Tutorial 3: Error Analysis and Conditioning

Floating-point arithmetic and error analysis (AFAE)

Sorbonne Université

Contents

1 Exercise 1: Summation

1.1 Question 1: Condition Number of Summation

Goal: Show that

$$\operatorname{cond}\left(\sum_{i=1}^{n} p_i\right) = \frac{\sum_{i=1}^{n} |p_i|}{|\sum_{i=1}^{n} p_i|} \tag{1}$$

Given Definition:

$$\operatorname{cond}\left(\sum_{i=1}^{n} p_i\right) := \lim_{\varepsilon \to 0} \sup \left\{ \frac{|\sum_{i=1}^{n} \tilde{p}_i - \sum_{i=1}^{n} p_i|}{\varepsilon |\sum_{i=1}^{n} p_i|} : |\tilde{p}_i - p_i| \leq \varepsilon |p_i| \text{ for } i = 1, \ldots, n \right\} \tag{2}$$

Proof:
Let $S = \sum_{i=1}^{n} p_i$ and $\tilde{S} = \sum_{i=1}^{n} \tilde{p}_i$.
The perturbation is:

$$\tilde{S} - S = \sum_{i=1}^{n} \tilde{p}_i - \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} (\tilde{p}_i - p_i) \tag{3}$$

Given the constraint $|\tilde{p}_i - p_i| \leq \varepsilon |p_i|$, we can write:

$$\tilde{p}_i - p_i = \delta_i |p_i| \tag{4}$$

where $|\delta_i| \leq \varepsilon$.
Therefore:

$$\tilde{S} - S = \sum_{i=1}^{n} \delta_i |p_i| \tag{5}$$

Taking absolute values:

$$|\tilde{S} - S| = \left| \sum_{i=1}^{n} \delta_i |p_i| \right| \leq \sum_{i=1}^{n} |\delta_i||p_i| \leq \varepsilon \sum_{i=1}^{n} |p_i| \tag{6}$$

The supremum is achieved when all δ_i have the same sign as $|p_i|$:

$$\sup |\tilde{S} - S| = \varepsilon \sum_{i=1}^{n} |p_i| \tag{7}$$

Dividing by $\varepsilon |S|$:

$$\operatorname{cond}(S) = \lim_{\varepsilon \to 0} \frac{\varepsilon \sum_{i=1}^{n} |p_i|}{\varepsilon |S|} = \frac{\sum_{i=1}^{n} |p_i|}{|\sum_{i=1}^{n} p_i|} \tag{8}$$

$$\boxed{\operatorname{cond}\left(\sum_{i=1}^{n} p_i\right) = \frac{\sum_{i=1}^{n} |p_i|}{|\sum_{i=1}^{n} p_i|}} \tag{9}$$

Interpretation

- If all p_i have the same sign, then cond $= 1$ (well-conditioned)
- If there is massive cancellation, $|\sum p_i| \ll \sum |p_i|$, the condition number is large (ill-conditioned)

1.2 Question 2: Backward Stability of Recursive Summation

Recursive Summation Algorithm:

```
 s    =  p
 s    =  s          p
 s    =  s          p
...
 s    =     s                p
```

Where $\oplus$ denotes floating-point addition: $a \oplus b = (a+b)(1+\delta)$ with $|\delta| \leq u$ (machine precision).

Backward Stability: An algorithm is backward stable if the computed result $\tilde{f}(x)$ satisfies:

$$\tilde{f}(x) = f(\tilde{x}) \tag{10}$$

where $\tilde{x}$ is a slightly perturbed input: $|\tilde{x} - x| = O(u)|x|$.

Proof:

For the recursive summation:

$$\tilde{s}_k = ((\tilde{s}_{k-1} + p_k)(1 + \delta_k)) \tag{11}$$

Expanding recursively:

$$\tilde{s}_n = ((p_1(1+\delta_1) + p_2)(1+\delta_2) + p_3)(1+\delta_3) \cdots + p_n)(1+\delta_n) \tag{12}$$

We can rewrite this as:

$$\tilde{s}_n = p_1 \prod_{j=1}^{n} (1+\delta_j) + p_2 \prod_{j=2}^{n} (1+\delta_j) + \cdots + p_n(1+\delta_n) \tag{13}$$

Let $\theta_i = \prod_{j=i}^{n}(1+\delta_j) - 1$. Using the fact that $\prod(1+\delta_j) \approx 1 + \sum \delta_j$ for small δ_j:

$$|\theta_i| \leq (n-i+1)u + O(u^2) \approx (n-i+1)u \tag{14}$$

Therefore:

$$\tilde{s}_n = \sum_{i=1}^{n} p_i(1+\theta_i) = \sum_{i=1}^{n} \tilde{p}_i \tag{15}$$

where $\tilde{p}_i = p_i(1+\theta_i)$ with $|\theta_i| \leq nu$.

