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Méthode DDFV : applications en mécanique des fluides et décomposition des domaines

DDFV method : applications to fluid mechanics and domain decomposition

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Introduction

The aim of this thesis is to study and develop the theory of Discrete Duality Finite Volume method ("DDFV" for short) for problems arising in fluid mechanics (namely Stokes and Navier-Stokes problem). In particular, my interests focus:

- on the study of different type of boundary conditions, such as mixed Dirichlet/Neumann or outflow boundary conditions;
- on the coupling between DDFV method and the algorithm of domain decomposition.

This dissertation will start with a brief *excursus* on what is the DDFV method and the reason why we choose it. I will then give an idea of the general results of the thesis, by describing the problems we treated, how they have been developed and the main difficulties related to them.

DDFV method

This method enters the class of finite volume methods, that are discretization methods in which volume integrals are converted to surface integrals, using the divergence theorem. An important feature of those methods is that they are locally conservative; in fact, surface integrals are evaluated as fluxes at the boundary of each finite volume and they are conserved from one discretization cell to its neighbor. This makes finite volume methods quite attractive when modeling problems arising from fluid mechanics, the derivation of which are precisely based on local balance principles.

Finite volume methods for Stokes and Navier-Stokes problem have been widely studied during the years. Concerning the Stokes flow, we refer to [EHL06] for a colocated and stabilized finite volume scheme, [BEH05] for a staggered finite volume scheme, [DE08] for a mixed finite volume/ finite element scheme, [Del07] for an alternative DDFV scheme with a different localization of the unknowns, [DO15] for a DDFV scheme for the vorticity-velocity-pressure formulation.

Concerning the Navier-Stokes flow, we refer to [BCH00] for a fractional step method combined with finite volume schemes, [EHL07] for a colocated finite volume scheme, [EH05] for a staggered finite volume scheme, [Del07] for an alternative DDFV scheme with a different localization of the unknowns, [DE08, LS17] for a combined finite volume/ finite element scheme, [GHL10] for a finite volume scheme with explicit time discretization, [CCML17] for a high order finite volume scheme based on polynomial reconstruction.

DDFV method has been developed since the early 2000's; the DDFV schemes have been first introduced and studied in [Her00] and [DO05] to approximate Laplace equation on a large class of 2D meshes including non-conformal and distorted meshes.

A way to consider general families of meshes is to add some unknowns to the problem: we require unknowns on vertices, centers and edges of control volumes; for this reason, DDFV method works on (three) staggered meshes. From an initial mesh, called the "primal mesh" (denoted with $\mathfrak{M}\cup\partial\mathfrak{M}$), we construct the "dual mesh" (denoted with $\mathfrak{M}^*\cup\partial\mathfrak{M}^*$), that is centered on the vertices of the primal mesh, and the "diamond mesh" (denoted with $\mathfrak{D}$) , which is centered on the edges of the primal mesh; see Fig. 1 for an illustration. The union of primal and dual mesh will be denoted by $\mathfrak{T}$.

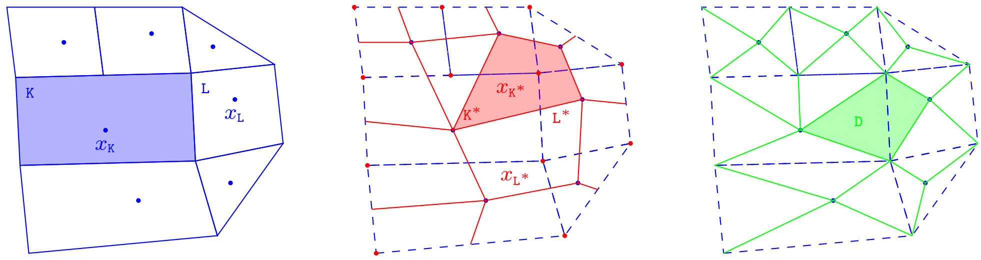


Fig. 1 DDFV meshes on a non conformal mesh: primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ (blue), dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ (red) and diamond mesh $\mathfrak{D}$ (green).

DDFV method for Stokes and Navier-Stokes problem leads naturally to locate the unknowns of velocity and pressure in different points; the velocity unknowns are associated to each primal and dual volume, while the pressure unknowns are located on each diamond. So DDFV enters the class of staggered methods. This is an usual technique for incompressible flows: one of the best known schemes is the MAC (Marker and Cell) scheme (see [HW65]), which is built for cartesian meshes; we can mention also [CR73] for triangular meshes. MAC schemes, as unknowns, consider the normal components of the velocity (located on the mesh edges) and the pressure (located at the centers of the cells); with DDFV, we generalize this schemes by considering all the components of the velocity and extending it to more general meshes. Moreover, it has been shown in [Kre10] that a DDFV scheme on a cartesian mesh is equivalent to two decoupled MAC schemes written on two different staggered meshes (except for the boundary).

With this kind of construction, DDFV has two important advantages:

1. it applies to general meshes, such as non-conformal and distorted meshes. It is useful, for instance, in the domain decomposition setting, where the subdomains can be meshed separately and non-conformal edges appear on the interface, or simply if one wants to locally refine the mesh or consider complex geometries;

2. it can reconstruct and mimic at the discrete level the dual properties of the continuous differential operators. In fact, thanks to this choice of unknowns and meshes, it is possible to obtain a full approximation of the gradient operator and to maintain the structure of the continuous problem, such as symmetry, that helps when dealing with nonlinearities.

Another important point is that the implementation of the method has the same difficulties compared to the classical finite volume schemes: the structure is in fact similar. Just to remark, all the simulations presented in this dissertation are done in a Fortran90 code.

DDFV method in 2D has been widely developed during the years; in the case of anisotropic scalar diffusion [DO05, Her00, Her03, BHK10a], convection-diffusion problems [CM10], Div-Rot systems [DDO07], Leray-Lions elliptic equations [ABH07, BH08], Stokes and Navier-Stokes problem [Del07, Kre11a, Kre11b, GKL17, GKL19], Maxwell [HLO08], and Cahn-Hilliard/Stokes phase field model [BN17].

Some works have been done even in 3D: for anisotropic linear diffusion problems [ABHK12, CPRT08, ABK08, Her07], for Leray-Lions elliptic equations [CH11], for Stokes problem [KG12].

This motivated us to extend and develop the theory on DDFV methods and this thesis specifically addresses the following issues

- mixed Dirichlet/Neumann boundary conditions for the Stokes problem
- outflow boundary condition for the Navier Stokes problem
- domain decomposition method for the Navier Stokes problem

In all three cases we establish a complete well-posedness theory of the discrete equations, and we perform a convergence analysis. The discussion relies on stability properties of the scheme (expressed by means of Inf-sup condition), and new functional inequalities (Korn's lemma, trace lemma...). Everytime, numerical test follow the theoretical analysis.

Stokes problem

We started our work by first considering the Laplace form of the Stokes system in a connected open bounded polygonal domain Ω of $\mathbb{R}^2$:

$$\begin{cases} -\Delta\mathbf{u}+\nabla\mathrm{p}=\mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u})=0 & \text{in } \Omega, \\ +\text{boundary conditions} & \text{on } \partial\Omega, \end{cases}$$

where the unknowns are the velocity $\mathbf{u}:\Omega\to\mathbb{R}^2$ and the pressure $\mathrm{p}:\Omega\to\mathbb{R}$; the data $\mathbf{f}\in(L^2(\Omega))^2$. The Stokes system is a PDEs system which arises in fluid mechanics: it is linear and its resolution is the preliminary step to handle more intricate models, like the evolution problem for the Navier-Stokes system. In the DDFV setting, this problem was studied in [Del07, Kre11a, BKN15] in the case of homogeneous Dirichlet boundary conditions, i.e.:

$$\mathbf{u}=0 \quad \text{on } \partial\Omega.$$

Our first goal was to extend the theory known for this problem to the case of mixed Dirichlet/ Neumann boundary conditions, i.e. to study the system:

$$\begin{cases} -\Delta\mathbf{u}+\nabla\mathrm{p}=\mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u})=0 & \text{in } \Omega, \\ \mathbf{u}=\mathbf{g} & \text{on } \Gamma_D, \\ (\nabla\mathbf{u}-\mathrm{pId})\,\vec{\mathbf{n}}=\Phi & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where the boundary of the domain Ω is split between $\partial\Omega=\Gamma_D\cup\Gamma_N$, $\Phi\,,\mathbf{g}\in(H^{\frac{1}{2}}(\partial\Omega))^2$ and $\vec{\mathbf{n}}$ is the unitary outer normal. We refer to [BF12] for the analysis of the continuous problem (1).

A first natural question is the well-posedness of (1): in the continuous case, this property is relied to the so-called Inf-sup stability inequality (or LBB). This will be a key point all along the dissertation; it is formulated as:

$$\inf_{\mathrm{p}\in L^2_0(\Omega)}\left(\sup_{\mathbf{v}\in(H^1_0(\Omega))^2}\frac{a(\mathbf{v},\mathrm{p})}{\|\mathbf{v}\|_{H^1}\|\mathrm{p}\|_{L^2}}\right)>0, \tag{2}$$

where $a(\mathbf{v},\mathrm{p})=\displaystyle\int_\Omega \mathrm{p}(\mathrm{div}(\mathbf{v}))$ and $L^2_0(\Omega)=\{\mathrm{p}\in L^2(\Omega): m(\mathrm{p})=\frac{1}{|\Omega|}\int_\Omega \mathrm{p}=0\}$; it is equivalent to the existence of a continuous right-inverse of the divergence operator (see [GR11, BF12]).

