Università degli Studi di Napoli Federico II

Scuola Politecnica e delle Scienze di Base

Ingegneria dell'Automazione e Robotica

Control of Complex System and Networks

How Network Topology Shapes the Opinion Dynamics Process

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October 2024

Contents

Introduction

In this study, a group of interacting individuals among whom the process of opinion generation takes place is considered. In order to explore the opinion dynamics within such groups, many models have been proposed over the past few decades. One of the fundamental models is the classic DeGroot model, introduced in the 1970s. This model examines how a group of individuals update their opinions over time and eventually, if the network satisfies some connection conditions, reach a group consensus.
Since then, various extensions and alternative models have been developed to capture the different shades of meaning in the opinion dynamics. The Expressed and Private Opinion Model is one of the most interesting one. This model examines how discrepancies in the public and private opinions of the same individual can arise and evolve separately.
In each of these models, opinion dynamics can lead to either a consensus or a disagreement among the agents, depending on the network structure and interaction rules. This project consists of two main chapters that, through theoretical analysis and simulation experiments, aim to answer several key questions regarding opinion dynamics.

- **Chapter 1** focuses on DeGroot model and it aims to understand if and under what conditions the individuals' opinions converge in different scenarios. In order to provide greater clarity, a practical and simplified example is included to illustrate the mechanics of the model.

- **Chapter 2** considers the EPO model and it investigates if there is a direct relationship between key model parameters and the structural characteristics of the network. Various network types are introduced to examine how their topological features influence the opinion dynamics process.

Finally, the project concludes with a summary of results drawn from the simulations presented in Chapter 1 and Chapter 2.

Chapter 1

DeGroot Model

This chapter will focus on the DeGroot model, a classical framework used to study how individuals in a network form and update their opinions through repeated interactions. The aim is to explore the process of opinion dynamics using the DeGroot model. Various scenarios will be presented in order to understand how network topologies influence the convergence of opinions and the development of ideas in collaborative environments. This will be exemplified through a case involving different groups of engineers within a company working on a joint project. The analysis will first consider a single group working together. Next, it will examine a group of interns who initially observe the work of a single group of senior engineers. Finally, it will explore the case where the interns observe two independent sector groups.

1.1 Model

The DeGroot model provides a mathematical representation of how social influence spreads through a network. In this framework, each individual's opinion is a real number. This model assumes that each individual's opinion evolves over time as they integrate learned opinion values of their neighbors with the individual's own opinion using a weighted averaging process to capture the concept of social influence. Eventually, a consensus is reached on the opinion value if the network satisfies some connection conditions. One of the key features of the model is its simplicity and mathematical tractability, allowing researchers to explore various aspects of opinion dynamics, such as consensus or disagreement.

A group of n individuals discussing a single topic is considered. Their interactions are modeled by a graph. The opinion of an individual i, denoted as $x_i(k) \in \mathbb{R}$, evolves over discrete time steps $k = 0, 1, \ldots$

According to the DeGroot model the opinions of all the individuals, record as $x = [x_1, ..., x_n]^T$, are described as follows:

$$x(k+1) = Ax(k) \tag{1.1}$$

where the elements a_{ij} of the matrix A indicate the level of weight that the agent i puts on the agent j. Each weight a_{ij} is nonnegative ($a_{ij} \geq 0$) and satisfies the condition $\sum_{j=1}^{n} a_{ij} = 1$, meaning that the total influence on agent i from all other agents sums to 1. This implies that the matrix A is nonnegative and row-stochastic.

1.2 Simulation

In this section, the DeGroot model is simulated in MATLAB to explore how opinions evolve. For this purpose, a group of engineers within company is considered, where each engineer collaborates on a joint project and continuously exchanges information with the others. The nodes represent the engineers and the relationships between them indicate the exchange of information. Specifically, the level of trust that the engineer i puts on the engineer j within the company is represented by the element a_{ij} of the adjacency matrix A. This scenario reflects an environment where decisions are made collectively and each engineer is influenced by the opinions of all the others, leading to a shared strategy. Depending on the network structure, the simulation will yield different results.

For the following simulation, the different types of networks are generated in MATLAB using the Erdős-Rényi probabilistic model algorithm. This approach facilitates the easy modification of the network structure according to specific needs for subsequent tests. In this model, an adjacency matrix A is created by connecting pairs of nodes with a fixed probability p. Each pair of nodes is considered independently, meaning that connections are made randomly based on the specified probability. By adjusting p and the total number of nodes n, the structure of the adjacency matrix A can be tailored to simulate different social interaction scenarios. This flexibility allows for the exploration of various network topologies and, consequentially, their effects on opinion dynamics. A probability of $p = 0.3$ is chosen to ensure a moderately connected network, which is suitable for observing the dynamics of opinion convergence.

1.2.1 Strongly Connected Aperiodic Graph

In the first scenario, each engineer is reachable through a directed path from every other engineer, so the network composed of the engineers is represented by a Strongly Connected Graph. Strong connectivity ensures that information spreads rapidly among all participants, leading to a unified perspective and facilitating the achievement of consensus on a joint project.
In this context the Strongly Connected Aperiodic Graphs theorem is applied. It requires the adjacency matrix A to be row-stochastic and associated with a Strongly Connected and Aperiodic graph G. Under these conditions, the solution to the equation $x(k+1) = Ax(k)$ converges to a consensus. In this particular case, the initial opinions influence the entire population over time, leading to a unified consensus among all individuals in the network. The consensus value can be expressed as:

$$\lim_{k \to +\infty} x(k) = (w^T x(0))\mathbf{1}_n \tag{1.2}$$

where w is the normalized left eigenvector associated to $\lambda = 1$ and $x(0)$ represents the vector of initial opinions of the engineers.
Addionally, if G is also weight-balance, then the average consensus is achieved, that is:

$$\lim_{k \to +\infty} x(k) = \frac{\mathbf{1}_n^T x(0)}{n}\mathbf{1}_n \tag{1.3}$$

Specifically, to ensure that the graph G is Strongly Connected and Aperiodic, it is sufficient to verify that the associated adjacency matrix A is primitive. For this purpose, a function ‘checkPrimitivity(A)‘ is implemented.

The matrix A is also guaranteed to be row-stochastic. This means that each engineer distributes their total influence among their colleagues, with the sum of influences they exert being equal to 1. However, the influence each engineer receives from others can vary, as some may receive more influence than others depending on their position within the network.

Once a graph that satisfies the previous condition is generated, an analysis is first conducted on a small group consisting of $n = 15$ engineers. This initial investigation allows for an exploration of the behavior of the DeGroot model in a more manageable setting, where individual interactions and opinion dynamics can be easily visualized and interpreted. The adjacency matrix A is primitive and the associated digraph is shown in Fig. 1.1.

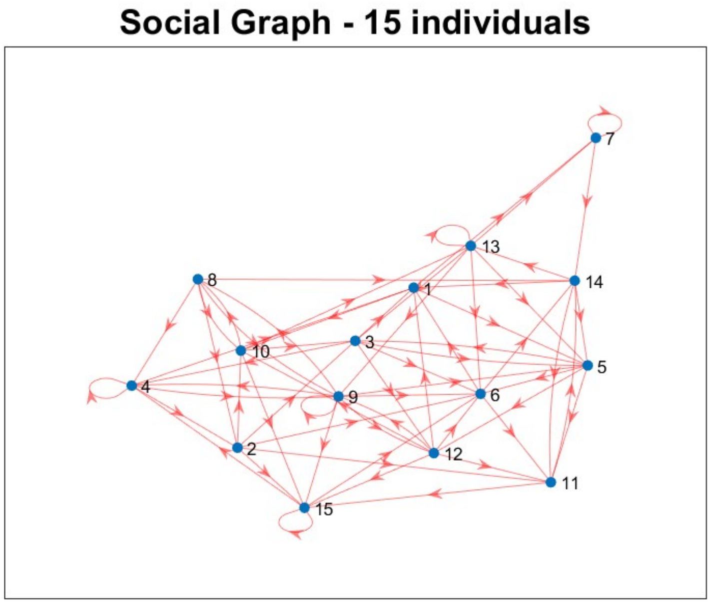

Figure 1.1: SC and Aperiodic Graph G of 15 individuals

As illustrated in Fig. 1.2, the opinions of all the engineers within the company converge towards a common opinion value over time. The consensus value is mathematically expressed by Equation 1.2, reflecting the influence of the initial opinions on the final outcome.

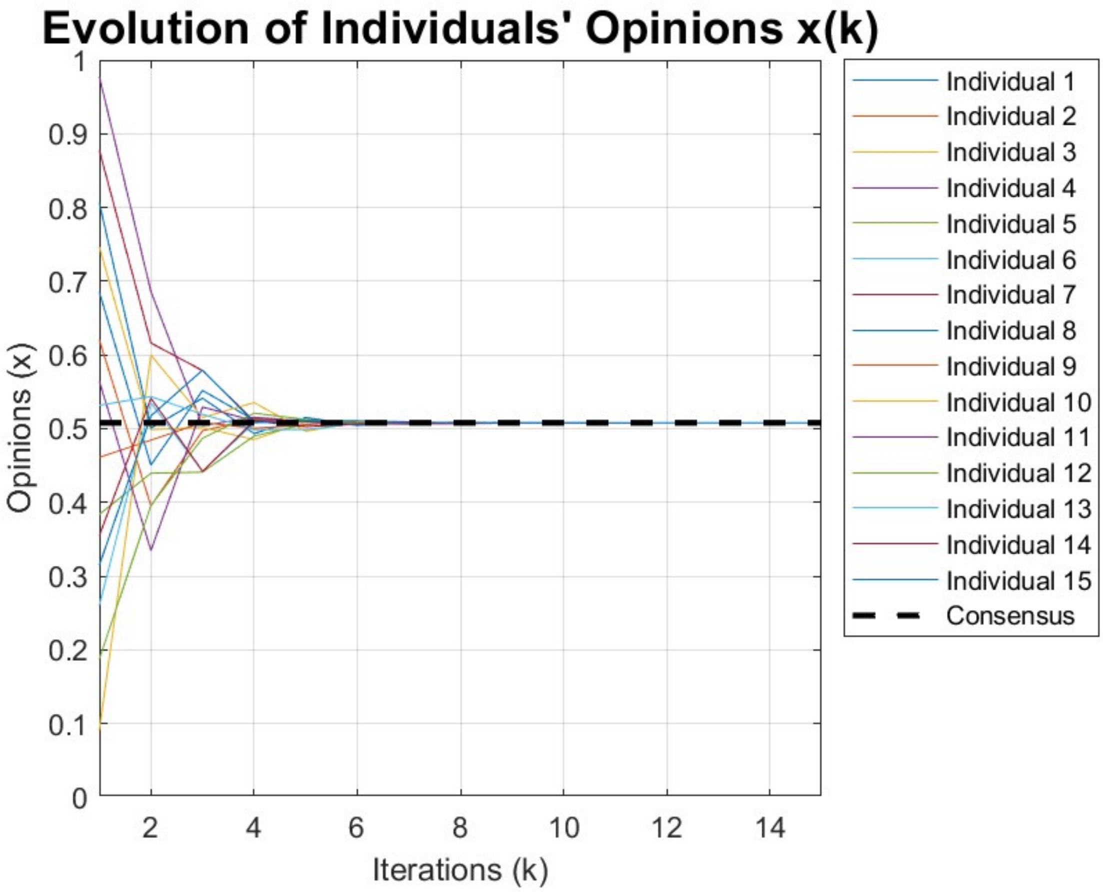

Figure 1.2: DeGroot Model - Opinion dynamics in a SC and Aperiodic Graph G of 15 individuals

Next, the same simulation will be analyzed with a larger group within the company of 100 interconnected engineers, as shown in Fig. 1.3. The associated adjacency matrix A is row-stochastic and primitive. The increased number of nodes and higher connectivity are expected to accelerate the convergence process.

