Equity Factors

"Financial Intermediaries and the Cross-Section of Asset Returns" by Adrian 2018

- Definition of broker-dealer leverage
  - Fed publishes the quarterly Flow of Funds dataset which has a breakdown of assets and liabilities for each broker-dealer
  - Leverage = Total Financial Assets / [Total Financial Assets – Total Liabilities]
  - Define the leverage factor as the change in aggregated broker-dealer leverage with seasonal adjustments i.e. regress the time series of leverage change onto dummies e.g. months, quarters and use the residual instead
- Procyclical vs countercyclical
  - Good times is when business is active, funding is not tight, risk-on sentiment is prevailing and marginal value of wealth is low while bad time is the opposite.
  - Procyclical is when something increases during good times and decreases during bad times e.g. stock market, inflation, wages
  - countercyclical is when something decreases during good times and increases during bad times e.g. volatility, gold price
- The broker-dealer leverage factor is procyclical
  - Correlation shows broker-dealers tend to deleverage when asset growth is low, when volatility is high, when credit spread is wide, and when financial sector stock return is low.
  - Household leverage is countercyclical because when asset growth is high your leverage will mechanically drop, but broker-dealer leverage is the opposite which means they are actively managing their balance sheets.
- Test the risk premium of broker-dealer leverage factor
  - First step is to define assets as 41 test portfolios including each quintile from double sort between size and BM, each quintile of MOM, and government bond with different maturities.
  - Second step is to estimate historical betas using time-series regression of asset return onto leverage change
  - Third step is to estimate the risk premium and unexplained cross-sectional alpha using this beta (for comparison do the same on factor models e.g. CAPM, FF3)
- Observations
  - Smaller cap and higher BM stocks have higher broker-dealer leverage beta.
  - Positive and significant risk premium for broker-dealer leverage factor.
  - The intercept from this single factor model is smaller and less significant than well-known factors model including CAPM, FF3, FFC4 and so on.
- Explanation
  - Changes in broker-dealer leverage represent shifts in risk appetites and funding constraints in the market
  - Stocks that have higher beta to this i.e. underperform when they deleverage due to risk-off and tight funding, are exposed to systematic risk and should be compensated a risk premium
  - Stocks that have lower beta to this i.e. can survive the risk-off and tight funding period are safer and thus should earn a smaller return.

"Replicating Anomalies" by Hou Xue Zhang 2018

- Empirically tested 400+ anomalies that have been published in the past, and concluded that 60% cannot be replicated

    - In particular, value anomalies and momentum anomalies replicate fairly okay, but liquidity / market micro-structure anomalies have 90% failure rate.
- For each anomaly, report the expected return and its t-stat using below methods
    - Single Sort (NYSE cutoff, EW)
    - Single Sort (NYSE cutoff, VW)
    - Single Sort (All cutoff, EW)
    - Single Sort (All cutoff, VW)
    - Fama-Macbeth (OLS)
    - Fama-Macbeth (WLS using VW)
- A lot of the failures are due to control for micro-caps (they have tiny market share but 60% in terms of the number of stocks)
    - They look good on paper but the market friction for these are extremely high
    - Need to always check VW to make sure your new alpha is not there just because of these microcaps.

"Betting against beta" by Frazzini 2010

- First estimate beta for each stock using time-series regression of stock returns onto market returns
    - Tried a few beta specifications e.g. shrinkage
- Single Sort on beta and report return, CAPM alpha, FF3 alpha and other factor model alphas for each decile.
    - Return is pretty close which is consistent with the fact that the slope of market factor is pretty flat
    - CAPM alpha decrease monotonically from low beta to high beta which is called the "Low Beta Anomaly" and is due to people crowding into high beta stocks looking for leverage
- Created BAB factor
    - Long low beta and short high beta with higher weights on low beta portfolio and lower weights on high beta portfolios
    - In this case the entire portfolio is beta neutral so we are not paying the beta risk premium
    - Factor is statistically significant both standalone and within other factor models

