Non-overlapping Schwarz algorithms for the Incompressible Navier-Stokes equations with DDFV discretizations

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Abstract

We propose and analyze non-overlapping Schwarz algorithms for the domain decomposition of the unsteady incompressible Navier-Stokes problem with Discrete Duality Finite Volume discretizations. The design of suitable transmission conditions for the velocity and the pressure is a crucial issue. We establish the well-posedness of the method and the convergence of the iterative process, pointing out how the numerical fluxes influence the asymptotic problem which is intended to be a discretization of the Navier-Stokes equations on the entire computational domain. Finally we perform some numerical tests to illustrate the behavior of the algorithm.

1 Introduction

The aim of this paper is to develop a non-overlapping iterative Schwarz algorithm for the incompressible Navier-Stokes problem with DDFV schemes. The problem we are interested in reads

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}(\sigma(\mathbf{u},\mathrm{p})) = \mathbf{f} & \text{in} \quad \Omega\times[0,T], \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega\times[0,T], \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega\times[0,T], \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where Ω is an open connected bounded polygonal domain of $\mathbb{R}^2$, $\mathbf{f}\in(L^2(\Omega))^2$ is a given force field, $\mathbf{u}_{init}\in(L^\infty(\Omega))^2$. The unknowns $\mathbf{u}:\Omega\times[0,T]\to\mathbb{R}^2$ and $\mathrm{p}:\Omega\times[0,T]\to\mathbb{R}$ are respectively the velocity and the pressure; $\sigma(\mathbf{u},\mathrm{p}) = \frac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$ stands for the stress tensor, and $\mathrm{Re}>0$ is the Reynolds number. Here and below, the strain rate tensor is defined by the symmetric part of the velocity gradient $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$.

Non-overlapping Schwarz algorithms enter the class of domain decomposition methods, in which a domain is decomposed into smaller subdomains. The main advantage is that, contrarily to direct methods, decomposition methods are naturally parallel; in fact, subdomains problems are connected by some transmission conditions on the interface, but they are uncoupled by an iterative procedure. This makes those methods interesting for high performance computing perspectives. The classical Schwarz algorithm, proposed in 1870 by H. A. Schwarz [Sch70] for the Laplace problem, is an iterative method that consists in transmitting the solution, or its normal derivative, from a subdomain to the other, in order to deal with complex domains. This method converges only if the subdomains overlap. Moreover, this convergence becomes slower as the overlap between the subdomains is smaller. In non-overlapping Schwarz algorithms, the subdomains intersect only on their interfaces and in order to obtain convergence, more elaborate transmission conditions should be defined on the interfaces. In 1990, P.-L. Lions [Lio90] showed that, with Fourier-Robin transmission conditions, Schwarz algorithm for the Laplace operator converges even without overlap. This method has been adapted to the discrete setting for many problems of isotropic diffusion, [AJNM02, CHH04, GJMN05], and for advection-diffusion-reaction problems, [GH07, HH14]. For the Navier-Stokes problem, different approaches have been proposed, with different design of the interface conditions, that depend also on the discretization framework. In the spirit of [HS89], [BCR16] derives optimal transparent boundary conditions for the Stokes equation; these conditions are tested with finite differences methods. In the finite element setting, [LMO01] proposes a non-overlapping domain decomposition algorithm of Robin-Robin type for the discretized Oseen equations (i.e. linearized Navier-Stokes). In [XCL05], still in the finite element setting, the authors build a Dirichlet-Neumann domain decomposition method for the nonlinear steady Navier-Stokes equations, under the hypothesis that the Reynolds number is sufficiently small and [GRW05] studies a family of discontinuous Galerkin finite element methods for Stokes and Navier-Stokes problems on triangular meshes. The Inf-Sup condition, which is a crucial ingredient of the stability analysis of numerical methods for Incompressible Navier-Stokes equations, has to be adapted to the domain decomposition formulation, in particular to satisfy the incompressibility constraint, and it might depend on the Reynolds number, see [GRW05, LMO01]. Therefore, our objective is to decompose the domain Ω of problem (1) into smaller subdomains, to solve the incompressible Navier-Stokes problem on those subdomains by imposing some transmission conditions on the interfaces, and to recover by an iterative Schwarz algorithm the discrete solution of (1) on the entire computational domain Ω. Since we are interested in the unsteady problem, we shall apply this iterative algorithm at each time iteration. Moreover, we want the interface conditions to be local and we wish the method to remain free of any restrictive condition on the Reynolds number. We address these issues in the framework of *finite volume* methods, and more specifically by using Discrete Duality Finite Volume discretizations.

The introduction of the DDFV formalism dates back to [CVV99, Her00, Her03, DO05], in order to approximate anisotropic diffusion problems on general meshes, including non-conformal and distorted meshes. Such schemes require unknowns on both the vertices and centers of primal control volumes and allow us to build two-dimensional discrete gradient and divergence operators that satisfy discrete duality relations analogous to the standard integration by parts formula. The DDFV scheme is extended in [ABH07] to general linear and nonlinear elliptic problems with non homogeneous Dirichlet boundary conditions, including the case of anisotropic elliptic problems. Applying the DDFV method for Stokes and Navier-Stokes problems leads naturally to locate the unknowns of velocity and pressure in different points; the velocity unknowns are associated to the vertices and centers of primal control volumes, while the pressure unknowns are located on the edges of the mesh [BKN15, Del07, DO15, GKL19, Kre10, Kre11a]. Hence, DDFV enters the class of staggered methods, reminiscent of the MAC scheme [HW65] constructed on Cartesian meshes for incompressible flows. The DDFV approach has, at least, two important advantages. First of all, it applies to very general meshes. It is useful, for instance, in the domain decomposition setting where the subdomains can be meshed separately and non-conformal edges appear on the interface, or simply if one wants to locally refine the mesh with cells adapted to complex geometries. Second of all, DDFV operators mimic at the discrete level the dual properties of the continuous differential operators, which leads to important properties for the numerical analysis of the schemes.

As a starting point of this study, we refer the reader to [GHHK18] and [HH14]: they both build a non-overlapping Schwarz algorithm in a finite volume framework with Fourier-like transmission conditions between subdomains, respectively for anisotropic diffusion with a DDFV discretization, see also [BHK10], and for advection-diffusion-reaction in a TPFA discretization. The case of the Navier-Stokes equations (1) is more demanding, since it combines further difficulties: the vectorial nature of the unknowns, the non-linear convection terms and the incompressibility constraint. Of course, there is no explicit interface conditions, and one should construct suitable transmission conditions between the subdomains, which have the shape of Fourier-like conditions on the velocity and account for the constraint by involving the divergence of the velocity and the pressure. Let us split the computational domain Ω into two smaller subdomains $\Omega = \Omega_1 \cup \Omega_2$. We denote by Γ the interface $\Omega_1 \cap \Omega_2$. The Schwarz algorithm defines a sequence of solutions $\left(\mathbf{u}_j^l\right)_{l\in\mathbb{N}}$ of the Navier-Stokes problem in Ω_j, with $j \in \{1,2\}$, endowed with the following two-fold transmission condition

$$\begin{aligned}\sigma(\mathbf{u}_j^l, \mathrm{p}_j^l) \cdot \vec{\mathbf{n}}_j - \frac{1}{2}(\mathbf{u}_j^l \cdot \vec{\mathbf{n}}_j)(\mathbf{u}_j^l) + \lambda \mathbf{u}_j^l = \sigma(\mathbf{u}_i^{l-1}, \mathrm{p}_i^{l-1}) \cdot \vec{\mathbf{n}}_i - \frac{1}{2}(\mathbf{u}_i^{l-1} \cdot \vec{\mathbf{n}}_i)(\mathbf{u}_i^{l-1}) + \lambda \mathbf{u}_i^{l-1}, \\ \mathrm{div}(\mathbf{u}_j^l) + \alpha \mathrm{p}_j^l = -\mathrm{div}(\mathbf{u}_i^{l-1}) + \alpha \mathrm{p}_i^{l-1},\end{aligned} \tag{2}$$

where $i \neq j$ and $\vec{\mathbf{n}}_j$ is the outward normal to Ω_j. The former condition, which involves the parameters $\alpha > 0$, $\lambda > 0$, is inspired by the classical Fourier condition; it linearly combines the values of the unknown and the values of its derivative; here, also the convection is included. The latter, which depends only on α, combines the divergence of the velocity with the pressure; it will be useful to satisfy the incompressibility constraint at the convergence of the algorithm. The first condition is comparable to the transmission conditions in [LMO01]. However they need to justify a modified Inf-Sup condition which induces Reynolds-dependent stability constraints. This can be relaxed by imposing the new condition for the pressure on the interface. Once the transmission condition fixed, it remains to establish the convergence of the iterative process: as $l \to \infty$, one expects to recover the solution of a discrete version of the Navier-Stokes equations on the entire domain Ω. We shall see that the asymptotic scheme depends on the details of the numerical fluxes of the domain decomposition method. To analyze this issue, it is convenient to discuss general discretizations of the convection terms, inspired by [CHD11, HH14].

Outline. This paper is organized as follows. In Section 2, we introduce the main elements of the DDFV framework. In Section 3, we set up the reference scheme for the Navier-Stokes problem on the entire domain Ω. The convection fluxes are seen as a centered discretization plus a diffusive perturbation, defined through a certain function B, as it appeared in [CHD11] when designing finite volume schemes for non-coercive elliptic problems with Neumann boundary conditions. We establish the well-posedness of such schemes, which generalize the mere upwind or centered discretizations. In Section 4, we introduce the composite meshes, i.e. the meshes on the subdomains, and we construct the DDFV Schwarz algorithm. The convergence issue is investigated in Sections 5 and 6, corresponding to the following discussion:

- starting with the “natural” domain decomposition approach, the limit problem — which can be proved to be well-posed — does not coincide with the reference scheme. Instead, some fluxes near the interface need to be modified.
- nevertheless, it is possible to recover the reference scheme, having unified fluxes over the entire domain Ω, at the price of modifying the fluxes in the original domain decomposition method.

This discussion motivates the need of a general analysis of B-schemes for Navier-Stokes equations. Finally, in Section 7 we illustrate the theoretical results with numerical simulations. In particular, we discuss the influence of the parameters λ, α of (2) and we apply the method to the simulation of flows past an obstacle.

2 DDFV framework

Here and below, we adopt the main definitions and notation introduced in [ABH07] and [Kre10].

2.1 Meshes

DDFV method requires unknowns on vertices, centers and edges of control volumes; for this reason, it works on (three) staggered meshes. From an initial mesh, called the “primal mesh” (denoted with $\mathfrak{M} \cup \partial\mathfrak{M}$), we construct the “dual mesh” (denoted with $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$), centered on the vertices of the primal mesh, and the “diamond mesh” (denoted with $\mathfrak{D}$), centered on the edges of the primal mesh; see Fig. 1. The union of the primal and dual meshes will be denoted by $\mathfrak{T}$.

More precisely, we consider a primal mesh $\mathfrak{M}$ consisting of open disjoints polygons κ such that $\bigcup_{\kappa \in \mathfrak{M}} \bar{\kappa} = \bar{\Omega}$. We denote $\partial\mathfrak{M}$ the set of edges of the primal mesh included in $\partial\Omega$, considered as degenerated primal cells. We associate to each κ a point x_κ, called center. For the volumes of the boundary, x_κ is situated at the mid point of the edge. When κ and L are neighboring volumes, we suppose that $\partial\kappa \cap \partial\mathrm{L}$ is a segment that we denote $\sigma = \kappa|\mathrm{L}$, edge of the primal mesh $\mathfrak{M}$. Let $\mathcal{E}$ be the set of all edges

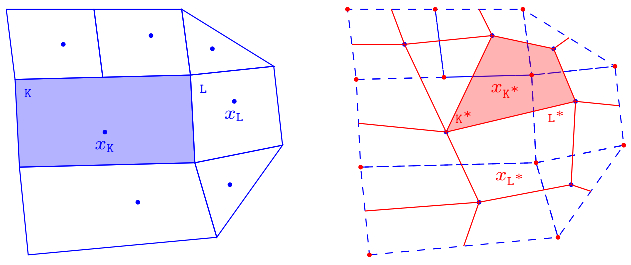


Fig. 1 DDFV meshes on a non conformal mesh: primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ (blue), dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ (red) and diamond mesh $\mathfrak{D}$ (green).

and $\mathcal{E}_{int} = \mathcal{E} \setminus \{\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega\}$. The DDFV framework is free of "admissibility constraint", in particular we do not need to assume the orthogonality of the segment $x_{\mathrm{K}}, x_{\mathrm{L}}$ with $\sigma = \mathrm{K}|\mathrm{L}$. Here we suppose:

Hp 2.1 *All control volumes* K *are star-shaped with respect to* x_{K}.

From the primal mesh, we build the associated dual mesh. A dual cell K^* is associated to a vertex x_{K^*} of the primal mesh. The dual cells are obtained by joining the centers of the primal control volumes that have x_{K^*} as vertex. We distinguish interior dual mesh, for which x_{K^*} does not belong to $\partial\Omega$, denoted by $\mathfrak{M}^*$ and the boundary dual mesh, for which x_{K^*} belongs to $\partial\Omega$, denoted by $\partial\mathfrak{M}^*$. We denote with $\sigma^* = \mathrm{K}^*|\mathrm{L}^*$ the edges of the dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\mathcal{E}^*$ the set of those edges. In what follows, we assume:

Hp 2.2 *All control volumes* K^* *are star-shaped with respect to* x_{K^*}.

The diamond mesh is made of quadrilaterals with disjoint interiors (thanks to Hp 2.1), such that their principal diagonals are a primal edge $\sigma = \mathrm{K}|\mathrm{L} = [x_{\mathrm{K}^*}, x_{\mathrm{L}^*}]$ and the dual edge $\sigma^* = [x_{\mathrm{K}}, x_{\mathrm{L}}]$. Hence, a diamond is a quadrilateral with vertices $x_{\mathrm{K}}, x_{\mathrm{L}}, x_{\mathrm{K}^*}$ and x_{L^*}, denoted with D or $\mathrm{D}_{\sigma,\sigma^*}$. We distinguish the diamonds on the interior and of the boundary:

$$\mathfrak{D}_{ext} = \{\mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}, \text{ such that } \sigma \subset \partial\Omega\}, \qquad \mathfrak{D}_{int} = \mathfrak{D} \setminus \mathfrak{D}_{ext}.$$

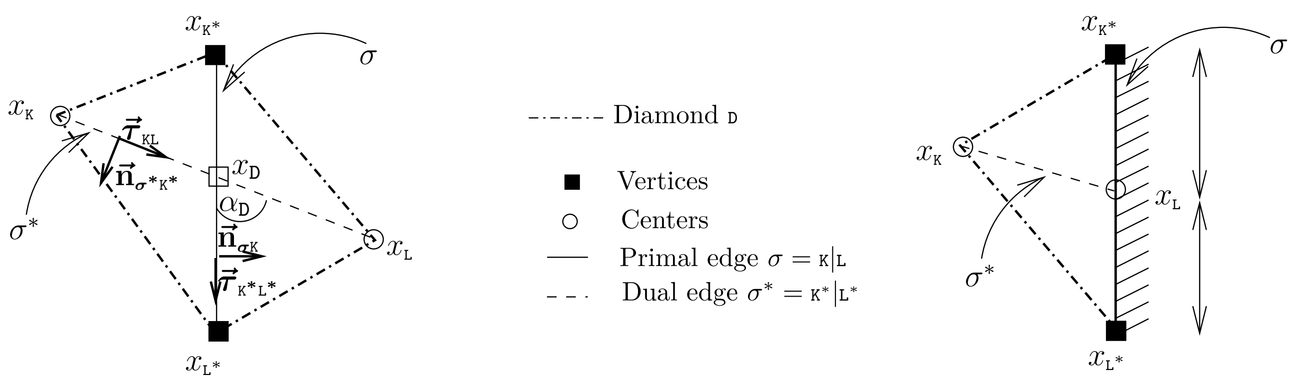


Fig. 2 A diamond $\mathrm{D} = \mathrm{D}_{\sigma,\sigma^*}$, on the interior (left) and on the boundary (right).

2.2 Notations

The following notation will be used throughout the paper. The reader familiar with DDFV may skip this section.

For a volume $\mathrm{V} \in \mathfrak{M} \cup \partial\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ we define:

- m_{V} the measure of the cell V,

- $\mathcal{E}_{\mathtt{V}}$ the set of edges of $\mathtt{v} \in \mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and the edge $\sigma = \mathtt{v}$ for $\mathtt{v} \in \partial\mathfrak{M}$,
- $\mathfrak{D}_{\mathtt{V}} = \{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}, \sigma \in \mathcal{E}_{\mathtt{V}}\}$,
- $\mathfrak{D}_{\mathtt{V}}^{int} = \left\{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{V}} \cap \mathfrak{D}_{int}\right\}$, $\mathfrak{D}_{\mathtt{V}}^{ext} = \left\{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{V}} \cap \mathfrak{D}_{ext}\right\}$,
- $d_{\mathtt{V}}$ the diameter of $\mathtt{v}$.

For a diamond $\mathtt{D}_{\sigma,\sigma^*}$ whose vertices are $(x_{\mathtt{K}}, x_{\mathtt{K}^*}, x_{\mathtt{L}}, x_{\mathtt{L}^*})$, we denote:

- $x_{\mathtt{D}}$ the center of the diamond $\mathtt{D}$: $x_{\mathtt{D}} = \sigma \cap \sigma^*$,
- m_σ the length of the edge σ,
- m_{σ^*} the length of σ^*,
- $m_{\mathtt{D}}$ the measure of the diamond $\mathtt{D}_{\sigma,\sigma^*}$,
- $d_{\mathtt{D}}$ the diameter of the diamond $\mathtt{D}_{\sigma,\sigma^*}$,
- $\alpha_{\mathtt{D}}$ the angle between σ and σ^*.

We introduce for every diamond two orthonormal basis $(\vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}, \vec{\mathbf{n}}_{\sigma\mathtt{K}})$ and $(\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}, \vec{\boldsymbol{\tau}}_{\mathtt{KL}})$, where:

- $\vec{\mathbf{n}}_{\sigma\mathtt{K}}$ the unit normal to σ going out from $\mathtt{K}$,
- $\vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}$ the unit tangent vector to σ oriented from $\mathtt{K}^*$ to $\mathtt{L}^*$,
- $\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}$ the unit normal vector to σ^* going out from $\mathtt{K}^*$,
- $\vec{\boldsymbol{\tau}}_{\mathtt{KL}}$ the unit tangent vector to σ^* oriented from $\mathtt{K}$ to $\mathtt{L}$.

We denote for each diamond:

- its sides $\mathfrak{s}$ (for example $\mathfrak{s} = [x_{\mathtt{K}}, x_{\mathtt{K}^*}]$),
- $\mathcal{E}_{\mathtt{D}} = \{\mathfrak{s}, \mathfrak{s} \subset \partial\mathtt{D} \text{ and } \mathfrak{s} \not\subset \partial\Omega\}$ the set of all interior sides of the diamond,
- $m_{\mathfrak{s}}$ the length of $\mathfrak{s}$,
- $\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}$ the unit normal to $\mathfrak{s}$ going out from $\mathtt{D}$,
- $\mathfrak{S} = \{\mathfrak{s} \in \mathcal{E}_{\mathtt{D}}, \forall \mathtt{D} \in \mathfrak{D}\}$ the set of interior edges of all diamond cells $\mathtt{D} \in \mathfrak{D}$,
- $\mathfrak{S}_{\mathtt{K}} = \{\mathfrak{s} \in \mathfrak{S}, \text{ such that } \mathfrak{s} \subset \mathtt{K}\}$ and $\mathfrak{S}_{\mathtt{K}^*} = \{\mathfrak{s} \in \mathfrak{S}, \text{ such that } \mathfrak{s} \subset \mathtt{K}^*\}$.

Every diamond is star-shaped with respect to x_D. It can happen that dual cells overlap; to avoid this inconvenient, we can either suppose that the diamonds are convex or consider the barycentric dual mesh, obtained by joining the centers $x_{\mathtt{K}}$ of the primal control volumes to the middle point of the edges that have $x_{\mathtt{K}^*}$ as a vertex. Thanks to Hypothesis 2.1, barycentryc dual cells have disjoint interiors.

Let $\operatorname{size}(\mathfrak{T})$ be the maximum of the diameters of the diamonds cells in $\mathfrak{D}$. The flattening of the triangles is measured by the angle $\alpha_{\mathfrak{T}} \in]0, \frac{\pi}{2}]$ such that $\sin(\alpha_{\mathfrak{T}}) := \min_{\mathtt{D}\in\mathfrak{D}} |\sin(\alpha_{\mathtt{D}})|$. We introduce a positive number $\operatorname{reg}(\mathfrak{T})$ that measures the regularity of the mesh:

$$\begin{aligned}\operatorname{reg}(\mathfrak{T}) \quad = \quad & \max\Big(\frac{1}{\sin(\alpha_T)}, \mathcal{N}, \mathcal{N}^*, \max_{\mathtt{D}\in\mathfrak{D}} \max_{\mathfrak{s}\in\mathcal{E}_{\mathtt{D}}} \frac{d_{\mathtt{D}}}{m_{\mathfrak{s}}}, \max_{\mathtt{K}\in\mathfrak{M}} \frac{d_{\mathtt{K}}}{\sqrt{m_{\mathtt{K}}}}, \max_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \left(\frac{d_{\mathtt{K}^*}}{\sqrt{m_{\mathtt{K}^*}}}\right), \\ & \max_{\mathtt{K}\in\mathfrak{M}} \max_{\mathtt{D}\in\mathfrak{D}_K} \left(\frac{d_{\mathtt{K}}}{d_{\mathtt{D}}}\right), \max_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \max_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} \left(\frac{d_{\mathtt{K}^*}}{d_{\mathtt{D}}}\right)\Big).\end{aligned}$$

where $\mathcal{N}$ and $\mathcal{N}^*$ are the maximal number of edges of each primal cell and the maximal number of edges incident to any vertex.

Hp 2.3 *The number* $\operatorname{reg}(\mathfrak{T})$ *is uniformly bounded form above and below as* $\operatorname{size}(\mathfrak{T}) \to 0$.

Accordingly, there exists two constants C_1 and C_2, which both depend on $\operatorname{reg}(\mathfrak{T})$, such that $\forall \mathtt{K} \in \mathfrak{M}, \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\forall \mathtt{D} \in \mathfrak{D}$ such that $\mathtt{D} \cap \mathtt{K} \neq 0$ and $\mathtt{D} \cap \mathtt{K}^* \neq 0$ we have:

$$\begin{aligned} C_1 m_{\mathtt{K}} \le m_{\mathtt{D}} \le C_2 m_{\mathtt{K}}, \quad C_1 m_{\mathtt{K}^*} \le m_{\mathtt{D}} \le C_2 m_{\mathtt{K}^*} \\ C_1\, d_{\mathtt{K}} \le d_{\mathtt{D}} \le C_2\, d_{\mathtt{K}}, \quad C_1\, d_{\mathtt{K}^*} \le d_{\mathtt{D}} \le C_2\, d_{\mathtt{K}^*}. \end{aligned}$$

2.3 Unknowns and meshes

The DDFV method for Navier-Stokes problem uses staggered unknowns. We associate to each primal volume $\kappa \in \mathfrak{M} \cup \partial\mathfrak{M}$ an unknown $\mathbf{u}_\kappa \in \mathbb{R}^2$ for the velocity, to every dual volume $\kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ an unknown $\mathbf{u}_{\kappa^*} \in \mathbb{R}^2$ for the velocity and to each diamond $\mathrm{D} \in \mathfrak{D}$ an unknown $\mathrm{p}^{\mathrm{D}} \in \mathbb{R}$ for the pressure. Those unknowns are collected in the families:

$$\mathbf{u}_{\mathfrak{T}} = \big((\mathbf{u}_\kappa)_{\kappa\in(\mathfrak{M}\cup\partial\mathfrak{M})}, (\mathbf{u}_{\kappa^*})_{\kappa^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)}\big) \in (\mathbb{R}^2)^{\mathfrak{T}} \quad \text{and} \quad \mathrm{p}_{\mathfrak{D}} = ((\mathrm{p}^{\mathrm{D}})_{\mathrm{D}\in\mathfrak{D}}) \in \mathbb{R}^{\mathfrak{D}}.$$

We define now two discrete average projections, for all functions $\mathbf{v}$ in $(H^1(\Omega))^2$:

- one on the interior:

$$\mathbb{P}_m^{\mathfrak{M}}\mathbf{v} = \left(\left(\frac{1}{m_\kappa}\int_\kappa \mathbf{v}(x)\mathrm{d}x\right)_{\kappa\in\mathfrak{M}}\right) \quad \mathbb{P}_m^{\mathfrak{M}^*}\mathbf{v} = \left(\left(\frac{1}{m_{\kappa^*}}\int_{\kappa^*} \mathbf{v}(x)\mathrm{d}x\right)_{\kappa^*\in\mathfrak{M}^*}\right),$$

- one on the boundary :

$$\mathbb{P}_m^{\partial\Omega}\mathbf{v} = \left(\left(\frac{1}{m_\kappa}\int_\kappa \mathbf{v}(x)\mathrm{d}x\right)_{\kappa\in\partial\mathfrak{M}}, \left(\frac{1}{m_{\kappa^*}}\int_{\kappa^*} \mathbf{v}(x)\mathrm{d}x\right)_{\kappa^*\in\partial\mathfrak{M}^*}\right).$$

We can collect them in a shorthand notation $\mathbb{P}_m^{\mathfrak{T}}\mathbf{v} = (\mathbb{P}_m^{\mathfrak{M}}\mathbf{v}, \mathbb{P}_m^{\mathfrak{M}^*}\mathbf{v}, \mathbb{P}_m^{\partial\Omega}\mathbf{v})$. We introduce also a centered projection on the mesh $\mathfrak{T}$:

$$\mathbb{P}_c^{\mathfrak{T}}\mathbf{v} = ((\mathbf{v}(x_\kappa))_{\kappa\in(\mathfrak{M}\cup\partial\mathfrak{M})}, (\mathbf{v}(x_{\kappa^*}))_{\kappa^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)}), \quad \forall \mathbf{v} \in (H^2(\Omega))^2.$$

Next, we define subsets of $(\mathbb{R}^2)^{\mathfrak{T}}$, which take in account Dirichlet boundary conditions. Let $\Gamma_D \subset \partial\Omega$, the boundary on which homogeneous Dirichlet conditions will be imposed. When $\Gamma_D \neq \partial\Omega$, we need to distinguish the subsets of the boundary mesh

$$\partial\mathfrak{M}_D = \{\kappa \in \partial\mathfrak{M} : x_\kappa \in \Gamma_D\}, \qquad \partial\mathfrak{M}_D^* = \{\kappa^* \in \partial\mathfrak{M}^* : x_{\kappa^*} \in \Gamma_D\},$$

and we set

$$\mathbb{E}_0^{\Gamma_D} = \{\mathbf{u}_{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}, \text{s. t. } \forall \kappa \in \partial\mathfrak{M}_D,\, \mathbf{u}_\kappa = 0 \text{ and } \forall \kappa^* \in \partial\mathfrak{M}_D^*,\, \mathbf{u}_{\kappa^*} = 0\}.$$

When $\Gamma_D = \partial\Omega$, we simply denote $\mathbb{E}_0$ the discrete space satisfying the Dirichlet condition.

2.4 Discrete operators

In this section we define the discrete operators of the DDFV scheme.

Definition 2.4 *We define the* discrete gradient of a vector field of $(\mathbb{R}^2)^{\mathfrak{T}}$ *as the operator*

$$\nabla^{\mathfrak{D}} : \mathbf{u}_{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \longmapsto (\nabla^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}})_{\mathrm{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$

such that for $\mathrm{D} \in \mathfrak{D}$ *:*

$$\nabla^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}} = \frac{1}{\sin(\alpha_{\mathrm{D}})}\left[\frac{\mathbf{u}_{\mathrm{L}} - \mathbf{u}_\kappa}{m_{\sigma^*}} \otimes \vec{\mathbf{n}}_{\sigma\kappa} + \frac{\mathbf{u}_{\mathrm{L}^*} - \mathbf{u}_{\kappa^*}}{m_\sigma} \otimes \vec{\mathbf{n}}_{\sigma^*\kappa^*}\right],$$

where $\otimes$ represents the tensor product. It can equivalently be written as

$$\nabla^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}} = \frac{1}{2m_{\mathrm{D}}}\left[m_\sigma(\mathbf{u}_{\mathrm{L}} - \mathbf{u}_\kappa) \otimes \vec{\mathbf{n}}_{\sigma\kappa} + m_{\sigma^*}(\mathbf{u}_{\mathrm{L}} - \mathbf{u}_{\kappa^*}) \otimes \vec{\mathbf{n}}_{\sigma^*\kappa^*}\right].$$

The discrete strain rate tensor $\mathrm{D}^{\mathfrak{D}} : \mathbf{u}_{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\mathrm{D}^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}})_{\mathrm{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ *is defined by:*

$$\mathrm{D}^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}} = \frac{\nabla^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}} + {}^t(\nabla^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}})}{2}, \quad \textit{for } \mathrm{D} \in \mathfrak{D}.$$

We define the discrete divergence of a vector field of $(\mathbb{R}^2)^{\mathfrak{T}}$ *as the operator*

$$div^{\mathfrak{D}} : \mathbf{u}_{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \longmapsto (div^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}})_{\mathrm{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$$

with $div^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}} = Tr(\nabla^{\mathrm{D}}\mathbf{u}_{\mathfrak{T}})$ *for any* $\mathrm{D} \in \mathfrak{D}$.

