Combinatorics

1 Permutation and Combination

An ordered selection or arrangements of r objects from a set of n objects is called a **r-permutation of n-objects**. It is denoted by $P(n,r)$ *or* ${}^nP_r$. If all elements are distinct and repetition is not allowed,then we have

$$P(n,r) = \frac{\lfloor n}{\lfloor (n-r)}$$

An unordered selection or arrangements of r objects from a set of n objects is called a **r-combination of n objects**. It is denoted by $C(n,r)$ *or* ${}^nC_r$. If all elements are distinct and repetition is not allowed,then we have

$$C(n,r) = \frac{\lfloor n}{\lfloor r \lfloor (n-r)}$$

1.1 Types of Permutation and Combination

1. **Permutation with repetition allowed**: When there is no restriction on the number of times a particular element may occur in the r-permutation of a set of n-objects is given by n^r

2. **Combinations with repetition allowed:** When out of n objects in a set, p objects are exactly alike of one kind, q objects exactly alike of the second kind and r objects exactly alike of the third kind and remaining objects are all different then the number of permutations of n objects taken all at a time is

$$= \frac{n!}{p!q!r!}$$

Ques. 1 *In how many ways can the letters of the word MONDAY be arranged? How many of them begin with M and end with Y? How many of them do not begin with M but end with Y?*

Sol. Given word is 'MONDAY'.

Total number of ways in which the letters of the word "MONDAY" is arranged = 6!

We have to find the number of arrangements so that words begin with M and end with Y.

Total no. of letters = 6

No. of choices for 1st position = 1
No. of choice for last position = 1

For 2nd position, no. of choice=4
For 3rd position, no. of choice=3
For 4th position, no. of choice=2
For 4th position, no. of choice=
Thus, number of permutation = 1×1×4×3×2×1=24.

We have to find the number of arrangements so that words donot begin with M but end with Y.

Total no. of letters = 6

No. of choices for 1st position = 4
No. of choice for last position = 1

For 2nd position, no. of choice=4
For 3rd position, no. of choice=3
For 4th position, no. of choice=2
For 4th position, no. of choice=
Thus, number of permutation = 4×1×4×3×2×1=96.

Ques. 2 *A man has 7 relatives, 4 of them are ladies and 3 gentlemen,his wife has 7 relatives and 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of men's relative and 3 of wife's relatives?*

Sol. Total number of invitees in the dinner party=3 ladies and 3 gentlemen
(3 of men's relative and 3 of wife's relatives)
The follwing are the way selection can be done:

From Man's relative	From wife's relative	
3 Ladies 0 Gentlemen	0 Ladies 3 Gentlemen	$({}^4C_3 \times {}^3C_0)({}^3C_0 \times {}^4C_3) = 4 \times 4 = 16$
0 Ladies 3 Gentlemen	3 Ladies 0 Gentlemen	$({}^4C_0 \times {}^3C_3)({}^3C_3 \times {}^4C_0) = 1$
2 Ladies 1 Gentlemen	1 Ladies 2 Gentlemen	$({}^4C_2 \times {}^3C_1)({}^3C_1 \times {}^4C_2) = (6 \times 3)(3 \times 6) = 324$
1 Ladies 2 Gentlemen	2 Ladies 1 Gentlemen	$({}^4C_1 \times {}^3C_2)({}^3C_2 \times {}^4C_1) = (4 \times 3)(3 \times 4) = 144$

Therefore the number of ways the invitation can be made $= 16 + 1 + 324 + 144 = 485$.

Ques. 3 *In how many ways can the letters of the word "DAUGHTER" be arranged so that the vowels never be separated.*

Sol. Given word "DAUGHTER". The vowels in the word are "AUE".
Thus number of ways the alphabets can be arranged considering "AUE" as one place = 6!
The number of ways the vowels can be arranged without seperating them = 3!
Hence the total number of ways alphabets of word can be arranged= $6! \times 3! = 4320$.

Ques. 4 *In how many ways the letter of the word "PETROL" be arranged.*

1. *How many of these arrangement do not begin with P.*
2. *How many of these arrangement begin with P but do not end in L.*
3. *Find the number of arrangements with O and L never together.*

Sol. Given word "PETROL".
Total no. of letters = 6

1. No. of choices for 1st position = 1
   For 2nd position, no. of choice=5
   For 3rd position, no. of choice=4
   For 4th position, no. of choice=3
   For 5th position, no. of choice=2
   For 6th position, no. of choice=1
   Thus, number of permutation = 1×5×4×3×2×1=120.

