Introduction to Differential Equations for Feynman Integrals

Oleksandr Gituliar
oleksandr@gituliar.net

II. Institut für Theoretische Physik
Universität Hamburg

Computer Algebra in Particle Physics 2017
DESY (Hamburg)

Introduction

Feynman Integrals Calculus — became in recent decades a science on its own.

$$\int \underbrace{\mathrm{d}^d l_1 \ldots \mathrm{d}^d l_n}_{\text{loops}} \underbrace{\mathrm{d}^d p_1 \delta(p_1^2) \ldots \mathrm{d}^d p_m \delta(p_m^2)}_{\text{legs}} \frac{1}{D_1^{n_1} \ldots D_k^{n_k}}$$

Numerical methods

- Sector Decomposition, Subtraction Schemes, . . .

Analytical methods

- Feynman/Schwinger/Mellin-Barnes parametrization
- $\boxed{\text{Integration-By-Parts}}$ reduction Chetyrkin, Tkachov '81
  - Laporta algorithm Laporta '00: `AIR`, `FIRE`, `Reduze`
  - Symbolic reduction: `LiteRed` Lee '12
  - private implementations
- $\boxed{\text{Method of Differential Equations}}$ Kotikov '91, Remiddi '97
  - Epsilon Form Henn '13
  - Lee algorithm Lee '14: `Fuchsia`, `Epsilon`
- . . .

Introduction

Feynman Integrals Calculus — became in recent decades a science on its own.

$$\int \underbrace{\mathrm{d}^d l_1 \ldots \mathrm{d}^d l_n}_{\text{loops}} \underbrace{\mathrm{d}^d p_1 \delta(p_1^2) \ldots \mathrm{d}^d p_m \delta(p_m^2)}_{\text{legs}} \frac{1}{D_1^{n_1} \ldots D_k^{n_k}}$$

$\boxed{\text{Integration-By-Parts}}$ reduction

- Integral Families
  - integration momenta
    - loop – $l_1, \ldots, l_n$ only
    - phase-space – $p_1, \ldots, p_m$ only
    - mixed
  - set of denominators (topology)
  - master integrals
- Reduction
  - any integral (from the family) in terms of masters
    - $\boxed{\text{including derivatives}}$
  - completely analytical
  - highly automated

Plan for Today

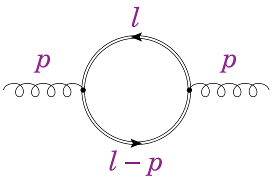
You will learn:

- Integration-by-Parts Reduction
  - `LiteRed`
- Differential Equations in Epsilon Form
  - `Fuchsia`
- Simple examples
  1. One-Loop Integral
  2. Two-Loop Phase-Space Integral
- Other tricks

Method of Differential Equations

1. Construct System of ODE (medium)
    - from definition (e.g. special functions)
    - from IBP rules
        - highly automated
        - `AIR`, `FIRE`, `LiteRed`, `Reduze2`
2. Find Epsilon Form (hard)
    - automated
    - Lee method: `Fuchsia`, `epsilon`
3. Solve System of ODE (easy)
4. Find Constants of Integration (medium)
    - depends on the problem

Example 1 — One-Loop Massive Self-Energy



$$= \Pi^{\mu\nu}_{ab}(p^2, m) = \delta_{ab}\left(p^\mu p^\nu - g^{\mu\nu} p^2\right)\Pi(p^2, m)$$

$$\Pi(p^2, m) = \int \mathrm{d}^n l \, F(p, l, m)$$

- **Arguments**: *from vectors to scalars*

$$F(p, l, m) \to F(l^2, \, l \cdot p, \, p^2, \, m)$$

- In general, the number of scalar integration variables is given by

$$N(L, E) = \frac{L(L+1)}{2} + LE \quad \sim \quad \mathcal{O}(L^2) \leftarrow$$ another source of growing complexity at higher orders

where E – number of *external momenta*, L – number of *loop momenta*

  – 1-loop propagator: $N(1, 1) = 2$

  – 4-loop propagator: $N(4, 1) = 14$ (ask Jos Vermaseren about details)