Definition 2.5 *We define the* discrete divergence of a tensor field of $(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ *as the operator*

$$\boldsymbol{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \longmapsto \boldsymbol{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}},$$

where $\boldsymbol{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} = (\boldsymbol{div}^{\mathfrak{M}}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\partial\mathfrak{M}}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\mathfrak{M}^*}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\partial\mathfrak{M}^*}\xi^{\mathfrak{D}})$,*with* $\boldsymbol{div}^{\mathfrak{M}}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathrm{K}}\xi^{\mathfrak{D}})_{\mathrm{K}\in\mathfrak{M}}$, $\boldsymbol{div}^{\partial\mathfrak{M}}\xi^{\mathfrak{D}} = 0$, $\boldsymbol{div}^{\mathfrak{M}^*}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathrm{K}^*}\xi^{\mathfrak{D}})_{\mathrm{K}\in\mathfrak{M}^*}$ *and* $\boldsymbol{div}^{\partial\mathfrak{M}^*}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathrm{K}^*}\xi^{\mathfrak{D}})_{\mathrm{K}^*\in\partial\mathfrak{M}^*}$ *and we have set*

$$\begin{aligned}
\mathbf{div}^{\mathrm{K}}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathrm{K}}}\sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathrm{K}}} m_\sigma \xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma\mathrm{K}}, \quad \forall \mathrm{K}\in\mathfrak{M}\\
\mathbf{div}^{\mathrm{K}^*}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathrm{K}^*}}\sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathrm{K}^*}} m_{\sigma^*} \xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*}, \quad \forall \mathrm{K}^*\in\mathfrak{M}^*\\
\mathbf{div}^{\mathrm{K}^*}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathrm{K}^*}}\left(\sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathrm{K}^*}} m_{\sigma^*} \xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*} + \sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathrm{K}^*}\cap\mathfrak{D}_{ext}} \frac{m_\sigma}{2}\xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma\mathrm{K}}\right) \quad \forall \mathrm{K}^*\in\partial\mathfrak{M}^*.
\end{aligned}$$

2.5 Scalar products and norms

Now we define the scalar products on the approximation spaces:

$$\begin{aligned}
&[[\mathbf{v}^{\mathfrak{T}}, \mathbf{u}_{\mathfrak{T}}]]_{\mathfrak{T}} = \frac{1}{2}\left(\sum_{\mathrm{K}\in\mathfrak{M}} m_{\mathrm{K}}\,\mathbf{u}_{\mathrm{K}}\cdot\mathbf{v}_{\mathrm{K}} + \sum_{\mathrm{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathrm{K}^*}\,\mathbf{u}_{\mathrm{K}^*}\cdot\mathbf{v}_{\mathrm{K}^*}\right) && \forall \mathbf{u}_{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}},\\
&(\Phi^{\mathfrak{D}}, \mathbf{v}^{\partial\mathfrak{M}})_{\partial\Omega} = \sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \Phi^{\mathrm{D}}\cdot\mathbf{v}^{\sigma} && \forall \Phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}, \mathbf{v}^{\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\partial\mathfrak{M}},\\
&(\xi^{\mathfrak{D}} : \Phi^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathrm{D}}(\xi^{\mathrm{D}} : \Phi^{\mathrm{D}}) && \forall \xi^{\mathfrak{D}}, \Phi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},\\
&(\mathrm{p}_{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathrm{D}}\mathrm{p}^{\mathrm{D}}\mathrm{q}^{\mathrm{D}} && \forall \mathrm{p}_{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}},
\end{aligned}$$

where $(\xi : \widetilde{\xi}) = \sum_{1\le i,j\le 2} \xi_{i,j}, \widetilde{\xi}_{i,j} = \mathrm{Tr}({}^t\xi\widetilde{\xi})$ for all $\xi, \widetilde{\xi} \in \mathcal{M}_2(\mathbb{R})$. They define respectively the norms $\|\mathbf{u}_{\mathfrak{T}}\|_2$, $\|\mathbf{v}^{\partial\mathfrak{M}}\|_{2,\partial\Omega}$, $|||\xi^{\mathfrak{D}}|||_2$ and $\|\mathrm{p}_{\mathfrak{D}}\|_2$.

We define the *trace operators*. Let $\gamma^{\mathfrak{T}} : \mathbf{u}_{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}} \mapsto \gamma^{\mathfrak{T}}(\mathbf{u}_{\mathfrak{T}}) = (\gamma_\sigma(\mathbf{u}_{\mathfrak{T}}))_{\sigma\in\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\partial\mathfrak{M}}$ be given by

$$\gamma_\sigma(\mathbf{u}_{\mathfrak{T}}) = \frac{\mathbf{u}_{\mathrm{K}^*} + 2\mathbf{u}_{\mathrm{L}} + \mathbf{u}_{\mathrm{L}^*}}{4}, \qquad \forall \sigma = [x_{\mathrm{K}^*}, x_{\mathrm{L}^*}] \in \partial\mathfrak{M}.$$

On the diamond mesh we set $\gamma^{\mathfrak{D}} : \Phi^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \to (\Phi^{\mathrm{D}})_{\mathrm{D}\in\mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$, which is the operator of restriction to the boundary diamonds.

The discrete gradient and divergence operators are linked by a discrete Stokes formula. This is precisely the duality property that gives its name to the method [DO05], see for instance [Kre10, Thm IV.9].

Theorem 2.6 (Discrete Green's formula) *For all* $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}, \mathbf{u}_{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$, *we have:*

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}_{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}_{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\boldsymbol{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}_{\mathfrak{T}}))_{\partial\Omega},$$

where $\vec{\boldsymbol{n}}$ *is the unitary outward normal.*

2.6 Brezzi-Pitkäranta stabilization

The Inf-Sup condition is a structure property crucial for the stability of a scheme for the simulation of incompressible viscous flows. A stabilization term involving the pressure can be added to enforce this condition. This idea dates back to [BP84] for finite element methods. It has been adapted to the finite volume framework too [EHL06, EHL07] and we refer the reader to [Kre11a] for the specific case of DDFV schemes. Note that the Inf-Sup condition actually holds for a large class of meshes, which do not require any stabilization [BKN15].

The stabilization term involves the second order discrete operator, denoted by $\Delta^{\mathfrak{D}} : \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \mapsto \Delta^{\mathfrak{D}}\mathrm{p}_{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, defined by

$$\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}} = \frac{1}{m_{\mathrm{D}}} \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}'\in\mathcal{E}_{\mathrm{D}}} \frac{\mathrm{d}_{\mathrm{D}}^2 + \mathrm{d}_{\mathrm{D}'}^2}{\mathrm{d}_{\mathrm{D}}^2}(\mathrm{p}^{\mathrm{D}'} - \mathrm{p}^{\mathrm{D}}), \quad \forall \mathrm{D} \in \mathfrak{D}.$$

It resembles an approximation of the Laplace's operator, however it is consistent only under orthogonality condition (as in the case of *admissible* meshes, see [EGH00, Dro14]); that is not true in general for diamond meshes obtained from $\mathfrak{M}$. In relation with this operator we define a semi-norm $|\cdot|$ on $\mathbb{R}^{\mathfrak{D}}$ that depends on the mesh:

$$|\mathrm{p}^{\mathfrak{D}}|^2 = \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}'\in\mathfrak{S}} (\mathrm{d}_{\mathrm{D}}^2 + \mathrm{d}_{\mathrm{D}'}^2)(\mathrm{p}^{\mathrm{D}'} - \mathrm{p}^{\mathrm{D}})^2, \quad \forall \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}.$$

Observe that

$$\begin{aligned} -(\mathrm{d}_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}p^{\mathfrak{D}}, p^{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathrm{D}\in\mathfrak{D}} \mathrm{p}^{\mathrm{D}} \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}'\in\mathcal{E}_{\mathrm{D}}} (\mathrm{d}_{\mathrm{D}}^2 + \mathrm{d}_{\mathrm{D}'}^2)(\mathrm{p}^{\mathrm{D}} - \mathrm{p}^{\mathrm{D}'}) \\ &= \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}'\in\mathfrak{S}} (\mathrm{d}_{\mathrm{D}}^2 + \mathrm{d}_{\mathrm{D}'}^2)(\mathrm{p}^{\mathrm{D}'} - \mathrm{p}^{\mathrm{D}}) = |\mathrm{p}^{\mathfrak{D}}|^2. \end{aligned} \tag{3}$$

3 DDFV scheme for the Navier-Stokes problem on Ω

This Section is concerned by the analysis of DDFV schemes for the Navier-Stokes problem with Dirichlet boundary conditions on the entire domain Ω. As far as the convection is treated by upwind discretization, the analysis has been performed in [Kre10]. As mentioned above, it is convenient to extend this analysis to general B-schemes where the convection term is approximated by a centered discretization plus a diffusive perturbation, which depends on a certain function B, see [CHD11, HH14]. In what follows, $\mathrm{D}_{\sigma,\sigma^*}$ will be denoted by D, to simplify the notations.

3.1 The scheme $(\mathcal{P})$

Let $N \in \mathbb{N}^*$ and $0 < T < \infty$. We note $\delta t = \frac{T}{N}$ and $t_n = n\delta t$ for $n \in \{0, \ldots, N\}$. We use an implicit Euler time-discretization, except for the nonlinear convection term, which is linearized by using a semi-implicit approximation. At each time step, we shall enforce the equality

$$\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^n) - \beta d_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}}\mathrm{p}^n = 0 \tag{4}$$

which takes into account the Brezzi-Pitkäranta stabilization, with a parameter $\beta > 0$.

We look for $\mathbf{u}^{\mathfrak{T},[0,T]} = (\mathbf{u}^n)_{n\in\{0,\ldots N\}} \in \left(\mathbb{E}_0\right)^{N+1}$ and $\mathrm{p}^{\mathfrak{D},[0,T]} = (\mathrm{p}^n)_{n\in\{0,\ldots N\}} \in (\mathbb{R}^{\mathfrak{D}})^{N+1}$, that we initialize with:

$$\begin{aligned} &\mathbf{u}^0 = \mathbb{P}_c^{\mathfrak{T}}\mathbf{u}_0 \in \mathbb{E}_0, \\ &\mathrm{p}^0 \in \mathbb{R}^{\mathfrak{D}} \text{ such that } \Delta^{\mathfrak{D}}\mathrm{p}^0 = \frac{1}{\beta d_{\mathfrak{D}}^2}\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^0) \text{ with } \sum_{\mathrm{D}\in\mathfrak{D}} m_{\mathrm{D}}\mathrm{p}_{\mathrm{D}}^0 = 0. \end{aligned}$$

The vector p^0 is well defined since it is solution of a square system, whose matrix is invertible. With those choices of $(\mathbf{u}^0, \mathrm{p}^0)$ we guarantee the property (4) at the initial time step. The discrete force term is also defined by a projection over $\mathfrak{T}$, with $\mathbb{P}_m^{\mathfrak{M}}\mathbf{f}$ and $\mathbb{P}_m^{\mathfrak{M}^*}\mathbf{f}$. From now on, to simplify the notations we will denote $(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})$ with $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ and $(\mathbf{u}^n, \mathrm{p}^n)$ with $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$.

Given $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$ satisfying (4) the update $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0$ and $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ is such that:

$$\begin{cases} m_{\mathtt{K}} \dfrac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \displaystyle\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_{\mathtt{K}}\mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} & \forall \mathtt{K} \in \mathfrak{M} \\ m_{\mathtt{K}^*} \dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \displaystyle\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^* \in \mathfrak{M}^* \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}_{\mathfrak{T}}) - \beta d_{\mathtt{D}}^2 \Delta^{\mathfrak{D}} \mathrm{p}_{\mathfrak{D}} = 0 & \\ \displaystyle\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} \mathrm{p}^{\mathtt{D}} = 0. & \end{cases} \tag{$\mathcal{P}$}$$

The fluxes are defined as a sum of a “diffusion” term and a “convection” term:

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_\sigma(\mathcal{F}^d_{\sigma\mathtt{K}} + \mathcal{F}^c_{\sigma\mathtt{K}}), \qquad m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\sigma^*}(\mathcal{F}^d_{\sigma^*\mathtt{K}^*} + \mathcal{F}^c_{\sigma^*\mathtt{K}^*}).$$

The *diffusion fluxes* are defined as:

$$\begin{aligned} m_\sigma \mathcal{F}^d_{\sigma\mathtt{K}} &= -\, m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} = -m_\sigma \left(\frac{2}{\mathrm{Re}} \mathrm{D}^{\mathtt{D}}\mathbf{u}_{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}}\mathrm{Id}\right) \vec{\mathbf{n}}_{\sigma\mathtt{K}}, \\ m_{\sigma^*} \mathcal{F}^d_{\sigma^*\mathtt{K}^*} &= -\, m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} = -m_{\sigma^*} \left(\frac{2}{\mathrm{Re}} \mathrm{D}^{\mathtt{D}}\mathbf{u}_{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}}\mathrm{Id}\right) \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}. \end{aligned}$$

The *convection fluxes* are expressed as the sum of a centered discretization and a diffusive perturbation

$$\begin{aligned} m_\sigma \mathcal{F}^c_{\sigma\mathtt{K}} =& m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}\, m_{\mathtt{D}}} B\left(\frac{2m_{\mathtt{D}}\mathrm{Re}}{m_\sigma} F_{\sigma\mathtt{K}}\right)(\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}), \\ m_{\sigma^*} \mathcal{F}^c_{\sigma^*\mathtt{K}^*} =& m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2}\right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re}\, m_{\mathtt{D}}} B\left(\frac{2m_{\mathtt{D}}\mathrm{Re}}{m_{\sigma^*}} F_{\sigma^*\mathtt{K}^*}\right)(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}). \end{aligned}$$

The diffusive part depends on the function B, which describes the different schemes that we can work with. The centered scheme corresponds to $B(s) = 0$ and the upwind scheme corresponds to $B(s) = \frac{1}{2}|s|$. However, for further purposes and the analysis of the domain decomposition method, it is relevant to consider a quite general framework where B can be matrix-valued. In what follows, we denote $B\left(\frac{2m_{\mathtt{D}}\mathrm{Re}}{m_\sigma} F_{\sigma\mathtt{K}}\right)$ with $B_{\sigma\mathtt{K}}$ and $B\left(\frac{2m_{\mathtt{D}}\mathrm{Re}}{m_{\sigma^*}} F_{\sigma^*\mathtt{K}^*}\right)$ with $B_{\sigma^*\mathtt{K}^*}$. The total fluxes then become:

$$\boxed{\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} &= -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}\, m_{\mathtt{D}}} B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}) \\ m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} &= -m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2}\right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re}\, m_{\mathtt{D}}} B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) \end{aligned}} \tag{5}$$

The definition of $F_{\sigma\mathtt{K}}$, $F_{\sigma^*\mathtt{K}^*}$ comes from [Kre10, Kre11b], up to the boundary terms. They are approximations of the fluxes: $\displaystyle\int_\sigma (\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}) \rightsquigarrow F_{\sigma\mathtt{K}}(\mathbf{u}_{\mathfrak{T}})$ and $\displaystyle\int_{\sigma^*} (\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}) \rightsquigarrow F_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathfrak{T}})$. Therefore, their expression come from the solenoidal constraint integrated on the diamond, (see Fig. 3), taking into account the stabilization:

$$m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \bar{\mathrm{p}}^{\mathfrak{D}} = \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}},$$

so that $\displaystyle\int_{\mathfrak{s}} \bar{\mathbf{u}}^{\mathfrak{T}} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} \mathrm{d}s$ is approximated by

$$m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} = m_{\mathfrak{s}} \frac{\bar{\mathbf{u}}_{\mathtt{K}} + \bar{\mathbf{u}}_{\mathtt{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} - \beta(d_{\mathtt{D}} + d_{\mathtt{D}'})(\bar{\mathrm{p}}^{\mathtt{D}} - \bar{\mathrm{p}}^{\mathtt{D}'})$$

for $\mathfrak{s} = [x_{\mathtt{K}}, x_{\mathtt{K}^*}] = \mathtt{D}|\mathtt{D}'$, $\mathfrak{s} \in \mathcal{E}_{\mathtt{D}}$. For $\mathtt{D} \in \mathfrak{D}_{ext}$, (see Fig. 3), a similar reasoning leads to

$$m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \bar{\mathrm{p}}^{\mathfrak{D}} = \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}' \in \mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} + m_\sigma \gamma^\sigma(\bar{\mathbf{u}}^{\mathfrak{T}}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}.$$

Based on these considerations, we impose:

► For the primal edges:

$$m_\sigma F_{\sigma\mathtt{K}} = -\sum_{\mathfrak{s} \in \mathfrak{S}_{\mathtt{K}} \cap \mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}}.$$

► For the dual edges:

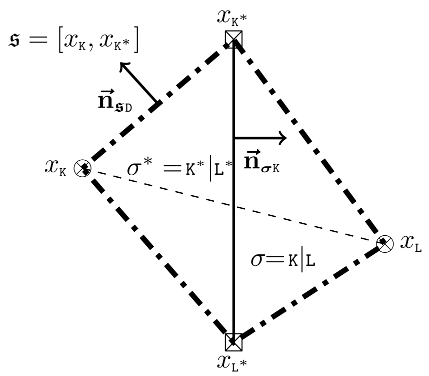
$$m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} = \begin{cases} -\displaystyle\sum_{\mathfrak{s} \in \mathfrak{S}_{\mathtt{K}^*} \cap \mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} & \text{if } \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*,\ \sigma^* \cap \partial\Omega = \emptyset \\ -m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} - \dfrac{1}{2} m_{\partial\Omega \cap \partial\mathtt{K}^*} H_{\mathtt{K}^*} & \text{if } \mathtt{K}^* \in \partial\mathfrak{M}^*,\ \sigma^* \cap \partial\Omega \neq \emptyset, \text{i.e. } \mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}^*}^{ext} \end{cases}$$

where $m_{\partial\Omega \cap \partial\mathtt{K}^*}$ indicates the measure of the intersection between $\partial\mathtt{K}^* \cap \partial\Omega$ and

$$m_{\partial\Omega \cap \partial\mathtt{K}^*} H_{\mathtt{K}^*} = \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}^{ext}} m_{\sigma \cap \partial\mathtt{K}^*} \bar{\mathbf{u}}_{\mathtt{K}^*} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}, \qquad \forall \mathtt{K}^* \in \partial\mathfrak{M}^*.$$

Remark that if (4) holds, then the fluxes $F_{\sigma\mathtt{K}}$ and $F_{\sigma^*\mathtt{K}^*}$ are conservative, that is to say

$$F_{\sigma\mathtt{K}} = -F_{\sigma\mathtt{L}}, \quad \forall \sigma = \mathtt{K}|\mathtt{L} \quad \text{and} \quad F_{\sigma^*\mathtt{K}^*} = -F_{\sigma^*\mathtt{L}^*}, \quad \forall \sigma^* = \mathtt{K}^*|\mathtt{L}^*.$$



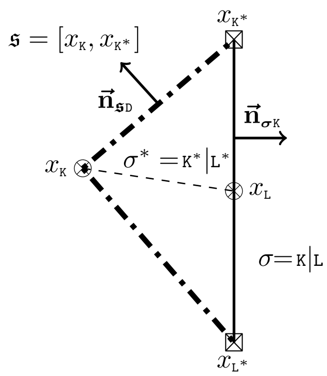


Fig. 3 *Left:* A diamond $\mathtt{D} = \mathtt{D}_{\sigma,\sigma^*}$ with $\sigma \subset \mathcal{E}_{int}$. *Right:* A diamond $\mathtt{D} = \mathtt{D}_{\sigma,\sigma^*}$ with $\sigma \in \partial\Omega$.

Proposition 3.1 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$, $\beta \in \mathbb{R}^*$ we have*

$$\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}} = 0 \qquad \forall \mathtt{K} \in \mathfrak{M},$$
$$\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} = 0 \qquad \forall \mathtt{K}^* \in \mathfrak{M}^*,$$
$$\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} = -m_{\partial\Omega \cap \partial\mathtt{K}^*} H_{\mathtt{K}^*} \quad \forall \mathtt{K}^* \in \partial\mathfrak{M}^*.$$

Proof For the interior mesh, we proceed as in [Kre10]. If $\mathtt{K} \in \mathfrak{M}$, by reorganizing the sum on the sides $\mathfrak{s} \in \mathfrak{S}_{\mathtt{K}}$ belonging to the primal cell $\mathtt{K}$, we obtain:

$$-\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}} \sum_{\mathfrak{s} \in \mathfrak{S}_{\mathtt{K}} \cap \mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} = -\sum_{\mathfrak{s} \in \mathfrak{S}_{\mathtt{K}}} m_{\mathfrak{s}} \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \cdot (\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} + \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}'}) = 0 \tag{6}$$

since $\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} = -\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}'}$, where $\mathtt{D}$ and $\mathtt{D}'$ denote the two neighboring diamonds which share the edge $\mathfrak{s}$, of vertices

x_{κ}, x_{κ^*}. In the same way,

$$-\sum_{D\in\mathfrak{D}_K}\sum_{\mathfrak{s}\in\mathfrak{S}_K\cap\mathcal{E}_D}(d_D^2+d_{D'}^2)(p^{D'}-p^{D}) = -\sum_{\mathfrak{s}\in\mathfrak{S}_K}(d_D^2+d_{D'}^2)(p^{D'}-p^{D}+p^{D}-p^{D'}) = 0. \tag{7}$$

We deduce that $\sum_{D\in\mathfrak{D}_K} m_\sigma F_{\sigma K} = 0$. The proof is similar for $\sum_{D\in\mathfrak{D}_{K^*}} m_{\sigma^*}F_{\sigma^*K^*} = 0$ if $K^*\in\mathfrak{M}^*$.
We now focus on the case in which $K^*\in\partial\mathfrak{M}^*$. By definition of $m_{\sigma^*}F_{\sigma^*K^*}$, we have

$$-\sum_{D\in\mathfrak{D}_{K^*}}\sum_{\mathfrak{s}\in\mathfrak{S}_{K^*}\cap\mathcal{E}_D}\left\{m_{\mathfrak{s}}\frac{\mathbf{u}_K+\mathbf{u}_{K^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}D}+(d_D^2+d_{D'}^2)(p^{D'}-p^{D})\right\}-\sum_{D\in\mathfrak{D}_{K^*}\cap\partial\Omega}\frac{1}{2}m_{\partial\Omega\cap\partial K^*}H_{K^*}$$
$$= 0 - m_{\partial\Omega\cap\partial K^*}H_{K^*}.$$

where the first sum vanishes thanks to (6),(7), and for the second term we use the fact that each vertex K^* is shared by two boundary diamonds. ■

3.2 Well-posedness of problem $(\mathcal{P})$

The well-posedness of the scheme $(\mathcal{P})$, which is known when the centered or upwind discretization is used, generalizes to a wide class of functions B. In what follows, for a $N\times N$ matrix A, we write $A\geq 0$ when the symmetric part of A is semi-definite positive, which means that $Az\cdot z\geq 0$ holds for any vector $z\in\mathbb{R}^N$.

Theorem 3.2 *Assume that*

$$\begin{aligned} &B_{\sigma K} = B_{\sigma L}, && B_{\sigma K}\geq 0\\ &B_{\sigma^*K^*} = B_{\sigma^*L^*}, && B_{\sigma^*K^*}\geq 0. \end{aligned} \tag{$\mathcal{H}p$}$$

Then the problem $(\mathcal{P})$ is well-posed.

Proof The scheme $(\mathcal{P})$ is a linear system in $(\mathbf{u}_{\mathfrak{T}}, p_{\mathfrak{D}})\in(\mathbb{R}^2)^{\mathfrak{T}}\times\mathbb{R}^{\mathfrak{D}}$. It corresponds to the specific case where $\mathbf{g}^{\partial\mathfrak{M}} = \mathbf{g}^{\partial\mathfrak{M}^*} = q^{\mathfrak{D}} = \phi = 0$, in

$$\begin{cases} m_K\dfrac{\mathbf{u}_K}{\delta t}+\displaystyle\sum_{D\in\mathfrak{D}_K}m_\sigma\mathcal{F}_{\sigma K} = m_K\mathbf{f}_K+m_K\dfrac{\bar{\mathbf{u}}_K}{\delta t} & \forall K\in\mathfrak{M}\\ m_{K^*}\dfrac{\mathbf{u}_{K^*}}{\delta t}+\displaystyle\sum_{D\in\mathfrak{D}_{K^*}}m_{\sigma^*}\mathcal{F}_{\sigma^*K^*} = m_{K^*}\mathbf{f}_{K^*}+m_{K^*}\dfrac{\bar{\mathbf{u}}_{K^*}}{\delta t} & \forall K^*\in\mathfrak{M}^*\\ \mathbf{u}^{\partial\mathfrak{M}} = \mathbf{g}^{\partial\mathfrak{M}} & \\ \mathbf{u}^{\partial\mathfrak{M}^*} = \mathbf{g}^{\partial\mathfrak{M}^*} & \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}_{\mathfrak{T}})-\beta d_D^2\Delta^{\mathfrak{D}}p_{\mathfrak{D}} = q^{\mathfrak{D}} & \\ \displaystyle\sum_{D\in\mathfrak{D}}m_D p^D = \phi. & \end{cases} \tag{$\mathcal{P}$}$$

Let us denote by N the dimension of $(\mathbb{R}^2)^{\mathfrak{T}}\times\mathbb{R}^{\mathfrak{D}}$. Equation $(\mathcal{P})$ is a linear system $Av=b$ with a rectangular matrix $A\in\mathcal{M}_{N+1,N}(\mathbb{R})$, $v\in\mathbb{R}^N$ and $b\in\mathbb{R}^{N+1}$. Let X be the following set:

$$X=\left\{(\mathbf{f}^{\mathfrak{M}},\mathbf{f}^{\mathfrak{M}^*},\mathbf{g}^{\partial\mathfrak{M}},\mathbf{g}^{\partial\mathfrak{M}^*},q^{\mathfrak{D}},\phi)\in\mathbb{R}^{N+1},\ \sum_{D\in\mathfrak{D}_{ext}}m_\sigma\gamma^\sigma(\mathbf{g}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma K}=\sum_{D\in\mathfrak{D}}m_D q^{\mathfrak{D}}\right\}.$$

We have $\dim(X)=N$, ${}^t(\mathbf{f}^{\mathfrak{M}},\mathbf{f}^{\mathfrak{M}^*},0,0,0)$ belongs to X and $\mathrm{Im}(A)\subset X$ as a consequence of the Green formula in Theorem 2.6. If we show that the matrix is injective, we conclude that $\dim(\mathrm{Im}(A))=N$ and that $\mathrm{Im}(A)=X$. We are going to show that if $\mathbf{f}^{\mathfrak{M}}=\mathbf{f}^{\mathfrak{M}^*}=0$, then $\mathbf{u}_{\mathfrak{T}}=0$ and $p_{\mathfrak{D}}=0$.
We multiply the equations on the primal and dual mesh of $(\mathcal{P})$ by $\mathbf{u}_{\mathfrak{T}}$ and we sum over all the control

volumes:

$$\frac{1}{2}\left[\frac{1}{\delta t}\left(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}\,|\mathbf{u}_{\mathtt{K}}|^2+\sum_{\mathtt{K}\in\mathfrak{M}^*} m_{\mathtt{K}^*}\,|\mathbf{u}_{\mathtt{K}^*}|^2\right)+\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*}\right]=0.$$

By definition of the scalar products we have $\frac{1}{2}\left[\frac{1}{\delta t}\left(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}\,|\mathbf{u}_{\mathtt{K}}|^2+\sum_{\mathtt{K}\in\mathfrak{M}^*} m_{\mathtt{K}^*}\,|\mathbf{u}_{\mathtt{K}^*}|^2\right)\right]=\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}}\|^2$ and, by replacing the definition of the fluxes, we get

$$\begin{aligned}\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}}\|^2&-\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}},\mathrm{p}_{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma\mathtt{K}}-\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}},\mathrm{p}_{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\\&+\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{L}}}{2}+\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}\frac{\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}}{2}\\&+\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})=0.\end{aligned}\tag{8}$$

We can consider separately the terms. By replacing the definition of the divergence operator and then by appying Green's formula (Theorem 2.6) for $\mathbf{u}_{\mathfrak{T}}\in\mathbb{E}_0$, we obtain

$$\begin{aligned}&-\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}},\mathrm{p}_{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma\mathtt{K}}-\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}},\mathrm{p}_{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\\&=-\left[\left[\mathbf{div}^{\mathfrak{T}}\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathfrak{T}}-\mathrm{p}_{\mathfrak{D}}\right),\mathbf{u}_{\mathfrak{T}}\right]\right]_{\mathfrak{T}}=\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathfrak{T}}\|_2^2-(\mathrm{p}_{\mathfrak{D}},\mathrm{div}^{\mathfrak{D}}\mathbf{u}_{\mathfrak{T}})_{\mathfrak{D}}=\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathfrak{T}}\|_2^2+\beta|\mathrm{p}_{\mathfrak{D}}|^2,\end{aligned}$$

where for the last equality we use that $\mathrm{div}^{\mathfrak{D}}(\mathbf{u}_{\mathfrak{T}})-\beta\mathrm{d}_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}\mathrm{p}_{\mathfrak{D}}=0$ and we apply (3).
For the convection terms, we sum over diamonds recalling that $\mathbf{u}_{\mathfrak{T}}\in\mathbb{E}_0$, so we do not have boundary terms. For the centered part, we apply Proposition 3.1, to conclude that

$$\begin{aligned}&\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{L}}}{2}+\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}\frac{\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}}{2}\\&=\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma F_{\sigma\mathtt{K}}(|\mathbf{u}_{\mathtt{K}}|^2-|\mathbf{u}_{\mathtt{L}}|^2)+\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}(|\mathbf{u}_{\mathtt{K}^*}|^2-|\mathbf{u}_{\mathtt{L}^*}|^2)\\&=\frac{1}{4}\sum_{\mathtt{K}\in\mathfrak{M}}|\mathbf{u}_{\mathtt{K}}|^2\underbrace{\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}}}_{=0}+\frac{1}{4}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}|\mathbf{u}_{\mathtt{K}^*}|^2\underbrace{\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}}_{=0}=0.\end{aligned}$$

For the diffusive perturbation, $(\mathcal{H}p)$ implies

$$\begin{aligned}&\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\\&=\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\geq 0.\end{aligned}$$

Putting all together, (8) becomes:

$$\frac{1}{\delta t}||\mathbf{u}_{\mathfrak{T}}||_2^2+\frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathfrak{T}}||_2^2+\beta|\mathrm{p}_{\mathfrak{D}}|^2\leq 0,$$

from which we deduce that $\mathbf{u}_{\mathfrak{T}}=0$ and $\mathrm{p}_{\mathfrak{D}}$ is a constant (we recall that $\beta>0$). Since $\mathrm{p}_{\mathfrak{D}}$ verifies $\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}}=0$, we have $\mathrm{p}_{\mathfrak{D}}=0$. ■

4 DDFV domain decomposition

We start by defining a discretization for the problem set on the subdomain Ω_j. As in Section 3, the nonlinear convection term will be approximated through B-schemes; we will see that the coefficients $B_{\sigma\kappa}, B_{\sigma^*\kappa^*}$ play an important role in the convergence of the Schwarz algorithm. We start by defining the meshes, and we analyse the scheme on each subdomain, denoted by $(\mathcal{P}_j)$, and we introduce the Schwarz algorithm for the domain decomposition. We present the study for two subdomains for the sake of simplicity, but it could be extended to a arbitrary number of adjacent subdomains. (Difficulties arise when more than two domains have common points on their interfaces.)

