DDFV method for Navier-Stokes problem with outflow boundary conditions

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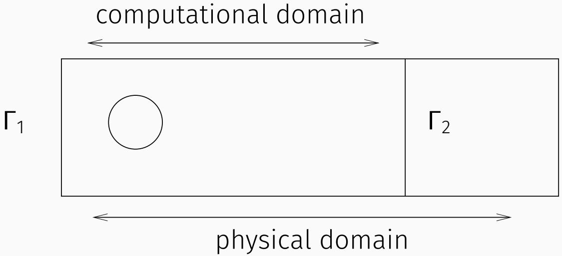
Outflow boundary conditions



- Γ_1: Dirichlet boundary conditions
- Γ_2: $\boxed{\text{Outflow boundary conditions}}$
- $\Gamma_0 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: no slip boundary conditions

Outflow boundary conditions

Find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $\mathrm{p} : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) = 0; & \text{in} \quad \Omega_T = \Omega \times [0, T] \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T, \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1 \times (0, T), \\ \mathbf{u} = 0 & \text{on} \quad \Gamma_0 \times (0, T), \\ \text{outflow boundary conditions} & \text{on} \quad \Gamma_2 \times (0, T) \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega \end{cases}$$

with $T > 0$, $\Omega \subset \mathbb{R}^2$, $\mathbf{u}_{init} \in (L^\infty(\Omega))^2$, $\mathbf{g}_1 \in (H^{\frac{1}{2}}(\Omega))^2$.

The stress tensor: $\sigma(\mathbf{u}, \mathrm{p}) = \dfrac{2}{\mathrm{Re}} \mathrm{D}\mathbf{u} - \mathrm{pId}$, with $\mathrm{Re} > 0$.

The strain rate tensor: $\mathrm{D}\mathbf{u} = \dfrac{(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})}{2}$.

Outflow boundary conditions

We choose a test function $\boldsymbol{\Psi}$ in the space
$V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

The weak formulation reads:

$$\int_\Omega \partial_t \mathbf{u} \cdot \boldsymbol{\Psi} + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \boldsymbol{\Psi} - \int_\Omega \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) \cdot \boldsymbol{\Psi} = 0$$

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The weak formulation reads:

$$\int_\Omega \partial_t \mathbf{u} \cdot \mathbf{\Psi} + \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{\Psi} - \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{\Psi} \cdot \mathbf{u}$$
$$- \int_\Omega \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) \cdot \mathbf{\Psi} = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u} \cdot \vec{\mathbf{n}})\mathbf{u} \cdot \mathbf{\Psi}$$

Outflow boundary conditions

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$V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \operatorname{div}(\psi) = 0\}$.

The weak formulation reads:

$$\int_\Omega \partial_t \mathbf{u} \cdot \boldsymbol{\Psi} + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\Psi} - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\boldsymbol{\Psi}\cdot\mathbf{u}$$
$$- \int_\Omega \operatorname{div}\left(\frac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - p\mathrm{Id}\right)\cdot\boldsymbol{\Psi} = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\boldsymbol{\Psi}$$

Outflow boundary conditions

We choose a test function Ψ in the space
$V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

The weak formulation reads:

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$+\frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}\mathbf{u} : \mathrm{D}\Psi - \int_{\Gamma_2} \sigma(\mathbf{u}, \mathrm{p})\cdot\vec{\mathbf{n}}\cdot\Psi = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\Psi$$

Outflow boundary conditions

$$\int_\Omega \partial_t \mathbf{u}\cdot\Psi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
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Let $\mathbf{u}_{ref}, \sigma_{ref}$ be a reference flow. We can impose:

$$\boxed{\sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}} = -\frac{1}{2}\Theta(\mathbf{u}\cdot\vec{\mathbf{n}})(\mathbf{u}-\mathbf{u}_{ref}) + \sigma_{ref}\cdot\vec{\mathbf{n}}} \qquad (1)$$

where

$$\Theta(a) = a, \quad \Theta(a) = -a^-, \quad \Theta(a) = |a|$$

with the notation $a = a^+ - a^-$.

(1): [Ch.-H. Bruneau and P. Fabrie, *Effective downstream boundary conditions for incompressible navier stokes equations*, 1994]

Outflow boundary conditions

We choose:

$$\boxed{\sigma(\mathbf{u}, p) \cdot \vec{n} = -\frac{1}{2}(\mathbf{u} \cdot \vec{n})^{-}(\mathbf{u} - \mathbf{u}_{ref}) + \sigma_{ref} \cdot \vec{n}}$$

so that the weak formulation becomes:

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Psi - \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\Psi \cdot \mathbf{u} + \frac{1}{2}\int_{\Gamma_2} (\mathbf{u} \cdot \vec{n})^{+}(\mathbf{u} \cdot \Psi)$$
$$+ \frac{2}{Re}\int_\Omega D(\mathbf{u}) : D(\Psi) = \frac{1}{2}\int_{\Gamma_2} (\mathbf{u} \cdot \vec{n})^{-}(\mathbf{u}_{ref} \cdot \Psi) + \int_{\Gamma_2} \sigma_{ref} \cdot \vec{n} \cdot \Psi$$

