2. CDF METHOD

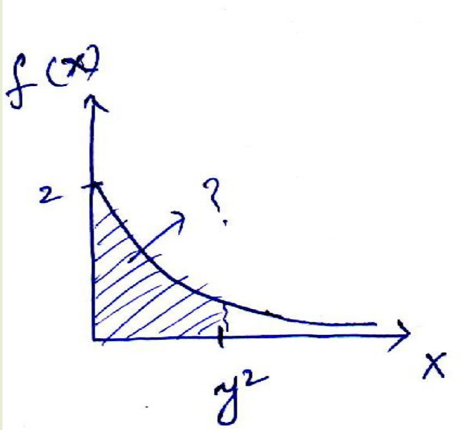
Finding pdf by using CDF method:

Qn-1:

Let X be a RV with pdf

$$f(x) = 2e^{-2x}, \quad x > 0.$$

Determine the pdf of $Y = \sqrt{X}$.



Soln:

$$F_Y(y) = P(Y \le y)$$

$$= P(\sqrt{x} \le y)$$

$$= P(x \le y^2)$$

$$= \int_{x=0}^{y^2} 2e^{-2x}\, dx$$

$$= 2 \cdot \left[\frac{e^{-2x}}{-2}\right]_0^{y^2}$$

$$= \left(e^{-2x}\right)_{y^2}^{0}$$

$$= 1 - e^{-2y^2}$$

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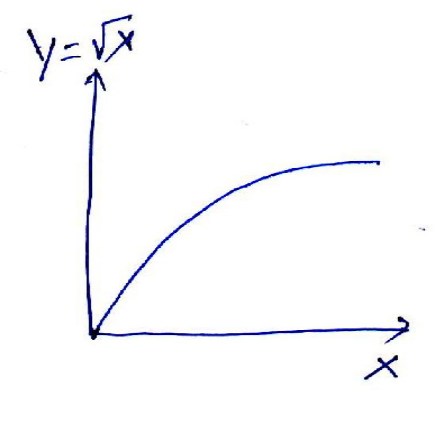


Now,

$$f_Y(y) = F'(y) = \frac{d}{dy}\left(1 - e^{-2y^2}\right)$$

$$= 2 \times 2\, e^{-2y^2}$$

$$= 4e^{-2y^2}; \quad y > 0$$

Qn-2:

Let X be a uniform RV on the interval $(0,1)$. Find the pdf of $\frac{1}{X}$.

Soln:

For Uniform Distribution,

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $Y = \frac{1}{X}$

$$F_Y(y) = P(Y \le y)$$
$$= P\left(\frac{1}{X} \le y\right) = P\left(X \ge \frac{1}{y}\right) = \int_{1/y}^{1} 1 \cdot dx = 1 - \frac{1}{y}$$

Function of Two random varia

Given two RVs X and Y. Now the function $g(X,Y)$ defined by

$$Z = g(X,Y)$$

defines another RV 'Z'.

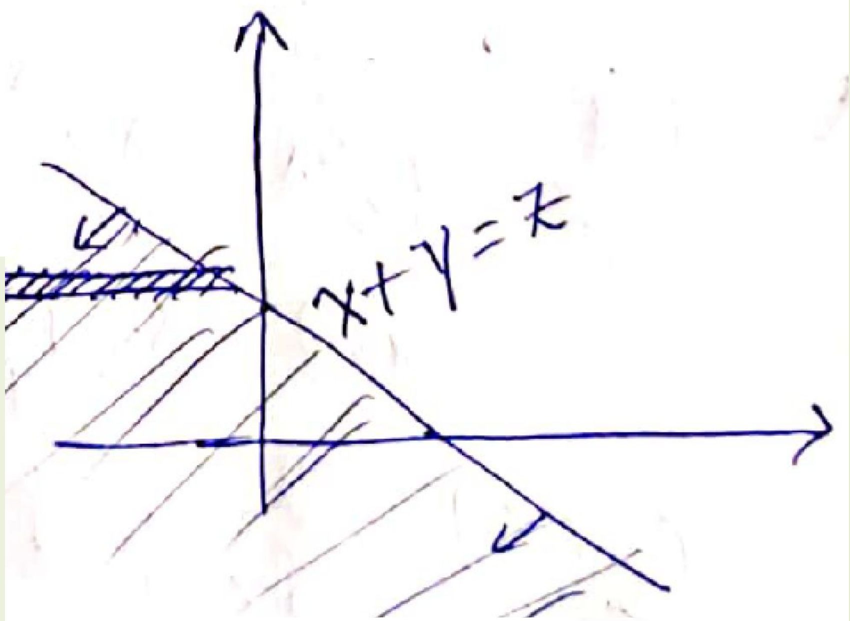
$$P(Z \leq z) = F_Z(z) = \iint_{D_z} f_{XY}(x,y)\, dx\, dy$$

