Monte Carlo simulation of Hull-White model and sensitivities computation

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Project Overview: The Hull-White Model

1. Estimation of $P(0, T)$ and $f(0, T)$

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW_t$$

- **Discrete simulation:** $r(t) = m_{s,t} + \Sigma_{s,t} G$
  - $m_{s,t}$ is the deterministic drift given $r(s)$;
  - $\Sigma_{s,t}$ is the accumulated volatility of $r(t)$ and $G \sim \mathcal{N}(0, 1)$.
- **Zero coupon bond price ($P(0, T)$) and forward rate ($f(0, T)$):** estimated by Monte Carlo from interest rate r.
- **Drift ($\theta(t)$):** term calibrated to fit the initial market data.

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2. Estimation of $\theta(t)$ and $\mathbf{ZBC}(5, 10, e^{-0.1})$

From $f(0, T)$ recover the drift $\theta(t)$ and from analytical expressions of $P(t, T)$ compute the value of the European call option $\mathbf{ZBC}(S_1, S_2, K)$, discounted expected payoff given S_1, S_2, K.

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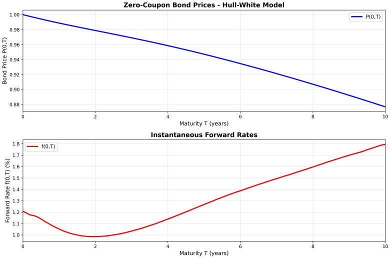
3. Estimation of $\partial_\sigma \mathbf{ZBC}(S_1, S_2, K)$

The sensitivity of $\mathbf{ZBC}(S_1, S_2, K)$ w.r.t. the parameter σ determining the volatility of $r(t)$, estimated by Monte Carlo.

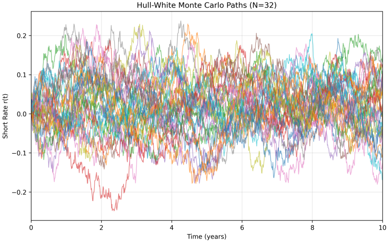
1. Monte Carlo Estimation of $P(0, T)$ and $f(0, T)$

Computed by 2^{21} paths:

- Generated bond prices $P(0, T)$ and forward rates $f(0, T)$ using 1000 timesteps between $[0, T]$.
- Simulation time $= 5.36$ ms
- Throughput $= 391$ M paths/sec
- $f(0, 0) = 1.21\%$ (Matches initial short rate r_0)



$P(0, T)$ and $f(0, T)$ curves



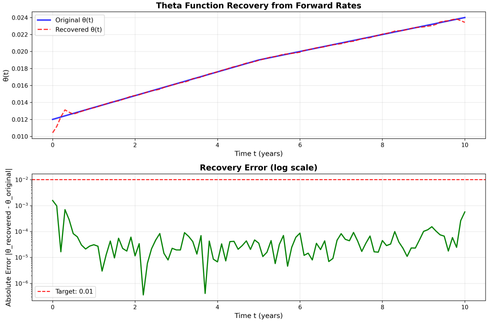
Sample Short Rate Paths $r(t)$

2a. $\theta(t)$ Calibration

- A thread for each t in the discretized interval $[0, T]$.
- Recover a given piecewise linear expression of $\theta(t)$ by

$$\theta(t) = \frac{\partial f(0,T)}{\partial T} + af(0,T) + \frac{\sigma^2(1-e^{-2aT})}{2a}$$

- Mean error $= 2.49 * 10^{-4}$



Recovered $\theta(t)$ and its error

2b. Monte Carlo Estimation of $\mathbf{ZBC}(5, 10, e^{-0.1})$

Computed by 2^{21} paths:

- **Idea:** Use underlying $P(S_1, S_2)$, whose price $P(0, S_2)$ is known, to correct the estimation of $\mathbf{ZBC}(S_1, S_2, K)$.
- **Target Payoff (X):**

$$X = e^{-\int_0^{S_1} r_t dt}(P(S_1, S_2) - K)_+$$

- **Control Variate (Y):**

$$Y = e^{-\int_0^{S_1} r_t dt} P(S_1, S_2), \text{with known } \mathbb{E}[Y] = P(0, S_2)$$

- $\mathbf{ZBC}_{CV} = \bar{X} - \beta^*(\bar{Y} - P(0, S_2))$ where $\beta^* = \frac{\widehat{\mathrm{Cov}}(X,Y)}{\widehat{\mathrm{Var}}(Y)}$
- Variance reduction of 20.4%
- Simulation time = 1.97 ms
- Throughput = 1064 M paths/sec

3. Sensitivity Analysis (∂_σ**ZBC**)

Methodologies

- **Finite Difference (FD):** Recompute **ZBC** with $\sigma \pm \epsilon$.
- **Pathwise Method (PW):** Differentiate the estimator directly from the analytical formula.

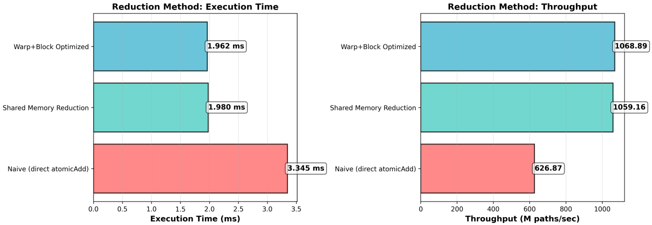
Results

- **Comparison:** Methods agree within **0.18**%.
- **PW performance:** 2.06 ms with a throughput of 509 M paths/sec.
- **Calibration Stability:** In FD need to recalibrate $\theta(t)$, since it depends on σ, and all the values computed from $\theta(t)$, i.e. $m_{s,t}$ and $r(t)$. Instead, a recalibration of $P(0, T)$ leads to an error of 127%

Reduction Benchmark

Comparison of reduction strategies tested for **ZBC** in question 2:

- **Naive approach:** every thread directly updates the global sum using `atomicAdd`;
- **Shared memory reduction:** Sums accumulated into a shared memory array first. Then, only one thread per block adds the result to global memory.
- **Warp+block reduction:** First, warp shuffle intrinsics, then via shared memory and finally to global memory.



Thank you for your attention!

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