Discrete Duality Finite Volume method

DDFV meshes

Primal mesh

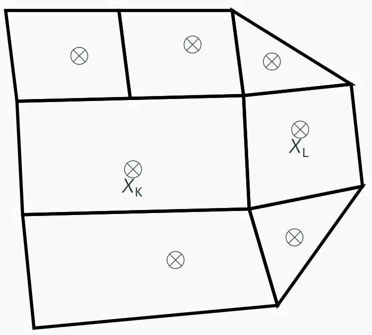
$$\rightsquigarrow \mathrm{u}^{\mathfrak{M}} = (\mathrm{u}_\kappa)_{\kappa \in \mathfrak{M}}$$

Dual mesh

$$\rightsquigarrow \mathrm{u}^{\mathfrak{M}^*} = (\mathrm{u}_{\kappa^*})_{\kappa^* \in \mathfrak{M}^*}$$

Diamond mesh

$$\rightsquigarrow \nabla^{\mathfrak{D}} \mathrm{u}^{\mathfrak{T}},\ \mathrm{p}^{\mathfrak{D}}$$



Our unknowns are:
$\mathrm{u}^{\mathfrak{T}} = (\mathrm{u}^{\mathfrak{M}}, \mathrm{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

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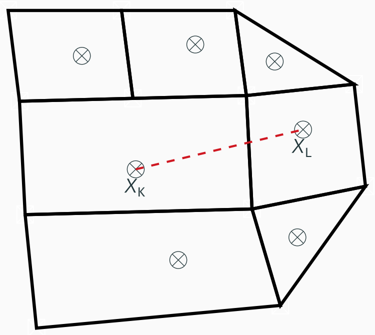

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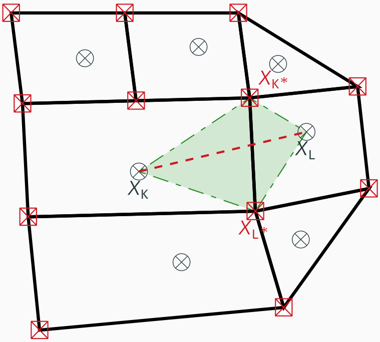

Our unknowns are:
$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

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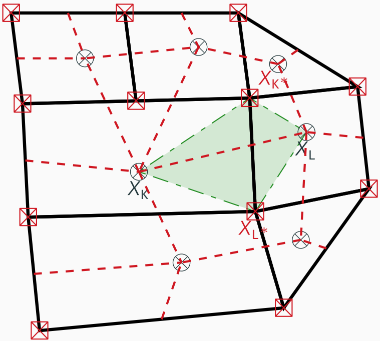
$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\kappa^*})_{\kappa^*\in\mathfrak{M}^*}$

Diamond mesh

$\rightsquigarrow \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, \ \mathrm{p}^{\mathfrak{D}}$



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$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

DDFV meshes

Primal mesh

$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_\kappa)_{\kappa \in \mathfrak{M}}$

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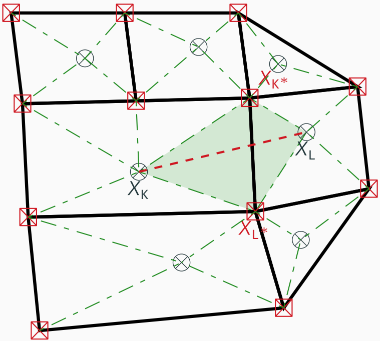
$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\kappa^*})_{\kappa^* \in \mathfrak{M}^*}$

Diamond mesh

$\rightsquigarrow \nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}, \ p^{\mathfrak{D}}$



Our unknowns are:
$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $p^{\mathfrak{D}}$

Discrete gradient

The operator $\nabla^{\mathfrak{D}} : (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ where

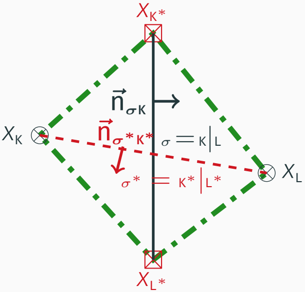
$$\nabla^{D}\mathbf{u}^{\mathfrak{T}}(x_L - x_K) = \mathbf{u}_L - \mathbf{u}_K,$$
$$\nabla^{D}\mathbf{u}^{\mathfrak{T}}(x_{L^*} - x_{K^*}) = \mathbf{u}_{L^*} - \mathbf{u}_{K^*}.$$

$$\nabla^{D}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_D}\left[m_\sigma(\mathbf{u}_L - \mathbf{u}_K) \otimes \vec{\mathbf{n}}_{\sigma K} + m_{\sigma^*}(\mathbf{u}_{L^*} - \mathbf{u}_{K^*}) \otimes \vec{\mathbf{n}}_{\sigma^* K^*}\right].$$

[Krell '11]

$$\rightsquigarrow \mathrm{div}^{D}\mathbf{u}^{\mathfrak{T}} = \mathrm{Tr}(\nabla^{D}\mathbf{u}^{\mathfrak{T}}).$$

$$\rightsquigarrow \mathrm{D}^{D}\mathbf{u}^{\mathfrak{T}} = \frac{\nabla^{D}\mathbf{u}^{\mathfrak{T}} + {}^{t}(\nabla^{D}\mathbf{u}^{\mathfrak{T}})}{2}.$$



