DDFV method for Navier-Stokes problem with outflow boundary conditions

Giulia Lissoni

Thesis supervisors: Thierry Goudon, Stella Krell

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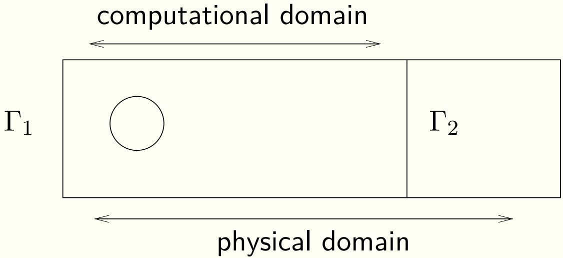

- Γ_1: Dirichlet boundary conditions
- Γ_2: $\boxed{\text{Outflow boundary conditions}}$
- $\Gamma_0 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: no slip boundary conditions

Outflow boundary conditions

Find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $\mathsf{p} : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u}, \mathsf{p})) = 0; & \text{in} \quad \Omega_T = \Omega \times [0, T] \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T, \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1 \times (0, T), \\ \mathbf{u} = 0 & \text{on} \quad \Gamma_0 \times (0, T), \\ \text{outflow boundary conditions} & \text{on} \quad \Gamma_2 \times (0, T) \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega \end{cases}$$

with $T > 0$, $\Omega \subset \mathbb{R}^2$, $\mathbf{u}_{init} \in (L^\infty(\Omega))^2$, $\mathbf{g}_1 \in (H^{\frac{1}{2}}(\Omega))^2$.

The stress tensor: $\sigma(\mathbf{u}, \mathsf{p}) = \dfrac{2}{\mathrm{Re}} \mathrm{D}\mathbf{u} - \mathsf{p}\mathrm{Id}$, with $\mathrm{Re} > 0$.

The strain rate tensor: $\mathrm{D}\mathbf{u} = \dfrac{(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})}{2}$.

Outflow boundary conditions

We choose a test function Ψ in the space
$V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

The weak formulation reads:

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Psi - \int_\Omega \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) \cdot \Psi = 0$$

Outflow boundary conditions

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The weak formulation reads:

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u} - \int_\Omega \mathrm{div}(\sigma(\mathbf{u},\mathrm{p}))\cdot\Psi = -\frac{1}{2}\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\Psi$$

Outflow boundary conditions

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The weak formulation reads:

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Outflow boundary conditions

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$V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

The weak formulation reads:

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$$+ \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}\mathbf{u} : \mathrm{D}\Psi - \int_{\Gamma_2} \sigma(\mathbf{u}, \mathrm{p})\cdot\vec{\mathbf{n}}\cdot\Psi = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\Psi$$

Outflow boundary conditions

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
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Let $\mathbf{u}_{ref}, \sigma_{ref}$ be a reference flow. We can impose:

$$\boxed{\sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}} = \frac{1}{2}\Theta(\mathbf{u}\cdot\vec{\mathbf{n}})(\mathbf{u}-\mathbf{u}_{ref}) + \sigma_{ref}\cdot\vec{\mathbf{n}}} \qquad (1)$$

where

$$\Theta(a) = a, \quad \Theta(a) = -a^-, \quad \Theta(a) = |a|$$

with the notation $a = a^+ - a^-$.

(1): [Ch.-H. Bruneau and P. Fabrie, *Effective downstream boundary conditions for incompressible navier stokes equations*, 1994]

Outflow boundary conditions

We choose:

$$\boxed{\sigma(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} = -\frac{1}{2}(\mathbf{u} \cdot \vec{\mathbf{n}})^{-}(\mathbf{u} - \mathbf{u}_{ref}) + \sigma_{ref} \cdot \vec{\mathbf{n}}}$$

so that the weak formulation becomes:

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Psi - \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\Psi \cdot \mathbf{u} + \frac{1}{2}\int_{\Gamma_2} (\mathbf{u} \cdot \vec{\mathbf{n}})^{+}(\mathbf{u} \cdot \Psi)$$
$$+ \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\Psi) = \frac{1}{2}\int_{\Gamma_2} (\mathbf{u} \cdot \vec{\mathbf{n}})^{-}(\mathbf{u}_{ref} \cdot \Psi) + \int_{\Gamma_2} \sigma_{ref} \cdot \vec{\mathbf{n}} \cdot \Psi$$

Discrete Duality Finite Volume method

DDFV meshes

<u>Primal mesh</u>

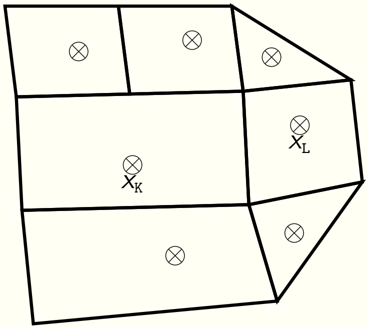
$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\kappa})_{\kappa \in \mathfrak{M}}$$

<u>Dual mesh</u>

$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\kappa^*})_{\kappa^* \in \mathfrak{M}^*}$$

<u>Diamond mesh</u>

$$\rightsquigarrow \nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}, \ \mathrm{p}^{\mathfrak{D}}$$



Our unknowns are:
$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

DDFV meshes

Primal mesh

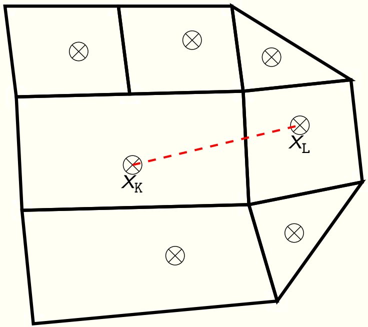
$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\kappa})_{\kappa \in \mathfrak{M}}$

Dual mesh

$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\kappa^*})_{\kappa^* \in \mathfrak{M}^*}$

Diamond mesh

$\rightsquigarrow \nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}, \; \mathrm{p}^{\mathfrak{D}}$



Our unknowns are:
$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

DDFV meshes

Primal mesh

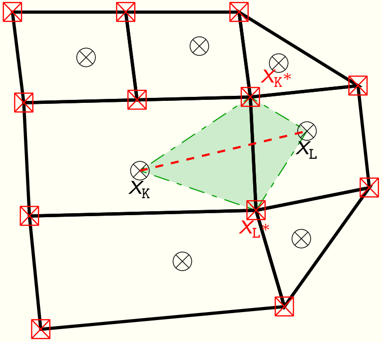
$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\mathrm{K}})_{\mathrm{K}\in\mathfrak{M}}$$

Dual mesh

$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\mathrm{K}^*})_{\mathrm{K}^*\in\mathfrak{M}^*}$$

Diamond mesh

$$\rightsquigarrow \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, \ \mathrm{p}^{\mathfrak{D}}$$



Our unknowns are:
$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

DDFV meshes

Primal mesh

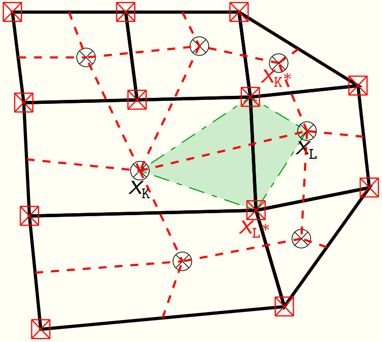
$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\kappa})_{\kappa \in \mathfrak{M}}$$

Dual mesh

$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\kappa^*})_{\kappa^* \in \mathfrak{M}^*}$$

Diamond mesh

$$\rightsquigarrow \nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}, \ \mathrm{p}^{\mathfrak{D}}$$



