Olympiad Book

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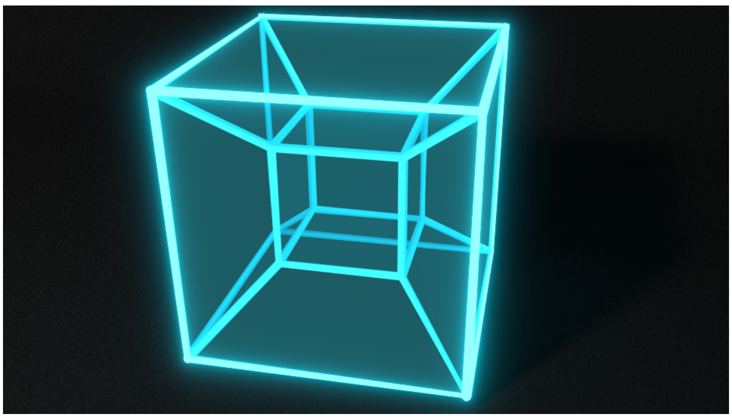

Contents

Chapter 1

Kinematics

1.1 Key Concepts and Formulae

1. Choose the most appropriate frame of reference. Depending on the problem, you may choose an inertial frame, center-of-mass frame, non-inertial frame in which most of the bodies are at rest, or any frame which simplifies the solution.

2. Before starting with the solution, choose the direction of co-ordinate axes and stick with the convention throughout the problem. A negative solution generally signifies that the direction of the quantity is opposite to the assumed direction.

3. Average velocity and acceleration over an interval are

$$\langle \mathbf{v} \rangle = \frac{\Delta \mathbf{r}}{t} \tag{1.1}$$

$$\langle \mathbf{a} \rangle = \frac{\Delta \mathbf{v}}{t} \tag{1.2}$$

   Bear in mind that these quantities are vectors and should be operated vectorically.

4. Instantaneous velocity and acceleration at a point are

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{dt} \tag{1.3}$$

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{dt} \tag{1.4}$$

$$a_x = v_x \frac{\mathrm{d}v}{\mathrm{d}x} \tag{1.5}$$

   Remember that a body can be accelerated not only by changing the magnitude of velocity but also by changing the direction of motion, which brings us to

5. It is often convenient to resolve acceleration vector in directions that are tangential and normal to a trajectory.

$$a_t = \frac{\mathrm{d}|\mathbf{v}|}{\mathrm{d}t} \tag{1.6}$$

$$a_n = \frac{v^2}{R} \tag{1.7}$$

   Here, a_t, the tangential acceleration, can be thought of as the component that measures the change in speed (magnitude of velocity) and a_n, the centripeta acceleration, as the component that measures the change in direction of motion

or the sharpness of turn.
And, R is the radius of curvature of the trajectory at the given point. For a trajectory $y = f(x)$, R at a point x is given by

$$R = \frac{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} \tag{1.8}$$

6. Distance covered by a body in an interval is given by

$$s = \int_{t_1}^{t_2} v_{(t)}\,\mathrm{d}t \tag{1.9}$$

7. Angular velocity and angular acceleration of a solid body are

$$\boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} \tag{1.10}$$

$$\boldsymbol{\alpha} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \tag{1.11}$$

The direction of θ is assumed to be perpendicular to the plane of rotation and its sense is taken according to the right hand screw rule.

8. Linear and rotational quantities are related in the following ways

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{1.12}$$

$$a_n = \omega^2 R \qquad \text{in the direction oppposite to } \mathbf{r} \tag{1.13}$$

$$a_t = \omega R \qquad \begin{array}{c}\text{along the direction of } \mathbf{v} \text{ if } \mathbf{v} \text{ is increasing}\\ \text{opposite the direction of } \mathbf{v} \text{ if } \mathbf{v} \text{ is decreasing}\end{array} \tag{1.14}$$

Here $\mathbf{r}$ is the radius vector of the considered point relative to an arbitrary point on the rotation axis, and R is the distance from the rotation axis.

Note: The constant-acceleration formulae are

$$v_x = v_{0x} + a_x t \tag{1.15}$$

$$x = x_0 + v_{0x} + \frac{1}{2}a_x t^2 \tag{1.16}$$

$${v_x}^2 = {v_{0x}}^2 + 2a_x(x - x_0) \tag{1.17}$$

$$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t \tag{1.18}$$

Although these expressions can easily be obtained from (3.4), (3.5) and (3.6), it is useful to remember them as they are used often. But, in using them, do not forget that they apply only when the acceleration is constant.

1.2 Bridging Problem

You fire a ball with an initial speed v_0 at and angle ϕ above the surface of an incline which is itself inclined at an angle θ above horizontal.

a) Find the distance, measured along the incline, from the launch point to the point where the ball strikes the incline.

b) What angle ϕ gives the maximum range measured along the incline?

SOLUTION:
First choose the co-ordinate axes. You could choose x-axis along the incline or along the horizontal. Using the incline is a bit easier, which is why we will choose x-axis along the line the incline the incline and y-axis perpendicular to it.
Our first task is to determine the acceleration along the x and y directions and check whether they are constant.
On resolution, we get

$$a_x = -g\sin\theta$$
$$a_y = -g\cos\theta$$

both of which are constant. hence, we may use the constant-acceleration formulae.
Before starting with the solution, it is often convenient to take stock of initial parameters.

$$x_0 = 0 \quad y_0 = 0 \quad v_{0x} = v_0\cos\phi \quad v_{0y} = v_0\sin\phi$$

Now using (3.16), we determine x and y as functions of time.

$$x_{(t)} = (v_0\cos\phi)t - \frac{g\sin\theta}{2}t^2 \tag{1.19}$$
$$\text{and} \quad y_{(t)} = (v_0\sin\phi)t - \frac{g\cos\theta}{2}t^2 \tag{1.20}$$

Next we determine the time τ it takes for the ball to hit the incline using $y_{(\tau)} = 0$.

$$\tau\left(\frac{g\cos\theta}{2}\tau - v_0\sin\phi\right) = 0$$

Since $\tau = 0$ signifies the launch, this solution is trivial.

$$\therefore \tau = \frac{2v_0\sin\phi}{g\cos\theta}$$

Now we obtain the x distance travelled by the ball in this time by plugging the value of τ in (1.19).

$$\text{Range} = R = x_{(\tau)} = v_0\cos\phi\left(\frac{2v_0\sin\phi}{g\cos\theta}\right) - \frac{g\sin\theta}{2}\left(\frac{2v_0\sin\phi}{g\cos\theta}\right)^2$$
$$\text{On simplification,} \quad R = \frac{2{v_0}^2\sin\phi\cos(\theta+\phi)}{g\cos^2\theta} \tag{1.21}$$

For part (b) of the problem, we use calculus. In order for R to be maximum, derivative of R with respect to ϕ must be zero.

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\big(\sin\phi\cos(\theta+\phi)\big) = 0$$
$$\cos(\theta+2\phi) = 0$$
$$\therefore \phi_c = 45^\circ - \frac{\theta}{2} \tag{1.22}$$

EVALUATION:

1. Check the dimensions of expression for R (1.21).

2. Check the answers in the limiting case of $\theta = 0$ and observe if they make sense.

$$R_{(\theta=0)} = \frac{{v_0}^2 \sin 2\phi}{g}$$
$$\phi_{c(\theta=0)} = 45^\circ$$

Have you seen these results elsewhere?

1.3 Level 1 Problems and Solutions

1. The velocity of a particle moving in the positive direction of the x-axis varies as $v = \alpha\sqrt{x}$ where α is a positive constant. Assuming that the particle was located at $x = 0$ at the moment $t = 0$, find

   (a) time dependence of velocity and acceleration of the particle.

   (b) mean velocity of particle averaged over the time that the particle takes to cover the first s meters of the path. *Irodov 1.22*

   SOLUTION:
   First find x as a function of time using the given expression and (3.4).

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha\sqrt{x}$$
$$\frac{\mathrm{d}x}{\sqrt{x}} = \alpha\,\mathrm{d}t$$

   Since $x = 0$ at $t = 0$, use these as limits of integration

$$\int_0^x x^{-1/2}\,\mathrm{d}x = \int_0^t \alpha\,\mathrm{d}t$$
$$x_{(t)} = \frac{\alpha^2 t^2}{4}$$
$$\therefore v_{(t)} = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\alpha^2 t}{2}$$
$$\text{and}\quad a_{(t)} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\alpha^2}{2}$$

   The following fact will be required to solve part (b)

   For a function $y = f(x)$ continuous over the interval $[a, b]$, the average value of y over $[a, b]$ is given by

$$y_{av[a,b]} = \frac{\int_a^b y_{(x)}\,\mathrm{d}x}{b-a} \tag{1.23}$$

   Using (1.23), average velocity over a time interval is given by

$$\langle v\rangle = \frac{\int_{t_1}^{t_2} v_{(t)}\,\mathrm{d}t}{t_2 - t_1}$$

For this particular problem

$$t_1 = 0 \quad \text{and} \quad t_2 = \frac{2\sqrt{s}}{\alpha} \qquad \left[\because x_{(t)} = \frac{\alpha^2 t^2}{4}\right]$$

$$\langle v \rangle = \frac{\int_0^{t_2} \frac{\alpha^2 t}{2}\,\mathrm{d}t}{t_2} = \frac{\alpha^2 t_2}{4}$$

Replace the value of t_2 to get

$$\langle v \rangle = \frac{\alpha\sqrt{s}}{2}$$

Check dimensions of all answers.