Discrete divergence

$\mathbf{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ where:

$$\mathbf{div}^{\kappa}\xi^{\mathfrak{D}} = \frac{1}{m_{\kappa}} \sum_{\sigma \subset \partial\kappa} m_{\sigma} \xi^{D} \vec{\mathbf{n}}_{\sigma\kappa}, \qquad \forall \kappa \in \mathfrak{M}$$

$$\mathbf{div}^{\kappa^*}\xi^{\mathfrak{D}} = \frac{1}{m_{\kappa^*}} \sum_{\sigma^* \subset \partial\kappa^*} m_{\sigma^*} \xi^{D} \vec{\mathbf{n}}_{\sigma^*\kappa^*}, \qquad \forall \kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$$

Discrete duality property

- On the continuous level: $\int_{\Omega} \mathrm{div}\xi \cdot \mathbf{u} = -\int_{\Omega} \xi : \nabla \mathbf{u} + \int_{\partial\Omega} \xi \overrightarrow{\mathbf{n}} \cdot \mathbf{u}$
- On the discrete level:

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\mathbf{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}$$

[S. Krell, *Stabilized DDFV schemes for the incompressible Navier-Stokes equations*, 2011]

DDFV for Navier-Stokes problem

The problem

Find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $\mathrm{p} : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) = 0; & \text{in} \quad \Omega_T = \Omega \times [0, T] \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T, \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1 \times (0, T), \\ \mathbf{u} = 0 & \text{on} \quad \Gamma_0 \times (0, T), \\ \sigma(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} + \frac{1}{2}(\mathbf{u} \cdot \vec{\mathbf{n}})^-(\mathbf{u} - \mathbf{u}_{ref}) = \sigma_{ref} \cdot \vec{\mathbf{n}} & \text{on} \quad \Gamma_2 \times (0, T) \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega \end{cases}$$

with $T > 0$, $\Omega \subset \mathbb{R}^2$, $\mathbf{u}_{init} \in (L^\infty(\Omega))^2$, $\mathbf{g}_1 \in (H^{\frac{1}{2}}(\Omega))^2$.

The stress tensor: $\sigma(\mathbf{u}, \mathrm{p}) = \dfrac{2}{\mathrm{Re}} \mathrm{D}\mathbf{u} - \mathrm{pId}$, with $\mathrm{Re} > 0$.

The strain rate tensor: $\mathrm{D}\mathbf{u} = \dfrac{(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})}{2}$.

Nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ (1/3)

We construct:

- $m_K \, \mathbf{b}^K(\mathbf{u}^n, \mathbf{u}^{n+1}) \rightsquigarrow \int_K (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}$ when $K \in \mathfrak{M}$,
- $m_{K^*} \, \mathbf{b}^{K^*}(\mathbf{u}^n, \mathbf{u}^{n+1}) \rightsquigarrow \int_{K^*} (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}$ when $K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*_N$.

We focus on the primal mesh and we observe:

$$\boxed{\int_K (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} = \sum_{\sigma \subset \partial K} \int_\sigma (\mathbf{u}^n \cdot \vec{\mathbf{n}}_{\sigma K})\mathbf{u}^{n+1}} \quad \forall K \in \mathfrak{M}$$

We define the fluxes:

$$\int_\sigma (\mathbf{u}^n \cdot \vec{\mathbf{n}}_{\sigma K}) \rightsquigarrow F_{\sigma,K}(\mathbf{u}^n)$$

Nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ (2/3)

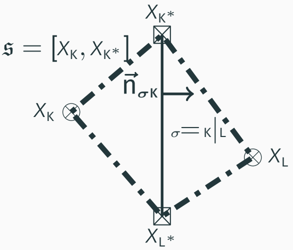
We impose: $F_{\sigma,\text{K}}(\mathbf{u}^n) = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in\mathfrak{S}_\text{K}\cap\mathcal{E}_\text{D}} G_{\mathfrak{s},\text{D}}(\mathbf{u}^n) & \text{if } \sigma \in \mathcal{E}_{int} \\ m_\sigma \gamma^\sigma(\mathbf{u}^n)\cdot \vec{\mathbf{n}}_{\sigma\text{K}} & \text{if } \sigma \in \partial\Omega \end{cases}$, where

$$G_{\mathfrak{s},\text{D}}(\mathbf{u}^n) = m_\mathfrak{s} \frac{\mathbf{u}_\text{K}^n + \mathbf{u}_{\text{K}*}^n}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\text{D}} \rightsquigarrow \int_\mathfrak{s} \mathbf{u}^n \cdot \vec{\mathbf{n}}_{\mathfrak{s}\text{D}},$$

$$\gamma_\sigma(\mathbf{u}^n) = \frac{\mathbf{u}_{\text{K}*}^n + 2\mathbf{u}_\text{L}^n + \mathbf{u}_{\text{L}*}^n}{4} \quad \forall \sigma = [x_{\text{K}*}, x_{\text{L}*}] \in \partial\mathfrak{M}.$$

