Find the mean and variance for both binomial and poisson distribution by using moment generating function.

Sol^n: Binomial Distribution:

Mgf : $f(x) = {}^nC_x\, p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$

$$mgf = M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (e^t p)^x q^{n-x}$$

$$= \left[{}^nC_0 \cdot 1 \cdot q^n + {}^nC_1 \cdot e^t p \cdot q^{n-1} + \cdots + {}^nC_n (e^t p)^n \cdot 1 \right]$$

$$\boxed{\Rightarrow M_x(t) = (q + e^t p)^n}$$

Mean $= \mu_1' =$ 1st moment about origin.

$$= \left[\frac{d}{dt} M_x(t) \right]_{t=0}$$

$$= \left[n (q + e^t p)^{n-1} \cdot p e^t \right]_{t=0}$$

$$= np$$

$$\text{Variance} = \sigma^2 = E(x^2) - \{E(x)\}^2 \longrightarrow (1)$$

Now,

$$E(x^2) = \left[\frac{d^2}{dt^2} M_x t \right]_{t=0}$$

$$= \left[np \frac{d}{dt} \{ e^t (q + e^t p)^{n-1} \} \right]_{t=0}$$

$$= \left[np \left\{ e^t (q+e^t p)^{n-1} + (n-1) e^t (q+e^t p)^{n-2} \cdot p e^t \right\} \right]_{t=0}$$

$$= np\{1+p(n-1)\} \cancel{= np}$$

$$\Rightarrow E(X^2) = np\{1+(n-1)\cdot p\}$$

$\therefore$ from ① $$\sigma^2 = np\{1+(n-1)p\} - (np)^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$= npq$$

Poisson distribution:

pmf: $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \; ; \; x = 0, 1, 2, \dots \infty \;\; \& \;\; \lambda = np$$

$$\text{mgf} = M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{e^t\lambda}{1!} + \frac{e^{2t}\cdot\lambda^2}{2!} + \frac{e^{3t}\cdot\lambda^3}{3!} + \dots\right] = e^{-\lambda}\cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

Mean $= \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$

$$= e^{-\lambda}\cdot\lambda\cdot\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\cdots\right)$$

$$= e^{-\lambda}\cdot\lambda\cdot e^{\lambda}$$

$$= \lambda$$

Variance $= \sigma^2 = E(x^2) - \{E(x)\}^2 \longrightarrow (2)$

Now, $E(x^2) = \left[\frac{d^2}{dt^2} M_x t\right]_{t=0}$

$$= \left[\frac{d}{dt}\left\{\lambda e^{-\lambda}\left(e^t + \lambda e^{2t} + \frac{\lambda^2 e^{3t}}{2!} + \cdots\cdots\right)\right\}\right]_{t=0}$$

$$= \left[\lambda e^{-\lambda}\left(e^t + 2\lambda e^{2t} + \frac{3\lambda^2 e^{3t}}{2!} + \frac{4\lambda^3 e^{4t}}{3!} + \cdots\cdots\right)\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left(1 + 2\lambda + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \cdots\cdots\right)$$

$$= \lambda e^{-\lambda}\left\{\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\cdots\right) + \left(\lambda + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} + \cdots\cdots\right)\right\}$$

$$= \lambda e^{-\lambda}\left\{e^{\lambda} + \lambda\left(1+\lambda+\frac{\lambda^2}{2!}+\cdots\cdots\right)\right\}$$

$$= \lambda e^{-\lambda}\left(e^{\lambda} + \lambda e^{\lambda}\right)$$

$$= \lambda(1+\lambda)$$

$$= \lambda + \lambda^2$$

$\therefore$ From (2);

$$\sigma^2 = \lambda + \lambda^2 - \lambda^2$$

$$= \lambda.$$

B

Ex: Determine the probability of getting a sum 9 exactly twice in 3 throws with a pair of fair dice.

