LINEAR ALGEBRA

STRANG, SECTION 5.1

1. The assignment

- Read section 5.1 of Strang (pages 244-251).
- Read the following.
- Prepare the items below for presentation.

2. The Determinant

Generally, the most interesting matrices to look at are the square ones. For square matrices, there is an important number called the *determinant* which helps us determine if the matrix is invertible or not.

Strang lists 10 important properties of determinants in this section, and verifies them for 2×2 matrices. The verifications for general matrices aren't any harder, but they sure are **longer**, so I am glad he skipped them. Anyway, these properties are enough to get by when it is time to compute. In fact, clever use of these properties can save you a lot of time.

2.1. Interpretation.

Suppose that an $n \times n$ matrix A is represented as a collection of its column vectors:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}.$$

Then the geometric significance of the determinant is this: The number $\det(A)$ represents the *signed* n-dimensional volume of the n-dimensional box in $\mathbb{R}^n$ with sides $v_1, v_2, \dots, v_n$.

This takes a bit of getting used to, and the hardest part is the choice of signs. We choose a positive sign if the vectors $v_1, v_2, \dots, v_n$ have the same orientation as the standard basis.

2.2. The Importance of the Determinant.

The real importance of the determinant is described in the following theorem. Note that this is a special result for *square* matrices. The shape is crucial for this result.

Theorem (The Invertible Matrix Theorem)**.** Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- The columns of A are linearly independent.
- The columns of A are a spanning set for $\mathbb{R}^n$.
- The colums of A are a basis for $\mathbb{R}^n$.
- The rows of A are linearly independent.
- The rows of A are a spanning set for $\mathbb{R}^n$.
- The rows of A are a basis for $\mathbb{R}^n$.
- For any choice of vector $b \in \mathbb{R}^n$, the system of linear equations $Ax = b$ has a unique solution.
- A is invertible.
- The transpose A^T is invertible.
- $\det(A) \neq 0$.
- $\det(A^T) \neq 0$.

3. Sage instructions

I have made a Sage worksheet file with some basic commands that you might find useful. The file is called `section5_1.sagews`.

4. Questions for Section 5.1

Exercise 136. Exercises 1 & 2 from section 5.1 of Strang.

Exercise 137. Exercise 12 from section 5.1 of Strang.

Exercise 138. Exercise 14 from section 5.1 of Strang.

Exercise 139. Exercise 15 from section 5.1 of Strang.

Exercise 140. Exercise 23 from section 5.1 of Strang.