Our unknowns are:
$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

DDFV meshes

Primal mesh

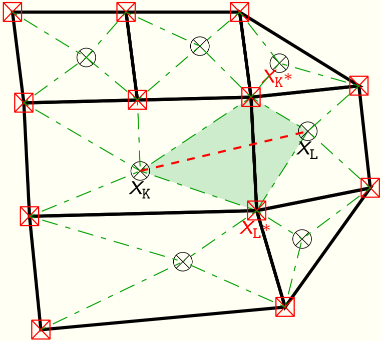
$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{K})_{K\in\mathfrak{M}}$$

Dual mesh

$$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{K^*})_{K^*\in\mathfrak{M}^*}$$

Diamond mesh

$$\rightsquigarrow \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, \ p^{\mathfrak{D}}$$



Our unknowns are:
$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $p^{\mathfrak{D}}$

DDFV operators

Discrete gradient

The operator $\nabla^{\mathfrak{D}} : (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ where

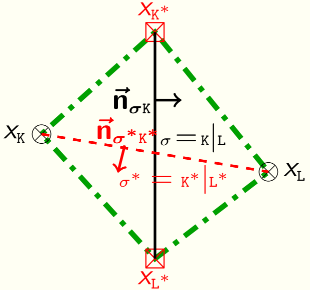
$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}(x_{\mathrm{L}} - x_{\mathrm{K}}) = \mathbf{u}_{\mathrm{L}} - \mathbf{u}_{\mathrm{K}},$$
$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}(x_{\mathrm{L}^*} - x_{\mathrm{K}^*}) = \mathbf{u}_{\mathrm{L}^*} - \mathbf{u}_{\mathrm{K}^*}.$$

$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_D}\left[m_\sigma(\mathbf{u}_{\mathrm{L}} - \mathbf{u}_{\mathrm{K}}) \otimes \vec{\mathbf{n}}_{\sigma\mathrm{K}} + m_{\sigma^*}(\mathbf{u}_{\mathrm{L}^*} - \mathbf{u}_{\mathrm{K}^*}) \otimes \vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*}\right].$$

[Krell '11]

$$\rightsquigarrow \mathrm{div}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \mathrm{Tr}(\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}).$$

$$\rightsquigarrow \mathrm{D}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \frac{\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} + {}^t\left(\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}\right)}{2}.$$



DDFV operators

Discrete divergence

$\mathbf{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ where:

$$\mathbf{div}^{\mathtt{K}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}}} \sum_{\sigma \subset \partial\mathtt{K}} m_\sigma \xi^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma\mathtt{K}}, \qquad \forall \mathtt{K} \in \mathfrak{M}$$

$$\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}^*}} \sum_{\sigma^* \subset \partial\mathtt{K}^*} m_{\sigma^*} \xi^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}, \qquad \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$$

Discrete duality property

- On the continuous level: $\int_\Omega \mathrm{div}\xi \cdot \mathbf{u} = -\int_\Omega \xi : \nabla \mathbf{u} + \int_{\partial\Omega} \xi \overrightarrow{\mathbf{n}} \cdot \mathbf{u}$
- On the discrete level:

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\mathbf{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}$$

[S. Krell, *Stabilized DDFV schemes for the incompressible Navier-Stokes equations*, 2011]

DDFV for Navier-Stokes problem

Outflow boundary conditions

Find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $\mathsf{p} : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div}(\sigma(\mathbf{u}, \mathsf{p})) = 0; & \text{in} \quad \Omega_T = \Omega \times [0, T] \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T, \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1 \times (0, T), \\ \mathbf{u} = 0 & \text{on} \quad \Gamma_0 \times (0, T), \\ \sigma(\mathbf{u}, \mathsf{p}) \cdot \vec{\mathbf{n}} + \dfrac{1}{2}(\mathbf{u} \cdot \vec{\mathbf{n}})^-(\mathbf{u} - \mathbf{u}_{ref}) = \sigma_{ref} \cdot \vec{\mathbf{n}} & \text{on} \quad \Gamma_2 \times (0, T) \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega \end{cases}$$

with $T > 0$, $\Omega \subset \mathbb{R}^2$, $\mathbf{u}_{init} \in (L^\infty(\Omega))^2$, $\mathbf{g}_1 \in (H^{\frac{1}{2}}(\Omega))^2$.