- The problem contains two denominators

$$D_1 = l^2 - m^2 \qquad D_2 = (l-p)^2 - m^2$$

which map into our integration invariants in a unique way

$$F(p,l,m) \to F(l^2,\, l\cdot p,\, p^2,\, m) \to F(D_1, D_2, p^2, m)$$

- One integral family

$$F(n_1,n_2) = \int \mathrm{d}^n l \, \frac{1}{D_1^{n_1} D_2^{n_2}}$$

```
<<LiteRed`

SetDim[n];
Declare[{m2}, Number, {l,p}, Vector];
NewBasis[$b, {sp[l]-m2, sp[l-p]-m2}, {l}, Directory->"b.ibp"];

GenerateIBP[$b];
AnalyzeSectors[$b];
FindSymmetries[$b];
```

In dimensional regularization the integral over a total derivative is zero.

$$\int \mathrm{d}^n l_i \frac{\mathrm{d}}{\mathrm{d}l_i^\mu} \left(q^\mu F(p_1,\ldots,l_1,\ldots)\right)$$

where q is arbitrary external or internal momenta.

```
IBP[$b]
```

```
SolvejSectors /@ UniqueSectors[$b]

MIs[$b]

> {j[$b,0,1], j[$b,1,1]}
```

- We obtain two master integrals

$$F_1 = F(0,1) = \int \mathrm{d}^n l \, \frac{1}{(l-p)^2 - m^2} \qquad F_2 = F(1,1) = \int \mathrm{d}^n l \, \frac{1}{\left(l^2 - m^2\right)\left((l-p)^2 - m^2\right)}$$

- Any other integral is a linear combination of only these two, e.g.,

$$F(2,1) = \frac{n-2}{2m^2(p^2-4m^2)} F_1 + \frac{n-3}{p^2-4m^2} F_2$$

- We can check that since we can do $l \to l + p$ transformation

$$F(0,1) = F(1,0)$$

```
$ds = Dinv[#,sp[p,p]]& /@ MIs[$b] // IBPReduce;
$ode = Coefficient[#, MIs[$b]]& /@ $ds;
```

- This code produces a system of differential equations

$$\frac{\mathrm{d}F_1}{\mathrm{d}p^2} = 0$$
$$\frac{\mathrm{d}F_2}{\mathrm{d}p^2} = \frac{2-2\epsilon}{p^2\,(p^2-4m^2)}F_1 + \frac{2m^2-\epsilon p^2}{p^2\,(p^2-4m^2)}F_2$$

  where we work in $n = 4-2\epsilon$ space-time dimensions

This system is simple and we could solve it right away using <*your favourite*> method. Today, I want to demonstrate you how this and many other systems can be solved throug using their ϵ-form. As you will see this is a highly automated task.

Exercise

Derive another system of differential equations, but this time in m^2.
(Hint: use `Fromj`, `D`, and `Toj` functions instead of `Dinv`).

Epsilon Form

- Classical Notation

$$\frac{\mathrm{d}F_1}{\mathrm{d}x} = A_{11}(x,\epsilon)F_1 + A_{12}(x,\epsilon)F_2$$
$$\frac{\mathrm{d}F_2}{\mathrm{d}x} = A_{21}(x,\epsilon)F_1 + A_{22}(x,\epsilon)F_2$$

- Matrix Notation

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}x} = A(x,\epsilon)\,\bar{F} \qquad \text{where} \qquad A = \begin{pmatrix} A_{11}(x,\epsilon) & A_{12}(x,\epsilon) \\ A_{21}(x,\epsilon) & A_{22}(x,\epsilon) \end{pmatrix} \quad \text{and} \quad \bar{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

It is very convenient to have our system in the epsilon form

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \epsilon\, B(x)\, G$$

since in this case we can easily find the solution to any order in ϵ parameter, as we will see on the next slide.
Some physical examples may lead to systems with ~ 500 equations. Hence, it is very important to make this task automatic.