Credit Factors

"Common Factors in Corporate Bond Returns" by Israel 2017

- Why does stock signals not directly translate into credit signals?
    - Given asset value change, they react different based on Merton model
    - Even if asset value does not change, some corporate actions such as leveraged buyout can benefit equity holders at the expense of debt holders
    - Stock and bond markets are segmented and run by two different sets of traders
- Tested a bunch of factors
    - Do not test size factor because corps are already very illiquid
    - Carry
        - BAML OAS (not using yield because do not want to include treasury component)
    - Defensive
        - Market leverage (net debt divided by net debt plus market cap)
        - Gross profitability

        - Low duration
    - Momentum
        - Trailing 6-month bond excess return
        - Trailing 6-month equity momentum
    - Value (a cheap bond has high spread relative to default risk)
        - Residual from OAS regressed onto Merton default probability
        - Residual from OAS regressed onto duration, rating and excess return vol
    - Combined
        - Weight each factor using inverse of return vol
    - Combined long-only
        - Solve linear optimization problem (not using Mean-Variance because this can avoid estimating the covariance matrix)
        - Can be later compared vs benchmark to see the smart-beta effect
- Fama-Macbeth
    - Every regression controls for a bunch of control variables
        - Market Beta
            - Slope from trailing 12-month regression of excess return onto value-weighted BAML universe excess return
        - Rating
            - S&P issuer rating
        - Duration
            - Rates sensitivity with optionality adjustment
        - Age percent
            - Time since issue date divided by original maturity
    - Carry is not significant standalone but marginally significant in kitchen sink
    - Defensive is significant standalone but insignificant in kitchen sink which means it is probably already explained by value factor
    - Momentum significant both standalone and in kitchen sink
    - Value significant both standalone and in kitchen sink
- Portfolio Sort
    - In many equity factors, the L-S portfolio does not have exactly zero beta but the magnitude is usually moderate
        - SMB has positive beta
        - HML has negative beta
        - Betting-Against-Beta has negative beta
    - Many credit factors are very correlated with beta, so it is important to condition on beta in the portfolio sort
        - Use duration times spread (DTS) as proxy for beta
        - Standardize the factors within each beta quintile which is similar to double sort with beta using 先并集，再做差
        - Calculate excess return series of L-S portfolio
        - Calculate excess return series of the 5% ex-ante vol scaled L-S portfolio using trailing 24-month realized vol
- Factor Model Alpha
    - Typical next step is to regress the vol scaled L-S portfolio of each factor onto famous factor model and see if the alpha is still significant.
        - Can do rolling regressions and plot time series of alpha and t-stat
    - Here we don’t have well-established credit factor model, so regress onto
        - Equity Risk Premium
        - Bond Risk Premium

        - Credit Risk Premium
        - SMB
        - HML
        - UMD
        - QMJ
    - All factors except for Carry remain robust after controlling for these risk factors, while Carry has strong positive correlation with CRP
    - Equity risk factors tend to be insignificant which means even for the same style e.g. value, the source of premium in corps is different from the source of premium in equities
- Explanation
    - Is it risk premium?
        - Regress factor return onto macro variables (VIX, CPI etc) and find the intercept remains robust for all factors except for Carry.
        - Regress factor return onto change in broker-dealer leverage and find Value makes money when leverage decreases (bad times).
    - Is it mispricing?
        - Calculate average analyst coverage, issue size, institutional ownership, and shorting cost for all quintiles and for each factor, and did not find enough evidence that the top and bottom quintile are sparse-covered, illiquid and limit-to-arb names.
        - Divide the universe into Low, Medium, High based on analyst coverage, issue size, institutional ownership and shorting cost. Calculate L-S factor returns within each sub-universe and find only momentum performs better in the less efficient sub-universe while other factors are inconclusive.
        - For each factor, plot the next 12-month cumulative EPS revisions for top quintile and for bottom quintile to see if top quintile tends to revise upward and bottom quintile tends to revise downward.
- Conclusion
    - Significant excess returns and alphas for Carry, Defensive, Value, and Momentum in both long-short and long-only settings.
    - Not driven by risk-premium and most except for momentum not driven by mispricing as well.

"Systematic Credit Investing" by Frieda 2015

- Decompose credit returns
    - Can be decomposed into rates return (duration matched treasury) and credit excess return (total return – rates return)
    - Credit excess return can be further decomposed into carry (spread at the beginning of period) and spread change (difference between end and beginning)
    - Pure credit play can be done through either treasury hedged corporate bonds or outright CDS.
- Cross-sectional credit plays
    - Tested value, momentum, carry and defensive on corporate bonds (similar results as Israel 2018)
    - Combine them together into cross-sectional CDS strategy and compared against passive risk premium, timing, and so on.

