BSCMA1004

STATISTICS II NOTES

WEEK 6 NOTES

IITM B.S Degree

PREPARED BY

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Stats II → Week VI Notes

★ Joint Density in 2D

A function $f(x,y)$ is said to be a joint density function if:
- $f(x,y) \geq 0$
- $\iint_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$
- $f(x,y)$ is piecewise continuous in each variable

- For every joint density $f(x,y)$, there ~~is~~ exists two jointly distributed variables (CRVs) X and Y s.t for any 2D region A:

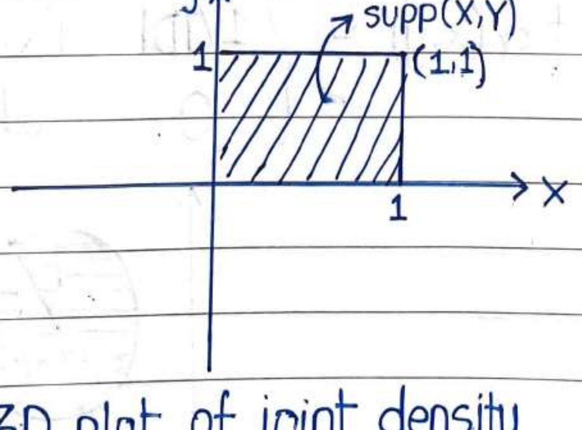
$$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy$$

$f(x,y)$ or $f_{XY}(x,y)$ is the joint density of X and Y.

- $\text{supp}(X,Y) = \{(x,y) : f_{XY}(x,y) > 0\}$

Eg uniform in the unit square

Let X and Y have joint density : $f_{XY}(x,y) = \begin{cases} 1 & 0<x<1, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$



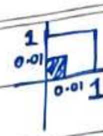
$$\int_0^1\int_0^1 1\,dx\,dy = 1$$

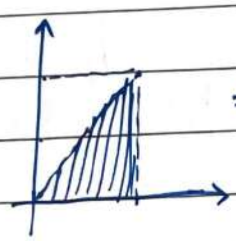
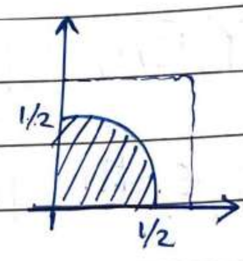
i.e $\int_0^1 x\Big|_0^1 dy$

$= \int_0^1 1\,dy = y\Big|_0^1$

$= \underline{\underline{1}}$

- Picture 3D plot of joint density
- To compute probability, find area of region.

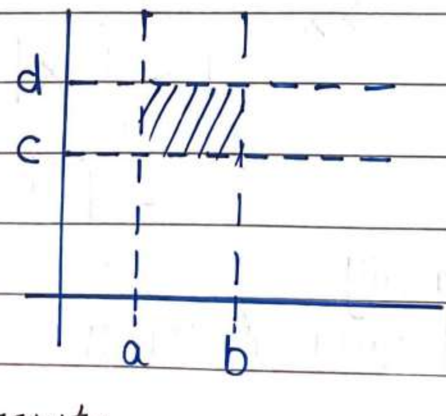
- $P(0<X<0.1,\ 0<Y<0.1) =$  $= 0.01$

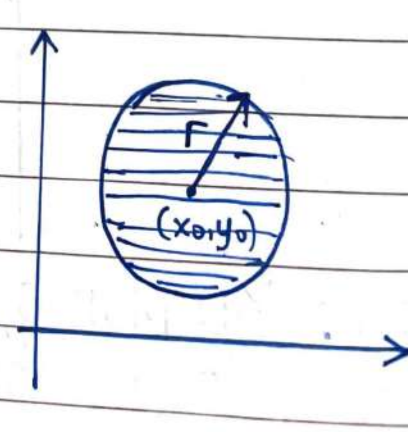
- $P(0.5<X<0.6,\ 0<Y<0.1) = 0.01$
- $P(0.9<X<1,\ 0.9<Y<1) = 0.01$
- $P(0<X<0.1) = 1\times 0.1 = 0.1$
- $P(0.5<Y<0.6) = 1\times 0.1 = 0.1$
- $P(X>Y) =$  $= \boxed{\frac{1}{2}}$
- $P(X^2+Y^2<0.25) =$  $= \boxed{\frac{\pi}{16}}$

