Example: Fourier Series Representation of a signal $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{5}{6+(5n\pi)^2} \cdot e^{jn\pi t} \quad \text{------} \ (1)$$

Find: (1) T_0, a_0 of the signal

(2) One term of the expression is $A_0 \cos 6\pi t$, then calculate A_0.

Solⁿ: Generic Eqnⁿ

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jnW_0 t} \quad \text{------} \ (2)$$

(1) comparing (1) & (2) we get

$W_0 = \pi$ $\qquad \therefore T = \frac{2\pi}{\pi} = 2$ Sec. (Ans)

We know,

$$a_0 = c_0 \quad \text{------} \ (3)$$

$$c_n = \frac{5}{6+(5n\pi)^2}$$

$$c_0 = \frac{5}{6+(5\cdot 0\cdot \pi)^2} = \frac{5}{6} = a_0 \ \text{(Ans)}$$

(2) $A_0 \cdot \cos 6\pi t = a_n \cdot \cos n\pi t$

$\Rightarrow 6\pi = n\pi$

$\Rightarrow n = 6$

$$\therefore A_0 = a_6 \quad \text{------} \ (4)$$

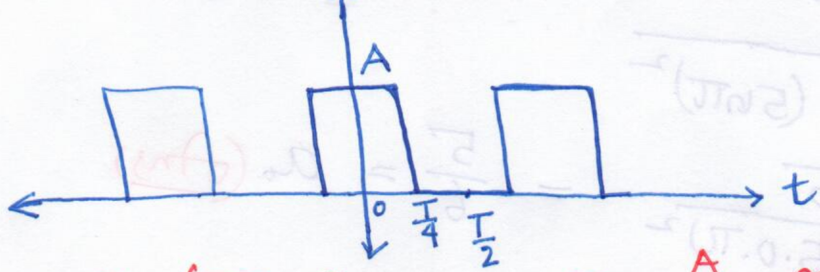
$$a_6 = 2 \cdot Re[C_6]$$

$$= 2 \cdot \frac{5}{6+(5\cdot 6\cdot \pi)^2} = \frac{10}{6+(30\pi)^2} \ \text{(Ans)}$$

Relationship $x(t)$ and C_n Pair

$x(t)$	C_n
Real (R)	Conjugate Symmetric (CS)
CS	R
I Imaginary	CAS Conjugate Anti Symmetric
R + E	R + E ($\uparrow$ Real $\uparrow$ Even)
I+E	I+E
R+O ($\uparrow$ Odd)	I+O
I+O	R+O

Q. Determine the FS coefficient for given periodic signal $x(t)$



(a) $\frac{A}{j\pi K}\sin\left(\frac{\pi}{2}\cdot K\right)$ (b) $\frac{A}{\pi j K}\cos\left(\frac{\pi}{2}\cdot K\right)$.

(c) $\frac{A}{\pi K}\sin\left(\frac{\pi}{2}K\right)$ (d) $\frac{A}{\pi K}\cos\left(\frac{\pi}{2}K\right)$

The signal is Real Signal and Even Signal

$\therefore$ $x(t)$ is R+E, C_n must be R+E.

(a) (b) are are imaginary, so <u>Not possible</u>

$c_K = \frac{A}{\pi K} \sin(\frac{\pi}{2} \cdot K).$

$c_{-K} = \frac{A}{\pi(-K)} \cdot \sin(\frac{\pi}{2} \cdot (-K)) = \frac{A}{-\pi K} \cdot (-1) \cdot \sin(\frac{\pi K}{2})$

$= +\frac{A}{\pi K} \cdot \sin(\frac{\pi}{2} K).$

· Multi Grid Computi
· AICTE Project.

$= c_K$

$=$ Even

$\therefore$ Option (3) is correct option.