"Value Investing in Credit Markets" by Correia in 2012

- P default probability models

  - Shumway 2008 Merton EDF (primary model in this paper)
    - Single factor
  - Beaver 2012 combined
    - Accounting + market factors
  - Shumway 2008 combined
    - Naïve Merton EDF + accounting + market factors
  - Moody KMV
    - Single factor
- Why convert from P default probability into Q default probability requires adding a risk premium?
  - Suppose real-world default frequency is 2% but people require 1% extra return to bear this risk, then the contingent claim will be priced at 3%
  - P default probability is the real-world default frequency i.e. 2%
  - Q default probability is what makes the contingent claim matches market price i.e. 3%
- Convert P default probability into CDS
  - First step is to calculate P default probability
    - Calculate Distance-To-Default not using Merton 1974 DD (Appendix A) as did in Shumway 2008, but instead use driftless DD (Appendix B)
      - Driftless DD = ln(V / F) / [sigma_V * sqrt(T)]
      - F is book value of short-term + 0.5 * long-term
      - V is the market value of equity + book value of debt
      - Sigma_V = sigma_E / (E + F) * E is the deleveraged equity vol
    - Then calculate Merton EDF = N(-Driftless DD)
  - Second step is to calculate cumulative P default probability
    - Assume flat term structure of P defaults
    - CPD = 1 – (1 – PD)^T
  - Third step is to convert cumulative P default probability into cumulative Q default probability
    - CQD = N[N^(-1)[CPD] + lambda * sqrt(r^2) * sqrt(T)] where the second term corresponds to the risk premium to go from P to Q
    - N is the cumulative normal density function
    - Lambda = 0.5 is market Sharpe ratio
    - r = 0.4 is the correlation between asset return market return
    - T is the duration
  - Forth step is to convert cumulative Q default probability into CDS
    - PV = e^[-r * T] * [1 * (1 – CQD) + R * CQD]
    - PV = e^[-(r + CDS) * T]
    - Equating the above gives CDS = -1/T * ln[1 – (1 – R) * CQD]
    - Note that if you want to convert to a 5-year spread then you need a 5-year CQD.
  - Fifth step is to construct value signal
    - CRV = ln(market CDS / model CDS)
    - The higher the CRV, the more the default risk is overstated and thus the bond is undervalued
- Results
  - P default probability
    - Time-series of cross-sectional average P default probability is comparable across the 4 models
    - Correlation is 60-70% across the 4 models

    - CDS
        - All 4 models have lower CDS than market CDS which is consistent with previous research
    - CRV
        - Run time-series regression of the change in CRV from t+k to t+k+1 for k=1, 2, … onto CRV at t for each bond, and then report cross-sectional distribution for the slope.
            - -10% mean-reversion with 5% Rsq for all 4 models and gradually become weaker as k increases
            - Not using Fama-Macbeth because LHS is not return
        - Run Fama-Macbeth regression of Lok 2011 corp bond return from t+k to t+k+1 onto CRV + equity factors at t
            - +5 t-stat with 6% Rsq for all 4 models and gradually become weaker as k increases
        - Single sort by CRV
            - All 4 models can achieve 1.5 Sharpe in the L-S portfolio
        - Run anomaly testing using time-series regression of L-S CRV return onto risk factor returns such as SMB, HML, MOM, dVIX etc
            - All 4 models have significant intercept
- Robustness Testing
    - Similar results between BAML 3-8 years universe with bond returns and 5-year CDS universe with CDS returns (both using Lok 2011 method)
    - Similar results if switching from Lok 2011 approximation to actual BAML returns
    - Stronger results if we de-mean CRV and returns within industry group
    - Similar results if using the residual of CRV regressing onto market CDS to test whether the predictability is just coming from the carry