The stress tensor: $\sigma(\mathbf{u}, \mathsf{p}) = \dfrac{2}{\mathsf{Re}} \mathsf{D}\mathbf{u} - \mathsf{pId}$, with $\mathsf{Re} > 0$.

The strain rate tensor: $\mathsf{D}\mathbf{u} = \dfrac{(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})}{2}$.

Nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$

We construct:

- $m_{\mathtt{K}}\, \mathbf{b}^{\mathtt{K}}(\mathbf{u}^n, \mathbf{u}^{n+1}) \rightsquigarrow \int_{\mathtt{K}} (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}$ when $\mathtt{K} \in \mathfrak{M}$,
- $m_{\mathtt{K}^*}\, \mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^n, \mathbf{u}^{n+1}) \rightsquigarrow \int_{\mathtt{K}^*} (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}$ when $\mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_N^*$.

We focus on the primal mesh and we observe:

$$\boxed{\int_{\mathtt{K}} (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} = \sum_{\sigma \subset \partial \mathtt{K}} \int_{\sigma} (\mathbf{u}^n \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}})\mathbf{u}^{n+1}} \quad \forall \mathtt{K} \in \mathfrak{M}$$

We define the fluxes:

$$\int_{\sigma} (\mathbf{u}^n \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}) \rightsquigarrow F_{\sigma,\mathtt{K}}(\mathbf{u}^n)$$

Nonlinear convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$

We impose: $$F_{\sigma,\mathrm{K}}(\mathbf{u}^n) = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathrm{K}}\cap\mathcal{E}_{\mathrm{D}}} G_{\mathfrak{s},\mathrm{D}}(\mathbf{u}^n) & \text{if } \sigma\in\mathcal{E}_{int} \\ m_\sigma\gamma^\sigma(\mathbf{u}^n)\cdot\vec{\mathbf{n}}_{\sigma\mathrm{K}} & \text{if } \sigma\in\partial\Omega \end{cases},$$

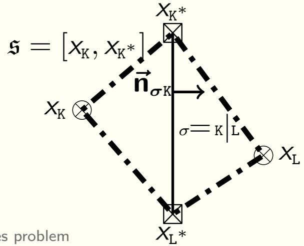
where

$$G_{\mathfrak{s},\mathrm{D}}(\mathbf{u}^n) = m_{\mathfrak{s}}\frac{\mathbf{u}^n_{\mathrm{K}}+\mathbf{u}^n_{\mathrm{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}} \rightsquigarrow \int_{\mathfrak{s}}\mathbf{u}^n\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}},$$

$$\gamma_\sigma(\mathbf{u}^n) = \frac{\mathbf{u}^n_{\mathrm{K}^*}+2\mathbf{u}^n_{\mathrm{L}}+\mathbf{u}^n_{\mathrm{L}^*}}{4} \quad \forall\sigma=[x_{\mathrm{K}^*},x_{\mathrm{L}^*}]\in\partial\mathfrak{M}.$$

We have conservativity:

$$F_{\sigma,\mathrm{K}} = -F_{\sigma,\mathrm{L}}, \quad \forall\sigma=\mathrm{K}|\mathrm{L}$$



Nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$

$$m_{\mathtt{K}}\, \mathbf{b}^{\mathtt{K}}(\mathbf{u}^n, \mathbf{u}^{n+1}) \rightsquigarrow \sum_{\sigma \subset \partial \mathtt{K}} \int_\sigma (\mathbf{u}^n \cdot \vec{\mathbf{n}}_{\sigma \mathtt{K}}) \mathbf{u}^{n+1}$$