2. No. of choices for 1st position = 1
For 6th position, no. of choice=4
For 2nd position, no. of choice=4
For 3rd position, no. of choice=3
For 4th position, no. of choice=2
For 5th position, no. of choice=1
Thus, number of permutation = $1\times4\times4\times3\times2\times1$=96.

3. Number of arrangements of letters in the word when O and L are always together $= 5! \times 2! = 240$
Number of arrangements of letters in the word when O and L are never together $= 6! - 240 = 480$

Ques. 5 *In how many ways the letters of the word "ENGINEERING" be arranged so that*

1. *vowels come together,*
2. *vowels do not come together.*

Sol. Given word "ENGINEERING".
The vowels in the word are "EIEEI" and consonant "NGNRNG".
Thus total number of alphabets are 11 with 3 E's, 2 I's, 3 N's and 2 G's
The number of ways the alphabets of the word can be arranged $= \frac{11!}{3!2!3!2!} = 277200$

If all the vowels are always kept together, there are 7 entities are arranged in following number of ways $= \frac{7!}{3!2!} = 420$
The 5 vowels are arranged in following number of ways $= \frac{5!}{3!2!} = 10$
Thus, the number of ways the alphabets of the word can be arranged so that the vowels are never seperated $= 420 \times 10 = 4200$

And the number of ways the alphabets of the word can be arranged so that the vowels are never together $= 277200 - 4200 = 273000$

Ques. 6 *A bit is either 0 or 1 and a byte is a sequence of 8 bits. Find (a) the number of bytes that can be formed from 8 bits, (b) the number of bytes that begin with 11 and end with 11,(c) the number of bytes that begin with 11 and do not end with 11 and (d) the number of bytes that begin with 11 or end with 11.*

Sol.
(a) For each byte there are 8 position, each having two choices 0 and 1 and repitition is allowed.
The number of possible different bytes that can be formed $= 2^8 = 256$

(b) If the bytes begin with 11 and end with 11. the first two position and last two position have only one possibility, the rest four position have two possibility each. Thus the number of bytes that begin with 11 and end with 11 $= 2^4 = 16$

(c) If the bytes begins with 11, the first two position have only one possibility, the rest six position have two possibility each. Thus the number of bytes that begin with 11 $= 2^6 = 64$
Hence the number of bytes that begin with 11 but donot end with 11 $= 2^6 - 2^4 = 48$

(d) The number of bytes that begin with 11 is$= 2^6 = 64$.
The number of bytes that end with 11 is $= 2^6 = 64$.
The number of bytes that begin with 11 and end with 11 is$= 2^4 = 16$.
The number of bytes that begin with 11 or end with 11 is$= 2^6 + 2^6 - 2^4 = 64 + 64 - 16 = 112$.

Ques. 7 *Show that* ${}^{2n+2}C_{n+1} = {}^{2n}C_{n+1} + 2\,{}^{2n}C_n + {}^{2n}C_{n-1}$.

Sol.

$$
\begin{aligned}
R.H.S &= {}^{2n}C_{n+1} + 2\,{}^{2n}C_n + {}^{2n}C_{n-1} \\
&= \frac{2n!}{(n-1)!(n+1)!} + 2\frac{2n!}{(n)!(n)!} + \frac{2n!}{(n-1)!(n+1)!} \\
&= \frac{2n!}{(n-1)!n!}\left(\frac{1}{(n+1)} + \frac{2}{n} + \frac{1}{(n+1)}\right) \\
&= \frac{2n!}{(n-1)!n!}\left(\frac{n+2(n+1)+n}{n(n+1)}\right) \\
&= \frac{2n!2(2n+1)}{(n+1)!(n)!} = \frac{2n!2(n+1)(2n+1)}{(n+1)!(n+1)(n)!} \\
&= \frac{(2n+2)!}{(n+1)!(n+1)!} = {}^{2n+2}C_{n+1}
\end{aligned}
$$

1.2 Pigeonhole Principle(Dirichlet Drawer Principle or Shoe Box Principle)

If n pigeons are assigned to $m(m < n)$ pigeonholes then at least one pigeonhole contains two or more pigeons.