We have conservativity:

$$F_{\sigma,\text{K}} = -F_{\sigma,\text{L}}, \quad \forall \sigma = \text{K}|\text{L}$$



Nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ (3/3)

$$m_K\, \mathbf{b}^\kappa(\mathbf{u}^n, \mathbf{u}^{n+1}) \rightsquigarrow \sum_{\sigma \subset \partial K} \int_\sigma (\mathbf{u}^n \cdot \vec{\mathbf{n}}_{\sigma\kappa})\mathbf{u}^{n+1}$$

So we define $\forall \kappa \in \mathfrak{M}$:

$$\boxed{m_K \mathbf{b}^\kappa(\mathbf{u}^n, \mathbf{u}^{n+1}) = \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K^{int}} F_{\sigma,\kappa}(\mathbf{u}^n)\mathbf{u}_{\sigma^+}^{n+1} + \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K^{ext}} F_{\sigma,\kappa}(\mathbf{u}^n)\gamma^\sigma(\mathbf{u}^{n+1})}$$

where

$$\mathbf{u}_{\sigma^+}^{n+1} = \begin{cases} \mathbf{u}_\kappa^{n+1} & \text{if } F_{\sigma,\kappa} \geq 0 \\ \mathbf{u}_L^{n+1} & \text{otherwise} \end{cases} \qquad \forall \sigma \in \mathcal{E}_{int}$$

and

$$\gamma_\sigma(\mathbf{u}^{n+1}) = \frac{\mathbf{u}_{\kappa^*}^{n+1} + 2\mathbf{u}_L^{n+1} + \mathbf{u}_{L^*}^{n+1}}{4} \quad \forall \sigma = [x_{\kappa^*}, x_{L^*}] \in \partial\mathfrak{M}.$$

Variational formulation

$$\int_\Omega \partial_t \mathbf{u}\cdot\Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\Psi) + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$= -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})^+(\mathbf{u}\cdot\Psi) + \frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}_{ref}\cdot\Psi) + \int_{\Gamma_2} \sigma_{ref}\cdot\vec{\mathbf{n}}\cdot\Psi$$

$$\Downarrow$$

$$[[\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}}$$
$$- \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} = -\frac{1}{2}\sum_{\mathrm{D}\in\mathrm{D}_{ext}\cap\Gamma_2} (F_{\sigma,\kappa}(\mathbf{u}^n))^+\gamma^\sigma(\mathbf{u}^{n+1})\gamma^\sigma(\Psi^{\mathfrak{T}})$$
$$+\frac{1}{2}\sum_{\mathrm{D}\in\mathrm{D}_{ext}\cap\Gamma_2} (F_{\sigma,\kappa}(\mathbf{u}^n))^-\gamma^\sigma(\mathbf{u}_{ref})\gamma^\sigma(\Psi^{\mathfrak{T}}) + \sum_{\mathrm{D}\in\mathrm{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma_{\mathrm{ref}}^{\mathfrak{D}}\cdot\vec{\mathbf{n}}_{\sigma\kappa})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}).$$

Theoretical Results

Well-posedness of the scheme

We obtain our scheme by projecting the discrete weak formulation on the DDFV mesh.

Theorem

(Well-posedness)

Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies inf-sup stability condition. The scheme we obtain has a unique solution $(\boldsymbol{u}^{\mathfrak{T},[0,T]}, p^{\mathfrak{D},[0,T]}) \in \left((\mathbb{R}^2)^{\mathfrak{T}}\right)^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$.

Proof.
The scheme is equivalent to a system $Av = b$ at each time step, with A square matrix and $v = (\mathbf{u}^{n+1}, \mathrm{p}^{n+1})$.

$$\Downarrow$$

We want to show that the square matrix is injective, i.e.

$$Av = 0 \implies v = 0$$

Proof of well-posedness (2/4)

If we multiply $Av = 0$ by a test function $\boldsymbol{\Psi}^{\mathfrak{T}}$ that satisfies

$$\begin{cases} \boldsymbol{\Psi}^{\mathfrak{T}} \in \mathbb{E}_0^D, \\ \mathrm{div}^{\mathfrak{D}}(\boldsymbol{\Psi}^{\mathfrak{T}}) = 0, \end{cases}$$

this is equivalent to consider the discrete variational formulation in the form:

$$\begin{aligned} \frac{1}{\delta t}[[\mathbf{u}^{n+1}, \boldsymbol{\Psi}^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\boldsymbol{\Psi}^{\mathfrak{T}})_{\mathfrak{D}} \\ + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \mathbf{u}^{n+1}), \boldsymbol{\Psi}^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \boldsymbol{\Psi}^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} \\ = -\frac{1}{2}\sum_{\mathrm{D} \in \mathrm{D}_{ext} \cap \Gamma_2} (F_{\sigma,\kappa}(\mathbf{u}^n))^+ \gamma^{\sigma}(\mathbf{u}^{n+1})\gamma^{\sigma}(\boldsymbol{\Psi}^{\mathfrak{T}}). \end{aligned}$$

Proof of well-posedness (3/4)