So we define $\forall \mathtt{K} \in \mathfrak{M}$:

$$\boxed{m_{\mathtt{K}} \mathbf{b}^{\mathtt{K}}(\mathbf{u}^n, \mathbf{u}^{n+1}) = \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}^{int}} F_{\sigma,\mathtt{K}}(\mathbf{u}^n) \mathbf{u}_{\sigma^+}^{n+1} + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}^{ext}} F_{\sigma,\mathtt{K}}(\mathbf{u}^n) \gamma^\sigma(\mathbf{u}^{n+1})}$$

where

$$\mathbf{u}_{\sigma^+}^{n+1} = \begin{cases} \mathbf{u}_{\mathtt{K}}^{n+1} & \text{if } F_{\sigma,\mathtt{K}} \geq 0 \\ \mathbf{u}_{\mathtt{L}}^{n+1} & \text{otherwise} \end{cases} \qquad \forall \sigma \in \mathcal{E}_{int}$$

and

$$\gamma_\sigma(\mathbf{u}^{n+1}) = \frac{\mathbf{u}_{\mathtt{K}^*}^{n+1} + 2\mathbf{u}_{\mathtt{L}}^{n+1} + \mathbf{u}_{\mathtt{L}^*}^{n+1}}{4} \quad \forall \sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}] \in \partial \mathfrak{M}.$$

Variational formulation

$$\int_\Omega \partial_t \mathbf{u}\cdot\Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\Psi) + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$= -\frac{1}{2}\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})^+(\mathbf{u}\cdot\Psi) + \frac{1}{2}\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}_{ref}\cdot\Psi) + \int_{\Gamma_2}\sigma_{ref}\cdot\vec{\mathbf{n}}\cdot\Psi$$

$$\Downarrow$$

$$[[\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n,\mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}}$$
$$-\frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n,\Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} = -\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2}(F_{\sigma,\mathtt{K}}(\mathbf{u}^n))^+\gamma^\sigma(\mathbf{u}^{n+1})\gamma^\sigma(\Psi^{\mathfrak{T}})$$
$$+\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2}(F_{\sigma,\mathtt{K}}(\mathbf{u}^n))^-\gamma^\sigma(\mathbf{u}_{ref})\gamma^\sigma(\Psi^{\mathfrak{T}}) + \sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma^{\mathfrak{D}}_{\mathrm{ref}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}).$$

The scheme

- For all $\mathtt{K} \in \mathfrak{M}$:

$$m_{\mathtt{K}} \frac{\mathbf{u}_{\mathtt{K}}^{n+1} - \mathbf{u}_{\mathtt{K}}^{n}}{\delta t} - m_{\mathtt{K}} \mathbf{div}^{\mathtt{K}}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1}, \mathsf{p}^{n+1})) + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}^{int}} F_{\sigma,\mathtt{K}}(\mathbf{u}^n) \frac{\mathbf{u}_{\mathtt{K}}^{n+1} + \mathbf{u}_{\mathtt{L}}^{n+1}}{2}$$
$$+ \frac{1}{2} \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}^{ext}} F_{\sigma,\mathtt{K}}(\mathbf{u}^n) \left(\mathbf{u}_{\mathtt{K}}^{n+1} + \gamma^{\sigma}(\mathbf{u}^{n+1}) \right) = 0;$$

- For all $\sigma \in \partial_O$:

$$m_{\sigma} \sigma^{\mathtt{D}}(\mathbf{u}^{n+1}, \mathsf{p}^{n+1})) \vec{\mathbf{n}}_{\sigma\mathtt{L}} - \frac{1}{4} F_{\sigma,\mathtt{L}}(\mathbf{u}^n) \left(\mathbf{u}_{\mathtt{K}}^{n+1} - \mathbf{u}_{\mathtt{L}}^{n+1}\right)$$
$$= -\frac{1}{2} (F_{\sigma,\mathtt{L}}(\mathbf{u}^n))^{-} \left(\gamma^{\sigma}(\mathbf{u}^{n+1}) - \gamma^{\sigma}(\mathbf{u}_{ref}) \right) + m_{\sigma} (\sigma_{\mathrm{ref}}^{\mathtt{D}} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}});$$

- For all $\mathtt{D} \in \mathfrak{D}$:

$$\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{n+1}) = 0.$$

Well-posedness of the scheme

We obtain our scheme by projecting the discrete weak formulation on the DDFV mesh.

