A related experiment

- Consider a 'bin' with red and green marbles.

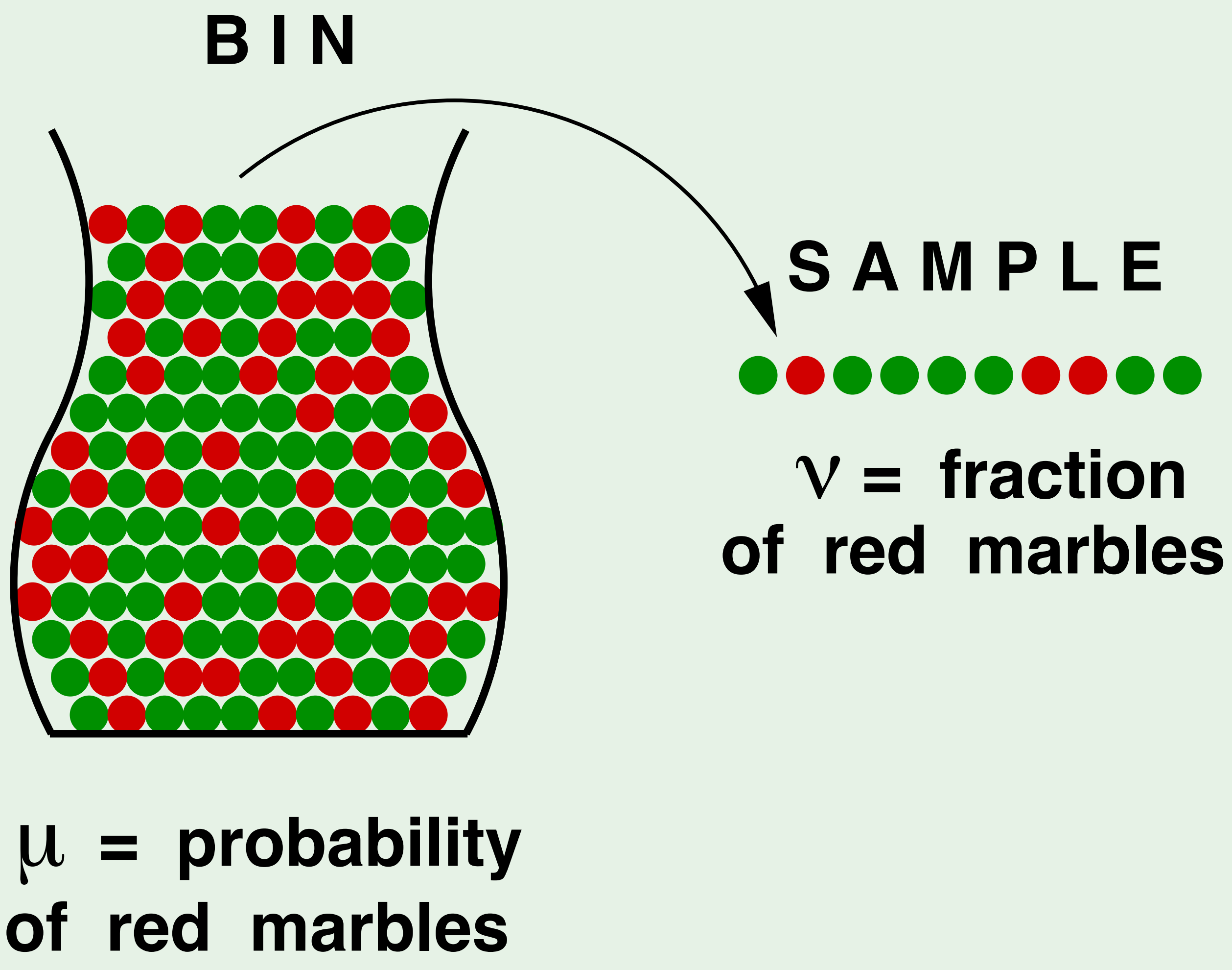
$$\mathbb{P}[\text{ picking a red marble }] = \mu$$

$$\mathbb{P}[\text{ picking a green marble }] = 1 - \mu$$

- The value of μ is unknown to us.

- We pick N marbles independently.

- The fraction of red marbles in sample $= \nu$



Does ν say anything about μ?