4.1 DDFV on composite meshes

For each subdomain Ω_j of Ω, $j = 1, 2$, we consider a DDFV mesh $\mathfrak{T}_j = (\mathfrak{M}_j \cup \partial\mathfrak{M}_j, \mathfrak{M}_j^* \cup \partial\mathfrak{M}_j^*)$ and the associated diamond mesh $\mathfrak{D}_j$. Note that the DDFV approach allows us to work with non conformal meshes, and the two subdomains can be meshed differently. Letting Γ be the interface between the two subdomains, we denote:

⋄ the diamond cells intersecting Γ :	$\mathfrak{D}_j^\Gamma := \{\mathtt{D} \in \mathfrak{D}_j, \mathtt{D} \cap \Gamma \neq \emptyset\}$;
⋄ the boundary primal cells intersecting Γ :	$\partial\mathfrak{M}_{j,\Gamma} := \{\kappa \in \partial\mathfrak{M}_j, \kappa \cap \Gamma \neq \emptyset\}$;
⋄ the boundary dual cells intersecting Γ :	$\partial\mathfrak{M}_{j,\Gamma}^* := \{\kappa^* \in \partial\mathfrak{M}_j^*, \kappa^* \cap \Gamma \neq \emptyset\}$;
⋄ the boundary primal cells intersecting $\partial\Omega$:	$\partial\mathfrak{M}_{j,D} := \{\kappa \in \partial\mathfrak{M}_j, \kappa \cap \partial\Omega \neq \emptyset\}$;
⋄ the boundary dual cells intersecting $\partial\Omega$:	$\partial\mathfrak{M}_{j,D}^* := \{\kappa^* \in \partial\mathfrak{M}_j^*, \kappa^* \cap \partial\Omega \neq \emptyset\}$;

see Fig. 4 for an example.

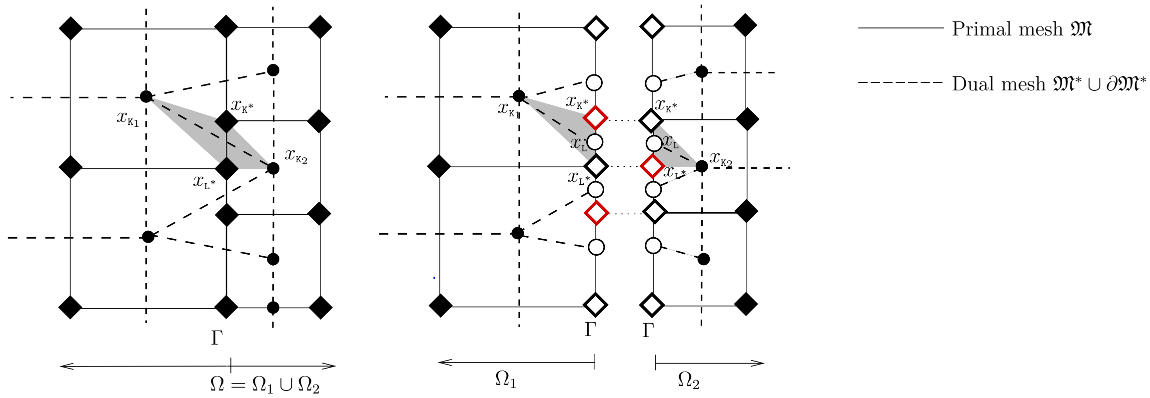


Fig. 4 DDFV meshes.

Definition 4.1 (Composite mesh) *We say that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are* compatible, *if the following conditions are satisfied:*

1. *the two meshes share the same vertices on Γ. This, in particular, implies that the two meshes have the same degenerate volumes on Γ, i.e. $\partial\mathfrak{M}_{1,\Gamma} = \partial\mathfrak{M}_{2,\Gamma}$.*
2. *The center $x_{\mathtt{L}}$ of the degenerate volumes of the interface $\mathtt{L} = [x_{\kappa^*}, x_{\mathtt{L}^*}] \in \partial\mathfrak{M}_{1,\Gamma} = \partial\mathfrak{M}_{2,\Gamma}$ is the intersection between $(x_{\kappa^*}, x_{\mathtt{L}^*})$ and $(x_{\kappa_1}, x_{\kappa_2})$, where $\kappa_1 \in \mathfrak{M}_1$ and $\kappa_2 \in \mathfrak{M}_2$ are the two primal cells such that $\mathtt{L} \in \partial\kappa_1$ and $\mathtt{L} \in \partial\kappa_2$ (see Fig. 4).*

Consider the composite mesh of Fig. 4; remark that:

- a diamond $\mathtt{D}$, of vertices $x_{\kappa_1}, x_{\kappa^*}, x_{\mathtt{L}^*}, x_{\kappa_2}$ that intersects Γ in the domain Ω can be written as the union of diamonds $\mathtt{D}_1$, of vertices $x_{\kappa_1}, x_{\kappa^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}}$, and $\mathtt{D}_2$, of vertices $x_{\kappa_2}, x_{\kappa^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}}$, respectively in Ω_1, Ω_2. Moreover, on the subdomain meshes we have additional unknowns on $x_{\mathtt{L}}$ on Γ with respect to the mesh on Ω;
- equivalently, a volume κ^* that intersects Γ in Ω is the union of κ_1^*, κ_2^* in Ω_1, Ω_2. In particular, an edge $\sigma^* = [x_{\kappa_1}, x_{\kappa_2}]$ can be split into $\sigma^* = \sigma_1^* \cup \sigma_2^* = [x_{\kappa_1}, x_{\mathtt{L}}] \cup [x_{\mathtt{L}}, x_{\kappa_2}]$;

- an edge $\sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}]$ on the interface Γ is shared by all the meshes.

Due to the fact that each dual cell on the original mesh that intersects Γ is split in two between the subdomains, it is necessary to introduce some additional unknowns fluxes $\Psi_{\mathtt{K}_j^*}$, for all $\mathtt{K}_j^* \in \partial\mathfrak{M}_{j,\Gamma}^*$, as in [GHHK18]. Those unknowns are intended to approximate the dual fluxes $\mathcal{F}_{\sigma^*\mathtt{K}^*}$ on the interface. For a diamond $\mathtt{D} \in \mathfrak{D}_j^\Gamma$, the unknowns are illustrated in Fig. 5.

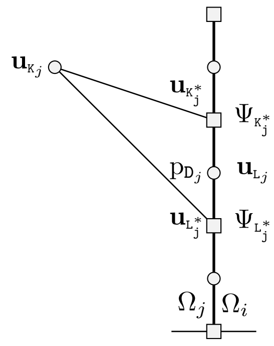


Fig. 5 The unknowns on a diamond on the interface for the subdomain Ω_j.

4.2 The subdomain problem: transmission conditions

On each subdomain Ω_j of Ω, we want to solve a Navier-Stokes system with mixed boundary conditions. On the fraction of the boundary that intersects $\partial\Omega$, we impose Dirichlet boundary conditions. On the the interface Γ between the two subdomains, we impose the discretized version of the transmission conditions (2).

To construct the scheme, we integrate the momentum equation over $\mathfrak{M}_j \cup \mathfrak{M}_j^* \cup \partial\mathfrak{M}_{j,\Gamma}^*$, we impose Dirichlet boundary conditions on $\partial\mathfrak{M}_{j,D} \cup \partial\mathfrak{M}_{j,D}^*$ and transmission conditions on $\partial\mathfrak{M}_{j,\Gamma} \cup \partial\mathfrak{M}_{j,\Gamma}^*$. The transmission conditions involve three positive parameters:

- λ which arises in the Fourier-like transmission condition for the velocity,
- α which arises in the Fourier-like transmission condition for the pressure,
- β that relies on the Brezzi-Pitkäranta stabilization.

Precisely, the solenoidal constraint is approximated on the diamond mesh $\mathfrak{D}_j$ and for the diamonds in $\mathfrak{D}_j^\Gamma$ a transmission term is added, controlled by the parameter α. We give now, formally, an hint of why it is necessary to add this condition on the interface diamonds $\mathfrak{D}_j^\Gamma$: our goal is to recover, at convergence of the Schwarz algorithm, the solenoidal constraint $\mathrm{div}^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}) = 0$ for all $\mathtt{D}$ that intersect Γ in Ω (that we write here for sake of simplicity without the stabilization term). As described in Definition 4.1, a diamond $\mathtt{D}$ in Ω that intersects Γ can be written as the union of diamonds $\mathtt{D}_1, \mathtt{D}_2$ in Ω_1, Ω_2. By definition of the discrete divergence, see Section 2.4, we can decompose $m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}\mathbf{u}_{\mathfrak{T}} = m_{\mathtt{D}_1}\mathrm{div}^{\mathtt{D}_1}\mathbf{u}_{\mathfrak{T}} + m_{\mathtt{D}_2}\mathrm{div}^{\mathtt{D}_2}\mathbf{u}_{\mathfrak{T}}$. This implies, in particular, that on Γ we would like:

$$m_{\mathtt{D}_1}\mathrm{div}^{\mathtt{D}_1}\mathbf{u}_{\mathfrak{T}} = -m_{\mathtt{D}_2}\mathrm{div}^{\mathtt{D}_2}\mathbf{u}_{\mathfrak{T}}. \tag{9}$$

Imposing a condition of this kind along the iterations of the Schwarz algorithm is however not sufficient to prove convergence of the algorithm, as we will show later in Theorem 5.8; in order to apply the analytical tools of the proof, it is necessary to add to (9) a Fourier-like term for the pressure, controlled by α.

The DDFV discretization leads to the following system on Ω_j:

$$\text{Find } (\mathbf{u}_{\mathfrak{T}_j}, \mathrm{p}_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}) \in \mathbb{E}_0^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}_j} \times \partial\mathfrak{M}_{j,\Gamma}^* \text{ such that}$$

$$\left\{\begin{array}{ll}
m_{\mathtt{K}}\dfrac{\mathbf{u}_{\mathtt{K}}}{\delta t}+\displaystyle\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_{\mathtt{K}}\mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} & \forall \mathtt{K}\in\mathfrak{M}_j\\
m_{\mathtt{K}^*}\dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t}+\displaystyle\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^*\in\mathfrak{M}^*_j\\
m_{\mathtt{K}^*}\dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t}+\displaystyle\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} + m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi_{\mathtt{K}^*} = m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}\\
-\mathcal{F}_{\sigma\mathtt{K}} + \dfrac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}} + \lambda\mathbf{u}_{\mathtt{L}} = \mathbf{h}_{\mathtt{L}} & \forall\sigma\in\partial\mathfrak{M}_{j,\Gamma}\\
-\Psi_{\mathtt{K}^*} + \dfrac{1}{2}H_{\mathtt{K}^*}\mathbf{u}_{\mathtt{K}^*} + \lambda\mathbf{u}_{\mathtt{K}^*} = \mathbf{h}_{\mathtt{K}^*} & \forall \mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}\\
m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \mathrm{p}_{\mathfrak{D}_j} = 0 & \forall \mathtt{D}\in\mathfrak{D}_j\setminus\mathfrak{D}_j^\Gamma\\
m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \mathrm{p}_{\mathfrak{D}_j} + \alpha m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}} = g_{\mathtt{D}} & \forall \mathtt{D}\in\mathfrak{D}_j^\Gamma,
\end{array}\right. \qquad (\mathcal{P}_j)$$

where $\bar{\mathbf{u}}_{\mathfrak{T}_j}$ the solution computed at the previous time step $t_{n-1}=(n-1)\delta t$ for $n\in\{1,\dots N-1\}$, and $\mathbf{h}$, g are certain boundary data in $(\mathbf{R}^2)^{\partial\mathfrak{M}_{j,\Gamma}}\times(\mathbf{R}^2)^{\partial\mathfrak{M}^*_{j,\Gamma}}$ and $\mathbf{R}^{\mathfrak{D}_j^\Gamma}$, respectively. Here, we denote $\mathbf{f}_{\mathfrak{T}}=\mathbb{P}_m^{\mathfrak{T}}\mathbf{f}$. We will refer to the system $(\mathcal{P}_j)$ in the shorthand form:

$$\mathcal{L}^{\mathfrak{T}_j}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j},\mathrm{p}_{\mathfrak{D}_j},\Psi_{\mathfrak{T}_j},\mathbf{f}_{\mathfrak{T}},\bar{\mathbf{u}}_{\mathfrak{T}_j},\mathbf{h}_{\mathfrak{T}_j},g_{\mathfrak{D}_j})=0.$$

Remark 4.2 *When we impose transmission conditions in Schwarz' algorithm, we are led to approximate the boundary term* $\displaystyle\int_\sigma\left(\sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}}-\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\right)$*, which keeps track of the anti-symmetrization of the convection term. Formally, at the continuous level, if* φ *is a test function in* $V=\{\varphi\in(H^1(\Omega))^2,\ \Psi|_{\Gamma_D}=0,\ div(\varphi)=0\}$*, the variational formulation of* (1) *reads:*

$$\int_\Omega\partial_t\mathbf{u}\cdot\varphi+\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi-\int_\Omega\mathrm{div}(\sigma(\mathbf{u},\mathrm{p}))\varphi=0. \qquad (10)$$

The convection term can be written as

$$\begin{aligned}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi&=\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi+\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi\\&=\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi-\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\varphi\cdot\mathbf{u}+\int_{\partial\Omega}\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\varphi,\end{aligned}$$

by integration by parts, since $\mathbf{u}$ *is divergence free. Coming back to* (10)*, we integrate by parts also the diffusion terms, and we end up with:*

$$\int_\Omega\partial_t\mathbf{u}\cdot\varphi+\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi-\frac{1}{2}\int_\Omega(\mathbf{u}\cdot\nabla)\varphi\cdot\mathbf{u}+\int_\Omega\sigma(\mathbf{u},\mathrm{p}):\nabla\varphi-\int_{\partial\Omega}\left(\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}}-\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\right)\cdot\varphi=0.$$

This is the reason why, when working with transmission conditions, we impose a condition on $\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}}-\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}$*, that contains just "half of the convection". Besides, the numerical flux* $\mathcal{F}_{\sigma\mathtt{K}}$ *is constructed to approximate the term*

$$\int_\sigma\left(-\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}}+(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\right).$$

This is why in the approximation it gives:

$$\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}}-\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}=\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}}-(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}+\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\approx-\mathcal{F}_{\sigma\mathtt{K}}+\frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}}.$$

Theorem 4.3 (Well-posedness of the DDFV subdomain problem) *Under the hypothesis* $(\mathcal{H}p)$ *and* $\lambda,\beta,\alpha>0$*, the problem* $(\mathcal{P}_j)$ *is well-posed.*

The proof relies on the following energy estimate (where we bear in mind that $B_{\sigma\mathtt{K}}$ and $B_{\sigma^*\mathtt{K}^*}$ can be matrix-valued).

Theorem 4.4 (Energy estimate on $(\mathcal{P}_j)$) *The scheme $(\mathcal{P}_j)$ satisfies the following relation*

$$\begin{aligned}\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}}\|D^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}\|_2^2 - (\mathrm{p}_{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j})_{\mathfrak{D}_j} \\ + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma(\mathcal{F}_{\sigma\mathtt{K}} - \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_\mathtt{L})\cdot\mathbf{u}_\mathtt{L} + \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Psi_{\mathtt{K}^*} - \frac{1}{2}H_{\mathtt{K}^*}\mathbf{u}_{\mathtt{K}^*})\cdot\mathbf{u}_{\mathtt{K}^*} \\ + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma\mathtt{K}}(\mathbf{u}_\mathtt{K} - \mathbf{u}_\mathtt{L})\cdot(\mathbf{u}_\mathtt{K} - \mathbf{u}_\mathtt{L}) + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) \\ = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}. \quad (11)\end{aligned}$$

Proof of Theorem 4.3. Let us explain how Theorem 4.4 can be used to justify the well-posedness of the equation with mixed conditions. We are going to prove that if $\mathbf{f}_{\mathfrak{T}_j} = 0 = \mathbf{h}_{\mathfrak{T}_j} = g_{\mathfrak{D}_j}$, then $\mathbf{u}_{\mathfrak{T}_j} = 0 = \Psi_{\mathfrak{T}_j}$ and $\mathrm{p}_{\mathfrak{D}_j} = 0$. Starting from (11), we apply:

- the transmission conditions on the sums over $\mathfrak{D}_j^\Gamma$ and $\partial\mathfrak{M}^*_{j,\Gamma}$:

$$-\mathcal{F}_{\sigma\mathtt{K}} + \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_\mathtt{L} + \lambda\mathbf{u}_\mathtt{L} = \mathbf{h}_\mathtt{L} \quad \forall\sigma\in\partial\mathfrak{M}_{j,\Gamma},$$

$$-\Psi_{\mathtt{K}^*} + \frac{1}{2}H_{\mathtt{K}^*}\,\mathbf{u}_{\mathtt{K}^*} + \lambda\mathbf{u}_{\mathtt{K}^*} = \mathbf{h}_{\mathtt{K}^*} \quad \forall\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma},$$

- the conditions on the equation of mass conservation:

$$\mathrm{div}^\mathtt{D}(\mathbf{u}_{\mathfrak{T}_j}) - \beta m_\mathtt{D} d_\mathtt{D}^2\Delta^\mathtt{D}\mathrm{p}_{\mathfrak{D}_j} = 0 \quad \forall\mathtt{D}\in\mathfrak{D}_j\setminus\mathfrak{D}_j^\Gamma,$$

$$\mathrm{div}^\mathtt{D}(\mathbf{u}_{\mathfrak{T}_j}) - \beta m_\mathtt{D} d_\mathtt{D}^2\Delta^\mathtt{D}\mathrm{p}_{\mathfrak{D}_j} + \alpha m_\mathtt{D}\mathrm{p}^\mathtt{D} = g_\mathtt{D} \quad \forall\mathtt{D}\in\mathfrak{D}_j^\Gamma.$$

This implies:

$$\begin{aligned}\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}\|_2^2 + \beta|\mathrm{p}_{\mathfrak{D}_j}|^2 \\ + \alpha\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\mathtt{D}|\mathrm{p}^\mathtt{D}|^2 + \frac{\lambda}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathbf{u}_\mathtt{L}|^2 + \frac{\lambda}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2 \\ + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma\mathtt{K}}(\mathbf{u}_\mathtt{K} - \mathbf{u}_\mathtt{L})\cdot(\mathbf{u}_\mathtt{K} - \mathbf{u}_\mathtt{L}) + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) \\ = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j} + (\mathrm{p}_{\mathfrak{D}_j}, g_{\mathfrak{D}_j})_{\mathfrak{D}_j^\Gamma} + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma\mathbf{h}_\mathtt{L}\cdot\mathbf{u}_\mathtt{L} + \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}\mathbf{h}_{\mathtt{K}^*}\cdot\mathbf{u}_{\mathtt{K}^*}. \quad (12)\end{aligned}$$

If now we impose $\mathbf{f}_{\mathfrak{T}_j} = 0 = \mathbf{h}_{\mathfrak{T}_j} = g_{\mathfrak{D}_j}$ in (12), we have:

$$\begin{aligned}\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}\|_2^2 + \beta|\mathrm{p}_{\mathfrak{D}_j}|^2 \\ \underbrace{+\alpha\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\mathtt{D}|\mathrm{p}^\mathtt{D}|^2 + \frac{\lambda}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathbf{u}_\mathtt{L}|^2 + \frac{\lambda}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2}_{\geq 0} \\ \underbrace{+ \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma\mathtt{K}}(\mathbf{u}_\mathtt{K} - \mathbf{u}_\mathtt{L})\cdot(\mathbf{u}_\mathtt{K} - \mathbf{u}_\mathtt{L}) + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*})}_{\geq 0} = 0,\end{aligned}$$

that leads to:

$$\frac{1}{\delta t}||\mathbf{u}_{\mathfrak{T}_j}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}||_2^2 + \beta|\mathrm{p}_{\mathfrak{D}_j}|^2 \leq 0,$$

from which we deduce that $\mathbf{u}_{\mathfrak{T}_j} = 0$ and $\mathrm{p}_{\mathfrak{D}_j}$ is a constant (since $\beta > 0$). Thanks to the transmission conditions on $\mathfrak{D}_j^\Gamma$, since $\alpha > 0$ and $\mathbf{u}_{\mathfrak{T}_j} = 0$, we obtain $\mathrm{p}_{\mathfrak{D}_j} = 0$. Finally, thanks to the transmission condition on $\partial\mathfrak{M}^*_{j,\Gamma}$ and $\mathbf{u}_{\mathfrak{T}_j}=0$, we also have $\Psi_{\mathfrak{T}_j}=0$. ■

Proof of Theorem 4.4. We multiply the equations on the primal and dual mesh of $(\mathcal{P}_j)$ by $\mathbf{u}_{\mathfrak{T}_j}$ and we sum over all the control volumes:

$$\frac{1}{2}\Bigg[\frac{1}{\delta t}\Bigg(\sum_{\mathtt{K}\in\mathfrak{M}_j} m_{\mathtt{K}}\,|\mathbf{u}_{\mathtt{K}}|^2+\sum_{\mathtt{K}\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j} m_{\mathtt{K}^*}\,|\mathbf{u}_{\mathtt{K}^*}|^2\Bigg)+\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi_{\mathtt{K}^*}\cdot\mathbf{u}_{\mathtt{K}^*}$$
$$+\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*}\Bigg]=[[\mathbf{f}_{\mathfrak{T}_j},\mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}. \quad (13)$$

By definition of the scalar products we have $\frac{1}{2}\Bigg[\frac{1}{\delta t}\Bigg(\sum_{\mathtt{K}\in\mathfrak{M}_j} m_{\mathtt{K}}\,|\mathbf{u}_{\mathtt{K}}|^2+\sum_{\mathtt{K}\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j} m_{\mathtt{K}^*}\,|\mathbf{u}_{\mathtt{K}^*}|^2\Bigg)\Bigg]=\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|^2$ and, by rewriting the fluxes as a sum of the diffusive and convective contribution we have:

$$\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|^2+\frac{1}{2}\Bigg[\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi_{\mathtt{K}^*}\cdot\mathbf{u}_{\mathtt{K}^*}+\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^d_{\sigma^*\mathtt{K}^*}$$
$$+\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^c_{\sigma^*\mathtt{K}^*}\Bigg]=[[\mathbf{f}_{\mathfrak{T}_j},\mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}.$$

We consider separately the two contributions. For the diffusion terms, we have, by the definition of the divergence operator

$$\frac{1}{2}\Bigg[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^d_{\sigma^*\mathtt{K}^*}\Bigg]$$
$$=-\left[\left[\mathrm{div}^{\mathfrak{T}_j}\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}-\mathrm{p}_{\mathfrak{D}_j}\mathrm{Id}\right),\mathbf{u}_{\mathfrak{T}_j}\right]\right]_{\mathfrak{T}_j}-\frac{1}{4}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}.$$

We can now apply Green's formula to the RHS, and remark that

$$\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}=\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot(\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}).$$

We thus find:

$$\frac{1}{2}\Bigg[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^d_{\sigma^*\mathtt{K}^*}\Bigg]$$
$$=\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}\|_2^2-(\mathrm{p}_{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j})_{\mathfrak{D}_j}+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot\gamma^\sigma(\mathbf{u}_{\mathfrak{T}_j})-\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot\frac{\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}}{4}.$$

By the definition of the trace operator, we obtain:

$$\frac{1}{2}\Bigg[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^d_{\sigma^*\mathtt{K}^*}\Bigg]$$
$$=\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}\|_2^2-(\mathrm{p}_{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j})_{\mathfrak{D}_j}+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}. \quad (14)$$

For the convection terms, we get

$$\frac{1}{2}\left[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}m_{\sigma^*}\mathcal{F}^c_{\sigma^*\mathtt{K}^*}\right]:=\frac{1}{2}(T_1+T_2)$$

We estimate the term T_1; we first integrate by parts:

$$\begin{aligned}T_1&=\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\\&=\sum_{\mathtt{D}\in\mathfrak{D}_j}m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j}m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}.\end{aligned}$$

We replace the definition of $\mathcal{F}^c_{\sigma\mathtt{K}}$ for all $\mathtt{D}\in\mathfrak{D}_j$:

$$\begin{aligned}T_1&=\sum_{\mathtt{D}\in\mathfrak{D}_j}m_\sigma F_{\sigma\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{L}}}{2}\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j}m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}\\&=\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}m_\sigma F_{\sigma\mathtt{K}}(|\mathbf{u}_{\mathtt{K}}|^2-|\mathbf{u}_{\mathtt{L}}|^2)+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j}m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}.\end{aligned}$$

Passing to the sum over primal cells $\mathtt{K}$ for the first term and applying Proposition 3.1 we get:

$$T_1=\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}_j}|\mathbf{u}_{\mathtt{K}}|^2\underbrace{\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}m_\sigma F_{\sigma\mathtt{K}}}_{=0}-\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j}m_\sigma F_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{L}}|^2+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j}m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}.$$

It can be rewritten as:

$$T_1=\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j}m_\sigma(\mathcal{F}^c_{\sigma\mathtt{K}}-\frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}})\cdot\mathbf{u}_{\mathtt{L}}+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}}).\tag{15}$$

