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NUMBER SYNTHESIS OF KINEMATIC CHAINS BASED ON PERMUTATION GROUPS

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Abstract. A systematic and precise approach is developed for enumerating non-isomorphic kinematic chains based on the theory of permutation groups. First, we define contracted link adjacency matrices of kinematic chains, and elements in the matrices are separated into four sets. We then propose an algorithm for assigning values to elements of these sets to generate non-isomorphic configurations according to their permutation groups. As a result, the numbers of simple kinematic chains with up to twelve links and seven degrees of freedom are listed.

INTRODUCTION

Number synthesis of kinematic chains has been the subject of numerous studies for a long time. They are based on intuition and ingenuity [1,2], Franke's notation [3,4], Baranov trusses [5], Assur group [6], graph theory [7–16], expansions from binary chains [17–20], base structures [21], and others [22,23].

In the past years, a major problem encountered in kinematic number synthesis is the elimination of isomorphic chains. Various codes, such as characteristic polynomials [24–29], linkage path codes [30], optimal code [31], min code [32], degree code [33], etc., are used for this purpose after kinematic chains are synthesized. In 1987, Tuttle and Peterson [21] applied the concept of symmetry group for generating non-isomorphic kinematic chains directly based on base structures. However, this method needs to generate non-isomorphic base structures in advance.

The purpose of this paper is to propose an algorithm for generating kinematic chains with required numbers of links and degrees of freedom based on permutation groups. This algorithm has the advantage of avoiding the generation of isomorphic configurations thoroughly in the process. Here, a kinematic chain refers only to the one which is planar, with simple revolute joints and without any rigid subchains.

CONTRACTED LINK ADJACENCY MATRIX

Traditionally, the topological structure of a kinematic chain is represented by *link adjacency matrix* (LAM) [8]. The LAM of a kinematic chain with N links is an $N \times N$ symmetry matrix with its elements $c_{ij} = 1$ if link i is adjacent to link j, and $c_{ij} = 0$ otherwise. For reasons that will become clear, we use *contracted link adjacency matrix* (CLAM) for representing the topological structure of kinematic chains.

A string of binary links in a kinematic chain is regarded as a *contracted link*. The diagonal element e_{ii}, called *link element*, of a CLAM indicates the type of link i, and its value is defined such that $e_{ii} = +u$ if link i is a multiple link with u joints, and $e_{ii} = -v$ if link i is a contracted link with v binary links. The off-diagonal element e_{ij}, called *joint element*, indicates the number

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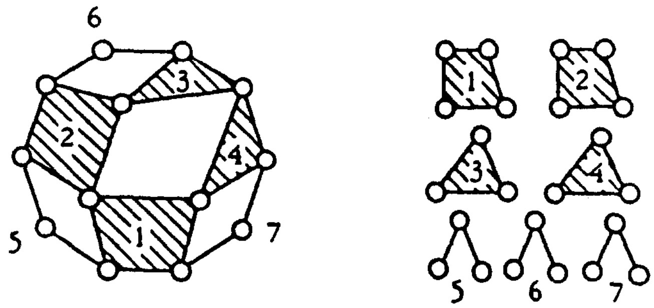

(a) (b)

Fig. 1. A (10,13) kinematic chain and its multiple and contracted links.

of joints incident between link i and link j, and its value is defined such that $e_{ij} = w$ if link i and link j are connected by w joints. Note that w can be 0, 1, or 2. The case of $w = 2$ happens only in the condition that one of these two adjacent links is a contracted link with three or more binary links, the other one is a multiple link, and both ends of the contracted link are connected to this multiple link. Without loss of generality, diagonal elements of a CLAM are rearranged in a nonincreasing sequence by exchanging the order of links. For example, the CLAM of the kinematic chain with ten links and thirteen joints shown in Figure 1(a) is expressed as

$$\text{CLAM} = \left[\begin{array}{cccc|ccc} 4 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & -2 \end{array}\right]$$

in which diagonal elements imply that the kinematic chain has two quarternary links (links 1 and 2), two ternary links (links 3 and 4), and three contracted links with two binary links (links 5, 6, and 7), as shown in Figure 1(b).

For the sake of simplicity, we divide a CLAM into four submatrices, MM, MC, CM, and CC, as follows:

$$CLAM = \left[\begin{array}{c|c} MM & MC \\ \hline CM & CC \end{array}\right]$$

MM is the upper-left square submatrix in the CLAM that expresses the configuration of multiple links. CC is the lower-right square submatrix in the CLAM that expresses the configuration of contracted links. Note that the value of off-diagonal elements in CC matrix are always zero because any two contracted links are nonadjacent. MC is the upper-right matrix in the CLAM that indicates the adjacent relation between multiple links and contracted links. CM is the lower-left matrix in the CLAM that is a transpose matrix of the MC matrix.