This shows that the computed sum is the exact sum of slightly perturbed values $\tilde{p}_i$.

$$\boxed{\text{Recursive summation is backward stable}} \tag{16}$$

Backward Stability

The computed result equals the exact sum of the inputs perturbed by at most $O(nu)$.

1.3 Question 3: Relative Error Bound for Summation

Combining backward stability with conditioning:

Backward Stability gives us:

$$\tilde{s}_n = \sum_{i=1}^{n} p_i(1+\theta_i) \tag{17}$$

with $|\theta_i| \leq nu$.

Forward Error:

$$\left|\frac{\tilde{s}_n - s_n}{s_n}\right| = \left|\frac{\sum_{i=1}^n p_i\theta_i}{\sum_{i=1}^n p_i}\right| \tag{18}$$

Using $|\theta_i| \leq nu$:

$$\left|\frac{\tilde{s}_n - s_n}{s_n}\right| \leq \frac{\sum_{i=1}^n |p_i||\theta_i|}{|\sum_{i=1}^n p_i|} \leq nu \cdot \frac{\sum_{i=1}^n |p_i|}{|\sum_{i=1}^n p_i|} \tag{19}$$

$$\boxed{\left|\frac{\tilde{s}_n - s_n}{s_n}\right| \leq nu \cdot \text{cond}\left(\sum_{i=1}^n p_i\right)} \tag{20}$$

Interpretation:

$$\text{Relative Error} \leq \text{Machine Precision} \times \text{Number of Operations} \times \text{Condition Number} \tag{21}$$

1.4 Question 4: Dot Product Analysis

Dot Product: $d = \sum_{i=1}^n x_i y_i$

1.4.1 (a) Condition Number of Dot Product

Let $d = \sum_{i=1}^n x_i y_i$ with perturbations $\tilde{x}_i, \tilde{y}_i$.

$$\tilde{d} = \sum_{i=1}^n \tilde{x}_i\tilde{y}_i = \sum_{i=1}^n (x_i + \Delta x_i)(y_i + \Delta y_i) \tag{22}$$

$$= \sum_{i=1}^n (x_i y_i + x_i\Delta y_i + y_i\Delta x_i + \Delta x_i\Delta y_i) \tag{23}$$

Neglecting second-order terms:

$$\tilde{d} - d \approx \sum_{i=1}^n (x_i\Delta y_i + y_i\Delta x_i) \tag{24}$$

With $|\Delta x_i| \leq \varepsilon|x_i|$ and $|\Delta y_i| \leq \varepsilon|y_i|$:

$$|\tilde{d} - d| \leq \varepsilon \sum_{i=1}^n (|x_i||y_i| + |y_i||x_i|) = 2\varepsilon \sum_{i=1}^n |x_i y_i| \tag{25}$$

Therefore:

$$\boxed{\text{cond}(x \cdot y) = \frac{\sum_{i=1}^n |x_i y_i|}{|\sum_{i=1}^n x_i y_i|}} \tag{26}$$

This has the same form as the summation condition number!

1.4.2 (b) Backward Stability of Dot Product

The dot product computation involves both multiplication and addition:

```
 t    =  x          y
 t    =  t        ( x          y   )
 t    =  t        ( x          y   )
...
```

Each multiplication: $x_i \otimes y_i = x_i y_i (1 + \delta_i^{\text{mult}})$ with $|\delta_i^{\text{mult}}| \leq u$
Each addition has error as before.
Following similar analysis to summation:

$$\tilde{d} = \sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i \tag{27}$$

where $\tilde{x}_i \tilde{y}_i = x_i y_i (1 + \theta_i)$ with $|\theta_i| \leq 2nu$.

$$\boxed{\text{Dot product computation is backward stable}} \tag{28}$$

1.4.3 (c) Relative Error Bound

$$\boxed{\left| \frac{\tilde{d} - d}{d} \right| \leq 2nu \cdot \text{cond}(x \cdot y) = 2nu \cdot \frac{\sum_{i=1}^{n} |x_i y_i|}{|\sum_{i=1}^{n} x_i y_i|}} \tag{29}$$

Orthogonal Vectors

When $x \perp y$ (nearly orthogonal), $\sum x_i y_i \approx 0$ while $\sum |x_i y_i|$ is not small. This makes the dot product ill-conditioned!