This inequality is satisfied by the continuous operators, cf [BBF08]; the analysis of numerical methods relies on the fact that the underlying discrete operators still satisfy an inequality of the same type. Thus, with the intention of writing a DDFV scheme for (1), it is necessary to understand if property (2) holds even in the discrete setting.

In the DDFV framework, we refer to [BKN15]; Inf-sup stability has been proven to hold unconditionally for conforming acute triangle meshes, non-conforming triangle meshes and chechkerboard meshes. For some conforming or non-conforming Cartesian meshes, it holds up to a single unstable pressure mode. Moreover, it has been proven numerically for many other families of meshes (it has still not been found a mesh that does not satisfy it).

Our first goal was to extend those results to the case of non-homogeneous Dirichlet boundary conditions (i.e. to the corresponding case of $\mathbf{v}\in(H^1_{\Gamma_D}(\Omega))^2$).

In the following, we denote by $\mathbf{v}^{\mathfrak{T}}$ the velocity unknowns on the centers and vertices of the mesh and $\mathrm{p}^{\mathfrak{D}}$ the pressure on the edges; $\nabla^{\mathfrak{D}}$ represents the discrete DDFV gradient operator and $\mathrm{div}^{\mathfrak{D}}$ the discrete DDFV divergence, both defined on the diamond mesh $\mathfrak{D}$, while $\mathbf{div}^{\mathfrak{T}}$ stands for the divergence on the primal and dual mesh (we recall that $\mathfrak{T}=\mathfrak{M}\cup\partial\mathfrak{M}\cup\mathfrak{M}^*\cup\partial\mathfrak{M}^*$). The space $\mathbb{E}_0^{\Gamma_D}$ corresponds to homogeneous Dirichlet boundary condition on Γ_D. By defining a scalar product on the primal and dual mesh, denoted by $[[\cdot,\cdot]]_{\mathfrak{T}}$, and one on the diamond mesh, $(\cdot,\cdot)_{\mathfrak{D}}$, we can deduce some discrete L^p norms, $\|\cdot\|_p$. We refer to Chap. I for all the detailed definitions.

The extension of the discrete Inf-sup condition to the non-homogeneous Dirichlet case reads:

Theorem 1 *For a given DDFV mesh $\mathfrak{T}$ that satisfies Inf-sup stability (see Def.I.6.1), there exists $\widetilde{\beta}_{\mathfrak{T}}$ such that:*

$$\widetilde{\beta}_{\mathfrak{T}}:=\inf_{\mathrm{p}^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}}\left(\sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0^{\Gamma_D}}\frac{a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2\|\mathrm{p}^{\mathfrak{D}}-m(\mathrm{p}^{\mathfrak{D}})\|_2}\right)>0, \tag{3}$$

where $a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})=(\mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}$ and $m(\mathrm{p}^{\mathfrak{D}})=\displaystyle\sum_{\mathtt{D}\in\mathfrak{D}}m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}}$.

This theorem ensures that $\forall \mathrm{p}^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}$:

$$\|\mathrm{p}^{\mathfrak{D}}-m(\mathrm{p}^{\mathfrak{D}})\|_2\leq\frac{1}{\widetilde{\beta}^2_{\mathfrak{T}}}\sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0^{\Gamma_D}}\frac{(\mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}}{\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2}; \tag{4}$$

this inequality is a useful tool to prove the wellposedness of the DDFV scheme associated to (1).

One can object that supposing Inf-sup stability property on the mesh can be restrictive, since it has not been proven uniformly for all meshes; there is a way to avoid this hypothesis, that is to stabilize the mass conservation equation. It can be done either by adding a linear stabilization or a stabilization term $\Delta^{\mathfrak{D}}$ inspired by the Brezzi-Pitkäranta method [BP84] in the finite element

framework. This latter strategy has been previously used in the finite volume framework by [EHL06, EHL07]; in particular, in the DDFV framework, it was proposed by the author of [Kre11a]. See Sec. I.7 for more details.
Motivated by the work in [Kre11a], we decided to adopt this approach when discretizing (1).

To obtain our scheme, we decided to integrate the momentum equation over all $\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}_N^*$. Thanks to the definition of the DDFV discrete operators, this is equivalent to replace the continuous with the corresponding discrete ones. We impose Dirichlet boundary conditions on $\partial\mathfrak{M}_D \cup \partial\mathfrak{M}_D^*$ and Neumann boundary conditions on $\partial\mathfrak{M}_N$. The mass conservation equation is directly approximated on the diamond mesh equation over $\mathfrak{D}$, and it is stabilized through two parameters $\beta \geq 0$, associated to the stabilization of Brezzi-Pitkäranta (see Sec. I.7) and $\mu \geq 0$, associated to a linear stabilization.

If $\mathbf{g}_\sigma, \Phi_\sigma$ denote the projection of the Dirichlet and Neumann data on the mesh, and h is the mesh size, the scheme we obtain is the following:

$$\text{Find } \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_{\mathbf{g}}^{\Gamma_D} \text{ and } \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that}$$

$$\begin{cases} \mathbf{div}^{\kappa}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\kappa} & \forall \kappa \in \mathfrak{M} \\ \mathbf{div}^{\kappa^*}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\kappa^*} & \forall \kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_N^* \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) + \mu h \mathrm{p}^{\mathfrak{D}} - \beta h^2 \Delta^{\mathfrak{D}} \mathrm{p}^{\mathfrak{D}} = 0 & \\ (\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathrm{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\kappa} = \Phi_\sigma & \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N. \end{cases} \tag{5}$$

The scheme (5) is well-posed on general meshes if $\mu + \beta > 0$ (see Thm. II.2.2). Moreover, if $\mu > 0$, we obtain a first rough estimate (see Thm. II.3.1) of order 0.5 for the velocity only; in order to obtain order 1, it is necessary to use the Brezzi-Pitkäranta stabilization (i.e. if we suppose $\beta > 0$). We first need to prove the stability result of Thm. 2. We point out that the number reg($\mathfrak{T}$), that will appear in the estimates, measures the regularity of the mesh and it is uniformly bounded as $h \to 0$.

Theorem 2 (Stability of the scheme) *Suppose that $\beta > 0$. There exist two constants $C_1, C_2 > 0$, depending only on Ω, β and reg($\mathfrak{T}$), such that for every pair $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_0^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$ with*

$$(-\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathrm{D}} Id)\vec{\mathbf{n}}_{\sigma\kappa} = \Phi_\sigma \quad \forall \sigma \in \Gamma_N,$$

there exist $\widetilde{\mathbf{u}}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and $\widetilde{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}}\|_2^2 + \|\widetilde{p}^{\mathfrak{D}}\|_2^2 \leq C_1(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \|\mathrm{p}^{\mathfrak{D}}\|_2^2), \tag{6}$$

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \|\mathrm{p}^{\mathfrak{D}}\|_2^2 \leq C_2 \left(B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathfrak{T}}, \widetilde{p}^{\mathfrak{D}}) + \left| \sum_{\mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma \Phi_\sigma \gamma^\sigma(\widetilde{\mathbf{u}}^\tau) \right| + \|\Phi_\sigma\|_2^2 \right), \tag{7}$$

where $\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})$ is a trace term and B is the bilinear form associated to (5)*:*

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \widetilde{\mathbf{u}}^{\mathfrak{T}}, \widetilde{p}^{\mathfrak{D}}) := [[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}), \widetilde{\mathbf{u}}^{\mathfrak{T}}]]_{\mathfrak{T}} \\ + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) + \mu h \mathrm{p}^{\mathfrak{D}} - \beta h^2 \Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}}, \widetilde{p}^{\mathfrak{D}})_{\mathfrak{D}}. \tag{8}$$

From this, we deduce the estimate of order 1 for the velocity, its gradient and the pressure:

Theorem 3 *We suppose that the solution of* (1) *satisfies* $(\mathbf{u}, p) \in (W^{2,\infty}(\mathfrak{D}))^2 \times W^{1,\infty}(\mathfrak{D})$. *Let* $\beta > 0$ *and* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ *be the solution of* (5). *Then there exists a constant* $C > 0$ *that depends on* reg$(\mathfrak{T}), \mu, \beta, \|\mathbf{u}\|_{W^{2,\infty}}$ *and* $\|p\|_{W^{1,\infty}}$ *such that*

$$\|\mathbf{u} - \mathbf{u}^{\mathfrak{T}}\|_2 + \|\nabla\mathbf{u} - \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq Ch,$$
$$\|p - \mathrm{p}^{\mathfrak{D}}\|_2 \leq Ch.$$

In the idea of moving to the discretization of the Navier Stokes problem, the second problem we considered is the Stokes problem in the divergence form:

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) = \mathbf{f} & \text{in} \quad \Omega, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on} \quad \Gamma_D, \\ \sigma(\mathbf{u}, \mathrm{p})\vec{\mathbf{n}} = \Phi & \text{on} \quad \Gamma_N. \end{cases} \tag{9}$$

The stress tensor is $\sigma(\mathbf{u}, \mathrm{p}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$, with $\mathrm{Re} > 0$. In particular, the strain rate tensor is defined by the symmetric part of the velocity gradient $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} +^t \nabla\mathbf{u})$.