“初探量化可转债策略” by 国金证券 2022

- Convert market in China-A
    - $100 billion notional outstanding in 500 converts
    - Majority is in banking industry and mostly small-to-mid caps
- Funds specialized in converts
    - 纽达投资 (1.7 Sharpe)
    - 达仁投资 (1.1 Sharpe)
    - 上海悬铃 (2.3 Sharpe)
- Popular strategies
    - Fundamental
        - Treat converts as stock
        - Even better because harder to trigger circuit break
    - Quant – Factor investing
        - Equity – High Net profit growth
        - Equity – Low PE Ratio
        - Equity – Small MktCap
        - Equity – High Momentum
        - Converts - Low Price
            - 95% of converts with end up being converted into shares
            - Assuming that is true, the cheaper you buy the more you make
        - Converts - Low Conversion Premium
            - Same as Low Price

        - Converts - High Notional Outstanding
            - Institutions tend to buy more liquid converts
        - Converts - Low Implied Vol Minus Realized Vol
            - Note this is the convert implied vol because there is no single name option
            - Lower vol spread means convert is undervalued
    - Quant – Option Pricing
        - Buy a basket of undervalued converts and hedge with index futures

Machine Learning

"Machine Learning and the Cross-Section of Emerging Market Corporate Bond Returns" by Mansouri 2023

- Survey past research on credit factors
    - MOM
        - Most found MOM in both IG and HY, but some argue IG tends to have no pattern or even reversal
        - Some show FX MOM can be used for EM bonds
    - Value
        - Classic paper is "Value Investing In Credit" by Correia in 2011 which uses Merton default probability to predict fair value spread
        - Most papers construct value through fair value regression of market spread onto predictors e.g. rating, leverage, vol
    - Carry
        - Less research than other factors and the most prominent is "Common Factors in Corporate Bond Returns" by Israel 2017
    - Defensive
        - Most papers show low duration bonds tend to outperform
    - Macroeconomic
        - A lot of studies on treasury term structure and other economic variables
        - Easier to look at the three-factor model in "Bond Return Predictability" by Gargano in 2019
- The use of ML in finance
    - Start with paper "Empirical Asset Pricing via Machine Learning" by Gu in 2020 which applies ML to stock market
    - A lot of later papers apply the method to credit and other markets e.g. "Predicting Corporate Bond Returns: Merton Meets Machine Learning" by Bali in 2020
- Method
    - Universe
        - EM corporate bond index constituents from 2008 to 2023
    - Features
        - First calculate multiple metrics for each of the above style categories which is total 73 features (Table 5),
        - Then calculate interaction variable of these features with rating and macroeconomic variables to obtain 1000 features.
    - Train-test split
        - Split the dataset into 3 periods i.e. first 3 year is for training, next 2 year is for validation and remaining is for testing
    - Model

    - Regularized linear models i.e. Lasso, Ridge, Enet
    - Dimension reduction models i.e. PCA, PLS
    - Regression trees i.e. RF, GBRT
  - Evaluation
    - Use OOS Rsq as the primary metric
    - Can aggregate returns into different length e.g. look at OOS Rsq of 3-month return vs 12-month returns
  - Backtesting
    - Run ML model to get forecast every month and solve optimization problem with a bunch of constraints e.g. long-only, turnover, rating restrictions etc
    - Do this for each ML model to get a bunch of backtests
- Empirical Results
  - Top 10 features based on reduction in OOS Rsq (set the feature to have zero values for all data points and see how much Rsq is lost) for each model (Fig 2)
  - Feature importance color map for each feature and for each model (Fig 3)
  - Feature importance color map for each category of features and for each model (Fig 4)
  - Cumulative P&L curve for each ML model (Fig 5)
  - Table of OOS Rsq, return, vol, SR, MDD for each model (Table 8, 9)
- Conclusions
  - OLS based portfolio do not beat value weighted benchmark, but complex ML models such as RF and GBRT beat the benchmark and can achieve 2.0 SR

Convert Pricing

“Valuing Convertible Bonds as Derivatives” by Bardhan 1994

- Goldman Sachs model summary
  - Build CRR stock price binomial tree and make sure the average stock price path matches the stock forward
  - At maturity, calculate payoff and conversion probability of the convert
  - Back-propagate the conversion probabilities (using up-or-down probability weighted average and ignore discount rate) and calculate the discount rate at each node as weighted average of riskless rate and risky rate. Back-propagate the theoretical price as the coupon + continuation value.
  - Handle call, put, conversion and other events by blending the payoff upon event with coupon + continuation value. Also need to adjust the conversion probabilities e.g. set to 0 if put and set to 1 if converted.