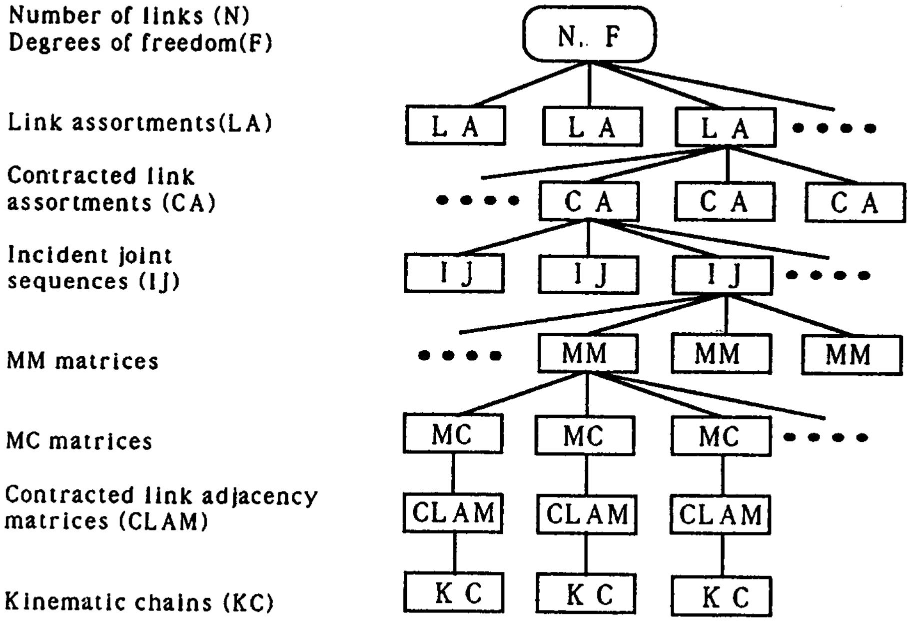

Fig. 2. Steps of the enumerating algorithm.

ENUMERATING ALGORITHM

Since a CLAM represents the topological structure of a kinematic chain, the enumeration of kinematic chains can be achieved by constructing all possible CLAMs. In the following, we describe briefly the algorithm for construction steps of all non-isomorphic CLAMs (Figure 2):

Step 1. Input the numbers of links N and degrees of freedom F.
Step 2. Find link assortments $LA = [n_2/n_3/\dots/n_i/\dots/n_q]$ in which n_i is the number of links with i joints.
Step 3. For each link assortment, find contracted link assortments $CA = [b_1/b_2/\dots/b_i/\dots/b_r]$ in which b_i is the number of contracted links with i binary links.
Step 4. For each contracted link assortment, find incident joint sequences $IJ = (a_1, a_2, \dots, a_i, \dots, a_{N_m})$ of multiple links, where N_m is the number of multiple links and a_i is the number of joints incident between multiple link i and the other multiple links.
Step 5. For each incident joint sequence, find all non-isomorphic MM matrices as follows:
  (1) Find the permutation groups of joint elements of MM matrix.
  (2) Based on this permutation group, assign "0"s and "1"s to joint elements to construct all non-isomorphic MM matrices.
Step 6. For each generated MM matrix, find all non-isomorphic MC matrices as follows:
  (1) Find the permutation group of joint elements of MC matrix.
  (2) Based on this permutation group, assign "0"s, "1"s, and "2"s to joint elements to construct all non-isomorphic MC matrices.
Step 7. Construct corresponding CLAMs from generated MC matrices. Those CLAMs with rigid subchains are deleted.
Step 8. Transform each constructed CLAM into its corresponding kinematic chain in graphic form.

In the following sections, we describe each step in detail.

Step 1

Input the numbers of links and degrees of freedom.

Since planar kinematic chains with revolute pairs and simple joints is our concern, the number of joints (J) can be calculated based on Gruebler's criterion [34] as follows:

$$J = [3(N-1) - F]/2 \tag{1}$$

where F is the degrees of freedom and N is the number of links.

For a kinematic chain with ten links ($N = 10$) and one degree of freedom ($F = 1$), its number of joints J is 13. Figure 1(a) shows such an example.

Step 2

Find link assortments.