2. A point moves along a circle with a velocity $v = at$, where $a = 0.50\ \mathrm{m\,s^{-1}}$. Find the total acceleration of the point at the moment when it covers the n^{th} $(n = 0.10)$ fraction of the circle after the beginning of motion. *Irodov 1.37*

   SOLUTION:
   Assume circle of radius R and location of point is $\theta = 0$ at $t = 0$.

$$\omega_{(t)} = \frac{v_{(t)}}{R} = \frac{at}{R}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{at}{R}$$

   When the circle covers n^{th} fraction of the circle, $\theta = 2\pi n$

$$\int_0^{2\pi n} R\,\mathrm{d}\theta = \int_0^{\tau} at\,\mathrm{d}t$$
$$\therefore \tau = \sqrt{\frac{4\pi n R}{a}}$$

   For a particle in circular motion, it is easier to think of its acceleration in terms of its tangential and normal components.Using (1.6) and (1.7), we get

$$a_t = \frac{\mathrm{d}}{\mathrm{d}t}(at) = a$$
$$a_n = \omega^2 R = \frac{a^2 t^2}{R}$$

   For $t = \tau$,

$$a_n = 4\pi a n$$

   So, the total acceleration at $t = \tau$ is

$$a_{tot} = \sqrt{{a_t}^2 + {a_n}^2}$$
$$a_{tot} = a\sqrt{1 + (4\pi n)^2}$$

   put $a = 0.50\ \mathrm{m\,s^{-2}}$ and $n = 0.10$,

$$a_{tot} = 0.81 \quad \mathrm{m\,s^{-2}}$$

   Note the use of significant figures.

3. An airplane pilot wishes to fly due west. A wind of 80.0 km/h is blowing toward the south.

(a) If the airspeed of the plane in still air is 320.0 km/h, in which direction should the pilot head?

(b) What is the speed of plane over the ground?

Draw a vector diagram. *UP 3.38*

SOLUTION:
$\mathbf{v}_{W/G}$ = velocity of wind w.r.t. ground
$\mathbf{v}_{P/G}$ = velocity of plane w.r.t. ground
$\mathbf{v}_{P/W}$ = velocity of plane w.r.t. wind (or velocity in still air)
The vector diagram of the situation described above is shown below

$$\mathbf{v}_{P/W} = \mathbf{v}_{P/G} + (-\mathbf{v}_{W/G})$$

So the direction of $\mathbf{v}_{P/W}$ or the direction pilot should head for is
Using Pythagorean theorem in the right triangle

$$\begin{aligned} {v_{P/W}}^2 &= {v_{P/G}}^2 + {v_{W/G}}^2 \\ v_{P/G} &= \sqrt{{v_{P/W}}^2 - {v_{W/G}}^2} \\ \text{And} \quad \sin\theta &= \frac{v_{W/G}}{v_{P/W}} \\ \theta &= \sin^{-1}\left(\frac{v_{W/G}}{v_{P/W}}\right) \end{aligned}$$

Use $v_{W/G}$=80.0 km/h and $v_{P/W}$=320.0 km/h,

$$v_{P/G} = 309.8\text{km/h} \quad \text{and} \quad \theta = 14.5°$$

The pilot should head 14.5°north of west and the speed of plane over the ground is 309.8 km/h.

4. A particle moves along a parabola $y = ax^2$ with velocity $\mathbf{v}$ whose modulus is constant. Find the acceleration of the particle at the point $x = 0$. *NePhO 2018*

SOLUTION:
Think of the particle's acceleration in terms of its tangential and normal components.

$$\begin{aligned} a_t &= \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}| \\ \text{But } |\mathbf{v}| \text{ is constant, so} \quad a_t &= 0 \\ a_n &= \frac{v^2}{R} \end{aligned}$$

Here, R is radius of curvature of the parabola $y = ax^2$ at $x = 0$. For R, use (3.9).

$$\begin{aligned} R &= \frac{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} \\ &= \frac{(1+2ax)^{\frac{3}{2}}}{2a} \\ &= \frac{1}{2a} \\ \therefore a_n &= 2av^2 \\ \text{Finally,} \quad a_{tot} &= a_n = 2av^2 \end{aligned}$$

Check dimensions of the final answer.

1.4 Level 2 Problems and Solutions

1. A ball is thrown at speed v from zero height on level ground. Let θ_0 be the angle at which the ball should be thrown so that the length of the trajectory is maximum. Show that θ_0 satisfies

$$\sin\theta_0 \ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right) = 1 \qquad \textit{Morin 3.19}$$

SOLUTION:
First let's obtain the equation of trajectory of the ball.

$$x_{(t)} = (v\cos\theta)t$$
$$y_{(t)} = (v\sin\theta)t - \frac{gt^2}{2}$$

Eliminating t from these equations, we get

$$y_{(x)} = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2$$

For length of the trajectory,

$$L = \int_0^R \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{1.24}$$

Here $R = \frac{v^2\sin 2\theta}{g}$ is the range of the ball. (1.24) is easily understood if you consider infinitesimal length $\mathrm{d}L = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$.

$$L = \int_0^{\frac{v^2\sin 2\theta}{g}} \sqrt{1+\left(\tan\theta - \frac{g}{v^2\cos^2\theta}x\right)^2}\,\mathrm{d}x$$

Substitute $\tan\theta - \frac{g}{v^2\cos^2\theta}x = u$,

$$= \frac{v^2\cos^2\theta}{g}\int_{-\tan\theta}^{\tan\theta}\sqrt{1+u^2}\,\mathrm{d}u$$

Since the integrand is an even function,

$$= \frac{2v^2\cos^2\theta}{g}\int_0^{\tan\theta}\sqrt{1+u^2}\,\mathrm{d}u$$
$$= \frac{2v^2\cos^2\theta}{g}\cdot\frac{1}{2}\left(u\sqrt{1+u^2} + \ln(u+\sqrt{1+u^2})\right)\Big|_0^{\tan\theta}$$
$$\therefore L = \frac{v^2}{g}\left(\sin\theta + \cos^2\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\right) \tag{1.25}$$

In order for L to be maximum, the derivative of L with respect to θ should be zero.

$$\cos\theta - 2\cos\theta\sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + \cos^2\theta\left(\frac{\cos\theta}{1+\sin\theta}\right)\frac{\cos^2\theta + (1+\sin\theta)\sin\theta}{\cos^2\theta} = 0$$

On simplification, this reduces to

$$\sin\theta \ln\left(\frac{1+\sin\theta}{\cos\theta}\right) = 1$$

On solving this equation, we get θ_0 for which length of trajectory is maximum.

Note: (a) Check the expression for L (1.25) in the limiting cases of $\theta = 0°$ and $\theta = 90°$ and observe if the results make sense.

(b) In some IPhO problems, you will be asked to solve equations like these numerically. We strongly recommend you learn at least one numerical method and as an exercise, we will leave it to you to verify that the solution for θ is $\theta_0 \approx 56.5°$.

2. A solid body rotates with a constant angular velocity $\omega_0 = 0.5$ rad/s about a horizontal axis AB. At the moment $t = 0$, the axis AB starts turning about the vertical with a constant angular acceleration $\beta_0 = 0.10$ rad/s^2. Find the angular velocity and angular acceleration of the body after $t = 3.5$ s. *Irodov 1.58*

SOLUTION:
Consider unit vectors **i** in the direction of axis AB, **k** perpendicular to AB in the plane of rotation and **j** perpendicular to the plane of rotation of AB as shown.

$$\boldsymbol{\omega_P} = \omega_0\mathbf{i} + \beta_0 t\mathbf{j}$$
$$|\boldsymbol{\omega_P}| = \sqrt{\omega_0{}^2 + \beta_0 t^2}$$

For $\alpha_{\mathbf{P}}$,

$$\boldsymbol{\alpha_P} = \frac{\mathrm{d}\boldsymbol{\omega_P}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\omega_0\mathbf{i} + \beta_0 t\mathbf{j})$$

Vector differentiation will be used to obtain the solution. Note that vector differentiation is equivalent to using product rule of differentiation on a vector assuming it to be a product of a scalar and a vector

$$\boldsymbol{\alpha_P} = \omega_0\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} + \mathbf{i}\frac{\mathrm{d}\omega_0}{\mathrm{d}t} + \beta_0 t\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}(\beta_0 t)}{\mathrm{d}t}$$
$$\text{Here,} \quad \frac{\mathrm{d}\omega_0}{\mathrm{d}t} = 0 \quad \frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} = 0 \quad \frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = \beta_0 t\mathbf{k}$$
$$\therefore \boldsymbol{\alpha_P} = \omega_0\beta_0 t\mathbf{k} + \beta_0\mathbf{j}$$
$$|\boldsymbol{\alpha_P}| = \beta_0\sqrt{1 + \omega_0{}^2 t^2}$$

Plug $\omega_0 = 0.50$ rad/s, $\beta_0 = 0.10$ rad/s^2 and $t = 3.50$ s.

$$\omega_P = 0.61\text{rad/s} \quad \text{and} \quad \alpha_P = 0.20\text{rad/s}^2$$

Note: The direction of $\frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t}$ is $\boldsymbol{k}$ for the same reason that direction of centripetal acceleration is perpendicular to velocity in uniform circular motion. Consider the figure where $\boldsymbol{i_1}$ is the initial direction of axis AB and $\boldsymbol{i_2}$ is the direction after time Δt. The direction of $\Delta\boldsymbol{i}$ is as shown. In the limit of $\Delta t \to 0$, $\boldsymbol{i_2} \to \boldsymbol{i_1}$ and hence the direction of $\mathrm{d}\boldsymbol{i}$ is perpendicular to $\boldsymbol{i}$ and along $\boldsymbol{k}$.