The main difficulty when dealing with this kind of problem is the so-called Korn inequality (see [BS07]), that relates the gradient with the strain rate tensor; in the continuous case it is formulated as:

$$\|\mathbf{u}\|_{(H^1(\Omega))^2} \leq C\|\mathrm{D}\mathbf{u}\|_{(L^2(\Omega))^{2\times 2}}.$$

In the DDFV setting, the equivalent discrete theorem was proved in [Kre10] in the case of homogeneous Dirichlet boundary conditions; in this case, the proof relies on the definition of the operators and thus the constant of the estimate can be explicitly computed.

If we add a part of the boundary with non-zero data, we introduce some difficulties: as in the continuous setting, we are able to prove the extension to inhomogeneous Dirichlet boundary condition by contradiction. So we proved the following theorem:

Theorem 4 *(Korn's inequality) Let* $\mathfrak{T}$ *be a mesh that satisfies Inf-sup stability condition. Then there exists* $C > 0$, *that depends only on* reg$(\mathfrak{T})$, *such that :*

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \qquad \forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}.$$

See Sec. I.8.2 for more details. It is important to mention that this theorem holds under the Inf-sup stability condition on the mesh; this is why it becomes superfluous to stabilize the mass conservation equation when working in the divergence form of Stokes problem (or Navier-Stokes problem).

The DDFV scheme corresponding to (9) then reads:

$$\text{Find } \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_{\mathbf{g}}^{\Gamma_D} \text{ and } \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that:}$$

$$\begin{cases} -\mathbf{div}^{\mathtt{K}}\left(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\right) = \mathbf{f}_{\mathtt{K}} & \forall \mathtt{K} \in \mathfrak{M} \\ -\mathbf{div}^{\mathtt{K}^*}\left(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\right) = \mathbf{f}_{\mathtt{K}^*} & \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_N^* \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) = 0 & \\ \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} = \Phi_\sigma & \forall \mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N, \end{cases} \tag{10}$$

with the discrete stress tensor defined by $\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathfrak{D}}\mathrm{Id}$, and the discrete strain rate tensor by $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \dfrac{\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} +^{t} (\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})}{2}$. This scheme is well-posed under the Inf-sup hypothesis on the mesh (see Thm. II.5.2). Moreover, all the results on the error estimates established for (5) can be extended to (10) thanks to Korn's inequality and the relation between the strain rate tensor and the gradient, illustrated in Sec. I.8.2.

At last, we extended (5) to the case of weak boundary conditions. In fact, if we decide not to stabilize the incompressibility constraint, problem (5) is well posed under the hypothesis that the mesh satisfies Inf-sup inequality; this inequality, in the simplest case of conformal square meshes, is valid up to an unstable mode for the pressure (see [BKN15]). A way of avoiding this inconvenient is thus to impose boundary conditions in "a weak sense"; the details can be found in Sec. II.6.

In Sec. II.4, II.5.2, II.6.2 we numerically tested (5), (10) and the formulation with weak boundary conditions on different meshes, by showing the convergence properties of the schemes and the influence of the stabilization parameters β, μ for (5).

Navier-Stokes problem

The step forward to our work was the study of the following 2D unsteady incompressible Navier-Stokes problem:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u},\mathrm{p})) = 0 & \text{in} \quad \Omega_T = \Omega \times [0,T], \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T, \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1 \times (0,T), \\ \sigma(\mathbf{u},\mathrm{p})\,\vec{\mathbf{n}} + \dfrac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{ref}) = \sigma_{ref}\,\vec{\mathbf{n}} & \text{on} \quad \Gamma_2 \times (0,T). \end{cases} \tag{11}$$

This problem arises when computing a flow whose velocity is prescribed at one part of the boundary and it flows freely on the other one. In this framework, we are often required to truncate the physical domain to obtain a reduced computational domain, either because we want to save computational ressources or because the physical domain is unbounded. We illustrate this setting in Fig. 2.

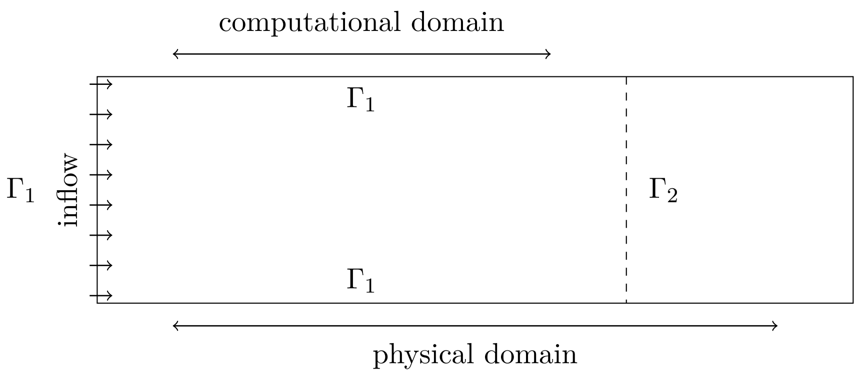


Fig. 2 Domain and notations.

This raises the question on what type of boundary conditions one should impose on the "artificial frontier", denoted by Γ_2; we choose to adopt the ones proposed in [BF94] and then further studied in [BF94, BF12]. Other techniques where proposed in the literature, we can mention for instance [HS89]; here, an artificial boundary condition is proposed for the Navier-Stokes problem under the hypothesis of small viscosity. The method consists into the approximation of the transparent boundary conditions, since they are non local.

We remark that in order to build the outflow boundary condition on Γ_2:

$$\sigma(\mathbf{u}, \mathrm{p})\,\vec{\mathbf{n}} + \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^{-}(\mathbf{u}-\mathbf{u}_{ref}) = \sigma_{ref}\,\vec{\mathbf{n}}, \tag{12}$$

we need to choose some *reference flow* $\mathbf{u}_{ref}$, which is any $\mathbf{u}_{ref} \in (H^1(\Omega))^2$ such that $\mathbf{u}_{ref} = \mathbf{g}_1$ on Γ_1, chosen so as to be a reasonable approximation of the expected flow near Γ_2, and a *reference stress tensor* σ_{ref} such that $\sigma_{ref}\,\vec{\mathbf{n}} \in (H^{-\frac{1}{2}}(\Omega))^2$. The choice of the reference flow is delicate and it will be widely discussed in Sec. III.6.

This nonlinear condition (12) is physically meaningful: if the flow is outward, we impose the constraint coming from the selected reference flow; if it is inward, we need to control the increase of energy, so we add a term that is quadratic with respect to velocity.

The analysis of the Navier-Stokes problem with the outflow boundary condition (12) is performed in [BF96] and [BF07] for the continuous equations and simulations are performed in [BF94] by the use of Finite Differences schemes in the case of Cartesian meshes. Within the framework of DDFV methods, we are able to reproduce those simulations by extending to the case of general meshes and we also offer a complete analysis of the discrete problem.

The first difficulty we faced when trying to write a DDFV scheme for (11) is the discretization of the nonlinear convection term: we will not detail the computations here, but we point out that it is necessary to construct the bilinear form $\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})$ and the numerical flux $F_{\sigma\mathrm{K}}$.

The bilinear form is built in order that $[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}}$ discretizes $\displaystyle\int_\Omega (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$, which is the trilinear form that appears when considering the variational formulation of (11); in particular, $\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})$ approaches $\int_\sigma (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{v}$ while the flux $F_{\sigma\mathrm{K}}$ is an approximation of $\int_\sigma \mathbf{u}\cdot\vec{\mathbf{n}}$. See Sec. III.1 for more details. We just state the following bound of the nonlinear convection term that we obtained, useful in order to prove the discrete energy estimate:

Proposition 5 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}, \mathbf{w}^{\mathfrak{T}}) \in \mathbb{E}_{\mathbf{g}_1}^{\Gamma_1} \times \mathbb{E}_{\mathbf{g}_1}^{\Gamma_1} \times \mathbb{E}_{\mathbf{g}_1}^{\Gamma_1}$, there exists a constant $C > 0$ that depens only on Ω and reg$(\mathfrak{T})$ such that:*

$$\begin{aligned}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} \leq & C\left(\|\mathbf{u}^{\mathfrak{T}}\|_3 + \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{3,\partial\Omega}\right)\|\mathbf{v}^{\mathfrak{T}}\|_6\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \\ & + C\,\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega}\|\widetilde{\gamma}^{\mathfrak{T}}(\mathbf{w}^{\mathfrak{T}})\|_{4,\partial\Omega}.\end{aligned}$$

where $\gamma^{\mathfrak{T}}, \widetilde{\gamma}^{\mathfrak{T}}$ are trace operators.

To write the DDFV scheme associated to (13), we choose to use an implicit Euler time discretization, except for the nonlinear term, which is linearized by using a semi-implicit approximation.
Let $N \in \mathbb{N}^*$, we note $\delta t = \frac{T}{N}$ and $t_n = n\delta t$ for $n \in \{0, \ldots, N\}$; we look for $\mathbf{u}^{\mathfrak{T},[0,T]} = (\mathbf{u}^n)_{n\in\{0,\ldots N\}} \in (\mathbb{E}_{\mathbf{g}_1}^{\Gamma_1})^{N+1}$ and $\mathrm{p}^{\mathfrak{D},[0,T]} = (\mathrm{p}^n)_{n\in\{0,\ldots N\}} \in (\mathbb{R}^{\mathfrak{D}})^{N+1}$. To simplify the notations we will denote $(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})$ with $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ and $(\mathbf{u}^n, \mathrm{p}^n)$ with $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$ that at each time step are known.