Link assortment, $LA = [n_2/n_3/\ldots/n_q]$, of a kinematic chain with N links and J joints can be obtained by solving the following three equations:

$$n_2 + n_3 + n_4 + \ldots + n_q = N \tag{2}$$

$$2n_2 + 3n_3 + 4n_4 + \ldots + qn_q = 2J \tag{3}$$

$$q = \begin{cases} (N-F+1)/2 & \text{if } F = 0 \text{ or } 1 \\ \min\{(N-F-1),(N+F-1)/2\} & \text{if } F \geq 2 \end{cases} \tag{4}$$

For kinematic chains with $N = 10$ and $F = 1$, and therefore $J = 13$, all possible link assortments are [4/6/0/0], [5/4/1/0], [6/2/2/0], [7/0/3/0], [6/3/0/1], [7/1/1/1], and [8/0/0/2]. The LA of the kinematic chain shown in Figure 1(a) is [6/2/2/0].

Step 3

Find contracted link assortments.

This step is equivalent to partition n_2 binary links into N_c parts, where N_c is the number of contracted links. Therefore, a contracted link assortment $CA = [b_1/b_2/\ldots/b_r]$ must satisfy the following equations:

$$b_1 + b_2 + \ldots + b_r = N_c \tag{5}$$

$$b_1 + 2b_2 + \ldots + rb_r = n_2 \tag{6}$$

It is obvious that a nondegenerate kinematic chain with F degrees of freedom cannot consist of a contracted link with $F+2$ or more binary links; therefore $r = F+1$. The range of N_c is [23]

$$J_m - J'_m \leq N_c \leq \min\{n_2, J_m\} \tag{7}$$

where

$$2\,J_m = 3n_3 + 4n_4 + \ldots + qn_q \tag{8}$$

$$2\,J'_m = \begin{cases} 0 & \text{if } N_m = 1 \\ 3(N_m-1)-1 & \text{if } N_m = 2,4,6,\ldots \\ 3(N_m-1)-2 & \text{if } N_m = 3,5,7,\ldots \end{cases} \tag{9}$$

Based on equations (5)–(9), all possible contracted link assortments can be generated for a given link assortment. For example, if $N = 10$, $F = 1$, and $LA = [6/2/2/0]$, then $r = 2$, $N_m = 4$, $J_m = 7$, $3 \leq N_c \leq 6$, and all possible contracted link assortments are [0/3], [2/2], [4/1], and [6/0].

Once contracted link assortments are obtained, the value of e_{ii} of the CLAM can be determined in a nonincreasing sequence. For example, if $LA = [6/2/2/0]$ and $CA = [2/2]$, then the two quarternary links must be labelled as links 1 and 2, the two ternary links as links 3 and 4, the two contracted links with one binary link as links 5 and 6, and the two contracted links with two binary links as 7 and 8. The sequence of the value of e_{ii} is $(4,4,3,3,-1,-1,-2,-2)$.

Step 4

Find incident joint sequences.

Before constructing the MM matrix, we must find all possible incident joint sequences $IJ = (a_1, a_2, \dots, a_i, \dots, a_{N_m})$ of multiple links, where a_i is the number of joints incident between multiple link i and the other multiple links. In other words, a_i is the sum of the values of off-diagonal elements of row i in MM matrix; i.e.,

$$\sum_{\substack{j=1 \\ j\neq 1}}^{N_m} e_{ij} = a_i, \quad i = 1, \dots, N_m \tag{10}$$

Equation (10) is called compatibility constraints of MM matrix. Furthermore, the sum of a_i, denoted by $2J_d$, can be obtained by the following expression:

$$2\ J_d = 2\ J_m - 2\ N_c \tag{11}$$

We see that the problem of determining incident joint sequences is equivalent to the problem of partitioning integer $2J_d$ into k parts, and assigning these k parts into N_m elements of IJ sequence. Here, $k \leq N_m$, since some multiple links may not be adjacent to the other multiple links.

In order to avoid the configuration of multiple links with rigid subchains, the k parts must satisfy the following constraints:

(1) The largest of parts, t, cannot be greater than the maximum number of joints of multiple links, i.e. $t \leq e_{11}$, and must be less than the number of parts, i.e. $t < k$.
(2) The degrees of freedom of the configuration of multiple links must be positive, i.e. $3(k-1) - 2J_d > 0$.

For example, if the sequence of link elements is $(4, 4, 3, 3, -1, -1, -2, -2)$, then $J_m = 7$, $N_m = 4$, $N_c = 4$, $J_d = 3$, and $k \leq 4$. From the second constraint, $k > 3$. Thus, $k = 4$. From the first constraint, $t \leq 4$. All possible partitions of $2J_d$, which is 6, into four parts are $3 + 1 + 1 + 1$ and $2 + 2 + 1 + 1$.