where D_Z is the subset of R_{XY} {Range of (X,Y)} such that $g(x,y) \leq z$.

<u>Note</u>:- Similar argument holds for Discrete Case.

Ex.1:

Let $Z = X+Y$.
If $f_{XY}(x,y)$ be the joint pdf of X and Y, then find $f_Z(z)$.



Sol.:

$$F_Z(z) = P(Z \le z)$$
$$= P(X+Y \le z)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y).\, dx\, dy$$

Alt

$$\int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\, dy\, dx$$

$$\int^{\infty} f_{XY}(x, z-x).\, dx$$

Leibnitz Rule:

$$G(x) = \int_{a(x)}^{b(x)} h(x,y)\,dy$$

$$\frac{d}{dx} G(x) = \frac{\partial}{\partial x} b(x) \cdot h(x, b(x)) - \frac{\partial}{\partial x} a(x) \cdot h(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial h(x,y)}{\partial x} \cdot dy$$

$$\therefore f_Z(z) = \frac{\partial}{\partial z} F_Z(z) = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial z}\int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\right)dy$$

$$= \int_{-\infty}^{\infty}\left[1\cdot f_{XY}(z-y,\,y)\right]dy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x,y)\cdot dx\,dy$$

In particular, if X and Y are independent, then

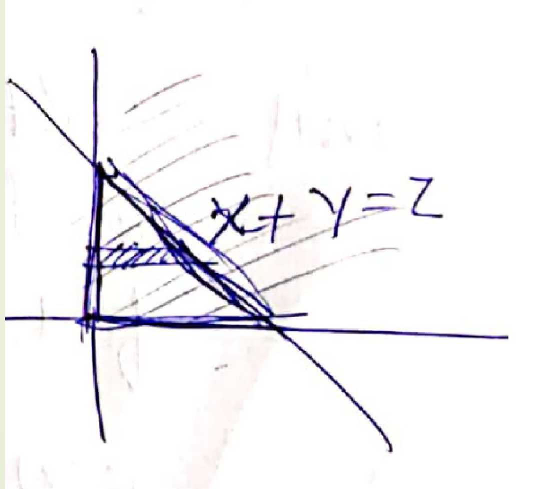
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\,f_Y(y)\,dy$$

OR

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-x)\,f_X(x)\cdot dx$$

Case:

$$X \geqslant 0,\ Y \geqslant 0$$



$$F_Z(z) = P(Z \leq z) = \int_0^z \int_0^{z-y} f_{XY}(x,y)\,.dx\,.dy.$$

$$\Rightarrow f_Z(z) = \int_0^z \left(\frac{\partial}{\partial z} \int_0^{z-y} f_{XY}(x,y)\,dx \right) dy$$

$$= \int_0^z \left(1 . f_{XY}(z-y, y) \right) . dy$$

Ex.2:

Let $Z = X + Y$.
If $f_{XY}(x,y)$ be the joint pdf of X and Y, then
find $f_Z(z)$.

where
,

$f_X(x) = \lambda e^{-\lambda x}$, $f_Y(y) = \lambda e^{-\lambda y}$, X, Y : Independent

Sol.:

Now, since X, Y are independent we can have

$$f_Z(z) = \int_0^z f_{XY}(z-y, y)\, dy$$

$$= \int_0^z f_X(z-y) \cdot f_Y(y)\, dy.$$

$$= \int_0^z \lambda e^{-\lambda(z-y)} \cdot \lambda e^{-\lambda y} \cdot dy$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dy \quad = \lambda^2 z e^{-\lambda z}; \text{ if } z \geq 0$$

Ex.3:

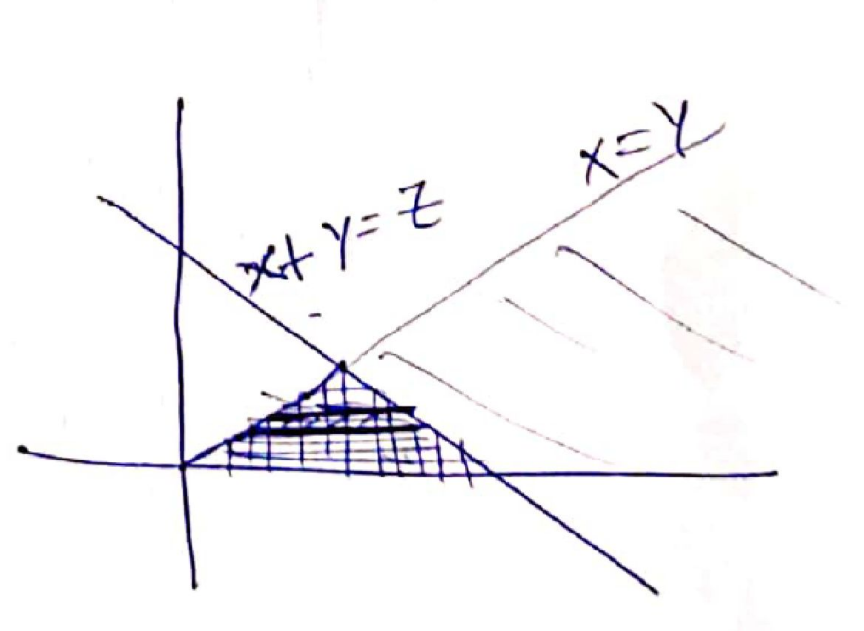
$$f_{XY} = \begin{cases} e^{-x}, & x > y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find $f_Z(z)$; where $z = X + Y$.

Sol.:

$$F_Z(z) = P(X + Y \le z)$$

$$= \int_0^{z/2} \int_y^{z-y} f_{XY}(x, y)\, dx \cdot dy$$



$$\Rightarrow f_Z(z) = \int_0^{z/2} 1 \cdot f_{XY}(z-y, y) \cdot dy$$

$$= \int_0^{z/2} 1 \cdot e^{-(z-y)} \cdot dy$$

$$= e^{-z} \int_0^{z/2} e^{y} \cdot dy = e^{-z} e^{y} \Big|_0^{z/2}$$

$$= e^{-z}\left(e^{z/2} - 1\right)$$

$$= e^{-z/2} - e^{-z} \quad ; \quad z \geq 0$$

Check the correctness:

$$\int_0^\infty \left(e^{-z/2} - e^{-z}\right) dz.$$

$$= \left[\frac{e^{-z/2}}{-1/2} - \frac{e^{-z}}{-1}\right]_0^\infty = 2 + (0-1) = 1$$

Ex.4: Let $Z = X/Y$.

If $f_{XY}(x,y)$ be the joint pdf of X and Y, then find $f_Z(z)$.

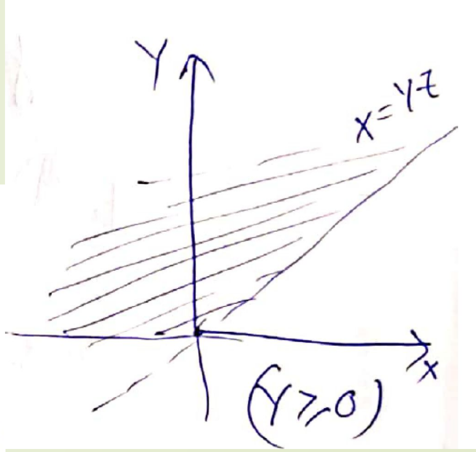
Sol.:

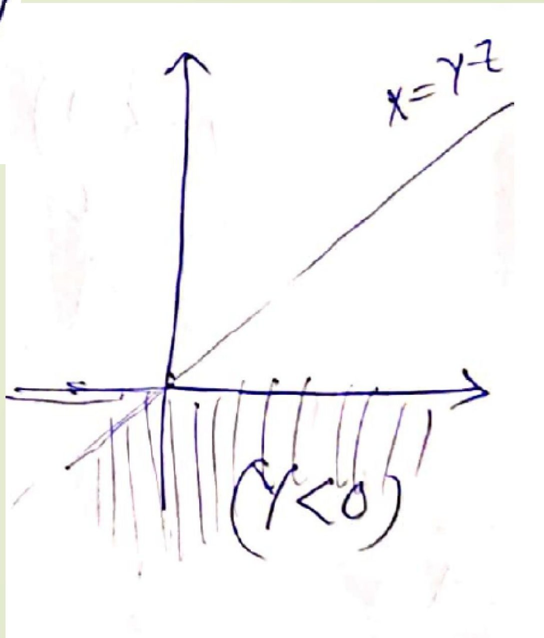
$$F_Z(z) = P(Z \le z)$$

$$= P\left(\frac{X}{Y} \le z\right)$$

$$= P\left(\frac{X}{Y} \le z,\ Y \ge 0\right) + P\left(\frac{X}{Y} \le z;\ Y < 0\right)$$

$$= \int_0^{\infty}\int_{-\infty}^{yz} f_{XY}(x,y)\,dx\,dy + \int_{-\infty}^{0}\int_{yz}^{\infty} f_{XY}(x,y)\,dx\,dy$$





$$\therefore f_Z(z) = \int_{0}^{\infty} y \cdot f_{XY}(yz, y)\, dy$$

$$+ \int_{-\infty}^{0} (-y)\, f_{XY}(yz, y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| \cdot f_{XY}(yz, y)\, dy$$

Ex.5: Let X and Y be independent uniform RVs over (0,1). Find and sketch the pdf of Z=X+Y.

Sol.:

Since X and Y are independent, we have

$$f_{XY}(x,y) = f_X(x)\,f_Y(y) = \begin{cases} 1 & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The range of Z is $(0, 2)$, and

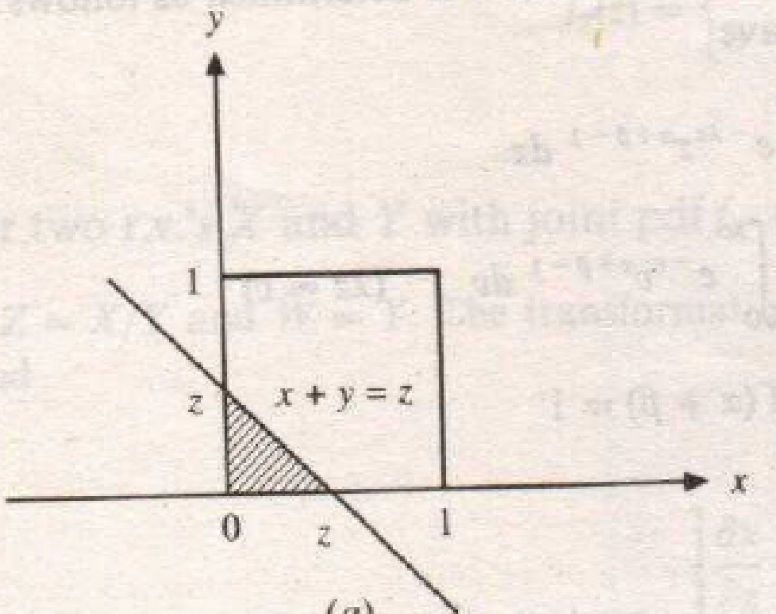
$$F_Z(z) = P(X + Y \le z) = \iint_{x+y\le z} f_{XY}(x,y)\,dx\,dy = \iint_{x+y\le z} dx\,dy$$