Then, our first guess for the scheme was to naively replace the continuous operators with the discrete ones (as done previously for Stokes). This would have given the following scheme:

$$\text{Find } \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_{\mathbf{g}_1}^{\Gamma_D} \text{ and } \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that:}$$

$$\begin{cases} m_{\mathtt{K}} \dfrac{\mathbf{u}_{\mathtt{K}} - \bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} - m_{\mathtt{K}} \mathbf{div}^{\mathtt{K}}(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})) + m_{\mathtt{K}} \mathbf{b}^{\mathtt{K}}(\bar{\mathbf{u}}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}) = 0 & \forall \mathtt{K} \in \mathfrak{M}, \\ m_{\mathtt{K}^*} \dfrac{\mathbf{u}_{\mathtt{K}^*} - \bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} - m_{\mathtt{K}^*} \mathbf{div}^{\mathtt{K}^*}(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})) + m_{\mathtt{K}^*} \mathbf{b}^{\mathtt{K}^*}(\bar{\mathbf{u}}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}) = 0 & \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_2^*, \\ m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \vec{\mathbf{n}}_{\sigma\mathtt{L}} + \dfrac{1}{2}(F_{\sigma\mathtt{L}}(\bar{\mathbf{u}}^{\mathfrak{T}}))^- (\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) - \gamma^\sigma(\mathbf{u}_{ref})) = m_\sigma \sigma_{\mathrm{ref}}^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma\mathtt{K}} & \forall \mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_2, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) = 0. \end{cases} \tag{13}$$

Unfortunately, we were not able to prove the wellposedness of (13) with the classical techniques. We thus had to change our strategy, by going back to the continuous problem: in fact, as presented in [BF12], the velocity $\mathbf{u}$ satisfies:

$$\begin{aligned} \int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{2}{\mathrm{Re}} \int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\Psi) + \frac{1}{2} \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Psi - \frac{1}{2} \int_\Omega (\mathbf{u} \cdot \nabla)\Psi \cdot \mathbf{u} \\ = -\frac{1}{2} \int_{\Gamma_2} (\mathbf{u} \cdot \vec{\mathbf{n}})^+ (\mathbf{u} \cdot \Psi) + \frac{1}{2} \int_{\Gamma_2} (\mathbf{u} \cdot \vec{\mathbf{n}})^- (\mathbf{u}_{ref} \cdot \Psi) + \int_{\Gamma_2} (\sigma_{ref} \vec{\mathbf{n}}) \cdot \Psi, \end{aligned} \tag{14}$$

where Ψ is a test function in the space

$$V = \{\Psi \in (H^1(\Omega))^2,\ \Psi|_{\Gamma_1} = 0,\ \mathrm{div}(\Psi) = 0\}.$$

If now we rewrite this weak formulation (14) in the DDFV framework we obtain:

$$\begin{aligned} [[\frac{\mathbf{u}^{\mathfrak{T}} - \bar{\mathbf{u}}^{\mathfrak{T}}}{\delta t}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\bar{\mathbf{u}}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\bar{\mathbf{u}}^{\mathfrak{T}}, \Psi^{\mathfrak{T}}), \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} \\ = -\frac{1}{2} \sum_{\mathtt{D} \in \mathtt{D}_{ext} \cap \Gamma_2} (F_{\sigma\mathtt{K}}(\bar{\mathbf{u}}^{\mathfrak{T}}))^+ \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}) + \frac{1}{2} \sum_{\mathtt{D} \in \mathtt{D}_{ext} \cap \Gamma_2} (F_{\sigma\mathtt{K}}(\bar{\mathbf{u}}^{\mathfrak{T}}))^- \gamma^\sigma(\mathbf{u}_{ref}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}) \\ + \sum_{\mathtt{D} \in \mathtt{D}_{ext} \cap \Gamma_2} m_\sigma (\sigma_{\mathrm{ref}}^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma\mathtt{K}}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}), \end{aligned} \tag{15}$$

where $\Psi^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ is a test function in the discrete space that satisfies similar properties compared to the continuous test function Ψ:

$$\begin{cases} \Psi^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}, \\ \mathrm{div}^{\mathfrak{D}}(\Psi^{\mathfrak{T}}) = 0. \end{cases} \tag{16}$$

At this point, we can project (15) on the meshes to obtain the scheme; we look for $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_{\mathbf{g}_1}^{\Gamma_D}$ and $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

- For all $\mathtt{K} \in \mathfrak{M}$:

$$\begin{aligned} m_{\mathtt{K}} \frac{\mathbf{u}_{\mathtt{K}} - \bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} - m_{\mathtt{K}} \mathbf{div}^{\mathtt{K}}(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})) + \frac{1}{2} m_{\mathtt{K}} \mathbf{b}^{\mathtt{K}}(\bar{\mathbf{u}}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}) \\ - \frac{1}{2} \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{int}} \left(F_{\sigma\mathtt{K}}^+(\bar{\mathbf{u}}^{\mathfrak{T}}) \mathbf{u}_{\mathtt{K}} - F_{\sigma\mathtt{L}}^-(\bar{\mathbf{u}}^{\mathfrak{T}}) \mathbf{u}_{\mathtt{L}} \right) = 0, \end{aligned} \tag{17}$$

- For all $\mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_2^*$:

$$m_{\mathtt{K}^*}\frac{\mathbf{u}_{\mathtt{K}^*}-\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} - m_{\mathtt{K}^*}\mathbf{div}^{\mathtt{K}^*}(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})) + \frac{1}{2}m_{\mathtt{K}^*}\mathbf{b}^{\mathtt{K}^*}(\bar{\mathbf{u}}^{\mathfrak{T}},\mathbf{u}^{\mathfrak{T}})$$
$$-\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}\left(F^+_{\sigma^*\mathtt{K}^*}(\bar{\mathbf{u}}^{\mathfrak{T}})\mathbf{u}_{\mathtt{K}^*} - F^-_{\sigma^*\mathtt{L}^*}(\bar{\mathbf{u}}^{\mathfrak{T}})\mathbf{u}_{\mathtt{L}^*}\right) = 0, \quad (18)$$

- For all $\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_2$:

$$m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma\mathtt{L}} + \frac{1}{2}(F_{\sigma\mathtt{L}}(\bar{\mathbf{u}}^{\mathfrak{T}}))^-(\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) - \gamma^\sigma(\mathbf{u}_{ref})) - \frac{1}{4}F_{\sigma\mathtt{L}}(\bar{\mathbf{u}}^{\mathfrak{T}})\,(\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}})$$
$$= m_\sigma\sigma^{\mathtt{D}}_{\mathrm{ref}}\vec{\mathbf{n}}_{\sigma\mathtt{K}}, \quad (19)$$

- For all $\mathtt{D} \in \mathfrak{D}$:

$$\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) = 0, \quad (20)$$

of which we can prove that there exists a unique solution (see Thm III.3.1). Remark that in the scheme (17)-(20) the anti-symmetrization of the convection term is taken into account, and that in (19) there is an additional term with respect to the equation on $\sigma \in \partial\mathfrak{M}_2$ in (13), due to the projection of the boundary terms in (15).

The open boundary condition (12) is derived from the weak formulation (14) which ensures an energy estimate, as presented in [BF96]; so we proved a discrete version of the energy estimate. In order to do so, it is necessary to consider the variational formulation (15) and select the solution as a test function. Since the solution $\mathbf{u}^{\mathfrak{T},[0,T]}$ is not zero on the Dirichlet boundary Γ_1, it does not satisfy the hypothesis (16). We decompose it as $\mathbf{u}^{\mathfrak{T},[0,T]} = \mathbf{v}^{\mathfrak{T},[0,T]} + \mathbf{u}^{\mathfrak{T}}_{ref}$ so that, thanks to the definition of $\mathbf{u}^{\mathfrak{T}}_{ref}$ (see (III.12)) , $\mathbf{v}^{\mathfrak{T},[0,T]}$ is a good candidate to be the test function.

Theorem 6 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies Inf-sup stability condition. Let $(\mathbf{u}^{\mathfrak{T},[0,T]}, p^{\mathfrak{D},[0,T]}) \in (\mathbb{E}^{\Gamma_1}_{\mathbf{g}_1})^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$ be the solution of the DDFV scheme* (17)-(20) *, where $\mathbf{u}^{\mathfrak{T},[0,T]} = \mathbf{v}^{\mathfrak{T},[0,T]} + \mathbf{u}^{\mathfrak{T}}_{ref}$.*
For $N > 1$, there exists a constant $C > 0$, depending on $\Omega, reg(\mathfrak{T}), \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{u}_0, Re$ and T such that:

$$\sum_{j=0}^{N-1}\|\mathbf{v}^{j+1}-\mathbf{v}^j\|_2^2 \le C, \quad \|\mathbf{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1}\delta t\frac{1}{Re}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{j+1}\|_2^2 \le C, \quad \delta t\frac{1}{Re}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1}\delta t\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma\mathtt{K}}(\mathbf{v}^j + \mathbf{u}^{\mathfrak{T}}_{ref}))^+|\gamma^\sigma(\mathbf{v}^{j+1})|^2 \le C.$$

To prove this result, it is mandatory to prove the following trace inequality:

Theorem 7 (Trace inequality) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, depending only on p,q, $\sin(\alpha_{\mathfrak{T}})$, reg($\mathfrak{T}$) and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and for all $s \ge 1$, $p > 1$:*

$$\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|^s_{s,\partial\Omega} \le C\|\mathbf{u}^{\mathfrak{T}}\|_{1,p}\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{p(s-1)}{p-1}}.$$

where $\sin(\alpha_{\mathfrak{T}})$ is a measure of the flattening of the mesh diamonds.