A few words on `Fuchsia`

Input

- System of Ordinary Differential Equations $A(x,\epsilon,\ldots)$, i.e.,

$$\frac{\mathrm{d}F}{\mathrm{d}x} = A(x,\epsilon,\ldots)\,F(x,\epsilon,\ldots)$$

Output

- Equivalent System in the Epsilon Form

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \epsilon\,B(x,\ldots)\,G(x,\epsilon,\ldots)$$

- Corresponding Basis Transformation

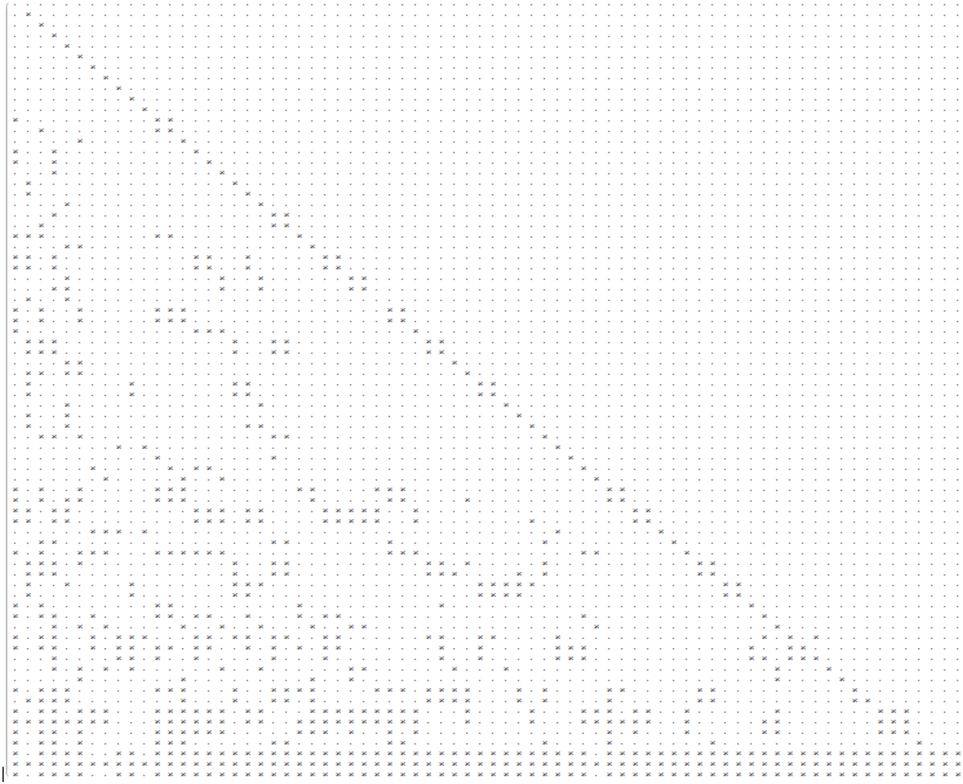
$$F(x,\epsilon,\ldots) = T(x,\epsilon,\ldots) \times G(x,\epsilon,\ldots)$$

- Other Operations
  - apply custom transformation
  - variable change
  - "sort" to block-diagonal form

A few words on `Fuchsia`

- Based on the *Lee algorithm* Lee '14
  - support additional symbols
  - alternative implementation: `epsilon`
- Open-Source and Free Gituliar, Magerya '16 '17
  - `http://github.com/gituliar/fuchsia`
- Implemented in `Python`
  - `SageMath`
  - `Maxima`
  - `Maple` (optional)
- Algorithm
  1. **Fuchsification** (Jordan form)
     Get rid of apparent singularities
  2. **Normalization** (eigenvalues, eigenvectors)
     Balance eigenvalues to $\alpha\,\epsilon$ form
  3. **Factorization** (solve linear equations)
     Reduce to the epsilon form

A few words on Fuchsia



Let us introduce a new variable y, such that

$$p^2 = -4m^2 \frac{y^2}{1-y^2}$$

The new equations look as

$$\frac{\mathrm{d}F_1}{\mathrm{d}y} = 0$$
$$\frac{\mathrm{d}F_2}{\mathrm{d}y} = \frac{1-\epsilon}{y\, m^2} F_1 + \left(\frac{\epsilon}{1-y} - \frac{\epsilon}{1+y} - \frac{1}{y} \right) F_2$$

