INVERSE KINEMATIC ANALYSIS OF ROBOT MANIPULATORS

By

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To those who fought for
the freedom and education
of all Algerians.

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Computer-controlled robot manipulators are becoming an important part of automated manufacturing plants thereby creating a need for reliable and fast control algorithms that can improve the performance of robot manipulators in industrial applications. An important part of such control algorithms is the inverse kinematics portion which consists of computing the values of the robotic joint variables corresponding to a desired end-effector position and orientation. This work is based on a new approach that uses orthogonality of rotation matrices to reduce the problem to a simpler form. The reduction techniques are first used to analyze the kinematics of four-degree-of-freedom (DOF) robots. The results obtained are then applied to the study of five- and six-degree-of-freedom manipulators. Fast one- and two-dimensional numerical techniques for solving five- and six-DOF arms of arbitrary geometry are developed. These new methods provide a large reduction in computational complexity and can be easily implemented in real-time applications. Another contribution of this work is a classification of robot geometries in terms of inverse kinematic complexity. Some new sufficient structural conditions for the possibility of closed-form solutions for five- and six-DOF robot manipulators are described. In the case of six-DOF arms, structural conditions for the applicability of a one-dimensional iterative technique are also provided. Finally, in the example applications of the techniques presented here, we describe a six-degree-of freedom manipulator capable of achieving a particular end-effector pose in sixteen distinct configurations.

CHAPTER 1
INTRODUCTION

An important part of computer control algorithms for open serial kinematic chains is the inverse kinematics section. In any robotic application, the hand or end-effector of the robot may move along a trajectory specified as a sequence of points at which the end-effector pose (orientation and position) is known. While this trajectory is specified in Cartesian coordinates, the motion of the robot is controlled through individual joint actuators that produce the necessary rotation in revolute joints, or the translation in prismatic joints. The robot controller must, therefore, be supplied the values of the joint variables corresponding to the end-effector pose, i.e., the coordinates in joint space of the robot hand for each point along the trajectory must be computed. The conversion of trajectory locations from Cartesian coordinates to joint coordinates is referred to as the inverse kinematics problem.

A desirable inverse kinematic algorithm is one capable of producing the joint coordinates in real-time. While the robot hand is at, or approaching, one location along the trajectory, the algorithm must be able to produce the joint coordinates for the next pose. In tasks where speed and precision are important, the real-time requirement puts heavy constraints on the computation time of the inverse kinematic algorithm.

The forward kinematics problem, the conversion from joint space to Cartesian space, is a much simpler problem that has a unique closed-form solution. In most cases a robot manipulator can achieve a desired end-effector pose in more than one configuration. The question of just how many distinct solutions there are to the inverse kinematics problem of general six-degree-of-freedom (DOF) robot manipulators has interested a few researchers. Roth, Rastegar, and Scheinman (1973) put an upper bound of 32 on the degree of a polynomial equation (in one joint variable) that can be derived from the inverse kinematics problem of six-DOF manipulators. A similar result was obtained by Duffy and Crane (1980), using the equivalence between an open 6-revolute-DOF kinematic chain and the 7-revolute single-loop spatial mechanism. Therefore, the number of inverse kinematic solutions for 6-revolute-DOF manipulators could be at most 32. More recently, Lee and Liang (in press), using Duffy's method, were able to reduce the degree of the inverse kinematic polynomial equation to 16, thereby reducing the upper bound on the number of inverse kinematic solutions to 16. Tsai and Morgan (1984), illustrating a new inverse kinematic method capable of producing all solutions, found a robot manipulator and an end-effector pose with 12 possible solutions. Manseur and Doty (in press) described a simple manipulator geometry capable of achieving a particular end-effector pose in 16 distinct configurations, thereby closing the proof that 16 is indeed the maximum achievable number of inverse kinematic solutions for six-DOF robot manipulators. The manipulator and the pose for which the sixteen solutions were found and the inverse kinematic solution search algorithm used are discussed in Chapter 9, Example 3 of this dissertation.

Another desirable property of an inverse kinematic algorithm is the capability of computing more than one solution, so that a solution can be chosen according to some optimality or collision avoidance criteria. Although for manipulators with simple geometries, such as the PUMA 560 industrial robot, several possible solutions can be obtained in closed-form, this multiple solution property conflicts with the real-time requirement discussed earlier for many other robots that must rely on iterative techniques.

After introducing the notation and some mathematical preliminaries and a brief discussion of existing inverse kinematic methods, we present a new approach to the inverse kinematic problem based on a reduced set of nonlinear equations. This new approach is then used to analyze the kinematics of four-, five-, and six-degree-of-freedom manipulators. Some simple and efficient iterative techniques are described and sufficient manipulator structural conditions for the applicability of these methods are determined. All the methods developed in this dissertation are illustrated by examples in Chapter 9. Chapter 10 summarizes the final results, discusses the contributions of this work to the field of robotics, and presents related topics and areas of future research.

CHAPTER 2
THE INVERSE KINEMATICS PROBLEM

Notation and Mathematical Preliminaries

The DH Parameters

A robot manipulator is modelled as an open kinematic chain of rigid bodies (links) connected by joints. A reference frame is assigned to each link along the chain starting with the base frame F_0, assigned to the fixed link, up to the end effector frame F_n, for a manipulator with n degrees of freedom (DOF). The position and orientation of frame $F_i=(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$, with respect to the preceding frame F_{i-1}, are entirely described by the four DH-parameters d_i, Θ_i, a_i and α_i (Denavit and Hartenberg 1955). These parameters are illustrated in Figure 2.1 and defined as:

d_i = distance between the common normal to axes $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ and the common normal to $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$ measured along axis $\mathbf{z}_i$.

Θ_i = the angle of rotation about $\mathbf{z}_i$ so that $\mathbf{x}_i$ becomes parallel to $\mathbf{x}_{i-1}$ when $\Theta_i = 0$.

a_i = the length of the common normal to axes $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$.

α_i = the angle of rotation about $\mathbf{x}_i$ so that $\mathbf{z}_i$ becomes parallel to $\mathbf{z}_{i-1}$ when $\alpha_i = 0$.

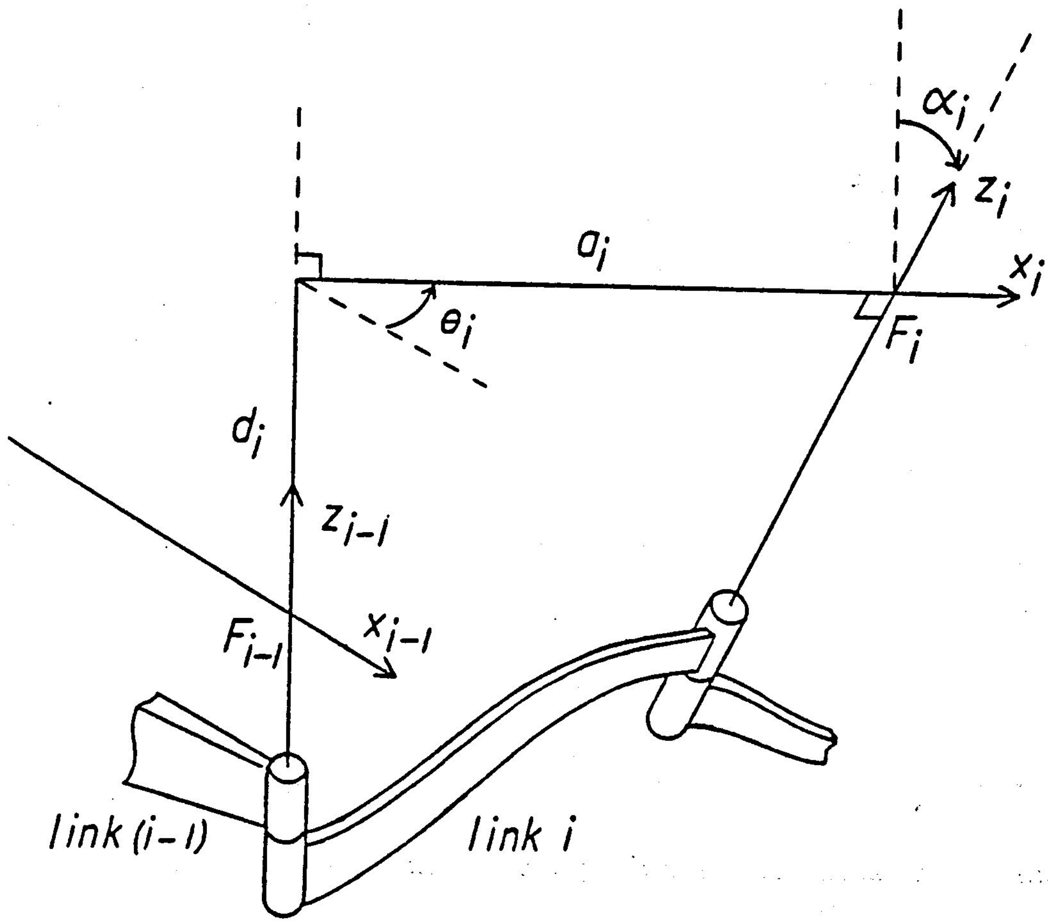


Figure 2.1. The DH-parameters.