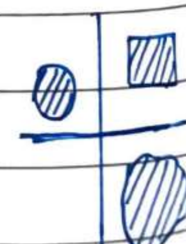
★ <u>2D Uniform Distribution</u>

~~For~~ Fix some region D in $\mathbb{R}^2$ with total area $|D|$.
We say $(X,Y) \sim \text{Uniform}(D)$ if they have joint density

$$f_{XY}(x,y) = \begin{cases} 1/|D| & (x,y)\in D \\ 0 & \text{otherwise} \end{cases}$$







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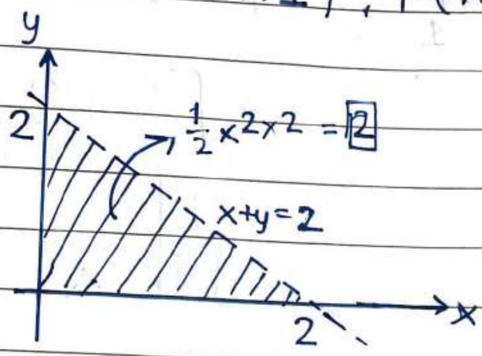
- For any sub-region A of D, $P((X,Y)\in A) = |A|/|D|$
- Uniform Distribution is a good approxmn. for flat histogs. (histograms)

Q. Let $(X,Y) \sim \text{Uniform}(D)$, $D = \{(x,y) : x+y \leq 2, x>0, y>0\}$
Find $P(X+Y<1)$, $P(X+2Y>1)$



$$f_{XY}(x,y) = \begin{cases} 1/2 & (x,y)\in D \\ 0 & \text{otherwise} \end{cases}$$

$$P(X+Y<1) = \frac{1/2}{2} = \boxed{1/4}$$

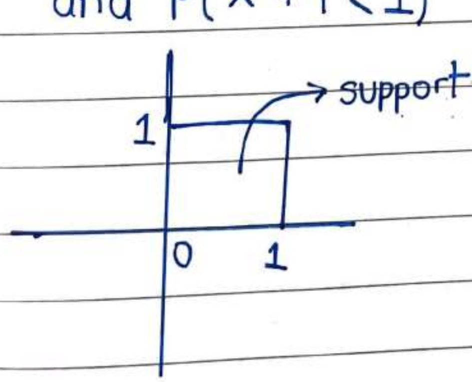
$$P(X+2Y>1) = 2 - \frac{1}{2}\times 1\times\frac{1}{2} = \frac{7}{4} \text{ (blue area)}$$
$$= \frac{7/4}{2} = \boxed{\frac{7}{8}}$$

★ 2D Non Uniform Distribution

Let (X,Y) have joint density:

$$f_{XY}(x,y) = \begin{cases} x+y, & 0<x,y<1 \\ 0, & \text{otherwise} \end{cases}$$

Show that this is a valid density. Find $P(X<1/2, Y<1/2)$ and $P(X+Y<1)$



$f_{XY} \geq 0$.

$$\int_{y=0}^{1}\underbrace{\int_{x=0}^{1}(x+y)\,dx}\,dy$$

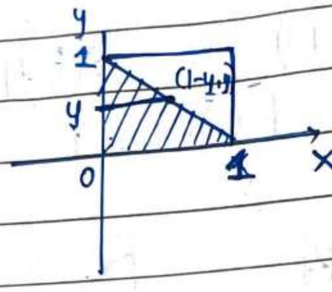
$$= \int_0^1 (x+y)\,dx = \frac{x^2}{2} + xy\Big|_0^1$$

$$= \frac{1}{2}+y \Rightarrow \int_{y=0}^{1}\left(y+\frac{1}{2}\right)dy = \frac{y}{2}+\frac{y^2}{2}\Big|_0^1$$

$$= \underline{\underline{1}}$$

$$P(X<1/2,\, Y<1/2) = \int_0^{1/2}\int_0^{1/2}(x+y)\,dx\,dy = \int_0^{1/2}\left(\frac{1}{8}+\frac{y}{2}\right)dy = \boxed{1/8}$$