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Soln:

$n = 3$

$x = 2$

$p = \frac{4}{36} = \frac{1}{9}$

$q = 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore f(x=2) = {}^3C_2 \cdot \left(\frac{1}{9}\right)^2 \cdot \left(\frac{8}{9}\right)^{3-2}$

$= \cancel{3} \cdot \frac{1}{9^2} \cdot \frac{8}{\cancel{9}_3}$

$= \frac{8}{243}$.

Ex: Out of 800 family with 5 children each, how many would expect to have

(a) 3 boys

(b) 5 girls

(c) Either 2 or 3 boys.

Assume equal probability of the birth of boys and girls.

Soln: (i) let

X : Getting boy.

Considering one family,

$P(\text{Getting exactly 3 boys}) = P(X=3)$

Here, $n = 5$

$x = 3$

$p : \frac{1}{2}$

$q : \frac{1}{2}$

$\therefore P(X=3) = {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{5-3}$

$= \frac{5 \cdot \cancel{4} \cdot \cancel{3}}{\cancel{3} \cdot 2 \cdot 1} \cdot \frac{1}{2^3} \cdot \frac{1}{\cancel{2^2}} = \frac{5}{16}$

The no. of families expected to have exactly three boys

$= 800 \times \frac{5}{16}$

$= 250$

(ii) Let X : getting girl

Considering one family ;

P (getting exactly 5 girls) $= P(X=5)$

Here ; $n = 5$

$x = 5$

$p = \frac{1}{2}$

$q = \frac{1}{2}$

$\therefore P(X=5) = {}^5C_5 \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^{5-5}$

$= 1 \cdot \frac{1}{32} \cdot 1$

$= \frac{1}{32}$

$\therefore$ The no. of families expected to have exactly 5 girls

$= 800 \times \frac{1}{32}$

$= 25$.

(iii) Let X : getting boy

Considering one family

P (Getting 2 or 3 boys) $= P(X=2) + P(X=3)$

Here $n = 5$

$x = 2$

$p = \frac{1}{2}$

$q = \frac{1}{2}$

$\therefore P(X=2) = {}^5C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{5-2}$

$= \frac{5 \cdot \cancel{4}}{1 \cdot 2} \cdot \frac{1}{\cancel{2^2}} \cdot \frac{1}{2^3}$

$= \frac{5}{16}$

Also, $p(x=3) = {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{5-3}$

$= \frac{5 \cdot \cancel{4} \cdot \cancel{3}}{\cancel{3} \cdot 2 \cdot 1} \cdot \frac{1}{2^3} \cdot \frac{1}{\cancel{2^2}}$

$= \frac{5}{16}$

$\therefore$ P(Getting 2 or 3 boys) $= \frac{5}{16} + \frac{5}{16}$

$= \frac{5}{8}$

$\therefore$ The no. of families expected to have either 2 or 3 boys

$= 800 \times \frac{5}{8}$

$= 500$

Ex: A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the probability that a particular packet will violate the guarantee. [0.0016]

Soln: Let X be a random variable of a seed not getting germinated.

$p = 5\% = 0.05$

$n = 200$

Average no. of seeds in a ~~packet~~ not germinating, $\lambda = np$

$= 200 \times 0.05$

$= 10$

Let ~~At~~ now X be the no. of seeds in a packets not getting germinated

A packet will violate guarantee if it contains more than 20 non-germinating seeds.

$$P(X \geqslant 20) = 1 - P(X \leqslant 20)$$

$$= 1 - \{P(X = 0) + P(X = 1) + \cdots\cdots + P(X = 20)\}$$

$$= 1 - \sum_{x=0}^{20} P(X = x)$$

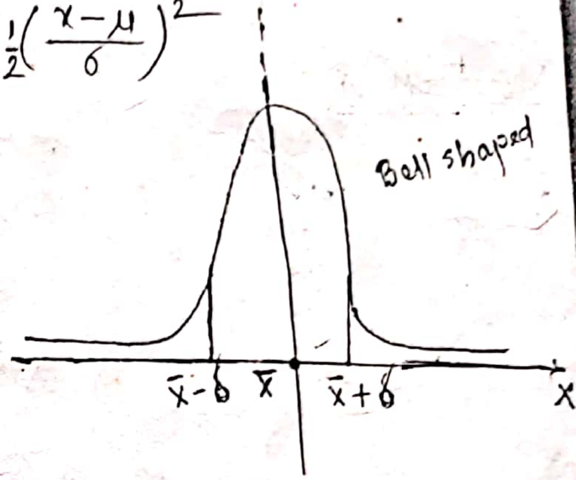
$$= 1 - \sum_{x=0}^{20} e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

$$= 1 - e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\cdots + \frac{\lambda^{20}}{20!}\right)$$