Generalized or Extended Pigeonhole Principle: If n pigeons are assigned to m pigeonholes where $m < n$, then one of the pigeonhole must contain at least $\lfloor\frac{(n-1)}{m}\rfloor + 1$ pigeons.

Ques. 8 *If 9 books are to be kept in 4 shelves, there must be at least one shelf which contain at least 3 books.*

Sol. Generalized or Extended Pigeonhole Principle: If n pigeons are assigned to m pigeonholes where $m < n$, then one of the pigeonhole must contain at least $\lfloor\frac{(n-1)}{m}\rfloor + 1$ pigeons.

Comparing the given problem with Extended Pigeonhole Principle, number of books(pigeons)=9.

Number of shelves(pigeon hole)=4

Then atleast one shelve contain atleast $= \lfloor\frac{9-1}{4}\rfloor + 1 = 3$ books.

Ques. 9 *Show that if any 20 people are selected, then we may choose a subset of 3 so that all 3 were born on the same day of the week.*

Sol. Generalized or Extended Pigeonhole Principle: If n pigeons are assigned to m pigeonholes where $m < n$, then one of the pigeonhole must contain at least $\lfloor\frac{(n-1)}{m}\rfloor + 1$ pigeons.

Comparing the given problem with Extended Pigeonhole Principle, number of people(pigeons)=20.

Distribution is to be done according to the days of a week, number of days(pigeon hole)=7

Then atleast on one day of the week will be the birth day of atleast $= \lfloor\frac{20-1}{7}\rfloor + 1 = 3$ people. Hence proved.-

Ques. 10 *Find the minimum number of students in a class to be sure that four out of them are born in the same month.*

Sol. Generalized or Extended Pigeonhole Principle: If n pigeons are assigned to m pigeonholes where $m < n$, then one of the pigeonhole must contain at least $\lfloor\frac{(n-1)}{m}\rfloor + 1$ pigeons.

Comparing the given problem with Extended Pigeonhole Principle, let the number of students(pigeons) be n.

Distribution is to be done according to the months of a year, number of days(pigeon hole)=12

Then given that atleast four students have their birth day in atleast one month, i.e.

$$\begin{aligned}
&\lfloor\frac{n-1}{12}\rfloor + 1 = 4 \\
\implies &\lfloor\frac{n-1}{12}\rfloor = 3 \\
\implies &\frac{n-1}{12} \geq 3 \\
\implies &n \geq 37
\end{aligned}$$

Thus atleast 37 students should be in a class for 4 students to have birthday in the sane month.

Ques. 11 *Show that among 1000 people, there are at least 84 people who are born in the same month*

Sol. Generalized or Extended Pigeonhole Principle: If n pigeons are assigned to m pigeonholes where $m < n$, then one of the pigeonhole must contain at least $\lfloor \frac{(n-1)}{m} \rfloor + 1$ pigeons.

Comparing the given problem with Extended Pigeonhole Principle, number of people(pigeons)=1000.

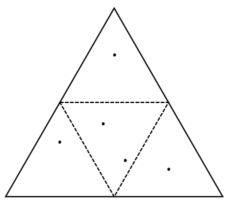
Distribution is to be done according to the months of a year, number of months(pigeon hole)=12

Then atleast on one month of the year will be the birth day of atleast $= \lfloor \frac{1000-1}{12} \rfloor + 1 = 84$ people. Hence proved.

Ques. 12 *Let any given 5 points to be placed in the interior of an equilateral triangle of side 1. Show that there exists two points within a distance of at most 1/2.*

Sol. Pigeonhole Principle: If n pigeons are assigned to $m(m < n)$ pigeonholes then at least one pigeonhole contains two or more pigeons.

Let the given equilateral triangle of side 1 is divided into four equilateral triangles by joining the midpoints



of the edges, so that length of edges of the smaller triangle(compartments) is 1/2.

Then we have, number of points to be placed in the interior of the triangle (pigeons) $n = 5$.
Number of compartments into which the points are to be distributed (pigeonhole) $m = 4$.