When joint i is revolute, parameter Θ_i is the joint variable and, if joint i is prismatic, the joint variable is d_i. When applicable, d_i measures the translation along axis $\mathbf{z}_{i-1}$.

Homogeneous Matrices

If a vector ${}^{i}\mathbf{u} = [{}^{i}u_x, {}^{i}u_y, {}^{i}u_z]^T$ is expressed in frame F_i, its expression with respect to frame F_{i-1}, ${}^{i-1}\mathbf{u}$, satisfies

$$\begin{bmatrix} {}^{i-1}u_x \\ {}^{i-1}u_y \\ {}^{i-1}u_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_i & -S_i\tau_i & S_i\sigma_i & a_iC_i \\ S_i & C_i\tau_i & -C_i\sigma_i & a_iS_i \\ 0 & \sigma_i & \tau_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{i}u_x \\ {}^{i}u_y \\ {}^{i}u_z \\ 1 \end{bmatrix} \qquad (2.1)$$

or in a compact notation,

$$\begin{bmatrix} {}^{i-1}\mathbf{u} \\ 1 \end{bmatrix} = \mathbf{A}_i \begin{bmatrix} {}^{i}\mathbf{u} \\ 1 \end{bmatrix} \qquad (2.2)$$

where $\tau_i=\cos(\alpha_i)$, $\sigma_i=\sin(\alpha_i)$, $C_i=\cos(\Theta_i)$, and $S_i=\sin(\Theta_i)$ and $\mathbf{A}_i$ is the homogeneous frame-transform matrix (Paul 1981). The leading superscript indicates the frame of expression.

The homogeneous matrix transform merely expresses the fact that frame F_i can be obtained from frame F_{i-1} by the following sequence of basic transforms:

1. Rotation about $\mathbf{z}_{i-1}$ of angle Θ_i whose homogeneous matrix is

$$R_z(\Theta_i) = \begin{bmatrix} C_i & S_i & 0 & 0 \\ -S_i & C_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.3)$$

2. Translation of d_i units along axis $\mathbf{z}_{i-1}$ described by the matrix

$$Tr_z(d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.4)$$

3. Translation of a_i units along axis $\mathbf{x}_i$, with homogeneous transform

$$Tr_x(a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.5)$$

4. Rotation about $\mathbf{x}_i$ of angle α_i,

$$R_x(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tau_i & -\sigma_i & 0 \\ 0 & \sigma_i & \tau_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.6)$$

The matrix $\mathbf{A}_i$ of Eq. (2.1) is obtained by the product (Paul, 1981),

$$\mathbf{A}_i = \mathbf{R}_z(\Theta_i)\ \mathbf{Tr}_z(d_i)\ \mathbf{Tr}_x(a_i)\ \mathbf{R}_x(\alpha_i).$$

A useful decomposition of matrix $\mathbf{A}_i$ is

$$\mathbf{A}_i = \underline{\mathbf{A}}_i\ \mathbf{B}_i \tag{2.7}$$

with the definitions

$$\underline{\mathbf{A}}_i = \mathbf{R}_z(\Theta_i) \tag{2.8}$$

and

$$\mathbf{B}_i = \mathbf{Tr}_z(d_i)\ \mathbf{Tr}_x(a_i)\ \mathbf{R}_x(\alpha_i). \tag{2.9}$$

Explicitly, matrix $\mathbf{B}_i$ is

$$\mathbf{B}_i = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \tau_i & -\sigma_i & 0 \\ 0 & \sigma_i & \tau_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_i & \mathbf{k}_i \\ 0\ 0\ 0 & 1 \end{bmatrix} \tag{2.10}$$

where $\mathbf{G}_i$ is the upper left 3 x 3 in $\mathbf{B}_i$ and $\mathbf{k}_i$ is the upper right 3 x 1 vector of $\mathbf{B}_i$. The upper left 3 x 3 matrix in $\mathbf{A}_i$ is the rotation matrix $\mathbf{R}_i$ necessary to align the unit vectors of F_i with their counterparts in F_{i-1}, while vector

$$\mathbf{l}_i = \begin{bmatrix} a_i C_i \\ a_i S_i \\ d_i \end{bmatrix}$$

positions the origin of F_i with respect to F_{i-1}.

A compact and useful expression for $\mathbf{A}_i$ is

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{R}_i & \mathbf{l}_i \\ 0\ 0\ 0 & 1 \end{bmatrix}. \tag{2.11}$$

Rotation matrices are orthogonal, so $R_i^{-1} = R_i^T$, where the superscript T denotes the transpose operation, and the inverse of matrix $\mathbf{A}_i$ can be expressed as

$$\mathbf{A}_i^{-1} = \begin{bmatrix} C_i & S_i & 0 & -a_i \\ -S_i\tau_i & C_i\tau_i & \sigma_i & -\sigma_i d_i \\ S_i\sigma_i & -C_i\sigma_i & \tau_i & -\tau_i d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i^T & (-\mathbf{R}_i^T\mathbf{l}_i) \\ 0\ 0\ 0 & 1 \end{bmatrix}. \tag{2.12}$$

Problem Definition

If the orientation of the end-effector is specified by the rotation matrix $\mathbf{R}$, necessary to align the unit vectors of the end-effector frame F_n with the corresponding vectors of base frame F_0, and the position of the origin of the end-effector frame is given as a vector $\mathbf{p}$ with respect to the base frame F_0, then the end-effector pose is adequately described by the 4 x 4 matrix

$$\mathbf{P} = \begin{bmatrix} n_x & b_x & t_x & p_x \\ n_y & b_y & t_y & p_y \\ n_z & b_z & t_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{b} & \mathbf{t} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ 0\ 0\ 0 & 1 \end{bmatrix}, \tag{2.13}$$

where

$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

and

$$\mathbf{R} = \begin{bmatrix} n_x & b_x & t_x \\ n_y & b_y & t_y \\ n_z & b_z & t_z \end{bmatrix}.$$

The inverse kinematics problem for a n-degree-of-freedom manipulator consists of finding a set of joint variables values, called a solution set, that will satisfy the equation

$$\mathbf{A}_1 \; \mathbf{A}_2 \; \mathbf{A}_3 \; \mathbf{A}_4 \; \mathbf{A}_5 \ldots \mathbf{A}_n = \mathbf{P}. \tag{2.14}$$

This matrix equation gives rise to a system of nonlinear equations whose complexity depends on the manipulator geometry, as described by the DH-parameters.

At least six degrees of freedom are required to arbitrarily position and orient a rigid body in space. Therefore, when n is larger than six, the manipulator is redundant and the system of equations implied by Eq. (2.14) is underconstrained. If n is less than 6, the system becomes overconstrained and when n is equal to 6, the inverse kinematic problem is exactly specified. In this research we will address the inverse kinematics problem of non-redundant robot manipulators.

Most existing industrial manipulators are 5- or 6-degree-of-freedom robots, hence, it is of practical importance to solve Eq. (2.14) for n=5 and n=6. The numerical techniques developed in this text are based on a complete inverse kinematic analysis of four-degree-of-freedom manipulators. Therefore, this research will aim at solving Eq. (2.14) for robots with four, five, and six joint axes.

Although the techniques described in this text can be applied to manipulators having prismatic joints (Manseur and Doty 1988), we concentrate on all-revolute six-DOF manipulators; n is six and all joints are assumed revolute.

CHAPTER 3
EXISTING SOLUTIONS

Closed-Form Architectures

The ability to compute the coordinates in joint space of an end-effector pose given in Cartesian space is an important criterion in the design of computer-controlled manipulators. A desirable property for an industrial manipulator is the possibility of computing the joint variables necessary to position and orient the end-effector as specified in Cartesian space, in closed-form. Pieper (1968) has shown that a closed-form solution is possible when the manipulator has three adjacent joint axes intersecting at a common point. The inverse kinematic problem reduces then to a quartic polynomial equation in one of the joint variables. Manipulators with the last three joint axes intersecting are said to be "wrist-partitioned". Computationally efficient methods for computing the position, velocity, and acceleration inverse kinematics for this type of manipulators have been presented by Featherstone (1983), Hollerbach and Sahar (1983), Paul and Zhang (1986), and Low and Dubey (1986). Several industrial six- and five-DOF manipulators such as the PUMA series robots are of the wrist-partitioned type. If, on top of having a wrist, the manipulator has some added structural feature such as two parallel or intersecting joint axes then closed-form solutions may be obtained in a simpler form than a quartic polynomial equation. This is the case of the PUMA 560 robot whose inverse kinematics are discussed in Example 1 of Chapter 9. An algebraic method for solving the inverse kinematics of the PUMA 560 can be found in Craig (1986) and a geometric approach is described in Fu, Gonzalez, & Lee (1987). Another sufficient condition for closed-form solutions is that three adjacent joint axes be parallel (Duffy 1980, Fu, Gonzalez, & Lee 1987).