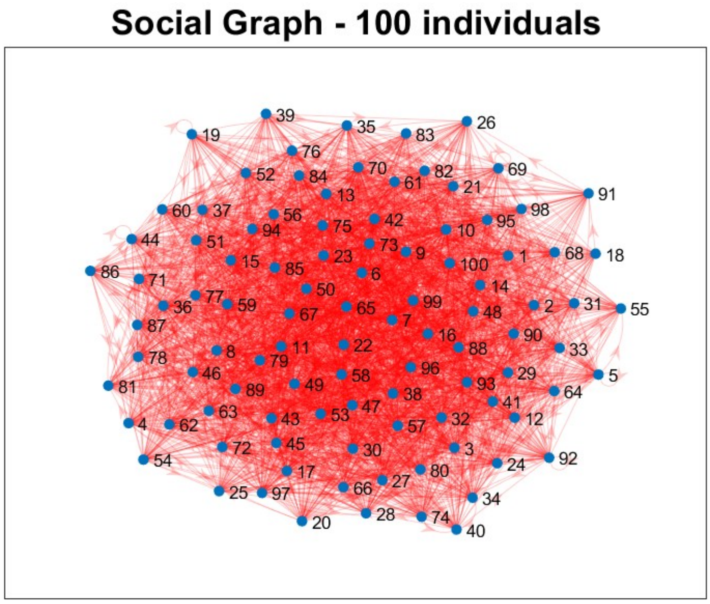

Figure 1.3: SC and Aperiodic Graph G of 100 individuals

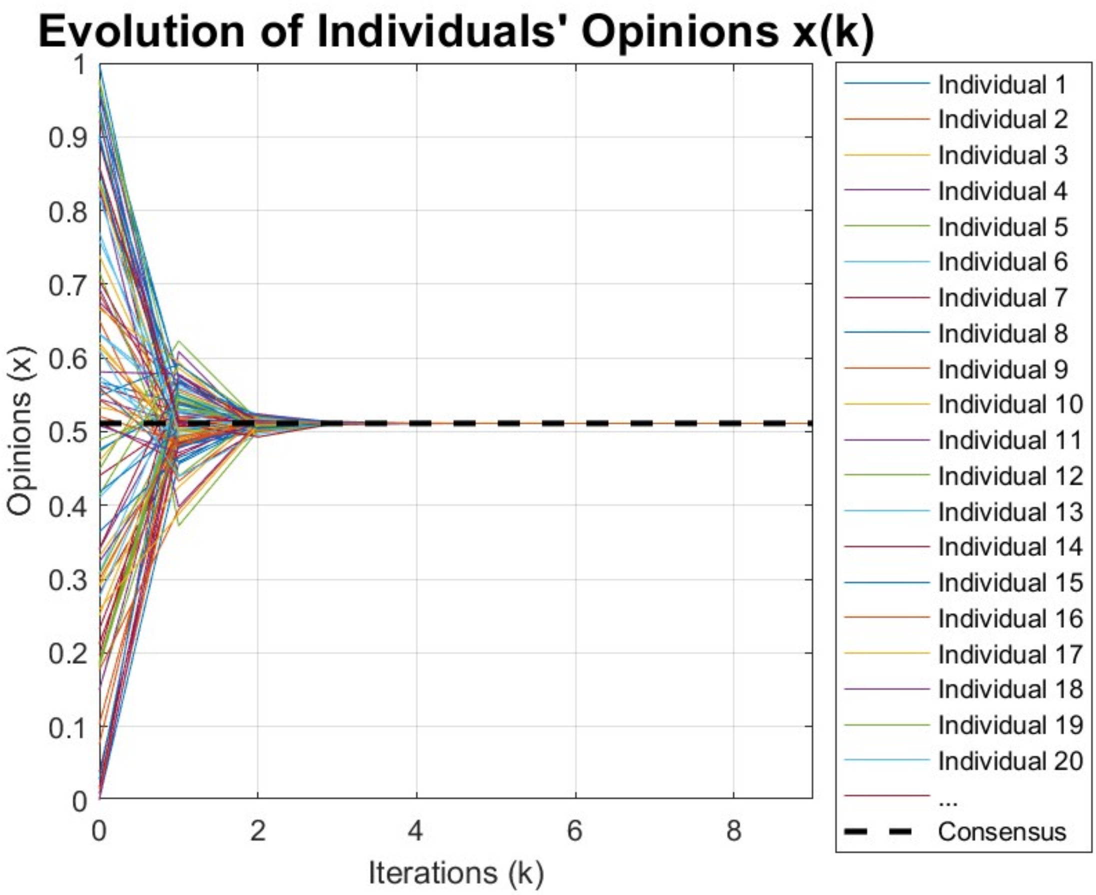

Figure 1.4: DeGroot Model - Opinion dynamics in a SC and Aperiodic Graph G of 100 individuals

The results, as shown in Fig. 1.4, illustrate how the opinion dynamics goes more rapidly to consensus in this more connected structure compared to the smaller network. In fact, after only $N = 10$ iterations, the disagreement vector, defined as $\delta(N) = x(N) - \text{consensus}$, is of the order of 10^{-7}.

Subsequently, both the influence exerted by each engineer on their colleagues and the influence received by each engineer are equally distributed, fostering a balanced exchange of information and collaboration among all team members. This type of interactions among the network can be modeled by a doubly stochastic matrix A. The matrix A is generated to be primitive, ensuring that the associated graph is strongly connected and aperiodic.
In this case, since A is doubly stochastic, the average consensus is achieved and is mathematically expressed by Equation 1.3. This ensures that the influence of each engineer is balanced and that the discussions remain equitable, promoting a more democratic exchange of ideas, as illustrated in Fig 1.5.

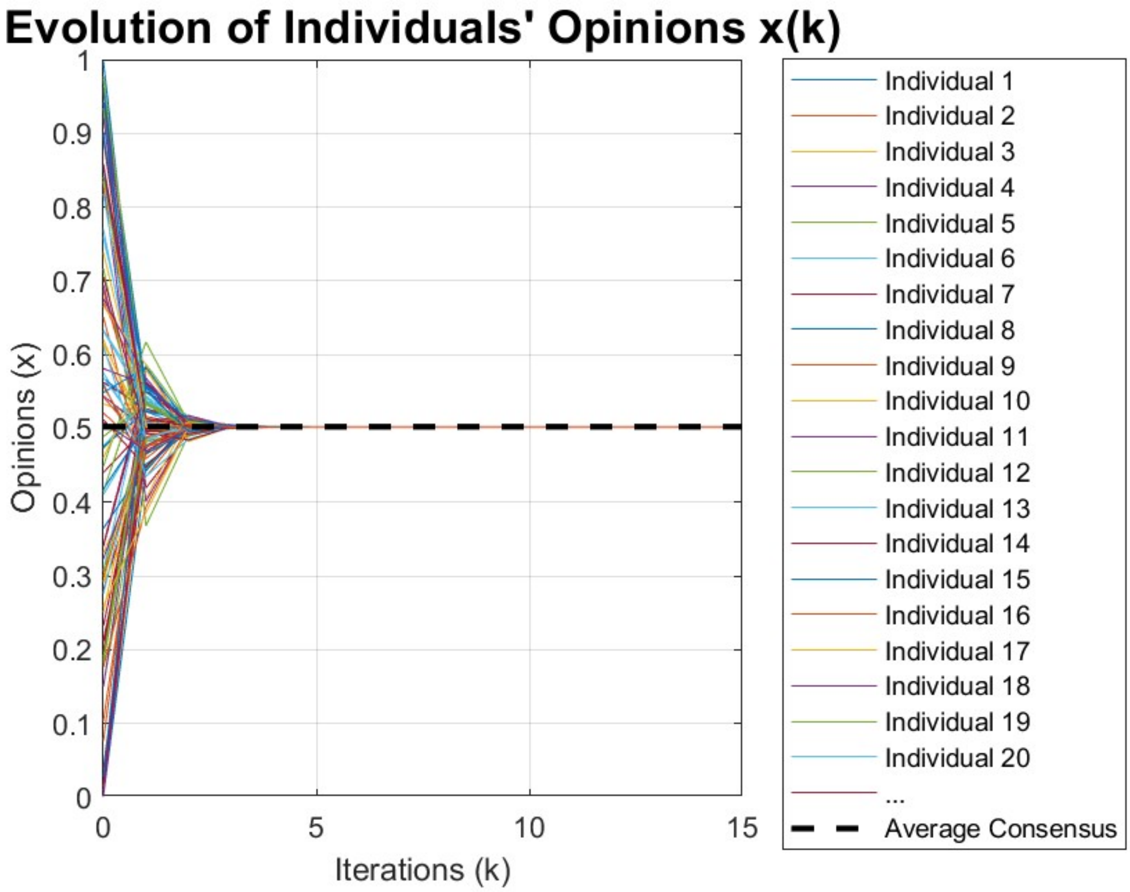

Figure 1.5: DeGroot Model - Opinion dynamics with a Doubly Stochastic A

1.2.2 Non-Strongly Connected Graph with One Destination

This section investigates the dynamics between a group of engineers and a group of intern engineers. The former group collaborates with each other. The latter also work together among themselves; however, since they hold a lower position, they are influenced by the engineers without having any influence over them. The graph describing this situation is a Non-Strongly Connected graph, characterized by a single destination, the senior engineers. For the sake of simplicity, the group of senior engineers is the network of 100 nodes previously used in Fig.1.3. It represents the condensation graph's destination. While the group of 10 intern engineers is randomly selected. It represents the condensation graph's source. Then the two groups are connected by an edge. Thanks to these measures, the entire network, shown in Fig. 1.6, has one aperiodic destination.

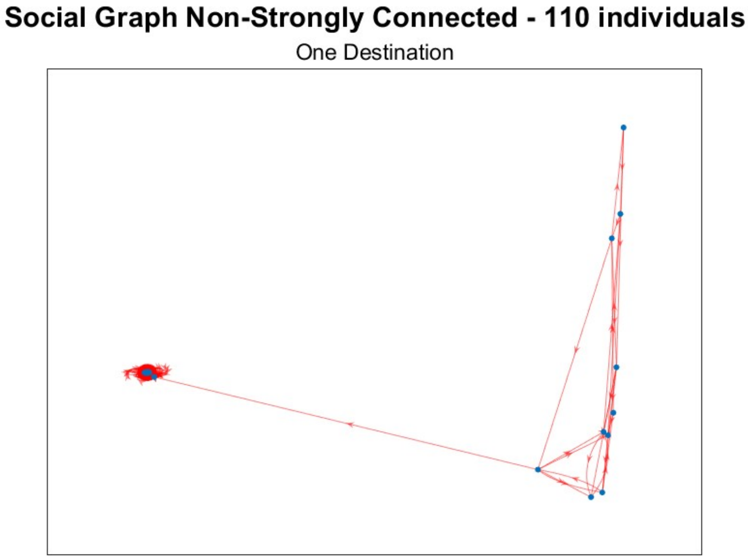

Figure 1.6: Non-SC Graph with One Destination

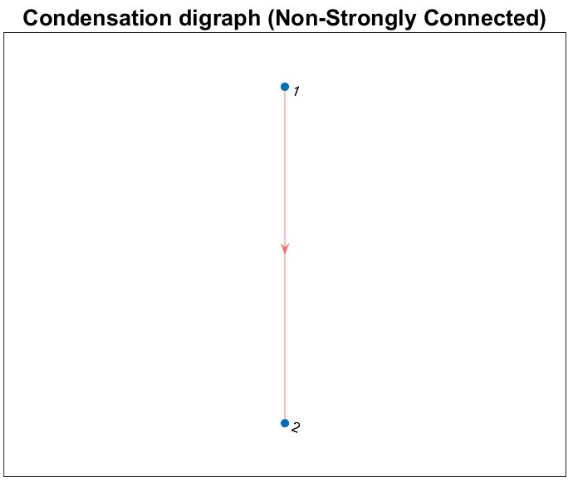

Figure 1.7: Connected Components of G

It's checked that the spectral radius of A is $\rho(A) = 1$ and it's a simple eigenvalue.
The left eigenvector associated with the spectral radius $\rho(A) = 1$ is scaled such that $w_1 + w_2 + \ldots + w_n = 1$, where:

$$\begin{cases} w_i > 0 & \text{if node } i \text{ belongs to the destination of } C(G) \\ w_i = 0 & \text{otherwise} \end{cases}$$

The following figures further illustrate how the opinion dynamics converge to consensus according to Equation 1.2. However, it is important to note that the final consensus value does not depend on all initial conditions; rather, it is influenced exclusively by the initial conditions of the destination nodes as shown in Fig. 1.8. So, in this case, the opinions of the intern engineers do not influence the joint project; instead, they converge to the consensus of the major engineers.