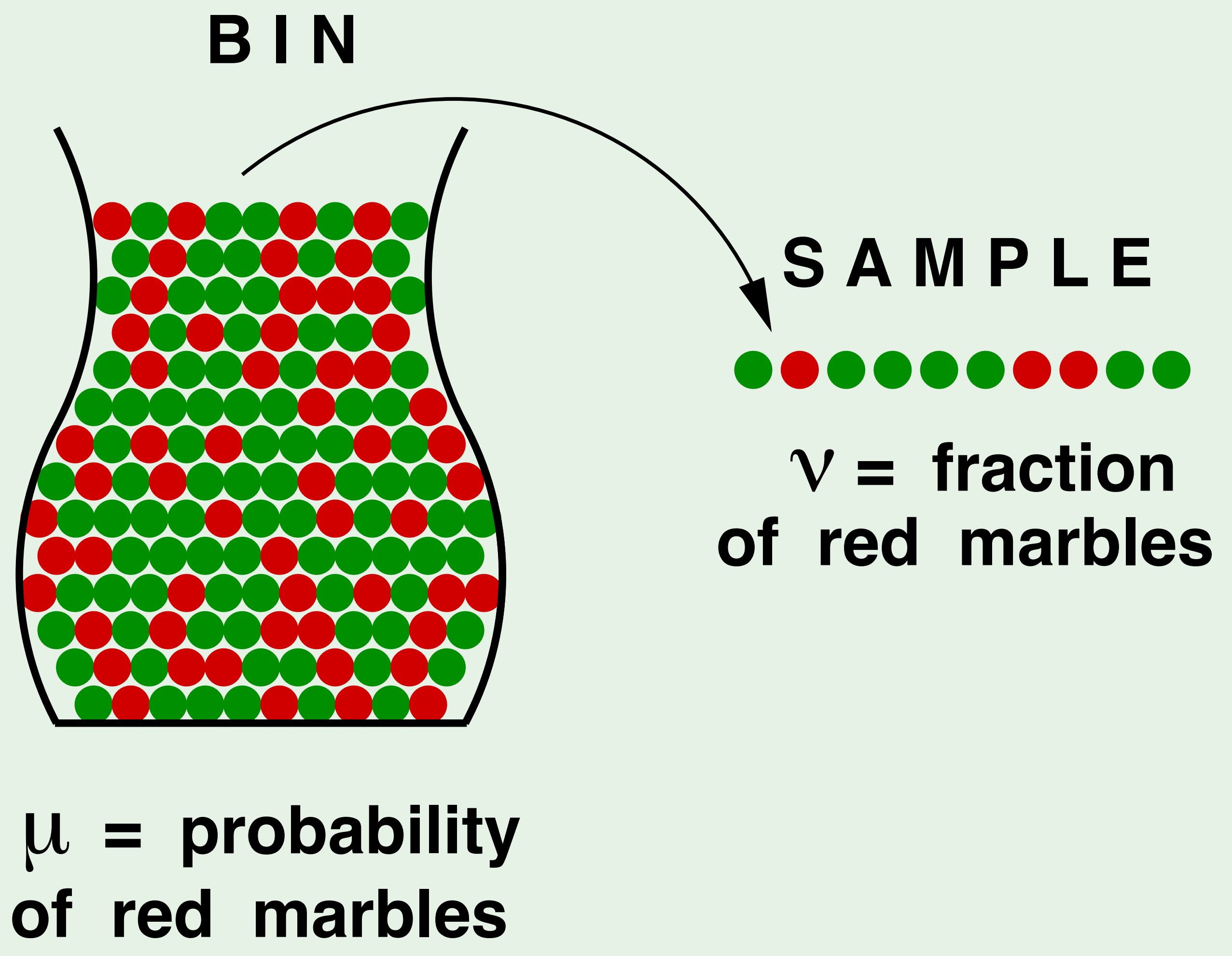
No!

Sample can be mostly green while bin is mostly red.

Yes!

Sample frequency ν is likely close to bin frequency μ.

possible versus **probable**



What <u>does</u> ν say about μ?

In a big sample (large N), ν is probably close to μ (within ϵ).