1.5 IPhO Problems

A ball, thrown with an initial speed v_0, moves in a homogeneous gravitational field in the x-z plane, where the x-axis is horizontal, and the z-axis is vertical and antiparallel to the free fall acceleration g . Neglect the effect of air drag.

i) By adjusting the launching angle for a ball thrown with a fixed initial speed v_0 from the origin, targets can be hit within the region given by

$$z \leq z_0 - kx^2 \tag{1.26}$$

You can use this fact without proving it. Find the constants z_0 and k.

ii) The launching point can now be freely selected on the ground level $z = 0$, and the launching angle can be adjusted as needed. The aim is to hit the topmost point of a spherical building of radius R (see fig.) with the minimal initial speed v_0. Bouncing off the roof prior to hitting the target is not allowed. Sketch qualitatively the shape of the optimal trajectory of the ball.

iii) What is the minimal launching speed v_{min} needed to hit the topmost point of a spherical building of radius R? *IPhO 2012 Estonia*

SOLUTION:

i The maximum heght z_0 is achieved when the ball is thrown vertically upwards.

$$z_0 = \frac{{v_0}^2}{2g} \tag{1.27}$$

The maximum horizontal distance is achieved when the ball is thrown at an angle of 45°and the distance covered in that case is $\frac{{v_0}^2}{g}$. But using (1.26), the maximum horizontal distance achievable is $\sqrt{\frac{z_0}{k}}$.

$$\sqrt{\frac{z_0}{k}} = \frac{{v_0}^2}{g}$$
$$k = \frac{g}{2{v_0}^2} \tag{1.28}$$

ii In kinematics, it is sometimes easier to analyze a process by reversing it. In this case, we can ask ourselves: What is the trajectory of the ball thrown from the roof of the building with minimum initial speed such that it does not hit the building on its way down?
With purely qualitative reasoning, we arrive at the conclusion that the ball must touch the building on its way down. For example, say a ball is thrown at an angle from the roof and its trajectory does not touch the building. Then we can argue that the ball can be thrown at the same angle by lowering its speed continually until the trajectory just touches the roof at some point. The speed can't be lowered any further as the ball would then hit the roof.
Now asume the ball is thrown horizontally from the roof. We can increase the throwing angle slightly upward while keeping the same speed. This trajectory would not touch the building at any point. Now we can use the same argument to conclude that the speed can still be lowered until the trajectory touches the building.
So the optimal trajectory must touch the building at some point as shown.

iii Before getting into the solution, it is essential to distinguish between targetable region and trajectory. Trajectory is the actual path the ball travels through when thrown from a point with a particular speed and at a particular angle. Targetable region, on the other hand, includes all possible trajectories the ball can trace when thrown from a point with a particular speed at different angles. We established in the solution to ii) that the optimal trajectory must touch the building. By similar analysis, we can also show that boundary of targetable region for minimum speed must also touch the building. If it were to not touch the building, it would be possible to throw the ball along any one trajectory that does not touch the building. Again it would be possible to further lower the speed hence we can conclude that the initial speed could not have been the minimum speed. So, the boundary of targetable region for minimum speed must touch the building. The equation of this boundary when a ball is thrown from origin with a speed v_0 is given below, using (1.26), (1.27) and (1.28):

$$z = \frac{{v_0}^2}{2g} - \frac{gx^2}{2{v_0}^2}$$

Now take origin at the top of spherical building, the equation of the spherical building in x-z plane is

$$x^2 + (z+R)^2 = R^2$$
$$x^2 + z^2 + 2zR = 0$$

Eliminating z from these equations, we get the following equation, biquadratic in x.

$$\frac{g^2}{4{v_0}^4}x^4 + \left(\frac{1}{2} - \frac{gR}{{v_0}^2}\right)x^2 + \left(\frac{{v_0}^4}{4g^2} + \frac{{v_0}^2 R}{g}\right) = 0 \tag{1.29}$$

Let's pause for a moment to understand the implications of (1.29). If the value of v_0 is less than the minimum speed, it will have four solutions because the boundary of targetable region and circle will intersect at four points. (Do not forget that the ball can be thrown in either direction from the top.) If the value of v_0 is too large, they will not intersect at all and (1.29) will have no real solutions. If v_0 is the minimum speed, then the boundary of targetable region touches the circle at two points and hence two real solutions are obtained.
Recall from your algebra classes that this transition between distinct real roots and imaginary roots takes place when the discriminant equals zero.

$$\left(\frac{1}{2} - \frac{gR}{{v_0}^2}\right)^2 - 4.\frac{g}{4{v_0}^4}.\left(\frac{{v_0}^4}{4g^2} + \frac{{v_0}^2 R}{g}\right) = 0$$

On solving for v_0, we get

$$v_0 = \sqrt{\frac{gR}{2}}$$

Now would be a great time to blunder by stopping here and recording this as your final answer. Remember that we have been considering the reverse process of throwing the ball off the roof while the problem is concerned with throwing the ball off the ground. So you have to find the speed of the ball when it reaches the ground and that is the required v_{min}.

$$v_{min} = \sqrt{{v_0}^2 + 2g(2R)}$$
$$\therefore v_{min} = 3\sqrt{\frac{gR}{2}} \tag{1.30}$$

Note: (a) The reverse process can be considered only if the principle of conservation of energy is valid. From an energy standpoint, it is the same thing to throw the ball from the roof as it is from the ground.

(b) At this point, you might be wondering why we used the formula of discriminant of quadratic equation in a biquadratic equation. Note that biquadratic equation is, in essence, a quadratic equation if you replace x^2 by y.

1.6 Supplementary Problems

1. Three points are located at the vertices of an equilateral triangle whose side equals a. They all start moving simultaneously with velocuty v constant in modulus, with the first heading continually for the second, the second for the third, and the third for the first. How soon will the points converge? *Irodov 1.12*

2. A balloon starts rising from the surface of the Earth. the ascension rate is constant and equal to v_0. Due to the wind the balloon gathers the horizontal velocity component $v_x = ay$, where a ia a constant and y is the height of ascent. Find how the following quantities depend on the height of ascent:

   (a) the horizontal drift of the balloon $x(y)$;

   (b) the total, tangential, and normal accelerations of the balloon. *Irodov 1.34*

3. A cylinder rolls without slipping over a horizontal plane The radius of the cylinder is equal to r. Find the curvature radii of trajectories traced out by points A and B (see fig.). *Irodov 1.54*

4. ...

Chapter 2

Newton's Laws

2.1 Key Concepts and Formulae

1. For problems involving dynamics of the system, check if the situation is an analogy of simple harmonic motion(SHM). If the situation can be modeled to SHM, we can easily find some unknown parameters required to solve the problem. For example, the system under a force proportional to x(displacement) can be compared with SHM.

2. When a body is supposed to leave a surface, ensure that the net normal force on the body is zero.

3. momentum equation vs. energy equation for scattering problems

4. In a perfectly elastic collision, the colliding particle scatter by double the initial angle, unless the collision is head-on. figure..

2.2 Bridging Problem

When viewed from the side, the cone in fig. subtends an angle 2θ at its tip. A block of mass m is connected to the tip by a massless string and moves in a horizontal circle of radius R around the surface. If the initial speed is v_0 and if the coefficient of kinetic friction between the block and the cone is μ, how much time does it take the block to stop? *Morin 3.70*

SOLUTION:
First draw a free body diagram of the block indicating all forces, their directions and choice of axes. Assume the block is moving into the plane of figure. Frictional force always opposes motion and hence points out of the plane of the figure. ($\odot$ signifies out of the plane of figure and $\otimes$ into the plane.)
Now we need to find the value of f_r and for that we need the value of N since $f_r = \mu N$. Before using Newton's Laws to find N, assess the values of a_x and a_y.

$$a_y = 0 \qquad \because \text{no vertical motion}$$
$$a_x = \frac{v^2}{R} \qquad \because \text{circular motion}$$

Using Newton's Laws,

$$\sum F_y = ma_y$$

$$T\cos\theta + N\sin\theta - mg = 0 \tag{2.1}$$

$$\sum F_x = ma_x$$

$$T\sin\theta - N\cos\theta = \frac{mv^2}{R} \tag{2.2}$$

Eliminating T from (2.1) and (2.2) and solving for N, we get,

$$N = mg\sin\theta - \frac{mv^2\cos\theta}{R} \tag{2.3}$$

As discussed earlier, friction opposes the motion and causes the block to slow down over time. Mathematically,

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\mu N$$

Using (2.3), we get

$$\frac{\mathrm{d}v}{gR\tan\theta - v^2} = -\frac{\mu\cos\theta}{R}\,\mathrm{d}t$$

If the total time taken by the block to stop is τ. use the limits

$$v = v_0 \quad \text{at} \quad t = 0$$
$$v = 0 \quad \text{at} \quad t = \tau$$

$$\int_{v_0}^{0} \frac{\mathrm{d}v}{gR\tan\theta - v^2} = -\int_0^{\tau} \frac{\mu\cos\theta}{R}\,\mathrm{d}t$$

$$-\frac{\mu\cos\theta}{R}\tau = \frac{1}{2\sqrt{gR\tan\theta}}\left|\ln\frac{\sqrt{gR\tan\theta}+v}{\sqrt{gR\tan\theta}-v}\right|_{v_0}^{0}$$

$$\therefore \tau = \frac{1}{2\mu}\sqrt{\frac{R}{g\sin\theta\cos\theta}}\ln\left(\frac{\sqrt{gR\tan\theta}+v_0}{\sqrt{gR\tan\theta}-v_0}\right)$$

Evaluation:

1. Check dimensions of final answer.
2. For an answer as messy as this, you must check limits. First thing you will notice is that $t = \infty$ for $v_0 = \sqrt{gR\tan\theta}$. This is to be expected because this is the speed of a body moving in a horizontal circle without contact with any surface. So, if $v_0 = \sqrt{gR\tan\theta}$, the normal force $N = 0$ and hence $f_r = 0$. So, the block will swing indefinitely.
3. For the limit $\theta \to \frac{\pi}{2}$, $\frac{v_0}{\sqrt{gR\tan\theta}} \ll 1$. So we can use the Taylor series approximation of $\ln(1+x)$ for $x \ll 1$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \tag{2.4}$$

For $x \ll 1$, $ln(1+x) \approx x$.

$$\therefore \tau_{(\theta\to\frac{\pi}{2})} = \frac{1}{2\mu}\sqrt{\frac{R}{g\sin\theta\cos\theta}}\left(\frac{v_0}{\sqrt{gR\tan\theta}} - \left(-\frac{v_0}{\sqrt{gR\tan\theta}}\right)\right)$$
$$= \frac{v_0}{\mu g}$$

This is a sensible result as deceleration due to friction on level ground is μg.