After partitions are obtained, we now assign these partitions to a_i. Since e_{ii} is the total number of joints belonging to multiple link i and a_i is only the number of joints incident to the other multiple links, $a_i \leq e_{ii}$. For the sake of avoiding having isomorphic configurations of multiple links, we set a rule that $a_i \leq a_j$, if $e_{ii} = e_{jj}$ and $i < j$. The resultant incident joint sequences IJ of the above example are: (3,1,1,1), (1,1,3,1), (2,2,1,1), (2,1,2,1), and (1,1,2,2).

Step 5

Construct MM matrices.

After incident joint sequences are obtained, the permutation group [35] of link elements of MM matrix can now be determined after the following terminologies are explained.

A one-to-one mapping from a finite set S onto itself is called a *permutation p*. For example, the sequence (s_2, s_3, s_1, s_4) is a permutation of the set $S = (s_1, s_2, s_3, s_4)$ in which $s_1 \rightarrow$ (is transformed into) s_2, $s_2 \rightarrow s_3$, $s_3 \rightarrow s_1$, and $s_4 \rightarrow s_4$. In this permutation, $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_1$ forms a cycle, denoted by $[s_1 s_2 s_3]$. $s_4 \rightarrow s_4$ forms another cycle $[s_4]$. The cyclic representation of this permutation is denoted by $[s_1 s_2 s_3][s_4]$. The usual composition of mappings provides a binary operation for permutations on the same set. Furthermore, whenever a collection of permutations is closed with respect to this composition, it is called a *permutation group G*. Two elements s_i and s_j of set S are similar, if there exists a permutation p of G which transforms s_i into s_j. Furthermore, set S can be partitioned into some *similar classes* by putting similar elements into the same class.

In our problem, let $ML = (L_1, L_2, \dots, L_{N_m})$ be the set of multiple links. The *attributes* of a link is some information about this link, such as e_{ii}, a_i, e_{ij}, ..., etc. The permutation group of ML called the *link-group* is the set of permutations which transform a link into another one without changing their attributes. Up to now, a multiple link has two given attributes, the number of its joints (e_{ii}) and the number of joints incident to the other multiple links (a_i). For example, if $ML(L_1, L_2, L_3, L_4)$, the values of e_{ii} are (4,4,3,3), and the values of a_i are (2,2,2,2), then the link-group G_L of the set ML is

$$G_L = \{p_1, p_2, p_3, p_4\}$$

where

$$\begin{aligned} p_1 &= [L_1][L_2][L_3][L_4] \\ p_2 &= [L_1\ L_2][L_3][L_4] \\ p_3 &= [L_1][L_2][L_3\ L_4] \\ p_4 &= [L_1\ L_2][L_3\ L_4] \end{aligned}$$

In order to determine the value of e_{ij} for each link i, we induce another permutation group, namely *joint-group* G_J, which acts on the set of e_{ij} of MM matrix. Each permutation of the link-group induces a corresponding permutation of the joint-group by mapping e_{ij} into e_{pq} if L_i is mapping into L_p and L_j into L_q. For example, the joint-group of the above example is

$$G_J = \{p'_1, p'_2, p'_3, p'_4\}$$

where

$$\begin{aligned} p'_1 &= [e_{12}][e_{13}][e_{14}][e_{23}][e_{24}][e_{34}] \\ p'_2 &= [e_{12}][e_{13}\ e_{23}][e_{14}\ e_{24}][e_{34}] \\ p'_3 &= [e_{12}][e_{13}\ e_{14}][e_{23}\ e_{24}][e_{34}] \\ p'_4 &= [e_{12}][e_{13}\ e_{24}][e_{14}\ e_{23}][e_{34}] \end{aligned}$$

There are three similar classes of G_J here: $\{e_{12}\}$, $\{e_{13}, e_{14}, e_{23}, e_{24}\}$, and $\{e_{34}\}$.

Now, we start to assign J_d "1"s to e_{ij} based on joint-group G_J. Unassigned elements are set to "0." In the assignment, the compatibility constraints, equation (10), must be satisfied. In this work, we propose a recursive procedure for our assignment problem. This procedure, which we call ASSIGNMENT, takes four arguments: $ESET$, $GROUP$, JD, and $PATH$. They are initially set equal to the set of elements, permutation group, number of "1"s to be assigned, and an empty list, respectively. What follows is the ASSIGNMENT.

Recursive procedure ASSIGNMENT(ESET,GROUP,JD,PATH)

1. if EMPTY (ESET), return;

   EMPTY is a predicate true for its argument that is an empty list. Upon the empty list, the procedure is returned.

2. CLASSLIST ← SIMILAR(ESET,GROUP);

   SIMILAR is a function that separates ESET into some similar classes based on GROUP and order them into a list.

3. CLASS ← ADDLAST(FIRST(CLASSLIST),CLASSEND);

   The first similar class is selected, and add a symbol, CLASSEND, to the last of the class.