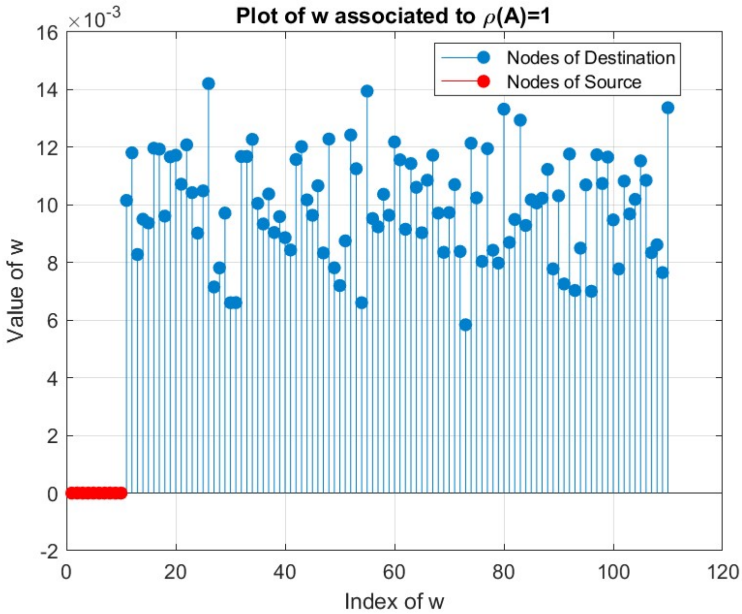

Figure 1.8: w's value associated to $\rho(A) = 1$

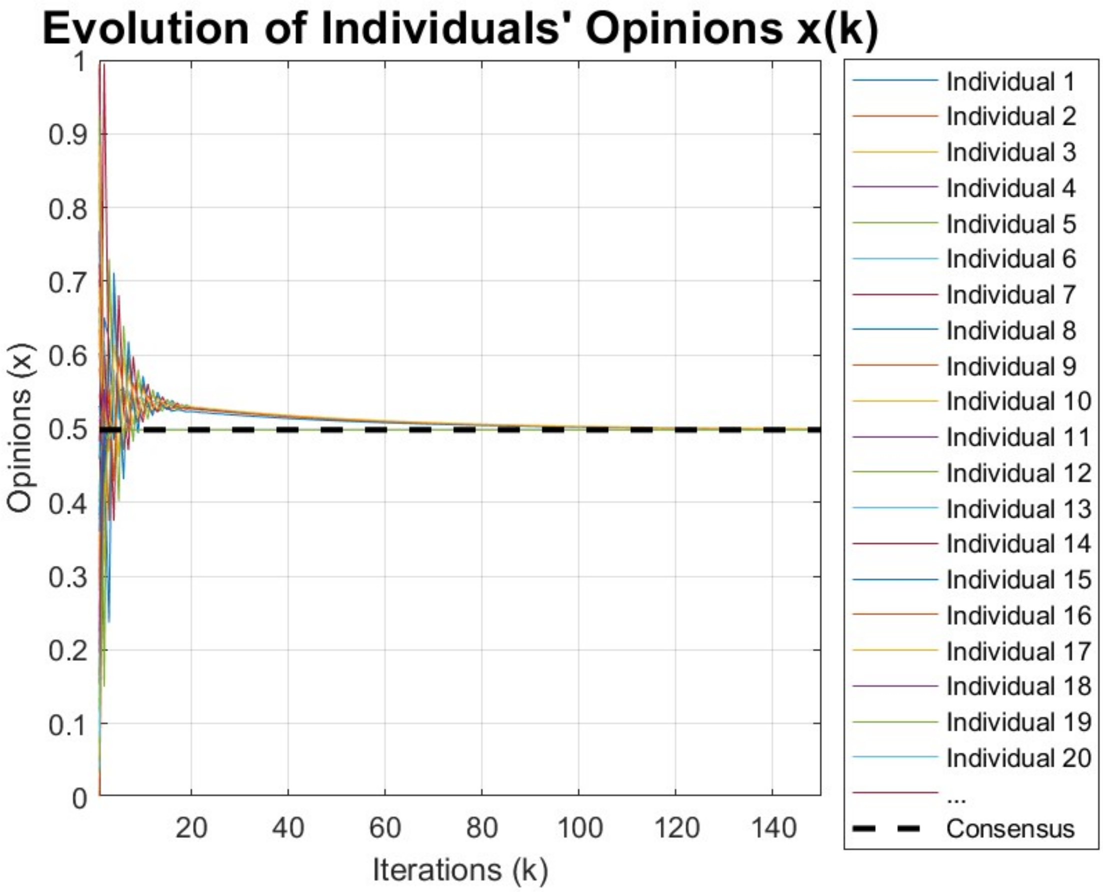

Figure 1.9: DeGroot Model - Opinion dynamics with N-SC graph

As can be observed in Fig. 1.9, while the group of engineers converges to consensus quickly, the group of interns converges to consensus at a notably slower rate. This difference in convergence speed can be attributed to the structural features of the graph. In Non-Strongly Connected graphs, certain nodes may remain isolated or have limited communication paths to other nodes, resulting in delayed information exchange and opinion dissemination. Consequently, the overall dynamics of opinion convergence are hindered, leading to a protracted consensus process.
In this case, after $N = 150$ iterations, the disagreement vector between the nodes' states and the consensus value is acceptable and it's on the order of 10^{-4}.

1.2.3 Non-Strongly Connected Graph with Multiple Destinations

In this last section, a network of 15 intern engineers is considered. They have to follow the work of two other groups of engineers employed for the same company but operating in two different sectors. Each group is developing distinct ideas for the same project, so they do not collaborate with each other. As before, the intern engineers can be influenced by the two groups separately without exerting any influence themselves due to their lower position. This scenario can be represented by a Non-Strongly Connected graph with two aperiodic destinations. These destinations represent the groups of senior engineers working in the two different sectors. The networks illustrated in Fig. 1.1 and 1.3 serve as the destination components, ensuring that they are Strongly Connected and Aperiodic.

While the group of intern engineers, which represents the condensation's graph source, is randomly selected as before. This configuration allows an analysis of the opinion dynamics convergence in the presence of multiple aperiodic destinations.
In the first scenario, a single intern engineer gathers information about the projects from both groups operating in different sectors. The entire network and its connected components are illustrated in the following figures.

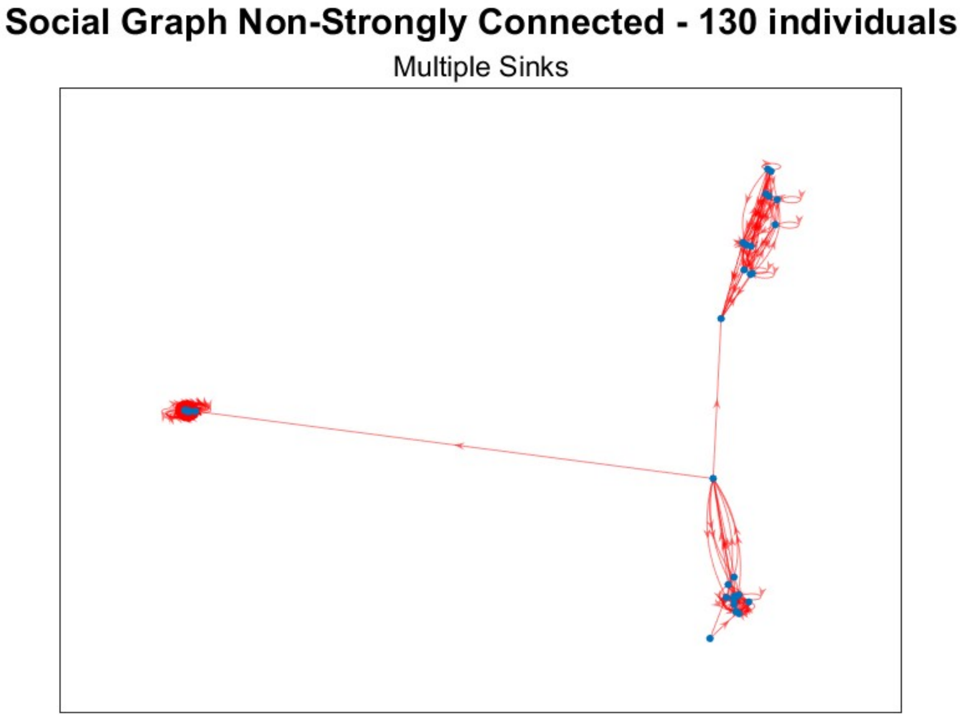

Figure 1.10: Non-SC Graph with Multiple Destinations

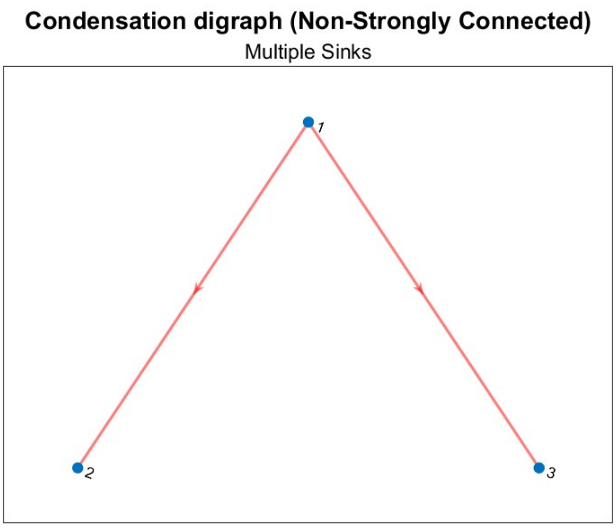

Figure 1.11: Connected Components of G

It is checked that the spectral radius of the matrix A is $\rho(A) = 1$ and that it is a semi-simple eigenvalue with multiplicity $M = 2$, corresponding to the number of destinations. In this scenario, the solution of the iterative equation $x(k+1) = Ax(k)$ satisfies:

$$\lim_{k \to +\infty} x(k) = \begin{cases} ((w^m)^T x(0))\mathbf{1}_n & \text{if } i \text{ belongs to destination } m \\ \sum_{m=1}^{M} z_{im}((w^m)^T x(0))\mathbf{1}_n & \text{otherwise} \end{cases} \tag{1.4}$$

where $m = 1, \ldots, M$.
The M left eigenvectors $w^1, \ldots, w^M$ associated with $\rho(A) = 1$ can be scaled such that $w^m \geq 0$ and $w_1^m + \ldots + w_n^m = 1$ for all $m = 1, \ldots, M$. Here,

$$\begin{cases} w_i^m > 0 & \text{if and only if node } i \text{ belongs to the } m\text{-th destination of } C(G) \\ w_i^m = 0 & \text{otherwise} \end{cases}$$

Additionally, $z_{i1} + \ldots + z_{iM} = 1$ with $z_{im} \geq 0$ for all m and $z_{im} > 0$ if and only if there exists a path from node i to destination m.

