Expectation: Let X be a random variable with a probability $f(x)$, then the expectation of the random variable X is nothing mean of the distribution.

i.e, $\boxed{E(X) = \mu}$.

Variance: $\boxed{\sigma^2 = E(X^2) - [E(X)]^2}$ **

Note: (i) $E[g(X)] = \sum_j g(x_j) \cdot f(x_j)$

EXPECTATION (Problems)

Ex: The pdf of a random variable X is ;

$$f(x) = \begin{cases} \frac{1}{2} ; & -1 \leq x \leq 1 \\ 0 ; & \text{Otherwise}. \end{cases}$$

find (i) $E(X)$ and (ii) $E(2X^3)$.

Sol: (i) $E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-1}^{1} x f(x)\, dx$$

$$= \int_{-1}^{1} x \cdot \frac{1}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2}\right]$$

$$= 0.$$

(ii) $E(2X^3) = \int_{-\infty}^{\infty} 2x^3 f(x)\, dx$

$$= \int_{-1}^{1} 2x^3 \cdot \frac{1}{2}\, dx$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{4}\left[1 - 1\right]$$

$$= 0.$$

Properties of expectation:

(i) $E(a) = a$; where a is a constant.

(ii) $E(aX) = a\,E(X)$; " "

(iii) $E(X \pm Y) = E(X) \pm E(Y)$, where X and Y are two different random variables.

(iv) $E(XY) = E(X)\cdot E(Y)$, where X, Y are independent.

Ex: A random no. is choosen from a set $\{1,2,\ldots\ldots,100\}$ and another from $\{1,2,\ldots\ldots,50\}$. What is the expection of the product?

Soln:

$$X : 1 \quad 2 \quad 3 \quad \ldots\ldots \quad 100$$

$$f(x) : 1/100 \quad 1/100 \quad 1/100 \quad \ldots\ldots \quad 1/100$$

$$Y : 1 \quad 2 \quad 3 \quad \ldots\ldots \quad 50$$

$$f(y) : 1/50 \quad 1/50 \quad 1/50 \quad \ldots\ldots \quad 1/50$$

$$E(XY) = E(X)\cdot E(Y)$$

$$E(X) = \sum_x x f(x)$$

$$= 1\cdot\frac{1}{100} + 2\cdot\frac{1}{100} + 3\cdot\frac{1}{100} + \ldots\ldots + 100\cdot\frac{1}{100}$$

$$= \frac{1}{100}(1+2+3+\ldots\ldots+100)$$

$$= \frac{1}{100}\cdot\frac{100\cdot(100+1)}{2}$$

$$= 101/2$$

$$E(Y) = \sum_y y f(y)$$

$$= 1\cdot\frac{1}{50} + 2\cdot\frac{1}{50} + 3\cdot\frac{1}{50} + \ldots\ldots + 50\cdot\frac{1}{50}$$

$$= \frac{1}{50}(1+2+3+\ldots\ldots+50)$$

$$= \frac{1}{50}\cdot\frac{50(50+1)}{2}$$

$$= 51/2$$

$$\therefore E(XY) = \frac{101}{2}\cdot\frac{51}{2} = 5151/4.$$

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A special die with $(n+1)$ faces is rolled. Its faces are marked $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}$. If X denotes the no. shown, then find

(i) $E(X)$

(ii) Standard deviation of X.

(iii) $E\left(X-\frac{1}{2}\right)^3$.

Soln:

$$X: \quad 0 \quad 1/n \quad 2/n \cdots\cdots \quad n/n$$

$$f(x): \quad \frac{1}{n+1} \quad \frac{1}{(n+1)} \quad \frac{1}{n+1} \cdots\cdots \quad \frac{1}{n+1}$$

(i) $\therefore E(X) = \sum_x x f(x)$

$$= 0 \cdot \frac{1}{n+1} + \frac{1}{n} \cdot \frac{1}{n+1} + \frac{2}{n} \cdot \frac{1}{n+1} + \cdots\cdots + \frac{n}{n} \cdot \frac{1}{n+1}$$

$$= \frac{1}{n(n+1)}\left[1+2+\cdots\cdots+n\right]$$

$$= \frac{1}{n(n+1)} \cdot \frac{n(n+1)}{2}$$

$$= \frac{1}{2}.$$

(ii) $\sigma^2 = E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_x x^2 f(x)$$

$$= 0 \cdot \frac{1}{n+1} + \frac{1}{n^2} \cdot \frac{1}{n+1} + \frac{4}{n^2} \cdot \frac{1}{n+1} + \cdots\cdots + \frac{n^2}{n^2} \cdot \frac{1}{n+1}$$

$$= \frac{1}{n^2(n+1)}\left(1^2+2^2+\cdots\cdots+n^2\right)$$

$$= \frac{1}{n^2(n+1)} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(2n+1)}{6n}$$

$$\therefore \sigma^2 = \frac{2n+1}{6n} - \frac{1}{4}$$

$$= \frac{4n+2-3n}{12n}$$

$$= \frac{n+2}{12n}$$

$$\therefore \sigma = \sqrt{\frac{n+2}{12n}}$$

(iii) $$E\left(X - \frac{1}{2}\right)^3 = E\left(X^3 - \frac{3}{2}X^2 + \frac{3}{4}X - \frac{1}{8}\right)$$

$$= E(X^3) - \frac{3}{2}E(X^2) + \frac{3}{4}E(X) -$$

$$E(X^3) = \sum_x x^3 f(x)$$

$$= 0\cdot\frac{1}{n+1} + \frac{1}{n^3}\cdot\frac{1}{n+1} + \frac{2^3}{n^3}\cdot\frac{1}{n+1} + \cdots + \frac{n^3}{n^3}\cdot\frac{1}{n+1}$$