Record and Playback

An industrial robot manipulator is usually equipped with sensors that can measure information such as joint variable values and rates of change of those values. A method that avoids the computational complexity of the inverse kinematic problem altogether consists of remotely guiding a robot end-effector trajectory by activating each joint separately while storing joint space coordinates and information from the sensors at selected points along the trajectory. The robot can then indefinitely repeat the recorded motion. Should the robot be needed for a different task or should a change in the workcell occur that requires different end-effector trajectories, the motion of the robot will have to be recorded again.

Numerical Techniques

Many six- and five-DOF kinematic structures lack the necessary architectural simplicity for closed-form inverse kinematic solutions. Solving such manipulators requires the use of numerical iterative techniques. For six-DOF robots, equation (2.14) can be expressed as a system of six nonlinear equations in the six joint variables of the form

$$f_1(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) = p_x$$
$$f_2(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) = p_y$$
$$f_3(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) = p_z$$
$$f_4(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) = \alpha$$
$$f_5(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) = \Theta$$
$$f_6(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) = \phi$$

where p_x, p_y, and p_z are the coordinates of the origin of the end-effector frame and α, Θ, and ϕ are either the Euler angles or the roll-pitch-yaw angles derived from the orientation matrix **R** of the end-effector frame (Paul 1981).

The six-dimensional equation is then solved by use of a direct or modified Newton-Raphson or similar technique. Multidimensional iterative techniques for solving the inverse kinematics problem of manipulators of arbitrary architecture are described by Angeles (1985, 1986), Goldenberg, Benhabib, & Fenton (1985), Goldenberg and Lawrence (1985). The computational efficiency of these methods is hindered by the need to compute the inverse of the manipulator Jacobian at several points.

Linares & Page (1984) and Kazerounian (1987) describe techniques that solve the inverse kinematic problem by varying one joint variable at a time so as to minimize the difference, measured by a defined norm, between the end-effector pose as computed from the current joint variables values and the desired pose. This technique has the advantages of guaranteed convergence and reliability even at a singular position. This method requires computation of the forward kinematics at each iteration and it has a computational complexity comparable to that of a Modified-multidimensional Newton-Raphson.

After reducing the problem to a polynomial system of four equations in only four of the joint variables, Tsai and Morgan (1984) used a homotopy map method, for solving systems of polynomial equations in several variables, to find the solutions of the inverse kinematics problem of revolute five- and six- degree-of-freedom manipulators of arbitrary architecture. The method finds all solutions but its computational complexity renders it impractical for many applications.

Lumelsky (1984) presented an iterative algorithm that finds estimates for three of the joint variables and solves in closed form for the remaining three variables at each iteration. The method applies to a particular type of arm geometry (that of the GP66 robot discussed in Example 2, Chapter 9, of this dissertation) and converges to an accurate end-effector position, but to a less accurate approximation of the end-effector orientation.

CHAPTER 4
NEW APPROACH

Link-Frames Assignment

Some simplification in the mathematical description of the inverse kinematics problem can be obtained if certain simple rules for assigning the link-frames are applied.

In selecting frame F_i, the direction of vector $\mathbf{z}_i$ is always chosen so that twist angle α_i is in the interval $[0,\pi)$. If $\alpha_i = 0$, then vectors $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ are parallel and the common normal can be arbitrarily located along both axes. In this case the position of Frame F_i should be chosen so that d_i is equal to zero.

For an n-DOF robot, frame F_n, attached to the end-effector, can be chosen so that it differs from link frame F_{n-1} only by a rotation of angle θ_n about $\mathbf{z}_{n-1}$. In other words, F_n can be selected so that $d_n = a_n = \alpha_n = 0$, without loss of generality. We prove this point for n = 6, but it is valid for any relevant value of n. Let us assume that Eq. (2.14) is to be solved with a 6-DOF manipulator for which d_6, a_6, or α_6 is not equal to 0, then the homogeneous matrix transform $\mathbf{A}_6$ decomposes into

$$\mathbf{A}_6 = \underline{\mathbf{A}}_6 \, \mathbf{B}_6$$

as given in Eq. (2.7). Equation (2.14) is then equivalent to

$$A_1\ A_2\ A_3\ A_4\ A_5\ \underline{A}_6\ = P\ B_6^{-1}$$

where the right hand side of this last equation is seen to be a constant pose matrix for a manipulator described by the left hand side (i.e. one for which $d_6=a_6=\alpha_6=0$ so that $A_6=\underline{A}_6$).

When joint 1 is not prismatic, d_1 is constant and the origin of the base frame F_0 can be positioned so that d_1 is equal to 0.

The Reduced System of Equations

For a 6-DOF arm, Eq. (2.14) becomes

$$A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ = P \qquad (4.1)$$

and it yields twelve non trivial scalar equations in the six unknown variables. It is desirable to reduce this system to a minimal number of equations involving as few of the joint variables as possible. For all-revolute, 6-DOF manipulators, Tsai and Morgan (1984) have established that with respect to frame F_3, the z-component of the position vector ${}^3\mathbf{p}$ and that of vector ${}^3\mathbf{t}$ along with the inner products $({}^3\mathbf{t}.{}^3\mathbf{p})$ and $({}^3\mathbf{p}.{}^3\mathbf{p})$ provide 4 equations in only 4 of the unknowns, thereby reducing the complexity of the problem. The process of obtaining these four equations involved multiplying the **A**-matrices and simplifying the expressions obtained for the elements of $^3\mathbf{t}$ and $^3\mathbf{p}$. Besides being lengthy, this method does not allow insight into the mechanisms that make the simplifications possible. The approach presented here provides the same results with much less effort and greater insight by taking advantage of the properties of rotation transformations.

By writing the product of two A matrices in the form

$$\mathbf{A}_i\ \mathbf{A}_j = \begin{bmatrix} \mathbf{R}_i\mathbf{R}_j & (\mathbf{R}_i\mathbf{l}_j + \mathbf{l}_i) \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

we divide Eq. (4.1) into a position equation

$$\mathbf{p} = \mathbf{R}_1(\mathbf{R}_2(\mathbf{R}_3(\mathbf{R}_4(\mathbf{R}_5\mathbf{l}_6+\mathbf{l}_5)+\mathbf{l}_4)+\mathbf{l}_3)+\mathbf{l}_2)+\mathbf{l}_1 \qquad (4.2)$$

and an orientation equation

$$\mathbf{R} = \mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{R}_5\ \mathbf{R}_6. \qquad (4.3)$$

With the frame assignment conventions discussed, $\mathbf{l}_6=\mathbf{0}$ when joint 6 is revolute. Equation (4.2) then simplifies to

$$\mathbf{p} = \mathbf{R}_1(\mathbf{R}_2(\mathbf{R}_3(\mathbf{R}_4\mathbf{l}_5+\mathbf{l}_4)+\mathbf{l}_3)+\mathbf{l}_2)+\mathbf{l}_1. \qquad (4.4)$$

Three independent scalar equations for p_x, p_y, and p_z can be obtained from Eq. (4.4) and more equations can be selected out of the 9 scalar equations implied by Eq. (4.3).

Since rotations are orthogonal transformations, they leave inner products invariant, hence

$$\mathbf{R}\,\mathbf{u} \cdot \mathbf{R}\,\mathbf{v} = \mathbf{u} \cdot \mathbf{v} \tag{4.5}$$

for any rotation matrix **R** and any vectors **u** and **v**. A special case of (4.5) that is very useful is

$$\mathbf{R}\,\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{R}^{-1}\mathbf{v}. \tag{4.6}$$

These properties are extremely efficient in eliminating algebraic terms and unnecessary joint variables when applied to Eqs. (4.3) and (4.4) if it is further recognized that

$$\mathbf{R}_i^{-1}\mathbf{l}_i = [a_i, d_i\sigma_i, d_i\tau_i]^T \tag{4.7}$$

and

$$\mathbf{R}_i^{-1}\mathbf{z} = [0, \sigma_i, \tau_i], \tag{4.8}$$

where $\mathbf{z} = [\,0,\ 0,\ 1]^T$, are always independent of Θ_i. Also, due to the frame assignments discussed above,

$$\mathbf{R}_6\ \mathbf{z} = \mathbf{R}_6^{-1}\ \mathbf{z} = \mathbf{z}$$

in all cases since frame F_6 is chosen to force $\alpha_6 = 0$.

