. Then Eqs. (5.42) and (5.43) form a linear system in S_3 and C_3 which can be solved uniquely for θ_3. With θ_1 and θ_3 computed, θ_2 and θ_4 can be uniquely obtained as shown earlier. Here again we find at most two solution sets.

Case 3: $\sigma_1=a_3=0$. First two joint axes are parallel and last two joint axes intersect. The reduced system becomes

$$a_1t_y\ S_1 + a_1t_x\ C_1 + a_2\sigma_3\ S_3 - \sigma_2\sigma_3d_2\ C_3 = r_1 \tag{5.44}$$

$$\sigma_2\sigma_3\ C_3 = r_2 \tag{5.45}$$

$$0 = r_3 \tag{5.46}$$

$$a_1p_y\ S_1 + a_1p_x\ C_1 = r_4. \tag{5.47}$$

From Eq. (5.19), the pose constraint $r_3 = 0$ translates to

$$p_z = d_1 + d_2 + \tau_2d_3.$$

For a pose matrix that satisfies this constraint, two possible values of θ_3 can be obtained from Eq. (5.45). For each of those θ_3 values, a unique value of θ_1 is computed from the linear system in S_1 and C_1 formed by Eqs. (5.44) and (5.47). The two solution sets are then completed as shown previously.

Case 4: $\sigma_2=\sigma_3=0$. The last three joint axes are parallel. The reduced system simplifies to

$$a_1t_y\ S_1 + a_1t_x\ C_1 = r_1 \tag{5.48}$$
$$\sigma_1t_x\ S_1 - \sigma_1t_y\ C_1 = r_2 \tag{5.49}$$
$$\sigma_1p_x\ S_1 - \sigma_1p_y\ C_1 = r_3 \tag{5.50}$$
$$a_1p_y\ S_1 + a_1p_x\ C_1 + a_2a_3\ C_3 = r_4. \tag{5.51}$$

Two out of the first three equations (Eqs. (5.48)-(5.50)) can be used to solve uniquely for θ_1. The third (unused equation) becomes a realizability constraint on the pose. With θ_1 known, Eq. (5.51) yields a value for C_3 which in turn gives two possible values for θ_3. Two solution sets can be obtained after computing θ_2 and θ_4.

Case 5: $\sigma_2=a_3=0$. The intermediate joint axes are parallel and the last two axes intersect. The reduced sytem becomes

$$a_1t_y\ S_1 + a_1t_x\ C_1 + a_2\sigma_3\ S_3 = r_1 \tag{5.52}$$
$$\sigma_1t_x\ S_1 - \sigma_1t_y\ C_1 = r_2 \tag{5.53}$$
$$\sigma_1p_x\ S_1 - \sigma_1p_y\ C_1 = r_3 \tag{5.54}$$
$$a_1p_y\ S_1 + a_1p_x\ C_1 = r_4. \tag{5.55}$$

Two out of the last three equations (5.53)-(5.55) can be solved uniquely for θ_1. The third equation becomes a realizability constraint on pose element t_z or p_z depending on the chosen equation. With Θ_1 known, two values of Θ_3 can be computed from the value of S_3 derived from Eq. (5.52).

Case 6: $\sigma_3 = a_1 = 0$. The last two joint axes are parallel and the first two axes intersect. The reduced system is

$$0 = r_1 \tag{5.56}$$

$$\sigma_1 t_x\, S_1 - \sigma_1 t_y\, C_1 = r_2 \tag{5.57}$$

$$\sigma_1 p_x\, S_1 - \sigma_1 p_y\, C_1 - \sigma_2 a_3\, S_3 = r_3 \tag{5.58}$$

$$\sigma_2 a_3 d_2\, S_3 + a_2 a_3\, C_3 = r_4. \tag{5.59}$$

Equation (5.56) is a realizability constraint on pose parameter t_z (Eq. (5.17)). For an end-effector pose that satisfies this constraint, Eq. (5.57) yields two values for Θ_1. Equations. (5.58) and (5.59) can then be solved for Θ_3 uniquely.

Case 7: $\sigma_3 = a_2 = 0$. The last two joint axes are parallel and the intermediate two axes intersect. The reduced system is

$$a_1 t_y\, S_1 + a_1 t_x\, C_1 = r_1 \tag{5.60}$$

$$\sigma_1 t_x\, S_1 - \sigma_1 t_y\, C_1 = r_2 \tag{5.61}$$

$$\sigma_1 p_x\, S_1 - \sigma_1 p_y\, C_1 - \sigma_2 a_3\, S_3 = r_3 \tag{5.62}$$

$$a_1 p_y\, S_1 + a_1 p_x\, C_1 + \sigma_2 a_3 d_2\, S_3 = r_4. \tag{5.63}$$

Here, Eqs. (5.60) and (5.61) yield a unique value for Θ_1, then one of Eqs. (5.62) or (5.63) can be used to solve for

S_3 thereby providing two values for θ_3, the remaining equation is a pose constraint.

Case 8: $a_1=a_2=d_2=0$. The first three joint axes intersect and the reduced system becomes

$$0 = r_1 \quad (5.64)$$

$$\sigma_1 t_x S_1 - \sigma_1 t_y C_1 + \sigma_2\sigma_3 C_3 = r_2 \quad (5.65)$$

$$\sigma_1 p_x S_1 - \sigma_1 p_y C_1 - \sigma_2 a_3 S_3 = r_3 \quad (5.66)$$

$$0 = r_4. \quad (5.67)$$

Equations (5.64) and (5.67) impose constraints on pose parameters t_z and p_z. Here again a solution can be obtained in form of a quartic polynomial equation in $t_1=\tan(\theta_1/2)$ by eliminating θ_3 from Eqs. (5.65) and (5.66) as we did earlier in case 1.

With θ_1 known, θ_3 can be uniquely obtained from Eqs. (5.65) and (5.66) and the solution set completed as usual. This method puts an upper bound of 4 on the number of solution sets since at most four distinct values of θ_1 can be obtained from the quartic polynomial equation in t_1.

An easier inverse kinematic analysis of this structure can be obtained if the roles of end-effector frame and base frame are reversed and the intermediate link-frames are reassigned accordingly. This will put the three intersecting axes at the end-effector position instead of at the base. The 4-DOF structure is seen to be equivalent to one that has $a_2 = a_3 = 0$ which is discussed in case 10.

With this analysis, we find that there can be at most two solution sets.

Case 9: a_1=a3=0. The first two joint axes intersect and the last two joint axes intersect. The reduced system is

$$a_2\sigma_3 S_3 - \sigma_2\sigma_3 d_2 C_3 = r_1 \tag{5.68}$$
$$\sigma_1 t_x S_1 - \sigma_1 t_y C_1 + \sigma_2\sigma_3 C_3 = r_2 \tag{5.69}$$
$$\sigma_1 p_x S_1 - \sigma_1 p_y C_1 = r_3 \tag{5.70}$$
$$0 = r_4. \tag{5.71}$$

Here, $r_4 = 0$ poses a constraint on pose parameter p_z. Equation (5.70) yields two distinct values of θ_1, then Eqs. (5.68) and (5.69) will form a linear system in S_3 and C_3 which can be solved for a unique value of θ_3 for each value of θ_1. Two solution sets are thus obtained.

Case 10: $a_2=a_3=0$. The intermediate two joint axes intersect and the last two joint axes intersect. The reduced system becomes

$$a_1 t_y S_1 + a_1 t_x C_1 - \sigma_2\sigma_3 d_2 C_3 = r_1 \tag{5.72}$$
$$\sigma_1 t_x S_1 - \sigma_1 t_y C_1 + \sigma_2\sigma_3 C_3 = r_2 \tag{5.73}$$
$$\sigma_1 p_x S_1 - \sigma_1 p_y C_1 = r_3 \tag{5.74}$$
$$a_1 p_y S_1 + a_1 p_x C_1 = r_4. \tag{5.75}$$

Equations (5.74) and (5.75) yield a unique solution for θ_1. The value of θ_1 obtained can be substituted in Eq. (5.72) or (5.73) to solve for C_3 which provides two possible values for θ_3. The unused equation is a constraint on the end-effector pose parameter t_z.

When $a_3 = 0$ or $d_2 = 0$, the conditions for $d_H=0$ obtained have all been already discussed, hence there are only ten minimal pose-independent arm geometry conditions for which the reduced system is singular. In all ten cases, we found at most two distinct inverse kinematic solution sets.

To summarize the above cases, we find that a four-DOF robot manipulator will in general have a unique inverse kinematic solution set. At most two solution sets can be found when the arm has one of the following special structures:

1. Three consecutive joint axes that are parallel.
2. Three consecutive joint axes intersect.
3. Two consecutive pairs of parallel or intersecting joint axes.
4. Three consecutive joint axes such that two intersect and two are parallel.
5. Three consecutive joint axes such that the first two intersect and the last two intersect.

CHAPTER 6
SOLVING FIVE-DOF MANIPULATORS

One-Dimensional Iterative Technique

With five degrees of freedom, Eq. (2.14) takes the form

$$A_1 A_2 A_3 A_4 A_5 = P. \tag{6.1}$$

and after multiplying both sides of this equation by A_1^{-1}, we obtain

$$A_2 A_3 A_4 A_5 = Q \tag{6.2}$$

with

$$Q = A_1^{-1} P. \tag{6.3}$$

When Θ_1 is known, matrix Q is fully determined and can be viewed as a pose matrix for a 4-DOF arm whose structure is described by the left hand side of Eq. (6.2) which merely expresses a 4-DOF problem. In Chapter 5, we have seen that a 4-DOF problem can always be solved in closed-form, hence the remaining joint variables can be computed as shown earlier.

Since we only need to know one of the joint variables to solve for the whole solution set, the inverse kinematics problem of five-DOF manipulators reduces to finding the value of the first joint variable only, and getting closed-form values for the remaining variables.