$P(X+Y<1) =$



$$\int_{y=0}^{1}\int_{x=0}^{1-y}(x+y)\,dx\,dy$$

$$= \int_0^1 \frac{x^2}{2}+xy\int_0^{1-y} dy$$

$$= \boxed{\frac{1}{3}}$$

★ Marginal Densities

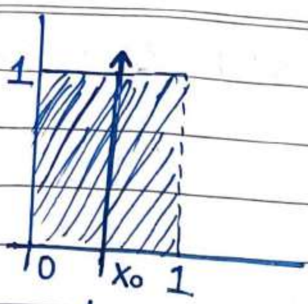
Suppose (X, Y) has joint density $f_{XY}(x,y)$. Then:→

- X has marginal density $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$

- Y has marginal density $f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$

- The joint density exactly determines marginal densities.

Eg. Uniform on unit square



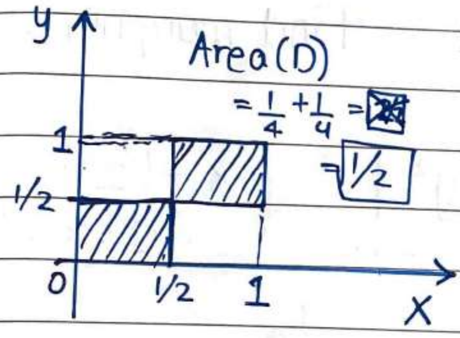
slice at $x = x_0$.

$$f_X(x_0) = \int_{y=0}^{1} f_{XY}(x_0, y)\,dy = 1, \quad 0 < x_0 < 1$$

$X \sim \text{Uniform}[0,1]$

$Y \sim \text{Uniform}[0,1]$

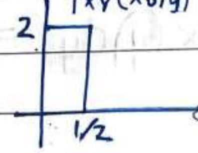
Eg. $(X,Y) \sim \text{Uniform}(D)$, where $D = [0, 0.5] \times [0, 0.5] \cup [0.5, 1] \times [0.5, 1]$



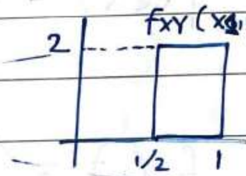
Area(D) $= \frac{1}{4} + \frac{1}{4} = \boxed{1/2}$

$$f_{XY}(x,y) = \begin{cases} 2, & (x,y) \in D \\ 0, & \text{otherwise} \end{cases}$$

$f_{XY}(x_0, y)$ $x_0 \in (0, 1/2)$, $f_X(x_0) = 1$



$f_{XY}(x_1, y)$ $x_1 \in [1/2, 1]$



$f_X(x_1) = 1$

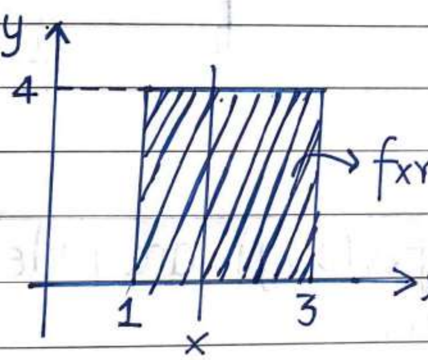
$X \sim \text{Uniform}[0,1]$

$Y \sim \text{Uniform}[0,1]$.

