Combinatorics

1 Recurrence Relation

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that relates a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, ..., a_{n-1}$ for all integers $n \geq n_0$, where n_0 is non-negative integer. The values $a_0, a_1, ..., a_{n-1}$ are explicitly given values. They are called initial condition or boundary conditions.
For example: The sequence $S = \{3, 3^2, 3^3, ..., 3^n, ...\}$ can be written using the recurrence relation

$$a_n = 3a_{n-1}, n \geq 2,\ for a_1 = 3$$

Linear Recurrence relation with constant coefficients: A linear Recurrence relation with constant coefficients is of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} = f(n)$$

where c_i's are constants. The order of the relation is k. If $f(n) = 0$, then the relation is homogeneous, otherwise it is not.

1.1 Solving Recurrence relation

1.1.1 The method of Characteristic roots:

The method of Characteristic roots is used to solve the Linear Recurrence relation with constant coefficients as the sum of two parts, the homogeneous solution, which satisfy the recurrence relation when $f(n) = 0$ and particular solution, which satisfies the relation with $f(n)$ on the right hand side.

Steps to get homogeneous solution:

1. To get **Characteristic equation** : Put $a_n = r^n$ and then divide the equation by r^{n-k}, we get the Characteristic equation:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - ...... - c_k = 0.$$

   The solution of this equation are called the **characteristic roots** say $r_1, r_2, ..., r_k$.

2. Then solution is given according to following rules:

S. No.	Characteristic roots	Solution a_n^h
1.	$r_1, r_2, ..., r_k$ are distinct	$b_1 r_1^n + b_2 r_2^n + ... + b_k r_k^n$, where $b_1, b_2, ..., b_k$ are constants
2.	r is repeated k times	$(b_1 + nb_2 + n^2 b_3 + ... + n^{k-1} b_k) r^n$
3.	combination of distinct and multiple roots	

Ques. 1 *Solve the recurrence relation* $a_n - 4a_{n-1} - 3a_{n-2} + 18a_{n-3} = 0$

Sol. Given recurrence relation

$$a_n - 4a_{n-1} - 3a_{n-2} + 18a_{n-3} = 0 \tag{1}$$

Homogeneous Solution: Put $a_n = r^n$

$$\begin{aligned} &r^n - 4r^{n-1} - 3r^{n-2} + 18r^{n-3} = 0 \\ \implies &r^3 - 4r^2 - 3r + 18 = 0 \\ \implies &r = -2, 3, 3 \end{aligned}$$

Then homogeneous solution $a_n^h = b_1(-2)^n + (b_2 + nb_3)3^n$

Non-homogeneous solution $a_n^p = 0$

Therefore the solution of the recurrence relation is

$$a_n = b_1(-2)^n + (b_2 + nb_3)3^n$$

Non-homogeneous Recurrence relation: A second order non-homogeneous linear recurrence relation with constant coefficients is of the form:

$$a_{n-2} + k_1 a_{n-1} + k_2 a_n = f(n)$$

The solution a_n of the non-homogeneous linear recurrence relation with constant coefficients is the sum of a particular solution $a_n^{(p)}$ and the homogeneous solution $a_n^{(h)}$.

Trial sequence method: For certain types of function $f(n)$, the solution can be obtained by trial sequence method based on the following conditions, where $A_0, A_1, ...$ represents unknown constants to be determined.

S. No.	Function $f(n)$	Trial function
1.	$f(n) = b^n$ where b is not a characteristic root	$= A_0 b^n$
2.	$f(n) = P(n)$, where $P(n)$ is a polynomial of degree m	$= A_0 + A_1 n + A_2 n^2 + ... + A_m n^m$
3.	$f(n) = b^n P(n)$, where $P(n)$ is a polynomial of degree m and b is not a characteristic root	$= \{A_0 + A_1 n + A_2 n^2 + ... + A_m n^m\} b^n$
4.	$f(n) = c^n$, where c is a characteristic root of multiplicity s	$= A_0 n^s c^n$
5.	$f(n) = c^n P(n)$, where $P(n)$ is a polynomial of degree m and c is a characteristic root of multiplicity s	$= \{A_0 + A_1 n + A_2 n^2 + ... + A_m n^m\} n^s c^n$