Example: $C_n = \begin{cases} 2 & n=0 \\ j(\frac{1}{2})^{|n|} & \text{, otherwise} \end{cases}$

Determine the Characteristics of $x(t)$ for various n Val

Answer: If we put $n=-n$, then

$C_{-n} = C_n$ because of $| \, |$ in otherwise and absence of n in $n=0$

$\therefore C_n$ is Even

For $n=0$

C_n is Real + Even, So $x(t)$ is Real + Even

For $n =$ otherwise

C_n is Img + Even, So $x(t)$ is Img + Even

Symmetricities in Fourier Series.

- Even Symmetry → $x(t) = x(-t)$.

  = Does Not contain Sine term

  = $b_n = 0$

- Odd Symmetry → $x(-t) = -x(t)$.

  = Does Not Contain Cosine term

  = $a_n = 0$

- Avg. Value = 0, then $a_0 = 0$

- Half Wave Symmetry (HWS):
  - $x(t) = -x\left(t + \frac{T_0}{2}\right)$
  - Only Odd Harmonics.

- Odd Symmetry + Half Wave Symmetry (HWS):

odd wave + HWS

HWS → only odd Harmonics

HWS → avg. Value $(a_0) = C_0 = 0$.

odd wave → $b_n \neq 0$

$a_n = 0$

$a_0 = 0$

∴ only Sine terms are present with odd Harmonics.

- Even + HWS:

Even + HWS

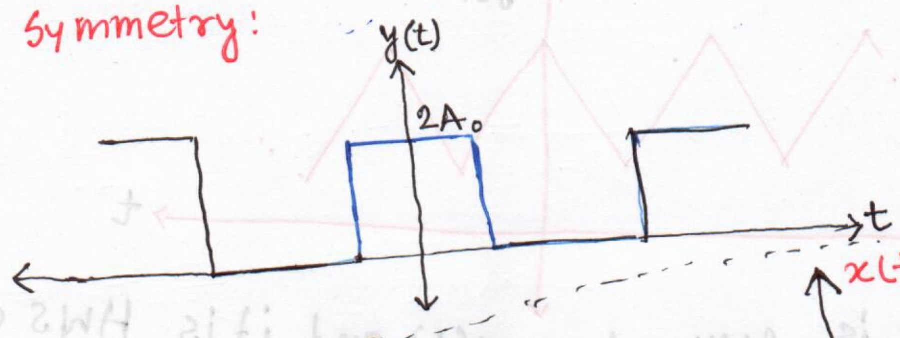
HWS ← $a_0 = 0$

HWS ← only odd Harmonics

Even → $b_n = 0$, $a_n \neq 0$

∴ Only cosine terms are present with odd Harmonics.

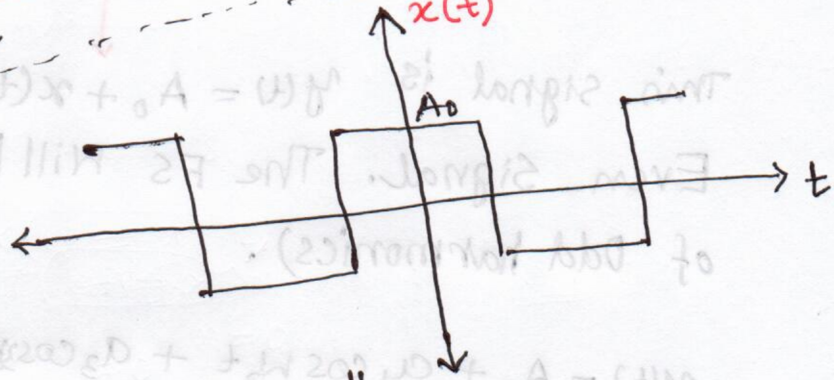
Hidden Symmetry:



$y(t) = A_0 + x(t)$

A_0 ↑ dc term

$x(t)$ ↑ Cosine terms (odd Harmonics)



