

# Foundations of Audio Signal Processing

## Assignment 6

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### Exercise 6.1

a.  $\hat{f}'(\omega) = 2\pi i\omega \hat{f}(\omega)$

Using the integration by parts:

$$u = e^{-2\pi i\omega t}$$

$$du = -2\pi i\omega e^{-2\pi i\omega t} dt$$

$$v = f(t)$$

$$dv = f'(t) dt$$

$$\begin{aligned}\hat{f}'(\omega) &= \int_{-\infty}^{\infty} f'(t) \cdot e^{-2\pi i\omega t} dt \\ &= [f(t) \cdot e^{-2\pi i\omega t}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -2\pi i\omega \cdot f(t) e^{-2\pi i\omega t} dt \\ &= [f(t) \cdot e^{-2\pi i\omega t}]_{-\infty}^{\infty} + 2\pi i\omega \int_{-\infty}^{\infty} f(t) e^{-2\pi i\omega t} dt\end{aligned}$$

Since  $f(t) \in L^2$  then:

$$= 2\pi i\omega \int_{-\infty}^{\infty} f(t) e^{-2\pi i\omega t} dt = 2\pi i\omega \hat{f}(\omega)$$

b.  $\hat{f}'(\omega) = -2\pi i\omega \hat{g}(\omega)$

$$\begin{aligned}\hat{f}'(\omega) &= \frac{d}{d\omega} \hat{f}(\omega) \\ &= \frac{d}{d\omega} \int_{-\infty}^{\infty} f(t) e^{-2\pi i\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{d}{d\omega} (f(t) e^{-2\pi i\omega t}) dt \\ &= \int_{-\infty}^{\infty} (-2\pi i t) f(t) e^{-2\pi i\omega t} dt \\ &= -2\pi i \int_{-\infty}^{\infty} t f(t) e^{-2\pi i\omega t} dt \\ &= -2\pi i \int_{-\infty}^{\infty} g(t) e^{-2\pi i\omega t} dt = -2\pi i\omega \hat{g}(\omega)\end{aligned}$$

c. We first prove that each function is composed by an even and odd part:

$$\begin{aligned}
 f(t) &= \frac{1}{2}(f(t) + f(t) + f(-t) - f(-t)) \\
 &= \frac{1}{2}(f(t) + f(-t)) + \frac{1}{2}(f(t) - f(-t)) \\
 &= f_e(t) + f_o(t)
 \end{aligned}$$

$f_e$  is even because:

$$\begin{aligned}
 f_e(t) &= f(t) + f(-t) \\
 &= f(-t) + f(t) \\
 &= f_e(-t)
 \end{aligned}$$

$f_o$  is odd because:

$$\begin{aligned}
 f_o(t) &= f(t) - f(-t) \\
 &= -(-f(t) + f(-t)) \\
 &= -f_o(-t)
 \end{aligned}$$

We now prove that the integral of the product of odd and even functions is 0:

$$\begin{aligned}
 \int_{-\infty}^{\infty} f_e(t)f_o(t)dt &= \int_{-\infty}^0 f_e(t)f_o(t)dt + \int_0^{\infty} f_e(t)f_o(t)dt \\
 &= \int_0^{\infty} f_e(-t)f_o(-t)dt + \int_0^{\infty} f_e(t)f_o(t)dt \\
 &= \int_0^{\infty} (f_e(-t)f_o(-t) + f_e(t)f_o(t))dt \\
 &= \int_0^{\infty} (f_e(-t)f_o(-t) - f_e(-t)f_o(-t))dt = 0
 \end{aligned}$$

Now we start proving that  $Re(\hat{f})$  is even and  $Im(\hat{f})$  is odd.

$$\begin{aligned}
 \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt \\
 &= \int_{-\infty}^{\infty} f(t)(\cos(2\pi\omega t) - i\sin(2\pi\omega t))dt \\
 &= \int_{-\infty}^{\infty} f(t)(\cos(2\pi\omega t))dt - \int_{-\infty}^{\infty} f(t)(i\sin(2\pi\omega t))dt
 \end{aligned}$$

Now we can substitute  $f(t)$  with  $f_e(t) + f_o(t)$  and we obtain:

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} (f_e(t) + f_o(t))(\cos(2\pi\omega t))dt - \int_{-\infty}^{\infty} (f_e(t) + f_o(t))(i\sin(2\pi\omega t))dt \\
 &= \int_{-\infty}^{\infty} f_e(t)(\cos 2\pi\omega t)dt + \int_{-\infty}^{\infty} f_o(t) \cos(2\pi\omega t)dt - \\
 &\quad \int_{-\infty}^{\infty} f_e(t)i\sin(2\pi\omega t)dt - \int_{-\infty}^{\infty} f_o(t)i\sin(2\pi\omega t)dt
 \end{aligned}$$

Since  $\cos$  is an even function and  $\sin$  is an odd function, then we can delete the two middle elements, since they are integrals of a product of an odd and an even function.

$$\begin{aligned} &= \int_{-\infty}^{\infty} f_e(t) \cos(2\pi\omega t) dt - \int_{-\infty}^{\infty} f_o(t) i \sin(2\pi\omega t) dt \\ &= \hat{f}_e(\omega) - \hat{f}_o(\omega) \end{aligned}$$

Since  $f(t)$  is real, then  $\hat{f}_e(\omega)$  is real too (because it does not contain any imaginary part) and  $\hat{f}_o(\omega)$  is imaginary because it has an imaginary component.

**d.** Assuming that  $f(\omega)$  is real and even, it holds that  $f(\omega) = f(-\omega)$ , and for  $\hat{f}$  to be even we need to prove that  $\hat{f}(\omega) = \hat{f}(-\omega)$ .

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt \\ &= \int_{-\infty}^{\infty} f(-t) e^{2\pi i \omega (-t)} dt \end{aligned}$$

We now substitute  $-t$  with  $u$ :

$$\begin{aligned} &= \int_{-\infty}^{\infty} f(u) e^{2\pi i \omega (u)} du \\ &= \int_{-\infty}^{\infty} f(u) e^{2\pi i (-\omega)(u)} du \\ &= \int_{-\infty}^{\infty} f(u) e^{-2\pi i (-\omega)(u)} du = \hat{f}(-\omega) \end{aligned}$$

We now have to prove that if  $f(\omega)$  is real and even,  $\hat{f}$  is real.

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) (\cos(2\pi\omega t) - i \sin(2\pi\omega t)) dt \\ &= \int_{-\infty}^{\infty} f(t) (\cos(2\pi\omega t)) dt \end{aligned}$$

which is real.

## Exercise 6.3

**a-b.** The solutions can be found inside the `code` folder.