We estimate the term T_2; we first integrate by parts:

$$\begin{aligned}T_2&=\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}m_{\sigma^*}\mathcal{F}^c_{\sigma^*\mathtt{K}^*}\\&=\sum_{\mathtt{D}\in\mathfrak{D}_j}m_{\sigma^*}\mathcal{F}^c_{\sigma^*\mathtt{K}^*}\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*}).\end{aligned}$$

We replace the definition of $\mathcal{F}^c_{\sigma\mathtt{K}}$ for all $\mathtt{D}\in\mathfrak{D}_j$:

$$\begin{aligned}T_2&=\sum_{\mathtt{D}\in\mathfrak{D}_j}m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}\frac{\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}}{2}\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\\&=\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}(|\mathbf{u}_{\mathtt{K}^*}|^2-|\mathbf{u}_{\mathtt{L}^*}|^2)+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*}).\end{aligned}$$

Passing to the sum over dual cells $\mathtt{K}^*$ for the first term we get:

$$T_2=\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}|\mathbf{u}_{\mathtt{K}^*}|^2\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*}).\tag{16}$$

From the definition of $F_{\sigma^*\mathtt{K}^*}$ and by Prop. 3.1 we have that $\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}=0$ for all $\mathtt{K}^*\in\mathfrak{M}^*_j$ and $\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}=-m_{\partial\Omega\cap\partial\mathtt{K}^*}H_{\mathtt{K}^*}$ for all $\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}$, that gives:

$$T_2=-\frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}}m_{\partial\Omega\cap\partial\mathtt{K}^*}H_{\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*}).$$

Gathering (14),(15) and (16) together, we find:

$$
\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|^2 + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}\|_2^2 - (\mathrm{p}_{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j})_{\mathfrak{D}_j} \\
&+ \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma(\mathcal{F}^d_{\sigma\mathtt{K}} + \mathcal{F}^c_{\sigma\mathtt{K}} - \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}})\cdot\mathbf{u}_{\mathtt{L}} + \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi_{\mathtt{K}^*}\cdot\mathbf{u}_{\mathtt{K}^*} - \frac{1}{4}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}H_{\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2 \\
+&\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}}) + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})\cdot(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*}) = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}.
\end{aligned}
$$

Since $\mathcal{F}^d_{\sigma\mathtt{K}} + \mathcal{F}^c_{\sigma\mathtt{K}} = \mathcal{F}_{\sigma\mathtt{K}}$, it leads to (11). ■

4.3 DDFV Schwarz algorithm

We can now introduce the iterative process that defines the Schwarz algorithm. Let $N \in \mathbb{N}^*$. We note $\delta t = \frac{T}{N}$ and $t_n = n\delta t$ for $n \in \{0, \dots N\}$. At each time step t_n we apply the following parallel DDFV Schwarz algorithm: for arbitrary initial guesses $\mathbf{h}^0_{\mathfrak{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$ and $g^0_{\mathfrak{D}_j} \in \mathbb{R}^{\mathfrak{D}_j}$, at each iteration $l = 1, 2, \dots$ and $i, j, \in \{1, 2\}$, $j \neq i$ we proceed with the following two steps:

1. Compute $(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}}$ solution to

$$\mathcal{L}^{\mathfrak{T}_j}_{\Omega_j,\Gamma}(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}_j}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}^{l-1}_{\mathfrak{T}_j}, g^{l-1}_{\mathfrak{D}^\Gamma_j}) = 0. \tag{$\mathcal{S}_1$}$$

2. Compute the new values of $\mathbf{h}^l_{\mathfrak{T}_J}$ and of $g^l_{\mathfrak{D}^\Gamma_j}$ by:

$$
\left\{
\begin{aligned}
\mathbf{h}^l_{\mathtt{L}_j} &= \mathcal{F}^l_{\sigma\mathtt{K}_i} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{u}^l_{\mathtt{L}_i} + \lambda\mathbf{u}^l_{\mathtt{L}_i}, && \forall \mathtt{L}_j = \mathtt{L}_i \in \partial\mathfrak{M}_{j,\Gamma} \\
\mathbf{h}^l_{\mathtt{K}^*_j} &= \Psi^l_{\mathtt{K}^*_i} - \frac{1}{2}H_{\mathtt{K}^*_j}\mathbf{u}^l_{\mathtt{K}^*_i} + \lambda\mathbf{u}^l_{\mathtt{K}^*_i}, && \forall \mathtt{K}^*_j \in \partial\mathfrak{M}^*_{j,\Gamma} \text{ such that } x_{\mathtt{K}^*_j} = x_{\mathtt{K}^*_i} \\
g^l_{\mathtt{D}_j} &= -\left(m_{\mathtt{D}_i}\mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^l_{\mathfrak{T}_i}) - \beta m_{\mathtt{D}_i}d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\mathrm{p}^l_{\mathfrak{D}_i}\right) + \alpha m_{\mathtt{D}_i}\mathrm{p}^l_{\mathtt{D}_i}, && \forall \mathtt{D}_j \in \mathfrak{D}^\Gamma_j \text{ such that } x_{\mathtt{D}_j} = x_{\mathtt{D}_i}.
\end{aligned}
\right. \tag{$\mathcal{S}_2$}
$$

5 Convergence analysis of the DDFV Schwarz algorithm

Bearing in mind the properties of the mesh discussed after Definition 4.1, we infer that the asymptotic fluxes as $l \to \infty$ should satisfy

$$m_\sigma\mathcal{F}_{\sigma\mathtt{K}} = m_\sigma\mathcal{F}_{\sigma\mathtt{K}_1} = -m_\sigma\mathcal{F}_{\sigma\mathtt{K}_2}, \quad \forall \mathtt{D} \in \mathfrak{D}^\Gamma \tag{17}$$

$$m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\sigma^*_1}\mathcal{F}_{\sigma^*_1\mathtt{K}^*} + m_{\sigma^*_2}\mathcal{F}_{\sigma^*_2\mathtt{K}^*}, \quad \forall\sigma^* = \sigma^*_1 \cup \sigma^*_2,\ \mathtt{K}^* \in \partial\mathfrak{M}^*_\Gamma. \tag{18}$$

In order to obtain these relations, it will become necessary to modify the fluxes on the interface, either for the limit or for the subdomain problem. For this reason, the convergence will be studied in two steps. In this Section, we shall identify the limit of the Schwarz algorithm defined in Section 4.3. We focus here on the natural situation where $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$ take *scalar values*, like with the upwind and centered discretizations. We will show that this limit is still a DDFV scheme for the problem (1), but with modified fluxes on Γ. We will then prove convergence to this limit scheme, to which we will refer to as $(\widetilde{\mathcal{P}})$. In the next Section 6, we will show that it is possible to obtain $(\mathcal{P})$ asymptotically, at the price of modifying the fluxes of the Schwarz algorithm $(\mathcal{S}_1)$, dealing with matrix coefficients $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$.

5.1 The limit problem $(\widetilde{\mathcal{P}})$

We consider the following DDFV scheme for (1), on the domain Ω: given $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$, satisfying (4), we look for $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0$ and $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\left\{\begin{aligned}
&m_{\mathrm{K}} \frac{\mathbf{u}_{\mathrm{K}}}{\delta t} + \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}}\setminus\mathfrak{D}_{\mathrm{K}}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathrm{K}} + \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}}^{\Gamma}} m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathrm{K}} = m_{\mathrm{K}} \mathbf{f}_{\mathrm{K}} + m_{\mathrm{K}} \frac{\bar{\mathbf{u}}_{\mathrm{K}}}{\delta t} && \forall \mathrm{K} \in \mathfrak{M}\\
&m_{\mathrm{K}^*} \frac{\mathbf{u}_{\mathrm{K}^*}}{\delta t} + \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}^*}\setminus\mathfrak{D}_{\mathrm{K}^*}^{\Gamma}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathrm{K}^*} + \sum_{\mathrm{D}\in\mathfrak{D}_{\mathrm{K}^*}^{\Gamma}} m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathrm{K}^*} = m_{\mathrm{K}^*} \mathbf{f}_{\mathrm{K}^*} + m_{\mathrm{K}^*} \frac{\bar{\mathbf{u}}_{\mathrm{K}^*}}{\delta t} && \forall \mathrm{K}^* \in \mathfrak{M}^*\\
&m_{\mathrm{D}} \mathrm{div}^{\mathrm{D}}(\mathbf{u}_{\mathfrak{T}}) - \beta m_{\mathrm{D}} d_{\mathrm{D}}^2 \Delta^{\mathrm{D}} \mathrm{p}_{\mathfrak{D}} = 0 && \forall \mathrm{D} \in \mathfrak{D}\\
&\sum_{\mathrm{D}\in\mathfrak{D}} m_{\mathrm{D}} \mathrm{p}^{\mathrm{D}} = 0.
\end{aligned}\right. \tag{$\widetilde{\mathcal{P}}$}$$

In the interior of the domain, the fluxes coincide with the fluxes in $(\mathcal{P})$, see (5). On the interface, they are defined as:

$$\boxed{\begin{aligned}
m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathrm{K}} &= -m_\sigma \sigma^{\mathrm{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathrm{K}} + m_\sigma F_{\sigma\mathrm{K}} \left(\frac{\mathbf{u}_{\mathrm{K}} + \mathbf{u}_{\mathrm{L}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathrm{D}}} \widetilde{B}_{\sigma\mathrm{K}} (\mathbf{u}_{\mathrm{K}} - \mathbf{u}_{\mathrm{L}}),\\
m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathrm{K}^*} &= -m_{\sigma^*} \sigma^{\mathrm{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*} + m_{\sigma^*} F_{\sigma^*\mathrm{K}^*} \left(\frac{\mathbf{u}_{\mathrm{K}^*} + \mathbf{u}_{\mathrm{L}^*}}{2}\right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathrm{D}}} \widetilde{B}_{\sigma^*\mathrm{K}^*} (\mathbf{u}_{\mathrm{K}^*} - \mathbf{u}_{\mathrm{L}^*}),
\end{aligned}}$$

where $\widetilde{B}_{\sigma\mathrm{K}}$ and $\widetilde{B}_{\sigma^*\mathrm{K}^*}$ are matrix-valued quantities that come from the transmission condition of the iterative process. Their expressions are established in Proposition 5.4 and 5.5.

5.2 Definition of $\widetilde{B}_{\sigma\mathrm{K}}$ and $\widetilde{B}_{\sigma^*\mathrm{K}^*}$

Let us start with some preliminary definition, bearing in mind that $B_{\sigma\mathrm{K}}, B_{\sigma^*\mathrm{K}^*}$ are supposed to be scalars.

Definition 5.1 *For $i = 1, 2$, and $\sigma \in \partial\mathfrak{M}_{i,\Gamma}$, we set $P = \mathrm{Id} + \vec{\mathbf{n}}_{\sigma\mathrm{K}} \otimes \vec{\mathbf{n}}_{\sigma\mathrm{K}}$ and*

$$A_i = \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathrm{D}_i}} (P + B_{\sigma\mathrm{K}_i} \mathrm{Id}),$$

where we recall that $B_{\sigma\mathrm{K}_i} = B\left(\frac{2m_{\mathrm{D}_i}\mathrm{Re}}{m_\sigma} F_{\sigma\mathrm{K}_i}\right)$. Next, we set $A = A_1 + A_2$.

Remark 5.2 *The matrix $A = A_1 + A_2$ is symmetric and definite positive, thus invertible, since it is the sum of two symmetric and definite positive matrices. In fact, with $\vec{\mathbf{n}}_{\sigma\mathrm{K}} = \begin{pmatrix} x \\ y \end{pmatrix}$, we have:*

$$A_i = \begin{pmatrix} 1 + B_{\sigma\mathrm{K}_i} + x^2 & xy \\ xy & 1 + B_{\sigma\mathrm{K}_i} + y^2 \end{pmatrix},$$

which is symmetric and for any $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ it holds:

$$\langle A_i v, v\rangle = (1 + B_{\sigma\mathrm{K}_i})(v_1^2 + v_2^2) + (x v_1 + y v_2)^2 \geq 0 \quad \text{and} \quad \langle A_i v, v\rangle = 0 \iff v = 0,$$

owing to Hypothesis $(\mathcal{H}p)$. For $i, j = 1, 2$, $i \neq j$, since A_i and A_j are polynomial in P, the following properties hold:

$$\begin{aligned} A_i A_j &= A_j A_i,\\ A_j A^{-1} &= A^{-1} A_j, \end{aligned}$$

since from Hypothesis $(\mathcal{H}p)$ we have $B_{\sigma\mathrm{K}_i} = B_{\sigma\mathrm{K}_j}$ for $\sigma \in \partial\mathfrak{M}_{i,\Gamma}$.

The fluxes $\widetilde{\mathcal{F}}_{\sigma\mathrm{K}}, \widetilde{\mathcal{F}}_{\sigma^*\mathrm{K}^*}$ are constructed in order to satisfy the properties (17)-(18). The system $(\widetilde{\mathcal{P}})$ is a scheme defined on the mesh $\mathfrak{T}$ on Ω; in particular, this means that there are no additional unknowns $\mathbf{u}_{\mathrm{L}}$ on the interface Γ, see Fig. 4. The following results apply for a general diamond:

Proposition 5.3 *Let* $\mathtt{D} \in \mathfrak{D}_\Gamma$ *be a diamond and let* $\mathtt{D}_1, \mathtt{D}_2$ *be the two semi-diamonds such that* $\mathtt{D} = \mathtt{D}_1 \cup \mathtt{D}_2$*, see Fig. 6 . We denote by* $(x_{\mathtt{K}}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}})$ *the vertices of* $\mathtt{D}$ *and by* $(x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_\sigma)$, $(x_{\mathtt{K}_2}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_\sigma)$ *the vertices of* $\mathtt{D}_1$ *and* $\mathtt{D}_2$*. Let* $\sigma = \mathtt{K}_1|\mathtt{K}_2$*, and let* A, A_1, A_2 *be as in Definition 5.1. Then, there exists a unique* $\mathbf{u}_{\boldsymbol{\sigma}}$*, given by*

$$\mathbf{u}_{\boldsymbol{\sigma}} = A^{-1}\left[A_1\mathbf{u}_{\mathtt{K}_1} + A_2\mathbf{u}_{\mathtt{K}_2} + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2})\right], \tag{19}$$

which satisfies

$$\mathcal{F}_{\sigma\mathtt{K}_1} = -\mathcal{F}_{\sigma\mathtt{K}_2}. \tag{20}$$

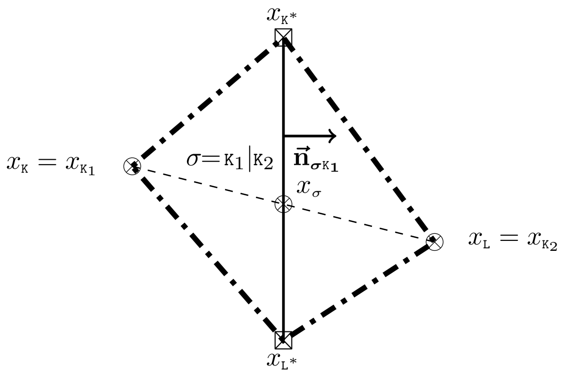


Fig. 6 A diamond $\mathtt{D}$, of vertices $x_{\mathtt{K}}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}}$ as a union of two semi-diamonds: $\mathtt{D}_1$ of vertices $x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_\sigma, x_{\mathtt{L}^*}$ and $\mathtt{D}_2$ of vertices $x_{\mathtt{K}_2}, x_{\mathtt{K}^*}, x_\sigma, x_{\mathtt{L}^*}$. In particular, $\sigma_1^* = [x_{\mathtt{K}_1}, x_{\mathtt{L}}]$ and $\sigma_2^* = [x_{\mathtt{L}}, x_{\mathtt{K}_2}]$.

Proof Condition (20) is a linear equation in $\mathbf{u}_{\boldsymbol{\sigma}}$, where $\mathcal{F}_{\sigma\mathtt{K}_1}$ is a flux on $\mathtt{D}_1$, and $\mathcal{F}_{\sigma\mathtt{K}_2}$ is a flux on $\mathtt{D}_2$. Inserting the definitions of the fluxes, (20) becomes:

$$\begin{aligned}
m_\sigma\mathcal{F}_{\sigma\mathtt{K}_1} &= -m_\sigma\sigma^{\mathtt{D}_1}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} + m_\sigma F_{\sigma\mathtt{K}_1}\left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\boldsymbol{\sigma}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}B_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\boldsymbol{\sigma}}) \\
&= -\mathcal{F}_{\sigma\mathtt{K}_2} = m_\sigma\sigma^{\mathtt{D}_2}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathtt{K}_2} - m_\sigma F_{\sigma\mathtt{K}_2}\left(\frac{\mathbf{u}_{\mathtt{K}_2} + \mathbf{u}_{\boldsymbol{\sigma}}}{2}\right) - \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}B_{\sigma\mathtt{K}_2}(\mathbf{u}_{\mathtt{K}_2} - \mathbf{u}_{\boldsymbol{\sigma}}).
\end{aligned} \tag{21}$$

The strain rate tensors can be written by using the matrix P as:

$$\begin{aligned}
&-m_\sigma\sigma^{\mathtt{D}_1}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} \\
&\quad= \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}P(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\boldsymbol{\sigma}}) + \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\sigma\mathtt{K}_1}\cdot\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\otimes\vec{\mathbf{n}}_{\sigma\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma\mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}_1},
\end{aligned} \tag{22}$$

$$\begin{aligned}
&m_\sigma\sigma^{\mathtt{D}_2}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}_2} \\
&\quad= -\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}P(\mathbf{u}_{\mathtt{K}_2} - \mathbf{u}_{\boldsymbol{\sigma}}) - \frac{m_\sigma m_{\sigma_2^*}}{2\mathrm{Re}m_{\mathtt{D}_2}}(\vec{\mathbf{n}}_{\sigma\mathtt{K}_2}\cdot\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\otimes\vec{\mathbf{n}}_{\sigma\mathtt{K}_2})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) - m_\sigma\mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}_2}.
\end{aligned} \tag{23}$$

Using (22), (23) in (21), since $\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} = -\vec{\mathbf{n}}_{\sigma\mathtt{K}_2}$ and $\frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}} = \frac{1}{\sin(\alpha_{\mathtt{D}})} = \frac{m_\sigma m_{\sigma_2^*}}{2\mathrm{Re}m_{\mathtt{D}_2}}$, the contributions of the pressure $\mathrm{p}^{\mathtt{D}}$ and of the velocity $\mathbf{u}_{\mathtt{K}^*}, \mathbf{u}_{\mathtt{L}^*}$ on the vertices cancel out. So (21) becomes:

$$\begin{aligned}
&\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}P(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\boldsymbol{\sigma}}) + m_\sigma F_{\sigma\mathtt{K}_1}\left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\boldsymbol{\sigma}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}B_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\boldsymbol{\sigma}}) = \\
&\qquad -\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}P(\mathbf{u}_{\mathtt{K}_2} - \mathbf{u}_{\boldsymbol{\sigma}}) - m_\sigma F_{\sigma\mathtt{K}_2}\left(\frac{\mathbf{u}_{\mathtt{K}_2} + \mathbf{u}_{\boldsymbol{\sigma}}}{2}\right) - \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}B_{\sigma\mathtt{K}_2}(\mathbf{u}_{\mathtt{K}_2} - \mathbf{u}_{\boldsymbol{\sigma}}).
\end{aligned}$$

We group the terms in $\mathbf{u}_{\boldsymbol{\sigma}}$ thanks to $F_{\sigma\mathtt{K}_1} = -F_{\sigma\mathtt{K}_2}$, and we obtain:

$$\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}\left(P + B_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\mathbf{u}_{\mathtt{K}_1} + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}\left(P + B_{\sigma\mathtt{K}_2}\mathrm{Id}\right)\mathbf{u}_{\mathtt{K}_2} + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) = \left(\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}\left(P + B_{\sigma\mathtt{K}_1}\mathrm{Id}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}\left(P + B_{\sigma\mathtt{K}_2}\mathrm{Id}\right)\right)\mathbf{u}_{\boldsymbol{\sigma}}. \quad (24)$$

By Definition 5.1, (24) becomes:

$$A_1\mathbf{u}_{\mathtt{K}_1} + A_2\mathbf{u}_{\mathtt{K}_2} + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) = A\mathbf{u}_{\boldsymbol{\sigma}}. \quad (25)$$

It is sufficient to show that this expression is injective; if $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ is equal to zero, we are going to show that $\mathbf{u}_{\boldsymbol{\sigma}}$ is zero. This is true because, if $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ vanishes, this means in particular $\mathbf{u}_{\mathtt{K}_1} = \mathbf{u}_{\mathtt{K}_2} = 0$ and (25) becomes $A\mathbf{u}_{\boldsymbol{\sigma}} = 0$. Since A is definite positive, see Remark 5.2, we deduce $\mathbf{u}_{\boldsymbol{\sigma}} = 0$. ∎

It is possible to obtain property (17), by adapting the fluxes on the interface.

Proposition 5.4 *Let* $\mathtt{D}$ *be a diamond and let* $\mathtt{D}_1, \mathtt{D}_2$ *be the two semi-diamonds such that* $\mathtt{D} = \mathtt{D}_1 \cup \mathtt{D}_2$*, see Fig. 6 . Then there exists a unique flux* $\widetilde{\mathcal{F}}_{\sigma\mathtt{K}}$ *on* $\sigma = \mathtt{K}_1|\mathtt{K}_2$ *such that*

$$m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = -m_\sigma \mathcal{F}_{\sigma\mathtt{K}_2}, \quad (26)$$

given by

$$m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}}\left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\mathtt{K}_2}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}\widetilde{B}_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}), \quad (27)$$

$$\widetilde{B}_{\sigma\mathtt{K}} = \frac{2\mathrm{Re}m_{\mathtt{D}}}{m_\sigma^2}\left(A_1 A_2 + \left(\frac{1}{2}m_\sigma F_{\sigma\mathtt{K}}\right)^2 \mathrm{Id}\right)A^{-1} - P. \quad (28)$$

Proof We consider $\mathcal{F}_{\sigma\mathtt{K}_1}$ and we refer the reader to Fig. 6: we recall that it is a flux on the semi-diamond $\mathtt{D}_1$ of vertices $x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_\sigma$. Thanks to (22), it can be written as:

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = A_1(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\boldsymbol{\sigma}}) + m_\sigma F_{\sigma\mathtt{K}_1}\left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\boldsymbol{\sigma}}}{2}\right) + \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} \cdot \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}_1}.$$

Definition (19) of $\mathbf{u}_{\boldsymbol{\sigma}}$ ensures (20), i.e. $m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = -m_\sigma \mathcal{F}_{\sigma\mathtt{K}_2}$. By grouping the terms in $\mathbf{u}_{\mathtt{K}_1}$ and $\mathbf{u}_{\boldsymbol{\sigma}}$ in $m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1}$, we are thus led to

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = \left(A_1 + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\mathbf{u}_{\mathtt{K}_1} + \left(-A_1 + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)A^{-1}\left[A_1\mathbf{u}_{\mathtt{K}_1} + A_2\mathbf{u}_{\mathtt{K}_2} + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2})\right] + \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} \cdot \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}_1}$$

that can be written as:

$$\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\left[\left(A_1 + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right) + \left(-A_1 + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)A^{-1}\left(A_1 + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\right]\mathbf{u}_{\mathtt{K}_1} \\ &+ \left(-A_1 + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)A^{-1}\left(A_2 - \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\right)\mathbf{u}_{\mathtt{K}_2} \\ &+ \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} \cdot \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}_1}. \end{aligned}$$

According to Remark 5.2, the matrices A and A_i commute, for $i = 1, 2$. Hence, we can write

$$
\begin{aligned}
m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\left(A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1} \mathrm{Id}\right) \left[\overbrace{A - A_1}^{=A_2} + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1} \mathrm{Id} \right] A^{-1} \mathbf{u}_{\mathtt{K}_1} \\
&+ \left(-A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1} \mathrm{Id}\right) \left(A_2 - \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1} \mathrm{Id}\right) A^{-1} \mathbf{u}_{\mathtt{K}_2} \\
&+ \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re} m_{\mathtt{D}_1}} (\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} \cdot \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma\mathtt{K}_1}.
\end{aligned}
$$

We develop the computations and we find:

$$
\begin{aligned}
m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\left[\left(A_1 A_2 + \left(\frac{1}{2} m_\sigma F_{\sigma\mathtt{K}}\right)^2 \mathrm{Id} \right) A^{-1} \right] (\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) \\
&+ \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re} m_{\mathtt{D}_1}} (\vec{\mathbf{n}}_{\sigma\mathtt{K}_1} \cdot \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma\mathtt{K}_1} + m_\sigma F_{\sigma\mathtt{K}_1} \left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\mathtt{K}_2}}{2} \right).
\end{aligned}
$$

Let $\widetilde{B}_{\sigma\mathtt{K}}$ be the matrix defined in (28). We get:

$$
\frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} (P + \widetilde{B}_{\sigma\mathtt{K}}) = \left(A_1 A_2 + \left(\frac{1}{2} m_\sigma F_{\sigma\mathtt{K}}\right)^2 \mathrm{Id} \right) A^{-1}.
$$

Since $m_\sigma F_{\sigma\mathtt{K}} = m_\sigma F_{\sigma\mathtt{K}_1} = -m_\sigma F_{\sigma\mathtt{K}_2}$, $\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \vec{\mathbf{n}}_{\sigma\mathtt{K}_1}$ and $\frac{m_{\sigma_1^*}}{m_{\mathtt{D}_1}} = \frac{m_{\sigma^*}}{m_{\mathtt{D}}}$ (see Fig. 6), we end up with:

$$
\begin{aligned}
m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} \left(P + \widetilde{B}_{\sigma\mathtt{K}}\right) (\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) \\
&+ \frac{m_\sigma m_{\sigma^*}}{2\mathrm{Re} m_{\mathtt{D}}} (\vec{\mathbf{n}}_{\sigma\mathtt{K}} \cdot \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \mathrm{Id} + \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\mathtt{K}_2}}{2} \right).
\end{aligned}
$$

We remark that now the expression of $m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1}$ depends only on the unknowns $\mathbf{u}_{\mathtt{K}_1}, \mathbf{u}_{\mathtt{K}_2}, \mathbf{u}_{\mathtt{K}^*}, \mathbf{u}_{\mathtt{L}^*}$; so it is a flux defined on the entire diamond $\mathtt{D}$ (see Fig. 6). It can be rewritten as:

$$
m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}}) \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\mathtt{K}_2}}{2} \right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} \widetilde{B}_{\sigma\mathtt{K}} (\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) := m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}}.
$$

so that we find (27). ■

We proceed similarly to obtain (18):

Proposition 5.5 *Let $\mathtt{D}$ be a diamond and let $\mathtt{D}_1, \mathtt{D}_2$ be the two semi-diamonds such that $\mathtt{D} = \mathtt{D}_1 \cup \mathtt{D}_2$, see Fig. 6 . Then, for $\mathtt{K}^* \in \partial\mathfrak{M}_\Gamma^*$, there exists a unique flux $\widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*}$ on $\sigma^* = \sigma_1^* \cup \sigma_2^* = [x_{\mathtt{K}_1}, x_{\mathtt{L}}] \cup [x_{\mathtt{L}}, x_{\mathtt{K}_2}]$ such that*

$$
m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = m_{\sigma_1^*} \mathcal{F}_{\sigma_1^*\mathtt{K}^*} + m_{\sigma_2^*} \mathcal{F}_{\sigma_2^*\mathtt{K}^*}, \tag{29}
$$

given by

$$
\widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = -m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}}) \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathtt{D}}} \widetilde{B}_{\sigma^*\mathtt{K}^*} (\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}), \tag{30}
$$

$$
\widetilde{B}_{\sigma^*\mathtt{K}^*} = \frac{m_{\sigma_1^*}}{m_{\sigma^*}} B_{\sigma_1^*\mathtt{K}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}} B_{\sigma_2^*\mathtt{K}^*}. \tag{31}
$$

Proof This is a direct consequence of the computation of (29). By definition,

$$
m_{\sigma_1^*} \mathcal{F}_{\sigma_1^*\mathtt{K}^*} = -m_{\sigma_1^*} \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}}) \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_1^*} F_{\sigma^*\mathtt{K}_1^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) + \frac{m_{\sigma_1^*}^2}{2\mathrm{Re} m_{\mathtt{D}_1}} B_{\sigma_1^*\mathtt{K}^*} (\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}),
$$

$$
m_{\sigma_2^*} \mathcal{F}_{\sigma_2^*\mathtt{K}^*} = -m_{\sigma_2^*} \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}}) \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_2^*} F_{\sigma^*\mathtt{K}_2^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) + \frac{m_{\sigma_2^*}^2}{2\mathrm{Re} m_{\mathtt{D}_2}} B_{\sigma_2^*\mathtt{K}^*} (\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}).
$$

Since $\sigma^* = \sigma_1^* \cup \sigma_2^*$, we have $m_{\sigma^*} = m_{\sigma_1^*} + m_{\sigma_2^*}$ and $\frac{m_{\sigma_2^*}}{m_{\mathrm{D}_2}} = \frac{m_{\sigma_1^*}}{m_{\mathrm{D}_1}}$. By definition, it holds $m_{\sigma^*} F_{\sigma^*\mathrm{K}^*} = m_{\sigma_1^*} F_{\sigma^*\mathrm{K}_1^*} + m_{\sigma_2^*} F_{\sigma^*\mathrm{K}_2^*}$. So if we take the sum of the two fluxes, we get:

$$\begin{aligned}m_{\sigma_1^*}\mathcal{F}_{\sigma_1^*\mathrm{K}^*} + m_{\sigma_2^*}\mathcal{F}_{\sigma_2^*\mathrm{K}^*} = -m_{\sigma^*}\sigma^{\mathrm{D}}(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*} + m_{\sigma^*}F_{\sigma^*\mathrm{K}^*}\left(\frac{\mathbf{u}_{\mathrm{K}^*} + \mathbf{u}_{\mathrm{L}^*}}{2}\right)\\ + \frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathrm{D}_1}}\underbrace{\left[\frac{m_{\sigma_1^*}}{m_{\sigma^*}}B_{\sigma_1^*\mathrm{K}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}}B_{\sigma_2^*\mathrm{K}^*}\right]}_{=\widetilde{B}_{\sigma^*\mathrm{K}^*}}(\mathbf{u}_{\mathrm{K}^*} - \mathbf{u}_{\mathrm{L}^*}),\end{aligned}$$

which ends the proof. ∎

5.3 Well-posedness of the limit problem $(\widetilde{\mathcal{P}})$

The expression of the new fluxes $\widetilde{\mathcal{F}}_{\sigma\mathrm{K}}, \widetilde{\mathcal{F}}_{\sigma^*\mathrm{K}^*}$ permits us to justify the well-posedness of $(\widetilde{\mathcal{P}})$.