With the help of `Fuchsia` we find a new basis G_1, G_2 given by the system

$$F_1 = \frac{4(1-2\epsilon)}{3(1-\epsilon)} G_1$$
$$F_2 = \frac{4}{3m^2} G_1 - \frac{2}{y} G_2$$

For this basis the differential equations are the epsilon form

$$\frac{\mathrm{d}G_1}{\mathrm{d}y} = 0$$
$$\frac{\mathrm{d}G_2}{\mathrm{d}y} = \frac{2}{3m^2} \left(\frac{\epsilon}{1+y} + \frac{\epsilon}{1-y} \right) G_1 - \left(\frac{\epsilon}{1-y} - \frac{\epsilon}{1+y} \right) G_2$$

Solutions

We are looking for the solution of a given system of ordinary differential equations in the epsilon form

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \epsilon\, B(x)\, G$$

as a Laurent series in ϵ

$$\boxed{G(x,\epsilon) = G_0(x) + G_1(x)\,\epsilon + G_2(x)\,\epsilon^2 + \ldots}$$

Let us put this "solution" into the initial equation

$$\frac{\mathrm{d}G_0}{\mathrm{d}x} + \frac{\mathrm{d}G_1}{\mathrm{d}x}\epsilon + \frac{\mathrm{d}G_2}{\mathrm{d}x}\epsilon^2 + \ldots = \epsilon\, B(x)\, G_0 + \epsilon^2\, B(x)\, G_1$$

we get

$$\frac{\mathrm{d}G_0}{\mathrm{d}x} = 0, \qquad \frac{\mathrm{d}G_1}{\mathrm{d}x} = B(x)G_0, \qquad \frac{\mathrm{d}G_2}{\mathrm{d}x} = B(x)G_1 \qquad \ldots \qquad \frac{\mathrm{d}G_n}{\mathrm{d}x} = B(x)G_{n-1}$$

This system can be easily solved (as promised)

$$G_0 = C_0, \qquad G_1 = C_1 + \int \mathrm{d}x B(x) C_0, \qquad G_2 = C_2 + \int \mathrm{d}x B(x)\left(C_1 + \int \mathrm{d}x B(x) C_0\right) \qquad \ldots$$

$$\boxed{G_n(x) = C_n + \int \mathrm{d}x B(x) G_{n-1}}$$

Solutions

My implementation of the solution algorithm, which I use to get results for the next slide.

```
SolveODE[m_, x_, ep_, n_, c_] := Module[
  {$i, $j, $n, $sol, $sol0, $sol1},

  $n = Length[m];
  $sol[0] = Table[c[$j,0], {$j,1,$n}];
  For[$i=1, $i<=n, $i++,
    $sol0 = Table[c[$j,$i], {$j,1,$n}];
    $sol1 = Integrate[Dot[#,$sol[$i-1]],x]& /@ m;
    $sol[$i] = $sol0 + $sol1;
  ];
  Sum[ep^$i*$sol[$i], {$i,0,n}]
];
```

Example 1

- Master #1

$$F_1(y,m^2) = \frac{4}{3}C_1^{(0)} + \frac{4}{3}\left(C_1^{(1)} - C_1^{(0)}\right)\epsilon + \ldots$$

- Master #2

$$F_2(y,m^2) = \frac{4C_1^{(0)}}{3m^2} - \frac{C_2^{(0)}}{y} + \frac{\epsilon}{3m^2 y}\left(4yC_1^{(1)} - 6m^2 C_2^{(1)} + \left(4C_1^{(1)} - 6m^2 C_2^{(0)}\right)\ln\left(\frac{1-y}{1+y}\right)\right)$$

- Finally, we need to find unknown integration constants whcih are functions of m^2 and ϵ, i.e.