By repeated use of Eqs. (4.5) and (4.6), we obtain four reduced equations from Eqs. (4.3) and (4.4).

<u>tz equation</u>.

$$t_z = \mathbf{t} \cdot \mathbf{z} = (\mathbf{R}\ \mathbf{z}) \cdot \mathbf{z}$$

$$t_z = (\mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{R}_5\ \mathbf{R}_6\ \mathbf{z}) \cdot \mathbf{z}$$

$$t_z = (\mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{R}_5\ \mathbf{z}) \cdot \mathbf{z}$$

$$t_z = \mathbf{z} \cdot (\mathbf{R}_5^{-1}\ \mathbf{R}_4^{-1}\ \mathbf{R}_3^{-1}\ \mathbf{R}_2^{-1}\ \mathbf{R}_1^{-1}\ \mathbf{z}) \tag{4.9}$$

pz equation

$$\mathbf{p} = R_1\ R_2\ R_3\ R_4\ \mathbf{q}$$

with

$$\mathbf{q} = \mathbf{l}_5 + R_4^{-1}(\mathbf{l}_4 + R_3^{-1}(\mathbf{l}_3 + R_2^{-1}(\mathbf{l}_2 + R_1^{-1}\mathbf{l}_1))),$$

so that

$$p_z = \mathbf{p} \cdot \mathbf{z} = \mathbf{q} \cdot (R_4^{-1}\ R_3^{-1}\ R_2^{-1}\ R_1^{-1})\ \mathbf{z}. \qquad (4.10)$$

p.t equation.

$$\mathbf{p} \cdot \mathbf{t} = R_5^{-1}\mathbf{q} \cdot \mathbf{z} \qquad (4.11)$$

p.p equation.

$$\mathbf{p}\cdot\mathbf{p} = \mathbf{p}^2 = \mathbf{q}\cdot\mathbf{q} = \mathbf{q}^2. \qquad (4.12)$$

Since $R_1^{-1}\mathbf{l}_1$ and $R_1^{-1}\mathbf{z}$ are independent of Θ_i (Eqs. (4.7) and (4.8)), vector $\mathbf{q}$ and Eqs. (4.9)-(4.12) are easily seen to be independent of the first and last joint variables and therefore form a system of 4 equations in 4 unknowns. Figure 4.1 illustrates this discussion. With $\mathbf{l}_6$=0, vector $\mathbf{t}$, which coincides with $\mathbf{z}_5$, and the position vector $\mathbf{p}$ of the origin of frame F_6 are invariant in the rotation R_6 (rotation about $\mathbf{z}_5$ which can only affect the end-effector orientation). Rotation about $\mathbf{z}_0$ has no effect on the z-component of any vector expressed in frame F_0. Hence, p_z and t_z are independent of Θ_1 as well. Finally, since rotation about $\mathbf{z}_0$ moves all the robotic structure as a bloc, it does not affect the length of vector $\mathbf{p}$ or the inner product of $\mathbf{t}$ and $\mathbf{p}$.

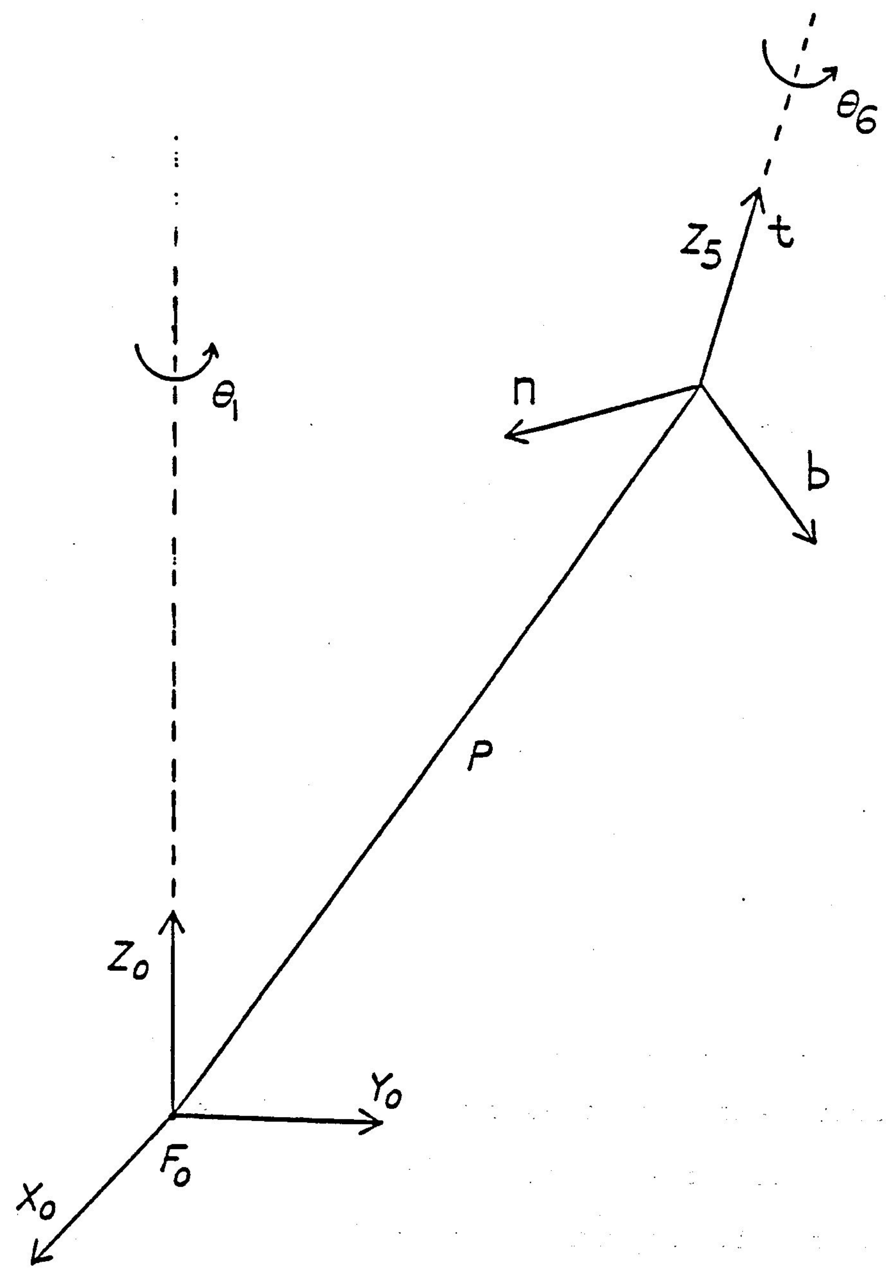


Figure 4.1. Rotations about $\mathbf{z}_0$ or $\mathbf{z}_5$ do not affect t_z, p_z, $\mathbf{t.p}$, and $\mathbf{p.p}$.

The reduced system of equations (4.9)-(4.12) determines candidate solutions for joint variables 2, 3, 4, and 5. Once this system of equations is solved, the remaining two variables can be found by using more equations from (4.1) and then tested for consistency. The power of this approach will become apparent for specific manipulators as further simplification using Eqs. (4.5)-(4.8) becomes obvious. Furthermore, simplification by use of rotation inner-product invariance is computationally economical and provides greater insight into the structure and properties of the inverse kinematic equations.

Additional Inverse Kinematics Equations

Equations (4.9)-(4.12) are necessary, but not sufficient. Although they are satisfied by all solution sets of Eq. (4.1), they are also, in general, satisfied by extraneous solutions. This problem was reported by Tsai and Morgan (1984) as well.

Another problem with considering Eqs. (4.9)-(4.12) alone is the presence of sign ambiguities. In many practical situations, one of the equations will allow a closed-form solution for either the sine or the cosine function of a revolute variable θ. The other function needs to be computed using the Pythagorean identity, which offers two values opposite in sign. Although both signs can be tried in the search for a solution, in some cases the number of sign ambiguities can be reduced by considering more constraints from Eqs. (4.3) and (4.4). Additional equations will also help filter out extraneous solutions and in some cases will ease the solution-finding process rather than complicate it. The x- and y-components of vectors **t** and **p** provide convenient additional constraints at the cost of introducing the variable Θ_1. Equations

$$t_x = \mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{R}_5\ \mathbf{z} \cdot \mathbf{x}, \tag{4.13}$$

$$t_y = \mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{R}_5\ \mathbf{z} \cdot \mathbf{y}, \tag{4.14}$$

$$p_x = (\mathbf{R}_1(\mathbf{R}_2(\mathbf{R}_3(\mathbf{R}_4\mathbf{l}_5+\mathbf{l}_4)+\mathbf{l}_3)+\mathbf{l}_2)+\mathbf{l}_1) \cdot \mathbf{x}, \tag{4.15}$$

and

$$p_y = (\mathbf{R}_1(\mathbf{R}_2(\mathbf{R}_3(\mathbf{R}_4\mathbf{l}_5+\mathbf{l}_4)+\mathbf{l}_3)+\mathbf{l}_2)+\mathbf{l}_1) \cdot \mathbf{y}, \tag{4.16}$$

where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are the usual canonical unit vectors, are still independent of Θ_6.