We want to point out that all these results are proved, for simplicity, in the case of a constant viscosity; they could be extended to the case of variable viscosity, by starting from the works of

[Kre11b, BF07].

To give an idea of application of the DDFV scheme we built to approximate (11), we refer to the numerical simulation illustrated in Fig. 3; the goal is to show that by adding an artificial boundary, thanks to condition (12), we do not introduce any perturbation to the flow. For this purpose, we first consider an original domain that we cut into smaller subdomains and we draw the streamlines of the respective solution. We observe that the recirculations are well located and that there is no spurious vortices. For more details on this test case and for further simulations, we refer to Sec. III.6.

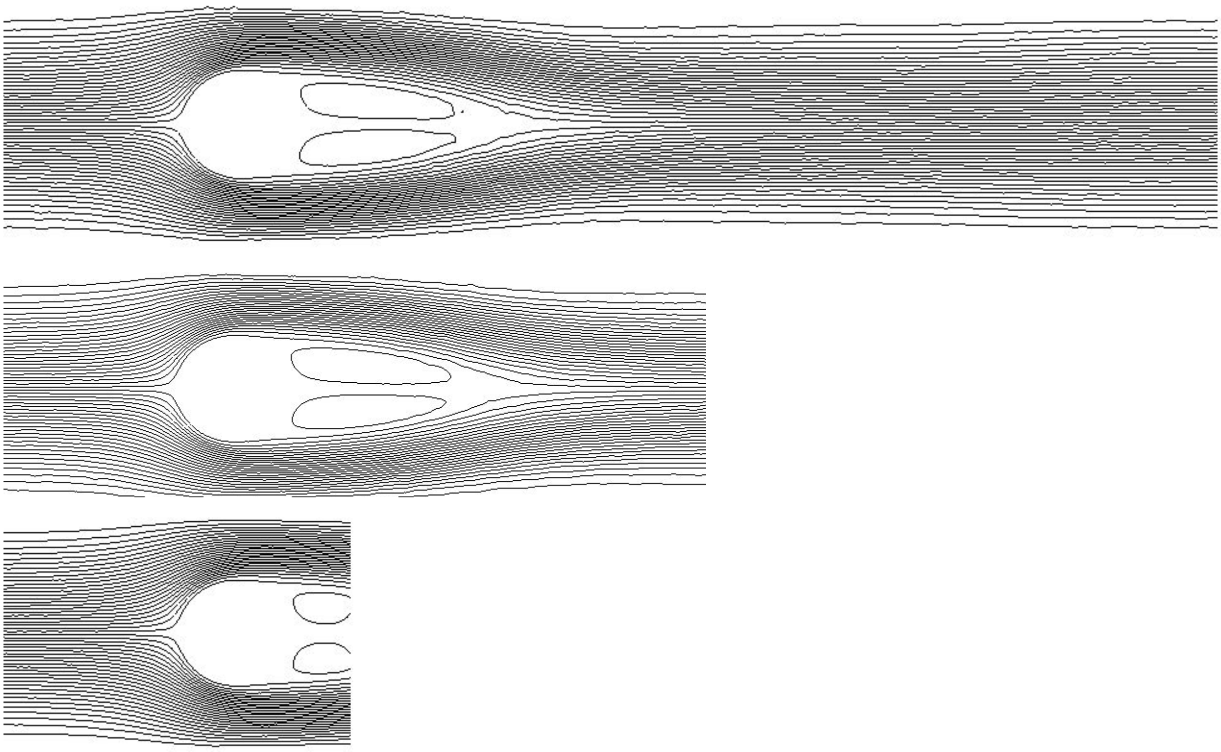


Fig. 3 Streamline of Test case 2 at $T = 3.5$, Re $= 250$. On the top: $\Omega = [0, 5] \times [0, 1]$, NbCell= 12118. In the middle: $\Omega' = [0, 3] \times [0, 1]$, NbCell=8636. On the bottom: $\Omega'' = [0, 1.5] \times [0, 1]$, NbCell=6534.

Domain decomposition method

Our next goal was to design a non-overlapping Schwarz algorithm for the Navier-Stokes problem. It is an iterative method that enters the class of domain decomposition methods, in which a domain is decomposed into smaller subdomains. The main advantage is that, contrary to direct methods, decomposition methods are naturally parallel; in fact, subdomains problems are related by some transmission conditions on the interface, but they are decoupled by the iterative procedure. This makes those methods interesting for high performance computing perspectives.

The main difficulty when dealing with the decomposition is that we introduce an "artificial" interface between the subdomains; it is important to understand what type of condition to impose in order to be able to recover the solution on the entire domain.

It was shown in 1990 by P.L.Lions [Lio90] that, with Fourier (i.e. Robin) transmission conditions, Schwarz algorithm for the Laplace operator converges even without overlap between subdomains. This method has been adapted to the discrete case for many problems of isotropic diffusion, [AJNM02, CHH04, GJMN05], for advection-diffusion-reaction problems, [GH07, HH14] and for anisotropic diffusion in a DDFV discretization, [BHK10b, GHHK18]. When moving to the Navier-Stokes problem, in the literature we can find many different approaches, in particular our focus is on the different design of the interface conditions. In the spirit of [HS89], [BCR16] derives optimal transparent boundary conditions for the Stokes equation, result of the discretization in time of the Navier-Stokes equation; these conditions are tested in the finite differences setting. In the finite element setting, [LMO01] proposes a non-overlapping domain decomposition algorithm of Robin–Robin type for the discretized Oseen equations (i.e. linearized Navier-Stokes); the transmission conditions they impose are equivalent to the ones that we finally chose, but in their case it was necessary to prove a modified Inf-sup condition whose stability constant depends on the Reynolds number: we will avoid this inconvenient by imposing a new condition for the pressure on the interface. In [XCL05], in the finite element setting, the authors build a Dirichlet-Neumann domain decomposition method for the nonlinear steady Navier-Stokes equations, under the hypothesis that the Reynolds number is sufficiently small and [GRW05] studies a family of discontinuous Galerkin finite element methods for Stokes and Navier-Stokes problems on triangular meshes and, as in [LMO01], they need to modify the Inf-sup condition in order to mantain the zero-divergence constraint in the decomposition .

Our objective was to write an algorithm for the complete incompressible Navier-Stokes system, with a local condition defined on the interface, without any condition on the Reynold's number. As a first guess, motivated by our previous work for the Navier-Stokes problem, we imagined that imposing a condition similar to outflow boundary condition of (12) would have worked: in fact, as a *reference flow*, we could have chosen the solution on the neighboring domain, computed at the previous iteration of the Schwarz algorithm. This is however not sufficient to prove the convergence, since it can be seen just as a "Neumann type" boundary condition; it is necessary to add a contribution that takes into account the velocity on the interface in order to recover a "Fourier-type" condition for this problem.

We further remarked that, in order to deal with the incompressibility constraint, another condition was required by the problem.

So, when decomposing the domain Ω into two (or more) smaller subdomains $\Omega = \Omega_1 \cup \Omega_2$, the Schwarz algorithm that we designed defines a sequence of solutions $\mathbf{u}_j^l$ of the Navier-Stokes problem in Ω_j, where the transmission condition on the interface between the subdomains (denoted by Γ) for $(j,i) = (1,2)$ or $(2,1)$, is defined by:

$$\begin{aligned} \sigma(\mathbf{u}_j^l, \mathrm{p}_j^l)\cdot\vec{\mathbf{n}}_j - \frac{1}{2}(\mathbf{u}_j^l\cdot\vec{\mathbf{n}}_j)(\mathbf{u}_j^l) + \lambda\mathbf{u}_j^l &= \sigma(\mathbf{u}_i^{l-1}, \mathrm{p}_i^{l-1})\cdot\vec{\mathbf{n}}_i - \frac{1}{2}(\mathbf{u}_i^{l-1}\cdot\vec{\mathbf{n}}_i)(\mathbf{u}_i^{l-1}) + \lambda\mathbf{u}_i^{l-1}. \\ \mathrm{div}(\mathbf{u}_j^l) + \alpha\mathrm{p}_j^l &= -\mathrm{div}(\mathbf{u}_i^{l-1}) + \alpha\mathrm{p}_i^{l-1}, \end{aligned} \tag{21}$$

where $\vec{\mathbf{n}}_j$ is the outer normal to Ω_j.

The first condition, which depends on λ, is inspired by the classical Fourier condition, which linearly combines the values of the unknown (in this case the velocity) and the values of its derivative; here, also the convection is included.

The second, which depends on α, combines the divergence of the velocity with the pressure; it will be useful to conserve the incompressibility constraint at the convergence of the algorithm.

This is the first time, to our knowledge, that this kind of condition appears.