Normal Distribution :

Let X be a continuous random variable with probability density function (pdf) $f(x)$ as ;

$$f(x) = P(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



X along with $f(x)$ is called normal distribution.

Properties :

(i) It is a continuous type of distribution

(ii) It is bell shaped in nature

(iii) Symmetric about Y - axis.

(iv) The +ve and -ve X-axis behave like asymptotic line to the normal curve.

* (v) Mean = Median = Mode

(vi) The area under the curve is always 1 i.e, $\int_{-\infty}^{\infty} f(x)dx = 1$.

* (vii) $\bar{X}+\sigma$ and $\bar{X}-\sigma$ are the points of inflection for a normal curve.

Distribution function $F(x)$:

W.K.T. $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

Let $\frac{t-\mu}{\sigma} = u$

$\Rightarrow \frac{dt}{\sigma} = du$

$\Rightarrow dt = \sigma du$

$t : -\infty \to x$

$u : -\infty \to z$

$$= \frac{1}{\sigma\sqrt{2\pi}}\, \sigma \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}u^2}\,du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2}\,du$$

$$= \phi(z), \text{ where } \phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2}\,du$$

~~Normal~~

Properties:

(i) $F(x) = \phi(z)$

$$= \phi\left(\frac{x-\mu}{\sigma}\right)$$

(ii) $P(a < X \leq b) = F(b) - F(a)$

$$= \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

(iii) $\phi(0) = \frac{1}{2}$

(iv) $\phi(\infty) = 1$.

(v) $\phi(-\infty) = 0$.

* (vi) $\phi(-z) = 1 - \phi(z)$.

Proof: (vi) $\phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-\frac{1}{2}u^2}\,du$

Let $u = -v$

$\Rightarrow du = -dv$

$u : -\infty \to -z$

$v : \infty \to z$

$$= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{+z} e^{-\frac{1}{2}v^2}\,dv$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{+z}^{\infty} e^{-\frac{1}{2}v^2} dv$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv - \int_{-\infty}^{z} e^{-\frac{1}{2}v^2} dv \right]$$

$$= 1 - \phi(z).$$

$P \to F \to \phi \to$ Table

Normalized Random Variable : $Z = \frac{X-\mu}{\sigma}$ (Th^m & Pf for Mean and Var)

Ex: find $P(\bar{x} - \sigma < x \le \bar{x} + \sigma)$. ; $\phi(1) = 0.8413$

Soln:

$$P(\bar{x} - \sigma < x \le \bar{x} + \sigma)$$

$$= F(\bar{x}+\sigma) - F(\bar{x} - \sigma)$$

$$= \phi\left(\frac{\cancel{\bar{x}} + \sigma - \cancel{\bar{x}}}{\sigma}\right) - \phi\left(\frac{\cancel{\bar{x}} - \sigma - \cancel{\bar{x}}}{\sigma}\right)$$

$$= \phi(1) - \phi(-1)$$

$$= \phi(1) - \{1 - \phi(1)\}$$

$$= 2\phi(1) - 1,$$

$$= (2 \times 0.8413) - 1 \; ; \because \phi(1) = 0.8413$$

$$\simeq 0.6826 \simeq 68\%$$

$\phi(2.44) = 0.9927$

Ex: Determine the probabilities where X is normal :— Mean = 0, Var = 1

MIP

(i) $P(X \le 2.44)$

(ii) $P(X \le -1.16)$

(iii) $P(X \ge 1)$

(iv) $P(2 \le X \le 10)$

$\phi(2.44) = 0.9927$	$\phi(0.82) = 0.7939$
$\phi(1.16) = 0.877$	$\phi(0.1) = 0.5398$
$\phi(1) = 0.8413$	$\phi(0.6) = 0.7257$
$\phi(2) = 0.9772$	$\phi(0.98) = 0.8365$

Mean = 0, Variance = 1.