Well-posedness

Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies inf-sup stability condition. The scheme we obtain has a unique solution $(\mathbf{u}^{\mathfrak{T},[0,T]}, \mathsf{p}^{\mathfrak{D},[0,T]}) \in \left((\mathbb{R}^2)^{\mathfrak{T}}\right)^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$.

Proof of well-posedness

The scheme is equivalent to a system $Av = b$ at each time step, with A square matrix and $v = (\mathbf{u}^{n+1}, \mathsf{p}^{n+1})$.

$$\Downarrow$$

We want to show that the square matrix is injective, i.e.

$$Av = 0 \implies v = 0$$

Proof of well-posedness

If we multiply $Av = 0$ by a test function $\Psi^{\mathfrak{T}}$ that satisfies

$$\begin{cases} \Psi^{\mathfrak{T}} \in \mathbb{E}_0^D, \\ \mathrm{div}^{\mathfrak{D}}(\Psi^{\mathfrak{T}}) = 0, \end{cases}$$

this is equivalent to consider the discrete variational formulation in the form:

$$\frac{1}{\delta t}[[\mathbf{u}^{n+1}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}}$$
$$+\frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{n}, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{n}, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}}$$
$$= -\frac{1}{2}\sum_{\mathtt{D} \in \mathtt{D}_{ext} \cap \Gamma_2} (F_{\sigma,\mathtt{K}}(\mathbf{u}^{n}))^{+} \gamma^{\sigma}(\mathbf{u}^{n+1})\gamma^{\sigma}(\Psi^{\mathfrak{T}}).$$

Proof of well-posedness

If in this weak formulation:

$$\frac{1}{\delta t}[[\mathbf{u}^{n+1}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}}$$
$$+\frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{n}, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{n}, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}}$$
$$= -\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma,\mathtt{K}}(\mathbf{u}^{n}))^{+}\gamma^{\sigma}(\mathbf{u}^{n+1})\gamma^{\sigma}(\Psi^{\mathfrak{T}}).$$

we choose $\Psi^{\mathfrak{T}} = \mathbf{u}^{n+1}$, we obtain:

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 + \underbrace{\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}} (F_{\sigma,\mathtt{K}}(\mathbf{u}^{n}))^{+}\left|\gamma^{\sigma}(\mathbf{u}^{n+1})\right|^2}_{\geq 0} = 0,$$

Proof of well-posedness

We end up with:

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathsf{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 \leq 0$$

from which we deduce that $\boxed{\mathbf{u}^{n+1} = 0}$.
To conclude the proof, we need to show that p^{n+1} is equal to zero too. This is true because of the inf-sup stability, that ensures an inequality of the type:

$$\|\mathsf{p}^{n+1} - m(\mathsf{p}^{n+1})\|_2 \leq C\|\nabla^{\mathfrak{D}}\mathbf{u}^{n+1}\|_2,$$

where $m(\mathsf{p}^{n+1}) = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} p^{\mathtt{D}}$. This inequality implies that p^{n+1} is constant.
Thanks to boundary condition on Γ_2, we conclude $\boxed{\mathsf{p}^{n+1} = 0}$.