As a first step to write the non-overlapping Schwarz algorithm in the DDFV framework, we proposed a DDFV discretization for the Navier-Stokes problem on the entire domain Ω with Dirichlet boundary conditions. Indeed, we find convenient to consider a general discretization of the convection term, seen as a centered discretization plus a diffusive perturbation, expressed through a certain function B. This is inspired from the work of [HH14], which handles scalar advection-diffusion-reaction equations with a classic finite volume discretization.

To obtain this scheme, we integrate the momentum equation over all $\mathfrak{M} \cup \mathfrak{M}^*$ and we impose Dirichlet boundary conditions on $\partial\mathfrak{M} \cup \partial\mathfrak{M}^*$. The equation of conservation of mass is directly approximated on the diamond mesh equation over $\mathfrak{D}$, and it is stabilized through a parameter $\beta > 0$ with a Brezzi-Pitkäranta stabilization (see Section I.7). It gives:

Given $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$, satisfying $\mathrm{div}^{\mathfrak{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta d_{\mathtt{D}}^2 \Delta^{\mathfrak{D}} \bar{\mathrm{p}}^{\mathfrak{D}} = 0$, we look for $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ and $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\begin{cases} m_{\mathtt{K}} \dfrac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_{\mathtt{K}} \mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} & \forall \mathtt{K} \in \mathfrak{M} \\ m_{\mathtt{K}^*} \dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^* \in \mathfrak{M}^* \\ \mathbf{u}^{\partial\mathfrak{M}} = 0 & \\ \mathbf{u}^{\partial\mathfrak{M}^*} = 0 & \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta d_{\mathtt{D}}^2 \Delta^{\mathfrak{D}} \mathrm{p}^{\mathfrak{D}} = 0 & \\ \displaystyle\sum_{\mathtt{D} \in \mathfrak{D}} m_{\mathtt{D}} \mathrm{p}^{\mathtt{D}} = 0 & \end{cases} \tag{$\mathcal{P}$}$$

with $\beta > 0$ and $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$ the solution computed at the previous time step $t_{n-1} = (n-1)\delta t$ for $n \in \{1, \dots N-1\}$. The total fluxes are thus approached by:

$$\boxed{\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} &= -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} B\left(\frac{2\mathrm{Re} m_{\mathtt{D}}}{m_\sigma} F_{\sigma\mathtt{K}}\right)(\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}), \\ m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} &= -m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2}\right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathtt{D}}} B\left(\frac{2\mathrm{Re} m_{\mathtt{D}}}{m_{\sigma^*}} F_{\sigma^*\mathtt{K}^*}\right)(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}), \end{aligned}} \tag{22}$$

where we denote the coefficients $B\left(\frac{2\mathrm{Re} m_{\mathtt{D}}}{m_\sigma} F_{\sigma\mathtt{K}}\right), B\left(\frac{2\mathrm{Re} m_{\mathtt{D}}}{m_{\sigma^*}} F_{\sigma^*\mathtt{K}^*}\right)$ by $B_{\sigma\mathtt{K}}$ and $B_{\sigma^*\mathtt{K}^*}$; they can be scalars, for instance if we want to recover an upwind scheme (for which $B(s) = \frac{1}{2}|s|$), or even matrices. Problem $(\mathcal{P})$ is well-posed under the hypothesis (formulated here for the scalar case):

$$\begin{aligned} B_{\sigma\mathtt{K}} &= B_{\sigma\mathtt{L}}, & B_{\sigma\mathtt{K}} &\geq 0 \\ B_{\sigma^*\mathtt{K}^*} &= B_{\sigma^*\mathtt{L}^*}, & B_{\sigma^*\mathtt{K}^*} &\geq 0 \end{aligned} \tag{23}$$

See Thm. IV.1.3 for more details.

The following step consists in defining the DDFV scheme for the Navier Stokes problem on the subdomain with transmission conditions (21). It is necessary to add some fluxes unknowns $\Psi_{\mathfrak{T}_j}$ on each dual cell that intersects Γ, which approximate the dual fluxes $\mathcal{F}_{\sigma^*\mathtt{K}^*}$ on the interface. The scheme reads:

$$
\text{Find } (\mathbf{u}_{\mathfrak{T}_j}, \mathrm{p}_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \partial\mathfrak{M}^*_{j,\Gamma} \text{ such that}
$$
$$
\begin{cases}
m_{\mathtt{K}} \dfrac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_{\mathtt{K}} \mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} & \forall \mathtt{K} \in \mathfrak{M}_j \\
m_{\mathtt{K}^*} \dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^* \in \mathfrak{M}^*_j \\
m_{\mathtt{K}^*} \dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} + m_{\partial\Omega\cap\partial\mathtt{K}^*} \Psi_{\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma} \\
-\mathcal{F}_{\sigma\mathtt{K}} + \dfrac{1}{2} F_{\sigma\mathtt{K}} \mathbf{u}_{\mathtt{L}} + \lambda \mathbf{u}_{\mathtt{L}} = \mathbf{h}_{\mathtt{L}} & \forall \sigma \in \partial\mathfrak{M}_{j,\Gamma} \\
-\Psi_{\mathtt{K}^*} + \dfrac{1}{2} H_{\mathtt{K}^*} \mathbf{u}_{\mathtt{K}^*} + \lambda \mathbf{u}_{\mathtt{K}^*} = \mathbf{h}_{\mathtt{K}^*} & \forall \mathtt{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma} \\
\mathbf{u}^{\partial\mathfrak{M}_{j,D}} = 0 & \\
\mathbf{u}^{\partial\mathfrak{M}^*_{j,D}} = 0 & \\
m_{\mathtt{D}} \mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \mathrm{p}^{\mathfrak{D}} = 0 & \forall \mathtt{D} \in \mathfrak{D}_j \setminus \mathfrak{D}^\Gamma_j \\
m_{\mathtt{D}} \mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \mathrm{p}^{\mathfrak{D}} + \alpha m_{\mathtt{D}} \mathrm{p}^{\mathtt{D}} = g_{\mathtt{D}} & \forall \mathtt{D} \in \mathfrak{D}^\Gamma_j
\end{cases}
\tag{24}
$$

with $\lambda, \beta, \alpha > 0$ and $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$ the solution computed at the previous time step $t_{n-1} = (n-1)\delta t$ for $n \in \{1, \dots N-1\}$. We show that (24) is wellposed. We will refer to the system (24) in the more compact form:

$$
\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j}, \mathrm{p}_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}_{\mathfrak{T}_j}, g_{\mathfrak{D}_j}) = 0. \tag{25}
$$

We then propose the following parallel DDFV Schwarz algorithm:

at each time step t_n, for arbitrary initial guesses $\mathbf{h}^0_{\mathfrak{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma} \cup \partial\mathfrak{M}^*_{j,\Gamma}}$ and $g^0_{\mathfrak{D}_j} \in \mathbb{R}^{\mathfrak{D}_j}$, the algorithm performs two steps on the iteration index $l = 1, 2, \dots$ and $i, j, \in \{1, 2\}$, $j \neq i$:

1. Compute $(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}}$ solution to

$$
\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}_j}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}^{l-1}_{\mathfrak{T}_j}, g^{l-1}_{\mathfrak{D}^\Gamma_j}) = 0. \tag{$\mathcal{S}_1$}
$$

2. Compute the new values of $\mathbf{h}^l_{\mathfrak{T}_J}$ and of $g^l_{\mathfrak{D}^\Gamma_j}$ by:

$$
\begin{aligned}
\mathbf{h}^l_{\mathtt{L_j}} &= \mathcal{F}^l_{\sigma\mathtt{K}_i} - \frac{1}{2} F_{\sigma\mathtt{K}_i} \mathbf{u}^l_{\mathtt{L_i}} + \lambda \mathbf{u}^l_{\mathtt{L_i}}, && \forall \mathtt{L}_j = \mathtt{L}_i \in \partial\mathfrak{M}_{j,\Gamma}, \\
\mathbf{h}^l_{\mathtt{K}^*_\mathtt{j}} &= \Psi^l_{\mathtt{K}^*_\mathtt{i}} - \frac{1}{2} H_{\mathtt{K}^*_\mathtt{j}} \mathbf{u}^l_{\mathtt{K}^*_\mathtt{i}} + \lambda \mathbf{u}^l_{\mathtt{K}^*_\mathtt{i}}, && \forall \mathtt{K}^*_\mathtt{j} \in \partial\mathfrak{M}^*_{j,\Gamma} \text{ such that } x_{\mathtt{K}^*_\mathtt{j}} = x_{\mathtt{K}^*_\mathtt{i}}, \\
g^l_{\mathtt{D}_j} &= -\Big(m_{\mathtt{D}_i} \mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^l_{\mathfrak{T}_i}) - \beta m_{\mathtt{D}_i} d^2_{\mathtt{D}_i} \Delta^{\mathtt{D}_i} \mathrm{p}^l_{\mathfrak{D}_i}\Big) + \alpha m_{\mathtt{D}_i} \mathrm{p}^l_{\mathtt{D}_i}, && \forall \mathtt{D}_j \in \mathfrak{D}^\Gamma_j \text{ such that } x_{\mathtt{D}_j} = x_{\mathtt{D}_i}.
\end{aligned}
\tag{$\mathcal{S}_2$}
$$

When proving the convergence as $l \to \infty$ of $(\mathcal{S}_1)$-$(\mathcal{S}_2)$, we realized that actually this algorithm converges to a modified version of $(\mathcal{P})$, where the fluxes on the interface depend on some coefficients $\widetilde{B}_{\sigma\mathtt{K}}, \widetilde{B}_{\sigma^*\mathtt{K}^*}$:

Theorem 8 *Let $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ be a solution of $(\widetilde{\mathcal{P}})$, where:*

- *On the primal mesh, the new discrete convective fluxes are defined by:*

$$
\begin{cases}
B_{\sigma\mathtt{K}} Id & \text{if } \sigma \notin \mathcal{E}_\Gamma, \\
\widetilde{B}_{\sigma\mathtt{K}} & \text{if } \sigma \in \mathcal{E}_\Gamma,
\end{cases}
$$

where we refer to (IV.28) *for the definition of* $\widetilde{B}_{\sigma\kappa}$.