$f_X(x) = 1$

$0 < x < 1$

Eg $(X,Y) \sim \text{Uniform}(D)$, $D = [1,3] \times [0,4]$



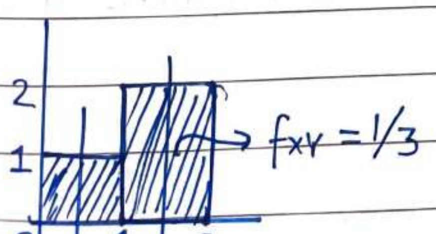
$f_X(x) = 1/2, \quad 1 \le x \le 3.$

(slice.... between 0 and 4, it will have height 1/8, length 4, so 1/2).

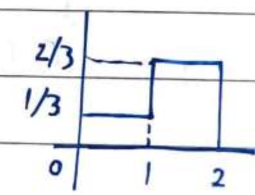
$X \sim \text{Uniform}[1,3]$

$Y \sim \text{Uniform}[0,4]$

Eg. $(X,Y) \sim \text{Uniform}(D)$, $D = [0,1] \times [0,1] \cup [1,2] \times [0,2]$



$$f_X(x) = \begin{cases} 1/3 & 0 < x < 1 \\ 2/3 & 1 < x < 2 \end{cases}$$



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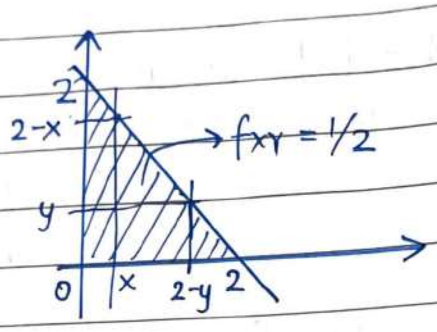
Eg. $(X,Y) \sim \text{Uniform}(D)$. $D = \{(x,y) : x+y<2,\ x,y>0\}$



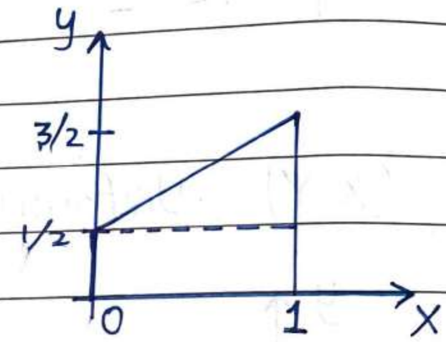
$$f_X(x) = \int_{y=0}^{2-x} \tfrac{1}{2}\,dy = 1 - \frac{x}{2} \quad 0<x<2$$

$$f_Y(y) = \int_{x=0}^{2-y} \tfrac{1}{2}\,dx = 1 - \frac{y}{2} \quad 0<y<2$$

Q. $f_{XY}(x,y) = \begin{cases} x+y, & 0<x,y<1 \\ 0, & \text{otherwise.} \end{cases}$ Find marginals.

$$f_X(x) = \int_{y=0}^{1} (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^1 = x + 1/2,\ 0<x<1$$

$$f_Y(y) = y + 1/2,\ 0<y<1.$$



★ Independence

(X,Y) with joint density $f_{XY}(x,y)$ are independent if $f_{XY}(x,y) = f_X(x)\,f_Y(y)$.

i.e $f_{XY}(x,y) = \underbrace{f_X(x)\,f_Y(y)}_{\text{Marginals}}$

- If independent, the marginals determine the joint density.

Eg. ✓ Uniform on unit square $\rightsquigarrow$ independent

✗ $(X,Y) \sim Unif(D)$ where $D = [0,\frac{1}{2}] \times [0,\frac{1}{2}] \cup [\frac{1}{2},1] \times [\frac{1}{2},1]$ NO

✓ $(X,Y) \sim Unif(D)$, $D = [1,3] \times [0,4]$ $\rightsquigarrow$ independent

✗ $(X,Y) \sim Unif(D)$, $D = [0,1] \times [0,1] \cup [1,2] \times [0,2]$ NO

✗ $f_{XY}(x,y) = \begin{cases} x+y, & 0<x,y<1 \\ 0, & \text{otherwise} \end{cases}$ NO

You can actually figure out from support (X,Y) itself.