Formally,

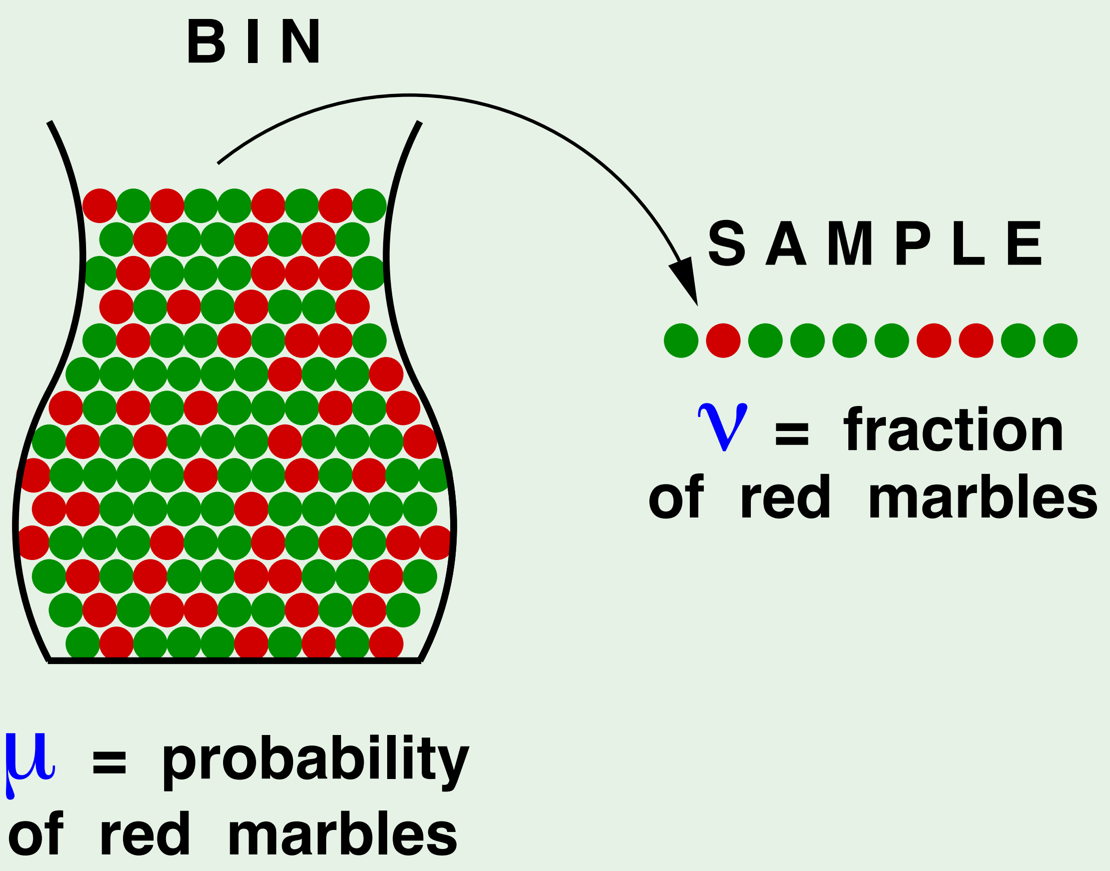
$$\mathbb{P}\left[\,|\nu - \mu| > \epsilon\,\right] \leq 2e^{-2\epsilon^2 N}$$

This is called **Hoeffding's Inequality**.

In other words, the statement "$\mu = \nu$" is P.A.C.