2.3 Level 1 Problems and Solutions

1. A rope rests on two platforms that are both inclined at an angle θ(which you are free to pick) as shown in the fig. The rope has uniform mass density and the coefficient of friction between it and the platforms is μ. The system has left-right symmetry. What angle θ allows the maximum fraction of rope that does not touch the platform? *Morin 2.25*

   SOLUTION:
   We will use the left-right symmetry of the system to simplify the problem. Consider only the left half of the system. Do not forget to add internal forces in the FBD at the points where you 'break off' a part of the system. Let the length of the left half of the rope is l and the fraction of this length that touches the platform is α. The linear mass density of the rope is λ.

   (a) FBD of the fraction that touches the platform ***Add figure later***

   (b) FBD of the left half of the system ***Add figure later***

   Use $\sum F_y = 0$ in FBD(a)

$$N - \alpha l \lambda g \cos\theta = 0$$
$$\therefore N = \alpha l \lambda g \cos\theta \tag{2.5}$$

   Use $\sum F_y = 0$ in FBD(b)

$$f_r \sin\theta + N\cos\theta - \lambda l g = 0 \tag{2.6}$$

   Apply (2.5) and $f_r = \mu N$ on (2.6) to get α as a function of θ

$$\alpha_{(\theta)} = \frac{1}{\mu \sin\theta\cos\theta + \cos^2\theta}$$

   For the minimum value of α, the denominator has to be maximized. So, set the derivative of denominator of α equal to zero for minimum α.

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\mu\sin\theta\cos\theta + \cos^2\theta) = 0$$
$$\mu\cos 2\theta - \sin 2\theta = 0$$
$$\therefore \theta_c = \frac{1}{2}\tan^{-1}\mu$$

   Note the clever choice of co-ordinate axes in order to avoid internal forces.

2. A small bar starts sliding down an inclined plane forming an angle α with the horizontal. The frictional coefficient depends on the distance x covered as $k = ax$, where a is a constant. Find the distance covered by the bar till it stops, and its maximum velocity over this distance. *Irodov 1.102*

   SOLUTION:
   Consider positive x-axis in the direction of descent of the bar. Draw the FBD of the bar.

   (a)

$$\sum F_y = 0$$
$$N = mg\cos\alpha$$

(b)

$$\sum F_x = mg\sin\alpha - kN$$

Let the bar has covered x distance since its release

$$F_{(x)} = mg(\sin\alpha - ax\cos\alpha)$$

Use $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$ to get

$$v\,\mathrm{d}v = g(\sin\alpha - ax\cos\alpha)\,\mathrm{d}x$$
$$\int_0^{v_{(x)}} v\,\mathrm{d}v = \int_0^x g(\sin\alpha - ax\cos\alpha)\,\mathrm{d}x$$
$$\therefore v_{(x)} = \sqrt{g(2x\sin\alpha - ax^2\cos\alpha)}$$

Plug $v_{(x)} = 0$ to get

$$x = 0 \qquad \& \qquad \frac{2}{a}\tan\alpha$$

The first solution is trivial, hence, the distance covered until it stops is $\frac{2}{a}\tan\alpha$. For v_{max}, set $v'_{(x)} = 0$ and you should arrive at

$$v_{max} = \sqrt{\frac{g}{a}\sin\alpha\tan\alpha} \qquad \text{at} \qquad x = \frac{\tan\alpha}{a}$$

Note: Do not make the mistake of using $F_{(x)} = 0$ to find the point where the bar stops. Remember that $\sum F = 0$ does not imply rest.

3. Prism 1 of mass m_1 with angle α(see fig.) rests on a horizontal surface. Bar 2 of mass m_2 is placed on the prism. Assuming the friction to be negligible, find the acceleration of the prism. *Irodov 1.81*

Solution:
Here, both the prism and the bar move. In such systems where components in contact with or connected with each other move separately, their motions are related by kinematic constraints. In order to determine this constraint, set up the co-ordinate frame and assume distances as shown.

$$\tan\alpha = \frac{H - y_1}{x_1 - x_2}$$
$$H - y_1 = (x_1 - x_2)\tan\alpha$$

Differentiate twice w.r.t. time.

$$-\ddot{y}_1 = (\ddot{x}_1 - \ddot{x}_2)\tan\alpha$$

But, $\ddot{y}_1 = a_{2y}$, $\ddot{x}_2 - a_{2x}$, & $\ddot{x}_1 = a_1$

$$\therefore -a_{2y} = (a_1 - a_{2x})\tan\alpha \tag{2.7}$$

This relation is a direct consequence of the fact that the bar is always in contact with the prism as it slides down. For more relations, draw FBD of bar and prism and use Newton's Laws using the same co-ordinate frame as defined at the start.

(a) From FBD(b)

$$\sum F_x = m_1 a_1$$
$$N\sin\alpha - = m_1 a_1 \tag{2.8}$$

(b) From FBD(a)

$$\begin{aligned} \sum F_x &= m_2 a_{2x} \\ -N\sin\alpha &= m_2 a_{2x} \end{aligned} \tag{2.9}$$

$$\begin{aligned} \sum F_y &= m_2 a_{2y} \\ N\cos\alpha - m_2 g &= m_2 a_{2y} \end{aligned} \tag{2.10}$$

We now have a system of four equations to solve for four unknowns $N, a_1, a_{2x}, \& \, a_{2y}$.We leave it to you to obtain

$$a_1 = \frac{g\sin\alpha\cos\alpha}{\sin^2\alpha + \dfrac{m_1}{m_2}}$$

Note: We took the liberty of replacing action-reaction pair N_{12} and N_{21} by the single value N. We hope there is no need to explain why they are equal and opposite in direction.

4. A "Pedagogical Machine" is illustrated in the sketch (see fig.). All the surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling? *Kleppner 2.19*

SOLUTION:
This system is another example of constrained/dependent motion. In such systems, it is easier to start by determining the kinematic constraint/s. The x-acceleration of M_3 and M_1 are equal as they are in contact with each other. If M_3 is stationary with respect to M_1, then $M-2$ is also stationary with respect to M_1. Hence, the x-accelerations of $M_1, M_2, \& \, M_1$ are equal(say a). ***Add figures later*** Using Newton's Laws, we get,

$$T = M_2 a \tag{2.11}$$
$$T = M_3 g \tag{2.12}$$
$$N_{31} = M_3 a \tag{2.13}$$
$$N_{Px} = T \tag{2.14}$$
$$F - N_{13} - N_{Px} = M_1 a \tag{2.15}$$

After necessary substitution, we arrive at

$$\begin{aligned} F &= (M_1 + M_3)\frac{M_3}{M_2}g + M_3 g \\ F &= \frac{M_3}{M_2}(M_1 + M_2 + M_3)g \end{aligned} \tag{2.16}$$

Note: You can also solve this problem by using Newton's Laws on the composite system of M_1, M_2 and M_3. All the reaction forces will cancel as they are internal forces and we get

$$F = (M_1 + M_2 + M_3)a \tag{2.17}$$

Using (2.11) and (2.12) on (2.17), we arrive at (2.16).

5. Body A in figure weighs 102 N and body B weighs 32 N. the coefficients of friction between A and the incline are $\mu_s = 0.56$ and $\mu_k = 0.25$. Angle θ is 40°. Sketch the free body diagrams and find the accelerations of A (use $g = 10\text{ms}^{-2}$) if it is initially

   (a) at rest.

   (b) moving up the incline.

   (c) moving down the incline. *NePhO 2016*

   SOLUTION:

   (a) Since $w_A \sin\theta > w_B$, A tends to slide down and hence static friction opposes the descent. Also, the acceleration of A and B are equal since the length of rope has to remain constant. From the FBDs and using Newton's laws, we arrive at the following system of equations

$$T - w_B = m_B a \tag{2.18}$$

$$N = w_A \cos\theta \tag{2.19}$$

$$w_A \sin\theta - T - f_S N = m_A a \tag{2.20}$$

   We know that static friction is defined as $f_s \leq \mu_s N$, where $f_s = \mu N$ is the limiting friction. If the combined effect of the weights of bodies A and B exceeds this limiting value, then the system moves. An easy way to check if this is the case is by using the limiting value to solve for a. If we get a positive value for a, the system moves and if we get a negative value, it is stationary. On solving for a using limiting friction, we get

$$a = \frac{w_A(\sin\theta - \mu_S \cos\theta) - w_B}{m_A + m_B}$$

   Replace the numerical values to get

$$a = -0.76\text{m/s}^2$$

   So, we conclude that the system does not overcome static friction. Hence, body A remains in rest i.e. $a = 0$.