4. RSET ← REMOVE(CLASS,ESET);

   REMOVE is a function that removes the elements of CLASS from ESET. The rest of the elements form a set, RSET, for the next recursion.

5. PATHS ← LIST(PATH);

   LIST is a function that produces a list of its argument. A path indicates a sequence of assigned elements.

6. until EMPTY(PATHS), do:
7. begin
8. PATHS $\leftarrow$ EXPAND(PATHS,CLASS);

   EXPAND is a function that expands the end node, n_i, of each path P_i in PATHS to generate a set of new paths. The successors of node n_i are the elements whose order in CLASS are lower than that of n_i.

9. for each path, P, in the PATHS, do:
10. begin
11. if LAST(P) = CLASSEND,

    then NGROUP $\leftarrow$ MODIFY(GROUP,P),

    ASSIGNMENT(REST,NGROUP,JD,REMOVELAST(P));

    If the path cannot be expanded in the current CLASS, then ASSIGNMENT is called recursively on the rest elements. MODIFY is a function that removes the destroyed permutations from GROUP based on path P, where a destroyed permutation is one in which the values of elements in the same cycle are distinct.

12. if NOT(MAXIMAL(P, GROUP)),

    then PATHS $\leftarrow$ REMOVE(P,PATHS), go end;

    MAXIMAL is a predicate true for its first argument, P, if P is a maximal path. A maximal path is defined that the highest order of elements of this path is greater than those of the other paths transformed by permutations of GROUP. If the highest orders are equal, then the second ones are compared, and so on. If P is not a maximal path, then it must be isomorphic to a maximal path and be removed from PATHS.

13. if LENGTH(P)=JD,

    then PATHS $\leftarrow$ REMOVE(P,PATHS),

    if COMPATIBLE(P), OUTPUT(P);

    If P contains J_d assigned elements, then remove P from PATHS. If P satisfies compatibility constraints, then it is a result and be output.

14. end
15. end
16. return

For the example mentioned previously, if the set of joint elements of MM matrix is $MJ = (e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$, incident joint sequence is $IJ = (2,2,2,2)$, $J_d = 4$, and joint group $G_J = \{p'_1, p'_2, p'_3, p'_4\}$, where

$$\begin{aligned}
p'_1 &= [e_{12}][e_{13}][e_{14}][e_{23}][e_{24}][e_{34}] \\
p'_2 &= [e_{12}][e_{13}\ e_{23}][e_{14}\ e_{24}][e_{34}] \\
p'_3 &= [e_{12}][e_{13}\ e_{14}][e_{23}\ e_{24}][e_{34}] \\
p'_4 &= [e_{12}][e_{13}\ e_{24}][e_{14}\ e_{23}][e_{34}]
\end{aligned}$$

We execute the assignment procedure with its initial arguments $ESET = MJ$, $GROUP = G_J$, $JD = 4$, and $PATH = nil$, where nil denotes an empty list. The process of the procedure is illustrated in Figure 3, which shows a tree generated to a depth of 4. In this tree, nodes depicted by a circle express assigned elements and nodes depicted by a box labeled by the modified group indicate the beginning of a recursive call. Isomorphic paths are cut off by a double circle, and the paths which cannot be expanded to a depth of 4 are terminated by a triple circle. There are six

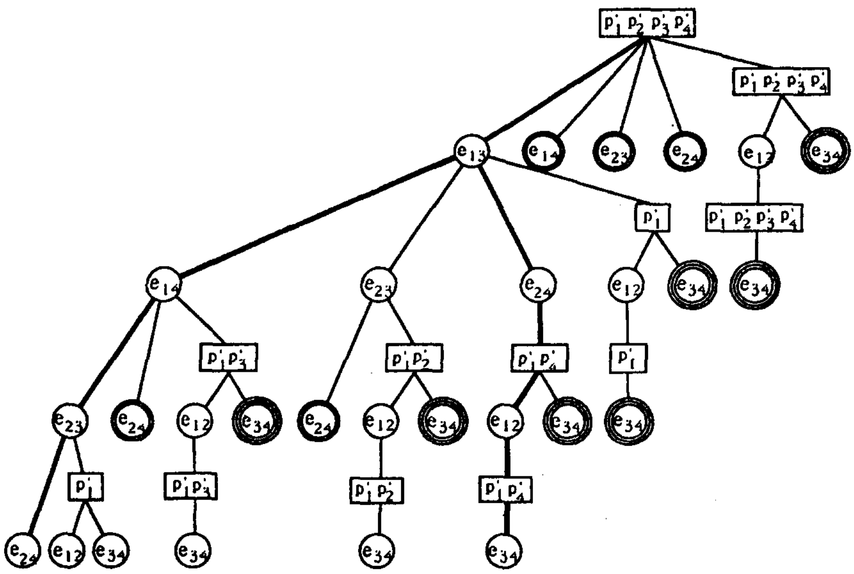

Fig. 3. Assignment tree for constructing MM matrices.

non-isomorphic paths of depth 4, but four of them do not satisfy the compatibility constraints. The two results are indicated by darkened branches and their MM matrices are

$$\begin{bmatrix} 4 & 0 & 1 & 1 \\ 0 & 4 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

The joint group of the first MM matrix is the same as the original group, but the joint group of the second one is modified by removing p'_2 and p'_3 from the original group. The corresponding link group of the second one is reduced as $G_L = \{p_1, p_4\}$.