$$\mathbb{P}\left[\,|\nu - \mu| > \epsilon\,\right] \le 2e^{-2\epsilon^2 N}$$

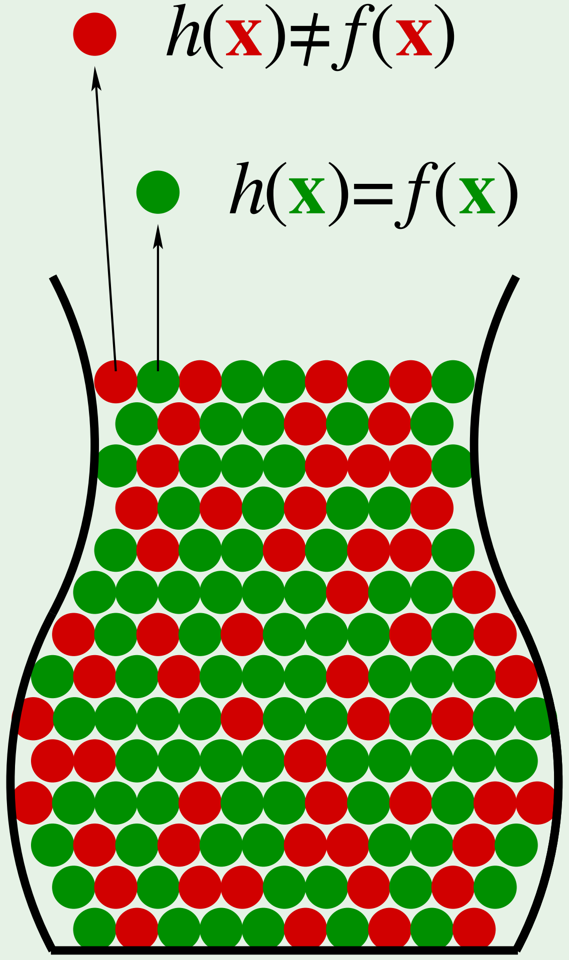
- Valid for all N and ϵ
- Bound does not depend on μ
- Tradeoff: N, ϵ, and the bound.
- $\nu \approx \mu \implies \mu \approx \nu$ ☺



Connection to learning

Bin: The unknown is a number μ

Learning: The unknown is a function $f : \mathcal{X} \to \mathcal{Y}$



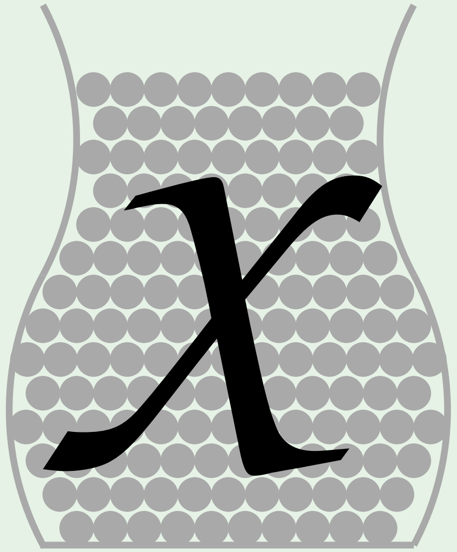
Each marble ● is a point $\mathbf{x} \in \mathcal{X}$

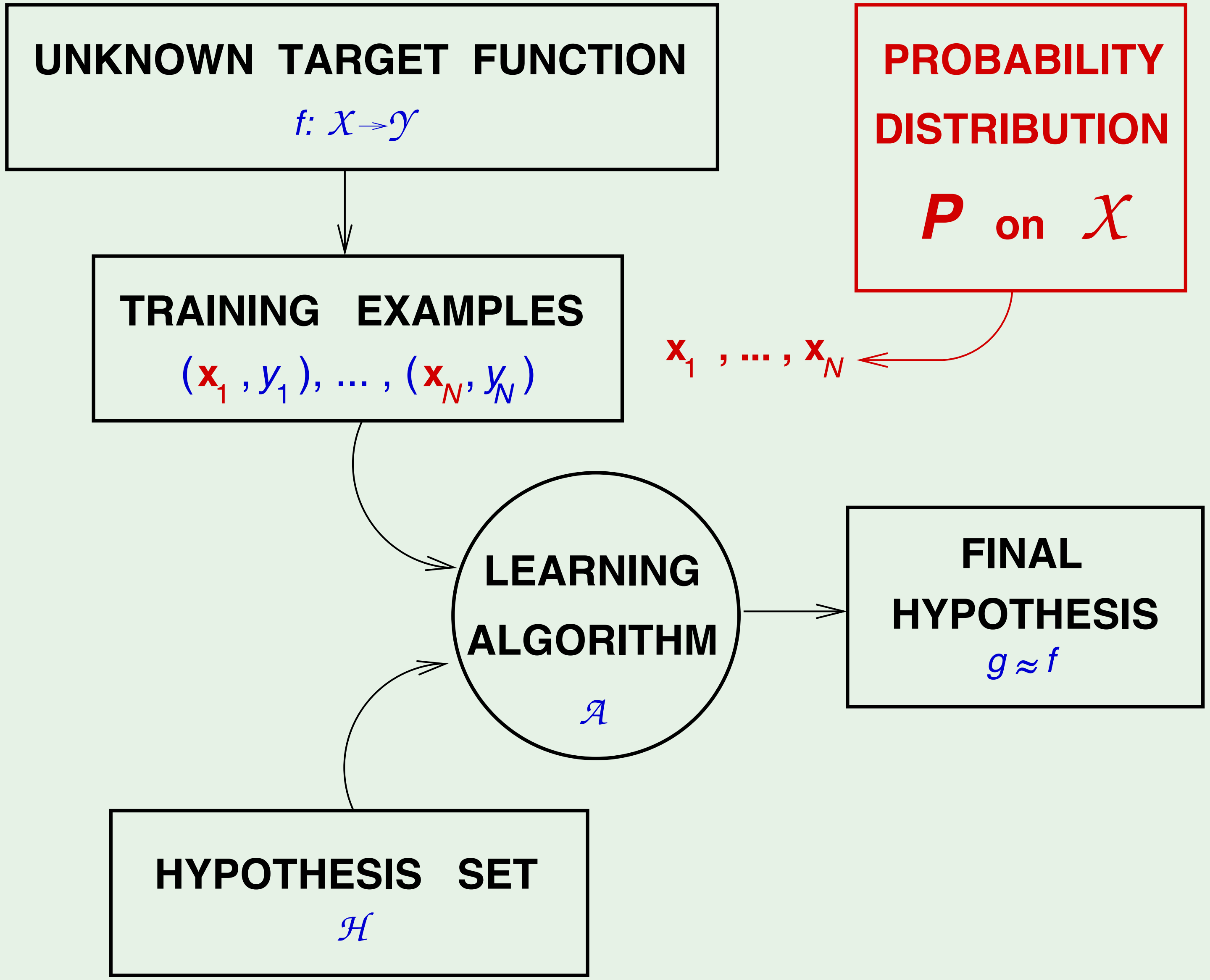
● : Hypothesis got it right $h(\mathbf{x}) = f(\mathbf{x})$