Q. Suppose $X \sim Exp(\lambda_1)$, $Y \sim Exp(\lambda_2)$ are independent random variables. Find joint density, compute $P(X>Y)$

$$f_X(x) = \lambda_1 e^{-\lambda_1 x}, \quad x>0 \qquad f_{XY}(x,y) = \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y}$$
$$f_Y(y) = \lambda_2 e^{-\lambda_2 y}, \quad y>0 \qquad x,y>0.$$

$$P(X>Y) = \int_0^\infty \left(\int_{y=0}^{\infty} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dy \right) dx$$

$$= \int_{x=0}^{\infty} \lambda_1 e^{-\lambda_1 x} \left(-e^{-\lambda_2 y} \Big|_0^x \right) dx$$

$$= \boxed{1 - \frac{\lambda_1}{\lambda_1+\lambda_2}} = \boxed{\frac{\lambda_2}{\lambda_1+\lambda_2}}$$

DATE

★ Conditional Density

Let (X,Y) be random variables with joint density $f_{XY}(x,y)$. Let $f_X(x)$ and $f_Y(y)$ be marginals.

- For a s.t $f_X(a) > 0$, the conditional density of Y given $X = a$, denoted $f_{Y|X=a}(y)$ is:→

$$f_{Y|X=a}(y) = \frac{f_{XY}(a,y)}{f_X(a)}$$

- For b such that $f_Y(b) > 0$, conditional density of X given $Y = a$, denoted $f_{X|Y=b}(x)$ is:→

$$f_{X|Y=b}(x) = \frac{f_{XY}(x,b)}{f_Y(b)}$$

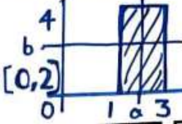
- Both conditional densities are valid densities in 1D. So the conditional RV's $(Y|X=a)$ and $(X|Y=b)$ are well defined.

- Joint = Marginal × Conditional, $x=a, y=b$ s.t. $f_X(a) > 0$, $f_Y(b) > 0$

$$\begin{aligned} f_{XY}(a,b) &= f_X(a)\, f_{Y|X=a}(b) \\ &= f_Y(b)\, f_{X|Y=b}(a) \end{aligned}$$

classmate

- $(X,Y) \sim \text{Unif}(D)$ where $D = [1,3]\times[0,4]$



$Y|X=a \sim \text{Uniform}[0,4]$
$X|Y=b \sim \text{Uniform}[1,3]$
$0<b<4,\ 1<a<3$

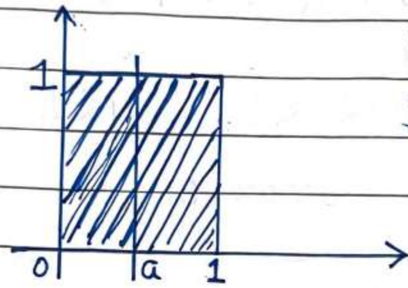
- $(X,Y) \sim \text{Unif}(D)$ where $D = [0,1]\times[0,1] \cup [1,2]\times[0,2]$

$Y|X=a \sim \text{Unif}[0,1]$
$0<a<1$
$1<a<2 \sim \text{Unif}[0,2]$

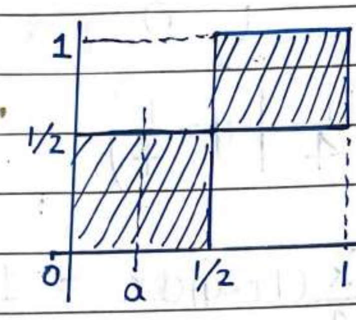
DATE

Eg. · Uniform on Unit^2 Square

·$(X,Y) \sim \text{Unif}(D)$ where: $\rightarrow$ $D = [0,0.5]\times[0,0.5] \cup [0.5,1]\times[0.5,1]$