Let the column vectors of pose matrix Q of Eq. (6.3) be given by $\mathbf{m}$, $\mathbf{c}$, $\mathbf{u}$, and $\mathbf{q}$, in order, so that

$$Q = \begin{bmatrix} m_x & c_x & u_x & q_x \\ m_y & c_y & u_y & q_y \\ m_z & c_z & u_z & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{m} & \mathbf{c} & \mathbf{u} & \mathbf{q} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (6.4)$$

then

$$\mathbf{u} = \mathbf{R}_1^{-1}\ \mathbf{R}\ \mathbf{z} = \mathbf{R}_1^{-1}\ \mathbf{t} \qquad (6.5)$$

and

$$\mathbf{q} = \mathbf{R}_1^{-1}\ \mathbf{p} - \mathbf{R}_1^{-1}\ \mathbf{l}_1 = \mathbf{R}_1^{-1}(\mathbf{p} - \mathbf{l}_1). \qquad (6.6)$$

From the left hand side of Eq. (6.2), two vectors $\mathbf{u}_L$ and $\mathbf{q}_L$, corresponding to vectors $\mathbf{u}$ and $\mathbf{q}$, are given by

$$\mathbf{u}_L = \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{z} \qquad (6.7)$$

and

$$\mathbf{q}_L = \mathbf{R}_2(\mathbf{R}_3\ \mathbf{l}_4 + \mathbf{l}_3) + \mathbf{l}_2. \qquad (6.8)$$

A nonlinear function of Θ_1 can be defined as a difference between corresponding quantities from the left and the right side of Eq. (6.2). For example, the difference between the inner-products $(\mathbf{u}_L.\mathbf{q}_L)$ and $(\mathbf{u}.\mathbf{q})$ yields the function

$$f(\Theta_1) = \mathbf{u}_L.\mathbf{q}_L - \mathbf{u}.\mathbf{q}. \qquad (6.9)$$

If the value of Θ_1 used to compute pose matrix Q in Eq. (6.3) does correspond to a solution set, then Eq. (6.2) will hold, vectors $\mathbf{u}_L$ and $\mathbf{q}_L$ will be exactly equal to $\mathbf{u}$ and $\mathbf{q}$, respectively, and function f will equal zero. In other words solution sets of Eq. (6.1) correspond to zeros of function f defined in Eq. (6.9). Hence, the inverse kinematics problem of 5-DOF robot manipulators reduces to solving the one-dimensional equation

$$f(\Theta_1) = 0.$$

The zeros of f can be found by use of any suitable one-dimensional technique such as Newton-Raphson or the secant method. Once Θ_1 is known, the solution set can be completed by solving Eq. (6.2) in closed form as we showed in Chapter 5. The solution set can then be checked for consistency with Eq. (6.1) to determine whether the one found is extraneous or not because the zeros of f are not always part of a solution set of the manipulator.

Computing $f(\Theta_1)$. Using Eqs. (6.7) and (6.8), the inner product $\mathbf{u}_L \cdot \mathbf{q}_L$ is given by

$$\mathbf{u}_L \cdot \mathbf{q}_L = (\mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{z}) \cdot (\mathbf{R}_2(\mathbf{R}_3\ \mathbf{l}_4 + \mathbf{l}_3) + \mathbf{l}_2).$$

If we apply properties (4.5) and (4.6) repeatedly, this last equation becomes

$$\mathbf{u}_L \cdot \mathbf{q}_L = \mathbf{z} \cdot (\mathbf{R}_4^{-1}\mathbf{l}_4) + \mathbf{z} \cdot (\mathbf{R}_4^{-1}\mathbf{R}_3^{-1}\mathbf{l}_3) + \mathbf{z} \cdot (\mathbf{R}_4^{-1}\mathbf{R}_3^{-1}\mathbf{R}_2^{-1}\mathbf{l}_2)$$

or after computing the z-components of the terms in parentheses,

$$
\begin{aligned}
\mathbf{u}_L \cdot \mathbf{q}_L = {} & \tau_4 d_4 + a_3 \sigma_4 \, S_4 - \sigma_3 d_3 \sigma_4 \, C_4 + \tau_4 d_3 \\
& + \sigma_4 \, S_4 \, (a_2 \, C_3 + \sigma_2 d_2 \, S_3) \\
& - \sigma_4 \, C_4 \, (-a_2 \tau_3 \, S_3 + \sigma_2 d_2 \tau_3 \, C_3 + \tau_2 d_2 \sigma_3) \\
& + \tau_4 \, (a_2 \sigma_3 \, S_3 - \sigma_2 d_2 \sigma_3 \, C_3 + \tau_2 d_2 \tau_2) . \qquad (6.10)
\end{aligned}
$$

This last equation shows that Θ_3 and Θ_4 must be known before we can compute $\mathbf{u}_L \cdot \mathbf{q}_L$. With Θ_1 known, Θ_3 and Θ_4 can be obtained by solving Eq. (6.2) as described in Chapter 5. The coordinates of vectors **u** and **q** and the inner-products **u.q**, and **q.q** are necessary for the 4-DOF inverse kinematic method of Chapter 5. Equation (6.5) yields

$$
\mathbf{u} = R_1^{-1} \, \mathbf{t} = \begin{bmatrix} t_x \, C_1 + t_y \, S_1 \\ -\tau_1 t_x \, S_1 + \tau_1 t_y \, C_1 + \sigma_1 t_z \\ \sigma_1 t_x \, S_1 - \sigma_1 t_y \, C_1 + \tau_1 t_z \end{bmatrix}, \qquad (6.11)
$$

and from Eq. (6.6),

$$
\mathbf{q} = \begin{bmatrix} p_x \, C_1 + p_y \, S_1 - a_1 \\ -\tau_1 p_x \, S_1 + \tau_1 p_y \, C_1 + \sigma_1 p_z \\ \sigma_1 p_x \, S_1 - \sigma_1 p_y \, C_1 + \tau_1 p_z \end{bmatrix}, \qquad (6.12)
$$

where we have assumed $(R_1^{-1} \, \mathbf{l}_1) = [a_1, \; 0, \; 0]^T$ since $d_1 = 0$ by proper positioning of frame F_o.

The inner-products **u.q** and **q.q** can then be easily computed by

$$\mathbf{u}\cdot\mathbf{q} = u_x q_x + u_y q_y + u_z q_z$$

and

$$\mathbf{q}\cdot\mathbf{q} = q_x^2 + q_y^2 + q_z^2,$$

when the numeric values of the components of **u** and **q** have been obtained. These inner products can be obtained from Eqs. (6.5) and (6.6) as well,

$$\mathbf{u}\cdot\mathbf{q} = (R_1^{-1}\,\mathbf{t})\ \cdot\ (R_1^{-1}(\mathbf{p} - \mathbf{l}_1)) = \mathbf{t}\ \cdot\ (\mathbf{p} - \mathbf{l}_1),$$

$$\mathbf{u}\cdot\mathbf{q} = t_x\ (p_x - a_1\ C_1) + t_y\ (p_y - a_1\ S_1) \qquad (6.13)$$

and

$$\mathbf{q}\cdot\mathbf{q} = R_1^{-1}(\mathbf{p} - \mathbf{l}_1)\ \cdot\ R_1^{-1}(\mathbf{p} - \mathbf{l}_1) = (\mathbf{p} - \mathbf{l}_1)\cdot(\mathbf{p} - \mathbf{l}_1),$$

$$\mathbf{q}\cdot\mathbf{q} = \mathbf{p}\cdot\mathbf{p} + \mathbf{l}_1\cdot\mathbf{l}_1 - 2(\mathbf{p}\cdot\mathbf{l}_1)$$

$$\mathbf{q}\cdot\mathbf{q} = \mathbf{p}\cdot\mathbf{p} + a_1^2 - 2\ (p_x a_1\ C_1 + p_y a_1\ S_1). \qquad (6.14)$$

Equations (6.11)-(6.14) clearly show that all components of **u** and **q**, and the inner-products **u.q** and **q.q** are linear functions of S_1 and C_1, a result that will prove useful in the next section.

To summarize, $f(\Theta_1)$ can be computed for a given value of Θ_1 according to the following steps:

Step 1. From the current estimate of Θ_1, Compute the components of **u** and **q** and the inner products **u.q** and **q.q** as shown in Eqs. (6.11)-(6.14).

Step 2. Compute Θ_2 and Θ_3 from the reduced system of equations

$$a_2u_y\ S_2 + a_2u_x\ C_2 + a_3\sigma_4\ S_4 - \sigma_3\sigma_4 d_3\ C_4 = r_1 \tag{6.15}$$

$$\sigma_2u_x\ S_2 - \sigma_2u_y\ C_2 + \sigma_3\sigma_4\ C_4 = r_2 \tag{6.16}$$

$$\sigma_2q_x\ S_2 - \sigma_2q_y\ C_2 - \sigma_3a_4\ S_4 = r_3 \tag{6.17}$$

$$a_2q_y\ S_2 + a_2q_x\ C_2 + \sigma_3a_4d_3\ S_4 + a_3a_4\ C_4 = r_4 \tag{6.18}$$

with

$$r_1 = \mathbf{q}\cdot\mathbf{u} - \tau_4d_4 - d_2u_z - \tau_3\tau_4d_3 \tag{6.19}$$

$$r_2 = \tau_3\tau_4 - \tau_2u_z \tag{6.20}$$

$$r_3 = \tau_2(d_2 - q_z) + d_3 + \tau_3d_4 \tag{6.21}$$

$$r_4 = (\mathbf{q}\cdot\mathbf{q}+a_2^2+d_2^2-a_3^2-d_3^2-a_4^2-d_4^2)/2 - d_2q_z - \tau_3d_3d_4, \tag{6.22}$$

derived from Eqs. (5.13)-(5.20) by proper index substitution (the indexes are incremented to fit the 4-DOF problem of Eq. (6.2)). Vectors **u** and **q** play the roles of vectors **t** and **p** respectively. The last system of equations gives the values of Θ_2 and Θ_4. Equations (5.23) and (5.24), with the proper index changes,

$$(\sigma_3d_4-\tau_3a_4S_4)\ S_3 + (a_3+a_4C_4)\ C_3 = C_2q_x + S_2q_y - a_2 \tag{6.23}$$

and

$$(a_3+a_4C_4)\ S_3 - (\sigma_3d_4-\tau_3a_4S_4)\ C_3 = -\tau_2S_2q_x + \tau_2C_2q_y + \sigma_2(q_z-d_2), \tag{6.24}$$

can be solved for Θ_3. Another way to obtain Θ_3 is by using the equations

$$(\tau_3\sigma_4C_4+\sigma_3\tau_4)\ S_3 + \sigma_4S_4\ C_3 = C_2u_x + S_2u_y \tag{6.25}$$

and

$$\sigma_4S_4\ S_3 - (\tau_3\sigma_4C_4+\sigma_3\tau_4)\ C_3 = -\tau_2S_2u_x + \tau_2C_2u_y + \sigma_2u_z, \tag{6.26}$$

derived from Eqs. (5.25) and (5.26) by incrementing the indexes. With Θ_1, Θ_2, and Θ_3 known, $\mathbf{u}_L\cdot\mathbf{q}_L$ can be computed as in Eq. (6.10) and $f(\Theta_1)$ is then given by Eq. (6.8).