● : Hypothesis got it wrong $h(\mathbf{x}) \neq f(\mathbf{x})$

Back to the learning diagram

The bin analogy:





Are we done?

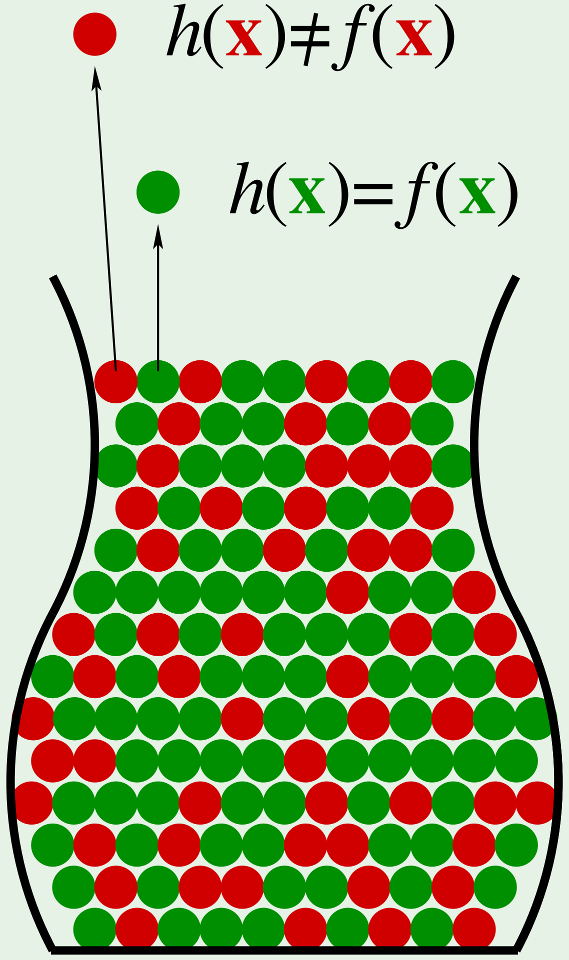
Not so fast! h is fixed.

For <u>this</u> h, ν generalizes to μ.

