Unit-5

Complex Analysis

Q. 1. Determine p such that the function $f(z) = \frac{1}{2}log(x^2+y^2) + i\,tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function.

Ans. Given $f(z) = \frac{1}{2}log(x^2+y^2) + i\,tan^{-1}\left(\frac{px}{y}\right)$

Here, $u = \frac{1}{2}log(x^2+y^2),\ v = tan^{-1}\left(\frac{px}{y}\right)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2}.\frac{1}{x^2+y^2}.2x = \frac{x}{x^2+y^2}; \qquad \frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}.2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{p^2x^2}{y^2}}\left(\frac{p}{y}\right) = \frac{py}{y^2+p^2x^2} \qquad \text{and } \frac{\partial v}{\partial y} = \frac{1}{1+\frac{p^2x^2}{y^2}}\left(-\frac{px}{y^2}\right) = \frac{-px}{y^2+p^2x^2}$$

The given function will be analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus,
$$\frac{x}{x^2+y^2} = \frac{-px}{y^2+p^2x^2} \qquad \ldots(1)$$

$$\frac{y}{x^2+y^2} = -\left(\frac{py}{y^2+p^2x^2}\right) \qquad \ldots(2)$$

Clearly (1) and (2) satisfies only if $p = -1$.

Q. 2. If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Ans. Let $f(z) = u(x,\ y) + iv(x,\ y)$

$\Rightarrow |f(z)|^2 = u^2 + v^2 = \varphi(x,\ y)$, (say)

$$\therefore \qquad \frac{\partial \varphi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

and
$$\frac{\partial^2 \varphi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly,
$$\frac{\partial^2 \varphi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we have

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]$$
$$+2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \ldots(1)$$

Since $u,\ v$ have to satisfy Cauchy-Riemann equations and the Laplace's equation.

$$\therefore \qquad \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2,\ \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial u}\right)^2$$

and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$

Thus eqn. (1) takes the form

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=4\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}$$

Hence

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2.$$

Q.3. Determine the analytic function $f(z)=u+iv$, if $u-v=(x-y)(x^2+4xy+y^2)$ and $f\left(\frac{\pi}{2}\right)=0$.

Ans. We have $\quad u-v=(x-y)(x^2+4xy+y^2)=x^3+3x^2y-3xy^2-y^3$

$\therefore \quad \frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=3x^2+6xy-3y^2 \qquad \ldots(1)$

and $\quad \frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=3x^2-6xy-3y^2$

$\Rightarrow -\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}=3x^2-6xy-3y^2 \qquad \ldots(2)$

Subtracting eqn. (2) from eqn. (1), we get

$$2\frac{\partial u}{\partial x}=12xy$$

Then

$$\frac{\partial u}{\partial x}=6xy$$

$\Rightarrow$ Adding eqn. (1) and eqn. (2), we have

$\Rightarrow -2\frac{\partial v}{\partial x}=6x^2-6y^2$

Then

$$\frac{\partial v}{\partial x}=3y^2-3x^2$$

Thus, using Milne Thomson's method

$$f'(z)=\left(\frac{\partial u}{\partial x}\right)_{(z,0)}+i\left(\frac{\partial u}{\partial x}\right)_{(z,0)}=6\cdot z\cdot 0+i\,(3\cdot 0-3z^2)=-3i\,z^2$$

$$\Rightarrow f(z)=-iz^3+c$$

Q. 4. Evaluate, using Cauchy's integral formula $\int_C \frac{e^{2z}dz}{(z-1)(z-2)}$, where C is circle $|z|=3$.

Ans. $f(z)=e^{2z}$ is analytic within the circle $C:\ |z|=3$.

Singular points: $(z-1)(z-2)=0$

$\Rightarrow z=1,\ z=2$

Two singular points $a=1$ and $a=2$ lie inside C.

$$\therefore \qquad \int_C \frac{e^{2z}dz}{(z-1)(z-2)} = \int_C e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz$$

$$= \int_C \frac{e^{2z}}{z-2} dz - \int_C \frac{e^{2z}}{z-1} dz$$

$$= 2\pi i e^{2\times 2} - 2\pi i e^{2\times 1} \qquad \text{(By Cauchy's Integral formula)}$$

$$= 2\pi i(e^4 - e^2).$$

Q. 5. Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region:

(a) $|z| < 1$ (b) $1 < |z| < 2$

(c) $|z| > 2$ (d) $0 < |z-1| < 1$

Ans. We have

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

(a) For $|z| < 1$

we have $\frac{|z|}{2} < 1$

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{-(1-z)}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots..\right) + (1+z+z^2+\ldots..)$$

$$= \frac{1}{2} + \frac{3z}{4} + \frac{7}{8}z^2 + \ldots..$$

(b) For $1 < |z| < 2$

we have $1 < |z|,\ |z| < 2$

$\Rightarrow \frac{1}{|z|} < 1,\ \frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots..\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots..\right)$$

$$= \left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\ldots..\right) - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \ldots..$$