The ability to compute $f(\Theta_1)$ when Θ_1 is given is sufficient to implement a practical Newton-Raphson algorithm for finding the zeros of function f. The algorithm can be programmed according to the following steps:

Step 1. Obtain an initial estimate for Θ_1. As for all iterative methods, the closer the initial estimate of Θ_1 is to a true solution, the faster the convergence will be. If the end-effector of the robot is tracking a trajectory given as a finite set of end-effector poses, a good estimate for finding the solution set for a pose along the trajectory is the value of Θ_1 corresponding to the preceding pose on the trajectory.

Step 2. Compute Θ_3 and Θ_4 and then $f(\Theta_1)$ as described earlier.

Step 3. Compute the derivative $df/d\Theta_1$ of f with respect to Θ_1. A numeric approximation of this derivative is given by

$$df/d\Theta_1 = [f(\Theta_1+\delta\Theta_1)-f(\Theta_1)]/\delta\Theta_1, \qquad (6.27)$$

where $\delta\Theta_1$ is a small increment of Θ_1. Note that this approximation requires another function evaluation at $(\Theta_1+\delta\Theta_1)$.

Step 4. Obtain a new estimate for Θ_1 by the one-dimensional Newton-Raphson method, i.e.

$$\Theta_{1(new)} = \Theta_1 - f(\Theta_1)/(df/d\Theta_1). \qquad (6.28)$$

Steps 2 to 4 must be repeated until Θ1 is obtained to the desired accuracy. The solution set can then be completed by using the values of Θ2, Θ3 and Θ4 as computed at the last iteration and by computing Θ5 uniquely from

$$R_5\ \mathbf{x} = \begin{bmatrix} C_5 \\ S_5 \\ 0 \end{bmatrix} = R_4^{-1}\ R_3^{-1}\ R_2^{-1}\ R_1^{-1}\ \mathbf{m}.$$

and

$$\Theta_5 = \mathrm{atan2}(S_5, C_5).$$

The one-dimensional method just described is flexible in terms of the choice of function f to be used. A different function can be implemented. The only requirements are that $f(\Theta_1)$ be computable for any value of

Θ_1 and some known relationship between the zeros of f and the solution sets of Eq. (6.1). For example, another choice of f may be

$$f(\Theta_1) = \mathbf{q}_L \cdot \mathbf{q}_L - \mathbf{q} \cdot \mathbf{q} \tag{6.29}$$

or any difference between corresponding quantities from the left and right side of Eq. (6.2). The function choice is important in terms of minimal computation complexity and filtering of extraneous solutions which are discussed next. In all practical experiments the function defined in Eq. (6.8) has given good results.

<u>Extraneous solutions</u>. An extraneous solution set is one that the iterative method converges to, i.e. it satisfies the reduced system of equations (6.15)-(6.18) and $f(\Theta_1) = 0$ but yet it is not a solution for Eq. (6.1). The iterative method just described may converge to such a set. This problem was also reported by Tsai and Morgan (1984) who developed a different inverse kinematic method that makes use of a similar reduced system of equations.

In deriving the reduced system of equations (5.13)-(5.16) in chapter 5, vectors **u** and **q**, and the inner-products **u.q** and **q.q** are the only pose related quantities that were involved. This means that a solution set obtained by convergence of the method just described does not necessarily satisfy other pose requirements from Eq. (6.1). Extraneous solutions can be filtered out by a choice of function f that constrains more of the end-effector pose elements at the expense of computation time or by checking all solutions found for consistency with one or more end-effector pose elements.

Iterating on θ_5. An equivalent one-dimensional iterative technique can be implemented based on a function of θ_5 instead of θ_1. Recall from Chapter 2 that the homogeneous matrix $\mathbf{A}_4$ decomposes into

$$\mathbf{A}_4 = \underline{\mathbf{A}}_4 \mathbf{B}_4$$

where $\mathbf{B}_4$ is a homogeneous matrix fully determined by parameters a_4, d_4, and α_4 and independent of Θ_4. Right-multiplication of Eq. (6.1) by $(\mathbf{A}_5{}^{-1}\mathbf{B}_4{}^{-1})$ yields

$$\mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 \underline{\mathbf{A}}_4 = \mathbf{Q} \tag{6.30}$$

with

$$\mathbf{Q} = \mathbf{P} \mathbf{A}_5{}^{-1}\mathbf{B}_4{}^{-1}. \tag{6.31}$$

When θ_5 is given, matrix $\mathbf{Q}$ becomes a known pose matrix for the 4-DOF problem expressed by equation (6.30). Vectors $\mathbf{u}$ and $\mathbf{q}$ are given by

$$\mathbf{u} = \mathbf{R} \mathbf{R}_5{}^{-1}\mathbf{G}_4{}^{-1} \mathbf{z} \tag{6.32}$$

and

$$\mathbf{q} = \mathbf{R} \mathbf{R}_5{}^{-1}(-\mathbf{G}_4{}^{-1}\mathbf{k}_4) + \mathbf{p}, \tag{6.33}$$

where $\mathbf{G}_4$ is the rotation part of homogeneous matrix $\mathbf{B}_4$,

$$\mathbf{B}_4 = \begin{bmatrix} 1 & 0 & 0 & a_4 \\ 0 & \tau_4 & -\sigma_4 & 0 \\ 0 & \sigma_4 & \tau_4 & d_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{k}_4 = \begin{bmatrix} a_4 \\ 0 \\ d_4 \end{bmatrix}$$ is the position vector

of $\mathbf{B}_4$, and $\mathbf{R}$ and $\mathbf{p}$ are the usual rotation matrix and position vector of end-effector pose $\mathbf{P}$. Explicitly, we get

$$\mathbf{u} = \begin{bmatrix} n_x\sigma_4 \; S_5 + b_x\sigma_4 \; C_5 + t_x\tau_4 \\ n_y\sigma_4 \; S_5 + b_y\sigma_4 \; C_5 + t_y\tau_4 \\ n_z\sigma_4 \; S_5 + b_z\sigma_4 \; C_5 + t_z\tau_4 \end{bmatrix} \tag{6.34}$$

and

$$\mathbf{q} = \begin{bmatrix} (-n_x\sigma_4 d_4 + b_x a_4) \; S_5 \\ \quad - (n_x a_4 + b_x\sigma_4 d_4) \; C_5 - t_x\tau_4 d_4 + p_x \\ (-n_y\sigma_4 d_4 + b_y a_4) \; S_5 \\ \quad - (n_y a_4 + b_y\sigma_4 d_4) \; C_5 - t_y\tau_4 d_4 + p_y \\ (-n_z\sigma_4 d_4 + b_z a_4) \; S_5 \\ \quad - (n_z a_4 + b_z\sigma_4 d_4) \; C_5 - t_z\tau_4 d_4 + p_z \end{bmatrix} \tag{6.35}$$

Expressions of inner-products $\mathbf{u}.\mathbf{q}$ and $\mathbf{q}.\mathbf{q}$ in terms of S_5 and C_5 can be obtained from Eqs. (6.32) and (6.33),

$$\mathbf{u}.\mathbf{q} = [\mathbf{R}\; \mathbf{R}_5^{-1}\mathbf{G}_4^{-1}\; \mathbf{z}] \cdot [\mathbf{R}\; (\mathbf{R}_5^{-1}(-\mathbf{G}_4^{-1}\mathbf{k}_4) + \mathbf{p}]$$

and

$$\mathbf{q}\cdot\mathbf{q} = [R\ (R_5^{-1}(-G_4^{-1}\mathbf{k}_4) + \mathbf{p}]\ \cdot\ [R\ (R_5^{-1}(-G_4^{-1}\mathbf{k}_4) + \mathbf{p}].$$

With the use of properties (4.5) and (4.6) as necessary and rearranging terms, the equations yield

$$\mathbf{u}\cdot\mathbf{q} = \sigma_4\ [(\mathbf{n}\cdot\mathbf{p})\ S_5 + (\mathbf{b}\cdot\mathbf{p})\ C_5] + \tau_4(\mathbf{t}\cdot\mathbf{p}) - d_4 \qquad (6.36)$$

and

$$\begin{aligned}\mathbf{q}\cdot\mathbf{q} = & -2[\sigma_4 d_4\ (\mathbf{n}\cdot\mathbf{p}) - a_4\ (\mathbf{b}\cdot\mathbf{p})]\ S_5 \\ & - 2[a_4\ (\mathbf{n}\cdot\mathbf{p}) + \sigma_4 d_4\ (\mathbf{b}\cdot\mathbf{p})]\ C_5 \\ & - 2\tau_4 d_4\ (\mathbf{t}\cdot\mathbf{p}) + a_4^2 + d_4^2 + \mathbf{p}\cdot\mathbf{p},\end{aligned} \qquad (6.37)$$

where we used the fact that

$$R^{-1}\ \mathbf{p} = \begin{bmatrix} \mathbf{n}\cdot\mathbf{p} \\ \mathbf{b}\cdot\mathbf{p} \\ \mathbf{t}\cdot\mathbf{p} \end{bmatrix} = \begin{bmatrix} n_x p_x + n_y p_y + n_z p_z \\ b_x p_x + b_y p_y + b_z p_z \\ t_x p_x + t_y p_y + t_z p_z \end{bmatrix}.$$

Here again, we note that u_z, p_z, $\mathbf{u}\cdot\mathbf{q}$, and $\mathbf{q}\cdot\mathbf{q}$ are linear functions of S_5 and C_5.

With the components of $\mathbf{u}$ and $\mathbf{q}$ and the inner products $\mathbf{u}\cdot\mathbf{q}$ and $\mathbf{q}\cdot\mathbf{q}$ computed, a one-dimensional iterative method can be implemented as described earlier with a function $f(\Theta_5)$ given by

$$f(\Theta_5) = \mathbf{u}_L\cdot\mathbf{q}_L - \mathbf{u}\cdot\mathbf{q} \qquad (6.38)$$

which will converge to a value of Θ_5.