Theorem 5.6 *Under Hypothesis $(\mathcal{H}p)$ for $B_{\sigma\mathrm{K}}, B_{\sigma^*\mathrm{K}^*}$, problem $(\widetilde{\mathcal{P}})$ is well-posed.*

Proof By Theorem 3.2, we need to verify that Hypothesis $(\mathcal{H}p)$ holds. Since we are supposing it for $B_{\sigma\mathrm{K}}, B_{\sigma^*\mathrm{K}^*}$, we just need to check it for $\widetilde{B}_{\sigma\mathrm{K}}, \widetilde{B}_{\sigma^*\mathrm{K}^*}$, the modified fluxes on the interface. As a direct consequence of (28) and (31), we have:

$$\widetilde{B}_{\sigma\mathrm{K}} = \widetilde{B}_{\sigma\mathrm{L}}, \qquad \widetilde{B}_{\sigma^*\mathrm{K}^*} = \widetilde{B}_{\sigma^*\mathrm{L}^*}.$$

In fact, if we consider a diamond on the interface between the two subdomains Ω_1, Ω_2, it can be seen as the one in Fig. 6. For $\mathrm{K} = \mathrm{K}_1$ and $\mathrm{L} = \mathrm{K}_2$, (28) reads

$$\widetilde{B}_{\sigma\mathrm{K}} = \widetilde{B}_{\sigma\mathrm{K}_1} = \frac{2\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2}\left(A_1A_2 + \left(\frac{1}{2}m_\sigma F_{\sigma\mathrm{K}_1}\right)^2\mathrm{Id}\right)A^{-1} - P,$$

$$\widetilde{B}_{\sigma\mathrm{L}} = \widetilde{B}_{\sigma\mathrm{K}_2} = \frac{2\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2}\left(A_2A_1 + \left(\frac{1}{2}m_\sigma F_{\sigma\mathrm{K}_2}\right)^2\mathrm{Id}\right)A^{-1} - P.$$

Observe that A, P do not depend on the index of the subdomain; moreover, we have $m_\sigma F_{\sigma\mathrm{K}_1} = -m_\sigma F_{\sigma\mathrm{K}_2}$, so that $(m_\sigma F_{\sigma\mathrm{K}_1})^2 = (m_\sigma F_{\sigma\mathrm{K}_2})^2$ and $A_1A_2 = A_2A_1$ from Remark 5.2. We conclude that $\widetilde{B}_{\sigma\mathrm{K}} = \widetilde{B}_{\sigma\mathrm{L}}$.
For the dual flux, (31) becomes

$$\widetilde{B}_{\sigma^*\mathrm{K}^*} = \frac{m_{\sigma_1^*}}{m_{\sigma^*}}B_{\sigma_1^*\mathrm{K}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}}B_{\sigma_2^*\mathrm{K}^*},$$

$$\widetilde{B}_{\sigma^*\mathrm{L}^*} = \frac{m_{\sigma_1^*}}{m_{\sigma^*}}B_{\sigma_1^*\mathrm{L}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}}B_{\sigma_2^*\mathrm{L}^*}.$$

Thanks to $(\mathcal{H}p)$, we have $B_{\sigma_1^*\mathrm{K}^*} = B_{\sigma_1^*\mathrm{L}^*}$ and $B_{\sigma_2^*\mathrm{K}^*} = B_{\sigma_2^*\mathrm{L}^*}$. So we get $\widetilde{B}_{\sigma^*\mathrm{K}^*} = \widetilde{B}_{\sigma^*\mathrm{L}^*}$.
We are left with the task of proving that $\widetilde{B}_{\sigma\mathrm{K}}, \widetilde{B}_{\sigma^*\mathrm{K}^*}$ are semi-definite positive. If $\vec{\mathbf{n}}_{\sigma\mathrm{K}} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $P = \mathrm{Id} + \vec{\mathbf{n}}_{\sigma\mathrm{K}} \otimes \vec{\mathbf{n}}_{\sigma\mathrm{K}} = \begin{pmatrix} 1+x^2 & xy \\ xy & 1+y^2 \end{pmatrix}$. Let us introduce the quantities

$$\mathrm{den} = 4m_\sigma^2(2 + 3B_{\sigma\mathrm{K}} + B_{\sigma\mathrm{K}}^2),$$

and

$$a = (m_{\mathrm{D}}\mathrm{Re}F_{\sigma\mathrm{K}})^2(1 + B_{\sigma\mathrm{K}}) + 8m_\sigma^2 B_{\sigma\mathrm{K}} + 12m_\sigma^2 B_{\sigma\mathrm{K}}^2 + 4m_\sigma^2 B_{\sigma\mathrm{K}}^3.$$

Coming back to (28), we have

$$\widetilde{B}_{\sigma\mathrm{K}} = \frac{1}{\mathrm{den}}\left[a\mathrm{Id} + (m_{\mathrm{D}}\mathrm{Re}F_{\sigma\mathrm{K}})^2\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}\right].$$

Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$; then:

$$\begin{aligned}\langle \widetilde{B}_{\sigma\kappa} v, v\rangle &= \frac{1}{\mathrm{den}} a\langle v, v\rangle + \frac{(m_{\mathrm{D}}\mathrm{Re}F_{\sigma\kappa})^2}{\mathrm{den}}\left(y^2v_1^2 - 2xyv_1v_2 + x^2v_2^2\right)\\ &= \frac{1}{\mathrm{den}} a\|v\|^2 + \frac{(m_{\mathrm{D}}\mathrm{Re}F_{\sigma\kappa})^2}{\mathrm{den}}\left(yv_1 - xv_2\right)^2 \geq 0.\end{aligned}$$

thanks to Hypothesis ($\mathcal{H}p$) on $B_{\sigma\kappa}$, that ensures $a \geq 0$ and den > 0. So $\widetilde{B}_{\sigma\kappa}$ is semi-definite positive.

For what concerns the dual flux, by (31) we obtain directly that $\widetilde{B}_{\sigma^*\kappa^*}$ is semi-definite positive since it is the sum of the two semi-definite positive matrices $B_{\sigma_1^*\kappa^*}$ and $B_{\sigma_2^*\kappa^*}$. ■

Further comments on problem ($\widetilde{\mathcal{P}}$) can be found in [GKL20].

5.4 Identification of the limit

In order to prove the convergence of the Schwarz algorithm towards the solution of ($\widetilde{\mathcal{P}}$), it is necessary to project this solution, that is defined on Ω, on the subdomains Ω_j, $j = 1, 2$.

Theorem 5.7 *Let $\mathfrak{T}$ be the composite mesh $\mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2$ and $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ be the solution of the DDFV scheme ($\widetilde{\mathcal{P}}$) on the domain Ω. For $j \in \{1, 2\}$, there exists a projection $(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j}, h^\infty_{\mathfrak{T}_j}, g^\infty_{\mathfrak{D}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}} \times \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j}$ of $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$, such that:*

$$\mathcal{L}^{\mathfrak{T}_j}_{\Omega_j,\Gamma}(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}_j}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}^\infty_{\mathfrak{T}_j}, g^\infty_{\mathfrak{D}_j}) = 0. \qquad (\widetilde{\mathcal{P}}^\infty)$$

Proof On the primal cells $\mathfrak{M}_j \cup \partial\mathfrak{M}_{j,D}$ and on the dual cells $\mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_{j,D} \cup \partial\mathfrak{M}^*_{j,\Gamma}$ we can simply define the values of $\mathbf{u}^\infty_{\mathfrak{T}_j}$ as the values of $\mathbf{u}_{\mathfrak{T}}$:

- for all $\kappa \in \mathfrak{M}_j$ and $\kappa^* \in \mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_{j,\Gamma}$, we set $\mathbf{u}^\infty_{\kappa_j} = \mathbf{u}_\kappa$ and $\mathbf{u}^\infty_{\kappa^*_j} = \mathbf{u}_{\kappa^*}$,
- for all $\kappa \in \partial\mathfrak{M}_{j,D}$ and $\kappa^* \in \partial\mathfrak{M}^*_{j,D}$, we set $\mathbf{u}^\infty_{\kappa_j} = 0$ and $\mathbf{u}^\infty_{\kappa^*_j} = 0$.
- for all $\mathrm{D} \in \mathfrak{D}_j$ such that $x_\mathrm{D} \notin \Gamma$, we set $\mathrm{p}^\infty_{\mathrm{D}_j} = \mathrm{p}^\mathrm{D}$.
- for all $\mathrm{D} \in \mathfrak{D}_j$ such that $x_\mathrm{D} \in \Gamma$, $\mathrm{D}_j \in \mathfrak{D}^\Gamma_j$ and $\mathrm{D}_i \in \mathfrak{D}^\Gamma_i$, we set $\mathrm{p}^\infty_{\mathrm{D}_j} = \mathrm{p}^\infty_{\mathrm{D}_i} = \mathrm{p}^\mathrm{D}$.

We then need to introduce new unknowns near the boundary Γ:

- for all $\mathrm{L} \in \partial\mathfrak{M}_{j,\Gamma}$, we impose (see Proposition 5.3):

$$\mathbf{u}^\infty_\mathrm{L} = \mathbf{u}^\infty_{\mathrm{L}_j} = \mathbf{u}^\infty_{\mathrm{L}_i} = A^{-1}\left[A_j\mathbf{u}_{\kappa_j} + A_i\mathbf{u}_{\kappa_i} + \frac{1}{2}m_\sigma F_{\sigma\kappa_1}(\mathbf{u}_{\kappa_j} - \mathbf{u}_{\kappa_i})\right]. \qquad (32)$$

- for all $\kappa^* \in \mathfrak{M}^*$ such that $x_{\kappa^*} \in \Gamma$, $\kappa^* = \kappa^*_j \cup \kappa^*_i$ with $\kappa^*_j \in \partial\mathfrak{M}^*_{j,\Gamma}$, we impose:

$$\Psi^\infty_{\kappa^*_j} = -\Psi^\infty_{\kappa^*_i} = -\frac{m_{\kappa^*_j}}{m_{\partial\Omega\cap\partial\kappa^*}}\frac{\mathbf{u}^\infty_{\kappa^*_j}}{\delta t} - \frac{1}{m_{\partial\Omega\cap\partial\kappa^*}}\sum_{\mathrm{D}\in\mathfrak{D}_{\kappa^*_j}}\mathcal{F}^\infty_{\sigma^*_j\kappa^*} + \frac{m_{\kappa^*_j}}{m_{\partial\Omega\cap\partial\kappa^*}}\mathbf{f}_{\kappa^*_j} + \frac{m_{\kappa^*_j}}{m_{\partial\Omega\cap\partial\kappa^*}}\frac{\bar{\mathbf{u}}_{\kappa^*_j}}{\delta t}. \qquad (33)$$

- for all $\mathrm{L} = \mathrm{L}_j \in \partial\mathfrak{M}_{j,\Gamma}$ and for all $\kappa^* \in \mathfrak{M}^*$ such that $x_{\kappa^*} \in \Gamma$, $\kappa^* = \kappa^*_j \cup \kappa^*_i$ with $\kappa^*_j \in \partial\mathfrak{M}^*_{j,\Gamma}$, $\kappa^*_i \in \partial\mathfrak{M}^*_{i,\Gamma}$, we impose:

$$\begin{aligned}\mathbf{h}^\infty_{\mathrm{L}_j} &= \mathcal{F}^\infty_{\sigma\kappa_i} - \frac{1}{2}F_{\sigma\kappa_i}\mathbf{u}^\infty_\mathrm{L} + \lambda\mathbf{u}^\infty_\mathrm{L},\\ \mathbf{h}^\infty_{\kappa^*_j} &= \Psi^\infty_{\kappa^*_i} - \frac{1}{2}H_{\kappa^*_i}\mathbf{u}^\infty_{\kappa^*_i} + \lambda\mathbf{u}^\infty_{\kappa^*_i}.\end{aligned}$$

- for all $\mathrm{D} \in \mathfrak{D}$ such that $x_\mathrm{D} \in \Gamma$, $\mathrm{D}_j \in \mathfrak{D}^\Gamma_j$ and $\mathrm{D}_i \in \mathfrak{D}^\Gamma_i$, we set

$$g^\infty_{\mathrm{D}_j} = -\left(m_{\mathrm{D}_i}\mathrm{div}^{\mathrm{D}_i}(\mathbf{u}^\infty_{\mathfrak{T}_i}) - \beta m_{\mathrm{D}_i}d^2_{\mathrm{D}_i}\Delta^{\mathrm{D}_i}\mathrm{p}^\infty_{\mathrm{D}_i}\right) + \alpha m_{\mathrm{D}_i}\mathrm{p}^\infty_{\mathrm{D}_i}.$$

Consequence on the equations.
We now show that from a solution $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ of the DDFV scheme $(\widetilde{\mathcal{P}})$ we built a solution to $(\widetilde{\mathcal{P}}^{\infty})$:

- $\forall \mathtt{K} \in \mathfrak{M}$, $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ satisfies:

$$m_{\mathtt{K}} \frac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}} \setminus \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = m_{\mathtt{K}} \mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}} \frac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t}.$$

If we look at the composite mesh (see Fig. 4), we remark that the primal cells $\mathtt{K} \in \mathfrak{M}$ correspond to $\mathtt{K}_{\mathtt{j}} \in \mathfrak{M}_j$ (or to $\mathtt{K}_{\mathtt{i}} \in \mathfrak{M}_i$). This implies that $m_{\mathtt{K}} = m_{\mathtt{K}_{\mathtt{j}}}$, $m_{\mathtt{K}}\mathbf{f}_{\mathtt{K}} = \int_{\mathtt{K}} \mathbf{f}(x)\mathrm{d}x = m_{\mathtt{K}_{\mathtt{j}}} \mathbf{f}_{\mathtt{K}_j}$ and $m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} = m_{\mathtt{K}_{\mathtt{j}}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}_{\mathtt{j}}}}{\delta t}$.
Moreover, for a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}} \setminus \mathfrak{D}_{\mathtt{K}}^{\Gamma}$, remark that the limit unknowns $\mathbf{u}_{\mathtt{K}_j}^{\infty}, \mathbf{u}_{\mathtt{K}_{\mathtt{j}}^*}^{\infty}, \mathrm{p}_{\mathtt{D}_j}^{\infty}$ on $\mathfrak{T}_j$ for $j = 1, 2$ coincide with $\mathbf{u}_{\mathtt{K}}, \mathbf{u}_{\mathtt{K}^*}, \mathrm{p}^{\mathtt{D}}$ on $\mathfrak{T}$; so if

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty} = -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}_j}^{\infty}, \mathrm{p}_{\mathfrak{D}_j}^{\infty})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}_j} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}_j}^{\infty} + \mathbf{u}_{\mathtt{L}_j}^{\infty}}{2} \right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}_j}^{\infty} - \mathbf{u}_{\mathtt{L}_j}^{\infty}),$$

we have:

$$\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}} \setminus \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}_{\mathtt{j}}} \setminus \mathfrak{D}_{\mathtt{K}_j}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty}.$$

For a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{\Gamma}$, if

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty} = -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}_j}^{\infty}, \mathrm{p}_{\mathfrak{D}_j}^{\infty})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}_j} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}_j}^{\infty} + \mathbf{u}_{\mathtt{L}}^{\infty}}{2} \right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} B_{\sigma\mathtt{K}_j}(\mathbf{u}_{\mathtt{K}_j}^{\infty} - \mathbf{u}_{\mathtt{L}}^{\infty}),$$

thanks to the choice (32) of $\mathbf{u}_{\mathtt{L}}^{\infty}$ for all $\mathtt{L} \in \partial\mathfrak{M}_{j,\Gamma}$ and thanks to Prop. 5.4, we have

$$m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty},$$

that implies:

$$\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}_j}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty}.$$

So in the end $(\mathbf{u}_{\mathfrak{T}_j}^{\infty}, \mathrm{p}_{\mathfrak{D}_j}^{\infty}, \Psi_{\mathfrak{T}_j}^{\infty})$ satisfies:

$$\boxed{m_{\mathtt{K}_{\mathtt{j}}} \frac{\mathbf{u}_{\mathtt{K}_j}^{\infty}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}_{\mathtt{j}}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty} = m_{\mathtt{K}_{\mathtt{j}}} \mathbf{f}_{\mathtt{K}_{\mathtt{j}}} + m_{\mathtt{K}_{\mathtt{j}}} \frac{\bar{\mathbf{u}}_{\mathtt{K}_{\mathtt{j}}}}{\delta t}, \qquad \forall \mathtt{K}_{\mathtt{j}} \in \mathfrak{M}_j.} \tag{34}$$

- $\forall \mathtt{K}^* \in \mathfrak{M}^*$, $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ satisfies:

$$m_{\mathtt{K}^*} \frac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*} \setminus \mathfrak{D}_{\mathtt{K}^*}^{\Gamma}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}^{\Gamma}} m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \frac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t}. \tag{35}$$

We need to distinguish two cases.

1. If $\partial\mathtt{K}^* \cap \Gamma = \emptyset$, equation (35) reduces to:

$$m_{\mathtt{K}^*} \frac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \frac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t},$$

and the cells $\mathtt{K}^* \in \mathfrak{M}^*$ correspond to $\mathtt{K}_{\mathtt{j}}^* \in \mathfrak{M}_j^*$ (or to $\mathtt{K}_{\mathtt{i}} \in \mathfrak{M}_i^*$). This implies that $m_{\mathtt{K}^*} = m_{\mathtt{K}_{\mathtt{j}}^*}$, $m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} = \int_{\mathtt{K}^*} \mathbf{f}(x)\mathrm{d}x = m_{\mathtt{K}_{\mathtt{j}}^*} \mathbf{f}_{\mathtt{K}_{\mathtt{j}}^*}$, $m_{\mathtt{K}^*} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} = m_{\mathtt{K}_{\mathtt{j}}^*} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}_{\mathtt{j}}^*}}{\delta t}$ and $m_{\sigma^*} = m_{\sigma_j^*}$.
Moreover, for a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*} \setminus \mathfrak{D}_{\mathtt{K}^*}^{\Gamma}$, (that is the case here since we are supposing $\partial\mathtt{K}^* \cap \Gamma = \emptyset$) remark that the limit unknowns $\mathbf{u}_{\mathtt{K}_j}^{\infty}, \mathbf{u}_{\mathtt{K}_{\mathtt{j}}^*}^{\infty}, \mathrm{p}_{\mathtt{D}_j}^{\infty}$ on $\mathfrak{T}_j$ for $j = 1, 2$ coincide with $\mathbf{u}_{\mathtt{K}}, \mathbf{u}_{\mathtt{K}^*}, \mathrm{p}^{\mathtt{D}}$ on $\mathfrak{T}$.

So if

$$m_{\sigma^*}\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*} = -m_{\sigma_j^*}\sigma^{\mathtt{D}_j}(\mathbf{u}^{\infty}_{\mathfrak{T}_j},\mathrm{p}^{\infty}_{\mathfrak{D}_j})\,\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_j^*}F_{\sigma_j^*\mathtt{K}^*}\left(\frac{\mathbf{u}^{\infty}_{\mathtt{K}_j^*}+\mathbf{u}^{\infty}_{\mathtt{L}_j^*}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_j}}B_{\sigma_j^*\mathtt{K}^*}(\mathbf{u}^{\infty}_{\mathtt{K}_j^*}-\mathbf{u}^{\infty}_{\mathtt{L}_j^*}),$$

we have:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_j^*}} m_{\sigma^*}\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*}.$$

So $(\mathbf{u}^{\infty}_{\mathfrak{T}_j},\mathrm{p}^{\infty}_{\mathfrak{D}_j},\Psi^{\infty}_{\mathfrak{T}_j})$ satisfies on the interior dual mesh:

$$\boxed{m_{\mathtt{K}_j^*}\frac{\mathbf{u}^{\infty}_{\mathtt{K}_j^*}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_j^*}} m_\sigma\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*} = m_{\mathtt{K}_j}\mathbf{f}_{\mathtt{K}_j^*} + m_{\mathtt{K}_j^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}_j^*}}{\delta t}, \qquad \forall \mathtt{K}_j^*\in\mathfrak{M}_j^*.} \tag{36}$$

2. If $\partial\mathtt{K}^*\cap\Gamma\neq\emptyset$, the cell $\mathtt{K}^*$ can be written as the union of $\mathtt{K}_j\in\partial\mathfrak{M}^*_{j,\Gamma}$ and $\mathtt{K}_i\in\partial\mathfrak{M}^*_{i,\Gamma}$. This implies that $m_{\mathtt{K}^*}=m_{\mathtt{K}_j^*}+m_{\mathtt{K}_i^*}$, $m_{\sigma^*}=m_{\sigma_j^*}+m_{\sigma_i^*}$, $m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*}=\int_{\mathtt{K}^*}\mathbf{f}(x)\mathrm{d}x = m_{\mathtt{K}_j^*}\mathbf{f}_{\mathtt{K}_j^*}+m_{\mathtt{K}_i^*}\mathbf{f}_{\mathtt{K}_i^*}$ and $m_{\mathtt{K}^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} = m_{\mathtt{K}_j^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}_j^*}}{\delta t} + m_{\mathtt{K}_i^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}_i^*}}{\delta t}$.
   Moreover, for a diamond $\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}\setminus\mathfrak{D}^{\Gamma}_{\mathtt{K}^*}$, remark that the limit unknowns $\mathbf{u}^{\infty}_{\mathtt{K}_j},\mathbf{u}^{\infty}_{\mathtt{K}_j^*},\mathrm{p}^{\infty}_{\mathtt{D}_j}$ on $\mathfrak{T}_j$ for $j=1,2$ coincide with $\mathbf{u}_{\mathtt{K}},\mathbf{u}_{\mathtt{K}^*},\mathrm{p}^{\mathtt{D}}$ on $\mathfrak{T}$. So if

$$m_{\sigma^*}\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*} = -m_{\sigma^*}\sigma^{\mathtt{D}}(\mathbf{u}^{\infty}_{\mathfrak{T}_j},\mathrm{p}^{\infty}_{\mathfrak{D}_j})\,\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*}F_{\sigma_j^*\mathtt{K}^*}\left(\frac{\mathbf{u}^{\infty}_{\mathtt{K}_j^*}+\mathbf{u}^{\infty}_{\mathtt{L}_j^*}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma_j^*\mathtt{K}^*}(\mathbf{u}^{\infty}_{\mathtt{K}_j^*}-\mathbf{u}^{\infty}_{\mathtt{L}_j^*}),$$

and

$$m_{\sigma^*}\mathcal{F}^{\infty}_{\sigma_i^*\mathtt{K}^*} = -m_{\sigma_i^*}\sigma^{\mathtt{D}_j}(\mathbf{u}^{\infty}_{\mathfrak{T}_i},\mathrm{p}^{\infty}_{\mathfrak{D}_i})\,\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_i^*}F_{\sigma_i^*\mathtt{K}^*}\left(\frac{\mathbf{u}^{\infty}_{\mathtt{K}_i^*}+\mathbf{u}^{\infty}_{\mathtt{L}_i^*}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_i}}B_{\sigma_i^*\mathtt{K}^*}(\mathbf{u}^{\infty}_{\mathtt{K}_i^*}-\mathbf{u}^{\infty}_{\mathtt{L}_i^*}),$$

we have:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}\setminus\mathfrak{D}^{\Gamma}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_j^*}\setminus\mathfrak{D}^{\Gamma}_{\mathtt{K}_j^*}} m_{\sigma_j^*}\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_i^*}\setminus\mathfrak{D}^{\Gamma}_{\mathtt{K}_i^*}} m_{\sigma_i^*}\mathcal{F}^{\infty}_{\sigma_i^*\mathtt{K}^*}.$$

For a diamond $\mathtt{D}\in\mathfrak{D}^{\Gamma}_{\mathtt{K}^*}$, thanks to (29), we have

$$m_{\sigma^*}\widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = m_{\sigma_j^*}\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*} + m_{\sigma_i^*}\mathcal{F}^{\infty}_{\sigma_i^*\mathtt{K}^*},$$

that implies:

$$\sum_{\mathtt{D}\in\mathfrak{D}^{\Gamma}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = \sum_{\mathtt{D}\in\mathfrak{D}^{\Gamma}_{\mathtt{K}_j^*}} m_{\sigma_j^*}\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*} + \sum_{\mathtt{D}\in\mathfrak{D}^{\Gamma}_{\mathtt{K}_i^*}} m_{\sigma_i^*}\mathcal{F}^{\infty}_{\sigma_i^*\mathtt{K}^*}.$$

We deduce from (35):

$$\begin{aligned}m_{\mathtt{K}_j^*}\frac{\mathbf{u}^{\infty}_{\mathtt{K}_j^*}}{\delta t} + m_{\mathtt{K}_i^*}\frac{\mathbf{u}^{\infty}_{\mathtt{K}_i^*}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_j^*}} m_{\sigma_j^*}\mathcal{F}^{\infty}_{\sigma_j^*\mathtt{K}^*} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_i^*}} m_{\sigma_i^*}\mathcal{F}^{\infty}_{\sigma_i^*\mathtt{K}^*}&\\ = m_{\mathtt{K}_j^*}\mathbf{f}_{\mathtt{K}_j^*} + m_{\mathtt{K}_i^*}\mathbf{f}_{\mathtt{K}_i^*} + m_{\mathtt{K}_j^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}_j^*}}{\delta t} + m_{\mathtt{K}_i^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}_i^*}}{\delta t}&.\end{aligned}$$

By definition, $\Psi^{\infty}_{\mathtt{K}_i^*}$ satisfies:

$$\Psi^{\infty}_{\mathtt{K}_i^*} = -\Psi^{\infty}_{\mathtt{K}_j^*} = -\frac{m_{\mathtt{K}_i^*}}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\frac{\mathbf{u}^{\infty}_{\mathtt{K}_i^*}}{\delta t} - \frac{1}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_i^*}} m_{\sigma_i^*}\mathcal{F}^{\infty}_{\sigma_i^*\mathtt{K}^*} + \frac{m_{\mathtt{K}_i^*}}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\mathbf{f}_{\mathtt{K}_i^*} + \frac{m_{\mathtt{K}_i^*}}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\frac{\bar{\mathbf{u}}_{\mathtt{K}_i^*}}{\delta t},$$

so $(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j})$ satisfies on the boundary dual mesh:

$$\boxed{m_{\mathtt{K}^*_{\mathtt{j}}} \frac{\mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{j}}}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*_{\mathtt{j}}}} m_{\sigma^*_j} \mathcal{F}^\infty_{\sigma^*_j\mathtt{K}^*} + m_{\partial\Omega\cap\partial\mathtt{K}^*} \Psi^\infty_{\mathtt{K}^*_{\mathtt{j}}} = m_{\mathtt{K}^*_{\mathtt{j}}} \mathbf{f}_{\mathtt{K}^*_{\mathtt{j}}} + m_{\mathtt{K}^*_{\mathtt{j}}} \frac{\bar{\mathbf{u}}_{\mathtt{K}^*_{\mathtt{j}}}}{\delta t}, \quad \forall \mathtt{K}^*_{\mathtt{j}} \in \partial\mathfrak{M}^*_{j,\Gamma}.} \tag{37}$$