Ques. 2 *Solve the recurrence relation* $a_n + 5a_{n-1} + 6a_{n-2} = 42.4^n$

Sol. Given recurrence relation

$$a_n + 5a_{n-1} + 6a_{n-2} = 42.4^n \tag{2}$$

<u>Homogeneous Solution:</u> Put $a_n = r^n$

$$\begin{aligned} & r^n + 5r^{n-1} + 6r^{n-2} = 0 \\ \implies & r^2 + 5r + 6 = 0 \\ \implies & r = -2, -3 \end{aligned}$$

Then homogeneous solution $a_n^{(h)} = b_1(-2)^n + b_2(-3)^n$

<u>Non-homogeneous solution</u>: For getting the particular solution $a_n^{(p)}$, we take the trial solution $a_n = A_0 4^n$ in (2)

$$\begin{aligned} & A_0 4^n + 5A_0 4^{n-1} + 6A_0 4^{n-2} = 42.4^n \\ \implies & A_0 4^{n-2}\left(4^2 + 5\cdot 4 + 6\right) = 42\cdot 4^2\cdot 4^{n-2} \\ \implies & A_0\,(42) = 42\cdot 16 \\ \implies & A_0 = 16 \end{aligned}$$

That is the particular solution $a_n^{(p)} = 16\cdot 4^n = 4^{n+2}$
Therefore the solution of the recurrence relation is

$$a_n = b_1(-2)^n + b_2(-3)^n + 4^{n+2}$$

Ques. 3 *Solve the recurrence relation* $a_{n+2} - 5a_{n+1} + 6a_n = n^2$

Sol. Given recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = n^2 \tag{3}$$

<u>Homogeneous Solution:</u> Put $a_n = r^n$

$$\begin{aligned} & r^{n+2} - 5r^{n+1} + 6r^n = 0 \\ \implies & r^2 - 5r + 6 = 0 \\ \implies & r = 2, 3 \end{aligned}$$

Then homogeneous solution $a_n^{(h)} = b_1(2)^n + b_2(3)^n$

<u>Non-homogeneous solution</u>: For getting the particular solution $a_n^{(p)}$, we take the trial solution $a_n = A_0 + A_1 n + A_2 n^2$ in (3)

$$\begin{aligned} & (A_0 + A_1(n+2) + A_2(n+2)^2) - 5(A_0 + A_1(n+1) + A_2(n+1)^2) + 6(A_0 + A_1 n + A_2 n^2) = n^2 \\ \implies & 2A_0 - 3A_1 - A_2 + (2A_1 - 6A_2)n + 2A_2 n^2 = n^2 \end{aligned}$$

Equating the coefficients of n on both sides

$$A_2 = \frac{1}{2},\ A_1 = \frac{3}{2},\ A_0 = \frac{5}{2}$$

That is the particular solution $a_n^{(p)} = \frac{1}{2}n^2 + \frac{3}{2}n + \frac{5}{2}$
Therefore the solution of the recurrence relation is

$$a_n = b_1(2)^n + b_2(3)^n + \frac{1}{2}n^2 + \frac{3}{2}n + \frac{5}{2}$$

Ques. 4 *Solve the recurrence relation* $a_n + a_{n-1} = 3n2^n$

Sol. Given recurrence relation

$$a_n + a_{n-1} = 3n2^n \tag{4}$$

Homogeneous Solution: Put $a_n = r^n$

$$\begin{aligned} & r^n + r^{n-1} = 0 \\ \implies & r + 1 = 0 \\ \implies & r = -1 \end{aligned}$$

Then homogeneous solution $a_n^{(h)} = b_1(-1)^n$

Non-homogeneous solution: For getting the particular solution $a_n^{(p)}$,
we take the trial solution $a_n = (A_0 + A_1 n)2^n$ in (4)

$$\begin{aligned} & (A_0 + A_1 n)2^n + (A_0 + A_1(n-1))2^{n-1} = 3n2^n \\ \implies & 2^{n-1}(2A_0 + 2A_1 n + A_0 + A_1 n - A_1) = 3n2^n \\ \implies & (3A_0 - A_1 + 3A_1 n) = 6n \end{aligned}$$