(c) $|z| > 2$

$\Rightarrow 2 < |z| \Rightarrow \frac{2}{|z|} < 1$

$\Rightarrow \frac{2}{|z|} < \frac{1}{2} < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots..\right) - \frac{1}{z}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots..\right)$$

$$= \frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\ldots.. - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \ldots..$$

$$= \frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\ldots..$$

(d) $0 < |z-1| < 1$

$\Rightarrow |z-1| < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -[1-(z-1)]^{-1} - (z-1)^{-1}$$

$$= -[1+(z-1)+(z-1)^2+(z-1)^3+\ldots..] - (z-1)^{-1}.$$

Q. 6. Find the Laurent's expansion of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in the region $1 < z+1 < 3$.

Ans. Region $1 < z+1 < 3$

We have

$$1 < z+1,\ z+1 < 3$$

$\Rightarrow \quad \frac{1}{z+1} < 1,\ \frac{z+1}{3} < 1$

Put $z+1 = u \Rightarrow z = u-1$

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-3)}$$

$$= \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

$$= -\frac{3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{2}{3\left(1-\frac{u}{3}\right)}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \ldots..\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{9} + \frac{u^3}{27} + \ldots..\right)$$

$$= \frac{-3}{2+1} + \frac{1}{z+1}\left(1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \ldots..\right) - \frac{2}{3}\left(1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \ldots..\right).$$

Q. 7. Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\int_C f(z)\ dz$, where C is the circle $|z| = 2.5$.

Ans. $f(z)$ has a simple at $z = -2$

$$\text{Res. } f(-2) = \lim_{z\to -2}(z+2).\frac{z^2}{(z-1)^2(z+2)}$$

$$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$$

Again $z = 1$ is a pole of order 2,

$f(z) = \frac{\varphi(z)}{(z-1)^2}$, where $\varphi(z) = \frac{z^2}{z+2}$

$$\text{Res. } f(1) = \frac{\varphi'(1)}{1\,!}$$

$$\varphi(z) = \frac{z^2}{z+2}$$

$$\varphi'(z) = \frac{z^2+4z}{(z+2)^2}$$

$$\varphi'(1) = \frac{5}{9}$$

$\therefore$ Res. $f(1) = \frac{\frac{5}{9}}{1\,!} = \frac{5}{9}$

Clearly $f(z)$ is analytic on $|z| = 2.5$ and at all points inside except the poles $z = -2$ and $z = 1$.

By Residue theorem,

$$\int_C f(z)\ dz = 2\pi i\ [\text{Res. } f(-2) + \text{Res. } f(1)]$$

$$= 2\pi i\ \left[\frac{4}{9} + \frac{5}{9}\right] = 2\pi i.$$

Q. 8. Evaluate: $\int_C \frac{e^z\ dz}{z^2+1},\ C : |z| = 2.$

Ans. Let

$$f(z) = \frac{e^z}{z^2+1} = \frac{e^z}{(z+i)(z-i)}$$

$z = i$ is a simple pole.

$z = -i$ is a simple pole.

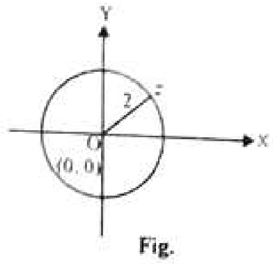

Fig.

$$C : |z| = 2$$

Both the poles $z = i$ and $z = -i$ lie inside circle.

$$C : |z| = 2$$

$$\text{Res. } f(i) = \lim_{z \to i} (z - i). \frac{e^z}{(z+i)(z-i)}$$

$$\text{Res. } f(i) = \frac{e^i}{2i}$$

$$\text{Res. } f(-i) = \lim_{z \to -i} (z + i). \frac{e^z}{(z+i)(z-i)}$$

By residue theorem,

$$\int_C f(z)\ dz = 2\pi i\ [\text{Res. } f(i) + \text{Res. } f(-i)]$$

$$= 2\pi i\ \left[\frac{e^i}{2i} - \frac{e^{-i}}{2i}\right]$$

$$= \pi\ [e^i - e^{-i}].$$

Q. 9. Show that $\int_0^{2\pi} \frac{cos\, 2\theta\ d\theta}{1-2a\, cos\, \theta + {}^2} = \frac{2\pi a^2}{1-a^2},\ (a^2 < 1)$.