By Pigeon-hole principle, atleast two points will lie inside atleast one compartment(smaller triangle). Since the sides of the triangles is 1/2, the maximum distance between these two points is 1/2.

1.3 Principle of Inclusion and Exclusion:

Theorem 1 *Let P and Q are two finite sets, then the number of elements in the set $P \cup Q$ is given by*

$$|P \cup Q| = |P| + |Q| - |P \cap Q|$$

Theorem 2 *Let A_1, A_2 and A_3 be finite sets, then the number of elements in the set $A_1 \cup A_2 \cup A_3$ is given by*

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| + |A_1 \cap A_2 + \cap A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$

Theorem 3 ***(The inclusion-exclusion principle)*** *Let $A_1, A_2, ..., A_n$ be n-finite sets. Then*

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| ... + (-1)^{n-1} |A_1 \cap A_2 \cap ... \cap A_n|$$

Theorem 4 ***(Alternative form of inclusion-exclusion principle)*** *Let $A_1, A_2, ..., A_n$ be n-finite subsets od set S and $A'_1, A'_2, ..., A'_n$ denotes the complements of the subsets. Then*

$$|A'_1 \cap A'_2 \cap ... \cap A'_n| = |S| - |A_1 \cup A_2 \cup ... \cup A_n|$$

Ques. 13 *Find the number of positive integer not exceeding by 500 which are divisible by 7 or 11.*

Sol. Let S be the set of positive integer not exceeding 500 that are divisible by 7 and E be the set of positive integer not exceeding 500 that are divisible by 11. Then $S \cup E$ is the set of integers divisible by either 7 or 11 and $S \cap E$ is the set of integers divisible by both 7 and 11.

Then number of integers not exceeding 500 that are divisible by 7, $|S| = \lfloor \frac{500}{7} \rfloor = 71$

Then number of integers not exceeding 500 that are divisible by 11, $|E| = \lfloor \frac{500}{11} \rfloor = 45$

Similarly, $|S \cap E| = \lfloor \frac{500}{L.C.M(7,11)} \rfloor = 6$

By principle of inclusion-exclusion, we have

$$\begin{aligned}|S \cup E| =&|S| + |E| - |S \cap E|\\ =&71 + 45 - 6 = 110\end{aligned}$$

Hence there are 110 positive integers not exceeding 500 that are divisible by 7 or 11.

Ques. 14 *Find the number of positive integer not exceeding by 250 which are divisible by 2, 6 or 9.*

Sol. Let T be the set of positive integer not exceeding 250 that are divisible by 2, S be the set of positive integer not exceeding 250 that are divisible by 6 and N be the set of positive integer not exceeding 250 that are divisible by 9. Then $T \cap S$ is the set of integers divisible by 2 & 6, $T \cap N$ is the set of integers divisible by 2 & 9, $N \cap S$ is the set of integers divisible by 6 & 6 and $S \cap T \cap N$ is the set of integers divisible by 2 & 6& 9.

Then number of integers not exceeding 250 that are divisible by 2, $|T| = \lfloor \frac{250}{2} \rfloor = 125$

Then number of integers not exceeding 500 that are divisible by 6, $|S| = \lfloor \frac{250}{6} \rfloor = 41$

Then number of integers not exceeding 500 that are divisible by 9, $|N| = \lfloor \frac{250}{9} \rfloor = 27$

Similarly, $|T \cap S| = \lfloor \frac{250}{L.C.M(2,6)} \rfloor = 41$

$|T \cap N| = \lfloor \frac{250}{L.C.M(2,9)} \rfloor = 13$

$|N \cap S| = \lfloor \frac{500}{L.C.M(9,6)} \rfloor = 13$

$|T \cap N \cap S| = \lfloor \frac{500}{L.C.M(2,9,6)} \rfloor = 13$

By principle of inclusion-exclusion, we have

$$\begin{aligned}|T \cup N \cup S| =&|T| + |S| + |N| - |T \cap S| - |T \cap N| - |N \cap S| + |T \cap N \cap S|\\ =&125 + 41 + 27 - 41 - 13 - 13 + 13 = 139\end{aligned}$$

Hence there are 139 positive integers not exceeding 250 that are divisible by 2,6 or 9.