Fig. 1.12 illustrates the convergence behavior of nodes within the network according to Equation 1.4. The groups of engineers of the two different sectors do not interact with each other, leading them to develop different ideas. Meanwhile, since the communication between the intern engineers and the two different groups is balanced, dictated by a single connection to each group, the opinions of the interns converge to an intermediate value between the two proposals from the groups in different sectors.

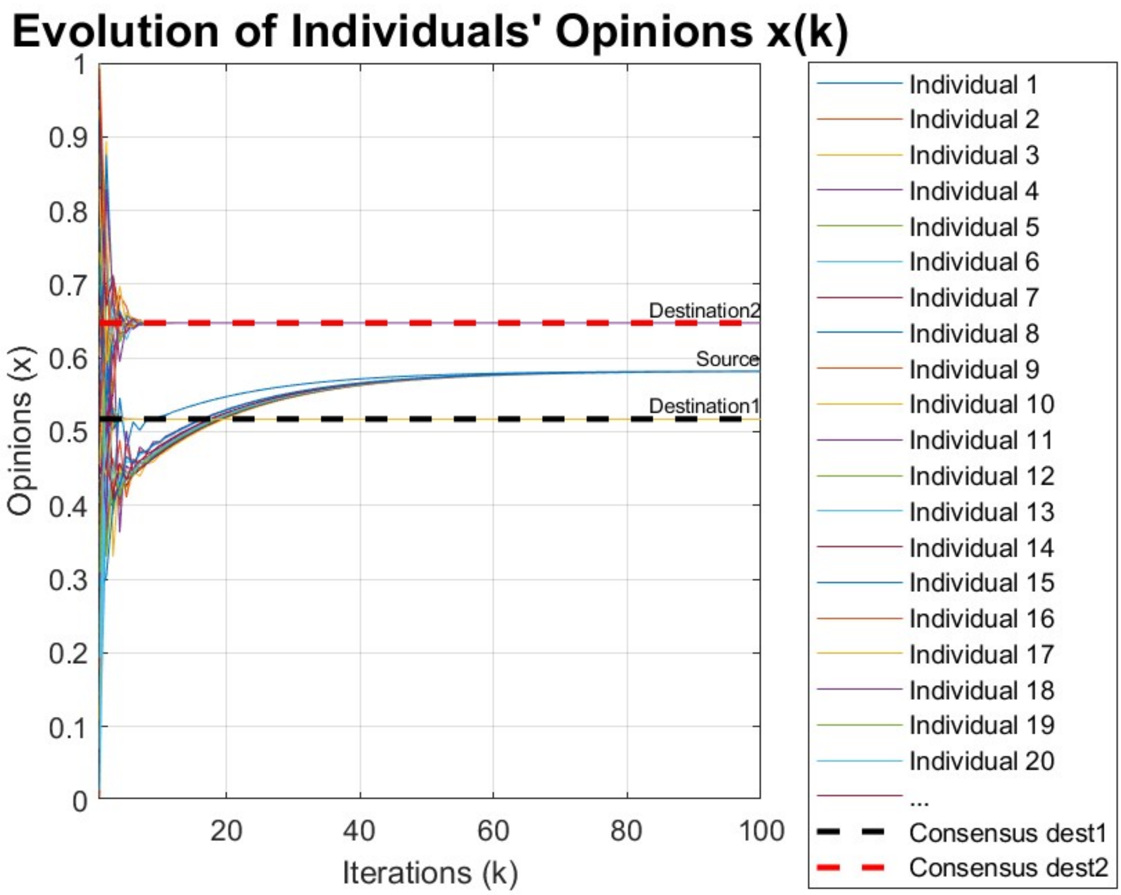

Figure 1.12: DeGroot Model - Opinion dynamics with N-SC graph with Multiple Destinations

Another scenario can be considered in which the groups of interns are more connected to one sector group than to the other, specifically to Group 1, as shown in Fig. 1.13.

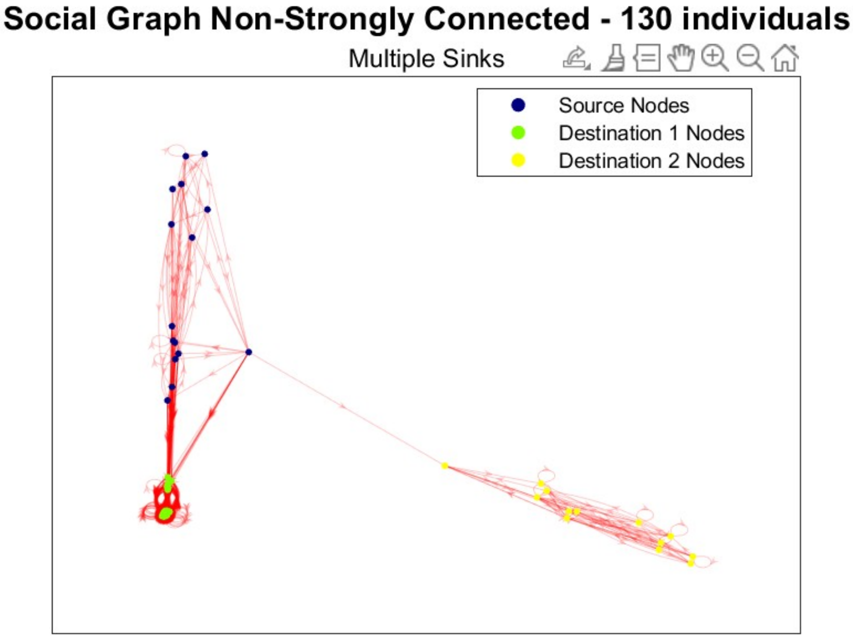

Figure 1.13: Non-SC Graph with Multiple Destinations: Source is more connected with Destination1

As before, the two groups of major engineers develop different ideas since they do not interact with each other. Meanwhile, the interns tend to converge towards the consensus value of the sector group to which they are more connected, in this case Group 1. This can be clearly analyzed in Fig. 1.14.

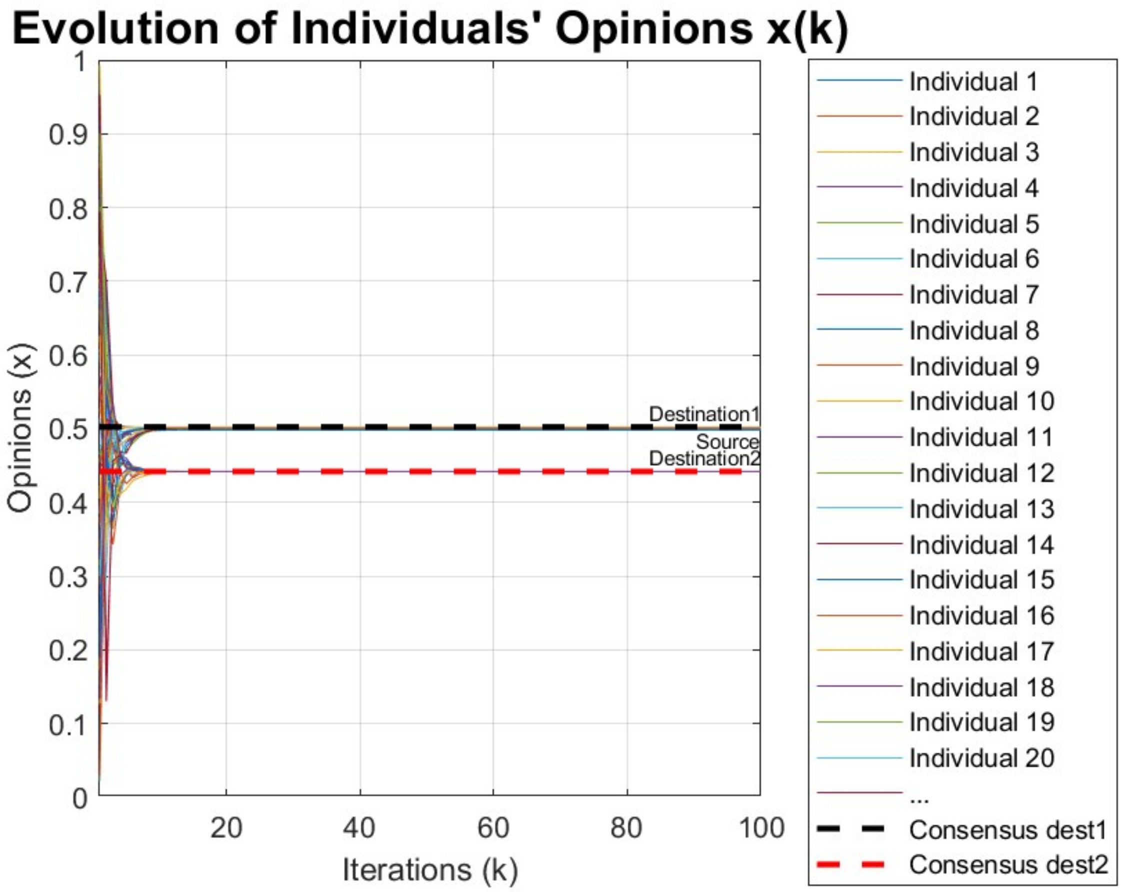

Figure 1.14: DeGroot Model - Opinion dynamics with N-SC graph with Multiple Destinations: Source is more connected with Destination1

Chapter 2

EPO Model

This chapter will explore the EPO (Expressed and Private Opinion) model. It investigates the factors that contribute to have discrepancies between an individual's publicly expressed opinions and their private beliefs. An individual's expressed opinion may be a modified version of their private opinion, depending on the key parameters of the model. The objective of this chapter is to establish a direct relationship between the key parameters and the features of the network. For this purpose, different types of networks, both deterministic and random, will be presented to analyze how their structures influence the model and, consequently, the opinions dynamics.

2.1 Model

The mathematical formulation of the EPO model serves as a subtle refinement of the Friedkin-Johnsen model. Specifically, in this model, an individual i at time instant k possesses a private opinion $x_i(k)$ as well as an expressed opinion $\hat{x}_i(k)$. The individual's private opinion evolves according to the Friedkin-Johnsen dynamics, but the model also incorporates the dynamics of their expressed opinion. On the other hand, the expressed opinion is influenced by the pressure to conform to the average public opinion. This can introduce a discrepancy between the private and expressed opinions. As mentioned earlier, considering a population of n individuals, the dynamics of the private and expressed opinions for each individual are described respectively by the following equations.

$$x_i(k+1) = \lambda_i \left[a_{ii} x_i(k) + \sum_{j \neq i}^{n} a_{ij} \hat{x}_j(k) \right] + (1 - \lambda_i) x_i(0) \tag{2.1}$$

$$\hat{x}_i(k) = \phi_i x_i(k) + (1 - \phi_i) \hat{x}_{avg}(k-1) \tag{2.2}$$

where the matrix A is nonnegative and row-stochastic and $\hat{x}_{avg}$ is called the public opinion.

The parameter $\lambda_i \in [0, 1]$ represents individual i's susceptibility to influence. For instance, if $\lambda_i = 1$, it indicates that individual i is fully susceptible to interpersonal influence, meaning i's opinions are entirely swayed by the opinions of others in their network. Conversely, if $\lambda_i = 0$, the individual remains completely steadfast in their private opinion, unaffected by external influences.

The parameter $\phi_i \in [0, 1]$ denotes individual i's ability to withstand group pressure. Notably, if $\phi_i = 1$, individual i exhibits complete resilience to social influence, resulting in perfect alignment between i's expressed opinion and private opinion. In contrast, if $\phi_i = 0$, the individual's own opinion is totally overwhelmed by network's average opinion. The time-shift of $\hat{x}_{avg}(k-1)$ ensures that Equation 2.2 is consistent with the discussion process. In this process, each individual i, at time step k, expressed opinion $\hat{x}_i(k)$ and learns of others' expressed opinions $\hat{x}_j(k), j \neq i$. Next, the privately held opinion $x_i(k+1)$ evolves according to 2.1. After this, individual i then determines the next $\hat{x}_i(k+1)$ to be expressed in the next round of discussion, according to 2.2.