⇓

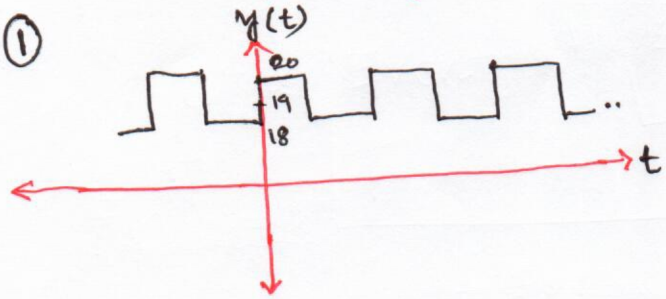
$x(t) = \text{Even} + \text{HWS}$

$= a_1 \cos W_0 t + a_3 \cos 3W_0 t$

$+ 5 \cos W_0 t + \cdots$

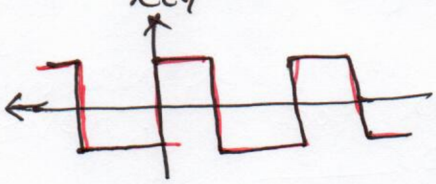
Hidden Symmetry Example

①



Solution:

$y(t) \xrightarrow[\text{Amp. Shifting}]{\text{Downward}}$



Odd + HWS.

↓

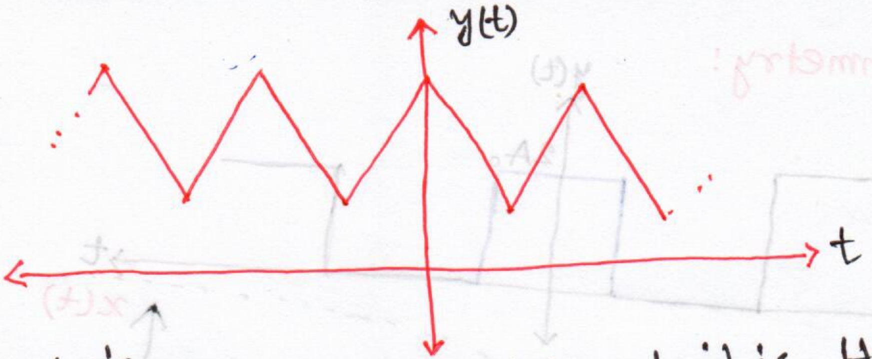
Only Sine terms with Odd Harmonics.

19 + (Sine Odd Harmonics

$= 19 + b_1 \sin W_0 t + b_3 \sin 3W_0 t$

$+ b_5 \sin 5W_0 t + \cdots$

Example:



This signal is $y(t) = A_0 + x(t)$ and it is HWS and Even Signal. The FS will be (DC + Cosine terms of Odd harmonics).

$$y(t) = A_0 + a_1 \cos w_0 t + a_3 \cos 3w_0 t + a_5 \cdot \cos 5W_0 t + \cdots$$

Properties of Fourier Series:

1) Linearity:

Let, $x_1(t) \rightleftharpoons C_{1n}$, $x_2(t) \rightleftharpoons C_{2n}$ & Common/Same Period T_0.

$$\underbrace{\alpha\, x_1(t) + \beta \cdot x_2(t)}_{\text{Composite Signal}} \rightleftharpoons \alpha \cdot C_{1n} + \beta \cdot C_{2n}$$

Homework: Establish the Equ^n. Linearity of FS

2) Conjugation:

let $x(t) \xrightleftharpoons{FS} C_n$.

then $x^*(t) \xrightleftharpoons{FS} C^*_{-n}$

3) Time Reversal:

Let, $x(t) \rightleftharpoons C_n$.

then $x(-t) \rightleftharpoons C_{-n}$.

Proof: For Signal $x(t)$, FS coefficients

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jnW_0t}\, dt$$

For Signal $x(-t)$, FS coefficients

$$C_n' = \frac{1}{T_0}\int_0^{T_0} x(-t)\cdot e^{-jnW_0t}\, dt$$

This is not going to change.