Step 6

Construct MC matrices.

In Step 5, we construct MM matrices. Now, we want to construct MC matrices for each MM matrix. Similar to the process of generating MM matrices, the process of generating MC matrices is transformed into the problem of assigning $2N_c$ joints into elements of MC matrices. Two differences between this step and Step 5 are as follows. (1) If the value of the element in MC matrix is "2," the assignment process must be executed for two times. At the first time, assign J_2 "2"s to the elements of MC matrix, and at the second time, assign $2N_c - 2J_2$ "1"s to the elements. Unassigned elements are set to "0." (2) Assigned elements must satisfy the following compatibility constraints of MC matrix:

$$\sum_{\substack{j=1 \\ j\neq i}}^{N_m+N_c} e_{ij} = e_{ii}, \quad i = 1, \dots, N_m \tag{11}$$

$$\sum_{j=1}^{N_m} e_{ij} = 2, \quad i = N_m + 1, \dots, N_m + N_c \tag{12}$$

Now, let $MBL = (L_1, L_2, \dots, L_n)$ be the set of multiple links and contracted links. The link group of multiple links with the given MM matrix is obtained in Step 5. The link group of contracted links is the set of permutations which transform the value of e_{ii} into itself. The link group of the set MBL can be obtained by combining link groups of multiple links and contracted links. For example, if $N = 10$, $F = 1$, $LA = [6/2/2/0]$, $CA = [0/3]$, and

$$CLAM = \begin{bmatrix} 4 & 1 & 1 & 0 & e_{15} & e_{16} & e_{17} \\ 1 & 4 & 0 & 1 & e_{25} & e_{26} & e_{27} \\ 1 & 0 & 3 & 1 & e_{35} & e_{36} & e_{37} \\ 0 & 1 & 1 & 3 & e_{45} & e_{46} & e_{47} \\ e_{51} & e_{52} & e_{53} & e_{54} & -2 & 0 & 0 \\ e_{61} & e_{62} & e_{63} & e_{64} & 0 & -2 & 0 \\ e_{71} & e_{72} & e_{73} & e_{74} & 0 & 0 & -2 \end{bmatrix}$$

then the link group of multiple links has two permutations: $[L_1][L_2][L_3][L_4]$ and $[L_1\ L_2][L_3\ L_4]$, which are obtained in Step 5. The link group of contracted links has six permutations: $[L_5][L_6][L_7]$, $[L_5][L_6\ L_7]$, $[L_6][L_5\ L_7]$, $[L_7][L_5\ L_6]$, $[L_5\ L_6\ L_7]$, and $[L_5\ L_7\ L_6]$. The link group G_L of MBL has 12 permutations by combining the above two link groups; i.e.,

$$G_L = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}\}$$

where

$$\begin{aligned}
p_1 &= [L_1][L_2][L_3][L_4][L_5][L_6][L_7] \\
p_2 &= [L_1][L_2][L_3][L_4][L_5][L_6\ L_7] \\
p_3 &= [L_1][L_2][L_3][L_4][L_6][L_5\ L_7] \\
p_4 &= [L_1][L_2][L_3][L_4][L_5\ L_6][L_7] \\
p_5 &= [L_1][L_2][L_3][L_4][L_5\ L_6\ L_7] \\
p_6 &= [L_1][L_2][L_3][L_4][L_5\ L_7\ L_6] \\
p_7 &= [L_1\ L_2][L_3\ L_4][L_5][L_6][L_7] \\
p_8 &= [L_1\ L_2][L_3\ L_4][L_5][L_6\ L_7] \\
p_9 &= [L_1\ L_2][L_3\ L_4][L_6][L_5\ L_7] \\
p_{10} &= [L_1\ L_2][L_3\ L_4][L_5\ L_6][L_7] \\
p_{11} &= [L_1\ L_2][L_3\ L_4][L_5\ L_6\ L_7] \\
p_{12} &= [L_1\ L_2][L_3\ L_4][L_5\ L_7\ L_6]
\end{aligned}$$