If in this weak formulation:

$$\frac{1}{\delta t}[[\mathbf{u}^{n+1}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}}$$
$$+\frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{n}, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{n}, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}}$$
$$= -\frac{1}{2}\sum_{\mathrm{D}\in\mathrm{D}_{ext}\cap\Gamma_2}(F_{\sigma,\kappa}(\mathbf{u}^n))^+\,\gamma^\sigma(\mathbf{u}^{n+1})\gamma^\sigma(\Psi^{\mathfrak{T}}).$$

we choose $\Psi^{\mathfrak{T}} = \mathbf{u}^{n+1}$, we obtain:

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 + \underbrace{\frac{1}{2}\sum_{\mathrm{D}\in\mathrm{D}_{ext}}(F_{\sigma,\kappa}(\mathbf{u}^n))^+\left|\gamma^\sigma(\mathbf{u}^{n+1})\right|^2}_{\geq 0} = 0,$$

Proof of well-posedness (4/4)

We end up with:

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 \leq 0$$

from which we deduce that $\boxed{\mathbf{u}^{n+1} = 0}$.

To conclude the proof, we need to show that p^{n+1} is equal to zero too. This is true because of the inf-sup stability, that ensures an inequality of the type:

$$\|\mathrm{p}^{n+1} - m(\mathrm{p}^{n+1})\|_2 \leq C\|\nabla^{\mathfrak{D}}\mathbf{u}^{n+1}\|_2,$$

where $m(\mathrm{p}^{n+1}) = \sum_{\mathrm{D}\in\mathfrak{D}} m_{\mathrm{D}} p^{\mathrm{D}}$. This inequality implies that p^{n+1} is constant.

Thanks to boundary condition on Γ_2, we conclude $\boxed{\mathrm{p}^{n+1} = 0}$.

$\square$

Discrete energy estimate

Theorem

Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies inf-sup stability condition.
Let $(\boldsymbol{u}^{\mathfrak{T},[0,T]}, p^{\mathfrak{D},[0,T]}) \in \left((\mathbb{R}^2)^{\mathfrak{T}}\right)^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$ be the solution of the DDFV scheme, where $\boldsymbol{u}^{\mathfrak{T},[0,T]} = \boldsymbol{v}^{\mathfrak{T},[0,T]} + \boldsymbol{u}_{ref}^{\mathfrak{T}}$.
For $N > 1$, there exists a constant $C > 0$, depending on $\Omega, \boldsymbol{u}_{ref}^{\mathfrak{T}}, \boldsymbol{u}_0, Re$ such that:

$$\sum_{j=0}^{N-1} \|\boldsymbol{v}^{j+1} - \boldsymbol{v}^j\|_2^2 \leq C, \quad \|\boldsymbol{v}^N\|_2^2 \leq C,$$

$$\sum_{j=0}^{N-1} \delta t \frac{1}{Re} \|\mathrm{D}^{\mathfrak{D}} \boldsymbol{v}^{j+1}\|_2^2 \leq C, \quad \delta t \frac{1}{Re} \|\mathrm{D}^{\mathfrak{D}} \boldsymbol{v}^N\|_2^2 \leq C,$$

$$\sum_{j=0}^{N-1} \delta t \sum_{\mathrm{D} \in \mathfrak{D}_{ext}} (F_{\sigma,\kappa}(\boldsymbol{v}^j + \boldsymbol{u}_{ref}^{\mathfrak{T}}))^+ (\gamma^\sigma(\boldsymbol{v}^{j+1}))^2 \leq C.$$

Proof.

1. Rewrite our discrete variational formulation for the unknown $\mathbf{v}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^{\mathfrak{T}}_{ref}$:

$$
\begin{aligned}
&[[\frac{\mathbf{v}^{n+1}-\mathbf{v}^{n}}{\delta t}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} \\
&+\frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^{n}+\mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{v}^{n+1}+\mathbf{u}^{\mathfrak{T}}_{ref}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^{n}+\mathbf{u}^{\mathfrak{T}}_{ref}, \Psi^{\mathfrak{T}}), \mathbf{v}^{n+1}+\mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}} \\
&\qquad + \frac{1}{2}\sum_{\mathrm{D}\in\mathfrak{D}_{ext}} (F_{\sigma,\kappa}(\mathbf{v}^{n}+\mathbf{u}^{\mathfrak{T}}_{ref}))^{+}\gamma^{\sigma}(\mathbf{v}^{n+1}+\mathbf{u}^{\mathfrak{T}}_{ref})\gamma^{\sigma}(\Psi^{\mathfrak{T}}) \\
&\qquad\qquad = -\frac{1}{2}\sum_{\mathrm{D}\in\mathfrak{D}_{ext}} F_{\sigma,\kappa}(\mathbf{v}^{n}+\mathbf{u}^{\mathfrak{T}}_{ref})^{-}\gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}}_{ref})\gamma^{\sigma}(\Psi^{\mathfrak{T}}).
\end{aligned}
$$

Sketch of the proof - Energy estimate (2/2)