5-DOF Robots with Closed-Form Solution

Certain Five-degree-of-freedom robots with simple geometries may yield inverse kinematic equations that can be solved directly and without need for a numeric technique such as Newton-Raphson. In Chapter 5, some particular 4-DOF robot structures were found for which the reduced system of equations (5.13)-(5.16) was overspecified i.e. the matrix **H** of the linear system was singular. The analysis of these special geometries showed that one or two of the four equations of the reduced system became constraint equations on pose elements, particularly, elements t_z, p_z, $\mathbf{t}\cdot\mathbf{p}$, and $\mathbf{p}\cdot\mathbf{p}$.

In the case of 5-DOF robots, the quantities u_z, q_z, $\mathbf{u}\cdot\mathbf{q}$, and $\mathbf{q}\cdot\mathbf{q}$ ($\mathbf{u}$ playing the role of $\mathbf{t}$ and $\mathbf{q}$ that of $\mathbf{p}$) are either linear functions of S_1 and C_1 as shown in Eqs. (6.11)-(6.14) or linear functions of S_5 and C_5 as shown in Eqs. (6.34)-(6.37). Either way, the constraint equations described in the ten cases of chapter 5 can be used to solve for the correct value of θ_1 or θ_5 directly without need for an iterative technique. This result means that if a 5-DOF robot manipulator has a 4-DOF section with one of the special geometries discussed in Chapter 5, then the arm can be solved in closed form. We now proceed to prove this point.

The 5-DOF inverse kinematics problem of Eq. (6.1) can be reduced to the 4-DOF one of Eq. (6.2). In this case, the reduced system of equations (6.15)-(6.18) must be solved. By substituting the expressions of u_z, q_z, $\mathbf{u}.\mathbf{q}$, and $\mathbf{q}.\mathbf{q}$ from Eqs. (6.11)-(6.14) into Eqs. (6.19)-(6.22) and rearranging, we get

$$r_1 = (-a_1 t_y - \sigma_1 d_2 t_x)\ S_1 + (-a_1 t_x + d_2 \sigma_1 t_y)\ C_1 + t_x p_x - t_y p_y - \tau_1 d_2 t_z - \tau_3 d_3 \tau_4 - \tau_4 d_4, \qquad (6.39)$$

$$r_2 = -\sigma_1 \tau_2 t_x\ S_1 + \sigma_1 \tau_2 t_y\ C_1 - \tau_1 \tau_2 t_z + \tau_3 \tau_4, \qquad (6.40)$$

$$r_3 = -\sigma_1 \tau_2 p_x\ S_1 + \sigma_1 \tau_2 p_y\ C_1 - \tau_1 \tau_2 p_z + \tau_2 d_2 + d_3 + \tau_3 d_4, \qquad (6.41)$$

and

$$r_4 = (-a_1 p_y - \sigma_1 d_2 p_x)\ S_1 + (-a_1 p_x + d_2 \sigma_1 p_y)\ C_1 - \tau_1 d_2 p_z + \tau_3 d_3 d_4 + (\mathbf{p}.\mathbf{p} + a_1^2 + a_2^2 + d_2^2 - a_3^2 - d_3^2 - a_4^2 - d_4^2). \qquad (6.42)$$

These last four equations prove that the terms r_1, r_2, r_3, and r_4 are all of the form

$$r_i = r_{i1}\ S_1 + r_{i2}\ C_1 + r_{i3},\ i=1,\ .\ .\ .\ ,\ 4,$$

where the constants r_{ij} are fully determined by the arm parameters and the end-effector pose elements.

Another choice is to use Eq. (6.30). The reduced system of equations is given by Eqs. (5.13)-(5.16) with all elements of pose matrix $\mathbf{P}$ replaced by corresponding elements of matrix Q of Eq. (6.31). The r_i quantities become linear expressions in S_5 and C_5 and have the form

$$r_i = r_{i1}\ S_5\ r_{i2}\ C_5 + r_{i3},\ i=1, \ . \ . \ . \ , 4.$$

Indeed, if we replace u_z, q_z, $\mathbf{u.q}$, and $\mathbf{q.q}$ by their expressions in terms of S_5 and C_5, as given by Eqs. (6.34)-(6.38) and substitute in Eqs. (5.17)-(5.20) (substitute for t_z, p_z, $\mathbf{t.p}$, and $\mathbf{p.p}$ and let d_1=0), we obtain

$$r_1 = \sigma_4(\mathbf{n.p})\ S_5 + \sigma_4(\mathbf{b.p})\ C_5 + \tau_4(\mathbf{t.p}) - \tau_3 d_3 - \tau_2 d_2 \tau_3 - d_4, \tag{6.43}$$

$$r_2 = -\tau_1\sigma_4 n_z\ S_5 - \tau_1\sigma_4 b_z\ C_5 - \tau_1\tau_4 t_z + \tau_2\tau_3, \tag{6.44}$$

$$r_3 = (\tau_1\sigma_4 d_4 n_z - \tau_1 a_4 b_z)\ S_5 + (\tau_1 a_4 n_z + \tau_1\sigma_4 d_4 b_z)\ C_5 + (\tau_1\tau_4 d_4 t_z - \tau_1 p_z) + d_2 + \tau_2 d_3, \tag{6.45}$$

and

$$\begin{aligned} r_4 = & -2[\sigma_4 d_4\ (\mathbf{n.p}) - a_4\ (\mathbf{b.p})]\ S_5 \\ & - 2[a_4\ (\mathbf{n.p}) + \sigma_4 d_4\ (\mathbf{b.p})]\ C_5 - 2\tau_4 d_4\ (\mathbf{t.p}) - \tau_2 d_2 d_3 \\ & + (\mathbf{p.p} + a_1^2 - a_2^2 - d_2^2 - a_3^2 \\ & - d_3^2 + a_4^2 + d_4^2)/2. \end{aligned} \tag{6.46}$$

In the analysis of special four-DOF geometries in Chapter 5, we found cases where the reduced system of equations included a constraint of the form $r_i = 0$. Such a constraint applied to one of Eqs. (6.39)-(6.46) will usually yield a value of θ_1 or θ_5 which in turn makes the 5-DOF inverse kinematics problem solvable in closed form.

<u>Case 1</u>: Three joint axes are parallel. When the parallel axes are the first three (i.e. axes 1, 2 and 3), Eq. (6.30) can be used. Case 1 of Chapter 5 shows that

$r_2=0$ and $r_3=0$. These two constraints and Eqs. (6.44) and (6.45) yield a system of equations in S_5 and C_5,

$$r_{21}\, S_5 + r_{22}\, C_5 = -r_{23}$$
$$r_{31}\, S_5 + r_{32}\, C_5 = -r_{33},$$

which can be solved for Θ_5 when the determinant given by $(r_{21}r_{32} - r_{31}r_{22})$ is not zero, otherwise there is no solution. With Θ_5 known, the remaining angles can be obtained in closed-form.

If the last three axes are parallel, a similar result is obtained by exchanging the roles of base and end-effector frames. When the intermediate axes are parallel, Eq. (6.2) should be used. The constraints $r_2 = r_3 = 0$ then yield a value of Θ_1 and the inverse kinematic problem can be solved in closed form as well.

<u>Case 2</u>: two consecutive sets of two parallel axes. If axes 1 and 2 are parallel and axes 3 and 4 are parallel, Eq. (6.30) and Chapter 5, case 2 yield $r_2 = 0$ which can be used to solve for Θ_5 from Eq. (6.44). If axes 2 and 3 are parallel and axes 4 and 5 are parallel, then using Eq. (6.2) and Eq. (6.40) will yield a value of Θ_1.

<u>Case 3</u>: Two parallel axes followed by two intersecting axes. When this special geometry concerns the first four joint axes of the 5-DOF arm, using Eq. (6.30) and Chapter 5, case 3 yields $r_3 = 0$. This constraint applied to Eq. (6.45) gives a value of Θ_5. If the upper part of the 5-DOF robot

has the special structure, Eq. (6.2) can be used and the constraint $r_3 = 0$ applies to Eq. (6.41). Angle θ_1 can be directly computed.

Case 4: Two intersecting axes followed by two parallel axes. This structure corresponds to Chapter 5, case 6. Here, the constraint is $r_1 = 0$ and, as in the preceding cases, θ_1 or θ_5 can be directly computed from Eq. (6.39) or (6.43), respectively.

Case 5: Three intersecting axes. Pieper (1968) has shown that a six-DOF manipulator with three intersecting axes can always be solved in closed form. This result applies to the simpler case of five-DOF robots. This structure corresponds to case 8 of Chapter 5 where the constraints are $r_1=0$ and $r_4=0$. If the three intersecting axes are the first three, Equation (6.30) should be used. With Eqs. (15) and (6.46), a value of θ_5 can be obtained directly. This same method can be used when the last three axes intersect by first exchanging the roles of end-effector and base frames. When the intermediate three axes are intersecting, use of Eqs. (6.2), (6.39) and (6.42) will yield a value of θ_1.

Case 6: Two consecutive sets of two intersecting axes. This structure is analyzed in case 9 of Chapter 5. The constraint equation is $r_4 = 0$. Depending on where this special structure is located along the five-DOF arm, θ_1 or

Θ_5 can be directly computed by use of Eqs. (6.42) or (6.46) respectively.

In the special geometries described in Chapter 5, cases 5, 7, and 10, we did not find a constraint of the form $r_i=0$, yet a five-DOF arm having one of these particular geometries can still be solved in closed form. We now study these special cases as they apply to five-degree-of-freedom robots.