If $0 < z < 1$ [Fig.(a)],

$$F_Z(z) = \iint_{x+y\le z} dx\,dy = \text{shaded area} = \frac{z^2}{2}$$

and

$$f_Z(z) = \frac{d}{dz} F_Z(z) = z$$

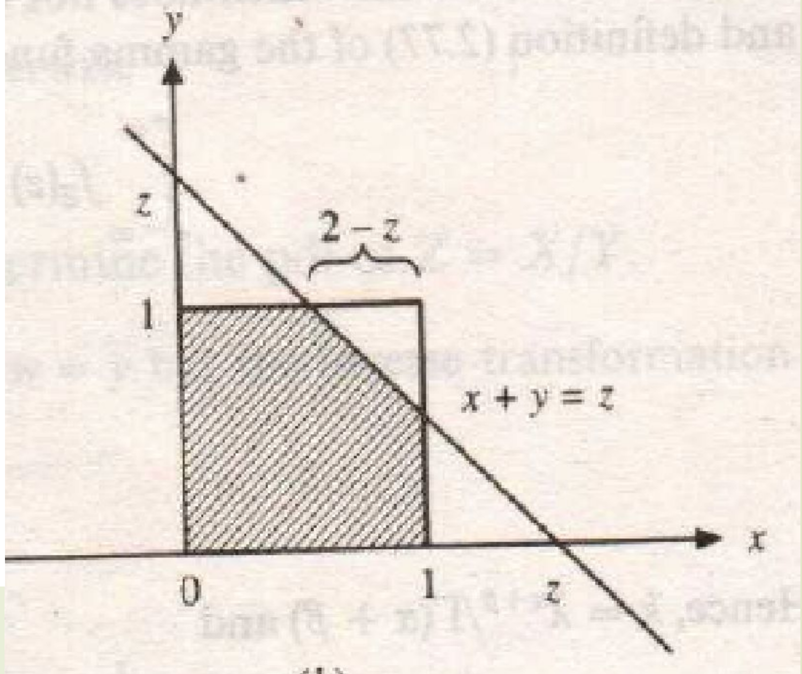


(a)

If $1 < z < 2$ [Fig.(b)],

$$F_Z(z) = \iint\limits_{x+y<z} dx\, dy = \text{shaded area} = 1 - \frac{(2-z)^2}{2}$$

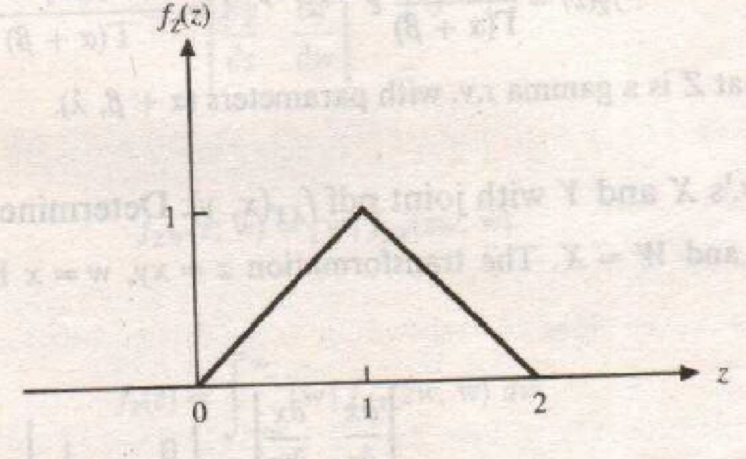
and
$$f_Z(z) = \frac{d}{dz} F_Z(z) = 2 - z$$



(b)

Hence,
$$f_Z(z) = \begin{cases} z & 0 < z < 1 \\ 2-z & 1 < z < 2 \\ 0 & \text{otherwise} \end{cases}$$

pdf sketch:



(c)

Expectation of a function of One RV

The expectation of $Y = g(X)$ is given by

$$E(Y) = E[g(X)] = \begin{cases} \sum_i g(x_i) p_X(x_i) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} g(x) f_X(x)\, dx & \text{(continuous case)} \end{cases}$$

Expectation of a function of more than one RV:

Let $X_1, \ldots, X_n$ be n r.v.'s, and let $Y = g(X_1, \ldots, X_n)$. Then

$$E(Y) = E[g(X)] = \begin{cases} \sum_{x_1} \cdots \sum_{x_n} g(x_1, \ldots, x_n) p_{X_1 \cdots X_n}(x_1, \ldots, x_n) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n & \text{(continuous case)} \end{cases}$$

Properties
:

i. The expectation operation is Linear.

   i. i.e. $$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

where a_i's are constants.

ii. If r.v.'s X and Y are independent, then we have

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

iii. In general to a mutually independent set of n r.v.'s $X_1, \ldots, X_n$:

$$E\left[\prod_{i=1}^{n} g_i(X_i)\right] = \prod_{i=1}^{n} E[g_i(X_i)]$$

Moments, Mean, Variance, MGF (2 RVs):