2 Exercise 2: Polynomial Evaluation

2.1 Question 1: Condition Number Formula

For $p(x) = \sum_{i=0}^{n} a_i x^i$, the condition number of evaluating p at x is:

$$\boxed{\text{cond}(p, x) = \frac{\sum_{i=0}^{n} |a_i||x|^i}{|p(x)|} = \frac{\tilde{p}(|x|)}{|p(x)|}} \tag{30}$$

where $\tilde{p}(x) = \sum_{i=0}^{n} |a_i| x^i$ is the polynomial with absolute value coefficients.
Derivation:
Consider perturbations $\tilde{a}_i = a_i(1 + \delta_i)$ with $|\delta_i| \leq \varepsilon$:

$$\tilde{p}(x) = \sum_{i=0}^{n} \tilde{a}_i x^i = \sum_{i=0}^{n} a_i (1 + \delta_i) x^i \tag{31}$$

$$\tilde{p}(x) - p(x) = \sum_{i=0}^{n} a_i \delta_i x^i \tag{32}$$

$$|\tilde{p}(x) - p(x)| \leq \varepsilon \sum_{i=0}^{n} |a_i||x|^i \tag{33}$$

$$\text{cond}(p, x) = \frac{\sum_{i=0}^{n} |a_i||x|^i}{|p(x)|} \tag{34}$$

2.2 Question 2: Backward Stability of Horner Scheme

Horner Scheme:

$$p(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + x \cdot a_n) \cdots)) \tag{35}$$

Algorithm:

```
b    =  a
   b        =     a               (x        b   )
   b        =     a               (x           b        )
...
b    =  a          (x        b   )
```

Proof of Backward Stability:
At each step, we have:

$$\tilde{b}_i = (a_i + x\tilde{b}_{i+1}(1 + \delta_i^{\text{mult}}))(1 + \delta_i^{\text{add}}) \tag{36}$$

Working backwards from $\tilde{b}_n = a_n$:

$$\tilde{b}_{n-1} = (a_{n-1} + xa_n(1 + \delta_n^{\text{mult}}))(1 + \delta_{n-1}^{\text{add}}) \tag{37}$$

After complete expansion, we can show:

$$\tilde{b}_0 = \sum_{i=0}^{n} \tilde{a}_i x^i \tag{38}$$

where $\tilde{a}_i = a_i(1 + \theta_i)$ with $|\theta_i| \leq 2nu$.
This means the computed value is the exact evaluation of a slightly perturbed polynomial.

$$\boxed{\text{Horner scheme is backward stable}} \tag{39}$$

Efficiency

Horner scheme uses only n multiplications and n additions (optimal!)

2.3 Question 3: Relative Error Bound

Combining backward stability with the condition number:

$$\boxed{\left|\frac{\tilde{p}(x) - p(x)}{p(x)}\right| \leq 2nu \cdot \text{cond}(p, x) = 2nu \cdot \frac{\tilde{p}(|x|)}{|p(x)|}} \tag{40}$$

When is polynomial evaluation ill-conditioned?

- When $|p(x)| \ll \tilde{p}(|x|)$, i.e., when there is significant cancellation in the sum
- Near roots of the polynomial: as $x \to \alpha$ where $p(\alpha) = 0$, we have $|p(x)| \to 0$

2.4 Question 4: Distance to Nearest Root

Given: Distance $d(p,q) = \max_i\{|a_i - b_i|/|a_i|\}$
To Show:

$$\min\{d(p,q) : q(z) = 0\} = \frac{1}{\operatorname{cond}(p,z)} \tag{41}$$

Proof:
We want to find the smallest relative perturbation $d(p,q)$ such that $q(z) = 0$.
Let $q(x) = \sum_{i=0}^{n} b_i x^i$ where $b_i = a_i(1+\delta_i)$.
For $q(z) = 0$:

$$\sum_{i=0}^{n} a_i(1+\delta_i)z^i = 0 \tag{42}$$

$$p(z) + \sum_{i=0}^{n} a_i\delta_i z^i = 0 \tag{43}$$

The minimum distance occurs when all δ_i are chosen to optimally cancel $p(z)$:

$$\sum_{i=0}^{n} a_i\delta_i z^i = -p(z) \tag{44}$$

The minimum value of $\max_i |\delta_i|$ needed to achieve this is:

$$d_{\min} = \frac{|p(z)|}{\sum_{i=0}^{n} |a_i||z|^i} = \frac{|p(z)|}{\tilde{p}(|z|)} = \frac{1}{\operatorname{cond}(p,z)} \tag{45}$$

$$\boxed{\min\{d(p,q) : q(z) = 0\} = \frac{1}{\operatorname{cond}(p,z)}} \tag{46}$$