‘**verification**’ of h, not **learning**

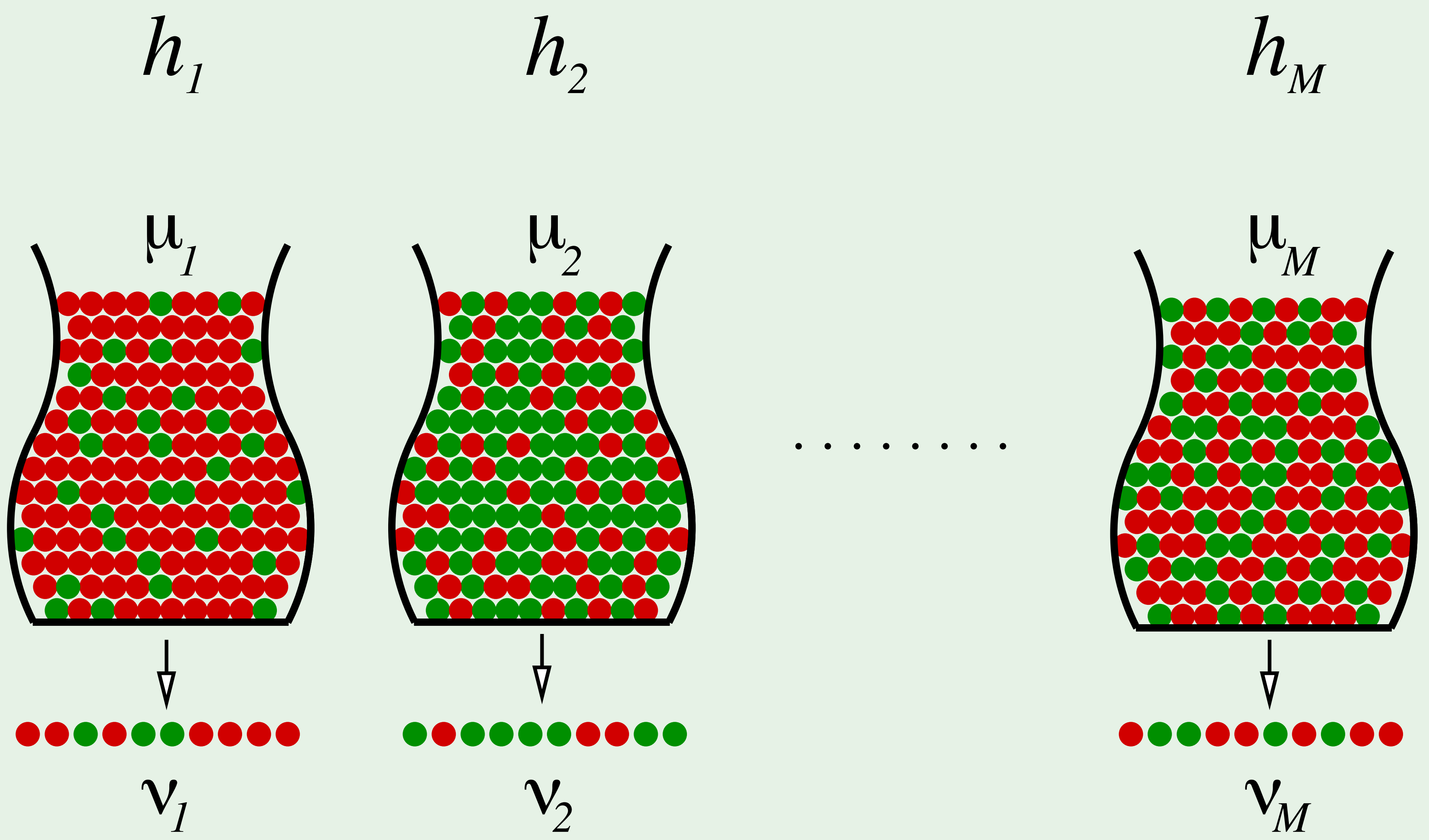
No guarantee ν will be small.

We need to **choose** from multiple h's.



Multiple bins

Generalizing the bin model to more than one hypothesis:



Notation for learning

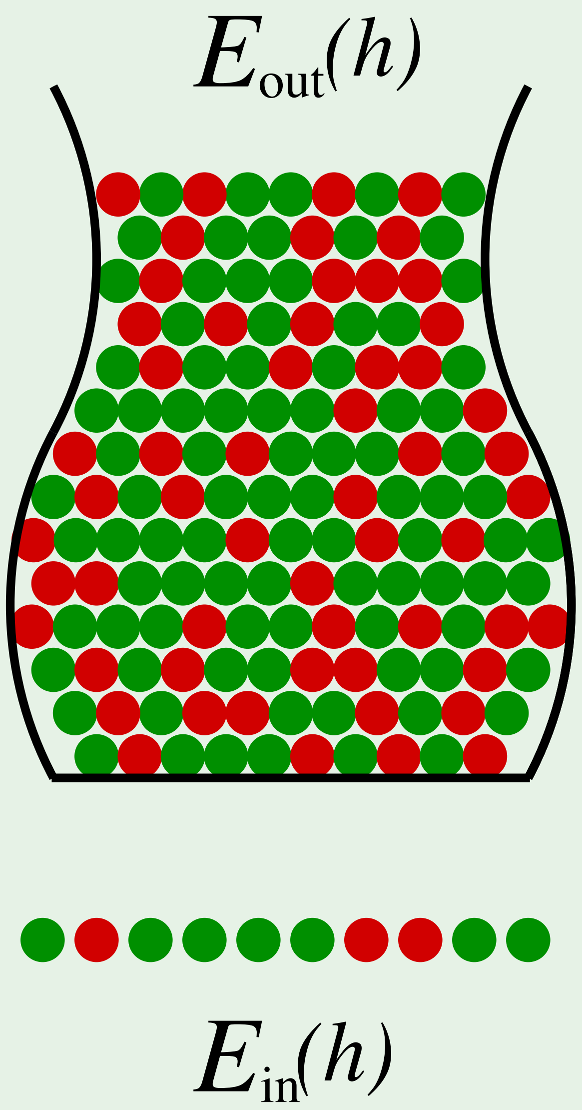
Both μ and ν depend on which hypothesis h

ν is '**in sample**' denoted by $E_{\text{in}}(h)$

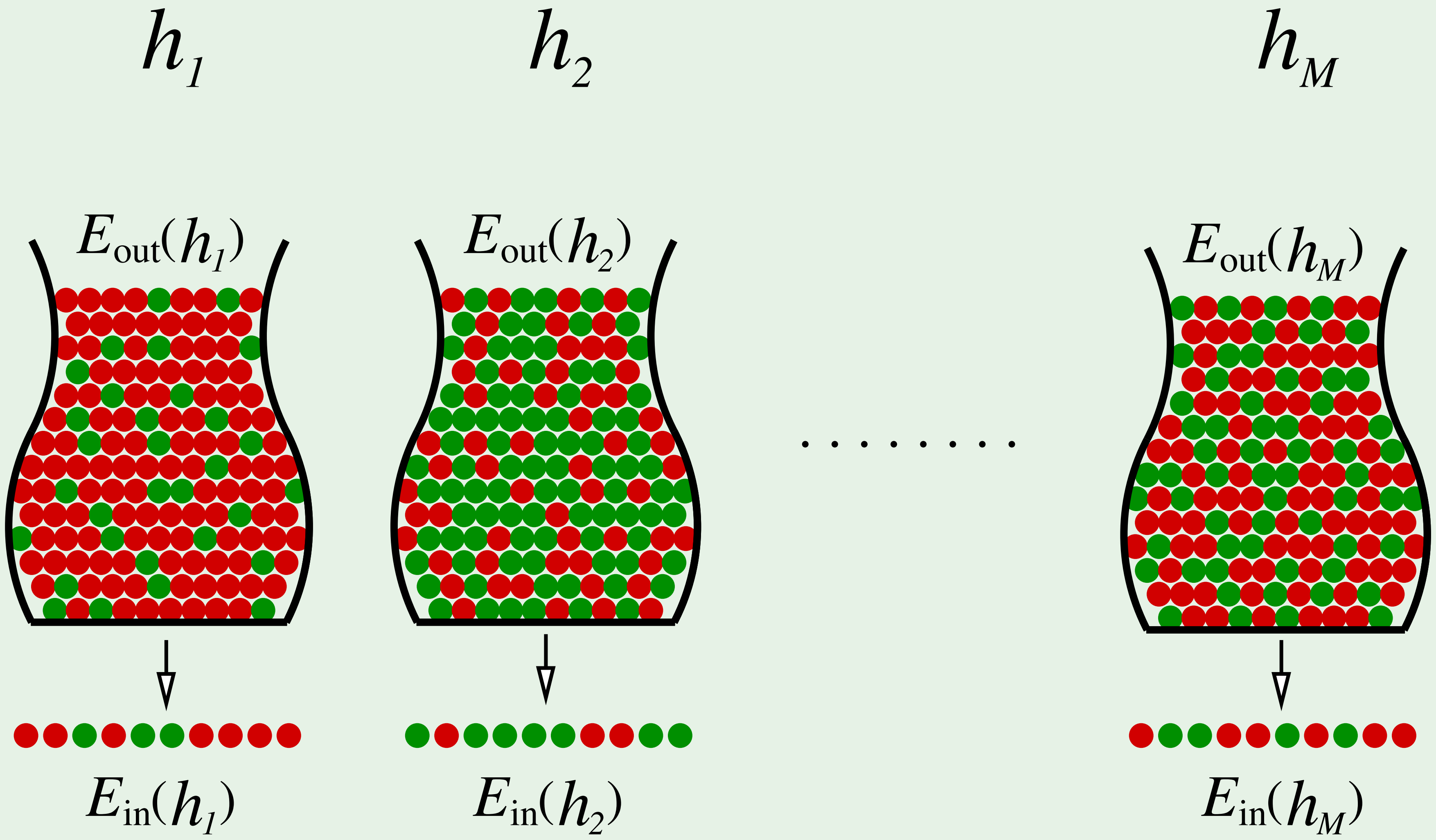
μ is '**out of sample**' denoted by $E_{\text{out}}(h)$

