Foundations of Audio Signal Processing
Assignment 2

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Exercise 2.1

a.

$$\begin{aligned}
2e^{\frac{\pi}{2}i}(1+i) &= 2(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))(1+i) \\
&= 2i(1+i) \\
&= -2+2i \\
a &= -2 \\
b &= 2 \\
r &= \sqrt{(-2)^2 + (2)^2} \\
&= 2\sqrt{2}
\end{aligned}$$

Knowing that $a = \cos(\phi)$ and $b = \sin(\phi)$:

$$\begin{aligned}
\cos(\phi) &= \frac{-2}{2\sqrt{2}} &&= -\frac{\sqrt{2}}{2} \\
\sin(\phi) &= \frac{2}{2\sqrt{2}} &&= \frac{\sqrt{2}}{2} \\
\phi &= \frac{3}{4}\pi \\
z &= 2\sqrt{2}e^{\frac{3}{4}\pi i}
\end{aligned}$$

b. Considering that $z = re^{\phi i}$, $\overline{z} = re^{-\phi i}$ and $|z| = r$

$$\begin{aligned}
z\overline{z} &= re^{\phi i}re^{-\phi i} \\
&= r^2e^0 &&= |z|^2
\end{aligned}$$

c.

$$\begin{aligned}
\frac{1}{2i}(e^{ia} - e^{-ia}) &= \frac{1}{2i}(\cos(a) + i\sin(a) - (\cos(-a) + i\sin(-a))) \\
&= \frac{1}{2i}(\cos(a) + i\sin(a) - \cos(a) + i\sin(a)) \\
&= \frac{1}{2i}(2i\sin(a)) &&= \sin(a)
\end{aligned}$$

Exercise 2.2

a. For $n = 4$:

$$\begin{aligned}
\Omega_4^1 &= \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i \\
\Omega_4^2 &= \cos(\pi) + i\sin(\pi) = -1 \\
\Omega_4^3 &= \cos(3\frac{\pi}{2}) + i\sin(3\frac{\pi}{2}) = -i \\
\Omega_4^4 &= 1
\end{aligned}$$

So the fourth roots of unity are $1, -1, i, -i$ and, since 4 is the smallest integer for which $\Omega = i, -i$, then these are its primitives.

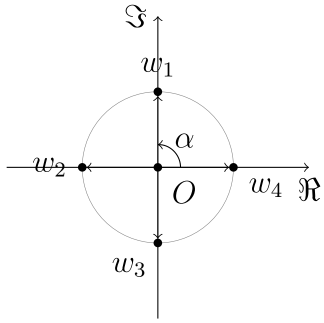

Figure 1: Illustration of all fourth roots of unity.

For $n = 6$:

$$\begin{aligned}
\Omega_6^1 &= \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}) = \frac{1}{2} + \frac{\sqrt{3}}{2}i \\
\Omega_6^2 &= \cos(2\frac{\pi}{3}) + i\sin(2\frac{\pi}{3}) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
\Omega_6^3 &= \cos(\pi) + i\sin(\pi) = -1 \\
\Omega_6^4 &= \cos(4\frac{\pi}{3}) + i\sin(4\frac{\pi}{3}) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\
\Omega_6^5 &= \cos(5\frac{\pi}{3}) + i\sin(5\frac{\pi}{3}) = \frac{1}{2} - \frac{\sqrt{3}}{2}i \\
\Omega_6^6 &= 1
\end{aligned}$$

So the sixth roots of unity are $1, -1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and, since 4 is the smallest integer for which $\Omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i$, then these are its primitives.

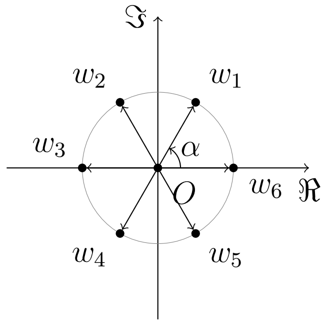

Figure 2: Illustration of all sixth roots of unity.

b. By definition of roots of unity

$$\sum_{k=0}^{n-1} \Omega_k = \omega^0 + \omega^1 + \cdots + \omega^{n-1}$$

where $\omega^k = e^{\frac{2k\pi i}{n}}$. Since $\omega^0 = 1$, we can write the definition as a finite geometric series

$$\sum_{k=0}^{n-1} e^{\frac{2k\pi i}{n}} = \frac{1 - e^{\frac{2n\pi i}{n}}}{1 - e^{\frac{2\pi i}{n}}}$$

since $e^{\frac{2n\pi i}{n}} = 1$, then

$$\sum_{k=0}^{n-1} \Omega_k = \frac{0}{1 - e^{\frac{2\pi i}{n}}} = 0$$

Exercise 2.3

a-c. The solutions can be found inside the `code` folder.

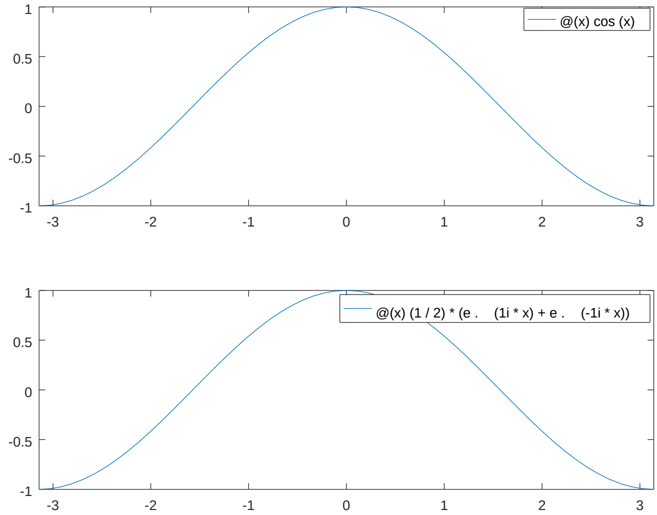

Figure 3: Visualization of the statement $\cos(\alpha) = \frac{1}{2} \cdot (e^{i\alpha} + e^{-i\alpha})$