   (b) Frictional force always opposes motion. So, the FBD of body A looks as shown in the figure. (2.18) and (2.19) apply here too. The final equation obtained from the FBD is

$$w_A \sin\theta + f_k - T = m_A a \tag{2.21}$$

   Use $f_k = \mu_k N$, (2.18) and (2.19) on (2.21)

$$a = \frac{w_A(\sin\theta + \mu_k \cos\theta) - w_B}{m_A + m_B}$$

$$\therefore a = 3.96\text{m/s}^2$$

   (c) Using the same methods as (b), we can easily arrive at

$$a = \frac{w_A(\sin\theta - \mu_k \cos\theta) - w_B}{m_A + m_B}$$

$$\therefore a = 1.05\text{m/s}^2$$

You could also solve these problems by considering the composite system of bodies A and B. We encourage you to try this approach and get to the same answer. Make sure you keep track of which forces are external and which ones are internal.

2.4 Level 2 Problems and Solutions

1. A chain with uniform mass density per unit length hangs between two supports located at the same height, a distance of $2d$ apart. What should the length of the chain be so that magnitude of force at supports is minimized? You may use the fact that a hanging chain takes the form, $y_{(x)} = (1/\alpha)\cosh(\alpha x)$. You will eventually need to solve an equation numerically. *Morin 2.9*

   SOLUTION:
   Start with a FBD as always. Let the force at supports be F. The forces at supports must be equal due to symmetry. ***Add figure later*** Here λ is the linear mass density of chain and l is its length. The forces at supports are tangential to the chain. Hence, θ can be determined by finding the slope/derivative of the chain the supports.

$$
\begin{aligned}
y_{(x)} &= \frac{1}{\alpha}\cosh(\alpha x) \\
\therefore y'_{(x)} &= \sinh(\alpha x) \\
y'_{(d)} &= \sinh(\alpha d) \\
\therefore \tan\theta &= \sinh(\alpha d)
\end{aligned}
\tag{2.22}
$$

   A closer inspection of the equation of $y_{(x)}$ should lead you to the conclusion that the length of chain depends on α. Now, to find the length,

$$
\begin{aligned}
l &= \int_{-d}^{d} \sqrt{1 + {y'_{(x)}}^2}\, \mathrm{d}x \\
\therefore l &= \frac{2}{\alpha}\sinh(\alpha d)
\end{aligned}
\tag{2.23}
$$

   Balancing the vertical forces on the chain, we get,

$$2F\sin\theta = \lambda l g$$

   Use (2.22), (2.23) and $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ to get

$$
\begin{aligned}
F &= \frac{\lambda l g}{2\tanh(\alpha d)} \\
F_{(\alpha)} &= \lambda g \frac{\cosh(\alpha d)}{\alpha}
\end{aligned}
$$

   In order to minimize F,

$$
\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}\alpha} &= 0 \\
\frac{\alpha d \sinh(\alpha d) - \cosh(\alpha d)}{\alpha^2} &= 0 \\
\therefore (\alpha d)\tanh(\alpha d) &= 1
\end{aligned}
$$

   Solve this equation numerically using αd as the variable and you will get

$$\alpha d \approx 1.997$$

   Replace the value of α in (2.23) for length

$$l \approx 2.52d$$

You can do further calculations to find θ and the height of the chain. We leave it to you to prove

$$\begin{aligned} & h \approx 0.675d \\ \text{and} \qquad & \theta \approx 56.5^\circ \end{aligned}$$

2. A ball is thrown with speed v_0 at an angle θ. Let the drag force from the air take the form $\boldsymbol{F_d} = -\beta \boldsymbol{v} = -m\alpha \boldsymbol{v}$.

   (a) Find $x_{(t)}$ and $y_{(t)}$.

   (b) Assume that the drag coefficient takes the value that makes the magnitude of the initial drag force equal to the weight of the ball. If your goal is to have x be as large as possible when y achieves its maximum value (you don't care what this maximum value actually is), show that θ should satisfy $\sin\theta = (\sqrt{5}-1)/2$, which just happens to be the inverse of the golden ratio. *Morin 3.53*

   SOLUTION:

   (a) Two forces are acting on the ball: drag force and gravity. Use Newton's Laws on x and y axes, which are assumed to be in horizontal and vertical direction respectively.

2.5 IPhO Problems

2.6 Supplementary Problems

Chapter 3

Electricity and Magnetism

3.1 Level 1 Problems and Solutions

1. Three point charges q, 2q and 8q are to be placed on a 9cm long straight line. Find the positions where the charges should be placed so that the potential energy of the system is minimum.

Solution:

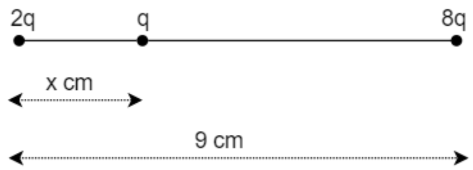

Figure 3.1: Conversion of soap bubble into spherical drop.

Since the potential energy is directly proportional to the product of charges and inversely proportional to the distance between them, we need to have a preliminary idea that the two greater charges should be placed as far as possible. In this case, they should be placed at two ends of a 9 cm line.
Let, the charge q be placed at a distance x from 2q. Potential energy of the system is given as,

$$U = \frac{1}{4\pi\epsilon_o}\left(\frac{2q \cdot q}{x} + \frac{2q \cdot 8q}{9} + \frac{q \cdot 8q}{9-x}\right) \tag{3.1}$$

For U to be minimum,

$$\begin{aligned}
\frac{dU}{dx} &= 0 \\
or, \frac{2q^2}{4\pi\epsilon_o}\left(\frac{dx^{-1}}{dx} + 0 + 4 \cdot \frac{d(9-x)^{-1}}{dx}\right) &= 0 \\
or, -\frac{1}{x^2} + \left(\frac{2}{9-x}\right)^2 &= 0 \\
or, \frac{2}{9-x} &= \frac{1}{x} \\
or, 2x &= 9 - x \\
\therefore x &= 3cm
\end{aligned}$$

Therefore, for the potential energy of the system to be minimum, the smallest charge q should be placed between 2q and 8q at a distance of 3 cm from 2q.

2. An electric dipole with dipole moment $\boldsymbol{p}$ is in a uniform electric field $\boldsymbol{E}$.
i. Find the operations of the dipole for which the torque on the dipole is zero.
ii. Which of the operations in part (i) is stable, and which is unstable?
iii. Show that for the stable orientation in part (ii), the dipole's own electric field tends to oppose the external field.

Solution:
Torque on a dipole $(\tau) = \boldsymbol{p} \times \boldsymbol{E}$ = pE $sin\theta$
i. So torque will be zero for $\theta = n\pi$, where n = 0,1,2,3..., i.e. torque will be zero if the direction of dipole moment is along $\boldsymbol{E}$, or if it is exactly opposite to $\boldsymbol{E}$.

(a) $\boldsymbol{p}$ in the direction of $\boldsymbol{E}$. (b) $\boldsymbol{p}$ opposite to $\boldsymbol{E}$.

Figure 3.2: Orientations of the dipole for which the torque is zero.

ii. Consider the orientation in 3.2a

(a) $\boldsymbol{p}$ in the direction of $\boldsymbol{E}$. (b) $\boldsymbol{p}$ opposite to $\boldsymbol{E}$.

Figure 3.3: Orientations of the dipole for which the torque is zero.

If we slightly shift the zero torque orientation in 3.2a either up or down as shown in **??**, then the torque will be produced in such a way that it rotates the dipole to acquire the initial configuration. So, it is stable.

Now, consider the orientation in 3.2b.
In this case a slight shifting of orientation will rotate the dipole further away from the equilibrium position. So, it is unstable.
iii. Since the electric field of the dipole is from +q to -q we can wasily see from above figures **??**, **??** and **??**, that the dipole's own electric field tends to oppose the external field.

(a) $\boldsymbol{p}$ in the direction of $\boldsymbol{E}$. (b) $\boldsymbol{p}$ opposite to $\boldsymbol{E}$.

Figure 3.4: Orientations of the dipole for which the torque is zero.

3.2 Level 2 Problems and Solutions

1. If the radius and surface tension of a spherical soap bubble be r and T respectively, calculate the charge required to double its radius.

Solution:

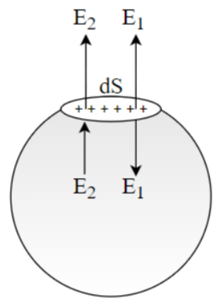

Figure 3.5: Electric fields due to dS and the remaining area of the soap bubble

Irrespective of the nature of concentrated charges in the soap bubble (either positive or negative), the electrostatic repulsion between them causes the soap bubble to expand.

Consider a soap bubble with uniform charge density σ distributed over the surface. Now, consider dS, a small area of the spherical surface, which acts as a thin sheet of charge. The electric field, E_1, due to this thin sheet of charge on its either direction is given as

$$E_1 = \frac{\sigma}{2\epsilon_o} \tag{3.2}$$

Now,
E_2 be the electric field due to remaining portion of the spherical surface (i.e. excluding dS). The direction of E_2 is as shown in the figure because of the spherical symmetry of the charge distribution. Inside the soap bubble, electric field is zero. So, E_1 and E_2 should be equal and opposite.

$$E_2 = E_1 = \frac{\sigma}{2\epsilon_o} \tag{3.3}$$

The force on the surface dS,

$$dF = \sigma.dS.E_2 \tag{3.4}$$

where $\sigma.dS$ is the small amount of charge on the surface dS.
From equation 3.3 and 3.4, we get

$$P_{elec} = \frac{\sigma^2}{2\epsilon_o} \tag{3.5}$$

The electric pressure given by 3.5 acts from inside to outside the soap bubble and expands it.