Solving Inverse Kinematic Equations

Once the reduced set of equations (4.9)-(4.12) and the additional equations (4.13)-(4.16) have been expanded, the problem becomes that of extracting the values of the joint angles from the equations which are in terms of the sines and cosines of the angles. In this section, we describe some of the techniques that can be used for this task.

Certain simple arm geometries allow a closed form solution. For such arms, one of the equations will have the form

$$a\, S + b\, C = d$$

where S and C are the sine and cosine, respectively, of some angle Θ. If the constants a, b, and d are known, then there are two possible solutions when $a^2 + b^2 \geq d^2$,

$$\Theta = \mathrm{atan2}[d, \pm\sqrt{(a^2+b^2-d^2)}] - \mathrm{atan2}(b,a)$$

where atan2(v,w) returns the angle arctan(v/w) adjusted to the proper quadrant according to the sign of the real numbers v and w.

A special case occurs when a = 0 or b = 0. The equation can then be solved for S or C separately. The other variable can be obtained from the Pythagorean identity

$$S^2 + C^2 = 1 \qquad (4.17)$$

with a sign ambiguity. Again, this leads to two possible values for the angle Θ,

$$\Theta = \mathrm{atan2}(S, \pm \sqrt{(1 - S^2)}) \qquad \text{if S is computed or}$$

$$\Theta = \mathrm{atan2}(\pm \sqrt{(1 - C^2)}, C) \qquad \text{if C is the known}$$

variable.

A value of Θ can be directly and uniquely obtained when two linear equations in the sine and cosines of one angle are obtained. In this case the values of S and C are computed and the angle Θ is then given by

$$\Theta = \operatorname{atan2}(S, C).$$

Exchanging Base and End-Effector Frames

The inverse kinematics problem consists of finding joint variables that realize a given relationship between two frames, the base frame F_0 and the end-effector frame F_n. The roles of these two frames are in fact interchangeable as we illustrate in Figure 4.2. This means that the problem can be viewed as finding the joint variables necessary for the robot to achieve the base frame as viewed from the end-effector frame. This problem reversal requires that the DH-parameters be rearranged and intermediate frames be reassigned as illustrated in Figure 4.1 but it can be useful in many ways. For example, several computationally efficient inverse kinematic techniques have been developed for robots with the last three joint axes intersecting at a common point (Featherstone 1983, Hollerbach and sahar 1983, Paul and Zhang 1986, Low and Dubey 1986). The same techniques can be used for a robot whose first three axes intersect by reversing the roles of end-effector and base frame. In the next chapters, we will use this problem reversal technique to avoid repetitious developments.

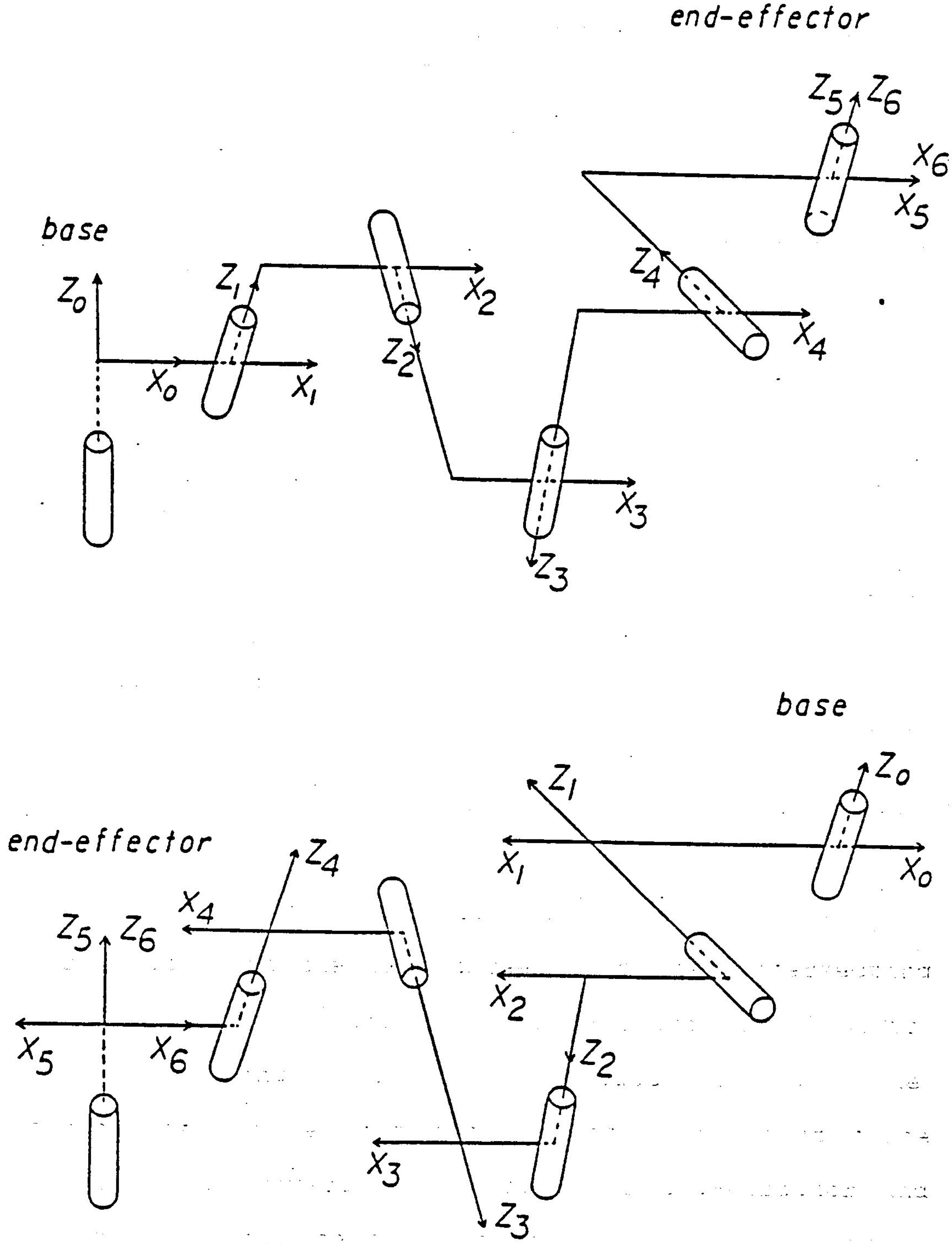


Figure 4.2. Interchanging base and end-effector frames.

CHAPTER 5
SOLVING 4-DOF MANIPULATORS

Reduced System of Equations

For 4-DOF robot arms, the inverse kinematic problem is solved when 4 joint values are found that satisfy the equation

$$\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4 = \mathbf{P}. \tag{5.1}$$

Equation (5.1) decouples into a position equation

$$\mathbf{R}_1\ (\mathbf{R}_2\ (\mathbf{R}_3\ \mathbf{l}_4 + \mathbf{l}_3) + \mathbf{l}_2) + \mathbf{l}_1 = \mathbf{p}, \tag{5.2}$$

and an orientation equation given by

$$\mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4 = \mathbf{R}. \tag{5.3}$$

When the fourth joint is revolute, $\mathbf{l}_4=\mathbf{0}$ is obtained by proper choice of frame F_4 and Eq. (5.2) simplifies to

$$\mathbf{R}_1\ (\mathbf{R}_2\ \mathbf{l}_3 + \mathbf{l}_2) + \mathbf{l}_1 = \mathbf{p}. \tag{5.4}$$

A reduced system of four equations in the sines and cosines of joint angles Θ_1 and Θ_3 can be derived by considering the quantities t_z, p_z and the inner products $\mathbf{t}.\mathbf{p}$ and $\mathbf{p}.\mathbf{p}$ expressed in frame F_1

Vector $\mathbf{t}$ is given by

$$\mathbf{t} = R\ \mathbf{z} = R_1\ R_2\ R_3\ R_4\ \mathbf{z}$$

where $\mathbf{z}$ is the third canonical unit vector $\mathbf{z} = [\ 0,\ 0,\ 1]^T$. Since twist angle α_4 is equal to 0,

$$R_4\ \mathbf{z} = [\sigma_4\ S_4,\ -\sigma_4\ C_4,\ \tau_4]^T = [0,\ 0,\ 1]^T = \mathbf{z},$$

and the expression for $\mathbf{t}$ simplifies to

$$\mathbf{t} = R_1\ R_2\ R_3\ \mathbf{z}. \qquad (5.5)$$

Multiplying by R_1^{-1} yields

$$R_1^{-1}\ \mathbf{t} = R_2\ R_3\ \mathbf{z}$$

and the inner product of each side of this equality with vector $\mathbf{z}$ provides

$$\mathbf{z}\ .\ (R_1^{-1}\ \mathbf{t}) = \mathbf{z}\ .\ (R_2\ R_3\ \mathbf{z}).$$

Eq. (4.5) applied to both sides of this last equation gives

$$R_1\ \mathbf{z}\ .\ \mathbf{t} = R_2^{-1}\ \mathbf{z}\ .\ R_3\ \mathbf{z}$$

or

$$(R_1\ \mathbf{z}).\mathbf{t} - (R_2^{-1}\ \mathbf{z}).(R_3\ \mathbf{z}) = 0. \qquad (5.6)$$

Since $R_2^{-1}\ \mathbf{z} = [0,\ \sigma_2,\ \tau_2]^T$ does not depend on θ_2, this last equation is independent of joint variables 2 and 4.

