Foundations of Audio Signal Processing
Assignment 3

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Exercise 3.1

a.

$$4 + i4\sqrt{3} = 4(1 + i\sqrt{3})$$

$a = 1$, $b = \sqrt{3}$

$$\begin{aligned} r &= \sqrt{1+3} = 2 \\ \cos\phi &= \frac{1}{2} \\ \sin\phi &= \frac{\sqrt{3}}{2} \\ \phi &= \frac{\pi}{3} \\ z &= 8e^{\frac{\pi i}{3}} \end{aligned}$$

b.

$$\begin{aligned} (-1 + i\sqrt{3})^4 &= (1 - i2\sqrt{3} - 3)^2 \\ &= (-2 - i2\sqrt{3})^2 \\ &= 4(1 + i2\sqrt{3} - 3) \\ &= 4(-2 + i2\sqrt{3}) \quad = -8 + i8\sqrt{3} \end{aligned}$$

$a = -8$, $b = 8\sqrt{3}$

$$\begin{aligned} r &= \sqrt{64 + 192} = 16 \\ \cos\phi &= -\frac{8}{16} = -\frac{1}{2} \\ \sin\phi &= \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2} \\ \phi &= \frac{2\pi}{3} \\ z &= 16e^{\frac{2\pi i}{3}} \end{aligned}$$

c. Here we use the solution from exercise b to solve the numerator.

$$
\begin{aligned}
\frac{(-1+i\sqrt{3})^4}{4+i4\sqrt{3}} &= \frac{-8+i8\sqrt{3}}{4+i4\sqrt{3}} \\
&= \frac{-2+i2\sqrt{3}}{1+i\sqrt{3}} \\
&= \frac{(-2+i2\sqrt{3})(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} \\
&= \frac{-2+i2\sqrt{3}+i2\sqrt{3}+6}{(1+3)} \\
&= \frac{4+i4\sqrt{3}}{4} & &= 1+i\sqrt{3}
\end{aligned}
$$

$a = 1$, $b = \sqrt{3}$, which are the same as in exercise a, and thus lead to same solution $z = 2e^{\frac{\pi i}{3}}$

d.

$$
\begin{aligned}
2e^{\frac{\pi}{2}i}(1+i) &= 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(1+i) \\
&= 2(0+i)(1+i) & &= -2+2i
\end{aligned}
$$

$a = -2$, $b = 2$

$$
\begin{aligned}
r &= \sqrt{4+4} = 2\sqrt{2} \\
\cos\phi &= -\frac{2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2} \\
\sin\phi &= \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\phi &= \frac{3\pi}{4} \\
z &= 2\sqrt{2}e^{\frac{3\pi i}{4}}
\end{aligned}
$$

Exercise 3.2

a. Figure 1 and 2 show the plots for $f_\omega(n) = e^{2\pi i\omega n}$ for $\omega = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}\}$. When $\omega \in \mathbb{Q}$, then this function corresponds to one that is evaluated in each of the k roots of unity, where $\omega = \frac{j}{k}$.

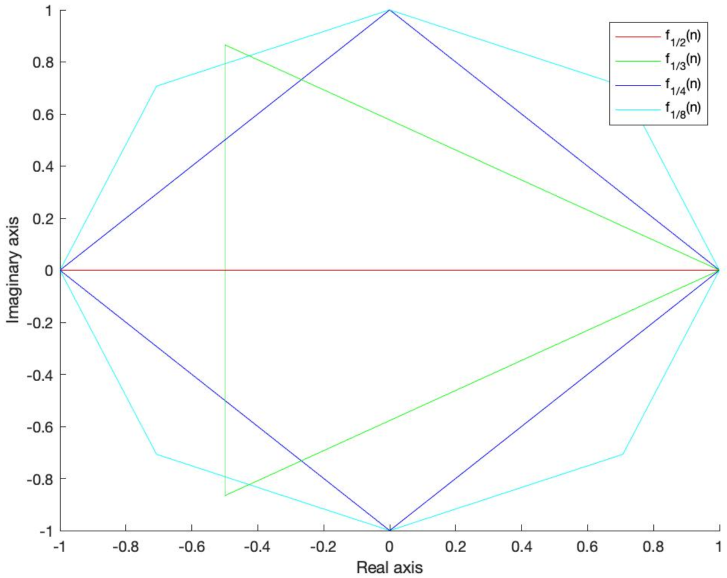

Figure 1: 2D plot of $f_\omega(n)$. It is possible to seen that each function is evaluated at the corresponding roots of unity.