<u>Case 7</u>: Three joint axes are such that they either intersect or they are parallel two at a time. This type of structure is studied in cases 5, 7, and 10 of Chapter 5. Assuming this geometry concerns axes 3, 4, and 5 of the five-DOF arm, Eq. (6.2) should be used. From Chapter 5, case 5 and case 10, we see that the last two equations of the reduced system, Eqs. (6.17) and (6.18) have the form

$$\sigma_2(q_x S_2 - q_y C_2) = r_3$$
$$a_2(q_y S_2 + q_x C_2) = r_4$$

where q_x, q_y, r_3, and r_4 are all linear expressions in S_1 and C_1. A quartic polynomial equation in $t_1 = \tan(\Theta_1/2)$ is readily obtained by squaring and adding the last two equations,

$$q_x^2 + q_y^2 = (r_3/\sigma_2)^2 + (r_4/a_2)^2,$$

and substituting $S_1 = 2t_1/(1+t_1^2)$ and $C_1 = (1-t_1^2)/(1+t_1^2)$. This polynomial can be solved for Θ_1 and the solution set completed as described earlier. Similarly, from case 7 of Chapter 5, we get the equations

$$a_2(u_y S_2 + u_x C_2) = r_1$$

$$\sigma_2(u_x S_2 - u_y C_2) = r_2$$

corresponding to Eqs.(6.15) and (6.16) of the reduced system of equations. Here again a quartic polynomial equation in t_1 is obtained by eliminating S_2 and C_2.

When the three axes with the special geometry are located elsewhere along the five-DOF structure, a similar result can be obtained by using equation (6.30) or by exchanging the roles of base and end-effector frames.

To summarize the above cases, we find that a 5-DOF robot manipulator will allow closed-form solutions if any of the following conditions is satisfied:

1. Three consecutive joint axes are parallel.
2. Three consecutive joint axes intersect.
3. There are two consecutive sets of two joint axes that are either parallel or intersecting.
4. Three consecutive joint axes are such that two intersect and two are parallel.
5. Three consecutive joint axes are such that the first two intersect and the last two intersect.

Note that these conditions are not exclusive of one another. For example, arms that satisfy condition 5 include those that satisfy condition 2.

CHAPTER 7
SOLVING SIX-DOF MANIPULATORS

Reduction to a 4-DOF Problem

At least six degrees of freedom are required for a robot manipulator to be able to arbitrarily position and orient its end-effector within its workspace. Equation (2.14), with n equal to six, yields

$$\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4\ \mathbf{A}_5\ \mathbf{A}_6 = \mathbf{P}. \tag{7.1}$$

If both sides of this equality are multiplied by $(\mathbf{A}_1\ \mathbf{A}_2)^{-1}$, the equation becomes

$$\mathbf{A}_3\ \mathbf{A}_4\ \mathbf{A}_5\ \mathbf{A}_6 = \mathbf{Q} \tag{7.2}$$

with

$$\mathbf{Q} = \mathbf{A}_2^{-1}\ \mathbf{A}_1^{-1}\ \mathbf{P}. \tag{7.3}$$

When θ_1 and θ_2 are known, matrix $\mathbf{Q}$ is fully determined and can be viewed as a pose matrix for a 4-DOF arm whose structure is described by the left hand side of Eq. (7.2) which merely expresses a 4-DOF problem. In Chapter 5, we have seen that a 4-DOF problem can always be solved in closed-form, hence the remaining joint variables can be computed from Eq. (7.2).

First we show that a similar result can be obtained if Θ_5 and Θ_6 are known or if Θ_1 and Θ_6 are known instead of the first two joint variables.

In the development of the 4-DOF inverse kinematics solution, we have used the simplifying assumption that the last frame had DH-parameters d, a, and α all equal to zero. Although this assumption is obviously correct in the case of Eq. (7.2), we must show that it can be obtained in other cases. As shown in Eq. (2.7), a homogeneous matrix $\mathbf{A}_i$ decomposes into $\mathbf{A}_i = \underline{\mathbf{A}}_i\ \mathbf{B}_i$ where $\underline{\mathbf{A}}_i$ and $\mathbf{B}_i$ are given by Eqs. (2.8) and (2.9) and $\underline{\mathbf{A}}_i$ is a homogeneous matrix for which DH-parameters a, d, and α are zero. If the values of Θ_1 and Θ_6 are known, Equation (7.1) now reduces to the 4-DOF problem

$$\mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4\ \underline{\mathbf{A}}_5 = \mathbf{Q} \tag{7.4}$$

where

$$\mathbf{Q} = \mathbf{A}_1^{-1}\ \mathbf{P}\ \mathbf{A}_6^{-1}\ \mathbf{B}_5^{-1}. \tag{7.5}$$

Similarly, If Θ_5 and Θ_6 are known, the inverse kinematic task becomes that of solving the 4-DOF case

$$\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \underline{\mathbf{A}}_4 = \mathbf{Q}, \tag{7.6}$$

with

$$\mathbf{Q} = \mathbf{P}\ \mathbf{A}_6^{-1}\ \mathbf{A}_5^{-1}\ \underline{\mathbf{B}}_4{}^{-1}. \tag{7.7}$$

In the following section, we will show how a two-dimensional iterative technique can be implemented to solve the inverse kinematics problem of six-DOF robot manipulators. Although this technique can equally be developed using Eqs. (7.4) or (7.6), it will be based on Eq. (7.2) for convenience.

Two-Dimensional Iterative Technique

Since we only need to know 2 of the joint variables to solve for the whole solution set, the inverse kinematics problem of six-DOF manipulators can be reduced to finding the values of the first two joint variables only, and getting closed-form values for the remaining variables. A numerical technique aimed at finding the values of Θ_1 and Θ_2 can be implemented by defining a system of two nonlinear equations in Θ_1 and Θ_2,

$$f(\Theta_1,\Theta_2) = 0 \tag{7.8}$$

$$g(\Theta_1,\Theta_2) = 0, \tag{7.9}$$

that can be solved using an iterative method such as a two-dimensional Newton-Raphson.

From the left hand side of Eq. (7.2), two vectors $\mathbf{u}_L$ and $\mathbf{q}_L$, corresponding to vectors $\mathbf{u}$ and $\mathbf{q}$, (vectors $\mathbf{u}$ and $\mathbf{q}$ relate to pose Q as shown in Eq. (6.4)), are given by

$$\mathbf{u}_L = \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{R}_5\ \mathbf{z} \tag{7.10}$$

and

$$\mathbf{q}_L = \mathbf{R}_3(\mathbf{R}_4\ \mathbf{l}_5 + \mathbf{l}_4) + \mathbf{l}_3. \tag{7.11}$$

We define two nonlinear functions of Θ_1 and Θ_2 as differences between the inner-products $\mathbf{u}_L \cdot \mathbf{q}_L$, $\mathbf{q}_L \cdot \mathbf{q}_L$ and the inner-products $\mathbf{u} \cdot \mathbf{q}$ and $\mathbf{q} \cdot \mathbf{q}$, respectively;

$$f(\Theta_1, \Theta_2) = \mathbf{u}_L \cdot \mathbf{q}_L - \mathbf{u} \cdot \mathbf{q}, \qquad (7.12)$$

$$g(\Theta_1, \Theta_2) = \mathbf{q}_L \cdot \mathbf{q}_L - \mathbf{q} \cdot \mathbf{q}. \qquad (7.13)$$

If the values of Θ_1 and Θ_2 used to compute pose matrix Q in Eq. (7.3) do correspond to a solution set, then Eq. (7.2) will hold and vectors $\mathbf{u}_L$ and $\mathbf{q}_L$ will be exactly equal to $\mathbf{u}$ and $\mathbf{q}$ forcing both functions f and g to be equal to zero. In other words solution sets of Eq. (7.1) correspond to zeros of the functions f and g defined in Eqs. (7.12) and (7.13).

Computing $f(\Theta_1, \Theta_2)$ and $g(\Theta_1, \Theta_2)$.

In order to compute the values of f and g for a given pair (Θ_1, Θ_2), the components of vectors $\mathbf{u}$, $\mathbf{q}$ and the inner products $\mathbf{u} \cdot \mathbf{q}$ and $\mathbf{q} \cdot \mathbf{q}$ are needed to solve the 4-DOF equation (7.2) which in turn allows computation of inner products $\mathbf{u}_L \cdot \mathbf{q}_L$ and $\mathbf{q}_L \cdot \mathbf{q}_L$ and finally the values of f and g.

Vectors $\mathbf{u}$ and $\mathbf{q}$, computed from Eq. (7.3), are

$$\mathbf{u} = R_2^{-1} R_1^{-1} R\, z = R_2^{-1} R_1^{-1}\, \mathbf{t} \qquad (7.14)$$

and

$$\mathbf{q} = R_2^{-1} [R_1^{-1} (p - l_1) - l_2]. \qquad (7.15)$$

If we consider the components of vector $\mathbf{t}$ as expressed with respect to frame F_1,

$$
{}^{1}\mathbf{t} = \begin{bmatrix} {}^{1}t_x \\ {}^{1}t_y \\ {}^{1}t_z \end{bmatrix} = R_1^{-1}\,\mathbf{t} = \begin{bmatrix} C_1\, t_x + S_1\, t_y \\ -\tau_1 S_1\, t_x + \tau_1 C_1\, t_y + \sigma_1\, t_z \\ \sigma_1 S_1\, t_x - \sigma_1 C_1\, t_y + \tau_1\, t_z \end{bmatrix}
$$

then vector $\mathbf{u}$ is given by

$$
\mathbf{u} = R_2^{-1}\,{}^{1}\mathbf{t} = \begin{bmatrix} C_2\,{}^{1}t_x + S_2\,{}^{1}t_y \\ -\tau_2 S_2\,{}^{1}t_x + \tau_2 C_2\,{}^{1}t_y + \sigma_2\,{}^{1}t_z \\ \sigma_2 S_2\,{}^{1}t_x - \sigma_2 C_2\,{}^{1}t_y + \tau_2\,{}^{1}t_z \end{bmatrix}. \tag{7.16}
$$

To obtain the components of vector $\mathbf{q}$, first we rewrite Eq. (7.15) as

$$
\mathbf{q} = R_2^{-1}(R_1^{-1}\,\mathbf{p} - R_1^{-1}\,\mathbf{l}_1) - R_2^{-1}\,\mathbf{l}_2
$$

and we define

$$
{}^{1}\mathbf{p} = \begin{bmatrix} {}^{1}p_x \\ {}^{1}p_y \\ {}^{1}p_z \end{bmatrix} = R_1^{-1}\,\mathbf{p} = \begin{bmatrix} C_1\, p_x + S_1\, p_y \\ -\tau_1 S_1\, p_x + \tau_1 C_1\, p_y + \sigma_1\, p_z \\ \sigma_1 S_1\, p_x - \sigma_1 C_1\, p_y + \tau_1\, p_z \end{bmatrix},
$$

vector $\mathbf{q}$ is then given by

$$
\mathbf{q} = \begin{bmatrix} C_2\,({}^{1}p_x - a_1) + S_2\,{}^{1}p_y - a_2 \\ -\tau_2 S_2\,{}^{1}p_x + \tau_2 C_2\,{}^{1}p_y + \sigma_2\,{}^{1}p_z - \sigma_2 d_2 \\ \sigma_2 S_2\,{}^{1}p_x - \sigma_2 C_2\,{}^{1}p_y + \tau_2\,{}^{1}p_z - \tau_2 d_2 \end{bmatrix}. \tag{7.17}
$$

