Key Results: Fast Verification Methods for Linear Systems

Extracted from AFAE Lecture Slides

1 Iterative Refinement

1.1 Iterative Refinement Algorithm

1. $\hat{x}_i \leftarrow$ computed solution of $Ax = b$
2. $\hat{r}_i \leftarrow$ computed residual $b - A\hat{x}_i$ **(with double precision)**
3. $\hat{c}_i \leftarrow$ computed solution of $Ac_i = \hat{r}_i$
4. $\hat{x}_{i+1} \leftarrow \mathrm{fl}(\hat{x}_i + \hat{c}_i)$
5. Repeat until stopping criterion satisfied

1.2 Condition Numbers

Definition 1 (Componentwise Condition Number)**.**

$$cond_{E,f}(A,x) := \lim_{\varepsilon\to 0} \sup_{\substack{|\Delta A|\le\varepsilon|E| \\ |\Delta b|\le\varepsilon|f|}} \left\{ \frac{\|\hat{x}-x\|_\infty}{\varepsilon\|x\|_\infty},\ (A+\Delta A)\hat{x} = b+\Delta b \right\}$$

Special cases:

- $E = |A|,\ f = |b|$: $\mathrm{cond}(A,x) = \frac{\||A^{-1}||A||x|\|_\infty}{\|x\|_\infty}$
- $E = |A|,\ f = 0$: $\mathrm{cond}(A) = \||A^{-1}||A|\|_\infty$
- Relation: $\mathrm{cond}(A,x) \le \mathrm{cond}(A) \le \kappa_\infty(A)$

1.3 Key Theorems

Theorem 1 (Backward Error - Thm 9.4, Higham)**.** *Let $A \in \mathbb{F}^{n\times n}$ and suppose GE produces computed LU factors $A \approx \hat{L}\hat{U}$, and computed solution $\hat{x}$ to $Ax = b$. Then*

$$(A+\Delta A)\hat{x} = b, \quad |\Delta A| \le \gamma_{3n}|\hat{L}||\hat{U}|$$

where $\gamma_k = \frac{ku}{1-ku}$.

Forward error bound:

$$\frac{\|\hat{x}-x\|_\infty}{\|x\|_\infty} \le \gamma_{3n}\frac{\||A^{-1}||\hat{L}||\hat{U}||\hat{x}|\|_\infty}{\|x\|_\infty}$$

Theorem 2 (Fixed Precision Refinement - Thm 12.2, Higham)**.** *Let iterative refinement in fixed precision be applied to $Ax = b$. Let*

$$\eta = u\||A^{-1}|(|A| + W(A,n))\|_\infty$$

Provided $\eta \ll 1$, iterative refinement reduces forward error by factor η at each stage, until

$$\frac{\|\hat{x}_k - x\|_\infty}{\|x\|_\infty} \le 2nu\, cond(A,x) + O(u^2)$$

For GE: $\eta = u\||A^{-1}|(|A| + 3n|\hat{L}||\hat{U}|)\|_\infty + O(u^2)$
If $|\hat{L}||\hat{U}| \approx |A|$, then $\eta \approx 3nu\,\mathrm{cond}(A)$.

Theorem 3 (Mixed Precision Refinement - Thm 12.1, Higham)**.** *Let iterative refinement be applied with* ***residuals computed in double the working precision****. Let*

$$\eta = u\||A^{-1}|(|A| + W(A,n))\|_\infty$$

Provided $\eta \ll 1$, iterative refinement reduces forward error by factor η at each stage until

$$\frac{\|\hat{x}_k - x\|_\infty}{\|x\|_\infty} \approx u$$

Key point: With double precision residuals, forward error $\approx u$ (independent of condition number, if $\mathrm{cond}(A) \ll u^{-1}$).