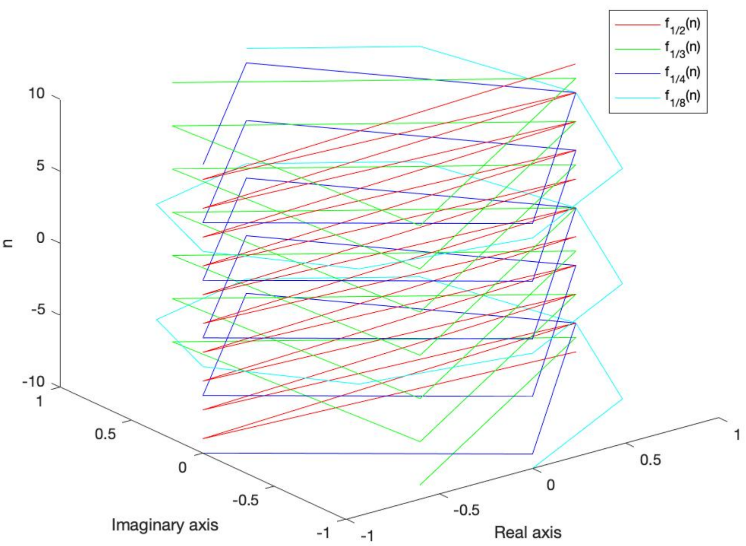

Figure 2: 3D plot of $f_\omega(n)$.

b A function $f(x)$ is said to be periodic if for a constant $D \in \mathbb{R}$ it follows that

$$f(x) = f(x+D), \qquad \forall x \in f$$

Then, for the concrete case of this exercise, $f_\omega(n)$ is periodic if:

$$f_\omega(n) = f_\omega(n+N), \qquad \forall n \in \mathbb{Z}$$

$$\implies e^{2\pi i\omega n} = e^{2\pi i\omega(n+N)}$$

$$e^{2\pi i\omega n} = e^{2\pi i\omega n} e^{2\pi i\omega N}$$

The latter implies that:

$$e^{2\pi i\omega N} = 1, \qquad \forall N \in \mathbb{Z}$$

Additionally, since $e^{2\pi ix} = 1, \qquad \forall x \in \mathbb{Z}$:

$$2\pi i\omega N = 2\pi ix$$

$$N\omega = x$$

$$\omega = \frac{x}{N}$$

Since x and $N \in \mathbb{Z}$, then ω has to be in $\mathbb{Q}$ so the function is periodic. $\square$

Exercise 3.3

a. Knowing that $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ and that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, then:

$$\begin{aligned}
\cos^3(x) &= \left(\frac{1}{2}\right)^3 (e^{ix} + e^{-ix})^3 \\
&= \frac{1}{8}\left(e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}\right) \\
&= \frac{1}{8}\left(e^{3ix} + e^{-3ix} + 3e^{ix} + 3e^{-ix}\right) \\
&= \frac{1}{8}\left(e^{3ix} + e^{-3ix}\right) + \frac{3}{8}\left(e^{ix} + e^{-ix}\right) \\
&= \frac{1}{4}\cdot\frac{1}{2}\left(e^{3ix} + e^{-ix3}\right) + \frac{3}{4}\cdot\frac{1}{2}\left(e^{ix} + e^{-ix}\right) \\
&= \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)
\end{aligned}$$

$\square$