2. The second step consists into selecting $\Psi^{\mathfrak{T}} = \mathbf{v}^{n+1} = (\mathbf{v}^{n+1} + \mathbf{u}^{\mathfrak{T}}_{ref}) - \mathbf{u}^{\mathfrak{T}}_{ref}$ as a test function. It follows:

$$
\begin{aligned}
E = [[\frac{\mathbf{v}^{n+1} - \mathbf{v}^{n}}{\delta t}, \mathbf{v}^{n+1}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^{n+1}\|_2^2 \\
+ \frac{1}{2} \sum_{\mathrm{D} \in \mathfrak{D}_{ext} \cap \Gamma_2} (F_{\sigma,\kappa}(\mathbf{v}^{n} + \mathbf{u}^{\mathfrak{T}}_{ref}))^{+} (\gamma^{\sigma}(\mathbf{v}^{n+1}))^2 \\
\leq \left| \frac{1}{2} [[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^{n} + \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{v}^{n+1}), \mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}} - \frac{1}{2} [[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^{n} + \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{u}^{\mathfrak{T}}_{ref}), \mathbf{v}^{n+1}]]_{\mathfrak{T}} \right| \\
+ \left| \frac{1}{2} \sum_{\mathrm{D} \in \mathfrak{D}_{ext} \cap \Gamma_2} F_{\sigma,\kappa}(\mathbf{v}^{n} + \mathbf{u}^{\mathfrak{T}}_{ref})^{-} \gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}}_{ref}) \gamma^{\sigma}(\mathbf{v}^{n+1}) \right|
\end{aligned}
$$

3. Estimate the RHS.

$\square$

Further results: Korn's and trace inequality

Korn's inequality

Let $\mathfrak{T}$ be a mesh that satisfies infsup stability condition. Then there exists $C > 0$ such that :

$$\boxed{\|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2 \leq C \|\mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2} \qquad \forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\mathbf{D}}$$

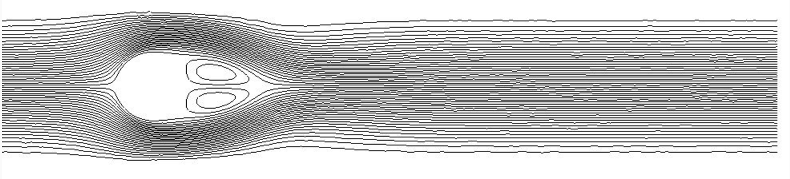
Trace theorem

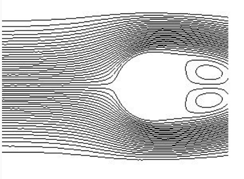
Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $p > 1$, there exists a constant $C > 0$, depending only on p, $\sin(\alpha_{\mathfrak{T}})$, $\mathrm{reg}(\mathfrak{T})$ and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\mathbf{D}}$ and for all $s \geq 1$:

$$\boxed{\|\gamma(\mathbf{u}^{\mathfrak{T}})\|_{s,\partial\Omega}^{s} \leq C \|\mathbf{u}^{\mathfrak{T}}\|_{1,p} \|\mathbf{u}^{\mathfrak{T}}\|_{\frac{p(s-1)}{p-1}}^{s-1}}$$

Numerical results

First numerical results (1/4)





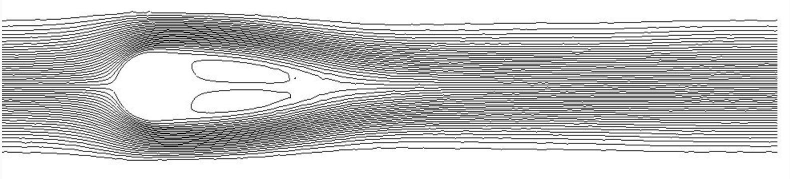
$$\mathbf{g}_1(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1$$

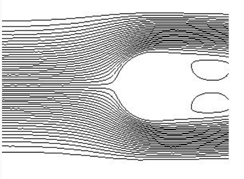
$$\mathbf{u}_{ref}(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_2$$

$$\sigma_{ref}(\mathbf{u},\mathrm{p}) \cdot \vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 6\eta(1-2y) \end{pmatrix} \qquad \text{on } \Gamma_2$$

with $T = 1.5s$, $Re = 100$, $\eta = 4 \times 10^{-3}$. On the top: $\Omega = [0,5] \times [0,1]$, NbCell= 12118. On the bottom: $\Omega' = [0,1.5] \times [0,1]$, NbCell=6534.

First numerical results (2/4)





$$\mathbf{g}_1(x, y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1$$

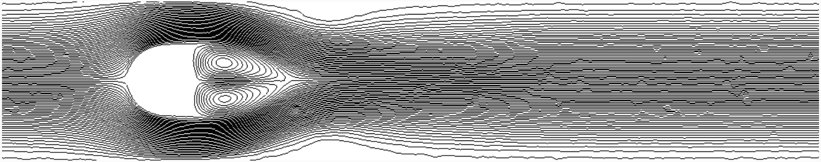
$$\mathbf{u}_{ref}(x, y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_2$$

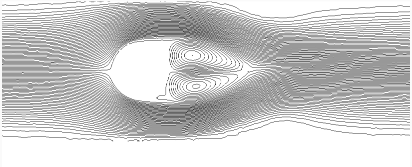
$$\sigma_{ref}(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 6\eta(1-2y) \end{pmatrix} \qquad \text{on } \Gamma_2$$

with $T = 3s$, $Re = 100$, $\eta = 4 \times 10^{-3}$. On the top: $\Omega = [0, 5] \times [0, 1]$,

NbCell= 12118. On the bottom: $\Omega' = [0, 1.5] \times [0, 1]$, NbCell=6534.