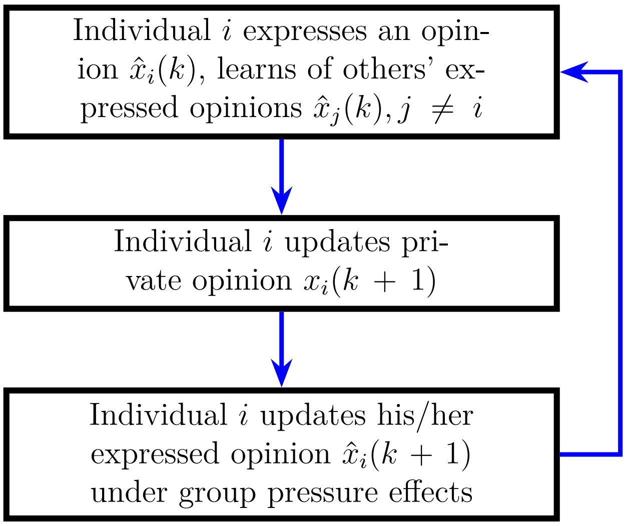

Figure 2.1: The discussion process.

In this context, a linear relationship between the parameters λ and ϕ and the key features of the network topology can be establish. Specifically, two important metrics are taking into account: the clustering coefficient and the closeness centrality of each node of the network.
Firstly, the clustering coefficient of a node i measures how interconnected its neighbors are and it is defined as:

$$c_i = \frac{2E_i}{k_i(k_i - 1)} \tag{2.3}$$

where E_i represents the number of links between the neighbours of node i and k_i the out-degree of the node i. A high clustering coefficient indicates that a node is part of a tightly-knit community, where most of its neighbors are also interconnected. Conversely, a low clustering coefficient suggests that a node's neighbors are more loosely connected.

The closeness centrality of a node i measures how close a node is to all other nodes in the network, based on the shortest paths between them. It is defined as:

$$centrality_i = \frac{n-1}{\sum_j d(i,j)} \tag{2.4}$$

where $d(i,j)$ is the shortest path distance between nodes i and j. A node with a high closeness centrality is centrally located, meaning it can quickly reach other nodes via relatively short paths. On the other hand, a node with low closeness centrality is more peripheral, indicating that it takes longer for influence to spread to or from that node.

Considering these networks properties, the parameters λ and ϕ for each individual i of the network can be expressed as linear functions of its clustering coefficient c_i and closeness centrality $centrality_i$, respectively:

$$\lambda_i = a \cdot c_i + b \tag{2.5}$$

$$\phi_i = 1 - (u \cdot \text{centrality}_i + v) \tag{2.6}$$

where $a, b, u, v \in \mathbb{R}$ are real constants.
These formulations allows to incorporate features of the network into the dynamics of the model. The susceptibility of an individual i to influence, λ_i, is linearly dependent on how tightly interconnected the node's neighborhood is, as captured by the clustering coefficient c_i. Specifically, a high clustering coefficient c_i corresponds to a high λ_i, so when a node is part of a closely knit community, it becomes more likely to conform to the opinions of its neighbors. On the other hand, the resistance of an individual i to group pressure, ϕ_i, is inversely related to its centrality closeness $centrality_i$ within the network. In other words, a higher value of closeness centrality leads to a lower ϕ_i. This means that individuals with high centrality are less resistant to external influence, as they are more connected and have shorter paths to the rest of the network. In contrast, individuals with lower centrality, and thus higher ϕ_i, are less susceptible to external influence and their expressed and private opinion coincide, potentially due to their more peripheral position in the network.

2.2 Simulation

In the following sections, EPO model simulations will explore how different network topologies impact the dynamics of private and expressed opinions. The aim is to illustrate how variations in network structure influence the parameters λ and ϕ, leading to different outcomes in opinion dynamics and revealing how susceptibility to influence and resistance to group pressure shape the processes of convergence or divergence. In all simulations, the following parameters are set as: $n = 100$, $a = 1$, $b = 0$, $c = 1$ and $d = 0$, expect where noted.

2.2.1 All-to-All Network

For the initial test, an All-to-All network is considered, where each node is connected to every other node, except itself. This setup reflects the dynamics of a small group of individuals who engage in constant discussion, where everyone has direct access to each other. To represent this situation, a number of individuals $n = 15$ is chosen. This fully connected topology ensures that every individual can directly influence all others in the network.

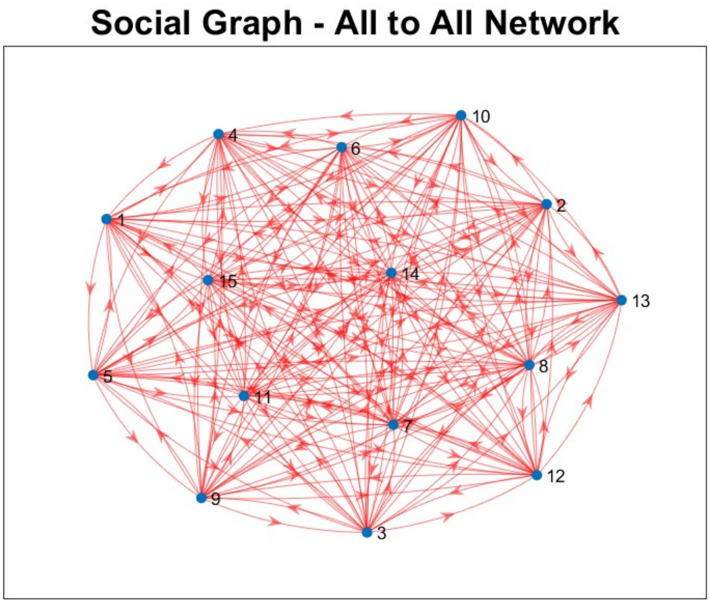

Figure 2.2: All-to-All Network

In such a configuration, both the Clustering Coefficient and Closeness Centrality are uniform across all nodes, as each node has the same number of neighbors and identical shortest paths to other nodes. As a result, every individual exhibits high susceptibility to influence and low resistance to group pressure, as illustrated in Fig. **??**.

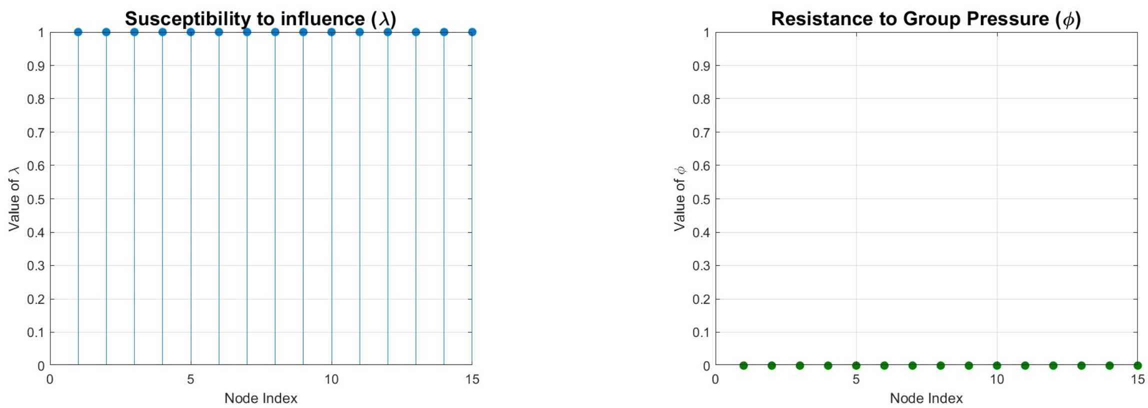

Figure 2.3: All-to-All Network - λ and ϕ

The results in Fig. 2.3 characterize the opinion dynamics illustrated below.

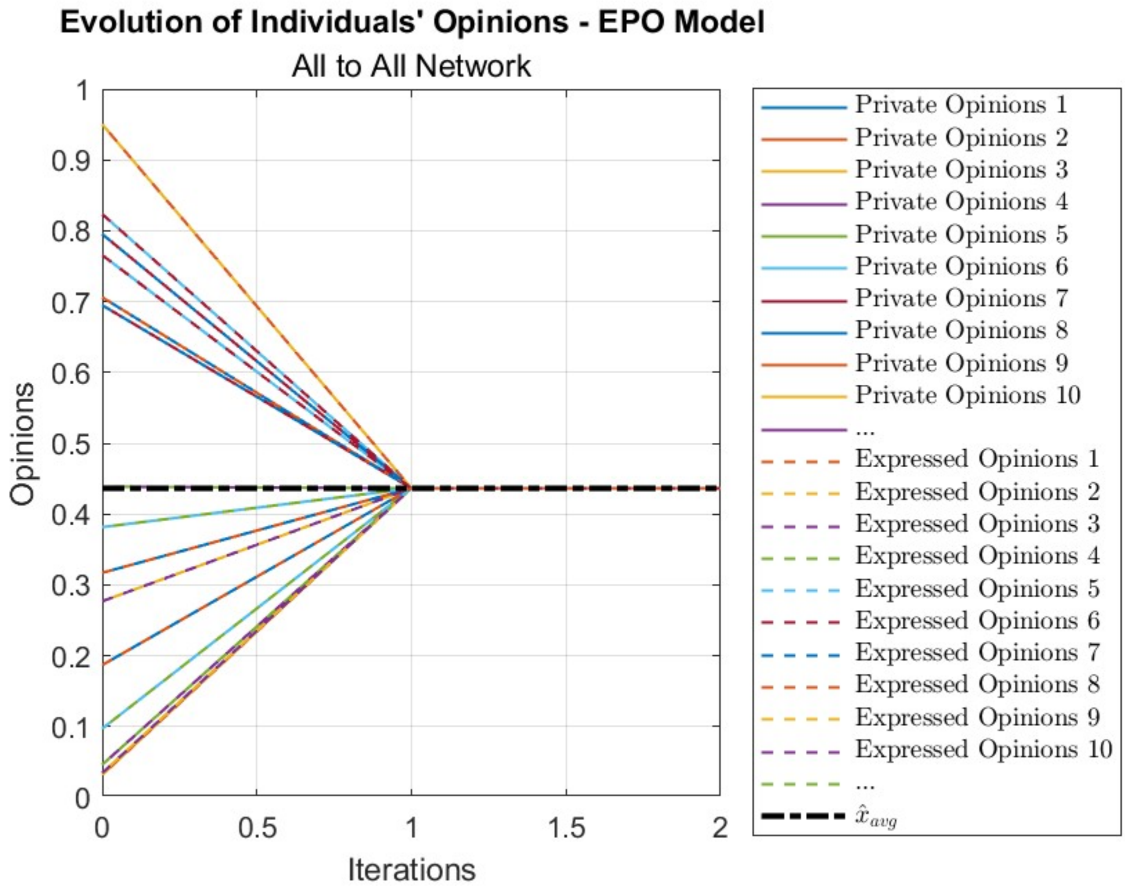

Figure 2.4: All-to-All Network - Private and Expressed Opinion dynamics

The features of the network reveal that all individuals are maximally susceptible to interpersonal influence, resulting in their private opinions conforming to the average public opinion of the network $\hat{x}_{\text{avg}}$, as $\lambda = 1$. Furthermore, their expressed opinions also converge to this value, as their resistance to group pressure is effectively null, $\phi = 0$. This demonstrates a rapid consensus among individuals, indicating no resistance to social influence and a complete inclination to adopt the collective opinion.