Ques. 15 *30 cars were assembled in the factory.The options available were a radio, an a.c. and white wall tyres. It is known that 15 of the cars have radio, 8 of them have a.c. and 6 of them have white wall tyres. Moreover 3 of them have all the three options. Find that at least how many cars do not have any options at all.*

Sol. Let the set of cars having the option "Radio" be R, the set of cars having the option "A.C" be A and the set of cars having the option "White wall tyre" be W. The set of all cars is C
Then the set of cars having the option "Radio" and "A.C" is $R \cap A$, the set of cars having the option "Radio" and "Whitewall tyre" is $R \cap W$, the set of cars having the option "Whitewall tyres" and "A.C" is $W \cap A$. The set of cars having all three options $R \cap A \cap W$ and the set of cars having any one option is $R \cup A \cup W$. Given that

$|R| = 15$, $|A| = 8$, $|W| = 6$, and $|A \cap R \cap W| = 3$
The number of cars having none of the option $= |C| - |A \cup R \cup W|$........................(1)

By Principle of inclusion-exclusion

$$\begin{aligned}
|A \cup R \cup W| =& |A| + |R| + |W| - |A \cap R| - |A \cap W| - |R \cap W| + |A \cap R \cap W| \\
\leq& |A| + |R| + |W| - |A \cap R \cap W| - |A \cap R \cap W| - |A \cap R \cap W| + |A \cap R \cap W| \\
& \quad [Since\ |A \cap R| \geq |A \cap R \cap W|,\ |A \cap W| \geq |A \cap R \cap W|,\ |W \cap R| \geq |A \cap R \cap W|] \\
\leq& 15 + 8 + 6 - 3 - 3 - 3 + 3 = 23 \qquad ...(2)
\end{aligned}$$

From equation (1) and (2)
The number of cars having none of the option $\geq |C| - |A \cup R \cup W| = 30 - 23 = 7$

Thus atleast 7 cars donot have any of the three option.

1.4 Mathematical Induction

Principle of Mathematical Induction: Suppose a given statement $S(n)$ involves $n \geq n_0$, where n_0 is a positive integer. Then $S(n)$ for all positive integer $n \geq n_0$ provided it follows the following steps:

1. **Inductive base.** If $S(n_0)$ is true.
2. **Inductive hypothesis.** Assume that $S(k)$ is true for an arbitrary value of k.
3. **Inductive Step.** Verify that $S(k+1)$ is true on the basis of Inductive hypothesis.

Ques. 16 *Show that*

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6},\ n \geq 1$$

by mathematical induction.

Given identity is

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6},\ n \geq 1 \qquad (1)$$

1. **Inductive base.** Here $n_0 = 1$ For $n = 1$
   L.H.S= $1^2 = 1$
   R.H.S= $\frac{1 \cdot 2 \cdot 3}{6} = 1$

   That is the identity holds for $n = 1$
2. **Inductive hypothesis.** Assume for an arbitrary value of $k \geq 1$.

$$1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad (2)$$

3. **Inductive Step.** For $n = (k+1)$

$$\begin{aligned}
1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 =& \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad [Using\ (2)] \\
=& (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right) \\
=& (k+1)\left(\frac{k(2k+1) + 6(k+1)}{6}\right) = (k+1)\left(\frac{2k^2 + 7k + 6}{6}\right) \\
=& \frac{(k+1)(2k+3)(k+2)}{6} \\
=& \frac{\overline{k+1}(\overline{k+1}+1)(2\,\overline{k+1}+1)}{6}
\end{aligned}$$

That is the identity holds for $n = k + 1$ whenever it is true for $k \geq 1$.

Thus, by Principle of mathematical induction, the given identity is true.

Ques. 17 *Show that*

$$3 + 33 + 333 + ... + 33...3 = \frac{(10^{n+1} - 9n - 10)}{27}, \ n \geq 1$$

by mathematical induction.