Let
$$t = -\tau$$
$$dt = -d\tau$$
$t=0 \quad t=T_0$

$\tau=0 \quad \tau=-T_0$

So, $$C_n' = \frac{1}{T_0}\int_0^{-T_0} x(\tau)\cdot e^{+jn\omega_0\tau}\cdot -d\tau$$

$$= \frac{1}{T_0}\int_{-T_0}^{0} x(\tau)\cdot e^{-j(-n)\cdot\omega_0\tau}\cdot d\tau$$

$$= \frac{1}{T_0}\int_{T_0} x(\tau)\cdot e^{-j(-n)\cdot\omega_0\tau}\cdot d\tau$$

$$= C_{-n}$$

4) Time Scaling:

Let $x(t) \xrightleftharpoons{FSC} C_n$ (period T_0)

then $x(at) \rightleftharpoons C_n$ (Period $\frac{T_0}{a}$)

5. Convolution in time:

Let $x_1(t) \rightleftharpoons C_{1n}$, $x_2(t) \rightleftharpoons C_{2n}$ $\Big\}$ Time Period T_0

then $x_1(t) * x_2(t) \rightleftharpoons T_0 \cdot (C_{1n}\cdot C_{2n})$

6. Multiplication in Time:-

let $x_1(t) \rightleftharpoons C_{1n}$

$x_2(t) \rightleftharpoons C_{2n}$

then $x_1(t)\cdot x_2(t) \rightleftharpoons C_{1n} * C_{2n}$

7. Time Shifting:

let $x(t) \rightleftharpoons C_n$ (Period T_0)

$$x(t \pm t_0) \rightleftharpoons e^{\pm jn\omega_0 \cdot t_0} \cdot C_n$$

8. Frequency Shifting:

let $x(t) \rightleftharpoons C_n$ (Period$=T_0$)

$$e^{jm\omega_0 t} \cdot x(t) \rightleftharpoons C_{n-m}$$

9. Differentiation in Time

$x(t) \rightleftharpoons C_n$ (Period$=T_0$, $\omega_0 = \frac{2\pi}{T_0}$)

$$\frac{dx(t)}{dt} \rightleftharpoons jn\omega_0 C_n$$

10. Integration in Time:

$$x(t) \rightleftharpoons C_n\ (T_0, \omega_0)$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \rightleftharpoons \frac{C_n}{jn\omega_0}$$

11. Parseval's Power Theorem:

$$x(t) \rightleftharpoons C_n \quad \text{Period} = T_0$$

** $$P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Proof: $$x(t) = \sum_{n=-\infty}^{\infty} C_n . e^{jn\omega_0 t}.$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} C_n^* . e^{-jn\omega_0 t} \quad \text{-------} \quad (1)$$

$$x(t) . x^*(t) = |x(t)|^2 \quad \text{-------} \quad (11)$$

We know,

$$P_{x(t)} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 . dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t) . x^*(t) . dt \quad \text{[Putting Value of (11)]}$$

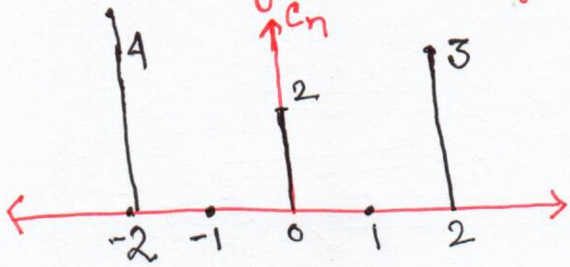
$$= \frac{1}{T_0}\int_{T_0} x(t) . \sum_{n=-\infty}^{\infty} C_n^* . e^{-jn\omega_0 t} dt . \quad \text{[Using Eq}^n\text{. (1)]}$$

$$= \sum_{n=-\infty}^{\infty} . C_n^* . \underbrace{\frac{1}{T_0}\int_{T_0} . x(t) . e^{-jn\omega_0 t} dt}$$

$$= \sum_{n=-\infty}^{\infty} C_n^* . C_n .$$

$$\boxed{P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2 .}$$

Example: Find the avg. Power of $x(t)$, when C_n is given



Answer: Using Parseval's theorem.