Subtracting vector $\mathbf{l}_1$ from both sides of Eq. (5.4) and multiplying by R_1^{-1} yields

$$R_2 \, l_3 + l_2 = R_1^{-1} \, (p - l_1) \tag{5.7}$$

and taking the inner-product with vector $\mathbf{z}$ provides

$$(\mathbf{z} \cdot R_2 \, l_3) + (\mathbf{z} \cdot l_2) = (\mathbf{z} \cdot R_1^{-1} \, p) - (\mathbf{z} \cdot R_1^{-1} \, l_1).$$

Applying (4.5) to the first term of both sides of this equation gives (after rearranging terms)

$$R_1 \, \mathbf{z} \cdot p - R_2^{-1} \, \mathbf{z} \cdot l_3 = \mathbf{z} \cdot R_1^{-1} \, l_1 + \mathbf{z} \cdot l_2. \tag{5.8}$$

The right hand side of Eq. (5.8) is constant since

$$R_i^{-1} \, l_i = \begin{bmatrix} a_i \\ d_i \sigma_i \\ d_i \tau_i \end{bmatrix} \quad \text{and } \mathbf{z} . l_i = d_i \text{ are independent of } \Theta_i.$$

Multiplying Eq. (5.7) by R_2^{-1} gives

$$l_3 + R_2^{-1} \, l_2 = R_2^{-1} \, R_1^{-1} \, (p - l_1) \tag{5.9}$$

and multiplication of Eq. (5.5) by $R_2^{-1} \, R_1^{-1}$ yields

$$R_3 \, \mathbf{z} = R_2^{-1} \, R_1^{-1} \, t. \tag{5.10}$$

The inner product of corresponding sides of equations (5.9) and (5.10) produces

$$(l_3 + R_2^{-1} \, l_2) . (R_3 \, \mathbf{z}) = [R_2^{-1} \, R_1^{-1} \, (p - l_1)] . [R_2^{-1} \, R_1^{-1} \, t].$$

Repeated use of properties (4.4) and (4.5) and reordering simplifies this last equation to

$$\mathbf{l}_1 \cdot \mathbf{t} - [\mathrm{R}_2^{-1}\, \mathbf{l}_2 \cdot \mathrm{R}_3\, \mathbf{z}] = \mathbf{t} \cdot \mathbf{p} - [\mathrm{R}_3^{-1}\, \mathbf{l}_3 \cdot \mathbf{z}]. \qquad (5.11)$$

Equation (5.11) is also independent of Θ_2 and Θ_4.

Using Eq. (5.4), the inner-product **p.p** satisfies

$$\mathbf{p} \cdot \mathbf{p} = [\mathrm{R}_1\, (\mathrm{R}_2\, \mathbf{l}_3 + \mathbf{l}_2) + \mathbf{l}_1] \cdot [\mathrm{R}_1\, (\mathrm{R}_2\, \mathbf{l}_3 + \mathbf{l}_2) + \mathbf{l}_1].$$

Expanding the left hand side, using inner-product invariance of rotations where needed, and rearranging terms yield

$$\mathbf{l}_3 \cdot \mathrm{R}_2^{-1}\, \mathbf{l}_2 + \mathbf{p} \cdot \mathbf{l}_1 = [\mathbf{p} \cdot \mathbf{p} + \mathbf{l}_1 \cdot \mathbf{l}_1 - \mathbf{l}_2 \cdot \mathbf{l}_2 - \mathbf{l}_3 \cdot \mathbf{l}_3]/2. \quad (5.12)$$

Equations (5.11), (5.2), (5.18), and (5.12) form a linear system in the variables S_1, C_1, S_3 and C_3. The four equations obtained are

$$a_1 t_y\, S_1 + a_1 t_x\, C_1 + a_2 \sigma_3\, S_3 - \sigma_2 \sigma_3 d_2\, C_3 = r_1 \qquad (5.13)$$

$$\sigma_1 t_x\, S_1 - \sigma_1 t_y\, C_1 + \sigma_2 \sigma_3\, C_3 = r_2 \qquad (5.14)$$

$$\sigma_1 p_x\, S_1 - \sigma_1 p_y\, C_1 - \sigma_2 a_3\, S_3 = r_3 \qquad (5.15)$$

$$a_1 p_y\, S_1 + a_1 p_x\, C_1 + \sigma_2 a_3 d_2\, S_3 + a_2 a_3\, C_3 = r_4 \qquad (5.16)$$

with

$$r_1 = \mathbf{t} \cdot \mathbf{p} - \tau_3 d_3 - d_1 t_z - \tau_2 \tau_3 d_2 \qquad (5.17)$$

$$r_2 = \tau_2 \tau_3 - \tau_1 t_z \qquad (5.18)$$

$$r_3 = \tau_1 (d_1 - t_z) + d_2 + \tau_2 d_3 \qquad (5.19)$$

$$r_4 = (\mathbf{p} \cdot \mathbf{p} + a_1^2 + d_1^2 - a_2^2 - d_2^2 - a_3^2 - d_3^2)/2 - d_1 p_z - \tau_2 d_2 d_3. \qquad (5.20)$$

The linear system of equations formed by Eqs. (5.13)-(5.16) will be referred to as the reduced system for a four-DOF manipulator. Although d_1 can be assumed zero without loss of generality for a 4-DOF manipulator, this system of equations will be used for 4-DOF sections of larger manipulators (next chapters) for which the parameter corresponding to d_1 will, in general, not be zero. Hence, d_1 is assumed not equal to zero at this point.

A unique solution to the reduced system is given by

$$\begin{bmatrix} S_1 \\ C_1 \\ S_3 \\ C_3 \end{bmatrix} = \mathbf{H}^{-1} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \tag{5.21}$$

where

$$\mathbf{H} = \begin{bmatrix} a_1 t_y & a_1 t_x & a_2 \sigma_3 & -\sigma_2 \sigma_3 d_2 \\ \sigma_1 t_x & -\sigma_1 t_y & 0 & \sigma_2 \sigma_3 \\ \sigma_1 p_x & -\sigma_1 p_y & -\sigma_2 a_3 & 0 \\ a_1 p_y & a_1 p_x & \sigma_2 a_3 d_2 & a_2 a_3 \end{bmatrix}, \tag{5.22}$$

when matrix **H** is nonsingular. Unique values of Θ_1 and Θ_3 are thus obtained from the values of S_1, C_1, S_3, and C_3. The case where **H** is not invertible is discussed in the next sections because of its interesting implications.

With Θ_1 and Θ_3 known, Eq. (5.7) provides a way to solve for Θ_2. Indeed, when expanded, the first 2 components yield

$$(\sigma_2 d_3 - \tau_2 a_3 S_3)\ S_2 + (a_2 + a_3 C_3)\ C_2 = C_1 p_x + S_1 p_y - a_1 \tag{5.23}$$

and

$$(a_2+a_3C_3)\ S_2 - (\sigma_2 d_3-\tau_2 a_3 S_3)\ C_2 =$$

$$-\tau_1 S_1 p_x + \tau_1 C_1 p_y + \sigma_1(p_z-d_1). \qquad (5.24)$$

When the determinant of this linear system of equations in S_2 and C_2 is not 0, a unique value of θ_2 can be computed. Otherwise, we can obtain θ_2 uniquely from another linear system of equations in S_2 and C_2,

$$(\tau_2\sigma_3 C_3+\sigma_2\tau_3)\ S_2 + \sigma_3 S_3\ C_2 = C_1 t_x + S_1 t_y \qquad (5.25)$$

and

$$\sigma_3 S_3\ S_2 - (\tau_2\sigma_3 C_3+\sigma_2\tau_3)\ C_2$$

$$= -\tau_1 S_1 t_x + \tau_1 C_1 t_y + \sigma_1 t_z, \qquad (5.26)$$

derived from Eq. (5.5). Note that θ_2 can also be computed using a system of two equations formed by Eq. (5.23) or Eq. (5.24) and one of Eqs. (5.25) and (5.26). The Appendix shows that the determinants of the two systems of equations above are simultaneously zero only when joint axis 2 aligns with another joint axis which puts the arm in a degenerate configuration.