Equating the coefficients of n on both sides

$$A_1 = 2,\ A_0 = \frac{2}{3}$$

That is the particular solution $a_n^{(p)} = \left(2n + \frac{2}{3}\right) 2^n$
Therefore the solution of the recurrence relation is

$$a_n = b_1(-1)^n + \left(n + \frac{1}{3}\right) 2^{n+1}$$

Ques. 5 *Solve the recurrence relation* $a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n$

Sol. Given recurrence relation

$$a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n \tag{5}$$

Homogeneous Solution: Put $a_n = r^n$

$$\begin{aligned} & r^{n+2} - 5r^{n+1} + 6r^n = 0 \\ \implies & r^2 - 5r + 6 = 0 \\ \implies & r = 2, 3 \end{aligned}$$

Then homogeneous solution $a_n^{(h)} = b_1(2)^n + b_2(3)^n$

Non-homogeneous solution: For getting the particular solution $a_n^{(p)}$,
we take the trial solution $a_n = A_0 n 2^n + (A_1 + A_2 n)$ in (4)

$$\begin{aligned} & A_0 n 2^n + (A_1 + A_2 n) - 5\left(A_0(n-1)2^{n-1} + (A_1 + A_2(n-1))\right) + 6\left(A_0(n-2)2^{n-2} + (A_1 + A_2(n-2))\right) \\ & = 2^n + n \\ \implies & A_0\left(n - \frac{5}{2}(n-1) + \frac{6}{4}(n-2)\right) 2^n + A_1 - 5A_1 + 6A_1 + 5A_2 - 12A_2 + (A_2 - 5A_2 + 6A_2)n = 2^n + n \\ \implies & A_0\left(-\frac{1}{2}\right) 2^n + 2A_1 - 7A_2 + 2A_2 n = 2^n + n \end{aligned} \tag{6}$$

Equating the coefficients of n on both sides

$$A_0\left(-\frac{1}{2}\right) = 1,\ 2A_1 - 7A_2 = 0,\ 2A_2 = 1$$

i.e.

$$A_0 = -2,\ A_2 = \frac{1}{2},\ A_1 = \frac{7}{4}$$

That is the particular solution $a_n^{(p)} = -2n2^n + (\frac{7}{4} + \frac{1}{2}n)$
Therefore the solution of the recurrence relation is

$$a_n = b_1(2)^n + b_2(3)^n - 2n2^n + (\frac{7}{4} + \frac{1}{2}n)$$

Ques. 6 *Solve the recurrence relation* $a_n - 2a_{n-1} + a_{n-2} = 5$*, given that* $a_0 = 1,\ a_1 = -2$

Sol. Given recurrence relation

$$a_n - 2a_{n-1} + a_{n-2} = 5 \tag{7}$$

<u>Homogeneous Solution:</u> Put $a_n = r^n$

$$\begin{aligned} & r^n - 2r^{n-1} + r^{n-2} = 0 \\ \Longrightarrow\ & r^2 - 2r + 1 = 0 \\ \Longrightarrow\ & r = 1, 1 \end{aligned}$$

Then homogeneous solution $a_n^{(h)} = (b_1 + b_2 n)(1)^n = b_1 + b_2 n$

<u>Non-homogeneous solution</u>: For getting the particular solution $a_n^{(p)}$,
we take the trial solution $a_n = A_0 n^2 (1)^n$ in (7)

$$\begin{aligned} & A_0 n^2 (1)^n - 2A_0(n-1)^2(1)^{n-1} + A_0(n-2)^2(1)^{n-2} = 5 \\ \Longrightarrow\ & A_0(n^2 - 2(n^2 - 2n + 1) + (n^2 - 4n + 4)) = 5 \\ \Longrightarrow\ & A_0\,(2) = 5 \end{aligned}$$