$$P_{x(t)} = |C_{-2}|^2 + |C_0|^2 + |C_2|^2$$
$$= 4^2 + 2^2 + 3^3$$
$$= 29 \text{ Watts}$$

Question: Find C_n' in terms of C_n, using FS properties.

$$x(t) \rightleftharpoons C_n$$
$$y(t) \rightleftharpoons C_n'$$

1. $y(t) = x(t+2) + x(t-3)$
2. $y(t) = \dfrac{dx(t)}{dt}$
3. $y(t) = \dfrac{d^2x(t)}{dt^2}$
4. $y(t) = \text{Real}[x(t)]$.
5. $y(t) = e^{-j2W_0t} \cdot x(t)$
6. $y(t) = \text{Even}[x(t)]$
7. $y(t) = \text{Odd}[x(t)]$.

Fourier Series of LTI System

$$x(t) \rightarrow \boxed{\text{LTI System}} \rightarrow y(t)$$

$x(t) \rightleftarrows C_n$, $\quad y(t) \rightleftarrows C_n'$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jnW_0t} \qquad y(t) = \sum_{n=-\infty}^{\infty} C_n' \cdot e^{jnW_0t}$$

$$\boxed{C_n' = H(jnW_0) \cdot C_n}$$

Example: Consider a causal LTI system whose i/p and o/p are related by following differential Equ^n.

$$\frac{dy(t)}{dt} + 4y(t) = x(t).$$

Find the o/p coefficient C_n' if i/p is $\cos 2\pi t + \sin 4\pi t$

Answer:-

$$x(t) = \cos \underbrace{2\pi}_{W_1} t + \sin \underbrace{4\pi}_{W_2} t.$$

$$W_0 = \text{HCF}(\underbrace{2\pi}_{W_1}, \underbrace{4\pi}_{W_2}).$$

$$= 2\pi$$

$$\frac{dy(t)}{dt} + 4y(t) = x(t)$$

$$\downarrow \text{LT} \qquad \downarrow \text{LT} \qquad \downarrow$$

$$S \cdot Y(S) + 4 \cdot Y(S) = X(S).$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{\frac{1}{4+S} \cdot X(S)}{X(S)} = \frac{1}{4+S}$$

Putting $s \rightarrow jn\omega_0$

$$H(jn\omega_0) = \frac{1}{4 + jn\omega_0} \quad \text{--- (1)}$$

$$x(t) = \cos 2\pi t + \sin 4\pi t$$

$$= \frac{1}{2} e^{j2\pi t} + \frac{1}{2} e^{-j2\pi t} + \frac{1}{2j} e^{j4\pi t} - \frac{1}{2j} e^{-j4\pi t}$$

$$= \frac{-1}{2j} e^{-j4\pi t} + \frac{1}{2} e^{-j2\pi t} + \frac{1}{2} e^{j2\pi t} + \frac{1}{2j} e^{j4\pi t}$$

-2   -1   1   2

$$= c_{-2} \cdot e^{-j2(2\pi)t} + c_{-1} \cdot e^{-j(2\pi)t} + c_1 \cdot e^{j(2\pi)t} + \frac{1}{2j} e^{j \cdot 2(2\pi) \cdot t}$$

$c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2j}$

$c_{-1} = \frac{1}{2}$, $c_{-2} = \frac{-1}{2j}$

$$c_1' = H(j2\pi) \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{4 + j2\pi}$$

$$c_{-1}' = H(j(-1)2\pi) \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{4 - j \cdot 2\pi}$$

$$c_2' = H(j \cdot 2 \cdot 2\pi) \cdot \frac{1}{2j} = \frac{1}{2j} \cdot \frac{1}{4 + j2 \cdot 2\pi}$$

$$c_{-2}' = H(j \cdot (-2) \cdot 2\pi) \cdot \frac{-1}{2j} = \frac{1}{-2j} \cdot \frac{1}{4 - j \cdot 2(2\pi)}$$

Ans.