Ans. Put

$$z = e^{i\ \theta} \Rightarrow dz = e^{i\ \theta}. i\ d\theta = zi\ d\theta \Rightarrow d\theta = \frac{dz}{zi}$$

$$cos\, 2\, \theta = \frac{e^{i\theta} + e^{-2i\theta}}{2} = \frac{z^2 + z^{-2}}{2}$$

$$cos\, \theta = \frac{e^{i\ \theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

$\therefore$ The given integral $I = \int_C \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{1-2a.\frac{1}{2}\left(z+\frac{1}{z}\right)+a^2} . \frac{dz}{zi}$, where C is the unit circle.

$$= \int_C \frac{\frac{1}{2x^2}(z^4+1)}{1 - a\ \left(\frac{z^2+1}{z}\right) + a^2} . \frac{dz}{zi}$$

$$= \int_C \frac{1}{2x^2} . \frac{(z^4+1)}{\frac{z - az^2 - a + a^2 z}{z}} . \frac{dz}{zi}$$

$$= \int_C \frac{1}{2x^2} . \frac{(z^4+1)}{[(z-a) - az(z-a)]} . \frac{dz}{i}$$

$$= \frac{1}{2i} \int_C \frac{(z^4 + 1)\ dz}{z^2(z-a)(1-az)}$$

Poles of $f(z)$:

$z = 0$ (order 2)

$z = a$ is a simple pole

$z = \frac{1}{a}$ is a simple pole.

$$C : \ |z| \ = 1,$$

Only $z = 0$ and $z = a$ lie inside the circle

$$C : \ |z| \ = 1$$

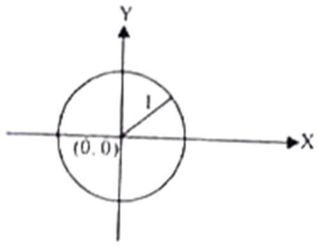

Fig.

$$\text{Res. } f(z) = \lim_{z \to a} (z-a). \frac{z^4+1}{z^2(z-a)(1-az)}$$

$$= \frac{a^4+1}{a^2(1-a^2)}$$

$z = 0$ is pole of order 2.

$f(z) = \frac{\varphi(z)}{z^2}$, where $\varphi(z) = \frac{z^4+1}{(z-a)(1-az)}$

$$\varphi(z) = \frac{z^4+1}{z - az^2 - a + a^2 z}$$

$$\text{Res. } f(0) = \frac{\varphi'(0)}{1\,!}$$

$$\varphi'(z) = \frac{(z - az^2 - a + a^2 z)(4z^3) - (z^4+1).(1 - 2az + a^2)}{(z - az^2 - a + a^2 z)^2}$$

$$\varphi'(0) = \frac{-(1+a^2)}{a^2}$$

By residue theorem,

$$\int_C f(z)\ dz = \frac{1}{2i}. 2\pi i\ [\text{Res. } f(a) + \text{Res. } f(0)]$$

$$= \pi \ \left[\frac{a^4+1}{a^2(1-a^2)} - \frac{(1+a^2)}{a^2}\right] = \frac{2\pi a^2}{1-a^2}. \qquad \textbf{Proved.}$$

Q. 10. Evaluate: $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$.

Ans. Consider

$$\int_C f(z)\ dz = \int_C \frac{z^2 dz}{(z^2+1)(z^2+4)},$$

Where C is the contour consisting of the semi circle C_R of radius R together with part of the real axis from $-R$ to R.

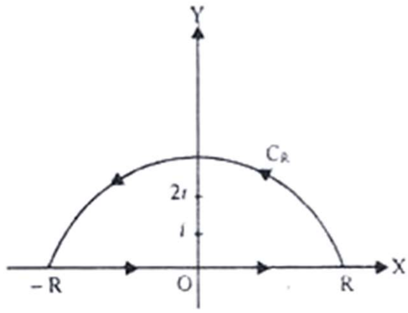

Fig.

Poles of $f(z)$:

$$z = \pm\ i,\ z = \pm\ 2iz = i,\ 2i \text{ lie inside C.}$$

$$\text{Res. } f(i) = \lim_{z \to i}\ (z-i).\frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)} = -\frac{1}{6i}$$

$$\text{Res. } f(2i) = \lim_{z \to 2i}\ (z-2i).\frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)} = \frac{1}{3i}$$

By Residue theorem,

$$\int_C f(z)\ dz = 2\pi i\ [\text{Res. } f(i) + \text{Res. } f(2i)] = 2\pi i\ \left[\frac{-1}{6i} + \frac{1}{3i}\right] = \frac{\pi}{3}$$

Also,

$$\int_C f(z)\ dz = \int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz.$$

Now let $R \to \infty$

For any point on C_R as $|z| \ \to \infty$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{z^2\left(1+\frac{1}{z^2}\right) z^2\left(1+\frac{4}{z^2}\right)}$$

$f(z) \to 0$ as $z \to \infty$

$\therefore \lim_{|z| \to \infty} \int_{C_R} f(z)\ dz = 0$

$\therefore$

$$\int_C f(z)\ dz = \int_{-\infty}^{\infty} f(x)\ dx \Rightarrow \int_{-\infty}^{\infty} \frac{x^2\ dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}.$$