LINEAR ALGEBRA

STRANG, SECTION 3.6

1. The assignment

- Read section 3.6 of Strang (pages 184-190).
- Read the following and complete the exercises below.

2. The Four Subspaces

This section summarizes a big tool for understanding the behavior of a matrix as a function. Recall that if A is an $m \times n$ matrix, then we can think of it as defining a function

$$\begin{aligned} T_A : \mathbb{R}^n &\to \mathbb{R}^m \\ v &\mapsto Av \end{aligned}$$

which takes as inputs vectors from $\mathbb{R}^n$ and has as outputs vectors in $\mathbb{R}^m$. We have also seen that properties of matrix multiplication translate into properties that make into a *linear transformation.*

We now have four fundamental subspaces associated to the matrix A.

- The column space, $\mathrm{col}(A)$, spanned by all of the columns of A. This is a subspace of $\mathbb{R}^m$.
- The row space, $\mathrm{row}(A)$, spanned by all of the rows of A. This is a subspace of $\mathbb{R}^n$. This also happens to be the column space of A^T.
- The nullspace (or kernel), $\mathrm{null}(A)$, consisting of all those vectors x for which $Ax = 0$. This is a subspace of $\mathbb{R}^n$.
- The left nullspace, which is just the nullspace of A^T. This is a subspace of $\mathbb{R}^m$.

And we have a big result:

Theorem. If A is an $m \times n$ matrix with rank $\mathrm{rank}(A) = r$, then

- $\dim(\mathrm{col}(A)) = \dim(\mathrm{row}(A)) = r$,
- $\dim(\mathrm{null}(A)) = n - r$, and
- $\dim(\mathrm{null}(A^T)) = m - r$.

(*Study Hint: Write that out in English, with no notation. It will help you remember it.*

We will have more to say about these spaces when we reconsider the uses of the dot product in chapter 4.

3. Sage instructions

I have made a Sage worksheet file with some basic commands that you might find useful in investigating matrices. The file is called `section3_6.sagews`.

4. Questions for Section 3.6

Note: We may not present all of these in class.

Exercise 111. Find the four subspaces, including a basis of each, for the matrix

$$A = \begin{pmatrix} 7 & -1 & 3 \\ -2 & 4 & -5 \\ 1 & 11 & -12 \end{pmatrix}.$$

Exercise 112. Find the four subspaces, including a basis of each, for the matrix

$$B = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}.$$

Exercise 113. Do Strang Exercise 3.6 number 12.

Exercise 114. Do Strang Exercise 3.6 number 14.

Exercise 115. Do Strang Exercise 3.6 number 16.

Exercise 116. Do Strang Exercise 3.6 number 24.