First numerical results (3/4)





$$\mathbf{g}_1(x, y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1$$

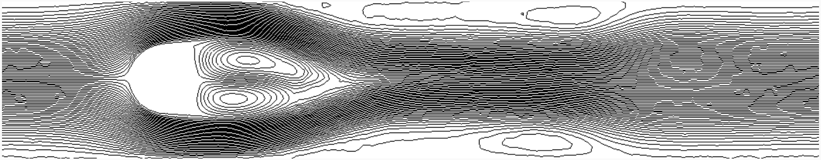
$$\mathbf{u}_{ref}(x, y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_2$$

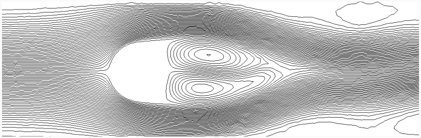
$$\sigma_{ref}(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 6\eta(1-2y) \end{pmatrix} \qquad \text{on } \Gamma_2$$

with $T = 1.5$s, Re $= 1000$, $\eta = 4 \times 10^{-3}$.On the top: $\Omega = [0, 5] \times [0, 1]$,

NbCell= 12118. On the bottom: $\Omega' = [0, 3] \times [0, 1]$, NbCell=8636.

First numerical results (4/4)





$$\mathbf{g}_1(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1$$

$$\mathbf{u}_{ref}(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_2$$

$$\sigma_{ref}(\mathbf{u},\mathrm{p}) \cdot \vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 6\eta(1-2y) \end{pmatrix} \qquad \text{on } \Gamma_2$$

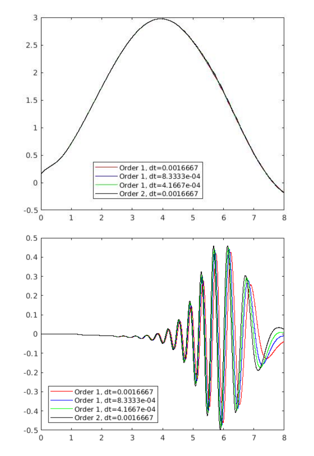
with $T = 3.5$s, Re $= 1000$, $\eta = 4 \times 10^{-3}$.On the top: $\Omega = [0,5] \times [0,1]$,

NbCell= 12118. On the bottom: $\Omega' = [0,3] \times [0,1]$, NbCell=8636.

Second numerical results

$c_{d,max} = 2.9754$

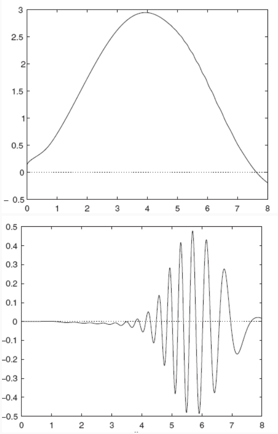
$c_{l,max} = 0.44902$



DDFV

$c_{d,max} = 2.9509$

$c_{l,max} = 0.47795$



Reference values [Volker, '04]

Conclusions

Conclusions and perspectives

Conclusions

- We built a well-posed DDFV scheme
- We proved a discrete energy estimate
- We obtained the expected numerical results

Perspectives

- investigate the choice of $\mathbf{u}_{ref}$ for higher Reynold's number
- application to Domain Decomposition

Grazie per l’attenzione!

The scheme (1/2)

- For all $\kappa \in \mathfrak{M}$:

$$m_\kappa \frac{\mathbf{u}_\kappa^{n+1} - \mathbf{u}_\kappa^n}{\delta t} - m_\kappa \mathbf{div}^\kappa(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})) + \frac{1}{2} m_\kappa \mathbf{b}^\kappa(\mathbf{u}^n, \mathbf{u}^{n+1}) - \frac{1}{2} \sum_{\mathrm{D} \in \mathfrak{D}_\kappa^{int}} \left(F_{\sigma,\kappa}^+(\mathbf{u}^n)\mathbf{u}_\kappa^{n+1} - F_{\sigma,\mathrm{L}}^-(\mathbf{u}^n)\mathbf{u}_\mathrm{L}^{n+1} \right) = 0; \quad (2)$$

- For all $\kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_O^*$:

$$m_{\kappa^*} \frac{\mathbf{u}_{\kappa^*}^{n+1} - \mathbf{u}_{\kappa^*}^n}{\delta t} - m_{\kappa^*} \mathbf{div}^{\kappa^*}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})) + \frac{1}{2} m_{\kappa^*} \mathbf{b}^{\kappa^*}(\mathbf{u}^n, \mathbf{u}^{n+1}) - \frac{1}{2} \sum_{\mathrm{D} \in \mathfrak{D}_{\kappa^*}} \left(F_{\sigma^*,\kappa^*}^+(\mathbf{u}^n)\mathbf{u}_{\kappa^*}^{n+1} - F_{\sigma^*,\mathrm{L}^*}^-(\mathbf{u}^n)\mathbf{u}_{\mathrm{L}^*}^{n+1} \right) = 0; \quad (3)$$