“Calibration and Implementation of Convertible Bond Models” by Andersen 2002

- The Bloomberg OVCV jump diffusion model

“Valuing American Options by Simulation: A Simple Least-Squares Approach” by Longstaff & Schwartz 2001

- Monte Carlo is good at handling path dependency because at time t you know the stock path between 0 and t (vs Tree or PDE which you do not know)
  - Easy handle CR adjustments, resets and so on.

- But Monte Carlo is bad at handling early exercise because at time t you do not know the continuation value i.e. the risk neutral expectation at t of payoff ay T (vs Tree or PDE which you do know)
    - Hard to handle events e.g. calls puts and so on.
    - Must note that the continuation of that single path is path-specific as opposed to the risk neutral expectation.
- Two ways to address early exercise which are least square (this paper) or parametric expression for early exercise boundaries.

"Simulation-based pricing of convertible bonds" by Ammann 2006

- Monte Carlo with parametric estimation (vs least square as in Longstaff and Schwartz) of early exercise boundaries
- Similar approach to "Convergence and biases of Monte Carlo estimates of American option prices using a parametric exercise rule" by Garcia 2003

"Forecasting Default with Merton Distance to Default Model" by Shumway 2008

- Two important equations from Merton 1974
    - E = V * N(d_1) – e^(-r * T) * F * N(d_2)
        - E is market value of equity
        - V is asset value of the firm
        - F is book value of the debt (estimated as current liabilities + 0.5 * long-term debt)
        - T is maturity of the debt
    - Sigma_E = (V / E) * N(d_1) * sigma_V
        - Sigma_E is equity vol (estimated as either realized in the previous year)
        - Sigma_V is asset vol
- Proxy for Merton default probability
    - Solve for V and sigma_V
        - Initialize sigma_V = sigma_E / (E + F) * E
        - Use the first equation and this initialization to calculate daily time series of V for the previous year
        - Calculate daily log return series of V for the previous year
        - Update the estimates for sigma_V and mu
        - Repeat until the update is less than 0.1 vol points
    - Calculate Merton Distance-To-Default
        - DD = [ln(V / F) + (mu – 0.5 * sigma_V^2) * T] / [sigma_V * sqrt(T)]
        - Derivation can be found in Appendix A
    - Calculate Expected-Default-Frequency
        - EDF = N(-DD)
- Key difference vs KMV
    - Firm model
        - Merton EDF is based on vanilla Merton 1974
        - KMV is based on KV model which is a generalized Merton 1974
    - Estimate book value of debt
        - Merton EDF look this up from public accounting data
        - KMV first makes proprietary adjustments to public accounting data and then calculate the book value of debt
    - How to convert from DD to default probability

        - Merton EDF uses inverse normal distribution
        - KMV uses proprietary database to estimate real-world distribution and use that to convert to default probability
- Conclusions
    - Merton EDF is as good as KMV for ranking but not for pricing purposes
        - Can achieve 70% rank correlation with KMV
        - Rsq from Log CDS regression onto Merton EDF is 20% which is significantly lower than 70% seen in same regression but with KMV
        - Slope from regression of CDS implied default probability onto Merton EDF is less than 1 which contradicts with default risk premium and suggests the magnitude in Merton EDF is off
    - Functional form is more important than ingredients
        - Feeding naïve inputs to Merton DD and EDF outperforms
    - Some parameters can slightly impact performance as well
        - Setting asset drift to risk-free rate underperforms
        - Using equity implied vol instead of equity realized vol outperforms

"An Examination of Corporate Call Policies on Convertible Securities" by Ingersoll 1977