- *On the dual mesh,*

$$\begin{cases} B_{\sigma^*\kappa^*} Id & \text{if } \sigma^* \cap \Gamma = \emptyset, \\ \widetilde{B}_{\sigma^*\kappa^*} & \text{if } \sigma^* \cap \Gamma \neq \emptyset. \end{cases}$$

Under the hypothesis that $m_{\sigma^*} = 2m_{\sigma_j^*} = 2m_{\sigma_i^*}$, *for* $j, i = 1, 2$, $j \neq i$, *the iterates of the Schwarz algorithm* $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ *converge as* l *tends to infinity to the solution of* $(\widetilde{\mathcal{P}})$.

We refer to Sec. IV.3.1 and Thm. IV.3.9 for the details. So we asked ourselves if it was possible to recover, at the limit, the solution of $(\mathcal{P})$. The answer is positive as long as we modify the fluxes on the interface $\mathcal{E}_\Gamma$ of the Schwarz algorithm, as shown in the following theorem (Thm. IV.4.1):

Theorem 9 *Let* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ *be a solution of* $(\mathcal{P})$ *for convective fluxes defined by a constant upwind flux* $B_{\sigma\kappa}(s) = \frac{1}{2}|s|$ *for all* $\sigma \in \mathcal{E}$, *and by the centered flux* $B_{\sigma^*\kappa^*}(s) = 0$ *for all* $\sigma^* \in \mathcal{E}^*$. *Define* $(\bar{S})$ *the Schwarz algorithm where*

- *On the primal mesh, the new discrete convective fluxes are defined as:*

$$\begin{cases} B_{\sigma\kappa}(s) Id & \text{if } \sigma \notin \mathcal{E}_\Gamma, \\ \bar{B}_{\sigma\kappa}(s) & \text{if } \sigma \in \mathcal{E}_\Gamma, \end{cases}$$

with:

$$\bar{B}_{\sigma\kappa}(s) = \frac{1}{2} Q \begin{pmatrix} |s| - 2 + 2\sqrt{1+|s|} & 0 \\ 0 & |s| - 1 + \sqrt{1 + 2|s|} \end{pmatrix} Q^{-1}, \tag{26}$$

and $Q = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$, *where* $\vec{\mathbf{n}}_{\sigma\kappa} = \begin{pmatrix} x \\ y \end{pmatrix}$ *is the outer normal to the interface* Γ.

- *On the dual mesh,* $B_{\sigma^*\kappa^*}(s) = 0$.

Under the hypothesis that $m_{\sigma^*} = 2m_{\sigma_j^*} = 2m_{\sigma_i^*}$, *for* $j, i = 1, 2$, $j \neq i$, $(\mathcal{P})$ *is the limit of the Schwarz algorithm* $(\bar{S})$.

We numerically tested and compared the convergences proved in Thm. 8 and Thm. 9. We also showed the influence of the parameters λ, α of the transmission condition (21); tipically, we observe the presence of an optimal choice for both λ and α. Moreover, the mesh, the test case and even the stabilization parameter affect this optimal values. For further details, we refer to Sec. IV.5.

Curved interface reconstruction

The last topic addressed in this dissertation is a work done during the CEMRACS project of 2018, with Igor Chollet, Théo Corot, Laurent Dumas, Philippe Hoch and Tomas Leroy.

We proposed a curved interface reconstruction procedure in the case of a 2D compressible flow made of two or more materials. Interface reconstruction (IR) methods are encountered in numerical simulation of multi-material or multi-fluid flows. If we suppose to consider the case of two materials, the objective of IR methods is to define a geometric interface separating material 1 and material 2 with the following properties:

- P1: volume fractions conservation,

- P2: continuity of the interface,
- P3: robustness,
- P4: low or moderate computational cost.

The first IR method that has been introduced in 1982 is due to D.L. Youngs [You82]. There exists many variants to Youngs method, for instance an order 2 reconstruction ([RK98]) or an extension to more than two materials ([SGFL09]). Some correction terms for the normal computation have also been proposed to reduce undesirable effects and to smooth the interface ([GDSS05]). Even though this method is still largely used up to now because of its simplicity and robustness, it suffers from the non continuity of the interface.

Recently, in [DGJM17], a new reconstruction method which ensures continuity of the interface and preserves volume fractions have been introduced. This new interface reconstruction method, called DPIR (*Dynamic Programming Interface Reconstruction*) has been used as a starting point for the presented work. It consists of two main steps:

1. the minimization of a suitable energy functional, which gives a continuous linear interface;
2. the addition of a control point in each cell in order to find the correct volume fractions.

The last step in usually made by searching the point in the normal direction of the interface, in the line passing through the center of this one. We had three main goals during our project that we detail in the following.

First, we extended the DPIR method for curved interfaces (Sec. V.3). It is of interest in particular in the case of curved meshes, where an exact reconstruction of the interface is expected. In order to be a real candidate for being used in multi-material hydrodynamic simulation using ALE remap methods, the DPIR method must be able to deal with distored meshes.

In order to obtain a curved interface, we chose to introduce rational Bezier curves in the local correction phase of DPIR.

Second, we proposed several improvements in order to deal with strongly distorded cells and small volume fraction issues.

In particular, we suggested to change the direction for the control point (the center of the cell instead of the perpendicular bisector to the interface), to pass from a uniform discretization of the segments crossing the interface to a Chebyshev one in order to obtain a finer discretization around the corners and to add a new penalty term to the energy functional that has to be minimized in step 1.

Finally, this work ended with a generalization of the method to the three materials case (Sec. V.5). Interface reconstruction for multi-material simulations is a complicated issue, and a comparison of several existing methods can be found in [KGSS10]. The proposed method applies the DPIR method for all the materials *without choosing any material ordering* and a suitable average is applied to obtain the final interfaces. The method has been tested on two test cases with triple point configuration on cartesian meshes, giving encouraging results for future unstructured meshes cases. We illustrate an example in Fig. 4. For more simulations and more details, that has been obtained with a *C++* code, we refer to Chap. V.

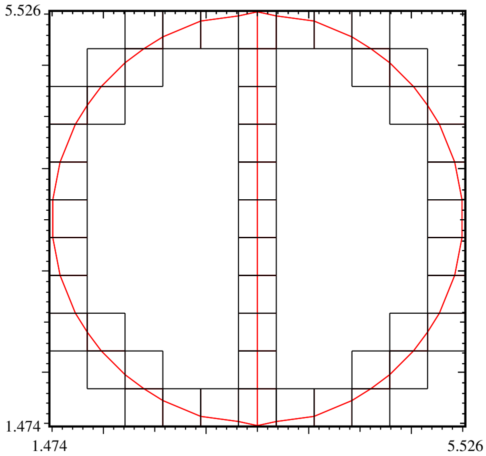


Fig. 4 Exemple of reconstruction of an interface between three materials, with two triple points configurations.

Publications

This thesis led to the following publications:

- [GKL17] *Numerical analysis of the DDFV method for the Stokes problem with mixed Neumann/Dirichlet boundary conditions*,Thierry Goudon, Stella Krell,Giulia Lissoni, FVCA8 2017 - International Conference on Finite Volumes for Complex Applications VIII, 2017, Lille, France.

- [GKL19] *DDFV method for Navier-Stokes problem with outflow boundary conditions*, Thierry Goudon, Stella Krell, Giulia Lissoni, Numerische Mathematik, 2019.

- *Reconstruction of curves interfaces for 2D compressible multi-material flows* (CEMRACS project), Igor Chollet, Théo Corot, Laurent Dumas, Philippe Hoch, Thomas Leroy, Giulia Lissoni, submitted to Esaim: proceedings and surveys.

- *Non-overlapping Domain Decomposition method for Navier-Stokes equation with DDFV discretizations*, Thierry Goudon, Stella Krell, Giulia Lissoni, in preparation.

Chapter I

DDFV method

Contents

The aim of this chapter is to introduce the DDFV method for Stokes and Navier-Stokes problem, the meshes and the notations that we will use along this dissertation; we adopt the main definitions introduced in [ABH07] and [Kre10].

This method enters the class of finite volume methods, that are discretization methods in which volume integrals are converted to surface integrals, using the divergence theorem. An important feature of those methods is that they are locally conservative; in fact, surface integrals are evaluated as fluxes at the boundary of each finite volume and they are conserved from one discretization cell to its neighbor. This makes finite volume methods quite attractive when modeling problems arising from fluid mechanics, the derivation of which are precisely based on local balance principles.