The inner product $\mathbf{u}.\mathbf{q}$ can be derived from Eqs. (7.14) and (7.15),

$$
\mathbf{u}.\mathbf{q} = (R_2^{-1}\,R_1^{-1}\,\mathbf{t}).\;R_2^{-1}R_1^{-1}\,[(\mathbf{p} - \mathbf{l}_1) - R_1\,\mathbf{l}_2].
$$

Using (4.4) and (4.5) as needed and expanding yields

$$\mathbf{u.q} = \mathbf{t.(p} - \mathbf{l}_1) - R_1^{-1}\mathbf{t} \cdot \mathbf{l}_2,$$

or

$$\mathbf{u.q} = \mathbf{t.p} - \mathbf{t.l}_1 - {}^1\mathbf{t.l}_2$$

which gives

$$\mathbf{u.q} = \mathbf{t.p} - a_1 t_x C_1 - a_1 t_y S_1 - a_2\ {}^1t_x C_2 \\ - a_2\ {}^1t_y S_2 - d_2\ {}^1t_z. \qquad (7.18)$$

Similarly, the square of the length of vector **q**, **q.q**, is given by

$$\mathbf{q.q} = R_2^{-1}R_1^{-1}\,[(\mathbf{p} - \mathbf{l}_1)-R_1\,\mathbf{l}_2] \cdot R_2^{-1}R_1^{-1}\,[(\mathbf{p} - \mathbf{l}_1)-R_1\,\mathbf{l}_2]$$

where we factored out $R_2^{-1}R_1^{-1}$ in the expression of **q** from Eq. (7.15). Using (4.5) and (4.6) and expanding again leads to

$$\mathbf{q.q} = \mathbf{p.p} + \mathbf{l}_1.\mathbf{l}_1 + \mathbf{l}_2.\mathbf{l}_2 - 2(\mathbf{p.l}_1 + R_1^{-1}\mathbf{p.l}_2 + R_1^{-1}\mathbf{l}_1.\mathbf{l}_2)$$

or

$$\mathbf{q.q} = -2a_2\,[(a_1 + {}^1p_x)\,C_2 + {}^1p_y\,S_2] - 2d_2\,{}^1p_z \\ - 2a_1\,{}^1p_x + \mathbf{p.p} + a_1^2 + a_2^2 + d_2^2. \qquad (7.19)$$

Equation (7.2) gives rise to a reduced system similar to that of Eqs. ((5.13)-(5.16) with the required shift in indexes,

$$a_3u_y\,S_3 + a_3u_x\,C_3 + a_4\sigma_5\,S_5 - \sigma_4\sigma_5 d_4\,C_5 = r_1 \qquad (7.20)$$

$$\sigma_3u_x\,S_3 - \sigma_3u_y\,C_3 + \sigma_4\sigma_5\,C_5 = r_2 \qquad (7.21)$$

$$\sigma_3 q_x\ S_3 - \sigma_3 q_y\ C_3 - \sigma_4 a_5\ S_5 = r_3 \qquad (7.22)$$

$$a_3 q_y\ S_3 + a_3 q_x\ C_3 + \sigma_4 a_5 d_4\ S_5 + a_4 a_5\ C_5 = r_4 \qquad (7.23)$$

with

$$r_1 = \mathbf{q}\cdot\mathbf{u} - \tau_5 d_5 - d_3 u_z - \tau_4 \tau_5 d_4 \qquad (7.24)$$

$$r_2 = \tau_4 \tau_5 - \tau_3 u_z \qquad (7.25)$$

$$r_3 = \tau_3 (d_3 - q_z) + d_4 + \tau_4 d_5 \qquad (7.26)$$

$$r_4 = (\mathbf{q}\cdot\mathbf{q} + a_3^2 + d_3^2 - a_4^2 - d_4^2 - a_5^2 - d_5^2)/2 - d_3 q_z - \tau_4 d_4 d_5. \qquad (7.27)$$

Solving this system of equations will yield the values of Θ_3 and Θ_5. The value of Θ_4 can then be computed from the two equations

$$(\sigma_4 d_5 - \tau_4 a_5 S_5)\ S_4 + (a_4 + a_5 C_5)\ C_4 = C_3 q_x + S_3 q_y - a_3 \qquad (7.28)$$

and

$$(a_4 + a_5 C_5)\ S_4 - (\sigma_4 d_5 - \tau_4 a_5 S_5)\ C_4 = -\tau_3 S_3 q_x + \tau_3 C_3 q_y + \sigma_3 (q_z - d_3) \ (7.29) \qquad (7.29)$$

derived from Eqs. (5.23) and (5.24), or from the equations

$$(\tau_4 \sigma_5 C_5 + \sigma_4 \tau_5)\ S_4 + \sigma_5 S_5\ C_4 = C_3 u_x + S_3 u_y \qquad (7.30)$$

and

$$\sigma_5 S_5\ S_4 - (\tau_4 \sigma_5 C_5 + \sigma_4 \tau_5)\ C_4 = -\tau_3 S_3 u_x + \tau_3 C_3 u_y + \sigma_3 u_z, \qquad (7.31)$$

corresponding to Eqs. (5.25) and (5.26).

We can now compute the inner products $\mathbf{u}_L \cdot \mathbf{q}_L$ and $\mathbf{q}_L \cdot \mathbf{q}_L$. By incrementing the indexes in Eq. (6.10), we derive

$$
\begin{aligned}
\mathbf{u}_L \cdot \mathbf{q}_L = {} & \tau_5 d_5 + a_4 \sigma_5 S_5 - \sigma_4 d_4 \sigma_5 C_5 + \tau_5 d_4 \\
& + \sigma_5 S_5 (a_3 C_4 + \sigma_3 d_3 S_4) \\
& - \sigma_5 C_5 (-a_3 \tau_4 S_4 + \sigma_3 d_3 \tau_4 C_4 + \tau_3 d_3 \sigma_4) \\
& + \tau_5 (a_3 \sigma_4 S_4 - \sigma_3 d_3 \sigma_4 C_4 + \tau_3 d_3 \tau_3) .
\end{aligned} \tag{7.32}
$$

Vector $\mathbf{q}_L$, obtained from the left hand side of Eq. (7.2), is

$$\mathbf{q}_L = \mathbf{R}_3 (\mathbf{R}_4 \mathbf{l}_5 + \mathbf{l}_4) + \mathbf{l}_3 \tag{7.33}$$

and the square of its length is given by

$$\mathbf{q}_L \cdot \mathbf{q}_L = (\mathbf{R}_3 (\mathbf{R}_4 \mathbf{l}_5 + \mathbf{l}_4) + \mathbf{l}_3) \cdot (\mathbf{R}_3 (\mathbf{R}_4 \mathbf{l}_5 + \mathbf{l}_4) + \mathbf{l}_3)$$

or

$$\mathbf{q}_L \cdot \mathbf{q}_L = (\mathbf{l}_5 + \mathbf{R}_4^{-1} \mathbf{l}_4 + \mathbf{R}_4^{-1} \mathbf{R}_3^{-1} \mathbf{l}_3) \cdot (\mathbf{l}_5 + \mathbf{R}_4^{-1} \mathbf{l}_4 + \mathbf{R}_4^{-1} \mathbf{R}_3^{-1} \mathbf{l}_3) ,$$

after factoring out $(\mathbf{R}_3 \ \mathbf{R}_4)$ and using inner product invariance of rotations. Multiplying out the terms in parentheses and using (4.5) and (4.6) where necessary, the last equation yields

$$
\begin{aligned}
\mathbf{q}_L \cdot \mathbf{q}_L = {} & 2 \, [a_5 C_5 (a_4 + a_3 C_4 + \sigma_3 d_3 S_4) \\
& + a_5 S_5 (-a_3 \tau_4 S_4 + \sigma_3 d_3 \tau_4 C_4 + \tau_3 d_3 \sigma_4 + \sigma_4 d_4) \\
& + d_5 (\tau_4 d_4 + a_3 \sigma_4 S_4 - \sigma_3 d_3 \sigma_4 C_4 + d_3 \tau_3 \tau_4) \\
& + a_3 a_4 C_4 + \sigma_3 d_3 a_4 S_4 + \tau_3 d_3 d_4] \\
& + a_3^2 + d_3^2 + a_4^2 + d_4^2 + a_5^2 + d_5^2 .
\end{aligned} \tag{7.34}
$$

Given a pair (Θ_1, Θ_2), the corresponding values of $f(\Theta_1,\Theta_2)$ and $g(\Theta_1,\Theta_2)$ are obtained by the following steps:

Step 1. For initial values of Θ_1 and Θ_2, compute the coordinates of vectors **u** and **q** as given by Eqs. (7.16) and (7.17). The inner products **u.q** and **q.q** can be computed using the regular inner product formula,

$$\mathbf{u}\cdot\mathbf{q} = u_x q_x + u_y q_y + u_z q_z$$

and

$$\mathbf{q}\cdot\mathbf{q} = q_x^2 + q_y^2 + q_z^2.$$

Step 2. Solve the reduced system of Eqs. (7.20)-(7.23) for Θ_3 and Θ_5.

Step 3. Compute the value of Θ_4 from Eqs. (7.28) and (7.29) or Eqs. (7.30) and (7.31).

Step 4. Compute the inner products $\mathbf{u}_L\cdot\mathbf{q}_L$ and $\mathbf{q}_L\cdot\mathbf{q}_L$, given by Eqs. (7.32) and (7.34), respectively, and compute the values of f and g as given by Eqs. (7.12) and (7.13).

