

# LINEAR ALGEBRA

## SECTION 1.2

### 1. THE ASSIGNMENT

- Read *Strang* Chapter 1, section 2, pages 11-18.
- (Optional) Read *Hefferon* Chapter One, Part II Linear Geometry, section 2 Length and Angle Measures, pages 39-43.
- Complete the exercises below before class.

### 2. DISCUSSION

The dot product is a wonderful tool for encoding the geometry of Euclidean space, but it can be a bit mysterious at first. As *Strang* shows, it somehow holds all of the information you need to measure lengths and angles.

The connection to linear algebra comes about through this: a dot product with a “variable vector” is a way of writing a linear equation. For example,

$$\begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7x + 3y - 2z.$$

Sometimes this will allow us to connect linear algebra to geometry, and use geometric thinking to answer algebraic questions.

### 3. SAGE INSTRUCTIONS

I have made a Sage .sagews file called *section1-2.sagews* which contains some basic commands for dealing with vectors and dot products, and a couple of interactive demos to play with dot products in two and three dimensions.

### 4. THE EXERCISES

**Exercise 1.** What shape is the set of solutions  $\begin{pmatrix} x \\ y \end{pmatrix}$  to the equation

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 5?$$

That is, if we look at all possible vectors  $\begin{pmatrix} x \\ y \end{pmatrix}$  which make the equation true, what shape does this make in the plane?

What happens if we change the vector  $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$  to some other vector? What happens if we change the number 5 to some other number?

What happens if we move up a dimension, and instead work in three dimensional space?

**Exercise 2.** Find an example of two 2-vectors  $v$  and  $w$  such that  $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot v = 0$  and  $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot w = 0$  and  $v \cdot w = 0$ , or explain why such an example is not possible.

**Exercise 3.** Let  $v = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ . Find an example of a pair of vectors  $u$  and  $w$  such that  $v \cdot u < 0$  and  $v \cdot w < 0$  and  $w \cdot u = 0$ , or explain why no such pair of vectors can exist.

**Exercise 4.** Find an example of three 2-vectors  $u$ ,  $v$ , and  $w$  so that  $u \cdot v < 0$  and  $u \cdot w < 0$  and  $v \cdot w < 0$ , or explain why no such example exists.

**Exercise 5.** Find an example of a number  $c$  so that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c$$

has the vector  $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$  as a solution, or explain why no such number exists.

**Exercise 6.** Let  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $w = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ . Find an example of a number  $c$  so that

$$v \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c, \text{ and } w \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c,$$

or explain why this is not possible.

**Exercise 7.** Let  $P = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ . Find an example of a numbers  $c$  and  $d$  so that

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot P = c, \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot P = d,$$

or explain why no such example is possible.

**Now, we move to three dimensions...**

**Exercise 8.** Let  $V = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ . Find an example of two vectors  $U$  and  $W$  such that

$$U \cdot V < 0, \quad U \cdot W < 0, \quad \text{and } V \cdot W < 0,$$

or explain why no such example exists.

**Exercise 9.** Let  $V = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ . Find a unit vector of the form  $X = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$  so that  $V \cdot X = \sqrt{2}$ , or explain why no such vector exists.

**Exercise 10.** Let  $V = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ . Find a unit vector of the form  $X = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$  so that  $V \cdot X = 10$ , or explain why no such vector exists.

**Exercise 11.** Find an example of numbers  $c$ ,  $d$ , and  $e$  so that there is no solution vector  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  which simultaneously satisfies the three equations

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot X = c, \quad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot X = d, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot X = e,$$

or explain why no such numbers exist.

**Exercise 12.** Find an example of numbers  $c$ ,  $d$ , and  $e$  so that there is no solution vector  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  which simultaneously satisfies the three equations

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot X = c, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot X = d, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot X = e,$$

or explain why no such numbers exist.