Equating the coefficients of n on both sides

$$A_0 = \frac{5}{2}$$

That is the particular solution $a_n^{(p)} = \frac{5}{2}n^2$
Therefore the solution of the recurrence relation is

$$a_n = b_1 + b_2 n + \frac{5}{2}n^2$$

Substituting $n = 0$, we get $a_0 = b_1$ and $n = 1$, we get $a_1 = b_1 + b_2 + \frac{5}{2}$. Given $a_0 = 1,\ a_1 = -2$, that is $b_1 = 1$ and $-2 = 1 + b_2 + \frac{5}{2}$, i.e. $b_2 = -\frac{11}{2}$
Thus solution of the recurrence relation is

$$a_n = 1 - \frac{11}{2}n + \frac{5}{2}n^2$$

1.2 Generating Function:

The generating function for the sequence $\{a_0, a_1, ...a_k, ...\}$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + ... + a_k x^k + ... = \sum_{k=0}^{\infty} a_k x_k$$

Some special Generating Functions:

S. No.	Sequence/ a_k	Generating function $G(z)$
1.	1,1,1,1,... 1	$\frac{1}{1-z}$
2.	1,2,3,4,... $k+1$	$\frac{1}{(1-z)^2}$
3.	0,1,2,3,....k	$\frac{z}{(1-z)^2}$
4.	1.2,2.3,3.4,....$k(k+1)$	$\frac{2z}{(1-z)^3}$
5.	$(k+1)(k+2)$	$\frac{2}{(1-z)^3}$
6.	^nC_k	$(1+z)^n$
7.	a^k	$\frac{1}{(1-az)}$

Ques. 7 *Find the generating function for the following numeric functions:*

1. $a_r = r.2^r$

2. $1, -2, 3, -4, 5, -6, ...(-1)^r(r+1), \cdots$

3. $1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \frac{5}{81}, ...$

Sol.

1. Given $a_r = r.2^r$, $r \geq 0$
   Then the sequence is $\{0, 1\cdot 2, 2\cdot 2^2, 3\cdot 2^3, \cdots\}$ Then the generating function $G(z)$

$$\begin{aligned} G(z) =& 1\cdot 2z + 2\cdot(2z)^2 + 3\cdot(2z)^3 + \cdots \\ =& u + 2u^2 + 3u^3 + \cdots \qquad [Take\ u = 2z] \\ =& u(1 + 2u + 3u^2 + \cdots) \\ =& u(1-u)^{-2} \\ =& \frac{2z}{(1-2z)^2} \end{aligned}$$

2. Given $1, -2, 3, -4, 5, -6, ...(-1)^r(r+1), \cdots$
   Given sequence is $\{1, -2, 3, -4, 5, -6, ...(-1)^r(r+1), \cdots\}$
   Then the generating function $G(z)$

$$\begin{aligned} G(z) =& 1 - 2z + 3z^2 - 4z^3 + \cdots \\ =& (1+z)^{-2} \\ =& \frac{1}{(1+z)^2} \end{aligned}$$

3. $1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \frac{5}{81}, ...$
   Given sequence is $\left\{1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \frac{5}{81}, \cdots, \frac{n+1}{3^n}, \cdots\right\}$.
   Then the generating function $G(z)$

$$\begin{aligned}
G(z) =& 1 + 2\frac{z}{3} + 3\left(\frac{z}{3}\right)^2 + 4\left(\frac{z}{3}\right)^3 + \cdots \\
=& 1 + 2u + 3u^2 + \cdots \qquad [Take\ u = \frac{z}{3}] \\
=& (1-u)^{-2} \\
=& \frac{1}{\left(1-\frac{z}{3}\right)^2} = \frac{9}{(3-z)^2}
\end{aligned}$$

Ques. 8 *Determine the numeric function corresponding to each of the following generating functions:*

1. $A(z) = \frac{1}{5-6z+z^2}$

2. $A(z) = \frac{z^4}{1-2z}$

Sol.

1. Given generating function

$$A(z) = \frac{1}{5-6z+z^2} = \frac{1}{(5-z)(1-z)}$$

   Using partial fraction, we get

$$\begin{aligned}
A(z) =& \frac{1}{4}\left(\frac{1}{1-z} - \frac{1}{5-z}\right) \\
=& \frac{1}{4}\left((1-z)^{-1} - \frac{1}{5}\left(1-\frac{z}{5}\right)^{-1}\right) \\
=& \frac{1}{4}\left((1+z+z^2+\cdots+z^n+\cdots) - \frac{1}{5}\left(1+\frac{z}{5}+\frac{z^2}{5^2}+\frac{z^3}{5^3}+\cdots+\frac{z^n}{5^n}+\cdots\right)\right) \\
=& \frac{1}{4}\sum_{n=0}^{\infty}\left(1-\frac{1}{5^{n+1}}\right)z^n
\end{aligned}$$