1. Moment:

The (k, n)th moment of a bivariate r.v. (X, Y) is defined by

$$m_{kn} = E(X^k Y^n) = \begin{cases} \sum_{y_j} \sum_{x_i} x_i^k y_j^n p_{XY}(x_i, y_j) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^n f_{XY}(x, y)\, dx\, dy & \text{(continuous case)} \end{cases}$$

If $n = 0$, we obtain the kth moment of X, and if $k = 0$, we obtain the nth moment of Y. Thus,

$$m_{10} = E(X) = \mu_X \quad \text{and} \quad m_{01} = E(Y) = \mu_Y$$

2. Mean and Variance:

MEAN

If (X, Y) is a discrete bivariate r.v., then

$$\mu_X = E(X) = \sum_{y_j} \sum_{x_i} x_i p_{XY}(x_i, y_j)$$

$$= \sum_{x_i} x_i \left[\sum_{y_j} p_{XY}(x_i, y_j) \right] = \sum_{x_i} x_i p_X(x_i)$$

$$\mu_Y = E(Y) = \sum_{x_i} \sum_{y_j} y_j p_{XY}(x_i, y_j)$$

$$= \sum_{y_j} y_j \left[\sum_{x_i} p_{XY}(x_i, y_j) \right] = \sum_{y_j} y_j p_Y(y_j)$$

Moreover,

$$E(X^2) = \sum_{y_j} \sum_{x_i} x_i^2 p_{XY}(x_i, y_j) = \sum_{x_i} x_i^2 p_X(x_i)$$

$$E(Y^2) = \sum_{y_j} \sum_{x_i} y_j^2 p_{XY}(x_i, y_j) = \sum_{y_j} y_j^2 p_Y(y_j)$$

MEAN

If (X, Y) is a discrete bivariate r.v., then

$$\mu_X = E(X) = \sum_{y_j} \sum_{x_i} x_i p_{XY}(x_i, y_j)$$

$$= \sum_{x_i} x_i \left[\sum_{y_j} p_{XY}(x_i, y_j) \right] = \sum_{x_i} x_i p_X(x_i)$$

$$\mu_Y = E(Y) = \sum_{x_i} \sum_{y_j} y_j p_{XY}(x_i, y_j)$$

$$= \sum_{y_j} y_j \left[\sum_{x_i} p_{XY}(x_i, y_j) \right] = \sum_{y_j} y_j p_Y(y_j)$$

Moreover,

$$E(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$E(Y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy$$

NOTE:

- Variance = E(X^2)- {E(X)}^2
- Covariance = E(XY)-E(X).E(Y)

Qn-1:

Let X and Y be defined by

$$X = \cos \varphi, \ Y = \sin \varphi$$

where φ is a random variable uniformly distributed over $(0, 2\pi)$.

(*a*) Show that X and Y are uncorrelated.

(*b*) Show that X and Y are not independent.

Sol.:

(a) We have

$$f_{\varphi}(\theta) = \begin{cases} \dfrac{1}{2\pi} & 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Then $E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{2\pi} \cos\theta\, f_{\varphi}(\theta)\, d\theta = \dfrac{1}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta = 0$

Similarly, $E(Y) = \dfrac{1}{2\pi}\int_0^{2\pi} \sin\theta\, d\theta = 0$

$$E(XY) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \sin\theta\, d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta\, d\theta = 0 = E(X)E(Y)$$

Thus, X and Y are uncorrelated.

(b)

$$E(X^2) = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{4\pi}\int_0^{2\pi} (1 + \cos 2\theta)\, d\theta = \frac{1}{2}$$

$$E(Y^2) = \frac{1}{2\pi}\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{4\pi}\int_0^{2\pi} (1 - \cos 2\theta)\, d\theta = \frac{1}{2}$$

$$E(X^2Y^2) = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta = \frac{1}{16\pi}\int_0^{2\pi} (1 - \cos 4\theta)\, d\theta = \frac{1}{8}$$

$$\therefore E(X^2Y^2) = \tfrac{1}{8} \neq \tfrac{1}{4} = E(X^2)E(Y^2)$$

If X and Y were independent, then we would have $E(X^2Y^2) = E(X^2)E(Y^2)$.

Hence X and Y are not independent.

(By Property-ii)