Interpretation:

- If $\operatorname{cond}(p,z)$ is large (ill-conditioned), then a tiny perturbation in coefficients can create a root at z
- This explains why roots of polynomials are sensitive to coefficient errors

3 Exercise 3: Roots of Polynomials

3.1 Question 1: Condition Number of a Simple Root

Given: $p(\alpha) = 0$ and $p'(\alpha) \neq 0$ (simple root)
Definition:

$$K(p,\alpha) := \lim_{\varepsilon\to 0} \sup_{|\Delta a_i|\le\varepsilon|a_i|} \left\{\frac{|\Delta\alpha|}{\varepsilon|\alpha|}\right\} \tag{47}$$

To Show:

$$K(p,\alpha) = \frac{\tilde{p}(|\alpha|)}{|\alpha||p'(\alpha)|} \tag{48}$$

Proof:
Consider the perturbed polynomial:

$$\tilde{p}(x) = \sum_{i=0}^{n} (a_i + \Delta a_i)x^i \tag{49}$$

Let $\tilde{\alpha} = \alpha + \Delta\alpha$ be the perturbed root: $\tilde{p}(\tilde{\alpha}) = 0$.

Taylor expansion of $\tilde{p}$ around α:

$$\tilde{p}(\tilde{\alpha}) = \tilde{p}(\alpha) + \tilde{p}'(\alpha)\Delta\alpha + O((\Delta\alpha)^2) = 0 \tag{50}$$

Since $\tilde{p}(\alpha) = p(\alpha) + \sum_{i=0}^{n} \Delta a_i \alpha^i = \sum_{i=0}^{n} \Delta a_i \alpha^i$ (as $p(\alpha) = 0$):

$$\sum_{i=0}^{n} \Delta a_i \alpha^i + p'(\alpha)\Delta\alpha + O(\varepsilon^2) = 0 \tag{51}$$

To first order:

$$\Delta\alpha \approx -\frac{\sum_{i=0}^{n} \Delta a_i \alpha^i}{p'(\alpha)} \tag{52}$$

Taking absolute values:

$$|\Delta\alpha| \leq \frac{\sum_{i=0}^{n} |\Delta a_i||\alpha|^i}{|p'(\alpha)|} \leq \frac{\varepsilon \sum_{i=0}^{n} |a_i||\alpha|^i}{|p'(\alpha)|} \tag{53}$$

Therefore:

$$\frac{|\Delta\alpha|}{\varepsilon|\alpha|} \leq \frac{\sum_{i=0}^{n} |a_i||\alpha|^i}{|\alpha||p'(\alpha)|} = \frac{\tilde{p}(|\alpha|)}{|\alpha||p'(\alpha)|} \tag{54}$$

$$\boxed{K(p,\alpha) = \frac{\tilde{p}(|\alpha|)}{|\alpha||p'(\alpha)|}} \tag{55}$$

3.2 Question 2: When is a Simple Root Ill-Conditioned?

A simple root α is **ill-conditioned** when $K(p,\alpha)$ is large.

From the formula: $K(p,\alpha) = \frac{\tilde{p}(|\alpha|)}{|\alpha||p'(\alpha)|}$
Ill-conditioning occurs when:

1. **Small derivative**: $|p'(\alpha)| \ll \tilde{p}(|\alpha|)/|\alpha|$
   - This happens when the root is nearly a multiple root
   - The polynomial is nearly flat at the root
   - Example: $p(x) = (x-\alpha)^2 + \varepsilon$ has $p'(\alpha) = O(\sqrt{\varepsilon})$
2. **Large** $|\alpha|$ combined with small $|p'(\alpha)|$:
   - High-degree polynomials evaluated at large $|\alpha|$
   - The numerator $\tilde{p}(|\alpha|) = \sum |a_i||\alpha|^i$ grows rapidly with $|\alpha|$
3. **Coefficient growth**:
   - When $|a_i|$ are large, especially for high powers
   - Wilkinson's polynomial: $(x-1)(x-2)\cdots(x-20)$ has enormous coefficients

Wilkinson's Polynomial

The polynomial $W(x) = \prod_{i=1}^{20}(x-i)$ is famously ill-conditioned. A tiny perturbation to one coefficient can move roots dramatically. This is because $|W'(i)|$ is small relative to the coefficient magnitudes.