However, the All-to-All structure may not reflect realistic social interactions, especially at a large scale, since such complete connectivity is rare. Furthermore, modeling opinion dynamics in a fully connected network can lead to rapid opinion convergence, which does not allow for the exploration of opinion diversity.

2.2.2 Star Network

Another test can be conducted on a Star network topology, where all nodes are connected to a central node, in this case Node 1. On the other hand, Node 1 is also connected to each of the other nodes. This structure can be represented as a scenario in which a party organizer acts as the central figure, while counselors communicate directly with the organizer, providing their individual advice. However, the counselors do not interact with one another.

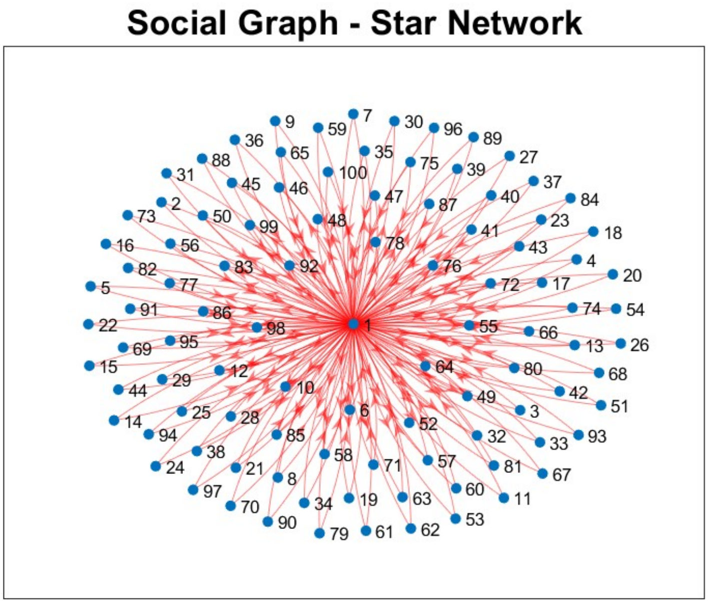

Figure 2.5: Star Network

Despite its central role, Node 1 exhibits a low Clustering Coefficient similar to that of the peripheral nodes. This is because the clustering coefficient measures the extent to which a node's neighbors are interconnected. In a Star Network, the peripheral nodes are only connected to Node 1 and not to each other, resulting in a low clustering coefficient for all nodes, including the central one.

On the other hand, Node 1 has a high Closeness Centrality, indicating that it can reach all other nodes in the network through the shortest possible paths. Consequently, this high centrality translates to a lower value for ϕ, which reflects the node's low ability to withstand the group pressure. As a result, the party organizer (Node 1)'s opinion, holding a central position, is highly influenced by the advice of the counselors.

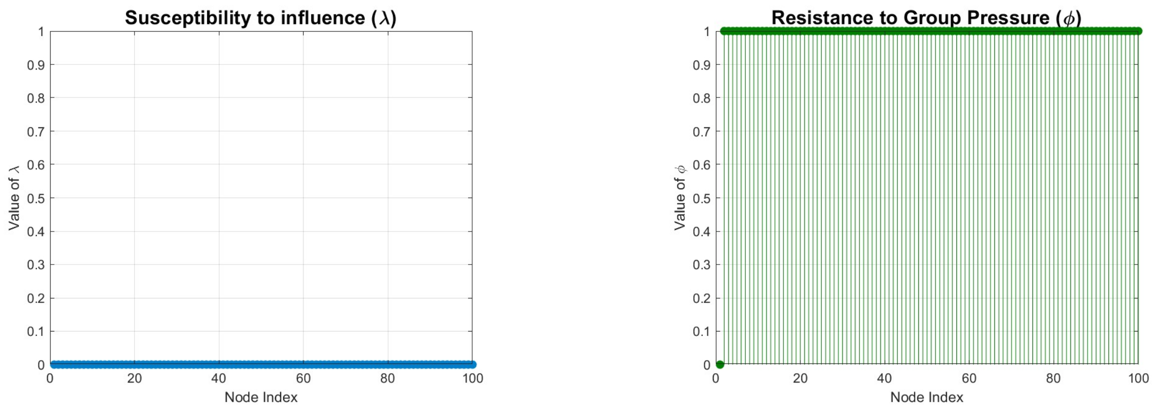

Figure 2.6: Star Network - λ and ϕ

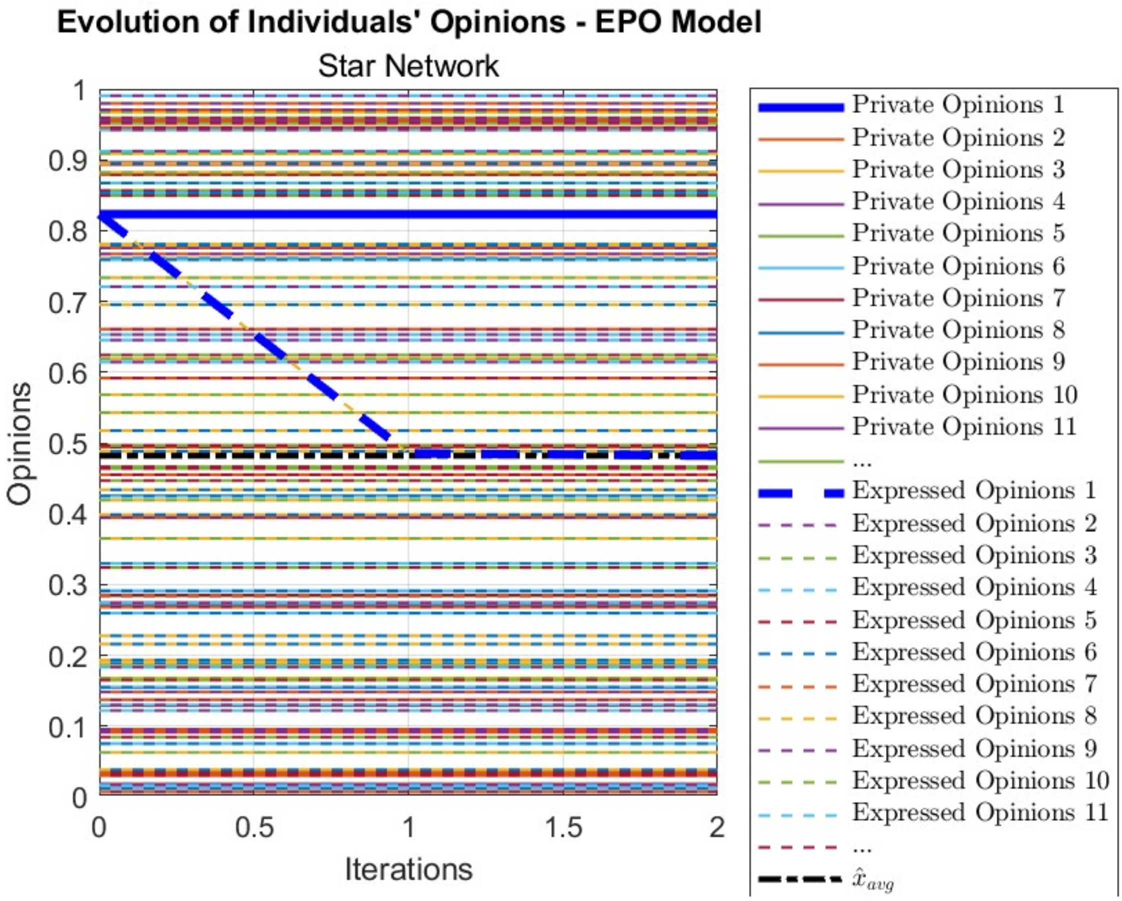

Figure 2.7: Star Network - Private and Expressed Opinion dynamics

As illustrated in Fig. 2.7, the low values of λ and high values of ϕ for peripheral nodes make these individuals maximally closed to interpersonal influence while being fully resilient to pressure. In contrast, the opinion dynamics of Node 1, highlighted in blue in the figure, exhibit high closeness centrality, which corresponds to low resistance to group pressure ϕ. Consequently, Node 1's expressed opinion converges to public opinion $\hat{x}_{\text{avg}}$. However, despite this convergence in expressed opinion, its low susceptibility λ shows how their private opinion remains invariant.

The Star Network does not fully capture real-world dynamics since it's improbable for 100 individuals to rely exclusively on a single central figure with minimal direct communication among themselves.

2.2.3 k-Nearest Neighbours Network

Another experiment can be conducted on a k-Nearest Neighbors network, where each node is connected exclusively to its k closest neighbors. This structure can be found in a local community, where individuals communicate only with their immediate neighbors, such as those living in adjacent houses. In this configuration, a number of neighbors equal to $k = 6$ is assigned to each node.

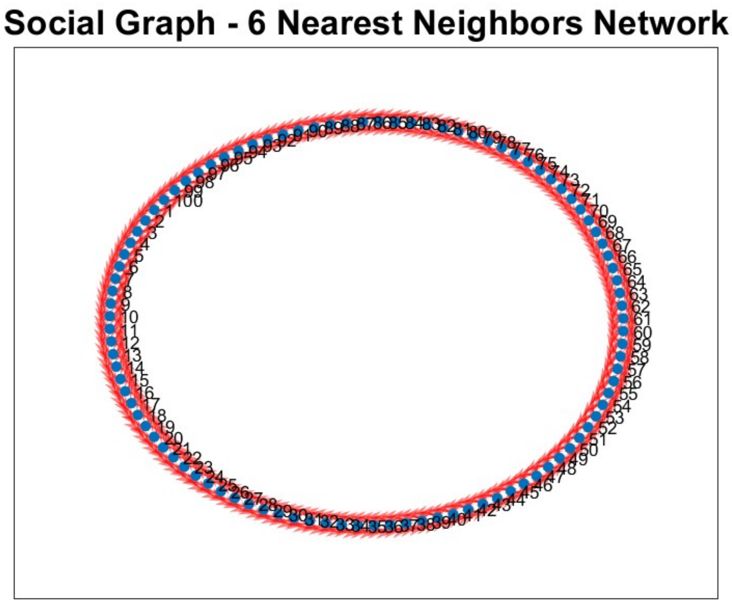

Figure 2.8: 6 Nearest Neighbours Network

Consequently, nodes in the Nearest Neighbors network exhibit a higher Clustering Coefficient compared to the Star network, as many of a node's neighbors are interconnected. This interconnectivity enables each node to efficiently reach other nodes within the network through the shortest paths, resulting in a closeness centrality of 1, leading to a low ϕ.

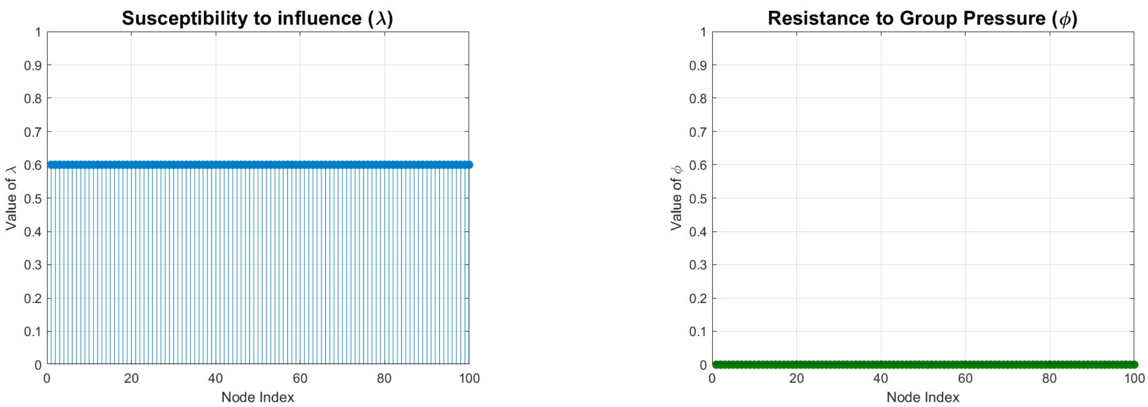

Figure 2.9: 6 Nearest Neighbours Network - λ and ϕ

As shown in Fig. 2.10, the high level of interconnectivity within the 6 Nearest Neighbors Network facilitates information flow. A λ value of 0.6 indicates a moderate susceptibility to influence, allowing nodes to balance their private opinions with the expressed opinions of their neighbors. In fact, the private opinions are closer to $\hat{x}_{\text{avg}}$ compared to their initial values, but they do not converge to that value. However, the low value of ϕ leads to the convergence of individuals' expressed opinions toward $\hat{x}_{\text{avg}}$.