- $\forall \mathtt{K} \in \mathfrak{M}$, with $\partial\mathtt{K} \cap \Gamma \neq \emptyset$, if we look at the composite mesh, the diamond $\mathtt{D} \in \mathfrak{D}^\Gamma_{\mathtt{K}}$ can be written as the union of $\mathtt{D}_j \in \mathfrak{D}^\Gamma_{\mathtt{K}_j}$ and $\mathtt{D}_i \in \mathfrak{D}^\Gamma_{\mathtt{K}_i}$. By definition, we have $F_{\sigma\mathtt{K}_j} = -F_{\sigma\mathtt{K}_i}$; moreover, thanks to the choice (32) of $\mathbf{u}^\infty_{\mathtt{L}}$ for all $\mathtt{L} \in \partial\mathfrak{M}_{j,\Gamma}$ and thanks to Prop. 5.4, we have $m_\sigma \mathcal{F}^\infty_{\sigma\mathtt{K}_j} = -m_\sigma \mathcal{F}^\infty_{\sigma\mathtt{K}_i}$.
From the definition of $\mathbf{h}^\infty_{\mathfrak{T}_i}$, we get the relation:

$$\mathbf{h}^\infty_{\mathtt{L}} = \mathcal{F}^\infty_{\sigma\mathtt{K}_i} - \frac{1}{2} F_{\sigma\mathtt{K}_i} \mathbf{u}^\infty_{\mathtt{L}} + \lambda \mathbf{u}^\infty_{\mathtt{L}} = -\mathcal{F}^\infty_{\sigma\mathtt{K}_j} + \frac{1}{2} F_{\sigma\mathtt{K}_j} \mathbf{u}^\infty_{\mathtt{L}} + \lambda \mathbf{u}^\infty_{\mathtt{L}}.$$

So $(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j})$ satisfies:

$$\boxed{\mathcal{F}^\infty_{\sigma\mathtt{K}_i} - \frac{1}{2} F_{\sigma\mathtt{K}_i} \mathbf{u}^\infty_{\mathtt{L}} + \lambda \mathbf{u}^\infty_{\mathtt{L}} = -\mathcal{F}^\infty_{\sigma\mathtt{K}_j} + \frac{1}{2} F_{\sigma\mathtt{K}_j} \mathbf{u}^\infty_{\mathtt{L}} + \lambda \mathbf{u}^\infty_{\mathtt{L}}.} \tag{38}$$

- $\forall \mathtt{K}^* \in \mathfrak{M}^*$, with $\partial\mathtt{K}^* \cap \Gamma \neq \emptyset$, the cell $\mathtt{K}^*$ can be written as the union of $\mathtt{K}_{\mathtt{j}} \in \partial\mathfrak{M}^*_{j,\Gamma}$ and $\mathtt{K}_{\mathtt{i}} \in \partial\mathfrak{M}^*_{i,\Gamma}$. By definition, we have $H_{\mathtt{K}^*_{\mathtt{j}}} = -H_{\mathtt{K}^*_{\mathtt{i}}}$ and $\Psi^\infty_{\mathtt{K}^*_{\mathtt{i}}} = -\Psi^\infty_{\mathtt{K}^*_{\mathtt{j}}}$. This leads, from the definition of $\mathbf{h}^\infty_{\mathfrak{T}_i}$, to the relation:

$$\mathbf{h}^\infty_{\mathtt{K}^*j} = \Psi^\infty_{\mathtt{K}^*_{\mathtt{i}}} - \frac{1}{2} H_{\mathtt{K}^*_{\mathtt{i}}} \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{i}}} + \lambda \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{i}}} = -\Psi^\infty_{\mathtt{K}^*_{\mathtt{j}}} + \frac{1}{2} H_{\mathtt{K}^*_{\mathtt{j}}} \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{j}}} + \lambda \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{j}}}.$$

So $(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j})$ satisfies:

$$\boxed{\Psi^\infty_{\mathtt{K}^*_{\mathtt{i}}} - \frac{1}{2} H_{\mathtt{K}^*_{\mathtt{i}}} \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{i}}} + \lambda \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{i}}} = -\Psi^\infty_{\mathtt{K}^*_{\mathtt{j}}} + \frac{1}{2} H_{\mathtt{K}^*_{\mathtt{j}}} \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{j}}} + \lambda \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{j}}}.} \tag{39}$$

- for all $\mathtt{D} \in \mathfrak{D}$, $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ satisfies:

$$m_{\mathtt{D}} \mathrm{div}^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}) - \beta m_{\mathtt{D}} d^2_{\mathtt{D}} \Delta^{\mathtt{D}} \mathrm{p}^{\mathtt{D}} = 0, \qquad \forall \mathtt{D} \in \mathfrak{D}. \tag{40}$$

We need to distinguish two cases:

1. If $\mathtt{D} \cap \Gamma = \emptyset$, the diamond $\mathtt{D}$ coincides with a diamond $\mathtt{D}_j \in \mathfrak{D}_j$ (or with a diamond $\mathtt{D}_i \in \mathfrak{D}_i$). For a diamond $\mathtt{D} \in \mathfrak{D} \setminus \mathfrak{D}^\Gamma$, remark that the limit unknowns $\mathbf{u}^\infty_{\mathtt{K}_j}, \mathbf{u}^\infty_{\mathtt{K}^*_{\mathtt{j}}}, \mathrm{p}^\infty_{\mathtt{D}_j}$ on $\mathfrak{T}_j$ for $j = 1, 2$ coincide with $\mathbf{u}_{\mathtt{K}}, \mathbf{u}_{\mathtt{K}^*}, \mathrm{p}^{\mathtt{D}}$ on $\mathfrak{T}$. Thus we can directly deduce that $(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j})$ satisfies $\forall \mathtt{D}_j \in \mathfrak{D}_j \setminus \mathfrak{D}^\Gamma_j$:

$$\boxed{m_{\mathtt{D}_j} \mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}_j} d^2_{\mathtt{D}_j} \Delta^{\mathtt{D}_j} \mathrm{p}^\infty_{\mathtt{D}_j} = 0.} \tag{41}$$

2. If $\mathtt{D} \cap \Gamma \neq \emptyset$, the diamond $\mathtt{D}$ can be written as the union of $\mathtt{D}_j \in \mathfrak{D}^\Gamma_j$ and $\mathtt{D}_i \in \mathfrak{D}^\Gamma_i$. This implies that the divergence can be split as : $m_{\mathtt{D}} \mathrm{div}^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}}) = m_{\mathtt{D}_j} \mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j}) + m_{\mathtt{D}_i} \mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^\infty_{\mathfrak{T}_i})$. From (40), the choice of unknowns $\mathrm{p}^\infty_{\mathfrak{D}}$ and from the definition of $g^\infty_{\mathtt{D}_j}$ we obtain:

$$\begin{aligned} g^\infty_{\mathtt{D}_j} = &- \left(m_{\mathtt{D}_i} \mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^\infty_{\mathfrak{T}_i}) - \beta m_{\mathtt{D}_i} d^2_{\mathtt{D}_i} \Delta^{\mathtt{D}_i} \mathrm{p}^\infty_{\mathtt{D}_i}\right) + \alpha m_{\mathtt{D}_i} \mathrm{p}^\infty_{\mathtt{D}_i} \\ &= \left(m_{\mathtt{D}_j} \mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}_j} d^2_{\mathtt{D}_j} \Delta^{\mathtt{D}_j} \mathrm{p}^\infty_{\mathtt{D}_j}\right) + \alpha m_{\mathtt{D}_j} \mathrm{p}^\infty_{\mathtt{D}_j}, \end{aligned}$$

   that implies for $(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j})$ that $\forall \mathtt{D}_j \in \mathfrak{D}^\Gamma_j$:

$$\boxed{\begin{aligned} -(m_{\mathtt{D}_i} \mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^\infty_{\mathfrak{T}_i}) - \beta m_{\mathtt{D}_i} d^2_{\mathtt{D}_i} \Delta^{\mathtt{D}_i} \mathrm{p}^\infty_{\mathtt{D}_i}) + \alpha m_{\mathtt{D}_i} \mathrm{p}^\infty_{\mathtt{D}_i} = \\ (m_{\mathtt{D}_j} \mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}_j} d^2_{\mathtt{D}_j} \Delta^{\mathtt{D}_j} \mathrm{p}^\infty_{\mathtt{D}_j}) + \alpha m_{\mathtt{D}_j} \mathrm{p}^\infty_{\mathtt{D}_j} \end{aligned}} \tag{42}$$

To recapitulate, (34), (36), (37), (38), (39),(41), (42) show that $(\mathbf{u}^\infty_{\mathfrak{T}_j}, \mathrm{p}^\infty_{\mathfrak{D}_j}, \Psi^\infty_{\mathfrak{T}_j})$ is a solution to $(\widetilde{\mathcal{P}}^\infty)$. ■

5.5 Convergence of the DDFV Schwarz algorithm towards $(\widetilde{\mathcal{P}})$

Theorem 5.8 (Convergence of the discrete Schwarz algorithm) *Under the hypothesis that $m_{\sigma^*} = 2m_{\sigma_i^*} = 2m_{\sigma_j^*}$ for $i,j=1,2$, $i \neq j$, the iterates of the Schwarz algorithm $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ converge as l tends to infinity to the solution of the DDFV scheme $(\widetilde{\mathcal{P}})$ (up to a constant for the pressure).*

Proof The iterates of $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ satisfy:

$$\mathcal{L}_{\Omega_j,\Gamma}^{\mathfrak{T}_j}(\mathbf{u}_{\mathfrak{T}_j}^l, \mathrm{p}_{\mathtt{D}_j}^l, \Psi_{\mathfrak{T}_j}^l, \mathbf{f}_{\mathfrak{T}}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}_{\mathfrak{T}_j}^{l-1}, g_{\mathfrak{D}_j}^{l-1}) = 0,$$

and $(\mathbf{u}_{\mathfrak{T}_j}^\infty, \Psi_{\mathfrak{T}_j}^\infty, \mathrm{p}_{\mathfrak{D}}^\infty)$, constructed from the solution of $(\widetilde{\mathcal{P}})$ is solution to:

$$\mathcal{L}_{\Omega_j,\Gamma}^{\mathfrak{T}_j}(\mathbf{u}_{\mathfrak{T}_j}^\infty, \mathrm{p}_{\mathfrak{D}_j}^\infty, \Psi_{\mathfrak{T}_j}^\infty, \mathbf{f}_{\mathfrak{T}_j}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}_{\mathfrak{T}_j}^\infty, g_{\mathfrak{D}_j}^\infty) = 0.$$

We define the errors

$$\mathbf{e}_{\mathfrak{T}_j}^l = \mathbf{u}_{\mathfrak{T}_j}^l - \mathbf{u}_{\mathfrak{T}_j}^\infty, \quad \Phi_{\mathfrak{T}_j}^l = \Psi_{\mathfrak{T}_j}^l - \Psi_{\mathfrak{T}_j}^\infty, \quad \Pi_{\mathfrak{D}_j}^l = \mathrm{p}_{\mathfrak{D}_j}^l - \mathrm{p}_{\mathfrak{D}_j}^\infty. \tag{43}$$

By linearity, they satisfy:

$$\mathcal{L}_{\Omega_j,\Gamma}^{\mathfrak{T}_j}(\mathbf{e}_{\mathfrak{T}_j}^l, \Pi_{\mathfrak{D}_j}^l, \Phi_{\mathfrak{T}_j}^l, 0, 0, \mathbf{H}_{\mathfrak{T}_j}^{l-1}, \mathbf{G}_{\mathfrak{D}_j}^{l-1}) = 0, \tag{44}$$

with

$$\begin{aligned}
\mathbf{H}_{\mathtt{L}_j}^{l-1} &= \mathcal{F}_{\sigma\mathtt{K}_i}^{l-1} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{e}_{\mathtt{L}_i}^{l-1} + \lambda\mathbf{e}_{\mathtt{L}_i}^{l-1}, && \forall \mathtt{L}_j = \mathtt{L}_i \in \partial\mathfrak{M}_{j,\Gamma}\\
\mathbf{H}_{\mathtt{K}_j^*}^{l-1} &= \Phi_{\mathtt{K}_i^*}^{l-1} - \frac{1}{2}H_{\mathtt{K}_i^*}\mathbf{e}_{\mathtt{K}_i^*}^{l-1} + \lambda\mathbf{e}_{\mathtt{K}_i^*}^{l-1}, && \forall \mathtt{K}_j^* \in \partial\mathfrak{M}_{j,\Gamma}^* \text{ such that } x_{\mathtt{K}_j^*} = x_{\mathtt{K}_i^*}\\
\mathbf{G}_{\mathtt{D}_j}^{l-1} &= -\big(m_{\mathtt{D}_i}\mathrm{div}^{\mathtt{D}_i}(\mathbf{e}_{\mathfrak{T}_i}^{l-1}) - \beta m_{\mathtt{D}_i}d_{\mathtt{D}_i}^2\Delta^{\mathtt{D}_i}\Pi_{\mathtt{D}_i}^{l-1}\big) + \alpha m_{\mathtt{D}_i}\Pi_{\mathtt{D}_i}^{l-1} && \forall \mathtt{D}_j \in \mathfrak{D}_j^\Gamma \text{ such that } x_{\mathtt{D}_j} = x_{\mathtt{D}_i}.
\end{aligned}$$

To prove the convergence of the iterates of Schwarz algorithm, it is sufficient to prove the convergence to 0 of the solution of (44). In the expanded form, (44) is written as:

$$\left\{\begin{aligned}
m_{\mathtt{K}}\frac{\mathbf{e}_{\mathtt{K}_j}^l}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^l &= 0 && \forall \mathtt{K}_j \in \mathfrak{M}_j\\
m_{\mathtt{K}^*}\frac{\mathbf{e}_{\mathtt{K}_j^*}^l}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^l &= 0 && \forall \mathtt{K}_j^* \in \mathfrak{M}_j^*\\
m_{\mathtt{K}^*}\frac{\mathbf{e}_{\mathtt{K}_j^*}^l}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^l + m_{\partial\Omega\cap\partial\mathtt{K}^*}\Phi_{\mathtt{K}_j^*}^l &= 0 && \forall \mathtt{K}_j^* \in \partial\mathfrak{M}_{j,\Gamma}^*\\
-\mathcal{F}_{\sigma\mathtt{K}_j}^l + \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}_{\mathtt{L}_j}^l + \lambda\mathbf{e}_{\mathtt{L}_j}^l &= \mathbf{H}_{\mathtt{L}_j}^{l-1} && \forall \sigma \in \partial\mathfrak{M}_{j,\Gamma}\\
-\Phi_{\mathtt{K}_j^*}^l + \frac{1}{2}H_{\mathtt{K}_j^*}\mathbf{e}_{\mathtt{K}_j^*}^l + \lambda\mathbf{e}_{\mathtt{K}_j^*}^l &= \mathbf{H}_{\mathtt{K}_j^*}^{l-1} && \forall \mathtt{K}_j^* \in \partial\mathfrak{M}_{j,\Gamma}^*\\
m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}_{\mathfrak{T}_j}^l) - \beta m_{\mathtt{D}_j}d_{\mathtt{D}_j}^2\Delta^{\mathtt{D}_j}\Pi_{\mathfrak{D}_j}^l &= 0 && \forall \mathtt{D}_j \in \mathfrak{D}_j \setminus \mathfrak{D}_j^\Gamma\\
m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}_{\mathfrak{T}_j}^l) - \beta m_{\mathtt{D}_j}d_{\mathtt{D}_j}^2\Delta^{\mathtt{D}_j}\Pi_{\mathtt{D}_j}^l + \alpha m_{\mathtt{D}_j}\Pi_{\mathtt{D}_j}^l &= \mathbf{G}_{\mathtt{D}_j}^{l-1} && \forall \mathtt{D}_j \in \mathfrak{D}_j^\Gamma.
\end{aligned}\right.$$

Thanks to the hypothesis $m_{\sigma^*} = 2m_{\sigma_i^*} = 2m_{\sigma_j^*}$, we have $m_{\mathtt{D}_i} = m_{\mathtt{D}_j}$; so, in the equation on $\mathtt{D}_j \in \mathfrak{D}_j^\Gamma$, we can simplify the measures and it becomes:

$$\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}_{\mathfrak{T}_j}^l) - \beta d_{\mathtt{D}_j}^2\Delta^{\mathtt{D}_j}\Pi_{\mathtt{D}_j}^l + \alpha\,\Pi_{\mathtt{D}_j}^l = -\big(\mathrm{div}^{\mathtt{D}_i}(\mathbf{e}_{\mathfrak{T}_i}^{l-1}) - \beta d_{\mathtt{D}_i}^2\Delta^{\mathtt{D}_i}\Pi_{\mathtt{D}_i}^{l-1}\big) + \alpha\Pi_{\mathtt{D}_i}^{l-1}.$$

We multiply the equations by $\mathbf{e}_{\mathfrak{T}_j}^l$ and we sum over all the control volumes, as in the proof of Theorem. 4.4.

We obtain, analogously to (11), the following:

$$
\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2+\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2-(\Pi^l_{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j}\\
&\qquad+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j}-\frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\mathbf{e}^l_{\mathtt{L}_j}+\frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_j}-\frac{1}{2}H_{\mathtt{K}^*_j}\mathbf{e}^l_{\mathtt{K}^*_j})\cdot\mathbf{e}^l_{\mathtt{K}^*_j}\\
&\qquad\qquad+\underbrace{\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m^2_\sigma}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma\mathtt{K}}|\mathbf{e}^l_{\mathtt{K}_j}-\mathbf{e}^l_{\mathtt{L}_j}|^2+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m^2_{\sigma^*}}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma^*\mathtt{K}^*}|\mathbf{e}^l_{\mathtt{K}^*_j}-\mathbf{e}^l_{\mathtt{L}^*j}|^2}_{\geq 0}=0.
\end{aligned}\tag{45}
$$

By the equations on $\mathfrak{D}^\Gamma_j$, we can split the scalar product into interior diamonds $\mathfrak{D}\setminus\mathfrak{D}^\Gamma_j$ and boundary diamonds $\mathfrak{D}^\Gamma_j$:

$$
-(\Pi^l_{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j}=-\sum_{\mathtt{D}_j\in\mathfrak{D}_j\setminus\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j});
$$

for the diamonds $\mathtt{D}_j\in\mathfrak{D}_j\setminus\mathfrak{D}^\Gamma_j$ we apply the equation of conservation of mass, for the diamonds $\mathtt{D}_j\in\mathfrak{D}^\Gamma_j$ we add and subtract the term $\sum_{\mathtt{D}_j\in\mathfrak{D}_j} m_{\mathtt{D}_j}\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}\cdot\Pi^l_{\mathfrak{D}^\Gamma_j}$:

$$
\begin{aligned}
-(\Pi^l_{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j}=&-\beta\sum_{\mathtt{D}_j\in\mathfrak{D}_j\setminus\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}\cdot\Pi^l_{\mathfrak{D}^\Gamma_j}-\beta\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}\cdot\Pi^l_{\mathfrak{D}^\Gamma_j}\\
&-\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\left(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}\right).
\end{aligned}
$$

We apply Remark 3 to the term $-\beta\sum_{\mathtt{D}_j\in\mathfrak{D}_j} m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}\cdot\Pi^l_{\mathfrak{D}^\Gamma_j}=-\beta\left(d^2_{\mathfrak{D}_j}\Delta^{\mathfrak{D}_j}\Pi^l_{\mathfrak{D}_j},\Pi^l_{\mathfrak{D}_j}\right)$; we then multiply and divide $\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\left(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}\right)$ by α to finally obtain:

$$
-(\Pi^l_{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j}=\beta\,|\Pi^l_{\mathfrak{D}_j}|^2-\frac{1}{\alpha}\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}).
$$

So (45) becomes:

$$
\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2+\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2+\beta\,|\Pi^l_{\mathfrak{D}_j}|^2-\frac{1}{\alpha}\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j})\\
&\qquad+\frac{1}{2\lambda}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j}-\frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\lambda\mathbf{e}^l_{\mathtt{L}_j}+\frac{1}{2\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_j}-\frac{1}{2}H_{\mathtt{K}^*_j}\mathbf{e}^l_{\mathtt{K}^*_j})\cdot\lambda\mathbf{e}^l_{\mathtt{K}^*_j}\leq 0,
\end{aligned}\tag{46}
$$

where we multiplied and divided by $\lambda>0$ the terms on the second line.
We start by considering $\frac{1}{2\lambda}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j}-\frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\lambda\mathbf{e}^l_{\mathtt{L}_j}$. By applying now the equality $-ab=\frac{1}{4}((-a+b)^2-(a+b)^2)$ we can write:

- $$
\begin{aligned}
\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j}-\frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\lambda\mathbf{e}^l_{\mathtt{L}_j}=&\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma|\mathcal{F}^l_{\sigma\mathtt{K}_j}-\frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j}+\lambda\mathbf{e}^l_{\mathtt{L}_j}|^2\\
&-\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma|\underbrace{-\mathcal{F}^l_{\sigma\mathtt{K}_j}+\frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j}+\lambda\mathbf{e}^l_{\mathtt{L}_j}}_{=\mathbf{H}^{l-1}_{\mathtt{L}_j}}|^2,
\end{aligned}
$$

Owing to the transmission conditions it becomes:

- $$\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L_j}})\cdot\lambda\mathbf{e}^l_{\mathtt{L_j}} = \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L_j}} + \lambda\mathbf{e}^l_{\mathtt{L_j}}|^2 - \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathcal{F}^{l-1}_{\sigma\mathtt{K}_i} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{e}^{l-1}_{\mathtt{L_i}} + \lambda\mathbf{e}^{l-1}_{\mathtt{L_i}}|^2.$$

Equivalently for $\frac{1}{2\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_\mathtt{j}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}})\cdot\lambda\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}}$, we obtain:

- $$\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_\mathtt{j}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}})\cdot\lambda\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}} = \frac{1}{4}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^l_{\mathtt{K}^*_\mathtt{j}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}} + \lambda\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}}|^2 - \frac{1}{4}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^{l-1}_{\mathtt{K}^*_\mathtt{i}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{e}^{l-1}_{\mathtt{K}^*_\mathtt{i}} + \lambda\mathbf{e}^{l-1}_{\mathtt{K}^*_\mathtt{i}}|^2.$$

If now we consider $m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j})$, thanks to the presence of the parameter α in the transmission conditions for the incompressibility constraint, we can treat it as the previous terms. In fact, thanks to the equality $-ab = \frac{1}{4}((-a+b)^2 - (a+b)^2)$ we can write:

- $$-\sum_{\mathtt{D}_j\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}) = \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j} - \alpha\,\Pi^l_{\mathtt{D}_j}|^2 - \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}|\underbrace{\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j} + \alpha\,\Pi^l_{\mathtt{D}_j}}_{=\frac{m_{\mathtt{D}_i}}{m_{\mathtt{D}_j}}\mathbf{G}^{l-1}_{\mathtt{D}_j}}|^2.$$

The hypothesis $m_{\sigma^*} = 2m_{\sigma^*_i} = 2m_{\sigma^*_j}$ implies $m_{\mathtt{D}_i} = m_{\mathtt{D}_j}$, so that this expression becomes:

- $$-\sum_{\mathtt{D}_j\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}) = \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j} - \alpha\,\Pi^l_{\mathtt{D}_j}|^2 - \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_i}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^{l-1}_{\mathfrak{T}_i}) - \beta d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\Pi^{l-1}_{\mathfrak{D}^\Gamma_i} - \alpha\,\Pi^{l-1}_{\mathtt{D}_i}|^2.$$

Replacing those results into (46), we have:

$$\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2 + \beta\,|\Pi^l_{\mathfrak{D}_j}|^2 \\
&+\frac{1}{4\alpha}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j} - \alpha\,\Pi^l_{\mathtt{D}_j}|^2 - \frac{1}{4\alpha}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^{l-1}_{\mathfrak{T}_i}) - \beta d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\Pi^{l-1}_{\mathfrak{D}^\Gamma_i} - \alpha\,\Pi^{l-1}_{\mathtt{D}_i}|^2 \\
&+\frac{1}{8\lambda}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L_j}} + \lambda\mathbf{e}^l_{\mathtt{L_j}}|^2 - \frac{1}{8\lambda}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathcal{F}^{l-1}_{\sigma\mathtt{K}_i} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{e}^{l-1}_{\mathtt{L_i}} + \lambda\mathbf{e}^{l-1}_{\mathtt{L_i}}|^2 \\
&+\frac{1}{8\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^l_{\mathtt{K}^*_\mathtt{j}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}} + \lambda\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}}|^2 - \frac{1}{8\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^{l-1}_{\mathtt{K}^*_\mathtt{i}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{e}^{l-1}_{\mathtt{K}^*_\mathtt{i}} + \lambda\mathbf{e}^{l-1}_{\mathtt{K}^*_\mathtt{i}}|^2 \le 0.
\end{aligned}$$

Summing over $l = 0, \dots l_{max}$ and $j = 1, 2$ we obtain:

$$\sum_{l=0}^{l_{max}} \sum_{j=1,2} \frac{1}{\delta t} \|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2 + \sum_{l=0}^{l_{max}} \sum_{j=1,2} \frac{2}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}_j} \mathbf{e}^l_{\mathfrak{T}_j}\|_2^2 + \sum_{l=0}^{l_{max}} \sum_{j=1,2} \beta\, |\Pi^l_{\mathfrak{D}_j}|^2$$
$$+ \frac{1}{4\alpha} \sum_{j=1,2} \|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^{l_{max}}_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j} \Delta^{\mathtt{D}_j} \Pi^{l_{max}}_{\mathfrak{D}^\Gamma_j} - \alpha\, \Pi^{l_{max}}_{\mathtt{D}_j}\|^2_{\mathfrak{D}^\Gamma_j} + \frac{1}{8\lambda} \sum_{j=1,2} \|\mathcal{F}^{l_{max}}_{\sigma\mathtt{K}_j} - \frac{1}{2} F_{\sigma\mathtt{K}_j} \mathbf{e}^{l_{max}}_{\mathtt{L}_j} + \lambda \mathbf{e}^{l_{max}}_{\mathtt{L}_j}\|^2_{\partial\mathfrak{M}_{j,\Gamma}}$$
$$+ \frac{1}{8\lambda} \sum_{j=1,2} \|m_{\partial\Omega\cap\partial\mathtt{K}^*} \Phi^{l_{max}}_{\mathtt{K}^*_j} - \frac{1}{2} H_{\mathtt{K}^*_j} \mathbf{e}^{l_{max}}_{\mathtt{K}^*_j} + \lambda \mathbf{e}^{l_{max}}_{\mathtt{K}^*_j}\|^2_{\partial\mathfrak{M}^*_{j,\Gamma}} \leq \frac{1}{4\alpha} \sum_{j=1,2} \|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^0_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j} \Delta^{\mathtt{D}_j} \Pi^0_{\mathfrak{D}^\Gamma_j} - \alpha\, \Pi^0_{\mathtt{D}_j}\|^2_{\mathfrak{D}^\Gamma_j}$$
$$+ \frac{1}{8\lambda} \sum_{j=1,2} \|\mathcal{F}^0_{\sigma\mathtt{K}_j} - \frac{1}{2} F_{\sigma\mathtt{K}_j} \mathbf{e}^0_{\mathtt{L}_j} + \lambda \mathbf{e}^0_{\mathtt{L}_j}\|^2_{\partial\mathfrak{M}_{j,\Gamma}} + \frac{1}{8\lambda} \sum_{j=1,2} \|m_{\partial\Omega\cap\partial\mathtt{K}^*} \Phi^0_{\mathtt{K}^*_j} - \frac{1}{2} H_{\mathtt{K}^*_j} \mathbf{e}^0_{\mathtt{K}^*_j} + \lambda \mathbf{e}^0_{\mathtt{K}^*_j}\|^2_{\partial\mathfrak{M}^*_{j,\Gamma}}$$

that shows how the total energy stays bounded as the iteration index l_{max} goes to infinity. The series $\sum_{l=0}^{l_{max}} \sum_{j=1,2} \frac{1}{\delta t} \|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2$ and $\sum_{l=0}^{l_{max}} \sum_{j=1,2} \beta\, |\Pi^l_{\mathfrak{D}_j}|^2$ converge, so their general term tends to zero, that implies the convergence to zero of the errors $\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2$, $|\Pi^l_{\mathfrak{D}_j}|^2$, defined in (43). Thus the algorithm converges.