Discrete energy estimate

Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies inf-sup stability condition.
Let $(\mathbf{u}^{\mathfrak{T},[0,T]}, \mathrm{p}^{\mathfrak{D},[0,T]}) \in \left((\mathbb{R}^2)^{\mathfrak{T}}\right)^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$ be the solution of the DDFV scheme, where $\mathbf{u}^{\mathfrak{T},[0,T]} = \mathbf{v}^{\mathfrak{T},[0,T]} + \mathbf{u}^{\mathfrak{T}}_{ref}$.
For $N > 1$, there exists a constant $C > 0$, depending on $\Omega, \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{u}_0, \mathrm{Re}$ such that:

$$\sum_{j=0}^{N-1} \|\mathbf{v}^{j+1} - \mathbf{v}^{j}\|_2^2 \leq C, \quad \|\mathbf{v}^N\|_2^2 \leq C,$$

$$\sum_{j=0}^{N-1} \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^{j+1}\|_2^2 \leq C, \quad \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^N\|_2^2 \leq C,$$

$$\sum_{j=0}^{N-1} \delta t \sum_{\mathtt{D} \in \mathfrak{D}_{ext}} (F_{\sigma,\mathtt{K}}(\mathbf{v}^j + \mathbf{u}^{\mathfrak{T}}_{ref}))^+ (\gamma^{\sigma}(\mathbf{v}^{j+1}))^2 \leq C.$$

Sketch of the proof

1. Rewrite our discrete variational formulation for the unknown $\mathbf{v}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}_{ref}^{\mathfrak{T}}$:

$$
\begin{aligned}
&[[\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} \\
&\quad + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}}, \mathbf{v}^{n+1} + \mathbf{u}_{ref}^{\mathfrak{T}}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}}, \Psi^{\mathfrak{T}}), \mathbf{v}^{n+1} + \mathbf{u}_{ref}^{\mathfrak{T}}]]_{\mathfrak{T}} \\
&\qquad + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma,\mathtt{K}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}}))^+ \gamma^{\sigma}(\mathbf{v}^{n+1} + \mathbf{u}_{ref}^{\mathfrak{T}})\gamma^{\sigma}(\Psi^{\mathfrak{T}}) \\
&\qquad\qquad = -\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} F_{\sigma,\mathtt{K}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}})^- \gamma^{\sigma}(\mathbf{u}_{ref}^{\mathfrak{T}})\gamma^{\sigma}(\Psi^{\mathfrak{T}}).
\end{aligned}
$$

2. The second step consists into selecting $\Psi^{\mathfrak{T}} = \mathbf{v}^{n+1} = (\mathbf{v}^{n+1} + \mathbf{u}_{ref}^{\mathfrak{T}}) - \mathbf{u}_{ref}^{\mathfrak{T}}$ as a test function. It follows:

$$
\begin{aligned}
E =& [[\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t}, \mathbf{v}^{n+1}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_2} (F_{\sigma,\mathtt{K}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}}))^+ (\gamma^{\sigma}(\mathbf{v}^{n+1}))^2 \\
\leq& \left| \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}}, \mathbf{v}^{n+1}), \mathbf{u}_{ref}^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}}, \mathbf{u}_{ref}^{\mathfrak{T}}), \mathbf{v}^{n+1}]]_{\mathfrak{T}} \right| \\
&+ \left| \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_2} F_{\sigma,\mathtt{K}}(\mathbf{v}^n + \mathbf{u}_{ref}^{\mathfrak{T}})^- \gamma^{\sigma}(\mathbf{u}_{ref}^{\mathfrak{T}})\gamma^{\sigma}(\mathbf{v}^{n+1}) \right|
\end{aligned}
$$

3. Estimate the RHS.

Further results: Korn's and trace inequality

Korn's inequality

Let $\mathfrak{T}$ be a mesh that satisfies infsup stability condition. Then there exists $C > 0$ such that :

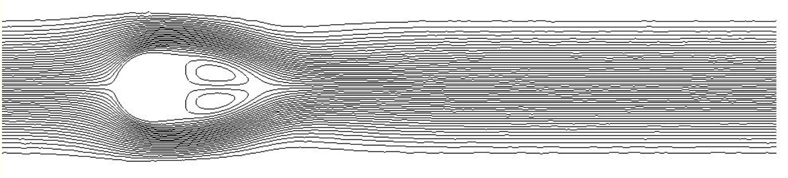
$$\boxed{\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2} \qquad \forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\mathtt{D}}$$