$$C_1^{(0)}(m^2,\epsilon), \quad C_1^{(1)}(m^2,\epsilon), \quad \ldots$$

$$C_2^{(0)}(m^2,\epsilon), \quad C_2^{(1)}(m^2,\epsilon), \quad \ldots$$

Master #1 (from `Fuchsia`)

$$F_1(y,m^2) = \frac{4}{3}C_1^{(0)} + \frac{4}{3}\left(C_1^{(1)} - C_1^{(0)}\right)\epsilon + \ldots$$

Closed-form solution from the literature (see Smirnov's book)

$$F(0,n) = (-1)^n \frac{\Gamma(n-2+\epsilon)}{\Gamma(n)} (m^2)^{2-\epsilon-n}$$

$$F_1(y,m^2) = F(0,1) = \frac{m^2}{\epsilon} + m^2\left(1-\gamma_E - \ln m^2\right) + \ldots$$

Result #1

$$\boxed{C_1^{(0)} = \frac{3m^2}{4\epsilon} \quad C_1^{(1)} = \frac{3m^2\left(2-\gamma_E-\ln m^2\right)}{4\epsilon}}$$

Result #1

$$C_1^{(0)} = \frac{m^2}{\epsilon} \quad C_1^{(1)} = \frac{m^2\left(1-\gamma_E - \ln m^2\right)}{\epsilon}$$

Master #2 (with Result #1 substituted)

$$F_2(y,m^2) = \frac{1}{\epsilon} + \frac{2y - \gamma_E\, y - 2C_2^{(0)} - y\, \ln m^2 + \ln\left(\frac{1-y}{1+y}\right)}{y} + \dots$$

We require that at the limit $y \to 0$ ($p^2 \to 0$) our result is regular. This leads to the solution

$$C_2^0 = 0$$

Result #2

$$\boxed{F_2(y,m^2) = \frac{1}{\epsilon} + 2 - \gamma_E\, - \ln m^2 + \frac{1}{y} \ln\left(\frac{1-y}{1+y}\right) + \dots}$$

This is in agreement with T.Riemann Monday's lecture!

We have seen how to

- generate IBP rules for a given graph
- construct differential equations
- find epsilon form
- solve differential equations
- find integration constants

Exercies

- using `LiteRed` choose some two-loop (massless and massive) propagator and find corresponding masters
- solve Example #1, but using equations in m^2 (for help see Smirnov's book)

Example 2 Splitting Functions from e^+e^--annihilation

In this example, I will show how to calculate a gluon-quark splitting function

$$P_{gq} = \frac{1+(1-x)^2}{x}$$

Using this technique you will be able to calculate remaining splitting functions P_{qq}, P_{qg}, and P_{gg} as well as higher-order corrections to these quantities.

$$e^+(q_1)+e^-(q_2) \to q(p_1)+\bar{q}(p_2)+g(p_3)$$

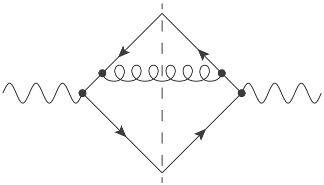
Mass-factorization theorem

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}x} = \frac{P_{iq}}{\epsilon}+a_i+b_i\epsilon+\ldots$$

where $q = q_1+q_2$ and

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}x} = \int \mathrm{d}^n p_1\delta(p_1^2)\,\mathrm{d}^n p_2\delta(p_2^2)\,\mathrm{d}^n p_3\delta(p_3^2)\,\delta\left(x-\frac{2q\cdot p_i}{q^2}\right)\sigma(q_1,q_2,p_1,p_2,p_3)$$

By their structure phase-space integrals are very similar to loop integrals (compare to the one-loop propagator from Example I), except that we apply on-shell conditions $\delta(p_i^2)$ to the cut lines as shown in the following **cut graph**



$$\frac{\mathrm{d}\sigma_g}{\mathrm{d}x} = \int \mathrm{d}^n p_1 \delta(p_1^2)\, \mathrm{d}^n p_2 \delta(p_2^2)\, \mathrm{d}^n p_3 \delta(p_3^2)\, \delta\left(x - \frac{2q\cdot p_3}{q^2}\right) \sigma(q_1, q_2, p_1, p_2, p_3)$$

where

$$\sigma(q_1, q_2, p_1, p_2, p_3) = N \frac{(p_1\cdot q_1)^2 + (p_2\cdot q_1)^2 + (p_1\cdot q_2)^2 + (p_2\cdot q_2)^2}{p_1\cdot p_3\; p_2\cdot p_3}$$