The limit is the solution of problem $(\widetilde{\mathcal{P}})$, that is problem $(\mathcal{P})$ with an appropriate choice of the flux on Γ; in fact, we can deduce that, as l_{max} goes to infinity:

- $\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2$ tends to zero implies $\mathbf{u}^l_{\mathfrak{T}_j} \to \mathbf{u}^\infty_{\mathfrak{T}_j}$ for $j = 1, 2$.
- $|\Pi^l_{\mathfrak{D}_j}|^2$ tends to zero implies: $\mathrm{p}^l_{\mathfrak{D}_j} \to \mathrm{p}^\infty_{\mathfrak{D}_j} + \mathrm{const}(\Omega_j)$ for $j = 1, 2$. Thus the pressure converges up to a constant that depends on the subdomain. In some cases we are able to determine $\mathrm{const}(\Omega_j)$.

■

Remark 5.9 *We can determine the constant const(Ω_j) if we suppose that the mesh satisfies the Inf-Sup inequality. In fact, this implies that* $\|\Pi^l_{\mathfrak{D}_j} - m(\Pi^l_{\mathfrak{D}_j})\|_2 \leq \|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2 \to 0$ *holds as* $l \to \infty$*, from which we deduce that* $\mathrm{p}^l_{\mathfrak{D}_j} - m(\mathrm{p}^l_{\mathfrak{D}_j}) \to \mathrm{p}^\infty_{\mathfrak{D}_j} - m(\mathrm{p}^\infty_{\mathfrak{D}_j})$ *for* $j = 1, 2$.

6 A modified DDFV Schwarz algorithm

We now investigate whether it is possible to construct a discrete Schwarz algorithm with modified fluxes that converges to the solution of $(\mathcal{P})$. We show that this is possible if we suppose an asymmetric discretization of (1), in the sense that we need to consider an upwind discretization of the convection term on the primal mesh and a centered scheme on the dual mesh, that corresponds to the choice

$$B_{\sigma\mathtt{K}}(s) = \frac{1}{2}|s| \qquad \text{and} \qquad B_{\sigma^*\mathtt{K}^*}(s) = 0$$

in $(\mathcal{P})$. We remind the reader that the convergence to $(\widetilde{\mathcal{P}})$ holds if and only if both (28) and (31) hold, which can be seen as a definition of $\widetilde{B}_{\sigma\mathtt{K}}$ (resp. $\widetilde{B}_{\sigma^*\mathtt{K}^*}$) as a function of $B_{\sigma\mathtt{K}_1}, B_{\sigma\mathtt{K}_2}$ (resp. $B_{\sigma^*_1\mathtt{K}^*}, B_{\sigma^*_2\mathtt{K}^*}$). The idea is to modify the Schwarz algorithm, so that it converges to the solution of $(\mathcal{P})$; this relies on the ability to invert these relations. Accordingly, the fluxes of the limit equation depend only on $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$ and a different definition of the fluxes is not required on the interface Γ.

Theorem 6.1 *Let* $(\mathbf{u}_{\mathfrak{T}}, \mathrm{p}_{\mathfrak{D}})$ *be a solution of* $(\mathcal{P})$ *for convective fluxes defined by a constant upwind flux* $B_{\sigma\mathtt{K}}(s) = \frac{1}{2}|s|$ *for all* $\sigma \in \mathcal{E}$*, and by the centered flux* $B_{\sigma^*\mathtt{K}^*}(s) = 0$ *for all* $\sigma^* \in \mathcal{E}^*$*. Define* $(\bar{S})$ *the Schwarz algorithm where*

- *On the primal mesh, the discrete convective fluxes are defined as:*

$$B_{\sigma\mathtt{K}}(s)\mathrm{Id} \text{ if } \sigma \notin \mathcal{E}_\Gamma, \qquad \bar{B}_{\sigma\mathtt{K}}(s) \text{ if } \sigma \in \mathcal{E}_\Gamma,$$

with:

$$\bar{B}_{\sigma\mathtt{K}}(s) = \frac{1}{2} Q \begin{pmatrix} |s| - 2 + 2\sqrt{1+|s|} & 0 \\ 0 & |s| - 1 + \sqrt{1 + 2|s|} \end{pmatrix} Q^{-1}, \tag{47}$$

and $Q = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$, *where* $\vec{\mathbf{n}}_{\sigma\kappa} = \begin{pmatrix} x \\ y \end{pmatrix}$ *is the outward normal to the interface* Γ.

- *On the dual mesh,* $B_{\sigma^*\kappa^*}(s) = 0$.

Assuming $m_{\sigma^*} = 2m_{\sigma_j^*} = 2m_{\sigma_i^*}$, *for* $j, i = 1, 2$, $j \neq i$, $(\mathcal{P})$ *is the limit of the Schwarz algorithm* $(\bar{S})$.

Proof The assumption $m_{\sigma^*} = 2m_{\sigma_j^*} = 2m_{\sigma_i^*}$ implies that $m_{\mathrm{D}_1} = m_{\mathrm{D}_2} = \frac{1}{2}m_{\mathrm{D}}$ and $B_{\sigma\kappa_1} = B_{\sigma\kappa_2} = B_{\sigma\kappa}$. This means that

$$A_1 = A_2 = \frac{m_\sigma^2}{\mathrm{Re}m_{\mathrm{D}}}(P + B_{\sigma\kappa}\mathrm{Id})$$

and

$$A = A_1 + A_2 = \frac{2m_\sigma^2}{\mathrm{Re}m_{\mathrm{D}}}(P + B_{\sigma\kappa}\mathrm{Id}) = 2A_1 = 2A_2.$$

Moreover, $A^{-1} = \dfrac{\mathrm{Re}m_{\mathrm{D}}}{2m_\sigma^2}(P + B_{\sigma\kappa}\mathrm{Id})^{-1}$. Therefore, we get

$$\widetilde{B}_{\sigma\kappa} = \frac{2\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2}\left(\frac{1}{4}A^2 + \left(\frac{1}{2}m_\sigma F_{\sigma\kappa}\right)^2 \mathrm{Id}\right)A^{-1} - P.$$

Expanding this expression, we get:

$$\begin{aligned}
\widetilde{B}_{\sigma\kappa} &= \left(\frac{\mathrm{Re}m_{\mathrm{D}}}{2m_\sigma^2}\right)A + \tfrac{2\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2}\left(\tfrac{1}{2}m_\sigma F_{\sigma\kappa}\right)^2 A^{-1} - P \\
&= P + B_{\sigma\kappa}\mathrm{Id} + \left(\frac{\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2}\right)^2 \left(\tfrac{1}{2}m_\sigma F_{\sigma\kappa}\right)^2 (P + B_{\sigma\kappa}\mathrm{Id})^{-1} - P.
\end{aligned}$$

by using the definition of A and A^{-1}. Let us set $s = \dfrac{\mathrm{Re}m_{\mathrm{D}}}{m_\sigma}F_{\sigma\kappa}$, We have $\left(\dfrac{\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2}\right)^2\left(\dfrac{1}{2}m_\sigma F_{\sigma\kappa}\right)^2 = \dfrac{1}{4}s^2$, so we end up with:

$$\widetilde{B}_{\sigma\kappa} = B_{\sigma\kappa}\mathrm{Id} + \frac{1}{4}s^2(P + B_{\sigma\kappa}\mathrm{Id})^{-1}.$$

If we make explicit the dependences of $\widetilde{B}_{\sigma\kappa}, B_{\sigma\kappa}$ as a function of s, since $B_{\sigma\kappa}$ is a function of $\frac{m_{\mathrm{D}}\mathrm{Re}}{m_\sigma}F_{\sigma\kappa}$ and $\widetilde{B}_{\sigma\kappa}$ a function of $\frac{2m_{\mathrm{D}}\mathrm{Re}}{m_\sigma}F_{\sigma\kappa}$, we are led to

$$\widetilde{B}_{\sigma\kappa}(2s) = B_{\sigma\kappa}(s)\mathrm{Id} + \frac{1}{4}s^2\left(P + B_{\sigma\kappa}(s)\mathrm{Id}\right)^{-1}, \quad \text{for } l = 1, 2.$$

We can rewrite this condition as:

$$\widetilde{B}_{\sigma\kappa} = \Upsilon(B_{\sigma\kappa}).$$

This relation implies that the Schwarz algorithm $(\mathcal{S}_1)$-$(\mathcal{S}_2)$, whose convection fluxes depend on $B_{\sigma\kappa}$, converges towards the solution of $(\widetilde{\mathcal{P}})$, whose convection fluxes depend on $\widetilde{B}_{\sigma\kappa}$ for $\sigma \in \mathcal{E}_\Gamma$.
We want to build a new Schwarz algorithm $(\bar{S})$ that converges toward $(\mathcal{P})$, whose fluxes are defined by $B_{\sigma\kappa}$; so we need to build $\bar{B}_{\sigma\kappa}$ such that:

$$B_{\sigma\kappa} = \Upsilon(\bar{B}_{\sigma\kappa}),$$

where $B_{\sigma\kappa}$ can be a full matrix. Since our goal is to converge towards the fluxes that define an upwind scheme, i.e. defined by $B(s) = \frac{1}{2}|s|$, $B_{\sigma\kappa}$ is actually a diagonal matrix, that will be denoted by $B_{\sigma\kappa}\mathrm{Id}$ to make its matrix nature clear.
Thus we need to invert the function Υ defined above to find the new coefficients $\bar{B}_{\sigma\kappa}$. The inverse of Υ does not exist for every $B_{\sigma\kappa}$. Given s and $B_{\sigma\kappa}(2s)$, we have a second-degree equation for $\bar{B}_{\sigma\kappa}(s)$:

$$\bar{B}_{\sigma\kappa}(s)^2 + \bar{B}_{\sigma\kappa}(s)\underbrace{\left(P - B_{\sigma\kappa}(2s)\mathrm{Id}\right)}_{T} + \underbrace{\frac{1}{4}s^2\mathrm{Id} - P\,B_{\sigma\kappa}(2s)\mathrm{Id}}_{V} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

that is:

$$\bar{B}_{\sigma\kappa}(s)^2 + \bar{B}_{\sigma\kappa}(s)T + V = 0.$$

Since the matrices T, V are symmetric and they commute (because they are polynomials in P), they can be diagonalized using the same basis of eigenvectors. The matrix $Q = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$ is an orthogonal matrix, and we can write:

$$T = Q\,\widetilde{T}\,Q^{-1}, \qquad V = Q\,\widetilde{V}\,Q^{-1},$$

with $\widetilde{T}$ and $\widetilde{V}$ diagonal matrices, whose expressions are:

$$\widetilde{T} = \begin{pmatrix} 2 - B_{\sigma\mathrm{K}} & 0 \\ 0 & 1 - B_{\sigma\mathrm{K}} \end{pmatrix}, \quad \widetilde{V} = \begin{pmatrix} \frac{1}{4}s^2 - 2B_{\sigma\mathrm{K}} & 0 \\ 0 & \frac{1}{4}s^2 - B_{\sigma\mathrm{K}} \end{pmatrix}.$$

We then look for $\bar{B}_{\sigma\mathrm{K}}(s)$ of the form $\bar{B}_{\sigma\mathrm{K}}(s) = Q\,\widetilde{M}\,Q^{-1}$, with $\widetilde{M}$ a diagonal matrix such that

$$\widetilde{M}^2 + \widetilde{M}\,\widetilde{T} + \widetilde{V} = 0.$$

Since we are supposing $B_{\sigma\mathrm{K}}(s) = \frac{1}{2}|s|$, the solution exists and is given by

$$\widetilde{M} = \frac{1}{2}\begin{pmatrix} |s| - 2 + 2\sqrt{1+|s|} & 0 \\ 0 & |s| - 1 + \sqrt{1+2|s|} \end{pmatrix},$$

that leads to our result (47).
For what concerns property (31), we would like to define a unique $B_{\sigma^*\mathrm{K}^*}(s^*)$ for $\sigma^* \in \mathcal{E}^*$ in the limit scheme $(\mathcal{P})$. With the assumption $m_{\sigma^*} = 2m_{\sigma_1^*} = 2m_{\sigma_2^*}$, we can define $s^* = \dfrac{\mathrm{Re}\,m_{\mathrm{D}}}{m_{\sigma^*}} F_{\sigma^*\mathrm{K}^*}$ and $s_j^* = \dfrac{\mathrm{Re}\,m_{\mathrm{D}j}}{m_{\sigma_j^*}} F_{\sigma_j^*\mathrm{K}^*}$ for $j = 1, 2$: remark that there is no relation between the s_j^*. We have $s^* = s_j^* + s_i^*$, since $m_{\sigma_j^*} F_{\sigma_j^*\mathrm{K}^*} + m_{\sigma_i^*} F_{\sigma_i^*\mathrm{K}^*} = m_{\sigma^*} F_{\sigma^*\mathrm{K}^*}$. This leads to the new expression for (31):

$$\widetilde{B}_{\sigma^*\mathrm{K}^*}(s_j^* + s_i^*) = \frac{1}{2}\left(B_{\sigma^*\mathrm{K}^*}(s_j^*) + B_{\sigma^*\mathrm{K}^*}(s_i^*) \right).$$

This is true only if $B_{\sigma^*\mathrm{K}^*} = \widetilde{B}_{\sigma^*\mathrm{K}^*} = 0$; in this way, property (30) is verified. So the dual flux for the algorithm $(\bar{S})$ and for the limit $(\mathcal{P})$ correspond to a centered discretization of the convection flux on the dual mesh.
The Schwarz algorithm $(\bar{S})$ is well posed, since $(\mathcal{H}p)$ is verified by its fluxes, and it converges towards $(\mathcal{P})$ with the choice of $B_{\sigma\mathrm{K}}(s) = \frac{1}{2}|s|$ for all $\sigma \in \mathcal{E}$ and $B_{\sigma^*\mathrm{K}^*}(s) = 0$ for all $\sigma^* \in \mathcal{E}$. ■

7 Numerical results

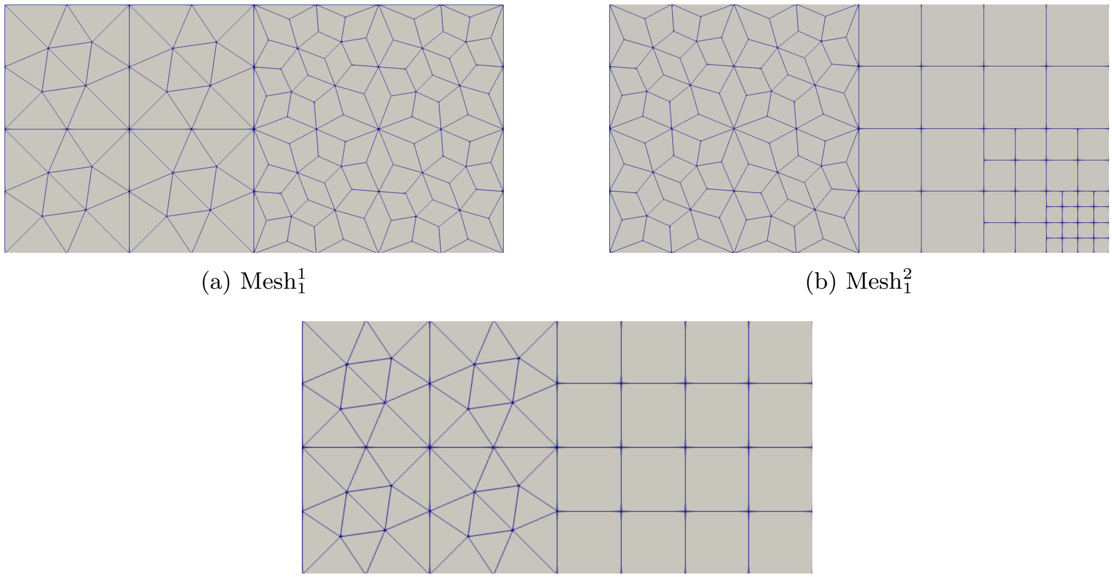
In this section, the objectives are the following:

- to show and compare the convergence properties of the Schwarz algorithms $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ (presented in Sec. 4.3)) and $(\bar{S})$ (presented in Sec. 6);
- to study on numerical grounds the influence of the parameters λ, α, β of (2) on the convergence;
- to further validate the method with the simulation of a benchmark of a flow past an obstacle.

We will refer to $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ as "**first Schwarz algorithm**", and to $(\bar{S})$ as "**second Schwarz algorithm**". We recall that the difference between the two algorithms relies in the definition of the fluxes at the interface; the former converges towards the solution of $(\widetilde{\mathcal{P}})$ (see Theorem 5.8), the latter towards the solution of $(\mathcal{P})$ (see Theorem. 6.1). For the first Schwarz algorithm, in all the following test cases, we will consider an upwind discretization of the convection flux, i.e. we set $B(s) = \frac{1}{2}|s|$.

7.1 Numerical study of the convergence

We recall that the domain decomposition algorithm is an iterative algorithm that is employed at each time step; this, in particular, implies that at each iteration of the Schwarz algorithm we solve a steady problem. In the following tests, we fix the time step ($\delta t = 10^{-4}$), and we apply the iterative method on the time interval $[0, \delta t]$. In all the test cases, the domain $\Omega = [-1, 1] \times [0, 1]$ will be divided into two subdomains $\Omega = \Omega_1 \cup \Omega_2$. The meshes we will consider are illustrated, in their first level of refinement, in Fig. 7.



(a) Mesh_1^1

(b) Mesh_1^2

(c) Mesh_1^3.

Fig. 7 Coarse level of refinement of the composite meshes on Ω, Mesh_1^k.

The sub-index in the name of the mesh (see Fig. 7) denotes the level of refinement, i.e. Mesh_1^k represents the coarse mesh of a family of refined meshes $(\text{Mesh}_m^k)_m$. More precisely, Mesh_m^k is obtained by dividing by two all the edges of Mesh_{m-1}^k. We consider the following exact solutions to (1):

Test 1:

$$\begin{aligned} \mathbf{u}(x,y,t) &= \begin{pmatrix} -2\pi\cos(\pi x)\sin(2\pi y)\exp(-5\eta t\pi^2), \\ \pi\sin(\pi x)\cos(2\pi y)\exp(-5\eta t\pi^2) \end{pmatrix}, \\ \mathrm{p}(x,y,t) &= -\frac{\pi^2}{4}(4\cos(2\pi x) + \cos(4\pi y))\exp(-10t\eta\pi^2). \end{aligned} \tag{48}$$

Test 2:

$$\begin{aligned} \mathbf{u}(x,y,t) &= \begin{pmatrix} \sin(2\pi x)\cos(2\pi y)\exp(-2\eta t), \\ -\cos(2\pi x)\sin(2\pi y)\exp(-2\eta t) \end{pmatrix}, \\ \mathrm{p}(x,y,t) &= -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))\exp(-4\eta t). \end{aligned} \tag{49}$$

The algorithms, in all the following simulations, are initialized with initial random guesses $\mathbf{h}^0_{\mathfrak{T}_j}$ and $g^0_{\mathfrak{D}_j}$ for $j = 1, 2$. As a stopping criterion, we impose:

$$\max\left(\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2, \|\Pi^l_{\mathfrak{D}_j}\|_2\right) < 10^{-5},$$

where the errors are defined in (43).

7.1.1 Error on the interface

In this first test case, we consider the first Schwarz algorithm; our goal is to point out that the error computed with respect to the solution of $(\widetilde{\mathcal{P}})$, along the iterations of the algorithm, stays localized at the interface between the two subdomains.

The domain Ω is meshed with Mesh_5^3, we fix the parameters $\lambda = 100, \alpha = 1, \beta = 10^{-2}$. In Fig. 8 we represent the error of the velocity on the entire domain at the initialization on the primal and dual mesh; the initialization assigns random values, and the initial error is 100 for both primal and dual mesh.

As we pass to the 1$^{\text{st}}$ iteration, we observe in Fig. 9 how it immediately locates on the interface between the subdomains; it decreases, passing from 100 to 1.9 on the primal mesh and to 6.9 on the dual mesh. Already at the 10$^{\text{th}}$ iteration we see in Fig. 10 how it has diminished, staying localized on the interface, passing from 1.9 to 0.52 on the primal mesh and from to 6.9 to 0.05 on the dual mesh.

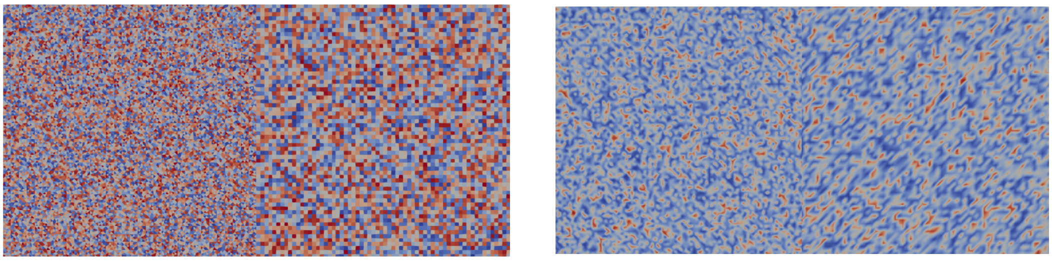


Fig. 8 Error $\mathbf{u}^0_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}$ at the initialization: $\|\mathbf{u}^0_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}\|_\infty = 100$. *Left:* Primal mesh. *Right:* Dual mesh.

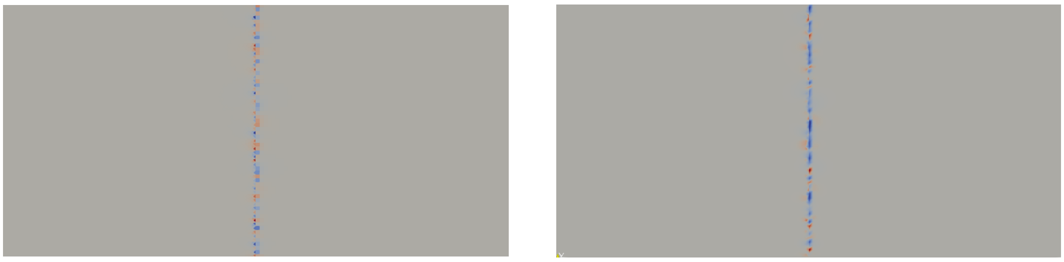


Fig. 9 Error $\mathbf{u}^1_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}$ at the 1$^{\text{st}}$ iteration. *Left:* Primal mesh, $\|\mathbf{u}^1_{\mathfrak{M}} - \mathbf{u}_{\mathfrak{M}}\|_\infty = 1.9$. *Right:* Dual mesh, $\|\mathbf{u}^1_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*} - \mathbf{u}_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\|_\infty = 6.9$.

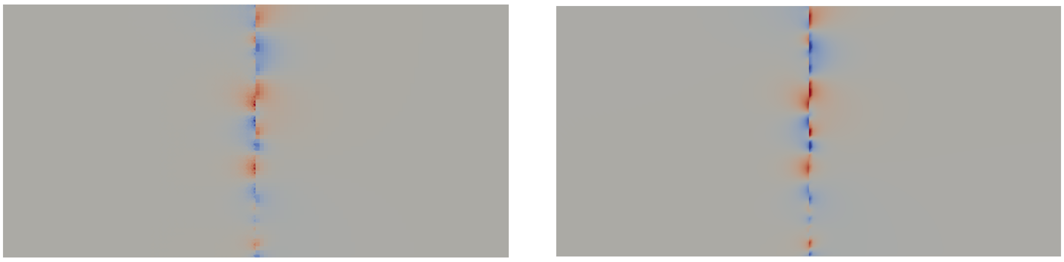


Fig. 10 Error $\mathbf{u}^{10}_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}$ at the 10th iteration. *Left:* Primal mesh, $\|\mathbf{u}^2_{\mathfrak{M}} - \mathbf{u}_{\mathfrak{M}}\|_\infty = 0.52$. *Right:* Dual mesh, $\|\mathbf{u}^2_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*} - \mathbf{u}_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\|_\infty = 0.05$.

7.1.2 Study of the parameters

In this section our goal is to study the influence of the parameters λ, α, β and of the mesh on the convergence of the first and second Schwarz algorithms. We recall that β is associated to the Brezzi-Pitkäranta stabilization (see Section 2.6) while the parameters λ and α are associated the transmission conditions between subdomains. In each test case we fix all parameters, but one. First, the value of β associated to the stabilization is set to 10^{-2}. In Fig. 11-13 we represent on the x-axis the number of iterations, on the y-axis the error.
We start by considering the first Schwarz algorithm; we can observe in Fig. 11 the convergence of the algorithm to the solution of Test 1 on Mesh$^1_1$.
In particular, on the *left* of Fig. 11, α is fixed to 1, and we observe how, as λ increases, the number of iterations necessary to converge decreases until $\lambda = 200$; passed this critical value, the number of iterations starts to increase again. This suggests that on Mesh$^1_1$, for $\alpha = 1$ and $\beta = 10^{-2}$, $\lambda = 200$ is a good choice to have a better convergence. On the *right* of Fig. 11, we set $\lambda = 100$ and we let α vary: we observe the same kind of behavior as the one of λ. We consider now the second Schwarz algorithm on the same test case, i.e. Test 1 on Mesh$^1_1$. We show its convergence in Fig. 12. This indicates that on Mesh$^1_1$, for $\lambda = 100$ and $\beta = 10^{-2}$, $\alpha = 0.25$ is a good choice to have a better convergence.

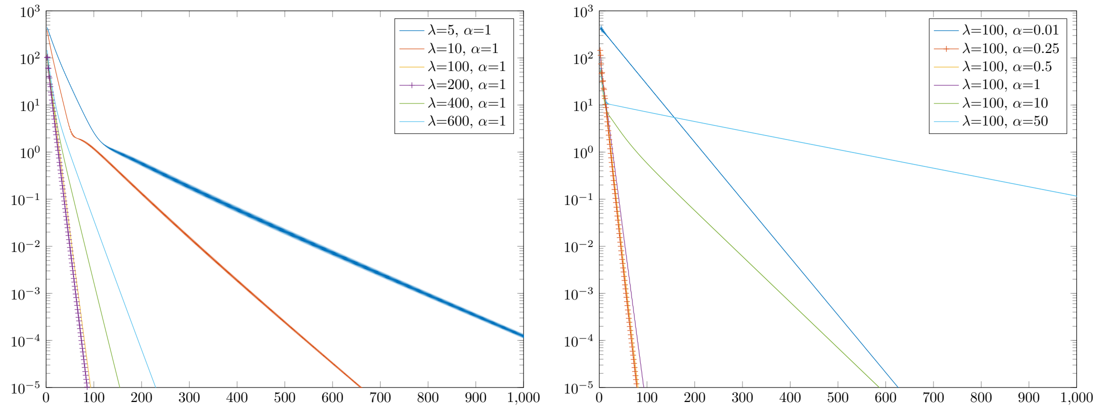


Fig. 11 Test 1, Mesh_1^1, first Schwarz algorithm. *Left:* optimization of λ, with $\alpha = 1$, $\beta = 10^{-2}$. *Right:* optimization of α, with $\lambda = 100$, $\beta = 10^{-2}$.

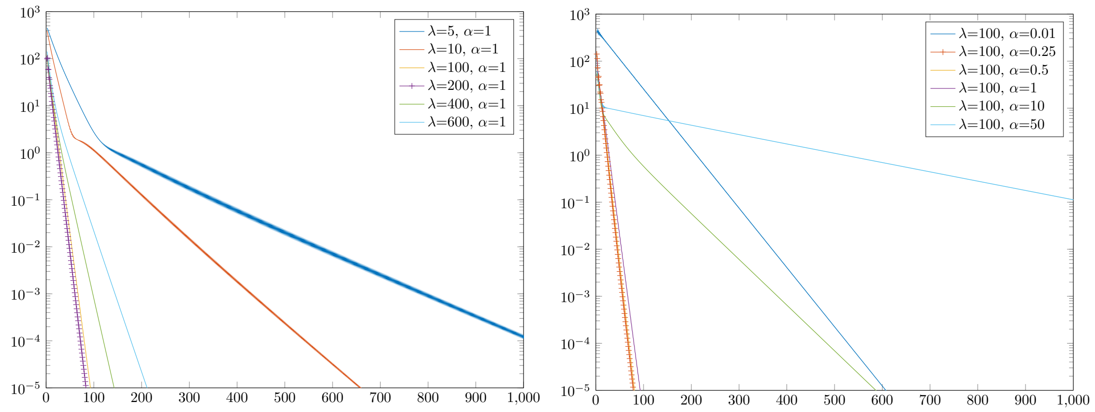


Fig. 12 Test 1, Mesh_1^1, second Schwarz algorithm.*Left:* optimization of λ, with $\alpha = 1$, $\beta = 10^{-2}$. *Right:* optimization of α, with $\lambda = 100$, $\beta = 10^{-2}$.

We remark that the second Schwarz algorithm behaves similarly to the first one, if we compare Fig. 11 and Fig. 12; thus, both algorithms converge and the speed of convergence is influenced by the choice of λ or α, once fixed the value of β and the mesh.