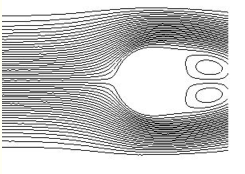
Trace theorem

Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $p > 1$, there exists a constant $C > 0$, depending only on p, $\sin(\alpha_{\mathfrak{T}})$, reg($\mathfrak{T}$) and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\mathtt{D}}$ and for all $s \geq 1$:

$$\boxed{\|\gamma(\mathbf{u}^{\mathfrak{T}})\|_{s,\partial\Omega}^{s} \leq C\|\mathbf{u}^{\mathfrak{T}}\|_{1,p}\|\mathbf{u}^{\mathfrak{T}}\|_{\frac{p(s-1)}{p-1}}^{s-1}}$$

Numerical results 1





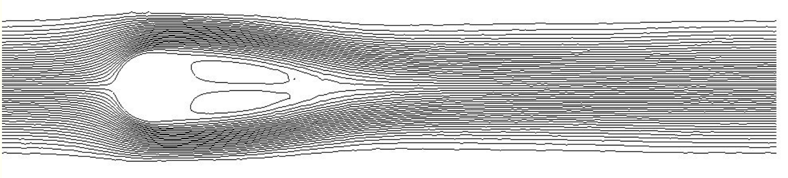
$$\mathbf{g}_1(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1$$

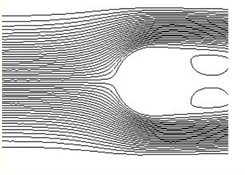
$$\mathbf{u}_{ref}(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_2$$

$$\sigma_{ref}(\mathbf{u},\mathrm{p}) \cdot \vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 6\eta(1-2y) \end{pmatrix} \qquad \text{on } \Gamma_2$$

with $T = 1.5s$, $\mathrm{Re} = 100$, $\eta = 4 \times 10^{-3}$.

Numerical results 1





$$\mathbf{g}_1(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1$$

$$\mathbf{u}_{ref}(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_2$$

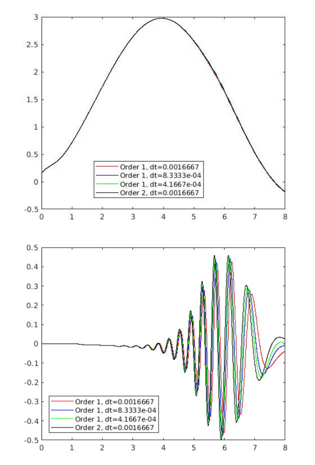
$$\sigma_{ref}(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 6\eta(1-2y) \end{pmatrix} \qquad \text{on } \Gamma_2$$

with $T = 3s$, $\mathrm{Re} = 100$, $\eta = 4 \times 10^{-3}$.

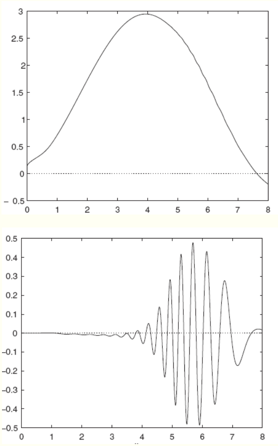
Numerical results 2

$$c_{d,max} = \color{red}{2.9}754$$
$$c_{l,max} = \color{blue}{0.4}4902$$

$$c_{d,max} = \color{red}{2.9}509$$
$$c_{l,max} = \color{blue}{0.4}7795$$



DDFV



Reference values [Volker, '04]

Conclusions and perspectives

Conclusions

- We built a well-posed DDFV scheme
- We proved a discrete energy estimate
- We obtained the expected numerical results

Perspectives

- investigate the choice of $\mathbf{u}_{ref}$ for higher Reynold's number
- **application to Domain Decomposition**

Grazie per l'attenzione!