# Latent Markov models for longitudinal data in R by LMest package

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## Outline

- Elements of theory:
  - Preliminary definitions
  - Latent Markov models
  - Estimation & model selection
  - Prediction of latent states
- ② LMest package:
  - Data processing
  - Specification of formulas
  - Example with continuous responses
  - Examples with categorical responses



#### Credits

- LMest package by:
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  - A.Farcomeni University of Rome-Tor Vergata, IT
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- Illustration of the LMest package by:
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## Preliminary definitions

- Given a set of response variables Y and a set of covariates X, latent variables (U) are unobservable variables supposed to exist and to affect Y; these may be correlated with X
- A latent variable model (LVM) formulates assumptions on:
  - the conditional distribution of Y given U and X, f(y|u,x) (measurement model)
  - the conditional distribution of  $\boldsymbol{U}$  given  $\boldsymbol{X}$ ,  $f(\boldsymbol{u}|\boldsymbol{x})$  (structural model)
- By marginalizing out the latent variables we obtain the *manifest* distribution f(y|x); by the Bayes theorem we obtain the posterior distribution f(u|x,y)



- A common assumption of LVMs is that of local independence, according to which the response variables are conditionally independent given the latent variables and the covariates
- Latent variables are typically included in a LVM with different aims, such as:
  - accounting for the unobserved heterogeneity among subjects
  - accounting for measurement errors
  - summarizing different measurements of the same (directly) unobservable characteristics
- We focus in particular on Latent (Hidden) Markov (LM) models for longitudinal data

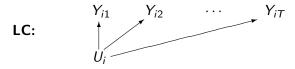
## Latent Markov models (basic version)

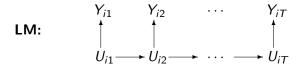
- These are models for the analysis of longitudinal categorical (or continuous) data; they are used in many contexts, e.g., economics, education, psychology, sociology
- Main references: Wiggins (1973), Bartolucci et al. (2013), Zucchini et al. (2016)
- For individual sequences of response variables at T time occasions,  $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{iT})'$ ,  $i = 1, \dots, n$ , the basic version of the LM model assumes that:
  - (*local independence*, LI) the response variables in  $Y_i$  are conditionally independent given a latent process  $U_i = (U_{i1}, \dots, U_{iT})'$
  - every latent process  $U_i$  follows a *first-order Markov chain* with state space  $\{1, \ldots, k\}$ , initial probabilities  $\pi_u$ , and transition probabilities  $\pi_{v|u}$ ,  $u, v = 1, \ldots, k$



## Possible interpretation

 The LM model may be seen as a generalization of the LC model in which subjects are allowed to move between latent classes







## Model parameters

- Each latent state u (u = 1, ..., k) corresponds to a *class of subjects* in the population, and is characterized by:
  - initial probability:

$$\pi_u = p(U_{i1} = u)$$

• *transition probabilities* (which may also be time-specific in the non-homogenous case):

$$\pi_{v|u} = p(U_{it} = v | U_{i,t-1} = u), \quad t = 2, ..., T, \ u, v = 1, ..., k$$

• distribution of the response variables (with categorical responses):

$$\phi_{y|u} = p(Y_{it} = y|U_{it} = u), \quad t = 1, ..., T, \ u = 1, ..., k, \ y = 0, ..., l-1$$

• The transition probabilities are collected in the *transition matrix*  $\Pi$  of size  $k \times k$ 



#### Manifest distribution

• LI implies that the *conditional distribution* of  $Y_i$  given  $U_i$  is:

$$p(\boldsymbol{y}_i|\boldsymbol{u}_i) = p(\boldsymbol{Y}_i = \boldsymbol{y}_i|\boldsymbol{U}_i = \boldsymbol{u}_i) = \prod_{t=1}^T \phi_{y_{it}|u_{it}}$$

- Distribution of  $U_i$ :  $p(u_i) = p(U_i = u_i) = \pi_{u_{i1}} \prod_{t>1} \pi_{u_{it}|u_{i,t-1}}$
- Manifest distribution of  $\mathbf{Y}_i$ :  $p(\mathbf{y}_i) = p(\mathbf{Y}_i = \mathbf{y}_i) = \sum_{\mathbf{u}} p(\mathbf{y}_i | \mathbf{u}) p(\mathbf{u})$
- This may be efficiently computed through suitable recursions known in the hidden Markov literature (Baum et al., 1970, Welch 2003)



#### Multivariate extension of the basic LM model

- We consider a vector of J response variables  $\mathbf{Y}_{it} = (Y_{i1t}, \dots, Y_{iJt})'$  for each individual i and time occasion t,  $i = 1, \dots, n$ ,  $t = 1, \dots, T$
- With categorical responses, the elements of  $Y_{it}$  are assumed to be conditionally independent given  $U_{it}$  (LI), so that

$$p(\mathbf{y}_{it}|u_{it}) = p(\mathbf{Y}_{it} = \mathbf{y}_{it}|U_{it} = u_{it}) = \prod_{j=1}^{J} \phi_{jy_{ijt}|u_{it}}$$
$$\phi_{jy|u} = p(Y_{ijt} = y|U_{it} = u)$$

• With continuous responses, it is typically assumed that

$$oldsymbol{Y}_{it}|u_{it}\sim N(oldsymbol{\mu}_{u_{it}},oldsymbol{\Sigma})$$

• The latent variables  $U_{i1}, \ldots, U_{iT}$  are assumed to follow a *first-order Markov chain* (possibly non-homogeneous)