Two-Dimensional Newton-Raphson

The zeros of f and g can be iteratively computed and checked for consistency with Eq. (7.1). If a computer program for evaluating the two functions is available, the partial derivatives of f and g with respect to Θ_1 and Θ_2, denoted f_1, f_2 and g_1, g_2 respectively, can be numerically approximated by

$$f_1(\Theta_1,\Theta_2) = \partial f/\partial\Theta_1 = [f(\Theta_1+\delta\Theta_1,\Theta_2)-f(\Theta_1,\Theta_2)]/\delta\Theta_1, \quad (7.35)$$

$$f_2(\Theta_1,\Theta_2)= \partial f/\partial\Theta_2 = [f(\Theta_1,\Theta_2+\delta\Theta_2)-f(\Theta_1,\Theta_2)]/\delta\Theta_2, \quad (7.36)$$

and

$$g_1(\Theta_1,\Theta_2)= \partial g/\partial\Theta_1 = [g(\Theta_1+\delta\Theta_1,\Theta_2)-g(\Theta_1,\Theta_2)]/\delta\Theta_1, \quad (7.37)$$

$$g_2(\Theta_1,\Theta_2)= \partial g/\partial\Theta_2 = [g(\Theta_1,\Theta_2+\delta\Theta_2)-g(\Theta_1,\Theta_2)]/\delta\Theta_2, \quad (7.40)$$

where $\delta\Theta_1$ and $\delta\Theta_2$ are small increments of Θ_1 and Θ_2 respectively.

The two-dimensional Newton-Raphson technique for solving the inverse kinematics problem for a six-revolute-DOF robot arm of arbitrary architecture proceeds according to the following steps:

Step 1. Assume an initial estimate of Θ_1 and Θ_2 and compute Θ_3, Θ_4, and Θ_5

Step 2. From the values of Θ_1, Θ_2, Θ_3, Θ_4, and Θ_5 compute $f(\Theta_1,\Theta_2)$ and $g(\Theta_1,\Theta_2)$ as in Eqs. (7.12) and (7.13).

Step 3. Compute the partial derivatives of f and g with respect to Θ_1 and Θ_2 by numeric approximations as shown earlier.

Step 4. Obtain a new estimate for Θ_1 and Θ_2 by the two-dimensional Newton-Raphson method, i.e.

$$\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}_{new} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} - \begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}^{-1} \begin{bmatrix} f(\Theta_1,\Theta_2) \\ g(\Theta_1,\Theta_2) \end{bmatrix}$$

Step 5. Repeat steps 2 to 5 until desired accuracy is attained.

<u>Step 6</u>. Complete the solution set by uniquely computing Θ_6 from

$$\mathbf{R}_6\ \mathbf{x} = \begin{bmatrix} C_6 \\ S_6 \\ 0 \end{bmatrix} = \mathbf{R}_5^{-1}\ \mathbf{R}_4^{-1}\ \mathbf{R}_3^{-1}\ \mathbf{R}_2^{-1}\ \mathbf{R}_1^{-1} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}. \qquad (7.41)$$

<u>Step 7</u>. Check the solution set for consistency with Eq. (7.1).

<u>Choice of functions f and g</u>. The functions f and g defined by Eqs. (7.12) and (7.13) are computationally economical since they do not require computation of the forward kinematics or the inverse jacobian of the manipulator. In fact, even the value of Θ_6 is not required to compute f and g since Eq. (7.41) is considered only after convergence. Unfortunately, a pair (Θ_1, Θ_2) for which both f and g are zero is not guaranteed to correspond to a solution set of Eq. (7.1). Extraneous solution sets can be converged to as well. These are joint values that will yield an end-effector pose that is not exactly the desired one.

Other functions that fully constraint the end-effector pose can be defined at the cost of greater computational complexity. Wu and Paul (1982) have shown that the difference between actual and desired end-effector poses can be expressed as a 6 x 1 vector $\mathbf{x}_e$ given by

$$\mathbf{x}_e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \mathbf{n}_L \cdot (\mathbf{p} - \mathbf{p}_L) \\ \mathbf{b}_L \cdot (\mathbf{p} - \mathbf{p}_L) \\ \mathbf{b}_L \cdot (\mathbf{p} - \mathbf{p}_L) \\ (\mathbf{t}_L.\mathbf{b} - \mathbf{t}.\mathbf{b}_L)/2 \\ (\mathbf{n}_L.\mathbf{t} - \mathbf{n}.\mathbf{t}_L)/2 \\ (\mathbf{b}_L.\mathbf{n} - \mathbf{b}.\mathbf{n}_L)/2 \end{bmatrix}, \tag{7.42}$$

where $\mathbf{n}$, $\mathbf{b}$, $\mathbf{t}$, and $\mathbf{p}$ are the vectors that describe the desired end-effector pose $\mathbf{P}$ as defined in (2.12) and vectors $\mathbf{n}_L$, $\mathbf{b}_L$, $\mathbf{t}_L$, and $\mathbf{p}_L$ are their corresponding vectors from the left hand side of equation (7.1). Two functions can be defined as

$$f(\Theta_1,\Theta_2) = x_1^2 + x_2^2 + x_3^2 \tag{7.43}$$

$$g(\Theta_1,\Theta_2) = x_4^2 + x_5^2 + x_6^2. \tag{7.44}$$

A pair (Θ_1,Θ_2) that is a zero of both f and g is guaranteed to correspond to a solution set of Eq. (7.1) so that the iterative method described above will only converge to a true solution. However, now, the forward kinematics must be computed at each iteration since the components of vectors $\mathbf{n}_L$, $\mathbf{b}_L$, $\mathbf{t}_L$, and $\mathbf{p}_L$ are all needed to evaluate functions f and g as defined by Eqs. (7.42) and (7.43).

One-Dimensional Method

The inverse kinematic problem for six-DOF manipulators reduces to a five-DOF one when the first or the last joint variable is known. Equation (7.1) becomes

$$A_2 \; A_3 \; A_4 \; A_5 \; A_6 = Q, \tag{7.45}$$

with

$$Q = A_1^{-1} \; P \tag{7.46}$$

when Θ_1 is known, and

$$A_1 \; A_2 \; A_3 \; A_4 \; \underline{A}_5 = Q, \tag{7.47}$$

with

$$Q = P \; A_6^{-1} B_5^{-1} \tag{7.48}$$

if Θ_6 is known. In both cases, a five-DOF problem is obtained. When the resulting five-DOF problem is solvable in closed form, knowledge of Θ_1 or Θ_6 is then sufficient to yield a complete solution set. The inverse kinematic problem then reduces to finding one joint angle which can be accomplished by a one-dimensional iterative technique.

In chapter 6, we found that a sufficient condition for closed form solutions of 5-DOF manipulators is that they have one of the special structures listed at the end of Chapter 6. Six-DOF arms that include a five-DOF segment with this type of geometry can then be solved using a one-dimensional iterative method. This iterative technique can be implemented in much the same way as described in Chapter

6 for five-degree-of-freedom arms. Assuming Eq. (7.45) is to be solved, we define a function f of Θ_1 by

$$f(\Theta_1) = \mathbf{u}_L \cdot \mathbf{q}_L - \mathbf{u} \cdot \mathbf{q} , \tag{7.49}$$

where vectors $\mathbf{u}_L$, $\mathbf{q}_L$, $\mathbf{u}$, and $\mathbf{q}$ are defined as earlier. Given a value of Θ_1, vectors $\mathbf{u}$ and $\mathbf{q}$ are computed from Eq. (7.46), the values of the remaining joint variables are computed in closed form from Eq. (7.45) as indicated in Chapter 6 and Appendix B, the inner product $\mathbf{u}_L \cdot \mathbf{q}_L$ can then be obtained as in Eq. (6.10) with the proper index adjustments, and the value of f is then given by Eq. (7.49). As we have seen before, the ability to compute the function f for a given value of Θ_1 allows the implementation of a practical one-dimensional Newton-Raphson algorithm. Therefore, we can conclude that a six-degree-of-freedom manipulator with two consecutive pairs of intersecting or parallel joint axes or three consecutive joint axes that are parallel and/or intersecting two at a time can be solved by use of a one-dimensional iterative technique instead of the two-dimensional method required for an arm of arbitrary architecture.

Closed-Form Solution

Some six-degree-of-freedom manipulators with simple geometries do not require any iterative method since they can be solved in closed-form. Pieper (1968) has shown that a sufficient condition for closed-form solutions is that three consecutive axes be intersecting. The inverse kinematics problem then reduces to finding the zeros of a quartic polynomial. In the literature, It seems to be common knowledge that three consecutive joint axes that are parallel is another sufficient geometric condition for closed form solutions.

The analysis of Chapter 5 and Appendix A showed that under certain conditions, the reduced system of equations (7.20)-(7.23) included constraint equations of the form

$$r_i = 0. \tag{7.50}$$

The quantities r_i, i=1, . . . ,4, are functions of Θ_1 and Θ_2, as we have seen earlier. By looking for conditions under which a joint variable value can be directly obtained from an equation having the form of Eq. (7.50), we find two more sufficient six-DOF robot structure conditions for closed-form solutions (excluding the already known conditions of three parallel or three intersecting axes).

When the first two joint axes of a manipulator are parallel so that α_1=0, then σ_1=0, τ_1=1, and the z-components of vectors **u** and **q**, given by Eqs. (7.16) and (7.17), become

$$u_z = \sigma_2 \; (-t_y \; C_{12} + t_x \; S_{12}) + \tau_2 t_z \tag{7.51}$$

and

$$q_z = \sigma_2 \; (p_x \; S_{12} - p_y \; C_{12}) + \tau_2 \; (p_z - d_2). \tag{7.52}$$

This shows that r_2 and r_3, as given in (7.25) and (7.26), become linear functions of S_{12} and C_{12}.

When joint axes 3 and 4 are parallel and joint axes 5 and 6 are parallel, the reduced system oe equations (7.20)-(7.23) becomes similar to that of case 2 of Chapter 5,

$$a_3u_y\ S_3 + a_3u_x\ C_3 = r_1 \qquad (7.53)$$

$$0 = r_2 \qquad (7.54)$$

$$-\ \sigma_4 a_5\ S_5 = r_3 \qquad (7.55)$$

$$a_3q_y\ S_3 + a_3q_x\ C_3 + \sigma_4 a_5 d_4\ S_5 + a_4 a_5\ C_5 = r_4. \qquad (7.56)$$

Equation (7.54) yields two possible values for Θ_{12}, each of which will provide two possible values of Θ_5 from Eq. (7.55) when substituted in the expression of r_3. The remaining joint values can then be computed in closed form. A similar development occurs when axes 3 and 4 are parallel and axes 5 and 6 intersect.