This integration is equivalent to the 2-loop propagator, since we can eliminate one of the integration momenta using momentum conservation

$$q_1 + q_2 = p_1 + p_2 + p_3$$

In order to integrate the cross-section we need a new IBP basis. Let us define one as

```
NewBasis[$a,{sp[p1], sp[p3], sp[q1+q2-p1-p3], s*x-2sp[q1+q2,p3], sp[p1,p3]},
            {p1, p3}, Append -> True];

GenerateIBP[$a];
AnalyzeSectors[$a, {___,0,0}, CutDs -> {1,1,1,1,0,0,0}];
FindSymmetries[];

SolvejRules /@ UniqueSectors[$a];
```

Note additional arguments in `AnalyzeSectors` routine:

- in `{___,0,0}` 0's represent invariants which appear in numerators only
- in `CutDs -> {1,1,1,1,0,0,0}` 1's represent "cut" propagators. It means that all integrals with at least one non-positive indices in these places vanish.

We get only one master integral

$$F_1(x,\epsilon) = \int \mathrm{d}^n p_1 \delta(p_1^2)\, \mathrm{d}^n p_2 \delta(p_2^2)\, \mathrm{d}^n p_3 \delta(p_3^2)\, \delta\left(x - \frac{2q\cdot p_3}{q^2}\right)$$

Partial Fractioning

Given a set of denominators, being a *linear combination* of the kinematic invariants s_{ij}, make a partial fraction such that

$$\frac{1}{D_1 \ldots D_n} \to \frac{a_1}{D_2 \ldots D_n} + \frac{a_2}{D_1 D_3 \ldots D_n} + \ldots + \frac{a_n}{D_1 \ldots D_{n-1}}$$

All we need is to solve a linear system of equations

$$a_1 D_1 + \ldots + a_n D_n = N$$

where the coefficient in front of every s_{ij} is zero and N is some number.

In particaulr, for

$$A = \frac{1}{(x+1)(y+1)(x+y+1)}$$

we write down

$$(a_1 + a_3)x + (a_2 + a_3)y + a_1 + a_2 + a_3 = N$$

the solution is

$$a_1 = -a_3 \quad a_2 = -a_3 \quad N = -a_3$$

which gives

$$A = \frac{1}{(y+1)(x+y+1)} + \frac{1}{(x+1)(x+y+1)} - \frac{1}{(x+1)(y+1)}$$

Example 2

Now we can convert the initial cross-section into the j-form and make IBP reduction.

```
M2 = (sp[p1,q1]^2+sp[p1,q2]^2+sp[p2,q1]^2+sp[p2,q2]^2)/(x*sp[q1,q2]*sp[p1,p3]

PS2 = x / (sp[p1]*sp[p3]*sp[q1+q2-p1-p3]*(s*x-2*sp[q1+q2,p3]));

jM2 = Toj[$a, PS2*M2];

jM2 = jM2 // IBPReduce

Pgq = Series[jM2 /. {m -> 4-2*eps}, {eps, 0, -1}]
```

This gives us

$$P_{gq} \sim \frac{2-2x+x^2}{x^2} F_1(x)$$

which contains one x factor more in the denominator than we expect.

Maybe $F_1(x) \sim x$? Let us check...

Example 2

```
F1 = j[$a, 1, 1, 1, 1, 0, 0, 0];

dF1 = Toj[$a, D[Fromj[$F1], x]] // IBPReduce;
```

This code produces the following equation

$$\frac{\mathrm{d}F_1}{\mathrm{d}x} = \left(\frac{\epsilon}{1-x} + \frac{1-2\epsilon}{x}\right) F_1$$

Of course we could use `Fuchsia` and find the ϵ-form, but we can solve this in a closed form

$$F_1 = C(\epsilon)\,(1-x)^{\epsilon}\,x^{1-2\epsilon}$$

which confirms our assumption from the previous slide.
The final result is

$$\boxed{P_{gq} \sim \frac{2-2x+x^2}{x}}$$

Example 2

Now you also now how to calculate phase-space integrals.

