Quantum Information and Computing 2021-2022

7^{th} week assignment
Transverse Field Ising Model
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- Given a system described by the wave-function

$$|\Psi\rangle = \sum_{\alpha_1,\dots,\alpha_N} C_{\alpha_1\dots\alpha_N} |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle ,$$

  the **mean-field approximation** consists of reducing the wave-function to a separable one

$$|\Psi_{MF}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots |\psi_N\rangle = \bigotimes_{i=1}^{N} \left(\sum_{j=1}^{d} A^{(i)}_{\alpha_j} |\alpha_j\rangle \right) = \bigotimes_{i=1}^{N} \left(\sum_{j=1}^{d} A_{\alpha_j} |\alpha_j\rangle \right) = \bigotimes_{i=1}^{N} |\psi\rangle ,$$

  where in the last two equalities we are imposing translational invariance to the system
- the transverse-field Ising model is described by the Hamiltonian

$$\hat{H} = -\sum_{i=1}^{N} \sigma_i^x \sigma_{i+1} + \lambda \sum_{i=1}^{N} \sigma_i^z , \qquad (1)$$

  where λ is the magnitude of the magnetic field applied along $\hat{z}$
- the notation used implies that

$$\sigma_i^x \sigma_{i+1} = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \sigma_i^x \otimes \sigma_{i+1} \otimes \mathbb{1}_{i+2} \cdots \otimes \mathbb{1}_N$$

$$\sigma_i^z = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \otimes \sigma_i^z \otimes \mathbb{1}_{i+1} \cdots \otimes \mathbb{1}_N$$

Theory: Mean-field Ising Model

- the Ising model Hamiltonian in 1 in the mean-field approximation for a translational invariant system is evaluated as:

$$E[\psi] = \langle \psi_M F| \hat{H} |\psi_M F\rangle = -\sum_{i=1}^{N} \left(\langle\psi| \sigma^x |\psi\rangle\right)^2 + \lambda \sum_{i=1}^{N} \langle\psi| \sigma^z |\psi\rangle$$

- the energy density is then

$$e[\psi] = \frac{E[\psi]}{N} \xrightarrow[N\to\infty]{} \left(\langle\psi| \sigma^x |\psi\rangle\right)^2 + \lambda \langle\psi| \sigma^z |\psi\rangle$$

- the **ground state** energy is obtained by the minimisation of $e[\psi]$
-

$$\min_{\psi} e[\psi] = \min_{\|r\|=1} \left(-r_x^2 + \lambda r_z\right),$$

with

- $r_z = tr(\rho\sigma^z) = \cos\theta, \quad r_x = tr(\rho\sigma^x) = \sin\theta$
- $r_x^2 + r_z^2 = 1$

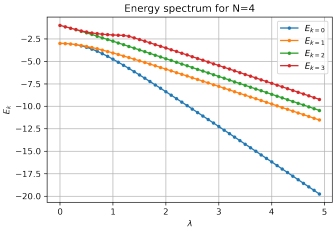
leading to

$$\min_{\psi} e[\psi] = min_\theta - \cos\theta^2 + \lambda\sin\theta = \begin{cases} e = -1 - \frac{\lambda^2}{4}, & \lambda \in [-2,2] \\ e = \quad \mod \lambda, & \lambda \notin [-2,2] \end{cases}$$

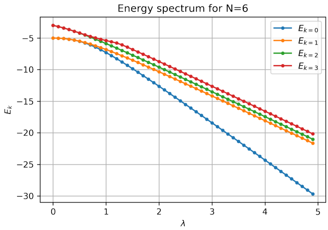
```fortran
function ising_ham (N, lambda) result(H)

   integer :: N, ii
   real(8) :: lambda
   double complex :: H(2**N, 2**N)

   do ii=1, N
       H = H + kronecker_product_cmplx(    &
               kronecker_product_cmplx(    &
               ID_ising( ii-1 ), s_z() ),  &
               ID_ising( N-ii ) )
   enddo

   H = lambda * H

   do ii=1, N

       H = H +     kronecker_product_cmplx(                       &
                   kronecker_product_cmplx( ID_ising( ii-1 ),     &
                   kronecker_product_cmplx( s_x(), s_x() ) )  ,   &
                   ID_ising( N -ii -1 ) )
   enddo


end function
```

Figure: Transverse Field Ising Hamiltonian initilisation function

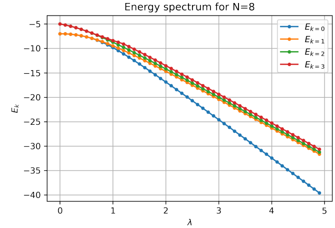
Results



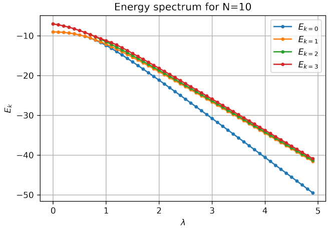
(a) N=4



(b) N=6



(c) N=8



(d) N=10

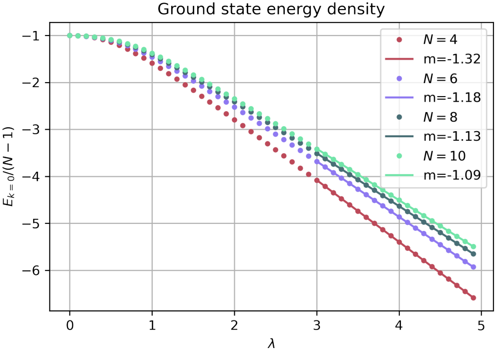


Figure: Linear fit on ground state energy density