- Whether to call the bond or not?
    - If equity is overvalued relative to debt, then should buy back the debt in exchange for shares.
        - This usually corresponds to Parity > Call Price + Accrued
    - If equity is undervalued relative to debt, then should buy back the shares in exchange for new debt issuance.
        - This usually corresponds to Parity < Call Price + Accrued
- Suppose issuer already decides to call, why he always prefers forced conversion?
    - One way is to force holders to convert
        - Only a paper transaction and no need to refinance
    - The other way is to pay call price
        - Need to raise the cash through new stock or debt issuance, which will incur underwriting fee
        - Or can hire market makers to buy all the outstanding bonds at a premium and let the market maker convert into stock, which will incur the premium at purchase
        - The underwriting fee from first method should be comparable to the premium from second method
    - So the issuer should prefer forced conversion to paying cash due to this transaction cost (either underwriting fee or premium)
- How do they make forced conversion happen?
    - If no notice period
        - Just need to leave a tiny cushion so that even the holders who prefer cash will be better off converting the stock and selling the stock with a commission fee
        - The tiny cushion should be comparable to this commission fee and is much smaller than 43.9% of call price observed in empirical data
    - With notice period
        - Need to have a much larger cushion to reduce the probability that stock will fall during the notice period
        - But leaving too large a margin has cost as well, which is to continue to pay the coupon for a longer time

- Optimal call policy
    - Parametrize call probability in stock price
        - When S < Effective Call Price / CR then assign to 0
        - When Effective Call Price / CR < S < S* then need to make it smooth
        - When S > S* assign to 1
    - How to determine S*
        - The method in paper is complex and requires solving PDE, so can apply the same intuition but use simpler method
        - The issuer wants to minimize Cost of Call = (1 – Prob Force Conv) * Transaction Cost + Expected Waiting Time * Coupon
        - Prob Force Conv is the probability the holder will convert at the end of notice period and can be approximated by N(d2)
        - Transaction Cost is either the underwriting fee or premium to purchase
        - Expected Waiting Time is expected time it takes for stock to rise from Effective Call Price / CR to S* and can be approximated by closed-form formula of first-passage time of a GBM
        - Intuition is that the issuer is standing at stock price = Effective Call Price / CR and trying to decide S*, if he set S* too high then he can guarantee the holder will convert and thus avoid paying transaction cost, but he risks the stock price will never get there so he has to keep paying coupon.

"To Call or Not to Call Convertible Debt" by Ederington 1997

- Notations
    - CV is conversion value
    - CP is effective call price which is call price + accrued
- Theories on optimal call decision
    - Issuer wants to destroy the embedded call option by calling as soon as possible
        - Brennan and Schwartz 1977 + Ingersoll 1977a
        - Should call immediately after CV > CP
    - Issuer prefers forced conversion so they delay the call until a safety premium has been reached to avoid stock price falling during notice period
        - Ingersoll 1977b + Asquith 1991 + Asquith 1995
        - Should call after CV > 1.2 * CP because investment bankers typically charge 20% premium
    - Investors interpret call as signal that stock will drop and the convertible bond will end up out of the money, so issuers delay calling to avoid sending this message
        - Investor's thought: if the convertible bond remains in the money until the maturity then they do not need to call it to force us to convert, and we will convert it optionally by ourselves
        - The fact that they call it now means they have private information that suggests the convertible bond will be out of the money at maturity, so they rush to call it now to avoid paying the face value at maturity
        - This can explain the typical stock drop after call announcements which can be caused partly by people expecting the dilution as well
    - Issuers have to pay bond holders coupon and pay stock holders dividend, so they delay call if coupon is less than dividend or they expect this in near future
        - Constantinides and Grundy 1987
    - Coupon are using pre-tax money and dividends are using after-tax money, so issuers delay call or avoid call to save tax
        - Mikkelson 1981 + Campbell 1991 + Asquith 1991

- Empirical results
    - Safety premium exists but 20% is not the consensus
        - Call is very rare when CV < 1.2 * CP but still not 100% after that
        - As you increase from 20% to 30% to 40% the probability keeps increasing
    - Yield advantage exists
        - Call is very rare if dividend > coupon
    - Best model is a combination of these but it still cannot predict call perfectly

