Foundations of Audio Signal Processing
Assignment 5

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Exercise 5.1

A signal f is said to belong to the vector space $L^p(\mathbb{R})$ if $\|f\|_p < \infty$.
a. The $L^1(\mathbb{R})$ norm of f:

$$\begin{aligned}\|f\|_1 &= \int_0^1 |f(t)|dt \\ &= \int_0^1 \frac{1}{\sqrt{t}}dt \\ &= \left|2\sqrt{t}\right|_0^1 \\ &= 2-0 \qquad = 2\end{aligned}$$

So $f \in L^1(\mathbb{R})$.
The $L^2(\mathbb{R})$ norm of f:

$$\begin{aligned}\|f\|_2 &= \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 \frac{1}{t}dt\right)^{\frac{1}{2}} \\ &= \left(|\log(t)|_0^1\right)^{\frac{1}{2}} \\ &= (0+\infty)^{\frac{1}{2}} \qquad = \infty\end{aligned}$$

So $f \notin L^2(\mathbb{R})$.
Thus $f \in L^1(\mathbb{R}) \backslash L^2(\mathbb{R})$.

b. The $L^1(\mathbb{R})$ norm of g:

$$\begin{aligned}\|g\|_1 &= \int_1^\infty |g(t)|dt \\ &= \int_1^\infty \frac{1}{t}dt \\ &= |\log(t)|_1^\infty \\ &= \infty - 1 \qquad = \infty\end{aligned}$$

So $f \notin L^1(\mathbb{R})$.
The $L^2(\mathbb{R})$ norm of g:

$$\begin{aligned}
\|g\|_2 &= \left(\int_1^\infty |g(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \left(\int_1^\infty \frac{1}{t^2} dt\right)^{\frac{1}{2}} \\
&= \left(\left|-\frac{1}{t}\right|_1^\infty\right)^{\frac{1}{2}} \\
&= \left(-\frac{1}{\infty}+1\right)^{\frac{1}{2}} \qquad = 1
\end{aligned}$$

So $f \in L^2(\mathbb{R})$.
Thus $f \in L^2(\mathbb{R}) \backslash L^1(\mathbb{R})$.

Exercise 5.2

a. $x(n) = e^n$
According to Jensen's Inequality we have that

$$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^3(\mathbb{Z}) \subset ... \subset \ell^\infty(\mathbb{Z})$$

Since the ℓ^∞ norm is

$$\begin{aligned}
\|x(n)\|_\infty &= \sup_{n\in\mathbb{Z}} |x(n)| \\
&= \sup_{n\in\mathbb{Z}} e^n \qquad = \infty
\end{aligned}$$

and the series $\sum_{n\in\mathbb{Z}}^\infty x(n)$ diverges because of the geometric series convergence test, then we know that there is no $p \in [1, \infty]$ for which $x \in \ell^p(\mathbb{Z})$.

b. $x(n) = e^{2\pi i n}$
Because of what we proved in (a), it holds that there is no $p \in [1, \infty]$ for which $x \in \ell^p(\mathbb{Z})$. In fact, multiplying the value n by other values (even i) will not change the divergence of this series.

c. $x(n) = \frac{1}{\sqrt{n}}, n > 0$
We know that the $\|x(n)\|_2 = \frac{1}{n}$, which is divergent, so also $\|x(n)\|_1 = \frac{1}{\sqrt{n}}$ is. Instead $\|x(n)\|_3 = \frac{1}{n^{\frac{3}{2}}}$ converges, because the numerator is a constant (and so does not increase) while the denominator is an increasing function, which means that it will eventually converge. Thus, according to Jensen's inequality, for $p \in [3, \infty]$ it holds that $x \in \ell^p(\mathbb{Z})$.

Exercise 5.3

a-b. The solutions can be found inside the `code` folder.