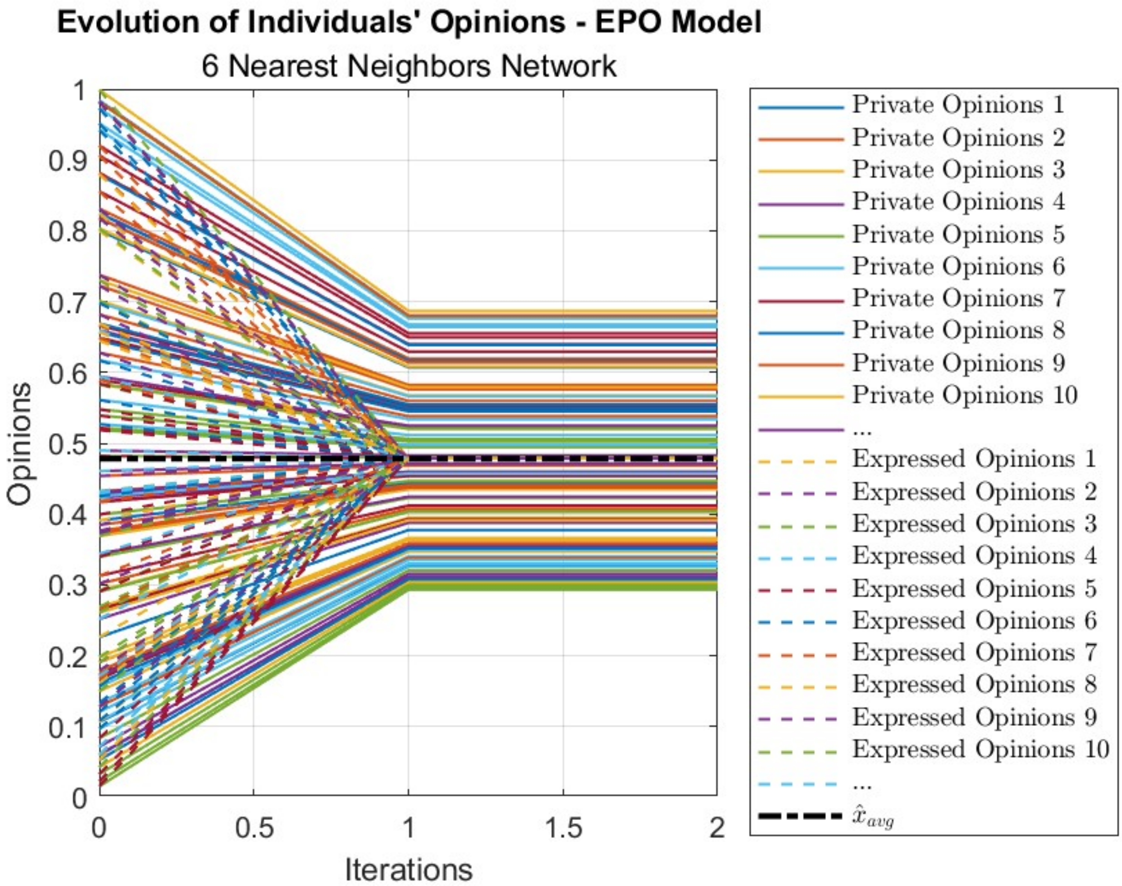

Figure 2.10: 6 Nearest Neighbours Network - Private and Expressed Opinion dynamics

While the 6 Nearest Neighbors Network provides an interesting perspective, its structural assumptions are unrealistic for modeling social graphs. In reality, human interactions are typically more complex and influenced by a variety of factors. Limiting individuals to communicate only with their close neighbors fails to capture the importance of long-distance connectivity, which is often present in social networks.

2.2.4 Small World Network

Another simulation can be performed on a Small-World network, which combines features of both deterministic and random networks, providing a more realistic model of social interactions. In this configuration, each node is initially connected to its k-Nearest Neighbors. Then some of these connections are rewired to random nodes with a probability p. To generate this type of network, the MATLAB function 'WattsStrogatz(n,k,p)' was used. This structure mirrors online social networks, such as Facebook, where most users are connected to a small group of friend, but occasional long-range connections significantly reduce the average distance between users. As a result, the Small-World network exhibits both local clusters and long-range connections. For this simulation, $k = 10$ and $p = 0.015$ are chosen.

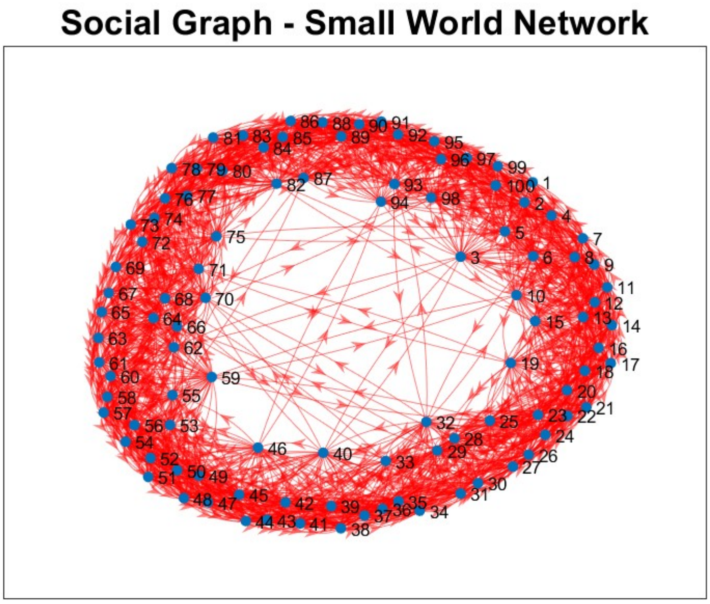

Figure 2.11: Small World Network

In this way, the number of neighbors k is fixed and so the clustering coefficient remains fairly consistent across all nodes. With $k = 10$, the clustering coefficient is relatively high, which in turn correlates with a higher value of λ. Most nodes exhibit moderate closeness centrality due to their position between nearby and distant nodes. However, certain nodes may achieve particularly high closeness centrality if they form key shortcuts, enabling faster access to distant parts of the network. This becomes evident when comparing the most central nodes in the network, which exhibit particularly high closeness centrality alongside their corresponding low values of ϕ. This behavior can be observed in specific nodes, such as nodes 3, 40, and 59. These nodes are connected to other nodes that are not closely related, which can lead to fewer tightly-knit clusters and consequently a lower, though still significant, value of λ. Despite this, the results in Fig. 2.12 demonstrate that the overall clustering coefficient and closeness centrality of the network remain elevated due to the presence of local clusters and connections among nodes.

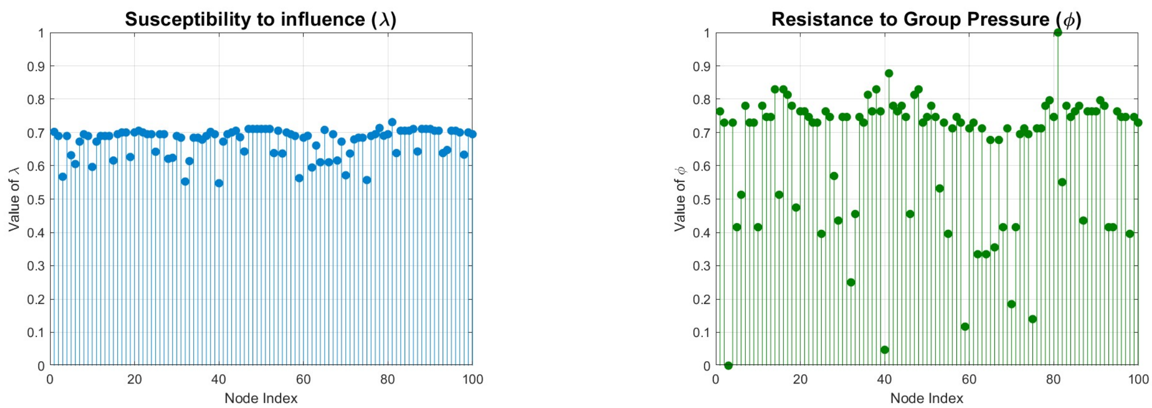

Figure 2.12: Small World Network - λ and ϕ

Due to lower resistance to group pressures, the expressed opinions of the previously mentioned nodes converge toward the average value $\hat{x}_{\text{avg}}$, particularly for nodes 3 and 40, while node 59 shows a slight threshold. In contrast, their private opinions remain stable and are not significantly altered. This stability is largely attributed to the overall high clustering coefficient in the network, indicating a moderate attachment to their own initial private opinion.

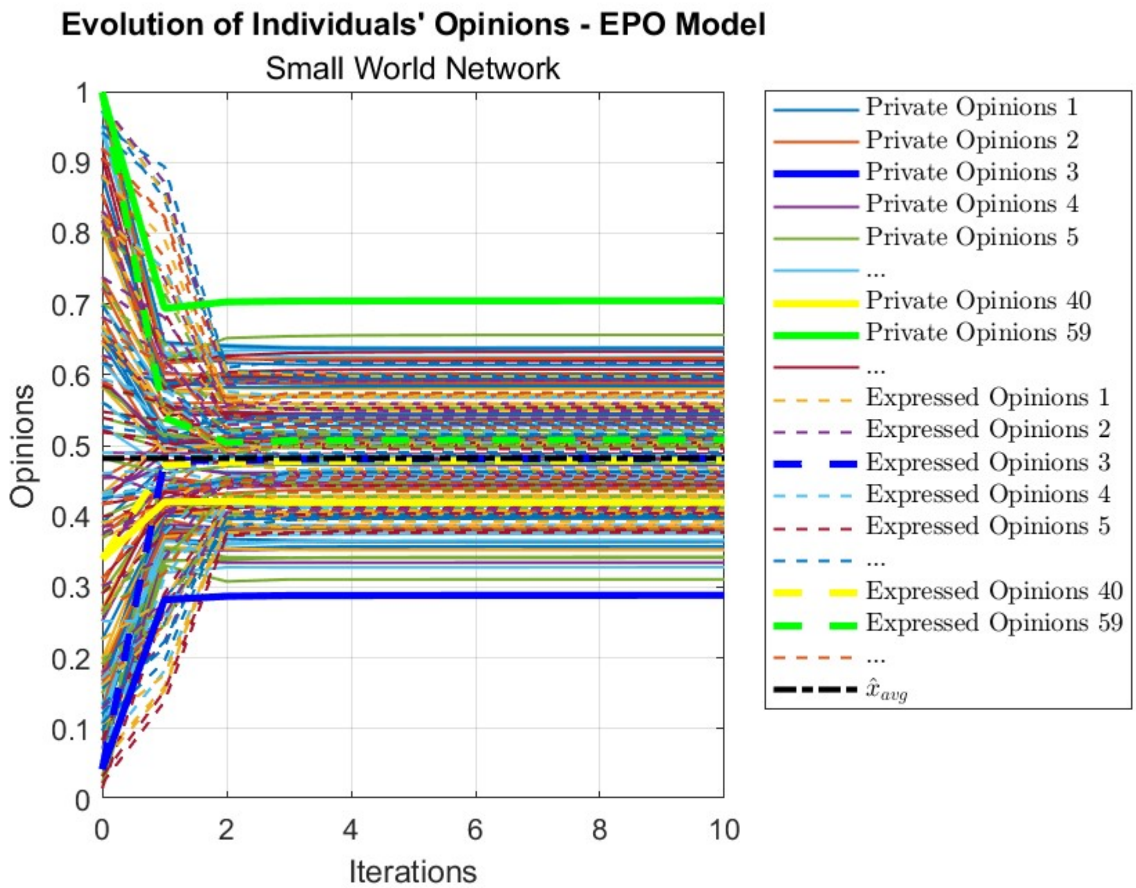

Figure 2.13: Small World Network - Private and Expressed Opinion dynamics

While it closely resembles real-world networks, Small-World Networks may lead to overly rapid opinion convergence due to the long-range links.