Before the application of charge, initial pressure acting inside the soap bubble is

$$P_{in} = P_a + \frac{4T}{r} \tag{3.6}$$

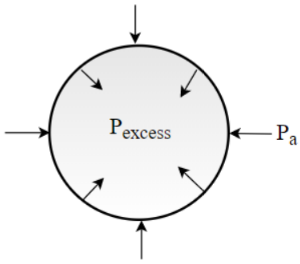

Figure 3.6: Point P_1 located outside the spherical volume, that is, r > R.

where,
P_a = Atmospheric pressure, and $\frac{4T}{r}$ = Excess pressure in a soap bubble of radius r.Note that the excess pressure is due to surface tension which intends to reduce the size of the bubble.
When the soap bubble is charged, it experiences outward pressure as discussed by equation 3.5. So with reference to 3.5 and 3.6, the final pressure acting inside the soap bubble is reduced and given as

$$P_f = P_a + \frac{4T}{R} - \frac{\sigma^2}{2\epsilon_o} \tag{3.7}$$

where R is the final radius after expansion. It is important to realize that this expansion is slow and isothermal so that we can use Boyle's law. Therefore,

$$P_{in}V_{in} = P_f V_f \tag{3.8}$$

$$\text{or, } \left(P_a + \frac{4T}{r}\right).\frac{4}{3}\pi r^3 = \left(P_a + \frac{4T}{R} - \frac{\sigma^2}{2\epsilon_o}\right).\frac{4}{3}\pi R^3$$
$$\text{or, } P_a(R^3 - r^3) + 4T(R^2 - r^2) = \frac{q^2}{32\epsilon_o \pi^2 R} \quad \left[\because \frac{\sigma^2}{2\epsilon_o} = \frac{q^2}{32\epsilon_o\pi^2 R^4}\right]$$

Given, R = 2r. So,

$$P_a(8r^3 - r^3) + 4T(4r^2 - r^2) = \frac{q^2}{32\epsilon_o\pi^2 2r}$$
$$\text{or, } 7P_a r^3 + 12Tr^2 = \frac{q^2}{64\epsilon_o\pi^2 r}$$
$$\text{or, } q^2 = 64\pi^2(7P_a\epsilon_o r^4 + 12T\epsilon_o r^3)$$

$\therefore q = 8\pi r\sqrt{\epsilon_o r(7P_a r + 12T)}$ is the amount of charge required to double the radius of the spherical soap bubble.

2. A soap bubble 10 cm in radius with a wall thickness of 3.3×10^{-6} **cm is charged to a potential of 100 V. The bubble bursts and falls as a spherical drop. Estimate the potential of the drop.**

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Solution:

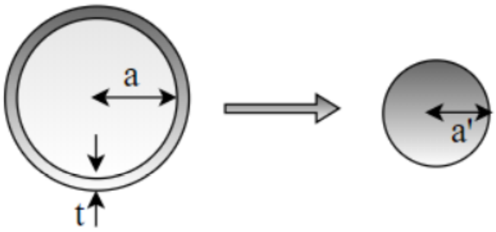

Figure 3.7: Conversion of soap bubble into spherical drop.

We solve this problem by using the concept of spherical capacitors.
Let Q be the charge supplied to the soap bubble of radius a and thickness t. If we draw a spherical Gaussian surface outside the soap bubble at any distance r from its center $(r > a)$,then Gauss's law gives,

$$E \cdot A = \frac{Q}{\epsilon_o}$$
$$\text{or, } E \cdot 4\pi r^2 = \frac{Q}{\epsilon_o}$$

$\therefore E = \dfrac{Q}{4\pi\epsilon_o r^2}$ is the electric field normal to the Gaussian surface.

Since we know that the electric field is the negative gradient of potential $\left(\text{i.e. } E = -\dfrac{dV}{dr}\right)$, we can find the potential of the soap bubble by integrating -E with respect to r from r = ∞ to r = a. Therefore,

$$V_a - V_\infty = \int_\infty^a E \cdot dr$$
$$\text{or, } V_a = -\frac{Q}{4\pi\epsilon_o a}\int_\infty^a \frac{dr}{r^2}$$
$$\text{or, } V_a = \frac{Q}{4\pi\epsilon_o a}$$

Therefore, the soap bubble of radius a and charged to potential V_a should carry the charge,

$$Q = 4\pi\epsilon_o a V_a \tag{3.9}$$

When the bubble bursts, volume of the new drop formed should be equal to the volume of liquid contained in the bubble. If a' be the radius of the new drop then,

$$\frac{4}{3}\pi(a+t)^3 - \frac{4}{3}\pi a^3 = \frac{4}{3}\pi a'^3$$
$$\text{or, } a^3 + 3a^2t + 3at^2 + t^3 - a^3 = a'^3$$

Since t is very small, the terms containing its higher powers are further smaller and can be neglected. So,

$$3a^2t = a'^3$$

$$\therefore a' = (3a^2t)^{\frac{1}{3}} \tag{3.10}$$

$\Big[$Or, we could have simply done $4\pi a^2 t = \dfrac{4}{3}\pi a'^3$, with the logic that volume is surface area ($4\pi a^2$) times thickness (t)$\Big]$

Since charge is conserved, the newly formed drop will act as a spherical capacitor of radius a' having charge Q $=4\pi\epsilon_o a V_a$ (from 3.9) uniformly distributed over its surface. As the drop of soap solution is a conductor, no charge rests inside.
Capacitance of the drop,

$$C = 4\pi\epsilon_o a' \tag{3.11}$$

Hence, the potential of the drop,

$$V = \frac{Q}{C}$$

Solving by using the expression for Q, C and a' from 3.9, 3.11 and 3.10 respectively, we get

$$V = \frac{a}{(3a^2t)^{\frac{1}{3}}} V_a$$

Putting a = 10 cm = 0.1 m, t= 3.3×10^{-8} m, and $V_a = 100$ V, we obtain the potential of the drop as 10 kV.

3. Calculate the amount of work needed to assemble a system at which -e charges are placed at each corner of a cube of length d and +2e at its center.

Solution:

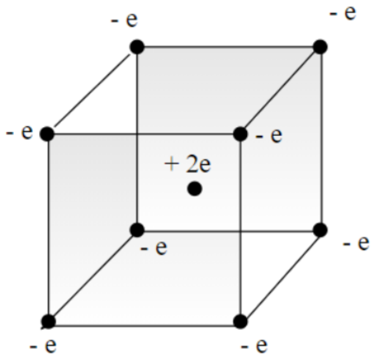

Figure 3.8: System of eight -e and one +2e charges arranged on a cube.

Initially assume that the charges are separated at infinite distance from one another such that the potential energy of the system is zero.
When the charges are then assembled as shown in figure 3.8, the amount of work done is equal to the final potential energy of the system.
Here, number of charges (n) = 9
So, the number of pairs that can be formed is given by

$$\frac{n(n-1)}{2} = 36$$

On analyzing, we get

i) 12 pairs of -e and -e separated by d (i.e adjacent ones)

ii) 12 pairs of -e and -e separated by $\sqrt{2}$d (i.e across the digonals of each face)

iii) 4 pairs of -e and e separated by $\sqrt{3}$d (i.e across the diagonals of the cube)

iv) 8 pairs of -e and +2e separated by $\frac{\sqrt{3}}{2}$d (i.e from each corner to the center of the cube)

Therefore the electric potential energy of the assembled system is given by

$$U = 12 \cdot \frac{(-e)(-e)}{4\pi\epsilon_o d} + 12 \cdot \frac{(-e)(-e)}{4\pi\epsilon_o \sqrt{2}d} + 4 \cdot \frac{(-e)(-e)}{4\pi\epsilon_o \sqrt{3}d} + 8 \cdot \frac{(-e)(+2e)}{4\pi\epsilon_o \frac{\sqrt{3}}{2} d}$$

On solving we get an expression for U which the required work done.

$$U = \frac{1.08e^2}{\pi\epsilon_o d}$$

4. Calculate the potential energy of a charge Q uniformly distributed throughout the sphere of radius R.

Solution: This problem can be solved by calculating the amount of work done in assembling a sphere of total charge Q uniformly distributed throughout its volume.
Let ρ be the volume charge density.
In the process of assembling a sphere, let, at any instant, the sphere has radius r and charge q.

$$q = \frac{4}{3}\pi\rho^3$$

Since charge distribution is uniform throughout the volume,

$$\frac{q}{Q} = \frac{\frac{4}{3}\pi\rho r^3}{\frac{4}{3}\pi\rho R^3}$$

$$\therefore q = \frac{Qr^3}{R^3}$$

If we add a spherical shell of thickness dr to the sphere of radius r, then the amount of charge added,

$$dq = \rho \cdot 4\pi r^2 \cdot dr$$

$$= \frac{Q}{\frac{4}{3}\pi R^3} \times 4\pi r^2 dr$$

$$\therefore dq = \frac{3Qr^2}{R^3}\, dr$$

So, the small amount of work done in assembling dq to a sphere of radius r, and carrying charge q,

$$\text{dW} = \text{V}\cdot \text{dq} \qquad [\because \text{Potential} = \text{Work Done / Charge}]$$

where, $V = \frac{1}{4\pi\epsilon_o}\frac{q}{r}$ is the potential of a sphere of radius r.
The total work done in assembling all the shells of charges to form a complete sphere of radius R and charge Q is

$$W = \int_0^R V \cdot dq = \int_0^R \frac{1}{4\pi\epsilon_o} \cdot \frac{q}{r} \cdot \frac{3Qr^2}{R^3} \cdot dr$$

$$\therefore W = \frac{3Q^2}{20\pi\epsilon_o R} \tag{3.12}$$

The work done is equal to the potential energy. So, equation 3.12, gives the expression for potential energy of a charge Q uniformly distributed throughout the sphere.