   Thus the numeric function or the nth term of the sequence

$$a_n = \frac{1}{4}\left(1-\frac{1}{5^{n+1}}\right)$$

2. Given generating function

$$A(z) = \frac{z^4}{1-2z}$$

   Expanding the function, we get

$$\begin{aligned}
A(z) =& \frac{z^4}{1-2z} \\
=& z^4(1-2z)^{-1} \\
=& z^4\left(1+2z+(2z)^2+\cdots+(2z)^n+\cdots\right) \\
=& z^4\sum_{n=0}^{\infty}2^n z^n \\
=& \sum_{n=0}^{\infty}2^n z^{n+4}
\end{aligned}$$

   Thus the numeric function or the nth term of the sequence

$$a_n = 2^{n-4}$$

Ques. 9 *Solve by the method of generating functions the recurrence relation*

1. $a_r - 4a_{r-1} + 3a_{r-2} = 0,\ r \geq 2$ with boundary conditions $a_0 = 2$ & $a_1 = 4$.

   Sol. Given recurrence relation

$$a_r - 4a_{r-1} + 3a_{r-2} = 0,\ r \geq 2 \tag{8}$$

Let the generating function for the given recursive relation be

$$G(z) = \sum_{r=0}^{\infty} a_r z^r \tag{9}$$

Multiply both sides of equation (8) by z^r and taking the sum from $r = 2$ to $r = \infty$, we obtain

$$\begin{aligned}
&\sum_{r=2}^{\infty} a_r z^r - 4\sum_{r=2}^{\infty} a_{r-1} z^r + 3\sum_{r=2}^{\infty} a_{r-2} z^r = 0\\
\implies &\left(a_2z^2 + a_3z^3 + a_4z^4 + \cdots\right) - 4\left(a_1z^2 + a_2z^3 + a_3z^4 + \cdots\right)\\
&\qquad + 3\left(a_0z^2 + a_1z^3 + a_2z^4 + \cdots\right) = 0\\
\implies &\left((a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots) - a_0 - a_1z\right) - 4z\left((a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots) - a_0\right)\\
&\qquad + 3z^2\left(a_0 + a_1z + a_2z^2 + \cdots\right) = 0\\
\implies &\left(G(z) - a_0 - a_1z\right) - 4z\left(G(z) - a_0\right) + 3z^2G(z) = 0\\
\implies &\left(G(z) - 2 - 4z\right) - 4z\left(G(z) - 2\right) + 3z^2G(z) = 0\\
\implies &\left(1 - 4z + 3z^2\right)G(z) - 2 + 4z = 0\\
\implies &G(z) = \frac{2 - 4z}{1 - 4z + 3z^2} = \frac{2 - 4z}{(1 - z)(1 - 3z)}
\end{aligned}$$

Using partial fraction

$$G(z) = \frac{1}{(1 - z)} + \frac{1}{(1 - 3z)}$$

Expanding the function as series, we get

$$\begin{aligned}
G(z) =& \frac{1}{(1 - z)} + \frac{1}{(1 - 3z)}\\
=&(1 - z)^{-1} + (1 - 3z)^{-1}\\
=&(1 + z + z^2 + z^3 + \cdots + z^n + \cdots) + (1 + 3z + 3^2z^2 + 3^3z^3 + \cdots + 3^nz^n + \cdots)\\
=&\sum_{n=0}^{\infty}(1 + 3^n)z^n
\end{aligned} \tag{10}$$

Thus on comparing equation (9) and (10), we get the solution of the recurrence relation

$$a_n = 1 + 3^n$$

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2. $a_r - 5a_{r-1} + 6a_{r-2} = 2,\ r \geq 2$ with boundary conditions $a_0 = 1$ & $a_1 = 2$.
   Sol. Given recurrence relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 2,\ r \geq 2 \tag{11}$$