"Determinants of corporate call policy of convertible bonds" by King 2009

- Factors that might impact optimal call policy
    - Call notice period
        - Ingersoll 1977b suggests the safety premium is needed
        - Should have larger safety premium if vol is higher
        - Should also have larger safety premium if liquidity is low and firm has difficulty raising enough cash to pay the redemption if stock falls e.g. see Emery 1989 and Shleifer 1990
    - Yield advantage
        - Asquith 1991 says lower call probability if dividend is more than coupon
    - Signaling
        - Harris 1985 says issuers with good private news will delay the call
    - Backdoor equity financing
        - Stein 1992 says a lot of firms want to issue stock but it is expensive and as a result they issue convertible bonds and plan to convert them into stocks as soon as they
        - In this case there should be little call delay
    - Event driven
        - If rating upgrade then issuer wants to call and issue another with lower coupon, if downgrade then it reduces call probability and increases call delay
- Empirical results
    - The median call delay after reaching a safety premium, among firms whose dividend is lower than after-tax coupon, is only 2 days

"Back to Basics: a new approach to the discrete dividend problem" by Haug 2003

- Three main ways to handle discrete dividends: Escrowed model, forward model and piecewise lognormal model
- The escrowed model assumes stock price less all future dividends follows a GBM, but in this case we need to adjust the vol
- This paper covers different vols to adjust the vol and found all of them can introduce bias

"Efficient Pricing of Derivatives on Assets with Discrete Dividend" by Vellekoop 2004

- Uses the piecewise lognormal model to handle discrete dividend (as opposed to the escrowed model in Haug 2003)
- Shows it does not have non-local dependencies (do not need to consider dividends paid after the maturity of the bond) and have no arbitrage unlike the Escrowed model.

Appendix A: Merton 1974 Distance-To-Default

To derive the **Merton Distance to Default (DD)** formula:

1. **Asset Dynamics**: Assume the firm's asset value V_t follows a geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma_V V_t dW_t.$$

2. **Log of Asset Value**: Using Itô's Lemma:

$$d\ln(V_t) = \left(\mu - \frac{1}{2}\sigma_V^2\right) dt + \sigma_V dW_t.$$

3. **Default Condition**: The firm defaults if $V_T \leq D$ (where D is the default threshold at time T).

   Taking the log of the default condition:

$$\ln(V_T) \leq \ln(D).$$

4. **Expected Value of $\ln(V_T)$**: From the dynamics of $\ln(V_t)$, the expected value at time T is:

$$\mathbb{E}[\ln(V_T)] = \ln(V_0) + \left(\mu - \frac{1}{2}\sigma_V^2\right) T.$$

5. **Standard Deviation of $\ln(V_T)$**:

$$\text{Volatility of } \ln(V_T) = \sigma_V \sqrt{T}.$$

6. **Z-Score (Distance to Default)**: Standardizing $\ln(V_T)$:

$$DD = \frac{\mathbb{E}[\ln(V_T)] - \ln(D)}{\sigma_V \sqrt{T}}.$$

7. **Substitute the Expected Value**:

$$DD = \frac{\ln(V_0) + \left(\mu - \frac{1}{2}\sigma_V^2\right) T - \ln(D)}{\sigma_V \sqrt{T}}.$$

This is the **Merton Distance to Default** formula:

$$DD = \frac{\ln\left(\frac{V_0}{D}\right) + \left(\mu - \frac{1}{2}\sigma_V^2\right) T}{\sigma_V \sqrt{T}}.$$

Appendix B: Driftless Distance-To-Default

If there is **no drift term in the** $\ln(V_t)$ SDE, the dynamics simplify to:

$$d\ln(V_t) = \sigma_V dW_t,$$

which implies $\ln(V_t)$ is purely stochastic with no deterministic drift. The derivation for the Distance to Default (DD) is as follows:

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1. **Asset Dynamics**: From the SDE:

$$\ln(V_t) = \ln(V_0) + \sigma_V W_t,$$

   where $\mathbb{E}[\ln(V_T)] = \ln(V_0)$ and $\mathrm{Std}[\ln(V_T)] = \sigma_V\sqrt{T}$.

2. **Default Condition**: The firm defaults if $\ln(V_T) \leq \ln(D)$.

3. **Distance to Default (DD)**: Standardizing $\ln(V_T)$:

$$DD = \frac{\mathbb{E}[\ln(V_T)] - \ln(D)}{\mathrm{Std}[\ln(V_T)]}.$$

   Substituting the values:

$$DD = \frac{\ln(V_0) - \ln(D)}{\sigma_V\sqrt{T}}.$$

Final Formula:

$$DD = \frac{\ln\left(\frac{V_0}{D}\right)}{\sigma_V\sqrt{T}}.$$