Foundations of Audio Signal Processing
Assignment 6

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Exercise 6.1

a. $\hat{f}'(\omega) = 2\pi i\omega \hat{f}(\omega)$
Using the integration by parts:

$$\begin{aligned} u &= e^{-2\pi i\omega t} \\ du &= -2\pi i\omega e^{-2\pi i\omega t}dt \\ v &= f(t) \\ dv &= f'(t)dt \end{aligned}$$

$$\begin{aligned} \hat{f}'(\omega) &= \int_{-\infty}^{\infty} f'(t) \cdot e^{-2\pi i\omega t}dt \\ &= \left[f(t) \cdot e^{-2\pi i\omega t}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -2\pi i\omega \cdot f(t)e^{-2\pi i\omega t}dt \\ &= \left[f(t) \cdot e^{-2\pi i\omega t}\right]_{-\infty}^{\infty} + 2\pi i\omega \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt \end{aligned}$$

Since $f(t) \in L^2$ then:

$$= 2\pi i\omega \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt = 2\pi i\omega \hat{f}(\omega)$$

b. $\hat{f}'(\omega) = -2\pi i\hat{g}(\omega)$

$$\begin{aligned} \hat{f}'(\omega) &= \frac{d}{d\omega}\hat{f}(\omega) \\ &= \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt \\ &= \int_{-\infty}^{\infty} \frac{d}{d\omega}(f(t)e^{-2\pi i\omega t})dt \\ &= \int_{-\infty}^{\infty} (-2\pi it)f(t)e^{-2\pi i\omega t}dt \\ &= -2\pi i\int_{-\infty}^{\infty} tf(t)e^{-2\pi i\omega t}dt \\ &= -2\pi i\int_{-\infty}^{\infty} g(t)e^{-2\pi i\omega t}dt \quad = -2\pi i\hat{g}(\omega) \end{aligned}$$

c. We first prove that each function is composed by an even and odd part:

$$\begin{aligned}
f(t) &= \frac{1}{2}(f(t) + f(t) + f(-t) - f(-t)) \\
&= \frac{1}{2}(f(t) + f(-t)) + \frac{1}{2}(f(t) - f(-t)) \\
&= f_e(t) + f_o(t)
\end{aligned}$$

f_e is even because:

$$\begin{aligned}
f_e(t) &= f(t) + f(-t) \\
&= f(-t) + f(t) \\
&= f_e(-t)
\end{aligned}$$

f_o is odd because:

$$\begin{aligned}
f_o(t) &= f(t) - f(-t) \\
&= -(-f(t) + f(-t)) \\
&= -f_o(-t)
\end{aligned}$$

We now prove that the integral of the product of odd and even functions is 0:

$$\begin{aligned}
\int_{-\infty}^{\infty} f_e(t) f_o(t) dt &= \int_{-\infty}^{0} f_e(t) f_o(t) dt + \int_{0}^{\infty} f_e(t) f_o(t) dt \\
&= \int_{0}^{\infty} f_e(-t) f_o(-t) dt + \int_{0}^{\infty} f_e(t) f_o(t) dt \\
&= \int_{0}^{\infty} (f_e(-t) f_o(-t) + f_e(t) f_o(t)) dt \\
&= \int_{0}^{\infty} (f_e(-t) f_o(-t) - f_e(-t) f_o(-t)) dt \ = 0
\end{aligned}$$

Now we start proving that $Re(\hat{f})$ is even and $Re(\hat{f})$ is odd.

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt \\
&= \int_{-\infty}^{\infty} f(t)(\cos(2\pi\omega t) - i \sin(2\pi\omega t)) dt \\
&= \int_{-\infty}^{\infty} f(t)(\cos(2\pi\omega t)) - \int_{-\infty}^{\infty} f(t)(i \sin(2\pi\omega t)) dt
\end{aligned}$$

