

# Foundations of Audio Signal Processing

## Assignment 3

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November 8, 2018

### Exercise 3.1

a.

$$4 + i4\sqrt{3} = 4(1 + i\sqrt{3})$$

$$a = 1, b = \sqrt{3}$$

$$r = \sqrt{1 + 3} = 2$$

$$\cos \phi = \frac{1}{2}$$

$$\sin \phi = \frac{\sqrt{3}}{2}$$

$$\phi = \frac{\pi}{3}$$

$$z = 8e^{\frac{\pi i}{3}}$$

b.

$$\begin{aligned} (-1 + i\sqrt{3})^4 &= (1 - i2\sqrt{3} - 3)^2 \\ &= (-2 - i2\sqrt{3})^2 \\ &= 4(1 + i2\sqrt{3} - 3) \\ &= 4(-2 + i2\sqrt{3}) = -8 + i8\sqrt{3} \end{aligned}$$

$$a = -8, b = 8\sqrt{3}$$

$$r = \sqrt{64 + 192} = 16$$

$$\cos \phi = -\frac{8}{16} = -\frac{1}{2}$$

$$\sin \phi = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2}$$

$$\phi = \frac{2\pi}{3}$$

$$z = 16e^{\frac{2\pi i}{3}}$$

c. Here we use the solution from exercise b to solve the numerator.

$$\begin{aligned}
 \frac{(-1 + i\sqrt{3})^4}{4 + i4\sqrt{3}} &= \frac{-8 + i8\sqrt{3}}{4 + i4\sqrt{3}} \\
 &= \frac{-2 + i2\sqrt{3}}{1 + i\sqrt{3}} \\
 &= \frac{(-2 + i2\sqrt{3})(1 - i\sqrt{3})}{(1 + i\sqrt{3})(1 - i\sqrt{3})} \\
 &= \frac{-2 + i2\sqrt{3} + i2\sqrt{3} + 6}{(1 + 3)} \\
 &= \frac{4 + i4\sqrt{3}}{4} = 1 + i\sqrt{3}
 \end{aligned}$$

$a = 1$ ,  $b = \sqrt{3}$ , which are the same as in exercise a, and thus lead to same solution  
 $z = 2e^{\frac{\pi i}{3}}$

d.

$$\begin{aligned}
 2e^{\frac{\pi}{2}i}(1 + i) &= 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})(1 + i) \\
 &= 2(0 + i)(1 + i) = -2 + 2i
 \end{aligned}$$

$a = -2$ ,  $b = 2$

$$\begin{aligned}
 r &= \sqrt{4 + 4} = 2\sqrt{2} \\
 \cos \phi &= -\frac{2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2} \\
 \sin \phi &= \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2} \\
 \phi &= \frac{3\pi}{4} \\
 z &= 2\sqrt{2}e^{\frac{3\pi i}{4}}
 \end{aligned}$$

## Exercise 3.2

a. Figure 1 and 2 show the plots for  $f_\omega(n) = e^{2\pi i \omega n}$  for  $\omega = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}\}$ . When  $\omega \in \mathbb{Q}$ , then this function corresponds to one that is evaluated in each of the  $k$  roots of unity, where  $\omega = \frac{j}{k}$ .

