CONVERGENCE RESULTS FOR SOLUTION OF STOCHASTIC HARD-SOFT CONSTRAINED CONVEX FEASIBILITY PROBLEM

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Abstract

This work considers the stochastic convex feasibility problem involving hard constraints (that must be satisfied) and soft constraints (whose proximity function should be minimized) in Hilbert space. Convergence in quadratic mean and almost surely was proved for the result of the solution. An alternating projection involving 1-lipschitzian and firmly non-expansive mapping was adopted.

1. Introduction

Sometimes, there arise cases where some of the constraints in a convex feasibility problem must be satisfied while others must not. Such scenario is called general convex feasibility problem [4] or Hard-constrained inconsistent signal feasibility problems [6]. In such case, we seek a point which satisfies some constraints while the remaining constraints in the system will have their proximity functions $f(x)$ minimized in some sense, where $f: H \to \mathbf{R}_+$.

Suppose that $\{C_i\}_{i\in I}$ is a family of closed convex subsets, let $C_H = \bigcap_{i\in I_H} C_i$ and $C_S = \bigcap_{i\in I_S} C_i$ be the hard and soft constraints respectively with I_H representing hard constraints index set and I_S representing soft constraints index set. Let the respective hard and soft constraints functions be given by: $\underset{i\in I_H}{c_i}(x) \le 0$ and $\underset{i\in I_S}{c_i}(x) \le 0$. If soft constraints proximity function $f(x)$ is convex and differentiable with $\tau = Inf_{x\in C_H} f(x)$, then the classical hard-soft constrained feasibility problem is to find a point: $z \in \phi = \{x \in C_H \mid f(x) = \tau\}$. The stochastic equivalent of the classical hard-soft constrained convex feasibility problem is to find a random point : $z(\omega) \in \phi = \{x(\omega) \in C_H \mid E(f(\omega, x)) = \tau(\omega)\}$, Where, $x(\omega)$ is a random variable defined on a probability space (Ω, Σ, P) for every $x(\omega) \in C$, $f(\omega, x)$ is an integrable random convex function on some measurable space, and E is the integral or expected value of $f(x,\omega)$ with respect to $p(\omega)$. The procedure for solution of classical hard-soft constrained convex feasibility problems would have served as a good substitute for the stochastic hard-soft constrained convex feasibility problems but the stochastic proximity function $E(f(\omega, x))$ is not fully observable and the knowledge of the random operator is scares for the following reasons as stated by [9]. For some discussions on iterative methods for convex feasibility problem see [1-10] and their references.

2. Main Result

Theorem 3.1: Let (Ω, Σ, P) be a complete probability measure space and $C = \{C_H, C_S\} \in H = \mathbf{R}^n$ be a nonempty closed convex subset of real finite dimensional Hilbert spaces with respective projectors $P_{C_H} : C_H \to H \text{ and } P_{C_S} : C_S \to H$. Suppose that $T : C \to H$, where $T = P_{C_H} P_{C_S}$ with both $P_{C_H} \text{ and } P_{C_S}$ being firmly non-expansive projection onto the nonempty close convex set $C_H \text{ and } C_S$ respectively. Let $\{\sigma_n(x_n(\omega))\} \ge 1$ where $\sigma : C_H \to (0, +\infty)$ such that $\lambda_n \in [\varepsilon, 2-\varepsilon]$ for an arbitrary $\varepsilon \in (0,1)$, with $\lim\inf \lambda_n(2-\lambda_n) > 0$ and $P_{C_H} \cap P_{C_S} \neq 0$ is not assumed, then the stochastic sequence $\{x_n(\omega)\}$ generated by: $x_{n+1}(\omega) = P_{C_H}\left(x_n(\omega) + \lambda_n \sigma_n(x_n(\omega))\left(P_{C_H} P_{C_S} x_n(\omega) - x_n(\omega)\right)\right),\ x_0(\omega) \in C_H$ (12)

Converges almost surely to a random point $z(\omega) \in \phi$

Theorem 3.2: Let (Ω, Σ, P) be a complete probability measure space and $C = \{C_H, C_S\} \in H = \mathbf{R}^n$ be a nonempty closed convex subset of real finite dimensional Hilbert spaces with respective projectors $P_{C_H} : C_H \to H \text{ and } P_{C_S} : C_S \to H$. Suppose that $T : C \to H$, where $T = \frac{1}{2}\left(R_{C_H} R_{C_S} + \mathbf{I}\right)$ with both $R_{C_H} = 2P_{C_H} - \mathbf{I}$ and $R_{C_S} = 2P_{C_S} - \mathbf{I}$ being firmly non-expansive projection onto the nonempty close convex set $C_H \text{ and } C_S$ respectively.

Let $\lim\inf \lambda_n(1-\lambda_n) > 0$ and $z(\omega) \in \left(R_{C_H} R_{C_S}\right) \neq 0$, then the stochastic sequence $\{x_n(\omega)\}$ generated by:

$x_{n+1}(\omega) = x_n(\omega) + \frac{\lambda_n}{2}\left(R_{C_H} R_{C_S} x_n(\omega) - x_n(\omega)\right), x_0(\omega) \in H$ converges in quadratic mean to $z(\omega) \in \phi$

Theorem 3.3: Let (Ω, Σ, P) be a complete probability measure space and $C = \{C_H, C_S\} \in H = \mathbf{R}^n$ be a nonempty closed convex subset of real finite dimensional Hilbert spaces with respective projectors $P_{C_H} : C_H \to H \text{ and } P_{C_S} : C_S \to H$. Suppose that the soft constraints proximity function is given by:

$f = \frac{1}{2}\sum_{i\in I_S} w_i d(.,C_i)^2 = \frac{1}{2}\sum_{i\in I_S} w_i \left\| P_i(x(\omega)) - x(\omega) \right\|^2$ where, $w_i \in [0,1] \text{ and } \sum_{i\in I_S} w_i = 1$. If we take $x_0 \in C_H$, $\beta_n \in [0,1]$ such that $\sum \beta_n(1-\beta_n) = +\infty$ and $k \in [0,2]$, then the stochastic sequence $\{x_n(\omega)\}$ generated by

$x_{n+1}(\omega) = (1-\beta_n)x_n(\omega) + \beta_n P_{C_H}\left((1-k)x_n(\omega) + k\sum_{i\in I_S} w_i P_i(x(\omega))\right)$, converges almost surely to a point $z \in \phi$, where, P_i is a projection onto C_i

3. **Conclusion:** We have considered in this paper the stochastic convex feasibility problem in the Hilbert space involving hard and soft constraints. Convergence in quadratic mean and almost surely was proved for the result of the solutions. An alternating projection of firmly non-expansive and 1-lipchitzian operator was adopted.

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