Let the generating function for the given recursive relation be

$$G(z) = \sum_{r=0}^{\infty} a_r z^r \tag{12}$$

Multiply both sides of equation (11) by z^r and taking the sum from $r = 2$ to $r = \infty$, we obtain

$$\begin{aligned}
&\sum_{r=2}^{\infty} a_r z^r - 5\sum_{r=2}^{\infty} a_{r-1} z^r + 6\sum_{r=2}^{\infty} a_{r-2} z^r = 2\sum_{r=2}^{\infty} z^r \\
\implies &\left(a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right) - 5\left(a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots\right) \\
&\qquad + 6\left(a_0 z^2 + a_1 z^3 + a_2 z^4 + \cdots\right) = 2\left(z^2 + z^3 + z^4 + \cdots\right) \\
\implies &\left((a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots) - a_0 - a_1 z\right) - 5z\left((a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots) - a_0\right) \\
&\qquad + 6z^2\left(a_0 + a_1 z + a_2 z^2 + \cdots\right) = 2\frac{z^2}{1-z} \\
\implies &\left(G(z) - a_0 - a_1 z\right) - 5z\left(G(z) - a_0\right) + 6z^2 G(z) = \frac{2z^2}{1-z} \\
\implies &\left(G(z) - 1 - 2z\right) - 5z\left(G(z) - 1\right) + 6z^2 G(z) = \frac{2z^2}{1-z} \\
\implies &\left(1 - 5z + 6z^2\right) G(z) - 1 + 3z = \frac{2z^2}{1-z} \\
\implies &G(z) = \frac{2z^2}{(1-z)(1-2z)(1-3z)} + \frac{1-3z}{(1-2z)(1-3z)} = \frac{2z^2 + (1-3z)(1-z)}{(1-z)(1-2z)(1-3z)} \\
\implies &G(z) = \frac{5z^2 - 4z + 1}{(1-z)(1-2z)(1-3z)}
\end{aligned}$$

Using partial fraction

$$G(z) = \frac{1}{(1-z)} - \frac{1}{(1-2z)} + \frac{1}{(1-3z)}$$

Expanding the function as series, we get

$$\begin{aligned}
G(z) =& \frac{1}{(1-z)} - \frac{1}{(1-2z)} + \frac{1}{(1-3z)} \\
=& (1-z)^{-1} - (1-2z)^{-1} + (1-3z)^{-1} \\
=& (1 + z + z^2 + z^3 + \cdots + z^n + \cdots) - (1 + 2z + 2^2 z^2 + 2^3 z^3 + \cdots + 2^n z^n + \cdots) \\
& + (1 + 3z + 3^2 z^2 + 3^3 z^3 + \cdots + 3^n z^n + \cdots) \\
=& \sum_{n=0}^{\infty} (1 - 2^n + 3^n) z^n
\end{aligned} \tag{13}$$

Thus on comparing equation (12) and (13), we get the solution of the recurrence relation

$$a_n = (1 - 2^n + 3^n)$$

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3. $a_{r+2} - 2a_{r+1} + a_r = 2^r$ with boundary conditions $a_0 = 2$ & $a_1 = 1$.
   Sol. Given recurrence relation

$$a_{r+2} - 2a_{r+1} + a_r = 2^r, \ r \geq 0 \tag{14}$$

Let the generating function for the given recursive relation be

$$G(z) = \sum_{r=0}^{\infty} a_r z^r \tag{15}$$

Multiply both sides of equation (14) by z^r and taking the sum from $r=0$ to $r=\infty$, we obtain