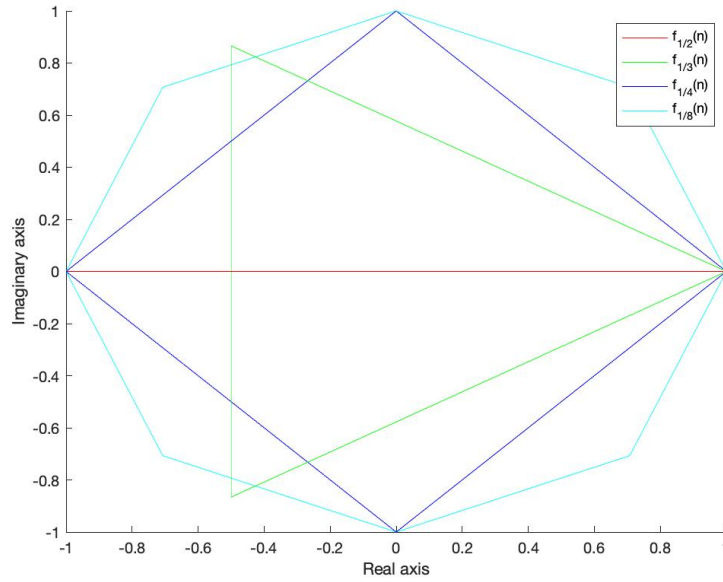


Figure 1: 2D plot of  $f_\omega(n)$ . It is possible to see that each function is evaluated at the corresponding roots of unity.

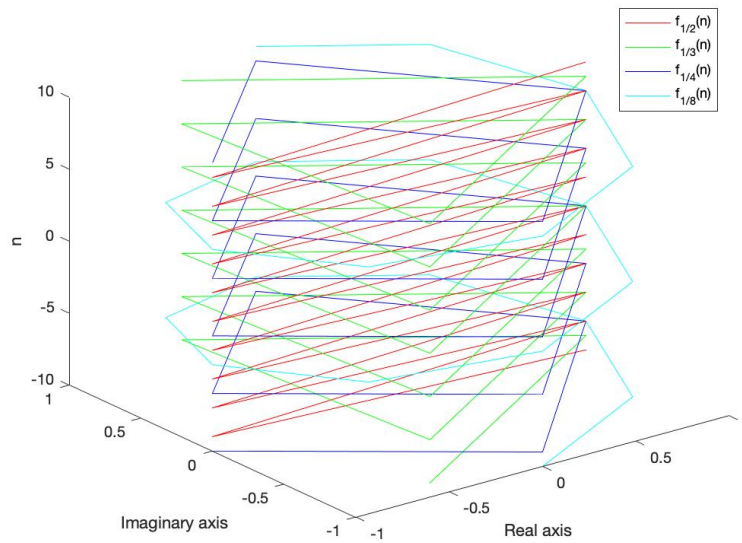


Figure 2: 3D plot of  $f_\omega(n)$ .

b. A function  $f(x)$  is said to be periodic if for a constant  $D \in \mathbb{R}$  it follows that

$$f(x) = f(x + D), \quad \forall x \in \mathbb{R}$$

Then, for the concrete case of this exercise,  $f_\omega(n)$  is periodic if:

$$f_\omega(n) = f_\omega(n + N), \quad \forall n \in \mathbb{Z}$$

$$\implies e^{2\pi i \omega n} = e^{2\pi i \omega (n+N)}$$

$$e^{2\pi i \omega n} = e^{2\pi i \omega n} e^{2\pi i \omega N}$$

The latter implies that:

$$e^{2\pi i \omega N} = 1, \quad \forall N \in \mathbb{Z}$$

Additionally, since  $e^{2\pi i x} = 1$ ,  $\forall x \in \mathbb{Z}$ :

$$2\pi i \omega N = 2\pi i x$$

$$N\omega = x$$

$$\omega = \frac{x}{N}$$

Since  $x$  and  $N \in \mathbb{Z}$ , then  $\omega$  has to be in  $\mathbb{Q}$  so the function is periodic.  $\square$

### Exercise 3.3

a. Knowing that  $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$  and that  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ , then:

$$\begin{aligned} \cos^3(x) &= \left(\frac{1}{2}\right)^3 (e^{ix} + e^{-ix})^3 \\ &= \frac{1}{8} (e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}) \\ &= \frac{1}{8} (e^{3ix} + e^{-3ix} + 3e^{ix} + 3e^{-ix}) \\ &= \frac{1}{8} (e^{3ix} + e^{-3ix}) + \frac{3}{8} (e^{ix} + e^{-ix}) \\ &= \frac{1}{4} \cdot \frac{1}{2} (e^{3ix} + e^{-ix3}) + \frac{3}{4} \cdot \frac{1}{2} (e^{ix} + e^{-ix}) \\ &= \frac{1}{4} \cos(3x) + \frac{3}{4} \cos(x) \end{aligned}$$

$\square$