- For all $\sigma \in \partial\mathfrak{M}_O$:

$$m_\sigma \sigma^{\mathsf{D}}(\mathsf{u}^{n+1}, \mathsf{p}^{n+1}))\vec{\mathsf{n}}_{\sigma\mathsf{L}} - \frac{1}{4} F_{\sigma,\mathsf{L}}(\mathsf{u}^n)\,(\mathsf{u}_{\mathsf{K}}^{n+1} - \mathsf{u}_{\mathsf{L}}^{n+1})$$
$$= -\frac{1}{2}(F_{\sigma,\mathsf{L}}(\mathsf{u}^n))^- \big(\gamma^\sigma(\mathsf{u}^{n+1}) - \gamma^\sigma(\mathsf{u}_{ref})\big) + m_\sigma(\sigma^{\mathsf{D}}_{\mathrm{ref}} \cdot \vec{\mathsf{n}}_{\sigma\mathsf{K}}); \quad (4)$$

- For all $\mathsf{D} \in \mathfrak{D}$:

$$\mathrm{div}^{\mathsf{D}}(\mathsf{u}^{n+1}) = 0. \quad (5)$$

Upwind or centered?

We remark that $\mathbf{b}^\kappa(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})$ on the interior diamonds $\mathfrak{D}_\kappa^{int}$ can be written as:

$$m_\kappa \mathbf{b}^\kappa(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}) = F_{\sigma,\kappa}(\mathbf{v}^{\mathfrak{T}}) \left(\frac{\mathbf{u}_\kappa + \mathbf{u}_\mathsf{L}}{2} \right) + B_{\sigma\kappa}(\mathbf{v}^{\mathfrak{T}})(\mathbf{u}_\kappa - \mathbf{u}_\mathsf{L}).$$

That allows us to generalize the results: if $B_{\sigma\kappa} = 0$ we get a centered approximation, if $B_{\sigma\kappa}(\mathbf{v}^{\mathfrak{T}}) = \frac{1}{2}|F_{\sigma,\kappa}(\mathbf{v}^{\mathfrak{T}})|$ it is an upwind scheme. The equation of $\kappa \in \mathfrak{M}$ becomes:

$$m_\kappa \frac{\mathbf{u}_\kappa^{n+1} - \mathbf{u}_\kappa^n}{\delta t} - m_\kappa \mathbf{div}^\kappa(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})) + \sum_{\mathsf{D} \in \mathfrak{D}_\kappa^{int}} F_{\sigma,\kappa}(\mathbf{u}^n) \frac{\mathbf{u}_\kappa^{n+1} + \mathbf{u}_\mathsf{L}^{n+1}}{2}$$

$$+ \frac{1}{2} \sum_{\mathsf{D} \in \mathfrak{D}_\kappa^{int}} (B_{\sigma\mathsf{L}}(\mathbf{u}^n) - B_{\sigma\kappa}(\mathbf{u}^n))\mathbf{u}_\mathsf{L}^{n+1} + \sum_{\mathsf{D} \in \mathfrak{D}_\kappa^{ext}} F_{\sigma,\kappa}(\mathbf{u}^n) \left(\frac{1}{2}\mathbf{u}_\kappa^{n+1} + \frac{1}{2}\gamma^\sigma(\mathbf{u}^{n+1}) \right) = 0.$$

Inf-sup Stability (1/2)

Theorem

A given DDFV mesh $\mathfrak{T}$ is said to satisfy the Inf-Sup stability if the following condition holds:

$$\beta_{\mathfrak{T}} := \inf_{p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}} \left(\sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0} \frac{a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}}, p^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2 \|p^{\mathfrak{D}} - m(p^{\mathfrak{D}})\|_2} \right) > 0,$$

where $a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}}, p^{\mathfrak{D}}) = (\mathrm{div}^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}, p^{\mathfrak{D}})_{\mathfrak{D}}$ *and* $m(p^{\mathfrak{D}}) = \sum_{\mathtt{D} \in \mathfrak{D}} m_{\mathtt{D}} p^{\mathtt{D}}$.

It follows:

- There exists $C > 0$, depending on $\beta_{\mathfrak{T}}$, such that $\forall p^{\mathfrak{D}} \in \mathfrak{D}$, $\forall \mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0$:
$$\|\mathrm{p}^{\mathfrak{D}} - m(\mathrm{p}^{\mathfrak{D}})\|_2 \leq C\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2, \tag{6}$$
- For every $p^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ such that $m(p^{\mathfrak{D}}) = 0$, there exists $\mathbf{w}^{\mathfrak{T}} \in \mathbb{E}_0$ such that:
$$\begin{aligned} \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}) &= \mathrm{p}^{\mathfrak{D}} \\ \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 &\leq C\|\mathrm{p}^{\mathfrak{D}}\|_2, \end{aligned} \tag{7}$$

  where

  $$\mathbb{E}_0 = \{\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}, \text{s t. } \forall \kappa \in \partial\mathfrak{M},\ \mathbf{u}_\kappa = 0 \text{ and } \forall \kappa^* \in \partial\mathfrak{M}^*,\ \mathbf{u}_{\kappa^*} = 0\}.$$