Given identity is

$$3 + 33 + 333 + ... + 33..n\ digits..3 = \frac{(10^{n+1} - 9n - 10)}{27}, \ n \geq 1 \tag{3}$$

1. **Inductive base.** Here $n_0 = 1$ For $n = 1$
   L.H.S= 3
   R.H.S= $\frac{10^2-9-10}{27} = 3$

   That is the identity holds for $n = 1$

2. **Inductive hypothesis.** Assume for an arbitrary value of $k \geq 1$.

$$3 + 33 + 333 + ... + 33..k\ digits..3 = \frac{(10^{k+1} - 9k - 10)}{27} \tag{4}$$

3. **Inductive Step.** For $n = (k + 1)$

$$\begin{aligned} &3+33 + 333 + ... + 33..k\ digits..3 + 33..(k + 1)\ digits..3 \\ =&\frac{(10^{k+1} - 9k - 10)}{27} + 33..(k + 1)\ digits..3 \ \ [Using\ (4)] \\ =&\frac{(10^{k+1} - 9k - 10)}{27} + \frac{99..(k + 1)\ digits..9}{3} \\ =&\frac{(10^{k+1} - 9k - 10)}{27} + \frac{10^{k+1} - 1}{3} \\ =&\frac{(10^{k+1} - 9k - 10)}{27} + \frac{9(10^{k+1} - 1)}{27} \\ =&\frac{(1 + 9)10^{k+1} - 9(k + 1) - 10}{27} = \frac{10^{k+2} - 9(k + 1) - 10}{27} \end{aligned}$$

   That is the identity holds for $n = k + 1$ whenever it is true for $k \geq 1$.

Thus, by Principle of mathematical induction, the given identity is true.

Ques. 18 *Show that*

$$n^2 > 2n + 1, \ for\ n \geq 3$$

by mathematical induction.

Given inequality is

$$n^2 > 2n + 1, \ n \geq 3 \tag{5}$$

1. **Inductive base.** Here $n_0 = 3$ For $n = 3$
   L.H.S= $3^2 = 9$
   R.H.S= $2 \cdot 3 + 1 = 7$

   $3^2 = 9 > 2 \cdot 3 + 1 = 7$ That is the inequality holds for $n = 3$

2. **Inductive hypothesis.** Assume for an arbitrary value of $k \geq 3$.

$$k^2 > 2k + 1, \; k \geq 3 \tag{6}$$

3. **Inductive Step.** For $n = (k + 1)$

$$\begin{aligned}
&(k+1)^2 \\
&\quad = k^2 + 2k + 1 \\
&\quad > 2k + 1 + 2k + 1 \;\; [Using\ (6)] \\
&\quad = 2(k+1) + 2k \\
&\quad > 2(k+1) + 1 \;\; [Since\ k \geq 3,\ 2k > 1]
\end{aligned}$$

That is the inequality holds for $n = k + 1$ whenever it is true for $k \geq 3$.

Thus, by Principle of mathematical induction, the given inequality is true.

Ques. 19 *Show that $n^4 - 4n^2$ is divisible by 3 for $n \geq 2$ by mathematical induction.*

Given $n^4 - 4n^2$ is divisible by 3 for $n \geq 2$.
That is there exist m such that

$$n^4 - 4n^2 = 3m \tag{7}$$

1. **Inductive base.** Here $n_0 = 2$ For $n = 2$
$2^4 - 4 \cdot 2^2 = 0 = 3 \cdot 0$

   That is the identity holds for $n = 2$

2. **Inductive hypothesis.** Assume for an arbitrary value of $k \geq 2$.

$$k^4 - 4k^2 = 3m, \; k \geq 2 \tag{8}$$

3. **Inductive Step.** For $n = (k + 1)$

$$\begin{aligned}
&(k+1)^4 - 4(k+1)^2 \\
&\quad = k^4 + 4k^3 + 6k^2 + 4k + 1 - 4(k^2 + 2k + 1) \\
&\quad = k^4 - 4k^2 + 4k^3 + 6k^2 - 4k - 3 \\
&\quad = 3m + 4k^3 - 4k + 3(2k^2 - 1) \quad [Using\ (8)] \\
&\quad = 4k(k^2 - 1) + 3(m + 2k^2 - 1) \\
&\quad = 4(k-1)k(k+1) + 3(m + 2k^2 - 1)
\end{aligned}$$

We note that the second term is a multiple of 3 and the first term is a product of three consecutive numbers, so that one of the three terms is multiple of 3.

That is $(k + 1)^4 - 4(k + 1)^2$ is divisible by 3.

Thus, by Principle of mathematical induction, the given statement is true.