2.2.5 Scale Free Network

Another test can be conducted using the Barabási-Albert (BA) model, which is designed to generate Scale-Free networks characterized by a power-law degree distribution. In this model, nodes are added to the network one at a time and each new node establishes connections to existing nodes with a probability proportional to their degree, embodying the principle of "preferential attachment". To generate this type of network, the function 'Scale_Free(n, mlinks)' is used. For this simulation, the initial matrix is generated using the Erdős-Rényi algorithm with $m = 35$ nodes and a probability $p = 0.6$. Each new node will then connect to $m_{\text{links}} = 2$ existing nodes. This mechanism ensures that nodes with higher connectivity have a greater chance of receiving additional links, leading to the emergence of hubs, which are highly connected nodes.
The parameters are set as follows: $m = 35$, $p = 0.6$ and $m_{\text{links}} = 2$.

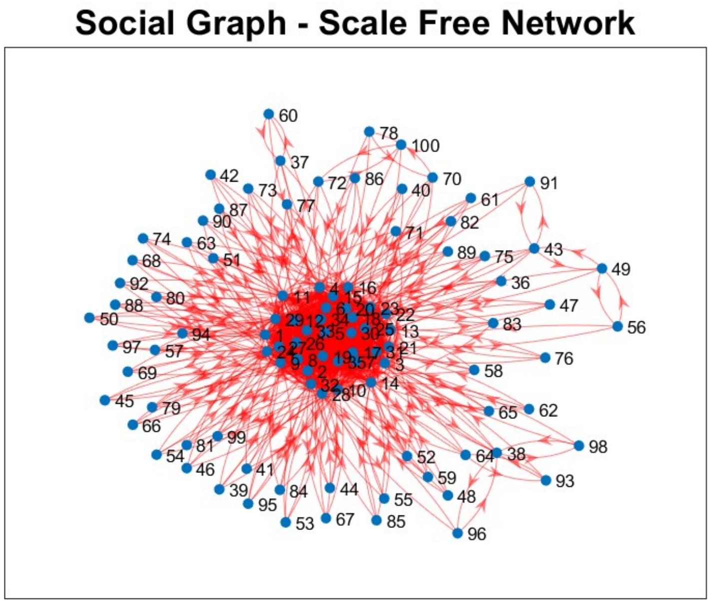

Figure 2.14: Scale Free Network

The nodes of the initial matrix, the Hubs, have a relatively low clustering coefficient due to the low m_{links}. On the other hand, the additional nodes, the Peripherals, exhibit a higher clustering coefficient since they connect to the hubs which present a connection rate of $p = 0.6$. However, some exceptions exist among the most distant nodes, whose neighbors do not form clusters.
Regarding closeness centrality, the Hubs will present high closeness centrality and, consequently, a lower resistance to group pressure, in contrast to the more Peripheral nodes.

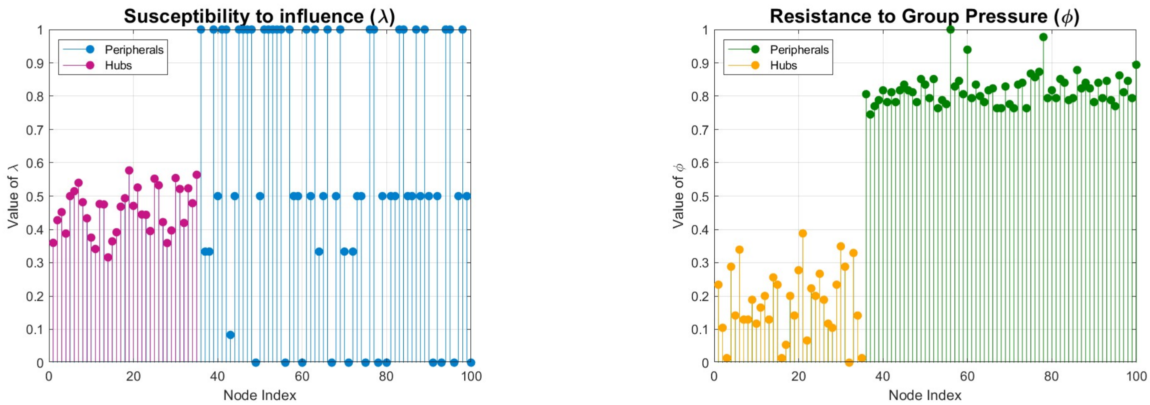

Figure 2.15: Scale Free Network - λ and ϕ

In this case, the simulation duration was extended because the system takes longer to stabilize compared to previous scenarios. This increased duration reflects the complexities inherent in a Scale-Free network, where the opinions dynamics are influenced by a mix of high-connection hubs and less connected peripheral nodes.
A real-world example of a Scale-Free network can be observed in Instagram. Influencers serve as the Hubs, possessing a vast number of connections. Due to their central position and low ϕ, their expressed opinions can be easily influenced by the network's average

opinion, converging towards the average public opinion $\hat{x}_{\text{avg}}$. While the relative low λ values make their private opinions moderate susceptible to interpersonal influence. In contrast, the followers of these influencers act as Peripheral nodes. As said before, most of followers exhibit high λ values, making their private opinions susceptible to group dynamics. However, their high ϕ enables them to resist group pressure in their expressed opinions. These results can be seen in the following Figure.

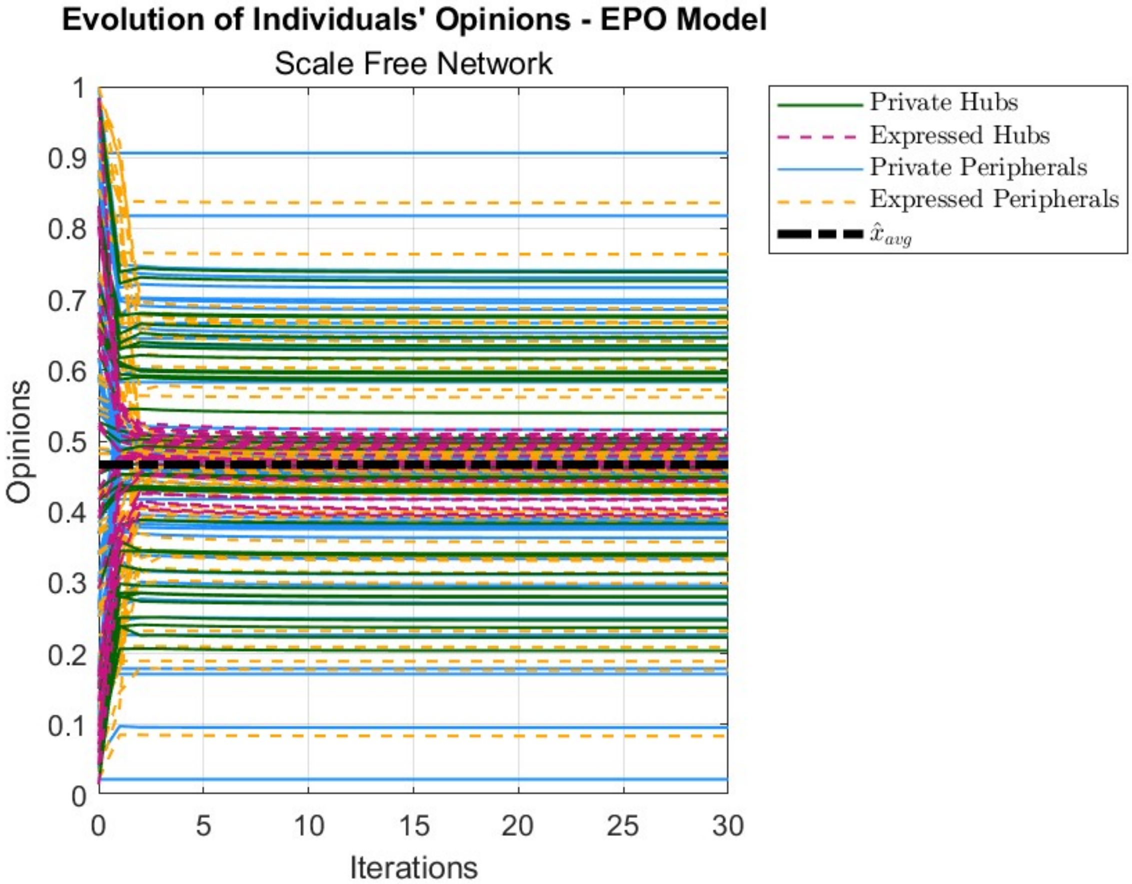

Figure 2.16: Scale Free Network - Private and Expressed Opinion dynamics

The Scale Free Network appears to be the closest to reality, although it still presents certain limitations. This structure illustrates how opinions can be influenced by the interconnectivity of nodes, but it may not fully account for other factors influencing human behavior, such as personal experiences, cultural backgrounds or emotional ties.

Conclusion

This project explored the dynamics of opinion formation through two models: the DeGroot model and the EPO model. The theoretical analysis, the simulation experiments and the examination of practical examples, conducted throughout this project, provided meaningful insights into the key questions regarding opinion dynamics.
In **Chapter 1** the DeGroot model was examined, focusing on its convergence properties. The analysis demonstrated that under certain conditions, such as in a Strongly Connected Network, the individuals' opinions converge to a consensus. However, in less connected or more complex networks, opinions might reach partial consensus, just convergence or not convergence at all. In the Strongly Connected Network with one Aperiodic Destination partial consensus was observed, while in the Strongly Connected Network with Multiple Destinations convergence has been found. The practical example provided in this chapter illustrated these dynamics, highlighting how the structure of the network plays a critical role in shaping the outcome of opinion exchanges.
In **Chapter 2** the EPO model was examined. The analysis investigated how network topology is correlated with the model's parameters. Specifically, the clustering coefficient and the closeness centrality of each node of the network significantly affect the susceptibility to influence and resistance to group pressure of each individual. The results were more clearly observed in Deterministic networks, where parameter values are relatively uniform across all nodes. However, these findings reflect less the real-world complexities observed in Random networks, which have more variety in parameter values. The tests demonstrated that in networks with high clustering coefficient, individuals are maximally susceptible to interpersonal influence, as seen in the All-to-All Network or among the Peripherals Nodes of the Scale Free Network. On the contrary, in networks with low clustering coefficient, individual's private opinions are less susceptible to social influence, as in the Star Network. Furthermore, the experiments also revealed that individuals with high closeness centrality are totally overwhelmed by the network's average opinion, as evident in the All-to-All Network, in the Nearest Neighbours Network or among the Hubs Nodes of the Scale Free Network. In contrast, individuals with lower closeness centrality exhibit a full resilience to group pressure, showing the alignment of their expressed and private opinion, as observed in the Star Network and most Nodes of the Small World Network.
Overall, the findings of this project emphasize the critical importance of the network topology in shaping opinion dynamics.

Future researches could explore additional variations of these models, incorporating dynamic network structures or more sophisticated interaction rules, to further investigate how opinions evolve in even more complex social environments. One interesting development could involve exploring how an individual can influence all others within the network.

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