Task Ex: Determine the probabilities in the previous example by assuming X is normal with mean 0.8 and variance 4.

Solution:

2. (i) $P(X \leq 2.44) = F(2.44)$

$$= \phi\left(\frac{2.44 - 0}{1}\right)$$

$$= \phi(2.44)$$

$$= 0.9927$$

(ii) $P(X \leq -1.16) = F(-1.16)$

$$= \phi\left(\frac{-1.16 - 0}{1}\right)$$

$$= \phi(-1.16)$$

$$= 1 - \phi(1.16)$$

$$= 1 - 0.877$$

$$= 0.123.$$

(iii) $P(X \geq 1) = 1 - P(X \leq 1)$

$$= 1 - F(1)$$

$$= 1 - \phi\left(\frac{1-0}{1}\right)$$

$$= 1 - \phi(1)$$

$$= 1 - 0.8413$$

$$= 0.1587.$$

(iv) $P(2 \leq X \leq 10) = F(10) - F(2)$

$$= \phi\left(\frac{10-0}{1}\right) - \phi\left(\frac{2-0}{1}\right)$$

$$= \phi(10) - \phi(2)$$

$$= 1 - 0.9772 = 0.0228$$

5. (i) $P(X \leq 2.44) = F(2.44)$

$$= \Phi\left(\frac{2.44 - 0.8}{2}\right)$$

$$= \Phi(0.82) \qquad \because \sigma^2 = 4 \therefore \sigma = 2$$

$$= 0.7939.$$

(ii) $P(X \leq -1.16) = F(-1.16)$

$$= \Phi\left(\frac{-1.16 - 0.8}{2}\right)$$

$$= \Phi(-0.98)$$

$$= 1 - \Phi(0.98)$$

$$= 1 - 0.8365$$

$$= 0.1635$$

(iii) $P(X \geq 1) = 1 - P(X \leq 1)$

$$= 1 - F(1)$$

$$= 1 - \Phi\left(\frac{1 - 0.8}{2}\right)$$

$$= 1 - \Phi(0.1)$$

$$= 1 - .5398$$

$$= 0.4602$$

(iv) $P(2 \leq X \leq 10) = F(10) - F(2)$

$$= \Phi\left(\frac{10 - 0.8}{2}\right) - \Phi\left(\frac{2 - 0.8}{2}\right)$$

$$= \Phi(4.6) - \Phi(0.6)$$

Ex: ~~Let~~ X be normal with mean 0 and variance 1. Deter the constant c such that

(i) $P(X \le c) = 5\%$ (ii) $P(-c \le X \le c) = 99\%$

Given

$\phi^{-1}(0.05) = -1.645$

$\phi^{-1}(0.995) = 2.576$

Soln:

(i) $P(X \le c) = 5\%$

$\Rightarrow F(c) = 0.05$

$\Rightarrow \phi\left(\frac{c-\mu}{\sigma}\right) = 0.05$

$\Rightarrow \phi(c) = 0.05$

$\Rightarrow c = \phi^{-1}(0.05)$

$\Rightarrow c = -1.645$

(ii) $P(-c \le X \le c) = F(c) - F(-c)$

$= \phi\left(\frac{c-\mu}{\sigma}\right) - \phi\left(\frac{-c-\mu}{\sigma}\right)$

$= \phi(c) - \phi(-c)$

$= \phi(c) - \{1 - \phi(c)\}$

$= 2\phi(c) - 1$

Given $P(-c \le X \le c) = 99\%$

$\Rightarrow 2\phi(c) - 1 = 0.99$

$\Rightarrow 2\phi(c) = 1.99$

$\Rightarrow \phi(c) = .995$

$\Rightarrow c = \phi^{-1}(0.995)$

$= 2.576.$