Furthermore, the joint-group G_J of the set $MBJ = (e_{15}, e_{16}, e_{17}, e_{25}, e_{26}, e_{27}, e_{35}, e_{36}, e_{37}, e_{45}, e_{46}, e_{47})$ is

$$G_J = \{p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8, p'_9, p'_{10}, p'_{11}, p'_{12}\}$$

where

$$\begin{aligned}
p'_1 =& [e_{15}][e_{16}][e_{17}][e_{25}][e_{26}][e_{27}][e_{35}][e_{36}] \\
& [e_{37}][e_{45}][e_{46}][e_{47}] \\
p'_2 =& [e_{15}][e_{16}\ e_{17}][e_{25}][e_{26}\ e_{27}][e_{35}] \\
& [e_{36}\ e_{37}][e_{45}][e_{46}\ e_{47}] \\
p'_3 =& [e_{16}][e_{15}\ e_{17}][e_{26}][e_{25}\ e_{27}][e_{36}] \\
& [e_{35}\ e_{37}][e_{46}][e_{45}\ e_{47}] \\
p'_4 =& [e_{15}\ e_{16}][e_{17}][e_{25}\ e_{26}][e_{27}][e_{35}\ e_{36}] \\
& [e_{37}][e_{45}\ e_{46}][e_{47}] \\
p'_5 =& [e_{15}\ e_{16}\ e_{17}][e_{25}\ e_{26}\ e_{27}][e_{35}\ e_{36}\ e_{37}] \\
& [e_{45}\ e_{46}\ e_{47}] \\
p'_6 =& [e_{15}\ e_{17}\ e_{16}][e_{25}\ e_{27}\ e_{26}][e_{35}\ e_{37}\ e_{36}] \\
& [e_{45}\ e_{47}\ e_{46}] \\
p'_7 =& [e_{15}\ e_{25}][e_{16}\ e_{26}][e_{17}\ e_{27}][e_{35}\ e_{45}] \\
& [e_{36}\ e_{46}][e_{37}\ e_{47}] \\
p'_8 =& [e_{15}\ e_{25}][e_{16}\ e_{27}][e_{17}\ e_{26}][e_{35}\ e_{45}] \\
& [e_{36}\ e_{47}][e_{37}\ e_{46}] \\
p'_9 =& [e_{15}\ e_{27}][e_{16}\ e_{26}][e_{17}\ e_{25}][e_{35}\ e_{47}] \\
& [e_{36}\ e_{46}][e_{37}\ e_{45}] \\
p'_{10} =& [e_{15}\ e_{26}][e_{16}\ e_{25}][e_{17}\ e_{27}][e_{35}\ e_{46}] \\
& [e_{36}\ e_{45}][e_{37}\ e_{47}] \\
p'_{11} =& [e_{15}\ e_{26}\ e_{17}\ e_{25}\ e_{16}\ e_{27}] \\
& [e_{35}\ e_{46}\ e_{37}\ e_{45}\ e_{36}\ e_{47}] \\
p'_{12} =& [e_{15}\ e_{27}\ e_{16}\ e_{25}\ e_{17}\ e_{26}] \\
& [e_{35}\ e_{47}\ e_{36}\ e_{45}\ e_{37}\ e_{46}]
\end{aligned}$$

Finally, the ASSIGNMENT procedure is executed with initial arguments $ESET = MBJ$, $GROUP = G_J$, $JD = 2N_c$, and $PATH = nil$. The resulting MC matrices are

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 7

Construct CLAMs.

This step is to construct corresponding CLAMs from MC matrices obtained in Step 6. For the three MC matrices generated in Step 6, their corresponding CLAMs are

$$\begin{bmatrix} 4 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Furthermore, each CLAM containing any construction of basic rigid chain should be deleted. A kinematic chain is degenerate with three- or five-link basic rigid chains if the upper-left submatrix

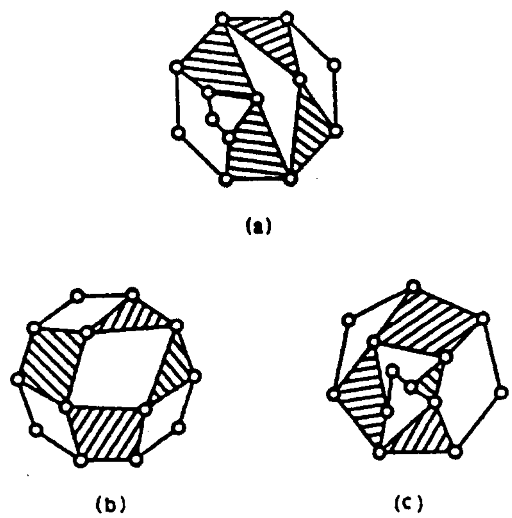

Fig. 4. Generated kinematic chains of an example.