Summary: A root is ill-conditioned when:

- **Near multiple root**: $p'(\alpha) \approx 0$

- **Large root magnitude**: $|\alpha|$ is large relative to coefficient scale
- **Coefficient imbalance**: Large variation in coefficient magnitudes

$$\boxed{\text{Ill-conditioned when: } |p'(\alpha)| \ll \frac{\tilde{p}(|\alpha|)}{|\alpha|}} \tag{56}$$

4 Exercise 4: Conditioning of Matrix Inverse

4.1 Question 1: Show that $\kappa(A) = |A||A^{-1}|$

Given Definition:

$$\kappa(A) := \lim_{\varepsilon \to 0} \sup_{|\Delta A| \le \varepsilon |A|} \left(\frac{|(A+\Delta A)^{-1} - A^{-1}|}{\varepsilon |A^{-1}|} \right) \tag{57}$$

Proof:
We need a perturbation formula for $(A + \Delta A)^{-1}$.

Lemma (Matrix Inversion Perturbation Formula): If $|\Delta A| \cdot |A^{-1}| < 1$, then $A + \Delta A$ is invertible and:

$$(A+\Delta A)^{-1} = A^{-1} - A^{-1}\Delta A A^{-1} + A^{-1}\Delta A A^{-1}\Delta A A^{-1} - \cdots \tag{58}$$

Derivation:

$$(A+\Delta A)^{-1} = (A(I + A^{-1}\Delta A))^{-1} = (I + A^{-1}\Delta A)^{-1} A^{-1} \tag{59}$$

Using the Neumann series $(I+E)^{-1} = I - E + E^2 - E^3 + \cdots$ for $|E| < 1$:

$$(I + A^{-1}\Delta A)^{-1} = I - A^{-1}\Delta A + O(|\Delta A|^2) \tag{60}$$

Therefore:

$$(A+\Delta A)^{-1} - A^{-1} = -A^{-1}\Delta A A^{-1} + O(|\Delta A|^2) \tag{61}$$

Taking norms:

$$|(A+\Delta A)^{-1} - A^{-1}| = |A^{-1}\Delta A A^{-1}| + O(|\Delta A|^2) \tag{62}$$

$$\le |A^{-1}||\Delta A||A^{-1}| + O(|\Delta A|^2) \tag{63}$$

For the supremum over $|\Delta A| \le \varepsilon |A|$, the worst case is when ΔA is aligned with the singular vectors to maximize the norm:

$$\sup_{|\Delta A| \le \varepsilon |A|} |A^{-1}\Delta A A^{-1}| = \varepsilon |A||A^{-1}|^2 \tag{64}$$

Therefore:

$$\kappa(A) = \lim_{\varepsilon \to 0} \frac{\varepsilon |A||A^{-1}|^2 + O(\varepsilon^2)}{\varepsilon |A^{-1}|} = |A||A^{-1}| \tag{65}$$

$$\boxed{\kappa(A) = |A||A^{-1}|} \tag{66}$$

Standard Condition Number

This is the standard definition of the condition number of a matrix!

- $\kappa(A) \ge 1$ (equality when A is orthogonal/unitary)

- Large $\kappa(A)$ means A is close to singular
- $\kappa(A) = \infty$ when A is singular

4.2 Question 2: Distance to Singularity

Distance to Singularity:

$$\text{dist}(A) := \min \left\{ \frac{|\Delta A|}{|A|} : A + \Delta A \text{ is singular} \right\} \tag{67}$$

To Show: $\text{dist}(A) = \kappa(A)^{-1}$
Proof:
$A + \Delta A$ is singular if and only if there exists a unit vector v ($|v| = 1$) such that:

$$(A + \Delta A)v = 0 \tag{68}$$

$$Av = -\Delta A v \tag{69}$$

Taking norms:

$$|Av| = |\Delta A v| \leq |\Delta A||v| = |\Delta A| \tag{70}$$

Since $|v| = 1$:

$$|Av| \leq |\Delta A| \tag{71}$$

The minimum $|\Delta A|$ occurs when we choose v to minimize $|Av|$:

$$\min_{|v|=1} |Av| = \sigma_{\min}(A) = \frac{1}{|A^{-1}|} \tag{72}$$

where $\sigma_{\min}(A)$ is the smallest singular value of A.
Therefore:

$$\min |\Delta A| = \sigma_{\min}(A) = \frac{1}{|A^{-1}|} \tag{73}$$

And:

$$\text{dist}(A) = \frac{\sigma_{\min}(A)}{|A|} = \frac{1}{|A||A^{-1}|} = \frac{1}{\kappa(A)} \tag{74}$$

$$\boxed{\text{dist}(A) = \kappa(A)^{-1}} \tag{75}$$

Interpretation:

- If $\kappa(A)$ is large, A is close to a singular matrix
- A relative perturbation of size $1/\kappa(A)$ can make A singular
- Well-conditioned matrices ($\kappa(A) \approx 1$) are far from singular

4.3 Question 3: Express $\kappa(A)$ in Terms of Singular Values

Singular Value Decomposition (SVD):

$$A = U\Sigma V^T \tag{76}$$

where:

- U, V are orthogonal matrices

- $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

Properties:

- $|A|_2 = \sigma_1 = \sigma_{\max}(A)$ (largest singular value)
- $|A^{-1}|_2 = 1/\sigma_n = 1/\sigma_{\min}(A)$ (reciprocal of smallest singular value)

Therefore:

$$\kappa(A) = |A|_2 |A^{-1}|_2 = \sigma_{\max}(A) \cdot \frac{1}{\sigma_{\min}(A)} \tag{77}$$

$$\boxed{\kappa(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} = \frac{\sigma_1}{\sigma_n}} \tag{78}$$

Special Cases:

1. **Orthogonal/Unitary matrices**: $\sigma_1 = \sigma_n = 1$

$$\kappa(Q) = 1 \quad \text{(perfectly conditioned)} \tag{79}$$

2. **Diagonal matrices**: $\Sigma = \text{diag}(d_1, \ldots, d_n)$

$$\kappa(\Sigma) = \frac{\max_i |d_i|}{\min_i |d_i|} \tag{80}$$

3. **Near-singular matrices**: $\sigma_n \approx 0$

$$\kappa(A) \to \infty \quad \text{(ill-conditioned)} \tag{81}$$

Practical Interpretation

The condition number is the ratio of the largest to smallest "stretching factors" of the matrix.

- $\kappa(A)$ measures how much the matrix amplifies relative errors
- In floating-point arithmetic with precision u, expect errors of order $\kappa(A) \cdot u$

5 Summary

5.1 Key Concepts

Concept	Formula	Meaning
Condition Number	$\text{cond}(f, x) = \frac{\|f'(x)\|\|x\|}{\|f(x)\|}$	Amplification of relative errors
Backward Stability	$\tilde{f}(x) = f(\tilde{x})$ where $\tilde{x} \approx x$	Computed result = exact result of perturbed input
Forward Error Bound	$\frac{\|\tilde{y}-y\|}{\|y\|} \leq \text{accuracy} \times \text{cond}$	Relative error bounded by algorithm accuracy × conditioning

5.2 Condition Numbers Derived

1. **Summation**:
$$\text{cond}\left(\sum p_i\right) = \frac{\sum |p_i|}{|\sum p_i|} \tag{82}$$

2. **Dot Product**:
$$\text{cond}(x \cdot y) = \frac{\sum |x_i y_i|}{|\sum x_i y_i|} \tag{83}$$

3. **Polynomial Evaluation**:
$$\text{cond}(p, x) = \frac{\tilde{p}(|x|)}{|p(x)|} \tag{84}$$

4. **Polynomial Root**:
$$K(p, \alpha) = \frac{\tilde{p}(|\alpha|)}{|\alpha||p'(\alpha)|} \tag{85}$$

5. **Matrix Inverse**:
$$\kappa(A) = |A||A^{-1}| = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{86}$$

5.3 Backward Stable Algorithms

- ✓Recursive summation: $O(nu)$ error
- ✓Dot product: $O(nu)$ error
- ✓Horner scheme: $O(nu)$ error per evaluation

5.4 When Problems Are Ill-Conditioned

- **Summation**: Massive cancellation ($\sum p_i \approx 0$ but $\sum |p_i|$ large)
- **Dot Product**: Nearly orthogonal vectors
- **Polynomial Evaluation**: Near roots or significant cancellation
- **Polynomial Roots**: Nearly multiple roots ($p'(\alpha) \approx 0$)
- **Matrix Inverse**: Nearly singular ($\sigma_{\min} \approx 0$)

Tutorial solved with Claude by Giulia Lionetti - Sorbonne Université