Exercise

Redefine x as

$$x = \frac{2q \cdot p_1}{q^2}$$

and find a well-known result

$$P_{qq} = \frac{1+x^2}{1-x}$$

for the quark-quark splitting function.

Holonomic Functions

A function $f = f(x)$ is called *holonomic* if there exist polynomials $a_n(x), \ldots, a_0(x)$ such that

$$a_n(x) f^{(n)} - a_{n-1}(x) f^{(n-1)} - \ldots - a_0(x) f = 0$$

holds for all x. Hence, the holonomic function is uniquely defined by

- the differential equation
- a number of initial values $f(x_0), f'(x_0), \ldots, f^{(n-1)}(x_0)$

Examples of holonomic functions:

- *all algebraic functions*
- *Generalized Hypergeometric functions*
  - *polylogarythms*
  - *Elliptic functions*
- *Bessel functions*
- *Airy functions*
- *Legendre and Chebyshev polynomials*
- *Heun functions*
- and many others that have no name and no closed form

Holonomic Functions

A function $f = f(x)$ is called *holonomic* if there exist polynomials $a_n(x), \ldots, a_0(x)$ such that

$$a_n(x)f^{(n)} - a_{n-1}(x)\, f^{(n-1)} - \ldots - a_0(x)\, f = 0$$

holds for all x. Hence, the holonomic function is uniquely defined by

- the differential equation
- a number of initial values $f(x_0)$, $f'(x_0)$, $\ldots$, $f^{(n-1)}(x_0)$

Conclusion

- simple representation
  - polynomials
  - ordinary differential equations
- define many complicated functions
  - no closed form
  - non-trivial integration representation
- represent Feynman integrals
- alternative for direct integration

Systems of ODE

We can easily rewrite a n^{th}-order linear ODE given by

$$y^{(n)} - a_1(x)\, y^{(n-1)} - \ldots - a_n(x)\, y = 0 \tag{1}$$

as an $n \times n$ system of the form

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}x} = A(x)\, \bar{y}$$

where

$$A(x) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_n(x) & a_{n-1}(x) & \cdots & a_2(x) & a_1(x) \end{bmatrix} \quad \text{and} \quad \bar{y} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{pmatrix}$$

However, the inverse opperation is not as easy anymore.

Expansion of Hypergeometric Functions

The Generalized Hypergeometric Function

$$ {}_{p+1}F_p\left(\begin{matrix} a_1,\, a_2,\, \ldots,\, a_{p+1} \\ b_1,\, b_2,\, \ldots,\, b_p \end{matrix};\, x\right) = \prod_{i=1}^{p} \frac{\Gamma(b_i)}{\Gamma(a_i)\Gamma(b_i - a_i)} \int_0^1 \frac{t_i^{a_i-1}(1-t_i)^{b_i-a_i-1}}{(1-x\,t_1\ldots t_p)^{a_{p+1}}} \mathrm{d}t_i $$

is a solution to the differential equation

$$ \left[D\,(D+b_1-1)\cdots(D+b_p-1) - x\,(D+a_1)\cdots(D+a_{p+1})\right]\,y = 0 $$

where

$$ D = x\frac{\mathrm{d}}{\mathrm{d}x} $$

Exercise

Using your favourite CAS write a routine which for a given Generalized Hypergeometric Function, defined by the list $\{a_1,\ldots,a_{p+1},b_1,\ldots,b_p\}$, returns a corresponding ODE, defined by the list $\{a_1(x),\ldots,a_p(x)\}$, in accordance with notation of eq. (1).

Reading List

- Feynman Integral Calculus by V. Smirnov
- Lectures on Differential Equations for Feynman Integrals by J. Henn
- Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations by W. Balser
- Computer Algebra in Particle Physics by S. Weinzierl
- Introduction to Loop Calculations by G. Heinrich
- Structure and Interpretation of Computer Programs by H. Abelson and G. Sussman with J. Sussman