5. Calculate the electric potential at
i. r > R
ii. r = R
iii. r < R
due to a uniformly charged non-conducting sphere of radius R containing charge Q.

Solution:
Solving i. and ii. should be a piece of cake, whereas iii. requires a little effort.
In a non-conducting sphere, charge is uniformly distributed throughout the volume with density,

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

The potential at infinity is zero.

i. Outside the spherical volume (r > R).

Consider a point P outside the spherical volume located at a distance r from the center. Magnitude of electric field at P_1,

$$E_1 = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{r^2}$$

Electrostatic potential at P_1,

$$V_1 = -\int_{\infty}^{r} E_1 \cdot dr = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{r^2} \cdot dr \therefore V_1 = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{r}$$

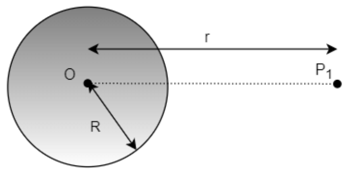

Figure 3.9: Point P_1 located outside the spherical volume, that is, r > R.

This is same as if the charge were placed at the center.
Thus, for external points the spherical charge behaves s if the whole charge were concentrated at the center of the spherical charge distribution.

ii. At the surface (r = R)

Following a similar approach as i.,we would integrate from ∞ to R to get the required potential,

$$V_R = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{R}$$

iii. Inside the spherical volume (r < R)

This is the major crux, and is the reason why the entire problem is included here in the first place. You would see how simply it can be solved.

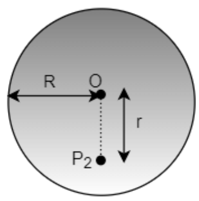

Figure 3.10: Point P_2 located inside the spherical volume, that is, r < R.

Consider a point P_2 inside the spherical charge at a distance r from the center O. To find the potential at P_2, we have to come from infinity to P_2 passing through regions having two different expressions for electric field.

At r > R,

$$E_1 = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{r^2}$$

At r < R,

$$E_2 = \frac{1}{4\pi\epsilon_o} \cdot \frac{Qr}{R^3}$$

Expression for E_2 can simply be calculated using Gauss's law, where $Q_{enclosed} = \frac{Qr^3}{R^3}$, resulting from uniform volume charge distribution.
Now,
Required potential at P_2 is given as,

$$\begin{aligned}
V_2 &= -\int_{\infty}^{r} E \cdot dr \\
&= -\left[\int_{\infty}^{R} E_1 \cdot dr + \int_{R}^{r} E_1 \cdot dr\right] \\
&= -\left[\int_{\infty}^{R} \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{r^2} \cdot dr + \int_{R}^{r} \frac{1}{4\pi\epsilon_o} \cdot \frac{Qr}{R^3} \cdot dr\right] \\
&= -\frac{Q}{4\pi\epsilon_o}\left[\left(-\frac{1}{r}\right)_{\infty}^{R} + \frac{1}{R^3}\left(-\frac{r^2}{2}\right)_{R}^{r}\right] \\
&= -\frac{Q}{4\pi\epsilon_o}\left[-\frac{1}{R} + \frac{r^2}{2R^3} - \frac{1}{2R}\right] \\
&= -\frac{Q}{4\pi\epsilon_o}\left[\frac{r^2}{2R^3} - \frac{3}{2R}\right] \\
&= \frac{Q}{4\pi\epsilon_o}\left[\frac{3R^2 - r^2}{2R^3}\right]
\end{aligned}$$

You can use the similar approach to find the gravitational potential of a planet at r < R. See I.E. Irodov 1.214 and give it a try. You will be pleased.

6. A thin wire ring of radius r has an electric charge q. What will be the increment of the force stretching the wire if a point charge q_o is placed at the ring's center?

Solution:
When the charge q_o is placed at the center of the ring then tension develops along the wire because of electrostatic force, which causes it to stretch. Let us consider a small segment of the ring which subtends $d\theta$ at the center and contains charge dq. Here,

$$dq = \frac{q}{2\pi r} \cdot r d\theta$$
$$\therefore dq = \frac{q}{2\pi} \cdot d\theta \tag{3.13}$$

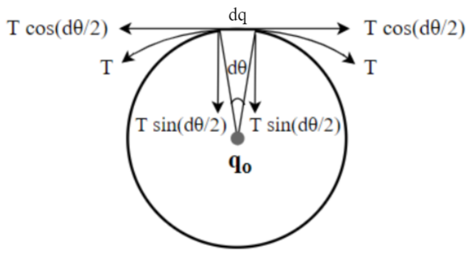

Figure 3.11: Point P_2 located inside the spherical volume, that is, r < R.

So, electrostatic force on dq,

$$dF_e = \frac{1}{4\pi\epsilon_o} \frac{q_o \cdot \frac{q}{2\pi} \cdot d\theta}{r^2}$$
$$= \frac{qq_o}{8\pi^2\epsilon_o r^2} \, d\theta \tag{3.14}$$

From figure 3.13, we can see that the horizontal forces due to tension cancel out, and the vertical tension forces are given as,

$$dF_t = T sin\frac{d\theta}{2} + T sin\frac{d\theta}{2}$$
$$= 2T sin\frac{d\theta}{2}$$
$$= T d\theta \tag{3.15}$$

$$\left(\because \text{for small angles, } \sin d\theta/2 \rightarrow d\theta/2.\right)$$

Here dF_e and dF_t are equal and opposite. So, equating equations 3.14 and 3.15 we obtain the required expression for the force.

$$T \cdot d\theta = \frac{qq_o}{8\pi^2\epsilon_o r^2} \, d\theta$$
$$\therefore T = \frac{qq_o}{8\pi^2\epsilon_o r^2}$$

Note: This approach is really important in solving several problems about ropes, strings or rings involving tension. Check out the following problem of Mechanics.

Q. A pole wraps at an angle θ around a pole. You grab one end and pull with tension T$_o$. The other end is attached to a large object, say, a boat. If the coefficient of static friction between the rope and the pole is μ, what is the largest force the rope can exert on the boat, if the rope is not to slip around the pole?

Solution:

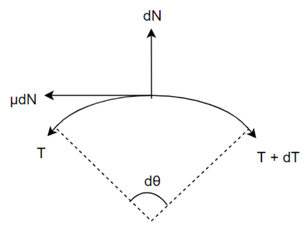

Figure 3.12: Tension, normal and frictional forces on a small segment of a rope wrapped around the pole.

Since the rope is wrapped around the frictional surface(pole), and is pulled from one end, tension varies over the length. Consider small piece of rope that subtends an angle $d\theta$. Let the tension at one end of this piece be T, which sightly varies(by dT) at the other end. The pole exerts a small outward normal force dN. To avoid slipping, friction should act in the direction of lesser tension.

Balancing forces in X-direction,

$$T\cos\frac{d\theta}{2} + \mu dN = (T + dT)\cos\frac{d\theta}{2}$$
$$\therefore \mu dN = dT \tag{3.16}$$

Balancing forces in Y-direction,

$$dN = (T + dT)sin\frac{d\theta}{2} + Tsin\frac{d\theta}{2}$$
$$\therefore \mu dN = dT \tag{3.17}$$

Dividing 3.16 by 3.17 and integrating from T$_o$ to T and 0 to θ we get,

$$T = T_o e^{\mu\theta}$$

which is the required expression for the largest force.

7. There is a spherical cavity inside a ball charged uniformly with volume density ρ. The center of cavity is displaced with respect to the center of the ball by $\boldsymbol{a}$. Find the field strength $\boldsymbol{E}$ inside the cavity, assuming the permittivity equal to unity.

Solution:
Let O and Q be the center of the ball and the spherical cavity respectively.
We know,
for a sphere with uniqform volume charge density, electric field at any point inside is given by

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_o} \cdot \frac{qr}{R^3}\,\hat{\mathbf{r}} \\ &= \frac{q}{\frac{4}{3}\pi R^3} \cdot \frac{r}{3\epsilon_o}\,\hat{\mathbf{r}} \\ &= \frac{\rho}{3\epsilon_o}\,\mathbf{r} \end{aligned} \tag{3.18}$$

$$\left[\because Unit\ vector(\hat{\mathbf{r}}) = \frac{Vector(\mathbf{r})}{Magnitude(r)} \right]$$

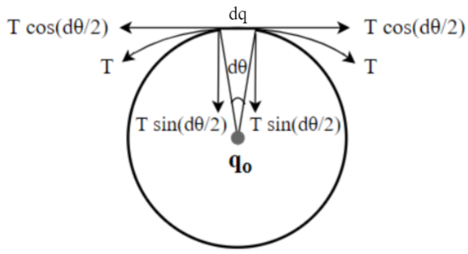

Figure 3.13: Point P_2 located inside the spherical volume, that is, r < R.