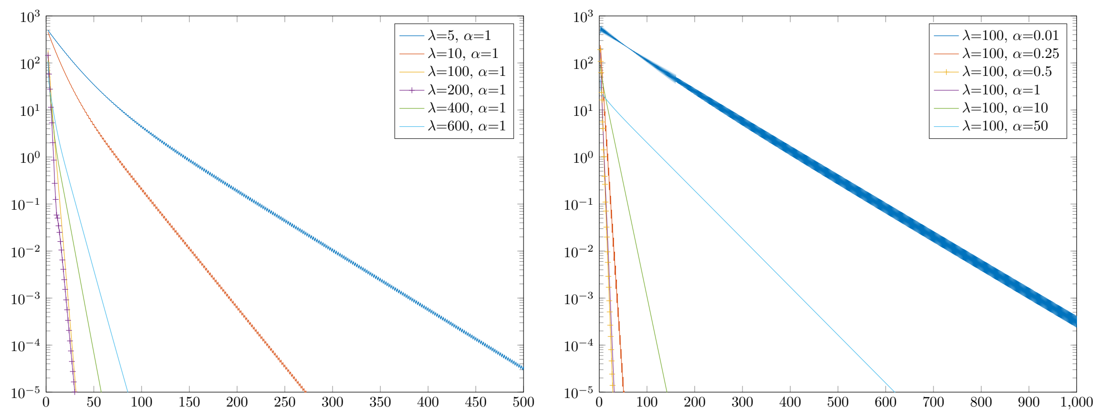


Fig. 13 Test 2, Mesh_1^2, first Schwarz algorithm. *Left:* optimization of λ, with $\alpha = 1$, $\beta = 10^{-2}$. *Right:* optimization of α, with $\lambda = 100$, $\beta = 10^{-2}$.

Since the parameters have the same behavior and the number of iterations necessary to the convergence is almost identical between the two algorithms, from now on we will focus just on the first one.

We consider now a different test case, i.e. Test 2 on Mesh_1^2 for the first Schwarz algorithm. We observe in Fig. 13 that we still have the same kind of behavior for the parameters λ, α; but we point out that the optimal value of the parameters depends on the mesh and on the test case. In fact, if we compare the optimal α in Fig. 11 and in Fig. 13, we remark that $\alpha = 0.25$ for the first case and $\alpha = 0.5$ for the second case.

In the first test case of Fig. 14, our goal is to show how the level of refinement of the mesh can influence the choice of the optimal parameter; we consider Test 1 on the family $(\text{Mesh}_m^1)_m$, $m = 1, 2, 3, 4$. As before, we set the value of β, we fix one of the two parameters between λ and α and we let the other vary; we represent on the x-axis the value of the parameter that changes, on the y-axis the number of iterations required to obtain an error of order 10^{-5}.

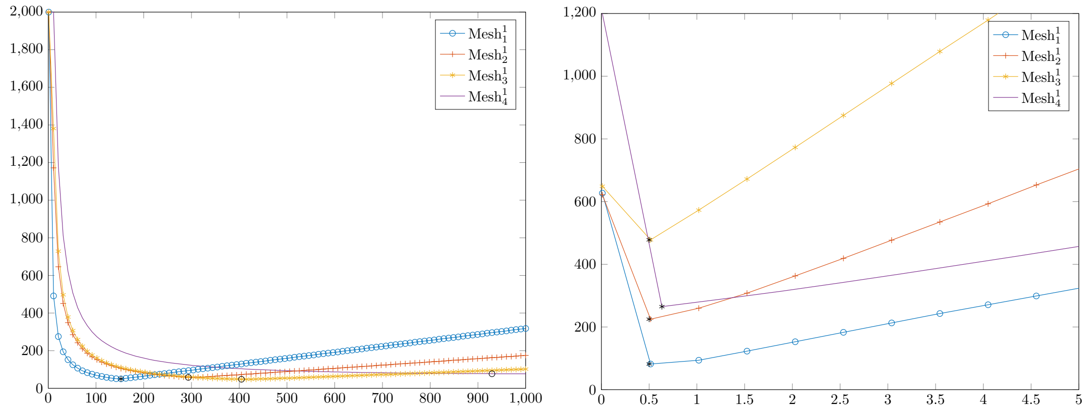


Fig. 14 Test 1, $(\text{Mesh}_m^1)_m$, $m = 1, 2, 3, 4$. *Left:* optimization of λ to obtain an error of order 10^{-5}, with $\alpha = 1, \beta = 10^{-1}$. *Right:* optimization of α to obtain an error of order 10^{-5}, with $\lambda = 100, \beta = 10^{-1}$.

Table 1 Test 1 on $(\text{Mesh}_m^1)_m$, $m = 1, 2, 3$. *First line:* Optimal value of λ for $\alpha = 1, \beta = 10^{-1}$. *Second line:* Optimal value of α for $\lambda = 100, \beta = 10^{-1}$.

	$\textbf{Mesh}_1^1$	$\textbf{Mesh}_2^1$	$\textbf{Mesh}_3^1$	$\textbf{Mesh}_4^1$
λ	152.36	293.36	404.63	929.36
α	0.5	0.5	0.5	0.6

As illustrated in Fig. 14 and summarized in Table 1, we observe different results for the two parameters; the mesh refinement has an impact on λ but not really on α. The mesh size h is divided by two at each level of refinement, and we see that it has an influence on the value of λ; unfortunately, we can not conclude by defining a relation between the two.

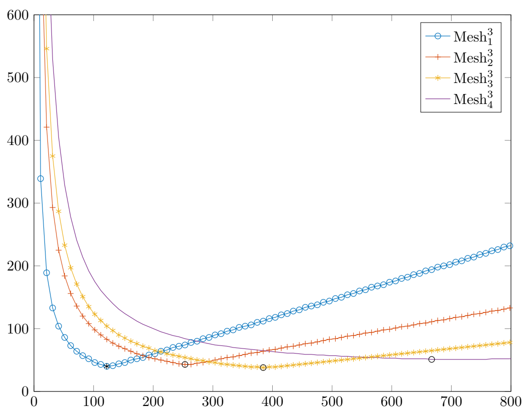


Table 2 Test 1 on $(\text{Mesh}_m^3)_m$, $m = 1, 2, 3, 4$. Optimal value of λ for $\alpha = 1, \beta = 10^{-1}$.

	$\textbf{Mesh}_1^3$	$\textbf{Mesh}_2^3$	$\textbf{Mesh}_3^3$	$\textbf{Mesh}_4^3$
λ	122	253.27	384.45	667.51

Fig. 15 Test 1, $(\text{Mesh}_m^3)_m$, $m = 1, 2, 3$. *Left:* optimization of λ to obtain an error of order 10^{-5}, with $\alpha = 1, \beta = 10^{-1}$. *Right:* Summary table of the optimal values of λ.

In Fig. 15 (*left*) and Table 2 we want to confirm the results obtained for λ on Fig. 14 (*left*) and Table 1, by considering the same test case (Test 1) on a different family of meshes, $(\text{Mesh}^3_m)_m$, $m = 1, 2, 3, 4$. As before, λ is influenced by the mesh discretization step but we can not conclude by defining a relation between the two; moreover, we remark that its optimal values change with respect to Table 1, due to the different meshes.

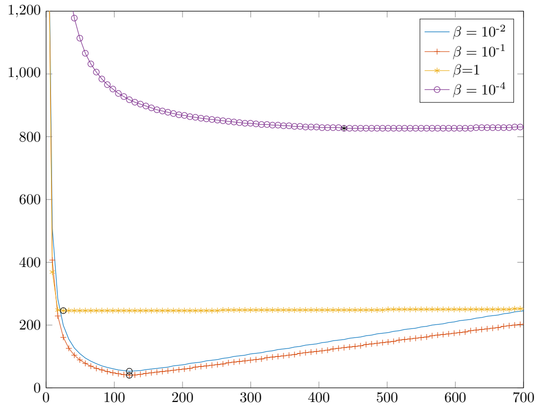


Table 3 Test 1 on Mesh^3_1. Optimal value of λ and the number of iterations for different values of β and for $\alpha = 1$.

β	10^{-4}	10^{-2}	10^{-1}	1
λ	436.81	122	122	25.2
# iter	818	53	40	246

Fig. 16 Test 1, Mesh^3_1. *Left:* optimization of λ with different values of β on Mesh^3_1; $\alpha = 1$. *Right:* Summary table of the optimal values of λ.

In Fig. 16 and in Table 3 we study the influence of the parameter β, associated to the Brezzi-Pitkäranta stabilization. We see how the choice of this parameter affects the convergence of the algorithm and how it affects the optimal value of λ: we pass from 818 iterations with $\lambda = 436.81$ (for $\beta = 10^{-4}$) to 40 iterations with $\lambda = 122$ (for $\beta = 10^{-1}$). There is then an optimal choice even for this parameter.

As last simulation, on Fig. 17 and Table 4 we compare the optimal values of λ for Test 1 on different meshes. We see that even the choice of the mesh influences the optimal choice of the parameter: for a Cartesian mesh, $\lambda = 105.91$ while for Mesh^2_1 $\lambda = 154.3$.

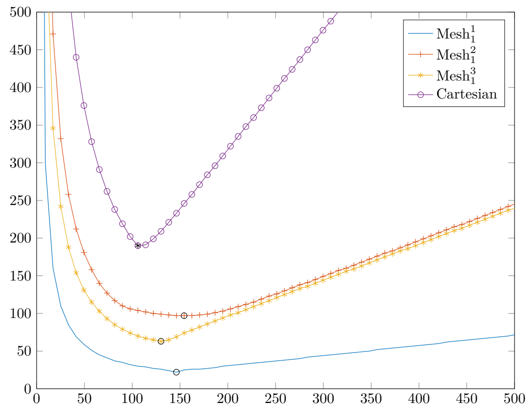


Table 4 Test 1. Optimal value of λ for $\alpha = 1, \beta = 10^{-1}$ on different meshes.

	Mesh^1_1	Mesh^2_1	Mesh^3_1	Cartesian
λ	146.2	154.3	130.1	105.91

Fig. 17 *Left:* Test1, optimization of λ for different meshes to obtain an error of order 10^{-6}, $\alpha = 1$ and $\beta = 10^{-1}$. *Right:* Summary table of the optimal values of λ.

To summarize, we observed that, for every test case, there are four factors that influence the convergence of the algorithm: the parameters λ and α associated to the transmission conditions, the parameter β of the Brezzi-Pitkäranta stabilization and the mesh choice (its geometry and its level of refinement). Once fixed three among the four factors, it is possible to optimize the remaining one in order to have a faster convergence of the algorithm.

7.2 Cylinder test case

In this section, we test the first Schwarz algorithm ($\mathcal{S}_1$)-($\mathcal{S}_2$), on a test case inspired by the benchmark of [STD+96] (we precisely use the detailed results in [Joh04]). In both [STD+96], [Joh04] the drag and lift coefficients of the flow past an obstacle are computed from simulations on a domain Ω, with Dirichlet boundary conditions. Our goal is to measure the quality of the DDFV solution obtained on Ω with a domain decomposition algorithm.
The benchmark is defined with dimensional equations, so we adopt the same framework, see Fig. 18. References [STD+96] and [Joh04] consider a long channel $\Omega = [0, 2.2\text{m}] \times [0, 0.41\text{m}]$ with a cylindrical obstacle S whose center is at $(0.2\text{m}, 0.2\text{m})$; we decompose the domain Ω into two subdomains Ω_1, Ω_2 and we place the interface Γ at $x = 0.56\text{m}$. On $\partial\Omega$ we impose Dirichlet boundary conditions, as in [Joh04].

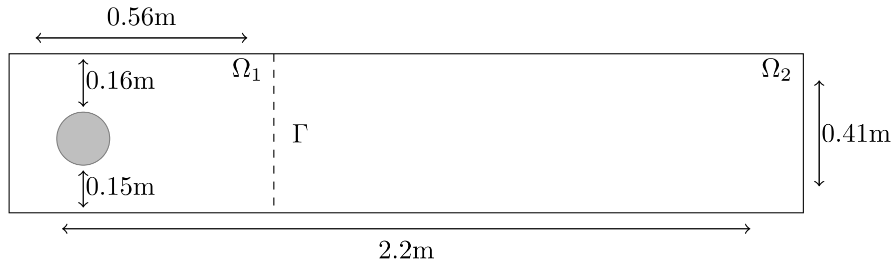


Fig. 18 Domain $\Omega = [0, 2.2] \times [0, 0.41]$, decomposed in $\Omega_1 = [0, 0.56] \times [0, 0.41]$ and $\Omega_2 = [0.56, 2.2] \times [0, 0.41]$.

The mesh that we consider on Ω is represented in Fig. 19; it is obtained with GMSH, it has 8560 cells and it is locally refined around the cylinder. Remark that on the left domain Ω_1 (the one with triangles) there are 4464 cells and on the right domain Ω_2 (the one with rectangles) there are 4096 cells.

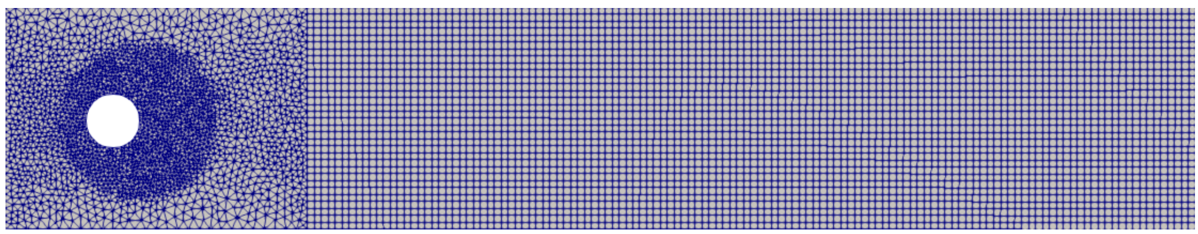


Fig. 19 Mesh on the domain Ω of Fig. 18. The number of primal cells is 8560; 4464 in the left domain, 4096 in the right one.

The viscosity of the fluid is set to $\eta = 10^{-3}\text{m}^2\text{s}^{-1}$ and the final time is $T = 8\text{s}$. The time-dependent inflow on $x = 0$ and outflow on $x = 2.2$ is:

$$\mathbf{g}_1(x, y, t) = 0.41^{-2} \sin(\pi t/8)(6y(0.41 - y), 0).$$

The initial condition is $\mathbf{u}_{init}(x, y) = (0, 0)$. The density of the fluid is given by $\rho = 1\text{kgm}^{-3}$, and the reference velocity is $\bar{U} = 1\text{ms}^{-1}$ (note that the maximum velocity is $\frac{3}{2}\bar{U}$). The diameter of the cylinder is $L = 0.1\text{m}$, so that the Reynold's number is $0 \leq \text{Re}(t) \leq 100$. The time step is $\delta t = 0.00166s$.

We use the limit scheme ($\widetilde{\mathcal{P}}$), but at some fixed times, we use instead the iterative Schwarz algorithm ($\mathcal{S}_1$)-($\mathcal{S}_2$), with the initial guesses $\mathbf{h}^0_{\mathfrak{T}_j}$ and $g^0_{\mathfrak{D}_j}$, for $j = 1, 2$, given by ($\widetilde{\mathcal{P}}$) at the previous time step. The stopping criterion is

$$\max\left(\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2, \|\Pi^l_{\mathfrak{D}_j}\|_2\right) < 10^{-3} \tag{50}$$

and the values of the parameters are set to $\lambda = 200$, $\alpha = 1$ and $\beta = 0.01$. To discretize the convection term, we choose $B = 0$ which gives a second accurate method.

To start with, we consider the profile of the first component of the velocity. The iterative algorithm is applied at time $T = 2s$ and $T = 6s$; we compare the solution of the limit problem ($\widetilde{\mathcal{P}}$) (Fig. 20,22) to the solution obtained by the iterative algorithm ($\mathcal{S}_1$)-($\mathcal{S}_2$) (Fig. 21,23). As we can see, the profile is the same and the domain decomposition does not introduce any spurious modes to the solution; the stopping criterion (50) is sufficient to obtain a fair approximation of the expected solution. The convergence of the algorithm is obtained in 299 iterations at $T = 2s$ and in 377 iterations at $T = 6s$; the number of iterations, as remarked in Sec. 7.1.2, can be optimized through the choice of the parameters λ, α, β (the parameters are fixed once for all and do not evolve in time).

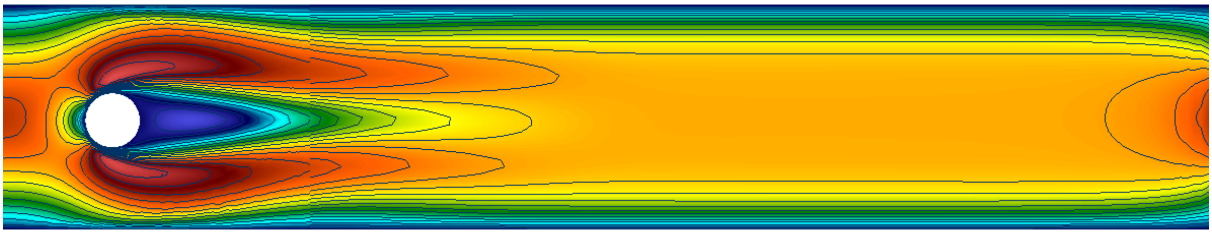


Fig. 20 First component of the velocity solution to the Navier-Stokes problem on Ω at $T = 2s$.

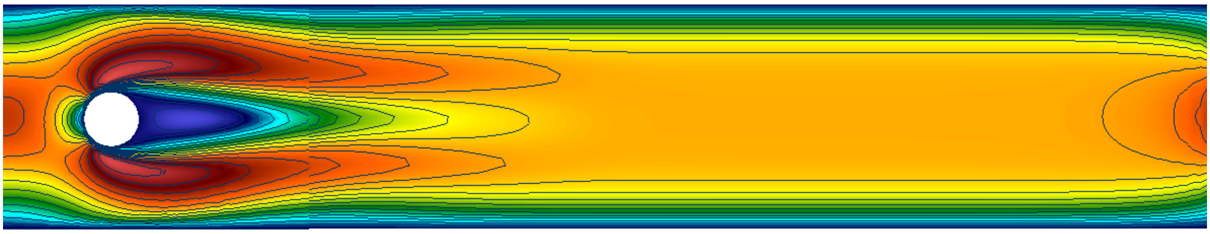


Fig. 21 First component of the velocity solution to the Navier-Stokes problem on Ω, obtained at the 299^{th} iteration of the Schwarz algorithm, at $T = 2s$.

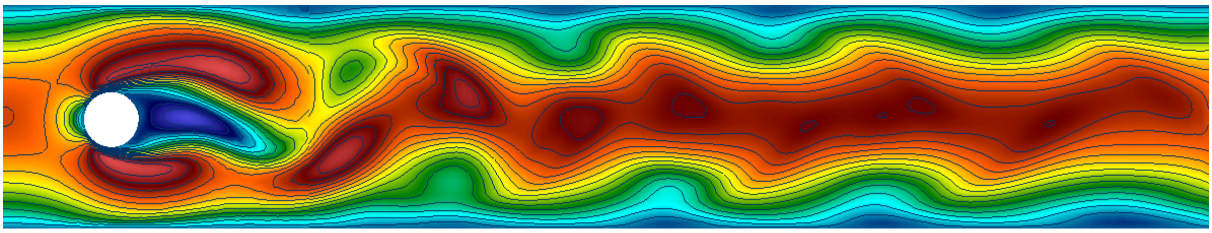


Fig. 22 First component of the velocity solution to the Navier-Stokes problem on Ω at $T = 6s$.

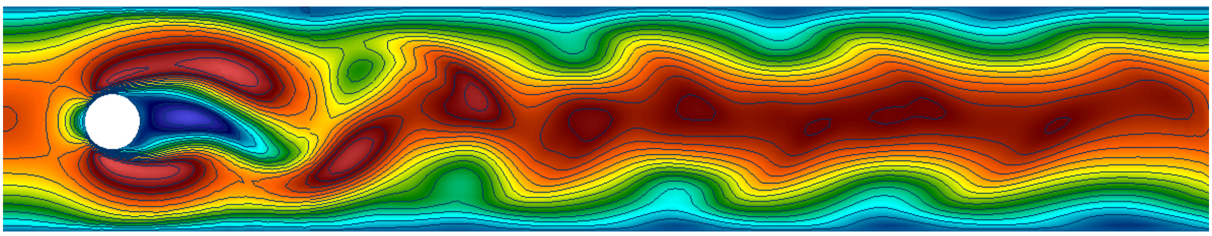


Fig. 23 First component of the velocity solution to the Navier-Stokes problem on Ω, obtained at the 377^{th} iteration of the Schwarz algorithm, at $T = 6s$.

In order to measure the quality of the obtained results, we focus on the computation of the drag and lift coefficients for the limit problem $(\widetilde{\mathcal{P}})$ and for the solution of $(\mathcal{S}_1)$-$(\mathcal{S}_2)$. We define the drag coefficient $c_d(t)$ and the lift coefficient $c_l(t)$ as

$$c_d(t) = \frac{2}{\rho L \bar{U}^2} \int_S \left(\rho\eta \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_y - p(t) n_x \right),$$
$$c_l(t) = -\frac{2}{\rho L \bar{U}^2} \int_S \left(\rho\eta \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_x + p(t) n_y \right),$$

where S stands for the boundary of the obstacle, $\vec{\mathbf{n}}_S = (n_x, n_y)$ is the normal vector on S pointing to Ω, $\mathbf{t}_S = (n_y, -n_x)$ the tangential vector and $\mathbf{u}_{t_S}$ the tangential velocity. In the DDFV setting,

$$c_d^n = \frac{2}{\rho L \bar{U}^2} \sum_{\mathtt{D} \in \mathfrak{D}_{ext} \cap S} m_\sigma \left(\rho\eta \nabla^{\mathtt{D}}(\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}\, n_y - \mathrm{p}^n n_x \right),$$
$$c_l^n = -\frac{2}{\rho L \bar{U}^2} \sum_{\mathtt{D} \in \mathfrak{D}_{ext} \cap S} m_\sigma \left(\rho\eta \nabla^{\mathtt{D}}(\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}\, n_x + \mathrm{p}^n n_y \right).$$

We study the evolution of the coefficients in Fig. 24 and their maximum value in Table 5, defined as:

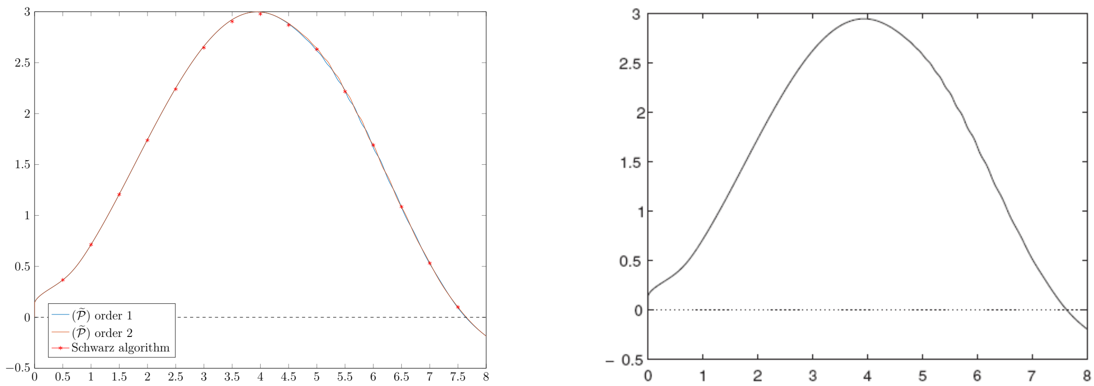
$$c_{d,max} = \max_{n \in \{0 \dots N\}} c_d^n, \qquad c_{l,max} = \max_{n \in \{0 \dots N\}} c_l^n.$$

The results shown in Table 5 and in Fig. 24 prove that the approximation given by the limit problem $(\widetilde{\mathcal{P}})$ and the results obtained with the Schwarz algorithm $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ are robust and quantitatively correct. The behavior of the drag and lift coefficients of $(\widetilde{\mathcal{P}})$ is coherent with the reference values from [Joh04], and the extreme values of both coefficients are similar, see Table 5. The slight discrepancy in the maximum value of the coefficients is due to level of refinement of the mesh and to the order of the scheme: we work with approximately 30 000 unknowns, for all velocity components and pressure, compared to the approximately 500 000 unknowns used in [Joh04]. Figure 24 shows that the lift coefficient is sensitive to the choice of the time discretization: the time step in [Joh04] is $\delta t = 0.00125s$ with a second order scheme in time. Our scheme is first order in time, and we work with $\delta t = 0.00166s$.

We have implemented a second order backward difference formula in time, as in [GKL17]: the first iteration in time remains unchanged, while for $n \in \{1, \dots N\}$ the term $\partial_t \mathbf{u}$ is discretized by $\frac{1}{\delta t}(\frac{3}{2}\mathbf{u}^{n+1} - 2\mathbf{u}^n + \frac{1}{2}\mathbf{u}^{n-1})$ instead of $\frac{1}{\delta t}(\mathbf{u}^{n+1} - \mathbf{u}^n)$ and the convection fluxes $F_{\sigma\mathtt{K}}$ depend on $(2\mathbf{u}^n - \mathbf{u}^{n-1})$ instead of $\mathbf{u}^n$. This approach indeed improves the quality of the approximation of the lift coefficient, see Fig. 24. The drag and lift coefficients associated to the domain decomposition method $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ have been computed with the second order scheme. The iterative process is applied on a finite number of steps, one per second for instance; we then compute the coefficients associated to the solution given by the algorithm. The results are illustrated in Fig. 24, where we can observe that the values of the coefficients associated to the Schwarz algorithm stay close to the curves given by the solution of $(\widetilde{\mathcal{P}})$; since we established that they are a coherent reproduction of the reference values in [Joh04], we can conclude that the algorithm produces a good approximation of the solution of the Navier-Stokes problem on the entire domain.

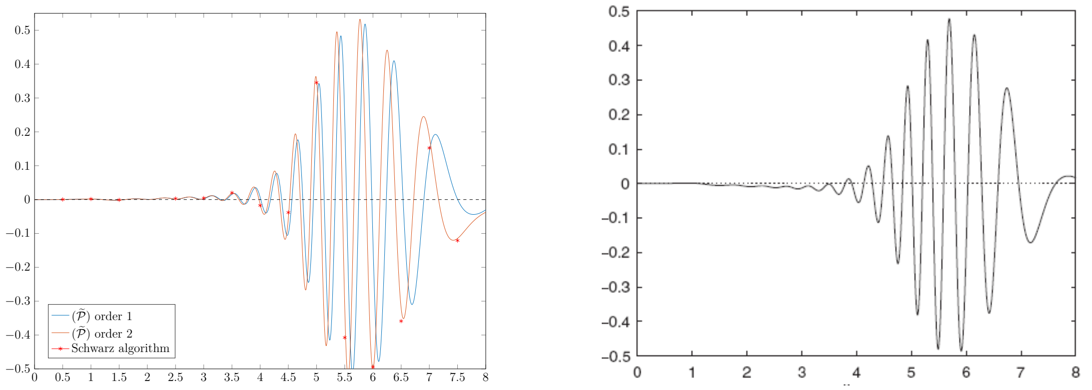
	$(\widetilde{\mathcal{P}})$ **order 1**	$(\widetilde{\mathcal{P}})$ **order 2**	**Reference**
$\mathbf{c_{d,max}}$	2.9985	2.9987	2.9509
$\mathbf{c_{l,max}}$	0.5183	0.53246	0.4779

Table 5 Comparison between the values of $c_{d,max}, c_{l,max}$ obtained with DDFV scheme $(\widetilde{\mathcal{P}})$ of order 1 and 2 in time and the reference values of [Joh04].



(a) DDFV. (b) Reference values.

Evolution in time of the drag coefficient



(c) DDFV. (d) Reference values.

Evolution in time of the lift coefficient

Fig. 24 Comparison between the evolution of c_d^n, c_l^n on the time interval $[0, 8]$ obtained with the DDFV scheme $(\widetilde{\mathcal{P}})$, of order 1 and 2 in time, and with the Schwarz algorithm $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ of order 2 (*left*) and the reference values of [Joh04] (*right*).

8 Conclusion

This paper establishes the well-posedness of DDFV schemes for solving the incompressible Navier-Stokes system on the entire domain Ω with general convection fluxes defined by means of B-schemes, and it proposes two non-overlapping DDFV Schwarz algorithms. DDFV discretizations are constructed with suitable transmission conditions, which are equally well-posed. When using standard convection fluxes in the domain decomposition method, the iterative process converges to a system with modified fluxes at the interface. However, it is possible to modify the fluxes of the domain decomposition algorithm so that it converges to the reference scheme on the entire domain. The algorithms are numerically tested on classical benchmarks, and the numerical experiments also shed some light on the role of the parameters of the method.

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