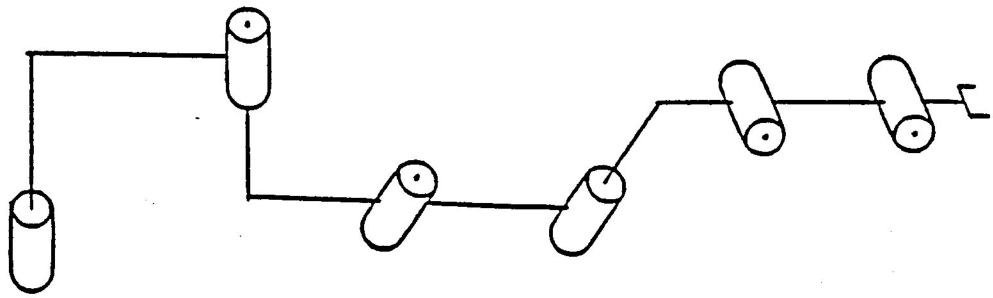
To summarize, a six-DOF manipulator has a closed-form solution if one of the following conditions is satisfied:

1. Three consecutive joint axes are parallel.

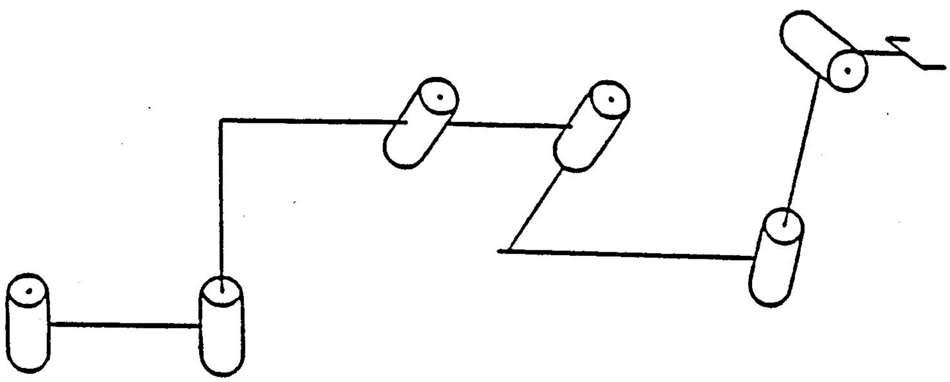
2. Three consecutive joint axes intersect at one point.

3. The arm is formed of three sets of two parallel axes. This structure is illustrated in Figure 7.1(a).

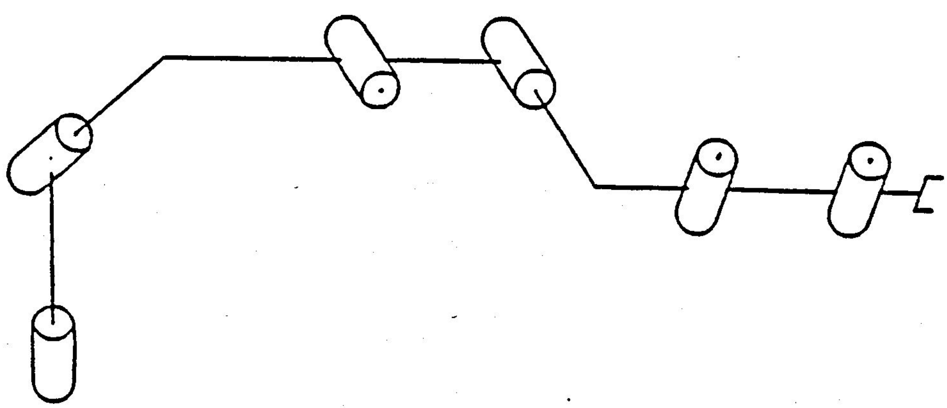
4. The arm has two sets of two parallel joint axes followed or preceded by two intersecting axes. These structures are illustrated in Figure 7.1(b) and 7.1(c).



a. 3 pairs of parallel joint axes.



b. 2 pairs of parallel joint axes followed by 2 intersecting joint axes.



c. 2 pairs of parallel joint axes preceded by 2 intersecting joint axes.

Figure 7.1. 6-DOF manipulators with closed-form inverse kinematics.

CHAPTER 8
ORTHOGONAL MANIPULATORS

Definition: An n-axes, serial kinematic chain of revolute or prismatic joints is orthogonal if all twist angles α_i, $i=1, \ldots, n$, along the chain are 0 or $\pi/2$. An open orthogonal kinematic chain will be called an orthogonal manipulator (Doty 1986).

Six-DOF orthogonal manipulators can be classified in terms of the values of their twist angles α_i, $i=1, \ldots, 5$. Twist angle α_6 is always assumed to be zero in this text. In fact, the value of α_6 can be chosen arbitrarily since z_6 is not a joint axis. Therefore, there are only $2^5 = 32$ distinct classes of orthogonal manipulators, as listed in Table 8-1, 8 of which have 4 or more adjacent parallel joint axes which reduces their capability to less than six degrees of freedom. As a result, there are only 24 types of six-joint orthogonal manipulators with full spatial position and orientation capability.

A convenient notation for this classification of orthogonal manipulators is obtained by assigning a 5-bit binary number to each of these 24 types in which bit i is 0 if $\alpha_i = 0$ and bit i is 1 if $\alpha_i = \pi/2$. For example, a manipulator with twist angles

$\alpha_5=\pi/2$, $\alpha_4=\pi/2$, $\alpha_3=0$, $\alpha_2=0$, and $\alpha_1=\pi/2$

belongs to the class 11-001 of orthogonal manipulators.

Since most industrial robot arms are orthogonal, it is worthwhile to consider the inverse kinematics problem with respect to these manipulators. The A-matrices associated with orthogonal arms have one of the two following forms

$$\mathbf{A}_i(\alpha=0) = \begin{bmatrix} C_i & -S_i & 0 & a_iC_i \\ S_i & C_i & 0 & a_iS_i \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (8.1)$$

or

$$\mathbf{A}_i(\alpha=\pi/2) = \begin{bmatrix} C_i & 0 & S_i & a_iC_i \\ S_i & 0 & -C_i & a_iS_i \\ 0 & 1 & 0 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (8.2)$$

Further computational simplification is obtained in the inverse kinematic equations with orthogonal manipulators since

$$\mathbf{R}_i\mathbf{z} = \mathbf{R}_i^{-1}\mathbf{z} = \mathbf{z} \text{ if } \alpha_i = 0$$

and

$$\mathbf{R}_i^{-1}\mathbf{z} = \mathbf{y} \quad \text{if } \alpha_i = \pi/2.$$

Doty (1986) has shown that, of the 24 classes of nontrivial orthogonal manipulators, those with 2 non-zero twist angles (classes 01-001, 01-010, 01-100, 10-100 and 10-010) have closed-form solutions. The inverse kinematic analysis of Chapter 7 shows that the most complex six-DOF robot structure can be solved by use of a two-dimensional iterative technique. Simpler structures only require a one-dimensional numerical technique and some even simpler structures can be solved in closed-form.

In Table 8-1, we provide a list of all thirty-two orthogonal manipulator classes in which we indicate the degenerate geometries and, for the twenty four non-degenerate classes, we indicate a suitable inverse kinematic method necessary for solving the most complex arm structure within that class. It must be understood that intersecting axes cannot be considered according to a classification based on the values of the twist angles alone. The choice of inverse kinematic method indicated in Table 8-1 is based solely on the presence of parallel axes within a given class. Simpler inverse kinematic methods can be used if any of the special structures discussed in chapters 5, 6, and 7 are present.

In Chapter 9, the inverse kinematics of four orthogonal manipulators are described in more detail.

Table 8-1. Inverse kinematics of orthogonal manipulators

	Class	Method	Justification
1	00-000	D	All six axes are parallel
2	00-001	D	Five consecutive parallel axes
3	00-010	D	Four consecutive parallel axes
4	00-011	D	Four consecutive parallel axes
5	00-100	CF	Three consecutive parallel axes
6	00-101	CF	Three consecutive parallel axes
7	00-110	CF	Three consecutive parallel axes
8	00-111	CF	Three consecutive parallel axes
9	01-000	D	Four consecutive parallel axes
10	01-001	CF	Three consecutive parallel axes
11	01-010	CF	Three pairs of parallel axes
12	01-011	1-D	Two pairs of parallel axes
13	01-100	CF	Three consecutive parallel axes
14	01-101	2-D	
15	01-110	2-D	
16	01-111	2-D	

Table 8-1. --Continued

	Class	Method	Justification
17	10-000	D	Five consecutive parallel axes
18	10-001	D	Four consecutive parallel axes
19	10-010	CF	Three consecutive parallel axes
20	10-011	CF	Three consecutive parallel axes
21	10-100	CF	Three consecutive parallel axes
22	10-101	1-D	Two pairs of parallel axes
23	10-110	2-D	
24	10-111	2-D	
25	11-000	D	Four consecutive parallel axes
26	11-001	CF	Three consecutive parallel axes
27	11-010	1-D	Two pairs of parallel axes
28	11-011	2-D	
29	11-100	CF	Three consecutive parallel axes
30	11-101	2-D	
31	11-110	2-D	
32	11-111	2-D	

Notation: D = Degenerate geometry

CF = Closed-Form

1-D = One Dimensional iterative method

2-D = Two-Dimensional iterative method

CHAPTER 9
APPLICATION EXAMPLES

Example 1: The PUMA 560

A popular orthogonal manipulator geometry, the PUMA 560, is described by the kinematic parameters given in Table 9-1 and illustrated in Figure 9.1. This manipulator has a spherical wrist and therefore allows closed-form solutions (Pieper 1968). Inverse kinematic solutions have been proposed by numerous authors for this type of arm (Lee and Ziegler 1984; Craig 1986; Paul and Zhang 1986).

Table 9-1. PUMA 560 kinematic parameters

Joint	d	Θ	a	α (rd)
1	0	Θ_1	0	$\pi/2$
2	0	Θ_2	a_2	0
3	d_3	Θ_3	a_3	$\pi/2$
4	d_4	Θ_4	0	$\pi/2$
5	0	Θ_5	0	$\pi/2$
6	0	Θ_6	0	0

This example is included here to demonstrate the utility of the approach already outlined and to contrast it