Rössler attractor

continuous deterministic chaos

Rössler system

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases}$$

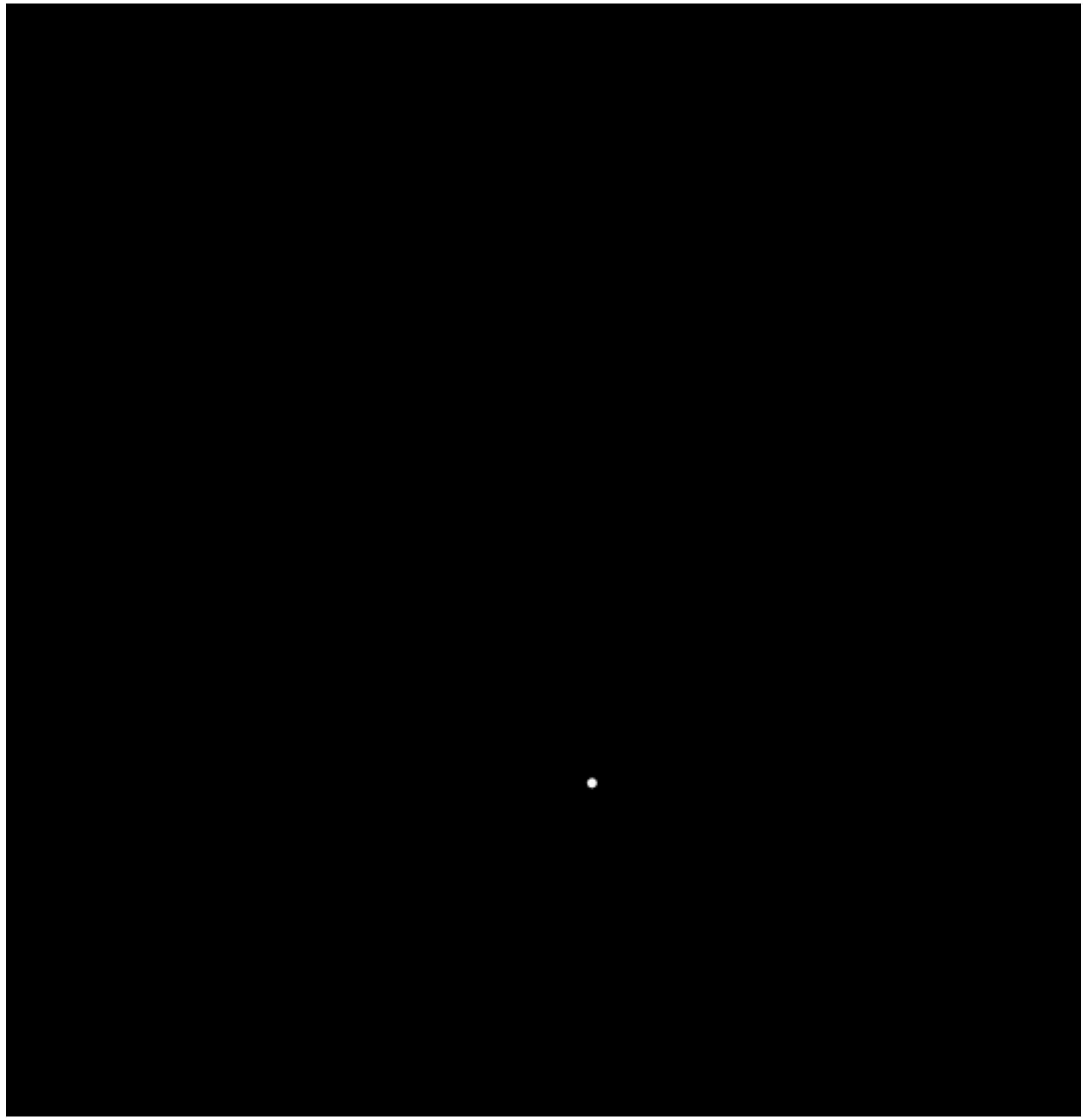
$\{x, y, z\}$ system state

$\{a, b, c\}$ parameters

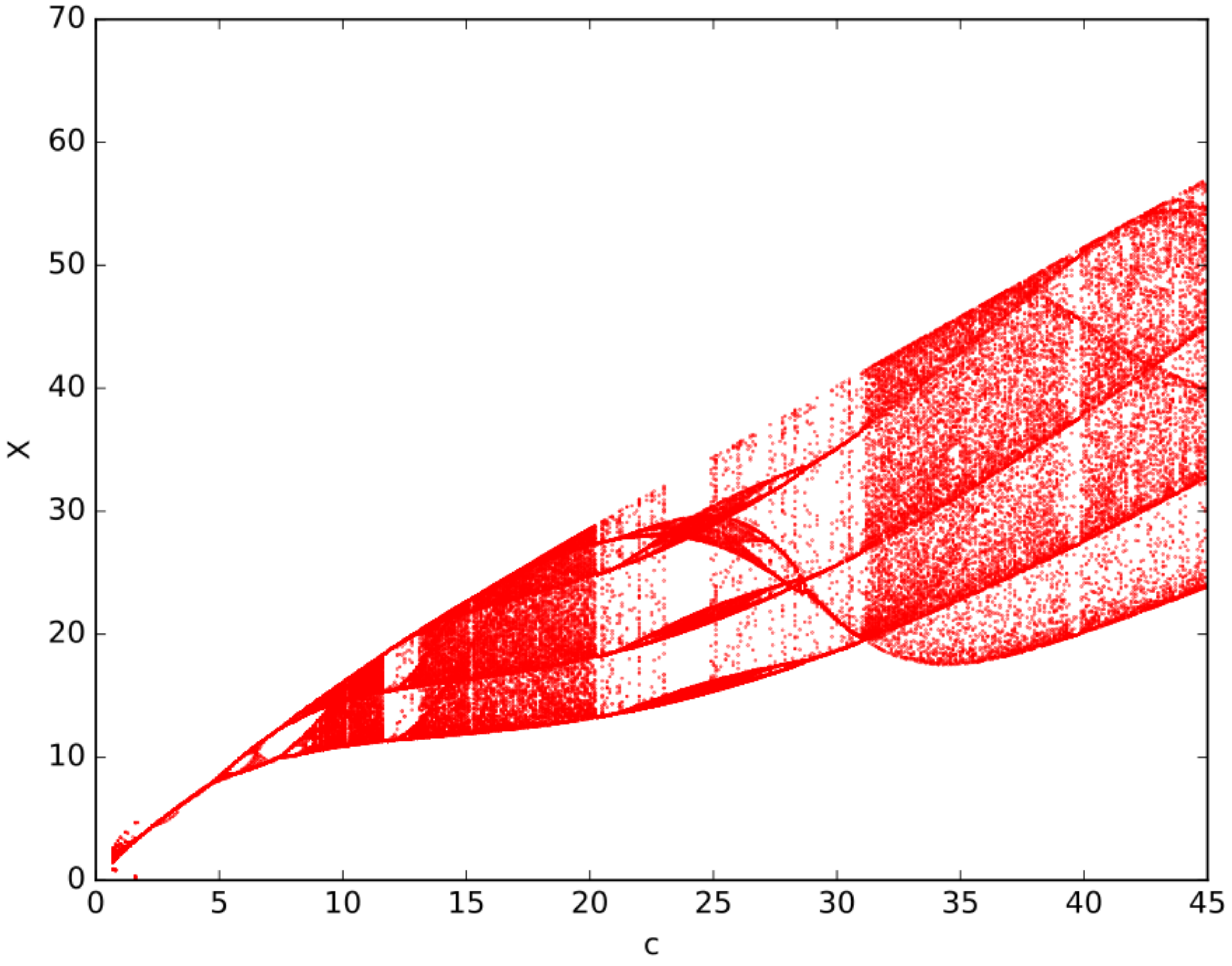
$\{\dot{x}, \dot{y}, \dot{z}\} = \{0,0,0\}$ if $\{x, y, z\}$ equal to:

$$\left(\frac{c + \sqrt{c^2 - 4ab}}{2}, \frac{-c - \sqrt{c^2 - 4ab}}{2a}, \frac{c + \sqrt{c^2 - 4ab}}{2a}\right)$$

$$\left(\frac{c - \sqrt{c^2 - 4ab}}{2}, \frac{-c + \sqrt{c^2 - 4ab}}{2a}, \frac{c - \sqrt{c^2 - 4ab}}{2a}\right)$$



Bifurcation diagram



$$a = b = 0.1$$
$$c \in (0,45]$$

References

Rössler, Otto E. "An equation for continuous chaos." *Physics Letters A* 57.5 (1976): 397-398.

This is the original paper. The Rossler attractor can be seen as a Lorentz attractor with one lobe. Differently from Lorentz, in Rossler there is just one non-linearity.

Letellier, Christophe, and Valérie Messager. "Influences on Otto E. Rössler's earliest paper on chaos." *International Journal of Bifurcation and Chaos* 20.11 (2010): 3585-3616.

