Floating-point arithmetic and error analysis (AFAE)

Practical n° 5 - Introduction to interval arithmetic

In this practical, we will use MATLAB and the INTLAB library that implements an interval arithmetic. First, one has to install the library that can be downloaded at the following address: `http://www-pequan.lip6.fr/~graillat/intlab.zip`

Some documentation about INTLAB and some links on articles explaining how it works are available at the following address: `www.ti3.tu-harburg.de/intlab/`

With INTLAB, the following functions make it possible to change the rounding mode:
- `setround(-1)`: rounding toward $-\infty$;
- `setround(1)`: rounding toward $+\infty$;
- `setround(0)`: rounding to the nearest.

To declare an interval, one can use the command `infsup(.,.)`.

Exercise 1 (Range of a function)**.** We consider the following function: $f(x) = x^2 - 4x$ on $\mathbf{X} := [1, 4]$

1. Using interval arithmetic, evaluate $f(\mathbf{X})$ using the formula of the definition of f but also use the following formulas: $f(x) = x(x-4)$ and $f(x) = (x-2)^2 - 4$.
2. Explain why one of the formulas gives a more accurate result than the others.

Exercise 2 (Invertibility of a matrix)**.** Let A be a matrix of size $n \times n$ with floating-point coefficients.

1. Show that if there existe a matrix R such that $\|I - RA\| < 1$ then A is invertible (nonsingular).
2. Using interval arithmetic and the `inv` function, give an algorithm certifying the invertibility of A.

Exercise 3 (Numerical solutions of linear systems)**.** Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ be respectively a matrix and a vector. Our aim is to solve the linear system $Ax = b$ by obtaining an enclosure of the exact solution (*i.e.* a interval containing the exact solution).

1. Implement the Gaussian Elimination (GE) algorithm with interval arithmetic to solve the linear system $Ax = b$. Test your program with the Hilbert matrix $H_{ij} = (1/(i + j - 1))$.
2. Let $R \in \mathbb{R}^{n\times n}$ be a matrix and I be the identity matrix in $\mathbb{R}^{n\times n}$. Assume that $Rb + (I - RA)\mathbf{X} \subset \operatorname{int}(\mathbf{X})$. Show that A and R are invertible and that we have $x = A^{-1}b \in Rb + (I - RA)\mathbf{X}$. Propose an algorithm that returns an inclusion for the exact solution x.
3. Using your implementation of the GE algorithm with interval arithmetic, propose an algorithm that returns an inclusion for the determinant of a matrix.
4. Propose a more accurate implementation by using the Gerschgorin circles.