#### Inclusion of covariates in the basic LM model

- Two possible *choices to include individual covariates* collected in  $X_i = (x_{i1}, \dots, x_{iT})$
- The first is in the measurement model so that we have random intercepts; for binary variables we could assume:

$$\lambda_{itu} = p(Y_{it} = 1 | U_{it} = u, \boldsymbol{X}_i),$$

$$\log \frac{\lambda_{itu}}{1 - \lambda_{itu}} = \alpha_u + \boldsymbol{x}'_{it}\boldsymbol{\beta}, \quad i = 1, \dots, n, \ t = 1, \dots, T, \ u = 1, \dots, k$$

 The latent variables are used to account for the unobserved heterogeneity and then the model may be seen as a "discrete version" of the logistic model for longitudinal data with random effects



- The second is in the structural model governing the distribution of the latent variables (via a multinomial logit parametrization):
  - initial probabilities:

$$\pi_{iu} = p(U_{i1} = u | \mathbf{x}_{i1}), \qquad \log \frac{\pi_{iu}}{\pi_{i1}} = \mathbf{x}'_{i1} \boldsymbol{\beta}_{u}, \quad u = 2, \dots, k$$

transition probabilities:

$$\begin{array}{rcl} \pi_{itv|u} & = & \rho(U_{it} = v | U_{i,t-1} = u, \boldsymbol{x}_{it}), \\ \log \frac{\pi_{itv|u}}{\pi_{itu|u}} & = & \boldsymbol{x}'_{it} \gamma_{uv}, \quad u, v = 1, \dots, k, \ u \neq v \end{array}$$

• The *main interest is on the latent variable* which is measured through the observable response variables (e.g., health status) and on how this latent variable evolves depending on the covariates



#### Mixed latent Markov models

- Additional random effects/latent variables may be included in an LM model to account for further sources of unobserved heterogeneity (van de Pol and Langeheine, 1990; Altman, 2007; Maroutti, 2011)
- Among the mixed LM models, we focus on that based on initial and transition probabilities of the individual latent processes defined conditional on a discrete latent variables  $V_i$ , i = 1, ..., n
- The model assumes that individuals are divided in *latent clusters*, with individuals in the same cluster following the same LM model, while the measurement model is common to all individuals
- Mixed LM models may be also used for multilevel longitudinal data with individuals collected in observable groups (Bartolucci et al., 2011)



## Maximum likelihood estimation of the basic LM model

- Model log-likelihood:  $\ell(\theta) = \sum_{i=1}^{n} \log p(\mathbf{y}_i) = \sum_{\mathbf{y}} n(\mathbf{y}) \log p(\mathbf{y})$ 
  - $\theta$ : vector of all model parameters  $(\pi_u, \pi_{v|u}, \phi_{v|u})$  for categorical data
  - n(y): frequency of the response configuration y
- $\ell(\theta)$  may be maximized by an *EM algorithm* (Baum *et al.*, 1970, Dempster *et al.*, 1977)
- The EM algorithms alternates two steps until convergence:
  - *E-step*: compute the *posterior distribution* of the latent states given the observed data and the current value of the parameters
  - *M-step*: maximize the posterior expected value of the *complete* data log-likelihood with respect to the model parameters



- Suitable *recursions* are required to compute the  $\ell(\theta)$  and to perform the E-step (Baum *et al.*, 1970; Welch, 2003)
- Being  $\ell(\theta)$  multimodal, different initializations (deterministic and random) of the algorithm are necessary to increase the chance to get its global maximum
- Extended models (multivariate, with covariates, and mixed) are still
  fitted by an EM algorithm in which the main adjustments are in the
  M-step
- If necessary, *selection of k* may be based on suitable statistical criteria (Akaike, 1973, Schwarz, 1978):

AIC = 
$$-2\ell(\hat{\boldsymbol{\theta}}) + 2\#\text{param}$$
.  
BIC =  $-2\ell(\hat{\boldsymbol{\theta}}) + \log(n)\#\text{param}$ .



# Prediction of latent states (dynamic pattern recognition)

• The posterior probabilities

$$p(u_t|\mathbf{y}_i) = p(U_{it} = u_t|\mathbf{Y}_i = \mathbf{y}_i)$$

may be used to assign a subject to a latent state at a given time occasion ( $local\ decoding$ ); state assigned to subject i at occasion t:

$$\hat{u}_{it}: p(\hat{u}_{it}|\boldsymbol{y}_i) = \max_{u_t} p(u_t|\boldsymbol{y}_i)$$

• More sophisticated is the problem of *path prediction*, i.e., finding the most likely sequence  $\tilde{\boldsymbol{u}}_i = (\tilde{u}_{i1}, \dots, \tilde{u}_{iT})'$  for subject i (*global decoding*):

$$\tilde{\boldsymbol{u}}_i: p(U_{i1} = \tilde{u}_{i1}, \dots, U_{iT} = \tilde{u}_{iT} | \boldsymbol{y}_i) = \max_{\boldsymbol{u}} p(\boldsymbol{u} | \boldsymbol{y}_i)$$

• For this aim we need an *iterative algorithm* due to Viterbi (1967) and further elaborated by Juan & Rabiner (1991)



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