with $e_{ii} \geq -1$ has the form of the following two matrices:

$$\begin{bmatrix} \cdot & & & & & & & & \\ & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \\ & & \cdot & & \cdot & & & \cdot & \\ & & & \cdot & \cdot & & & \cdot & \\ & & & & \cdot & \cdot & \cdot & 1 & \\ & & & & & \cdot & & \cdot & \\ & & & & & & \cdot & \cdot & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \end{bmatrix} \quad \text{with the three-link basic rigid chain}$$

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ & \cdot & & & \cdot & & & \cdot & \\ & \cdot & & & \cdot & & & \cdot & \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \end{bmatrix} \quad \text{with the five-link basic rigid chain}$$

As for detecting CLAMs with seven-or-more-link basic rigid chains, we use the method proposed in reference [23] by removing binary links in sequence.

Step 8

Transform CLAMs to kinematic chains.

The last step is to transform each CLAM obtained in Step 7 into its corresponding kinematic chain in graphic form. This is done by using a sketching algorithm developed by authors in reference [26]. Figure 4 shows the corresponding kinematic chains of the three CLAMs obtained in Step 7.

RESULTS

A computer code is developed based on the algorithm described. As simple as inputing the required numbers of links (N) and degrees of freedom (F), all desired non-isomorphic kinematic chains are be generated automatically. As a result, for one degree of freedom, there are 1 for

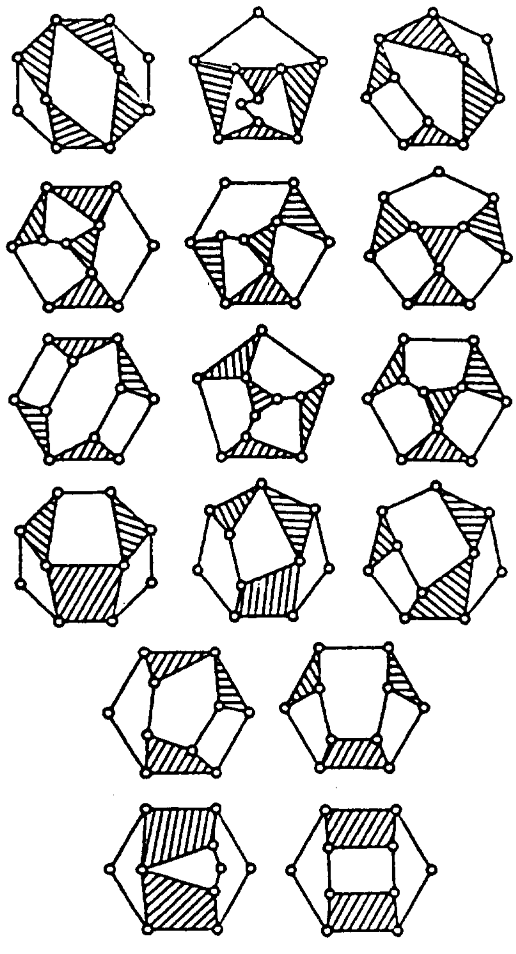

Fig. 5. Computer generated sketches of (8,10) kinematic chains.

4 links, 2 for 6 links, 16 for 8 links, 230 for 10 links, and 6862 for 12 links. For two degrees of freedom, there are 4 for 7 links, 40 for 9 links, and 839 for 11 links. For three degrees of freedom, there are 7 for 8 links, 98 for 10 links, and 2442 for 12 links. For four degrees of freedom, there are 10 for 9 links and 189 for 11 links. For five degrees of freedom, there are 14 for 10 links and 354 for 12 links. For six degrees of freedom, there are 19 for 11 links. For seven degrees of freedom, there are 24 for 12 links. Table 1 summarizes the results. Figure 5 shows the results of $N = 8$ and $F = 1$.

Table 1. The number of kinematic chains with 6 to 12 links.

N \ F	1	2	3	4	5	6	7
6	2						
7		4					
8	16		7				
9		40		10			
10	230		98		14		
11		839		189		19	
12	6862		2442		354		24

CONCLUSION

In conclusion, we present a systematic and precise algorithm for enumerating kinematic chains based on permutation groups. This approach guarantees that isomorphic kinematic chains will not be generated in the process of synthesis. Based on this proposed algorithm, a computer program is developed such that the catalogues of planar kinematic chains with required numbers of links and degrees of freedom can be obtained automatically. In addition, the efforts of this work can be further applied to the number synthesis of kinematic chains with other types of joints rather than revolute pairs. The result of this work is of helpful to mechanism designers in conceptual stage for selecting better configurations.

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