To complete the 4-DOF solution set, we use Eq. (5.2) which can be rewritten as

$$\mathbf{R}_4 = \mathbf{R}_1^{-1}\ \mathbf{R}_2^{-1}\ \mathbf{R}_1^{-1}\ \mathbf{R}.$$

The first column vector of $\mathbf{R}_4$, obtained by multiplying both sides by the first canonical unit vector $\mathbf{x} = [1,0,0]^T$,

$$R_4\ \mathbf{x} = \begin{bmatrix} c_4 \\ s_4 \\ 0 \end{bmatrix} = R_3^{-1}\ R_2^{-1}\ R_1^{-1}\ \mathbf{R}\ \mathbf{x} = R_3^{-1}\ R_2^{-1}\ R_1^{-1} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

can be used to compute the last variable Θ_4. This shows that a 4-DOF inverse kinematic problem will, in general (general in the sense that matrix $\mathbf{H}$ is nonsingular), yield a unique solution set. However, for some manipulator geometries and/or some particular end-effector poses, the problem may have more than one solution.

Special 4-DOF Manipulator Geometries

Equation (5.21) is valid only when matrix $\mathbf{H}$ is invertible. The determinant of matrix $\mathbf{H}$, computed from Eq. (5.22), is given by

$$d_H = \sigma_1\ \sigma_2\ a_1[a_2\ (a_3^2\ w_1 + \sigma_3^2\ w_2) + 2\ \sigma_2\ \sigma_3\ a_3\ d_2\ w_3] + \sigma_3\ a_3\ [\sigma_1^2\ (a_2^2 + \sigma_2^2\ d_2^2) + \sigma_2^2\ a_1^2]\ w_4 \qquad (5.27)$$

where the quantities w_1, w_2, w_3, and w_4 are defined, in terms of the components of pose vectors $\mathbf{t}$ and $\mathbf{p}$, as

$$w_1 = t_x^2 + t_y^2$$
$$w_2 = p_x^2 + p_y^2$$
$$w_3 = p_x\ t_x$$
$$w_4 = p_x\ t_y - p_y\ t_x.$$

<u>Analyzing d_H</u>

Equation (5.27) shows that the value of d_H depends on the seven robot parameters σ_1, σ_2, σ_3, a_1, a_2, a_3, and d_2 as well as the pose quantities w_1, w_2, w_3, and w_4. However, for certain robot structures d_H is equal to zero no matter what the end-effector pose is. The expression of d_H above provides us with a way to find all such 4-DOF robot geometries. Due to our link frames assignment, the only robot parameter in the expression of d_h that can be negative is d_2. By expanding Eq. (5.27) we get

$$\begin{aligned} d_H = {} & \sigma_1 \sigma_2 a_1 a_2 a_3^2 w_1 + \sigma_1 \sigma_2 \sigma_3^2 a_1 a_2 w_2 \\ & + 2 \sigma_1 \sigma_2^2 \sigma_3 a_1 a_3 d_2 w_3 + \sigma_1^2 \sigma_3 a_2^2 a_3 w_4 \\ & + \sigma_1^2 \sigma_2^2 \sigma_3 a_3 d_2^2 w_4 + \sigma_2^2 \sigma_3 a_1^2 a_3 w_4 \end{aligned} \qquad (5.28)$$

where only the quantities d_2, w_3, and w_4 can be negative. If an arm structure is such that d_H is zero for every possible end-effector pose, then d_H will be zero even for a pose with positive w_3 and w_4.

If we assume w_3, and w_4 non negative, then with d_2 negative, d_H can be zero if the equality

$$\begin{aligned} -2 \sigma_1 \sigma_2^2 \sigma_3 a_1 a_3 d_2 w_3 = {} & \\ & \sigma_1 \sigma_2 a_1 a_2 a_3^2 w_1 + \sigma_1 \sigma_2 \sigma_3^2 a_1 a_2 w_2 \\ + \sigma_1^2 \sigma_3 a_2^2 a_3 w_4 + \sigma_1^2 \sigma_2^2 \sigma_3 a_3 d_2^2 w_4 + \sigma_2^2 \sigma_3 a_1^2 a_3 w_4 & \end{aligned}$$

holds. However, such an equality is actually a condition on pose quantities w_1, w_2, w_3, and w_4. We conclude that robot structures for which d_H is always zero (independent of the end-effector pose) have DH-parameters σ_1, σ_2, σ_3, a_1, a_2, a_3, and d_2 for which each of the six terms in Eq. (5.28) is individually zero. These terms, in turn, are zero when particular structure parameters are equal to zero. For example, $\sigma_1=\sigma_2=0$ will make the determinant zero. In order to enumerate the minimum number of distinct combinations of zero parameters that make $d_H = 0$, we examine all possible cases when a particular parameter is zero. We get seven simpler expressions of d_H, listed in Table 5-1., by separately assuming each relevant parameter to be equal to zero.

Table 5-1 provides a simple mean for finding all (pose-independent) 4-DOF robot geometries for which matrix **H** will be singular. In the next section, we show that the inverse kinematics problem for such robots can still be solved by use of the reduced system of equations (5.13)-(5.16).

<u>Special 4-DOF Arm Structures</u>

A trivial condition occurs when two consecutive joint axes coincide somewhere along the arm. Such a degenerate condition is detected by $a_i = \alpha_i = 0$ for some joint i. In this case, the manipulator loses one degree of freedom and becomes a redundant 3-DOF arm. If a solution set exists for such an arm, there will be an infinite number of solution sets. A careful analysis of Table 5-1 shows that there are only ten minimal, non-trivial, conditions on the arm

Table 5-1. Special expressions for d_H.

Condition	d_H
1 $\sigma_1 = 0$	$\sigma_2^2 \sigma_3 a_1^2 a_3 w_4$
2 $\sigma_2 = 0$	$\sigma_1^2 \sigma_3 a_2^2 a_3 w_4$
3 $\sigma_3 = 0$	$\sigma_1 \sigma_2 a_1 a_2 a_3^2 w_1$
4 $a_1 = 0$	$\sigma_1^2 \sigma_3 a_3 (a_2^2 + \sigma_2^2 d_2^2) w_4$
5 $a_2 = 0$	$\sigma_2^2 \sigma_3 a_3[2 \sigma_1 a_1 d_2 w_3 + (a_1^2 + \sigma_1^2 d_2^2) w_4]$
6 $a_3 = 0$	$\sigma_1 \sigma_2 \sigma_3^2 a_1 a_2 w_2$
7 $d_2 = 0$	$\sigma_1 \sigma_2 a_1 a_2 (\sigma_3^2 w_2 + a_3^2 w_1)$ $+ \sigma_3 a_3 (\sigma_2^2 a_1^2 + \sigma_1^2 a_2^2) w_4$

geometry (pose-independent) for $d_H = 0$. All ten conditions are listed and described in Table 5-2 and illustrated in Figure 5.1.

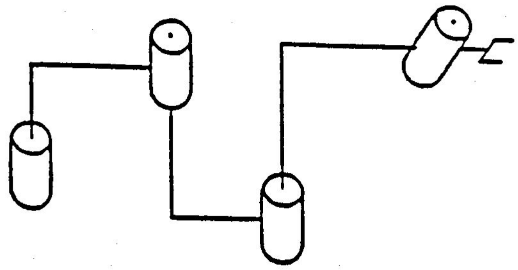
The first three conditions in Table 5-2 follow from the first entry of Table 5-1. Conditions 4 and 5 are derived from the second entry in Table 5-1 after dropping duplicate conditions already established. Continuing in this fashion, all of Table 5-2 can be completed. Observe that entry 7 in Table 5-1 does not add any new conditions into Table 5-2 since all minimal sets of zero parameters implied by $d_H=0$ in entry 7 have already been accounted for.