Now we can substitute $f(t)$ with $f_e(t) + f_o(t)$ and we obtain:

$$\begin{aligned}
&= \int_{-\infty}^{\infty} (f_e(t) + f_o(t))(\cos(2\pi\omega t)) - \int_{-\infty}^{\infty} (f_e(t) + f_o(t))(i \sin(2\pi\omega t)) dt \\
&= \int_{-\infty}^{\infty} f_e(t)(\cos 2\pi\omega t) dt + \int_{-\infty}^{\infty} f_o(t) \cos(2\pi\omega t) dt - \\
&\int_{-\infty}^{\infty} f_e(t) i \sin(2\pi\omega t) dt - \int_{-\infty}^{\infty} f_o(t) i \sin(2\pi\omega t) dt
\end{aligned}$$

Since cos is an even function and sin is an odd function, then we can delete the two middle elements, since they are integrals of a product of an odd and an even function.

$$\begin{aligned}&= \int_{-\infty}^{\infty} f_e(t)\cos(2\pi\omega t)dt - \int_{-\infty}^{\infty} f_o(t) i \sin(2\pi\omega t)dt\\&= \hat{f}_e(\omega) - \hat{f}_o(\omega)\end{aligned}$$

Since $f(t)$ is real, then $\hat{f}_e(\omega)$ is real too (because it does not contain any imaginary part) and $\hat{f}_o(\omega)$ is imaginary because it has an imaginary component.

d. Assuming that $f(\omega)$ is real and even, it holds that $f(\omega) = f(-\omega)$, and for $\hat{f}$ to be even we need to prove that $\hat{f}(\omega) = \hat{f}(-\omega)$.

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt\\&= \int_{-\infty}^{\infty} f(-t)e^{2\pi i\omega(-t)}dt\end{aligned}$$

We now substitute $-t$ with u:

$$\begin{aligned}&= \int_{-\infty}^{\infty} f(u)e^{2\pi i\omega(u)}du\\&= \int_{-\infty}^{\infty} f(u)e^{2\pi(-i)(-\omega)(u)}du\\&= \int_{-\infty}^{\infty} f(u)e^{-2\pi i(-\omega)(u)}du \ = \hat{f}(-\omega)\end{aligned}$$

We now have to prove that if $f(\omega)$ is real and even, $\hat{f}$ is real.

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt\\&= \int_{-\infty}^{\infty} f(t)(\cos(2\pi\omega t) - i\sin(2\pi\omega t))dt\\&= \int_{-\infty}^{\infty} f(t)(\cos(2\pi\omega t))\end{aligned}$$

which is real.

Exercise 6.2

a.

$$\langle f|A_k\rangle = \sqrt{2}\int_0^1 f(t)cos(2\pi kt)dt$$

$$= \sqrt{2}\left(\int_0^{1/2} -cos\,(t2\pi kt)\,dt + \int_{1/2}^1 cos\,(2\pi kt)\,dt \right)$$

$$= \sqrt{2}\left[\frac{-sin(2\pi kt)}{2\pi k}\Bigg|_0^{1/2} + \frac{sin(2\pi kt}{2\pi k}\Bigg|_{1/2}^1 \right]$$

$$= \sqrt{2}\left[\frac{sin(2\pi k)}{2\pi k} - \frac{sin(\pi k)}{\pi k} \right]$$

$$\langle f|B_k\rangle = \sqrt{2}\int_0^1 f(t)sin(2\pi kt)dt$$

$$= \sqrt{2}\left[\int_0^{1/2} -sin(2\pi kt)dt + \int_{1/2}^1 sin(2\pi k)dt \right]$$

$$= sqrt2\left[\frac{cos(2\pi k)}{2\pi k}\Bigg|_0^{1/2} - \frac{cos(2\pi k)}{2\pi k}\Bigg|_{1/2}^1 \right]$$

$$= \frac{\sqrt{2}}{nk}\left[\frac{cos(\pi k)}{2\pi k} - \frac{1}{2\pi k} - \frac{cos(2\pi k)}{2\pi k} + \frac{cos(\pi k)}{2\pi k} \right]$$

$$= \frac{\sqrt{2}}{\pi k}\left[cos(\pi k) - \frac{1}{2} - \frac{cos(2\pi k)}{2} \right]$$

b-c. The code can be found inside the `code` folder. The phenomenon that can be observed when increasing k is that it will create more *sin* and *cos* functions and also increase the frequency of them, so it will approximate to the original function better.