Historical overview on the Rossler system and its main influence in physics.

Delage, Olivier, and Alain Bourdier. "Selection of Optimal Embedding Parameters Applied to Short and Noisy Time Series from Rössler System." *Journal of Modern Physics* 8.09 (2017): 1607.

All you need to smartly succeed in this TP.

Observability

$\dot{W} = \mathbf{A}W$ $\quad$ $\mathbf{A} \in \mathbb{R}^{m \times m}$ Jacobian $\quad$ $\mathbf{C} \in \mathbb{R}^{r \times m}$ measure matrix

$X = \mathbf{C}W$ $\quad$ $W \in \mathbb{R}^{m}$ state $\quad$ $X \in \mathbb{R}^{r}$ measure

The system is observable in X
if rank$(\mathbf{Q}) = m$

for Rossler $m = 3$ and observing
one coordinate $r = 1$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{m-r} \end{bmatrix}$$

Observability in Rossler

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases} \qquad \mathbf{A} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{bmatrix} \qquad \mathbf{A}^2 = \begin{bmatrix} -1 - z & -a & c - x \\ a & a^2 - 1 & -1 \\ z(x - c) & -z & -z + (x - c)^2 \end{bmatrix}$$

$$\mathbf{C}_y = [0 \quad 1 \quad 0] \qquad \mathbf{Q}_y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & a^2 - 1 & -1 \end{bmatrix} \qquad \det(\mathbf{Q}_y) = 1$$

$$\mathbf{C}_x = [1 \quad 0 \quad 0] \qquad \mathbf{Q}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 - z & -a & c - x \end{bmatrix} \qquad \det(\mathbf{Q}_x) = 0 \textbf{ if } x = a + c$$

$$\mathbf{C}_z = [0 \quad 0 \quad 1] \qquad \mathbf{Q}_z = \begin{bmatrix} 0 & 0 & 1 \\ z & 0 & x - c \\ z(x - c) & -z & -z + (x - c)^2 \end{bmatrix} \qquad \det(\mathbf{Q}_z) = 0 \textbf{ if } z = 0$$

Goals for this TP

- Generate the data (time series) with $a = b = 0.2, c = 5.7$
- Learn the discrete or continuous dynamical system from the time series

$$W_{n+1} = NN(W_n) \qquad\qquad \dot{W} = NN(W)$$

70% of the note: recover statistics

- PDF
- Time correlations
- Spectral density
- …

30% of the note: recover dynamics

- Equilibrium point (at least one)
- Lyapunov exponents (the largest one) $\lambda \approx 7 \times 10^{-2}$

differences

Discrete		Continuous
$W_{t+\Delta t} = M(W_t)$	dynamicsl system	$\dot{W} = F(W)$
$w_{t+\Delta t} = \mathbf{J} w_t$	linearized system	$\dot{w} = \mathbf{A} w$
$\mathbf{J}$	Jacobian	$\mathbf{A}$
	Note that $\mathbf{J} \approx e^{\mathbf{A}\Delta t}$	
$W_0 - M(W_0) = 0$	Equilibrium state	$F(W_0) = 0$

Constraints

- Temporal embedding or memory, discrete system: **just ONE coordinate**

$$y_1 = NN(y_0, y_{-1}, y_{-2,\ldots})$$
$$y_1 = NN(y_0, \dot{y}_0, \ddot{y}_0, \ldots)$$
$$y_1 = NN(y_0, [h_0])$$

h_0 memory in RNN

- No Temporal embedding and no memory, discrete system: **the whole state**

$$W_1 = NN(W_0)$$

- Continuous system: **no constraints**

$$\dot{W} = NN(W)$$

Be smart

To find the equilibrium point and to evaluate the Lyapunov exponent, the Jacobian has to be computed. Introduce a penalization onto the sensibility of your model wrt the inputs!

$$loss = \|W - \hat{W}\| + \lambda(?)$$

$$\|\dot{W} - \hat{\dot{W}}\|$$

$$\|A - \hat{A}\|$$

$$\|\hat{A}\|_F$$

Attention: with a RNN the total Jacobian needs to be computed!

Appendix D in: "Backpropagation Algorithms and Reservoir Computing in Recurrent Neural Networks for the Forecasting of Complex Spatiotemporal Dynamics"

deliver

- small report

  Justify the loss function (e.g. MSE, L_∞, penalizations, etc) and the performed analysis to validate your model

- codes to reproduce the pictures in the report

- code to generate a time series with your model

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