The Hoeffding inequality becomes:

$$\mathbb{P}\left[\, |E_{\text{in}}(h) - E_{\text{out}}(h)| > \epsilon \,\right] \;\leq\; 2e^{-2\epsilon^2 N}$$



Notation with multiple bins



A simple solution

$$
\begin{aligned}
\mathbb{P}[\ |E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon\ ] \ \leq\ \mathbb{P}[\quad & |E_{\text{in}}(h_1) - E_{\text{out}}(h_1)| > \epsilon \\
& \textbf{or } |E_{\text{in}}(h_2) - E_{\text{out}}(h_2)| > \epsilon \\
& \cdots \\
& \textbf{or } |E_{\text{in}}(h_M) - E_{\text{out}}(h_M)| > \epsilon\ ] \\
\leq\ & \sum_{m=1}^{M} \mathbb{P}\left[|E_{\text{in}}(h_m) - E_{\text{out}}(h_m)| > \epsilon\right]
\end{aligned}
$$

The final verdict

$$\mathbb{P}[\ |E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon\ ] \ \leq \ \sum_{m=1}^{M} \mathbb{P}\left[|E_{\text{in}}(h_m) - E_{\text{out}}(h_m)| > \epsilon\right]$$

$$\leq \ \sum_{m=1}^{M} 2e^{-2\epsilon^2 N}$$

$$\mathbb{P}[|E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon] \leq 2{\color{red}M}e^{-2\epsilon^2 N}$$

feasibility of learning

- can we make $E_{out}(g)$ close enough to $E_{in}(g)$?
- can we make $E_{in}(g)$ small enough ?

the complexity of $\mathcal{H}$

- can we make $E_{out}(g)$ close enough to $E_{in}(g)$?

$$\mathcal{P}(|E_{in}(g) - E_{out}(g)| > \epsilon) \leq 2Me^{-2\epsilon^2 N}$$

- *M* is a measure of “complexity” of $\mathcal{H}$
- we would like to keep *M* small
- **but ...**

the complexity of $\mathcal{H}$

- can we make $E_{out}(g)$ close enough to $E_{in}(g)$?

$$\mathcal{P}(|E_{in}(g) - E_{out}(g)| > \epsilon) \leq 2Me^{-2\epsilon^2 N}$$

- can we make $E_{in}(g)$ small enough ?
  - a richer $\mathcal{H}$ helps to get $E_{in}(g) \simeq 0$
  - richer means more, and more complex *h*
  - ...so we want a large M

the complexity of $\mathcal{H}$

generalization

- can we make $E_{out}(g)$ close enough to $E_{in}(g)$?
- can we make $E_{in}(g)$ small enough ?

approximation

the complexity of *f*

- can we make $E_{out}(g)$ close enough to $E_{in}(g)$?

$$\mathcal{P}(|E_{in}(g) - E_{out}(g)| > \epsilon) \leq 2Me^{-2\epsilon^2 N}$$

  - this does not depend on *f*

- can we make $E_{in}(g)$ small enough ?
  - approximating a complex *f* is harder
  - we will need a $\mathcal{H}$ with more complex h
  - ...i.e., a larger M