LINEAR ALGEBRA

SECTION 1.2

1. The Assignment

- Read *Strang* Chapter 1, section 2, pages 11-18.
- (Optional) Read *Hefferon* Chapter One, Part II Linear Geometry, section 2 Length and Angle Measures, pages 39-43.
- Complete the exercises below before class.

2. Discussion

The dot product is a wonderful tool for encoding the geometry of Euclidean space, but it can be a bit mysterious at first. As *Strang* shows, it somehow holds all of the information you need to measure lengths and angles.

The connection to linear algebra comes about through this: a dot product with a "variable vector" is a way of writing a linear equation. For example,

$$\begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7x + 3y - 2z.$$

Sometimes this will allow us to connect linear algebra to geometry, and use geometric thinking to answer algebraic questions.

3. Sage Instructions

I have made a Sage .sagews file called *section1-2.sagews* which contains some basic commands for dealing with vectors and dot products, and a couple of interactive demos to play with dot products in two and three dimensions.

4. The Exercises

Exercise 1. What shape is the set of solutions $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ to the equation

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 5?$$

That is, if we look at all possible vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ which make the equation true, what shape does this make in the plane?

What happens if we change the vector $\left(\begin{smallmatrix} 3 \\ 7 \end{smallmatrix}\right)$ to some other vector? What happens if we change the number 5 to some other number?

What happens if we move up a dimension, and instead work in three dimensional space?

Exercise 2. Find an example of two 2-vectors v and w such that $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \cdot v = 0$ and $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \cdot w = 0$ and $v \cdot w = 0$, or explain why such an example is not possible.

Exercise 3. Let $v = \left(\begin{smallmatrix} 3 \\ -1 \end{smallmatrix}\right)$. Find an example of a pair of vectors u and w such that $v \cdot u < 0$ and $v \cdot w < 0$ and $w \cdot u = 0$, or explain why no such pair of vectors can exist.

Exercise 4. Find an example of three 2-vectors u, v, and w so that $u \cdot v < 0$ and $u \cdot w < 0$ and $v \cdot w < 0$, or explain why no such example exists.

Exercise 5. Find an example of a number c so that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c$$

has the vector $\left(\begin{smallmatrix} 4 \\ 7 \end{smallmatrix}\right)$ as a solution, or explain why no such number exists.

Exercise 6. Let $v = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} -3 \\ 4 \end{smallmatrix}\right)$. Find an example of a number c so that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot v = c, \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot w = c,$$

or explain why this is not possible.

Exercise 7. Let $P = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$. Find an example of a numbers c and d so that

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = c, \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix} = d,$$

or explain why no such example is possible.

Now, we move to three dimensions...

Exercise 8. Let $V = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$. Find an example of two vectors U and W such that

$$U \cdot V < 0, \qquad U \cdot W < 0, \qquad \text{and } V \cdot W < 0,$$

or explain why no such example exists.

Exercise 9. Let $V = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$. Find a unit vector of the form $X = \left(\begin{smallmatrix} x \\ y \\ 0 \end{smallmatrix}\right)$ so that $V \cdot X = \sqrt{2}$, or explain why no such vector exists.

Exercise 10. Let $V = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$. Find a unit vector of the form $X = \left(\begin{smallmatrix} x \\ y \\ 0 \end{smallmatrix}\right)$ so that $V \cdot X = 10$, or explain why no such vector exists.

Exercise 11. Find an example of numbers c, d, and e so that there is no solution vector $X = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ which simultaneously satisfies the three equations

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot X = c, \qquad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot X = d, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot X = e,$$

or explain why no such numbers exist.

Exercise 12. Find an example of numbers c, d, and e so that there is no solution vector $X = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ which simultaneously satisfies the three equations

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot X = c, \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot X = d, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot X = e,$$

or explain why no such numbers exist.