1.4 Stopping Criterion for Verified Solution

Assume we have upper bound $\overline{\delta}$ such that $\|x - \hat{x}\|_\infty \leq \overline{\delta}$.
From $\overline{\delta} \geq \|\hat{x}\|_\infty - \|x\|_\infty$, we get $\|x\|_\infty \geq \|\hat{x}\|_\infty - \overline{\delta}$.
If $\overline{\delta} < \|\hat{x}\|_\infty$, then:

$$\frac{\|x - \hat{x}\|_\infty}{\|x\|_\infty} \leq \frac{\overline{\delta}}{\|\hat{x}\|_\infty - \overline{\delta}}$$

Stop when:

- $\frac{\|x-\hat{x}\|_\infty}{\|x\|_\infty} \leq \tau$ (tolerance), OR
- 3 refinement steps completed

2 Inversion of Ill-Conditioned Matrices

2.1 Problem Statement

Challenge: If $\kappa(A) > u^{-1}$, standard verification ($\|RA - I\| < 1$) likely fails.
Kahan-Gastinel Theorem:

$$\kappa(A)^{-1} = \min\left\{\frac{\|\Delta A\|}{\|A\|} : A + \Delta A \text{ singular}\right\}$$

If $\kappa(A) > u^{-1}$, perturbation of norm $O(u)$ can make A singular.

2.2 Rump's Algorithm

Notation:

- $P = \mathrm{fl}_{k,1}(AB)$: product computed in precision u_k, rounded to precision u
- $\{P\} = \mathrm{fl}_{k,k}(AB)$: product in precision u_k, stored as unevaluated sum $\{P_1, \ldots, P_k\}$

Error bounds:

$$\|\mathrm{fl}_{k,1}(AB) - AB\| \leq u\|AB\| + nu_k\|A\|\|B\| + O(u_{k+1})$$
$$\|\mathrm{fl}_{k,k}(AB) - AB\| \leq nu_k\|A\|\|B\| + O(u_{k+1})$$

Theorem 4 (Rump's Inversion Algorithm)**.** ***Input:*** *$A \in \mathbb{F}^{n\times n}$ with $\kappa(A) \gg u^{-1}$*

Algorithm:

1. $\{R^{(0)}\} = fl(\|A\|^{-1}) \cdot I,\ k = 0$
2. ***repeat***
3. $\quad k = k + 1$
4. $\quad P^{(k)} = fl_{k,1}(\{R^{(k-1)}\} \cdot A)$
5. $\quad X^{(k)} = inv(P^{(k)})$
6. $\quad \{R^{(k)}\} = fl_{k,k}(X^{(k)} \cdot \{R^{(k-1)}\})$
7. ***until*** $cond(P^{(k)}) < (100u)^{-1}$

Heuristic behavior:

$$\mathrm{cond}(\{R^{(k)}\}A) \approx u^{k-1}\mathrm{cond}(A)$$

Condition number decreases by factor u at each iteration.
Algorithm terminates when $\|\{R^{(k)}\}A - I\| \leq 1/100$, proving A nonsingular.

3 Key Formulas Summary

3.1 Error Bounds

- **Unit roundoff:** $u = 2^{-53} \approx 1.11 \times 10^{-16}$ (double precision)
- γ_n **notation:** $\gamma_n = \frac{nu}{1-nu} \approx nu$ (for small nu)
- **Backward error (GE):** $|\Delta A| \leq \gamma_{3n}|\hat{L}||\hat{U}|$
- **Forward error (fixed precision):** $\frac{\|\hat{x}_k - x\|_\infty}{\|x\|_\infty} \leq 2nu\,\mathrm{cond}(A, x)$
- **Forward error (mixed precision):** $\frac{\|\hat{x}_k - x\|_\infty}{\|x\|_\infty} \approx u$

3.2 Condition Number Relations

$$\mathrm{cond}(A, x) \leq \mathrm{cond}(A) \leq \kappa_\infty(A)$$

3.3 Convergence Factor (Fixed Precision)

$$\eta = u\||A^{-1}|(|A| + W(A, n))\|_\infty$$

For GE with $|\hat{L}||\hat{U}| \approx |A|$:

$$\eta \approx 3nu\,\mathrm{cond}(A)$$

4 Important Notes

- **Fixed precision refinement:** Best achievable error $\sim u \cdot \mathrm{cond}(A, x)$
- **Mixed precision refinement:** Best achievable error $\sim u$ (if $\mathrm{cond}(A) \ll u^{-1}$)
- **Cost of certified algorithms:**
  - certifLSV1: $6n^3 + O(n^2)$ flops

  - certifLSV4: $\frac{4}{3}n^3 + O(n^2)$ flops

- **When to use Rump:** $\kappa(A) > u^{-1}$ (extremely ill-conditioned)
- **Rump convergence:** Condition number reduced by factor u per iteration