It must be noted that d_H can still be zero for 4-DOF arm geometries not listed in the preceding Table. However from the discussion above, we see that such a situation can only happen at particular end-effector poses whereas d_H will

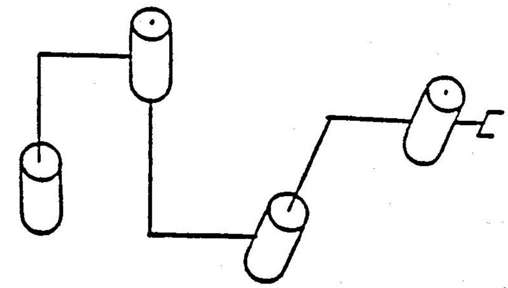
Table 5-2. Special structures of 4-DOF manipulators

Condition		Description
1	$\sigma_1=\sigma_2=0$	First 3 joint axes (1,2,3) are parallel
2	$\sigma_1=\sigma_3=0$	First 2 axes (1,2) are parallel and Last 2 axes (3,4) are parallel
3	$\sigma_1=a_3=0$	First 2 axes (1,2) are parallel and last 2 axes (3,4) intersect
4	$\sigma_2=\sigma_3=0$	Last three axes (2,3,4) are parallel
5	$\sigma_2=a_3=0$	Middle axes (2,3) are parallel and last 2 axes (3,4) intersect
6	$\sigma_3=a_1=0$	First 2 axes (1,2) intersect & last 2 axes (3,4) are parallel
7	$\sigma_3=a_2=0$	Middle axes (2,3) intersect & last 2 axes (3,4) are parallel
8	$a_1=a_3=0$	First 2 axes (1,2) intersect & last 2 axes (3,4) intersect
9	$a_1=a_2=d_2=0$	First 3 axes (1,2,3) intersect
10	$a_2=a_3=0$	Middle 2 axes (2,3) intersect and last 2 axes (3,4) intersect

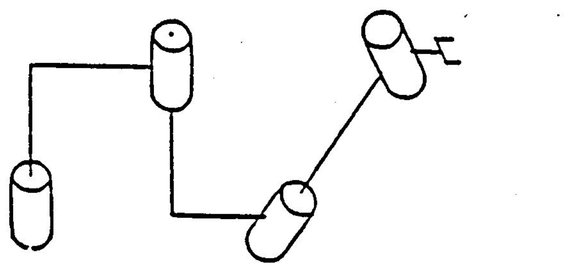
be zero for the geometries described in Table 5-2 at any pose. We now examine in detail the inverse kinematics of each of the special 4-DOF robot architectures described in Table 5-2 and illustrated in Figure 5.1.



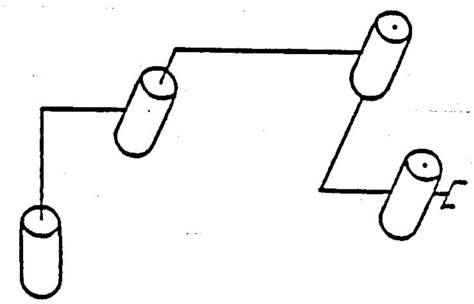
1. First three joint axes are parallel.



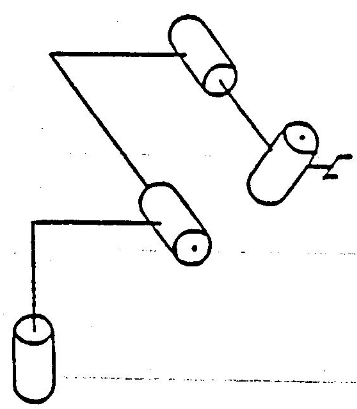
2. Axes 1 and 2 are parallel, 3 and 4 are parallel.



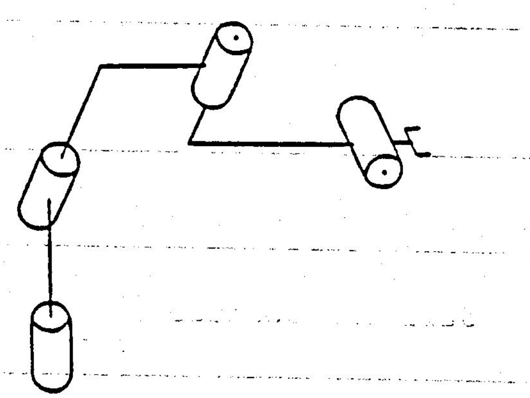
3. Axes 1 and 2 are parallel, axes 3 and 4 intersect.



4. Axes 2, 3, and 4 are parallel.

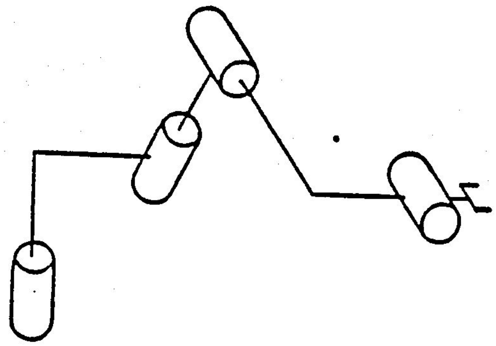


5. Axes 2 and 3 are parallel, 3 and 4 intersect.

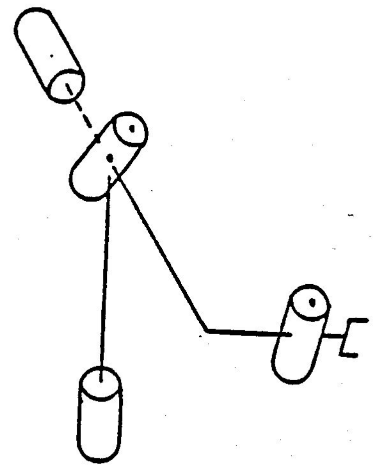


6. Axes 1 and 2 intersect 2 and 3 are parallel.

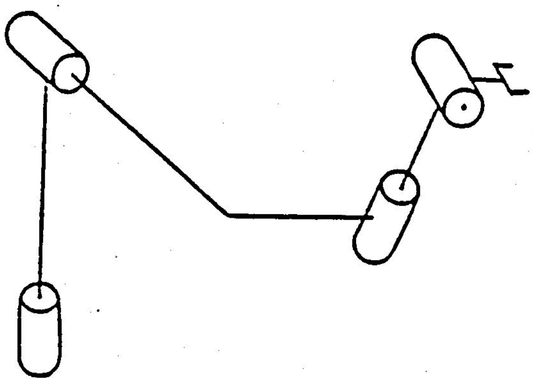
Figure 5.1. Special 4-DOF structures.



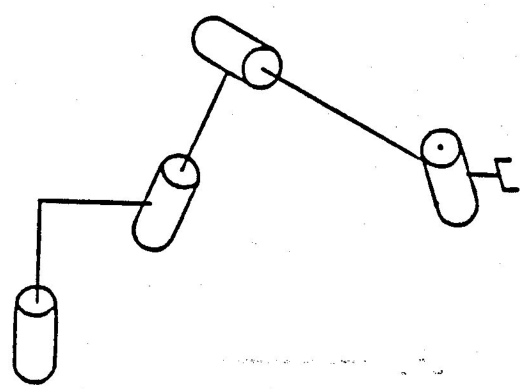
7.Axes 2 and 3 intersect,
3 and 4 are parallel.



8.Axes 1, 2, and 3 intersect.



9.Axes 1 and 2 intersect,
3 and 4 intersect.



10.Axes 2 and 3 intersect,
3 and 4 intersect.

Figure 5.1--Continued.

The reduced system of equations (5.13)-(5.16) can still be efficiently used to find all the solution sets of Eq. (5.1) when matrix **H** is singular.

<u>Case 1</u>: $\sigma_1=\sigma_2=0$. First three joint axes are parallel (Entry 1 in Table 5-2).The reduced system of equations becomes

$$a_1t_y\ S_1 + a_1t_x\ C_1 + a_2\sigma_3\ S_3 = r_1 \qquad (5.29)$$

$$0 = r_2 \qquad (5.30)$$

$$0 = r_3 \qquad (5.31)$$

$$a_1p_y\ S_1 + a_1p_x\ C_1 + a_2a_3\ C_3 = r_4. \qquad (5.32)$$

Equations (5.30) and (5.31) are constraints on pose parameters t_z and p_z respectively. Only end-effector poses that satisfy $p_z = d_1 + d_2 + d_3$ and $t_z = \tau_3$ (Eqs. (5.18) and (5.19)) are solvable with this arm geometry.

Equations (5.29) and (5.32) still allow a solution in the style of Pieper (1968) by first eliminating S_3 and C_3 from the equations. This can be done by solving for S_3 and C_3 and substituting in the Pythagorean identity (4.17) to get

$$\{[r_1 - (a_1t_y\ S_1 + a_1t_x\ C_1)]/a_2\sigma_3\}^2 +$$
$$\{[r_1 - (a_1p_y\ S_1 + a_1p_x\ C_1)]/a_2a_3\}^2 = 1. \qquad (5.33)$$

With the trigonometric identities

$$S_1 = 2\ t_1/(1 + t_1^2) \text{ and } C_1 = (1 - t_1^2)/(1 + t_1^2)$$