$$= \frac{1}{n^3(n+1)}\left[1^3 + 2^3 + \cdots + n^3\right]$$

$$= \frac{1}{n^3(n+1)}\left\{\frac{n(n+1)}{2}\right\}^2$$

$$= \frac{1}{n^3(n+1)}\cdot\frac{n^2(n+1)^2}{4}$$

$$= \frac{n+1}{4n}$$

$$\therefore E\left(X-\frac{1}{2}\right)^3 = \frac{n+1}{4n} - \frac{3}{2}\cdot\frac{2n+1}{6n} + \frac{3}{4}\cdot\frac{1}{2} - \frac{1}{8}$$

$$= \frac{1}{4n}\left(n+1-2n-1\right) + \frac{2}{8}$$

$$= -\frac{n}{4n} + \frac{1}{4}$$

$$= -\frac{1}{4} + \frac{1}{4}$$

$$= 0$$

A random variable X has pdf

$$f(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; \text{otherwise.} \end{cases}$$

Find (i) $E(X)$ (ii) $E(X^2)$ (iii) $E[(X-1)^2]$ (iv) $E(e^{2X/3})$.

Soln:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x \cdot e^{-x}\, dx.$$

$$= \left[-x e^{-x}\right]_0^{\infty} - \int_0^{\infty} 1 \cdot (-e^{-x})\, dx.$$

$$= \left[-x e^{-x}\right]_0^{\infty} - \left[e^{-x}\right]_0^{\infty}$$

$$= [0-0] - [0-1]$$

$$= 1.$$

Alternative:

$$E(X) = \int_0^{\infty} x \cdot e^{-x}\, dx$$

$$= \Gamma(2)$$

$$= (2-1)!$$

$$= 1$$

$\left[\because \Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1}\, dx. = \lfloor n-1 \right]$

(ii)

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\infty} x^2 \cdot e^{-x}\, dx$$

$$= \Gamma(3)$$

$= (3-1)!$

$= 2!$

$= 2.$

(iii) $E[(X-1)^2] = E(X^2 - 2X + 1)$

$= E(X^2) - 2E(X) + E(1)$

$= 2 - 2.1 + 1$

$= 1.$

(iv) $E(e^{2X/3}) = \int_{-\infty}^{\infty} e^{2x/3} f(x)\, dx$

$= \int_{0}^{\infty} e^{2x/3} . e^{-x}\, dx$

$= \int_{0}^{\infty} e^{-x/3}\, dx$

$= \left[-3 e^{-x/3}\right]_0^{\infty}$

$= -3[0-1]$

$= 3.$

Moments and Moment Generating Function:

(i) r-th order moment (about origin):

$$\mu_r' = E(X^r) = \sum_x x^r . f(x) \quad [\text{for discrete } X]$$

$$= \int_{-\infty}^{\infty} x^r f(x)\, dx \quad [\text{for continuous } X]$$

(i) Mean $= E(x) =$ 1st order moment about origin.

(ii) r th order moment about any point c :

$$E\left[(x-c)^r\right] = \sum_x (x-c)^r f(x) \quad ; \text{for discrete}$$

$$= \int_{-\infty}^{\infty} (x-c)^r f(x)\, dx \quad ; \text{for continuous.}$$

(iii) r th order moment about the mean (μ) :

OR

r th order central moment :

$$M_r' = E\left[(x-\mu)^r\right] = \sum_x (x-\mu)^r f(x) \quad ; \text{for discrete}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx \; ; \text{for continuous.}$$

Q. Find the 1st & 2nd order central moment.

Soln:

$$M_1' = E(x-\mu) = E(x) - E(\mu)$$

$$= \mu - \mu$$

$$= 0.$$

$$M_2' = E(x-\mu)^2 = E(x^2 - 2x\mu + \mu^2)$$

$$= E(x^2) - 2\mu\, E(x) + E(\mu^2)$$

$$= E(x^2) - 2\{E(x)\}^2 + \mu^2$$

$$= E(x^2) - 2\{E(x)\}^2 + \{E(x)\}^2$$

$$= E(x^2) - \{E(x)\}^2$$

$$= \text{Variance } (= \sigma^2)$$

** Note: 2nd order central moment is variance $= E(x^2) - \{E(x)\}^2$

Moment Generating Function:

(It is a technique to generate the moments order.)

The mgf of a random variable X, about the origin with probability distribution $f(x)$ is denoted by $M_X(t)$ and is defined

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx}.f(x) \;;\; \text{for } X = \text{discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \;;\; \text{for } X = \text{continuous} \end{cases}$$

Theorem: Let X be a random variable with the probability distribution $f(x)$ and mgf $M_X(t)$. Then the r^{th} moments about the origin can be generated as;

$$\mu_r' = \left[\frac{d^r M_X(t)}{dt^r}\right]_{t=0}$$

Proof: The mgf of the random variable X is;

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \end{cases}$$

Diff w.r.t. 't' r times;

$$\frac{d^r}{dt^r} M_X(t) = \begin{cases} \sum_x x^r. e^{tx} f(x) \\ \int_{-\infty}^{\infty} x^r e^{tx} f(x)\, dx \end{cases}$$

At $t=0$

$$\left[\frac{d^r M_X(t)}{dt^r}\right]_{t=0} = \begin{cases} \sum_x x^r . f(x) \\ \int_{-\infty}^{\infty} x^r f(x)\, dx \end{cases}$$