$$
\begin{aligned}
&\sum_{r=0}^{\infty} a_{r+2}z^r - 2\sum_{r=0}^{\infty} a_{r+1}z^r + \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 2^r z^r \\
\implies &\left(a_2 + a_3 z + a_4 z^2 + \cdots\right) - 2\left(a_1 + a_2 z + a_3 z^2 + \cdots\right) \\
&\qquad + \left(a_0 + a_1 z + a_2 z^2 + \cdots\right) = \left(1 + 2z + (2z)^2 + \cdots\right) \\
\implies &\frac{1}{z^2}\left((a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots) - a_0 - a_1 z\right) - 2\frac{1}{z}\left((a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots) - a_0\right) \\
&\qquad + \left(a_0 + a_1 z + a_2 z^2 + \cdots\right) = (1-2z)^{-1} \\
\implies &\frac{1}{z^2}\left(G(z) - a_0 - a_1 z\right) - \frac{2}{z}\left(G(z) - a_0\right) + G(z) = \frac{1}{1-2z} \\
\implies &\frac{1}{z^2}\left(G(z) - 2 - z\right) - \frac{2}{z}\left(G(z) - 2\right) + G(z) = \frac{1}{1-2z} \\
\implies &\frac{1}{z^2}\left((1 - 2z + z^2)G(z) - 2 - z + 4z\right) = \frac{1}{1-2z} \\
\implies &(1 - 2z + z^2)G(z) - 2 + 3z = \frac{z^2}{1-2z} \\
\implies &G(z) = \frac{z^2}{(1-2z)(1-z)^2} + \frac{2-3z}{(1-z)^2} \\
\implies &G(z) = \frac{7z^2 - 7z + 2}{(1-2z)(1-z)^2}
\end{aligned}
$$

Using partial fraction

$$G(z) = \frac{A}{(1-2z)} + \frac{B}{(1-z)} + \frac{Cz}{(1-z)^2} \ (say)$$

Then

$$
\begin{aligned}
&A(1-z)^2 + B(1-z)(1-2z) + Cz(1-2z) = 7z^2 - 7z + 2 \\
&\text{Put } z = 1 \\
&-C = 2 \implies C = -2 \\
&\text{Put } z = 1/2 \\
&A(1/2)^2 = 7/4 - 7/2 + 2 \implies A = 1 \\
&\text{Put } z = 0 \\
&A + B = 2 \implies B = 1
\end{aligned} \tag{16}
$$

Thus we get

$$G(z) = \frac{1}{(1-2z)} + \frac{1}{(1-z)} - \frac{2z}{(1-z)^2} \ (say)$$

Expanding the function as series, we get

$$
\begin{aligned}
G(z) =& \frac{1}{(1-2z)} + \frac{1}{(1-z)} - \frac{2z}{(1-z)^2} \\
=&(1-2z)^{-1} + (1-z)^{-1} - 2z(1-z)^{-2} \\
=&(1 + z + z^2 + z^3 + \cdots + z^n + \cdots) + (1 + 2z + 2^2z^2 + 2^3z^3 + \cdots + 2^n z^n + \cdots) \\
&- 2(z + 2z^2 + 3z^3 + 4z^4 + \cdots + (n+1)z^{n+1} + \cdots) \\
=&\sum_{n=0}^{\infty}(1 + 2^n - 2n)z^n
\end{aligned} \tag{17}
$$

Thus on comparing equation (15) and (17), we get the solution of the recurrence relation

$$a_n = (1 + 2^n - 2n)$$

4. $a_n = a_{n-1} + n$ with boundary conditions $a_0 = 1$
Sol. Given recurrence relation

$$a_n - a_{n-1} = n, \ n \geq 1 \tag{18}$$

Let the generating function for the given recursive relation be

$$G(z) = \sum_{n=0}^{\infty} a_n z^n \tag{19}$$

Multiply both sides of equation (18) by z^n and taking the sum from $n = 1$ to $r = \infty$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n z^n - \sum_{n=1}^{\infty} a_{n-1} z^n = \sum_{n=1}^{\infty} n z^n \\
\implies &\left(a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right) - \left(a_0 z + a_1 z^2 + a_2 z^3 + \cdots\right) \\
&\qquad\qquad\qquad\qquad\qquad = \left(z + 2z^2 + 3z^3 + \cdots\right) \\
\implies &G(z) - a_0 - z\left(G(z)\right) = \frac{z}{(1-z)^2} \\
\implies &(1-z)G(z) = \frac{z}{(1-z)^2} + 1 \\
\implies &G(z) = \frac{z}{(1-z)^3} + \frac{1}{(1-z)}
\end{aligned}$$

Expanding the function as series, we get

$$G(z) = \frac{1}{2} \sum_{n=0}^{\infty} (n(1+n)) z^n + \sum_{n=0}^{\infty} z^n \tag{20}$$

Thus on comparing equation (19) and (20), we get the solution of the recurrence relation

$$a_n = \frac{1}{2}(n(1+n) + 1)$$