Now, let P be the any point inside the spherical cavity such that $\boldsymbol{OP} = \boldsymbol{r_1}$ and $\boldsymbol{QP} = \boldsymbol{r_2}$.
Had there not been a cavity, then, with reference to equation 3.18, the electric field at P due to whole sphere would be,

$$E_1 = \frac{\rho}{3\epsilon_o}\boldsymbol{r_1} \tag{3.19}$$

Similarly, had there been material on the cavity, the electric field at P only due to material of the cavity would be

$$E_2 = \frac{\rho}{3\epsilon_o}\boldsymbol{r_2} \tag{3.20}$$

Since there is a cavity we need to subtract the effect due to material of the cavity ($\boldsymbol{E_2}$) from that of the entire sphere ($\boldsymbol{E_1}$) to obtain the electric field at point P in the cavity ($\boldsymbol{E}$). So,

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E_1} - \boldsymbol{E_2} \\ &= \frac{\rho}{3\epsilon_o}\boldsymbol{r_1} - \frac{\rho}{3\epsilon_o}\boldsymbol{r_2} \\ &= \frac{\rho}{3\epsilon_o}(\boldsymbol{r_1} - \boldsymbol{r_2}) \\ &= \frac{\rho}{3\epsilon_o}\boldsymbol{a} \end{aligned}$$

Since the expression is independent of $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ we can say that the electric field inside the cavity is uniform.

8. Two thin rods of length L lie along the x-axis, one between x = a/2 and x = a/2 + L, and the other between x = -a/2 and x = -a/2 - L. Each rod has positive charge Q distributed uniformly along its length.
i. Calculate the electric field produced by the second rod at points along the positive x-axis.
ii. Calculate the magnitude of the force one rod exerts on the another.
iii. Reduce the result of part ii. for a>>L.
Hint:
Use the expansion ln(1+z) = z - $\frac{z^2}{2}$+$\frac{z^3}{3}$ ...., valid for $| z | \ll 1$. Carry all expansions to at least order $\frac{L^2}{a^2}$.

Solution:
i. Consider a point P in positive x-axis at a distance x from the origin.

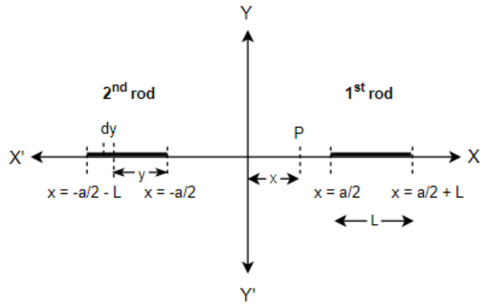

Figure 3.14: System of two rods placed on x-axis.

Consider a small segment dy of 2^{nd} rod at a distance y from its free end as shown in figure 3.14.
Charge per unit length, (λ) = Q/L
Charge within dy (dq) = (Q/L) · dy
Electric field at P due to dq,

$$dE = \frac{1}{4\pi\epsilon_o} \cdot \frac{(Q/L){\cdot}dy}{\{y + (a/2) + x\}^2}$$

Hence, the required electric field due to second rod at points along +ve x-axis,

$$\begin{aligned}
E &= \frac{Q}{4\pi\epsilon_o L}\int_0^L \frac{dy}{\{y + (a/2) + x\}^2} \\
&= \frac{Q}{4\pi\epsilon_o L}\int_0^L \{y + (a/2) + x\}^{-2} dy \\
&= \frac{Q}{4\pi\epsilon_o L}\left[\frac{\{y + (a/2) + x\}^{-2+1}}{-2+1}\right]_0^L \\
&= -\frac{Q}{4\pi\epsilon_o L}\left[\frac{1}{\{L + (a/2) + x\}} - \frac{1}{x + (a/2)}\right] \\
&= \frac{Q}{2\pi\epsilon_o L}\left[\frac{1}{2x + a} - \frac{1}{2x + a + 2L}\right]
\end{aligned}$$

ii. Let us consider a small segment dx on the first rod at a distance x from the origin. Then, charge within dx (dq') = (Q/L)dx

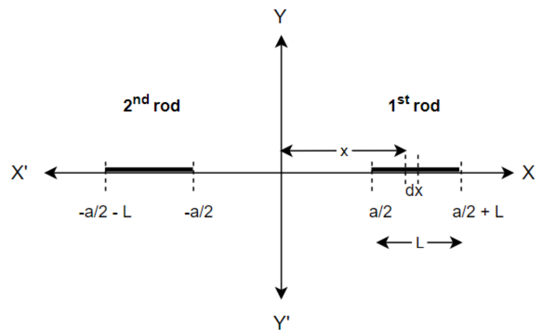

Figure 3.15: System of two rods placed on x-axis.

The force on dq due to the second rod is given by,

$$\begin{aligned}
dF &= dq \cdot E \\
&= \frac{Q}{L}dx \cdot \frac{Q}{2\pi\epsilon_o L}\left[\frac{1}{2x+a} - \frac{1}{2x+a+2L}\right] \\
&= \frac{Q^2}{2\pi\epsilon_o L^2}\left[\frac{dx}{2x+a} - \frac{dx}{2x+a+2L}\right]
\end{aligned}$$

The total force exerted on the first rod,

$$\begin{aligned}
F &= \frac{Q^2}{2\pi\epsilon_o L^2}\left[\int_{a/2}^{(a/2)+L}\frac{dx}{2x+a} - \int_{a/2}^{(a/2)+L}\frac{dx}{2x+a+2L}\right] \\
&= \frac{Q^2}{2\pi\epsilon_o L^2}\left[\frac{1}{2}\left\{ln(a+2x)\right\}_{a/2}^{(a/2)+L} - \frac{1}{2}\left\{ln(2x+a+2L)\right\}_{a/2}^{(a/2)+L}\right] \\
&= \frac{Q^2}{4\pi\epsilon_o L^2}\left[ln\frac{2a+2L}{2a} - ln\frac{2a+4L}{2a+2L}\right] \\
&= \frac{Q^2}{4\pi\epsilon_o L^2}\left[ln\frac{(a+L)^2}{a(a+2L)}\right]
\end{aligned}$$

iii. The reduced expression for force can be obtained as

$$\begin{aligned}
F &= \frac{Q^2}{4\pi\epsilon_o L^2}\cdot ln\frac{\left[(a/a)+(L/a)\right]^2}{\left[(a^2/a^2)+(2aL/a^2)\right]} \\
&= \frac{Q^2}{4\pi\epsilon_o L^2}\cdot ln\frac{\left[1+(L/a)\right]^2}{\left[(1+(2L/a)\right]} \\
&= \frac{Q^2}{4\pi\epsilon_o L^2}\left[2ln(1+L/a) - ln(1+2L/a)\right] \\
&= \frac{Q^2}{4\pi\epsilon_o L^2}\left[\frac{2L}{a} - \frac{2L^2}{2a^2} - \frac{2L}{a} + \frac{4L^2}{2a^2}\right] \quad [\because a \ll L] \\
&= \frac{Q^2}{4\pi\epsilon_o L^2}\cdot\frac{2L^2}{2a^2} \\
&= \frac{Q}{4\pi\epsilon_o a^2}
\end{aligned}$$

This shows that if a $\ll$ L, then the two rods interact as point charges, which clearly makes sense.

9. An uncharged shell of radius a is surrounded by another concentric shell of radius b carrying a charge Q. Determine the potential of each shell. What will be the situation when inner shell is grounded? Observe the potential difference and capacitance.

Solution:
For a metallic shell of radius R and charge Q, potential on the surface and at any point inside is same and given by

$$V = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{R}$$

Any conducting object inside such shell will also be at same potential.
When a conducting body is connected to the earth its potential becomes zero.

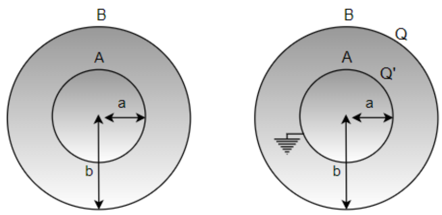

Figure 3.16: Before and after grounding the inner shell.

Before grounding inner shell

As there is no charge in the inner shell, potential is due to charge on outer shell only.

Potential on the outer shell $(V_B) = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{b}$

From above discussion,

Potential on the inner shell $(V_A) = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{b}$

This shows that there is no any potential difference between shell A and shell B.

After grounding inner shell

After grounding the inner shell, charges (electrons) flow from earth (zero potential) to inner shell (higher potential) and the potential of inner shell becomes zero. Let Q' be the amount of charge developed on inner shell.

The potential due to Q' $(V'_A) = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q'}{a}$

Here V_A and V'_A should combine to give zero potential.

$$V_A + V'_A = 0 \text{or}, \ \frac{1}{4\pi\epsilon_o} \cdot \frac{Q'}{a} \qquad = -\frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{b} \therefore Q' = -\frac{a}{b}Q$$

This is the amount of charge induced on the inner shell. Because of this charge at A, the potential at B will also change.
Potential of B after grounding,
(V'_B) = Potential due to Q on shell B + Potential due to Q' on shell A

$$\therefore V'_B = \frac{1}{4\pi\epsilon_o} \cdot \frac{Q}{b} + \frac{1}{4\pi\epsilon_o} \cdot \frac{Q'}{b}$$

Putting $Q' = -\frac{a}{b}Q$ we get,

$$V'_B = \frac{Q}{4\pi\epsilon_o} \cdot \frac{b-a}{b^2}$$

Potential difference between the shells (P.D) $= V'_B - 0 = V_B$

The capacitance of the system (C) $= \dfrac{Q}{P.D} = \dfrac{Q}{\dfrac{Q(b-a)}{4\pi\epsilon_o b^2}} = \dfrac{4\pi\epsilon_o b^2}{b-a}$

3.3 Practice Problems

1. A soap bubble of radius r and surface tension T is given a potential V volts. Show that the radius R of the bubble is related to its initial radius by the equation

$$P(R^3 - r^3) + 4T(R^2 - r^2) - \frac{\epsilon_o V^2 R}{2} = 0,$$

where P is the atmospheric pressure.

2. A sphere of uniform density ρ has within it a spherical cavity whose centers is at a distance a from the sphere. Show that the gravitational field within the cavity is uniform and determine its magnitude. (Ans : -4/3 πGρa)