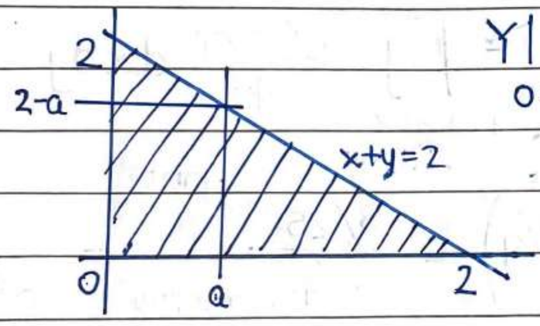


$Y|X=a \sim \text{Uniform}[0,1],\quad 0<a<1$
$X|Y=b \sim \text{Uniform}[0,1],\quad 0<b<1.$



$Y|X=a \sim \text{Unif}[0, 1/2]$
$0<a<1/2$
$\text{Unif}[1/2,1],\ 1/2<a<1$
$X|Y=b \sim \text{Unif}[0,1/2)$
$0<b<1/2$
$\sim \text{Unif}[1/2,1]$
$1/2<b<1$

Eg. $(X,Y) \sim \text{Unif}(D)$ where: $D = \{(x,y) : x+y<2,\ x,y>0\}$



$Y|X=a \sim \text{Uniform}[0, 2-a]$
$0<a<2$

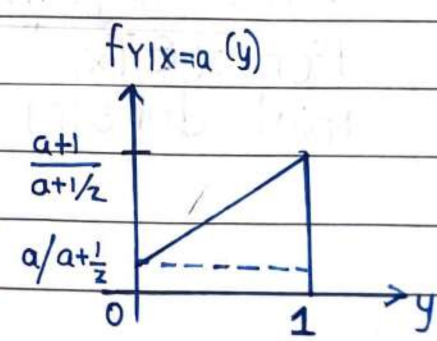
$$f_{Y|X=a}(y) = \begin{cases} 1/(2-a), & 0<y<2-a \\ 0 & \text{else.} \end{cases}$$

Eg. $f_{XY}(x,y) = \begin{cases} x+y & 0<x,y<1 \\ 0, & \text{otherwise.} \end{cases}$ Find conditionals.

$$f_X(x) = \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)\Big|_0^1 = x + \frac{1}{2},\quad 0<x<1$$

$$f_Y(y) = y + 1/2 \quad 0<y<1$$

$$f_{Y|X=a}(y) = \frac{a+y}{a+1/2},\quad 0<y<1$$
$0<a<1$



AQ 6.4 Question 2

Let the joint PDF of 2 RV's X and Y be:→

$$f_{XY}(x,y) = \begin{cases} \frac{x(1+2y)}{4} , & 0<x<2, 0<y<1 \\ 0 , & \text{otherwise} \end{cases}$$

$$P\left(\frac{1}{8} < X < \frac{1}{4} \,\middle|\, Y = \frac{1}{4}\right)$$

$$f_Y(y) = \int_0^2 \frac{x}{4}(1+2y)\,dx = \frac{1+2y}{2} , \ 0<y<1$$

For $y = 1/4$, $f_Y(1/4) = 3/4$

For $P(1/8<x<1/4, \ y=1/4) = \int_{y=0}^{1}\int_{x=1/8}^{x=1/4} \frac{x(1+2y)}{4}\,dx\,dy = \frac{3}{256}$

over the entire y range

And $\underbrace{P(1/8<x<1/4 \mid y=1/4)}_{\text{conditional}} = \frac{\overset{\text{joint}}{3/256}}{\underset{\text{marginal}}{3/4}} = \frac{1}{64}$

$\approx \underline{\underline{0.015}}$ ✓

Joint dist. with 1 discrete, 2 continuous variables?

- (X,Y,Z) : X discrete, (Y,Z) continuous
- X has range T_X and PMF p_X
- For $x \in T_X$, "conditional" joint density $f_{YZ|X=x}(y,z)$.
- Joint density $f_{YZ}(y,z) = \sum_{x \in T_X} p_X(x) f_{YZ|X=x}(y,z)$