LINEAR ALGEBRA

STRANG, SECTION 1.3

1. The assignment

- Read section 1.1 of Strang (pages 22-27).
- Read the following and complete the exercises below.

2. Matrices

A *matrix* is a two-dimensional array of numbers like this:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Sometimes it helps to think of a matrix as a collection of its *rows* which are read across:

$$M = \begin{pmatrix} \longrightarrow \\ \longrightarrow \end{pmatrix}$$

and sometimes it helps to think of a matrix as a collection of its *columns* which are read down:

$$M = \begin{pmatrix} \downarrow & \downarrow \end{pmatrix}.$$

It is often more clear to describe a matrix by giving the sizes of its rows and columns. An m by n matrix is one having m rows and n columns. It is really easy to get these reversed, so be careful. For example, this is a 2×3 matrix, because it has two rows and three columns:

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \end{pmatrix}$$

A matrix is called a *square* matrix when the number of rows and the number of columns is equal. The matrix A that I wrote down above is square because it is a 2×2 matrix.

2.1. **Multiplying Matrices and Vectors.**

It is possible to multiply a matrix by a vector like this:

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \end{pmatrix} \begin{pmatrix} 13 \\ 21 \\ 34 \end{pmatrix} = \begin{pmatrix} 102 \\ 416 \end{pmatrix}$$

For this to work, it is absolutely crucial that the sizes match up properly. If the matrix is m by n, then the vector must have size n. In the above example $m = 2$ and $n = 3$.

Later, we shall see that the word "multiplication" is not really the best choice here. It is better to think of the matrix as "acting on" the vector and turning it into a new vector. For now, the word multiplication will serve.

How exactly does one define this matrix-vector multiplication?

2.1.1. *Linear Combination of Columns Approach.*

The first way to perform the matrix-vector multiplication is to think of the vector as holding some coefficients for forming a linear combination of the columns of the matrix. In our example, it looks like this:

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \end{pmatrix} \begin{pmatrix} 13 \\ 21 \\ 34 \end{pmatrix} = 13 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 21 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 34 \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 102 \\ 416 \end{pmatrix}.$$

2.1.2. *Dot Products with the Rows Approach.* The second way is to think of the matrix as a bundle of vectors lying along the rows of the matrix, and use the dot product. In our example above, this means that we consider the vectors

$$r_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}, \text{ and } v = \begin{pmatrix} 13 \\ 21 \\ 34 \end{pmatrix}$$

(notice I've rewritten the rows as columns) and then perform this kind of operation:

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \end{pmatrix} \begin{pmatrix} 13 \\ 21 \\ 34 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} v = \begin{pmatrix} r_1 \cdot v \\ r_2 \cdot v \end{pmatrix} = \begin{pmatrix} 102 \\ 416 \end{pmatrix}.$$

Two important remarks:

- Note that these operations only make sense if the sizes match up properly.
- Note that the two versions of the operation give you the same results.

2.2. **Matrix Equations.** There are many situations in linear algebra that can be rewritten in the form of an equation that looks like this:

$$Av = b \qquad (*)$$

where A is a matrix, and v and b are vectors. The interesting case is when we know A and b, but we want to find the unknown v.

Let's consider the case where you are given some <u>square</u> matrix A. **Sometimes** one can find another matrix B so that no matter what b is chosen in (*), the solution vector takes the form $v = Bb$. When this happens, we say that A is *invertible* and call B the *inverse* of A. It is common to use the notation A^{-1} in place of B. This is a wonderful situation to be in! Eventually, we will want to figure out some test for when a given matrix is invertible, and find some ways to compute the inverse.

2.3. **A Note about Vectors.** This reading also has a brief introduction to the idea of a set of vectors being *linearly depedent* or *linearly independent*. Strang is coy about the precise definition, so here it is:

Definition. A set of vectors $v_1, v_2, \dots, v_n$ is called *linearly depdendent* when there is some choice of numbers $a_1, a_2, \dots, a_n$ which are not all zero so that the linear combination

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

A set of vectors which is not linearly dependent is called *linearly independent.*

This is a little funny the first time you read it. Note that for any set of vectors, you can make a linear combination of those vectors come out as 0. Simply choose all of the coefficients to be zero. But that is so easy to do we call it *trivial.* What the definition is asking is that we find a *nontrivial linear combination of the vectors to make zero.*

3. Sage instructions

I have made a Sage worksheet file with some basic commands that you might find useful in investigating with matrices. The file is called `section1_3.sagews`. It also has some interactive demonstrations about how to deal with matrices.

4. Questions for Section 1.3

Exercise 25. Make an example of a matrix $\begin{pmatrix} 1 & \bullet \\ -1 & \bullet \end{pmatrix}$ so that the equation

$$\begin{pmatrix} 1 & \bullet \\ -1 & \bullet \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

has exactly one solution, or explain why this is not possible.

Interpret this as a statement about 2-vectors and draw the picture which corresponds.

Exercise 26. Make an example of a matrix $\begin{pmatrix} 4 & 8 & \bullet \\ 3 & 6 & \bullet \\ 1 & 2 & \bullet \end{pmatrix}$ so that the equation

$$\begin{pmatrix} 4 & 8 & \bullet \\ 3 & 6 & \bullet \\ 1 & 2 & \bullet \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}$$

has exactly one solution, or explain why this is not possible.

Interpret this as a statement about 3-vectors and draw the picture which corresponds.

Exercise 27. Make an example of a matrix $\begin{pmatrix} 2 & -1 \\ \bullet & \bullet \end{pmatrix}$ so that the equation

$$\begin{pmatrix} 2 & -1 \\ \bullet & \bullet \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

has exactly one solution, or explain why this is not possible.

Interpret this as a statement about a pair of lines in the plane and draw the picture which corresponds.

Exercise 28. Make an example of a matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ \bullet & \bullet & \bullet \end{pmatrix}$ so that the equation

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ \bullet & \bullet & \bullet \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

has no solutions, or explain why this is not possible.

Interpret this as a statement about a planes in space and draw the picture which corresponds.

Exercise 29. Find a triple of numbers x, y, and z so that the linear combination

$$x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + z \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

yields the zero vector, or explain why this is not possible.

Rewrite the above as an equation which involves a matrix.

Plot the three vectors and describe the geometry of the situation.

Exercise 30. The vectors

$$r_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \qquad \text{and} \qquad r_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

are linearly dependent because they lie in a common plane (through the origin). Find a normal vector to this plane.

Since the vectors are linearly dependent, there must be (infinitely) many choices of scalars x, y, and z so that $xr_1 + yr_2 + zr_3 = 0$. Find two sets of such numbers.

Exercise 31. Consider the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We are interested in being able to solve this for x and y for any given choice of the numbers b_1 and b_2. Figure out a way to do this by writing x and y in terms of b_1 and b_2.

Rewrite your solution in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = b_1 \begin{pmatrix} \bullet \\ \bullet \end{pmatrix} + b_2 \begin{pmatrix} \bullet \\ \bullet \end{pmatrix}.$$

How is this related to the inverse of the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$?

Exercise 32. Find an example of a number c and a vector $\left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$ so that the equation

$$\begin{pmatrix} 3 & 51 \\ c & 17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

does not have a solution, or explain why no such example exists.

Explain your solution in terms of

- lines in the plane,
- 2-vectors and linear combinations, and
- invertibility of a matrix.