LINEAR ALGEBRA

STRANG, SECTION 3.1

1. The assignment

- Read section 3.1 of Strang (pages 120-127).
- Read the following and complete the exercises below.

2. Vector Spaces and Subspaces

The key concepts in this section are those of a *vector space* and of a *subspace*. The basic idea is that a vector space is a kind of place where the basic operations involved in a linear combination make sense. There is a set of rules for being a vector space, but they are all aimed at the fact that there are two operations (addition and scalar multiplication) and we can form linear combinations with them.

The biggest thing is that we possibly enlarge the kinds of things we call "vectors." My favorites are things like this:

- The set $M_{2,2}$ of 2×2 matrices is a vector space. But now the things we call vectors are actually matrices.
- The set $\mathcal{C}(\mathbb{R})$ of continuous functions with domain and range both equal to the set of real numbers is a vector space. But now the things we call vectors are actually functions.
- The set $\ell(\mathbb{R})$ of sequences $(x_1, x_2, x_3, x_4, \ldots)$ of real numbers is a vector space. But now the things we call vectors are actually sequences.

The idea of a subspace is some subset, some part, of a vector space which is a vector space in its own right. The prototype is the xy-plane inside of $\mathbb{R}^3$.

For now, the most important subspaces we see will be derived from individual matrices. Our first example is the column space of a matrix A. If A is an $m \times n$ matrix, then the column space $\text{col}(A)$ of A is the collection of n-vectors which can be expressed as linear combinations of the columns of A. This is our first exposure to the idea of a *span*. The column space of A is the subspace of $\mathbb{R}$ spanned by the columns of A.

3. Sage instructions

I have made a Sage worksheet file with some basic commands that you might find useful in investigating linear systems. The file is called `section3_1.sagews`.

4. Questions for Section 3.1

Exercise 78. Find an example of a vector which is not in the column space of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

or explain why that is not possible.

Exercise 79. Find an example of a vector which is not in the column space of the matrix

$$B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix},$$

or explain why it is not possible.

Exercise 80. Let $\mathcal{P}$ be the set of polynomials of degree 3 or less. Explain why $\mathcal{P}$ is a vector space, or explain why it is not.

Exercise 81. Consider the vector space $\mathbb{R}^2$. Explain why the following are not subspaces:

- The unit circle.
- The line $x + y = 4$.
- The union of lines $2x + 3y = 0$ and $x - y = 0$.
- The first quadrant where $x \geq 0$ and $y \geq 0$.

Exercise 82. Let W be the set of functions f in $\mathcal{C}(\mathbb{R})$ which satisfy the differential equation $f''(x) = 0$. Decide if W is a subspace of $\mathcal{C}(\mathbb{R})$, and explain your thinking.

Exercise 83. Design a 3×2 matrix whose column space does not contain the vector $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.