Finite volume methods for Stokes and Navier-Stokes problem have been widely studied during the years. Concerning the Stokes flow, we refer to [EHL06] for a colocated and stabilized finite volume scheme, [BEH05] for a staggered finite volume scheme, [DE08] for a mixed finite volume/ finite element scheme, [Del07] for an alternative DDFV scheme with a different localization of the unknowns, [DO15] for a DDFV scheme for the vorticity-velocity-pressure formulation.

Concerning the Navier-Stokes flow, we refer to [BCH00] for a fractional step method combined with finite volume schemes, [EHL07] for a colocated finite volume scheme, [EH05] for a staggered finite volume scheme, [Del07] for an alternative DDFV scheme with a different localization of the unknowns, [DE08, LS17] for a combined finite volume/ finite element scheme, [GHL10] for finite volume scheme with explicit time discretization, [CCML17] for a high order finite volume scheme based on polynomial reconstruction.

DDFV method has been developed since the early 2000's; the DDFV schemes have been first introduced and studied in [Her00] and [DO05] to approximate Laplace equation on a large class of 2D meshes including non-conformal and distorted meshes.

A way to consider general families of meshes is to add some unknowns to the problem: we require unknowns on vertices, centers and edges of control volumes; for this reason, DDFV method works on (three) staggered meshes. From an initial mesh, called the "primal mesh" (denoted with $\mathfrak{M} \cup \partial\mathfrak{M}$), we construct the "dual mesh" (denoted with $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$), that is centered on the vertices of the primal mesh, and the "diamond mesh" (denoted with $\mathfrak{D}$) , which is centered on the edges of the primal mesh; see Fig. I.1 for an illustration. The union of primal and dual mesh will be denoted by $\mathfrak{T}$.

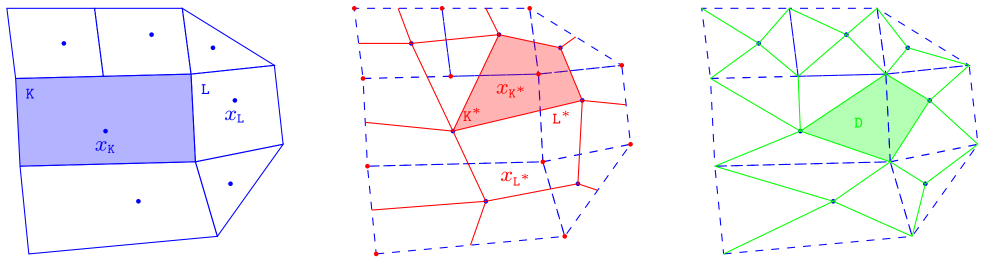


Fig. I.1 DDFV meshes on a non conformal mesh: primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ (blue), dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ (red) and diamond mesh $\mathfrak{D}$ (green).

DDFV method for Stokes and Navier-Stokes problem leads naturally to locate the unknowns of velocity and pressure in different points; the velocity unknowns are associated to each primal and dual volume, while the pressure unknowns are located on each diamond. So DDFV enters the class of staggered methods. This is an usual technique for incompressible flows: one of the best known schemes is the MAC (Marker and Cell) scheme (see [HW65]), which is built for cartesian meshes; we can mention also [CR73] for triangular meshes. MAC schemes, as unknowns, consider the normal components of the velocity (located on the mesh edges) and the pressure (located at the centers of the cells); with DDFV, we generalize this schemes by considering all the components of the velocity and extending it to more general meshes. Moreover, it has been shown in [Kre10] that a DDFV scheme on a cartesian mesh is equivalent to two decoupled MAC schemes written on two different staggered meshes (made exception for the boundary).

With this kind of construction, DDFV has two important advantages:

1. it applies to general meshes, such as non-conformal and distorted meshes;

2. it can reconstruct and mimic at the discrete level the dual properties of the continuous differential operators.

The second point is what gives the terms "Discrete Duality" to the name of the method: the discrete gradient $\nabla^{\mathfrak{D}}$ (see Def. I.3.1) is proven to be in duality with the discrete divergence $\mathbf{div}^{\mathfrak{T}}$ (see Def. I.3.5) which is naturally associated to the finite volume setting. In particular, the duality consists into verifying a discrete Green's formula, see Thm. I.5.1.

Outline. This chapter is organized as follows. In Sec. I.1 we recall the description of the DDFV meshes, followed by the approximation spaces and projections in Sec. I.2. The discrete operators are introduced in Sec. I.3 and in Sec. I.4 we detail the definition of the associated scalar products and norms. In Sec. I.5 the duality property, i.e. the discrete Green's formula, is stated. A reader that is familiar with DDFV can easily skip those sections, which maintain the notations and the structure of the presentation of the method done for instance in [Kre10, GKL19]; our intention is to give continuity to the previous works on the subject.
In Sec. I.6 we recall the main results on Inf-sup stability for DDFV, property that will be crucial all along the dissertation; in this section, we extend the existing results to the case of inhomogeneous Dirichlet boundary conditions. In Sec. I.7 we define the stabilization of Brezzi-Pitkäranta, a useful tool in order to deal with general meshes. In Sec. I.8 we study the relation between the discrete gradient and the discrete strain rate tensor; an important result of this section is the proof of Korn inequality in the case of inhomogeneous Dirichlet boundary conditions. In Sec. I.9 we extend to general L^p norms some discrete trace inequalities; we conclude by recalling some properties of the discrete operators in Sec. I.10.

I.1 DDFV meshes

A DDFV mesh $\mathfrak{T}$ is constituted by a primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ and a dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$, see Fig. I.2.

Construction of the primal mesh

We consider a primal mesh $\mathfrak{M}$ consisting of open disjoints polygons $\mathtt{K}$ called primal cells, such that $\bigcup_{\mathtt{K}\in\mathfrak{M}} \bar{\mathtt{K}} = \bar{\Omega}$. We denote $\partial\mathfrak{M}$ the set of edges of the primal mesh included in $\partial\Omega$, that are considered as degenerated primal cells. We associate to each $\mathtt{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$ a point $x_{\mathtt{K}} \in \mathtt{K}$, called center. For the volumes of the boundary, the point $x_{\mathtt{K}}$ is situated at the mid point of the edge. When $\mathtt{K}$ and $\mathtt{L}$ are neighboring volumes, we suppose that $\partial\mathtt{K} \cap \partial\mathtt{L}$ is a segment that we denote $\sigma = \mathtt{K}|\mathtt{L}$, edge of the primal mesh $\mathfrak{M}$. When $\mathtt{K} \in \mathfrak{M}$ and $\mathtt{L} \in \partial\mathfrak{M}$, we denote σ the segment $\partial\mathtt{K} \cap \partial\mathtt{L}$ that coincides with $\mathtt{L}$. We denote with $\mathcal{E}$ the set of all edges and with $\mathcal{E}_{int} = \mathcal{E} \setminus \{\sigma \in \mathcal{E} \text{ such that } \sigma \subset \partial\Omega\}$.
The DDFV framework is free of further "admissibility constraint", in particular we do not need to assume the orthogonality of the segment $x_{\mathtt{K}}, x_{\mathtt{L}}$ with $\sigma = \mathtt{K}|\mathtt{L}$, that is the case for instance in TPFA schemes (see [EGH00, Dro14]). Here we suppose:

Hp I.1.1 *All control volumes $\mathtt{K}$ are star-shaped with respect to $x_{\mathtt{K}}$.*

Construction of the dual mesh

From the primal mesh, we build the associated dual mesh. A dual cell $\mathtt{K}^*$ is associated to a vertex $x_{\mathtt{K}^*}$ of the primal mesh. The dual cells are obtained by joining the centers of the primal control volumes that have $x_{\mathtt{K}^*}$ as vertex. Then, the point $x_{\mathtt{K}^*}$ is called center of $\mathtt{K}^*$. We will distinguish interior dual mesh, for which $x_{\mathtt{K}^*}$ does not belong to $\partial\Omega$, denoted by $\mathfrak{M}^*$ and the boundary dual mesh, for which $x_{\mathtt{K}^*}$ belongs to $\partial\Omega$, denoted by $\partial\mathfrak{M}^*$. We denote with $\sigma^* = \mathtt{K}^*|\mathtt{L}^*$ the edges of the dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\mathcal{E}^*$ the set of those edges. In what follows, we assume:

Hp I.1.2 *All control volumes $\mathtt{K}^*$ are star-shaped with respect to $x_{\mathtt{K}^*}$.*

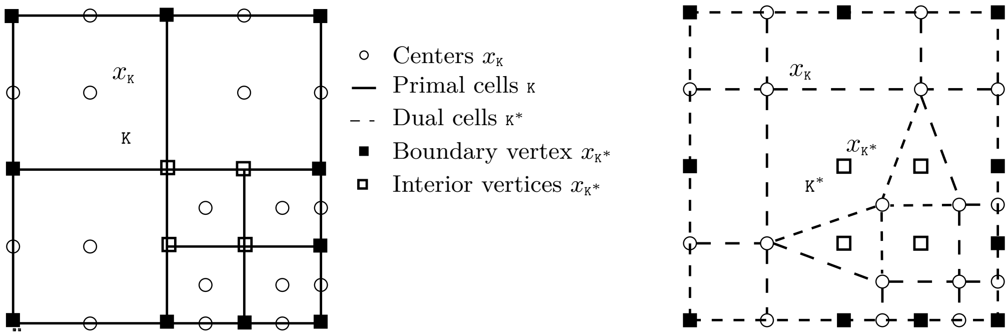


Fig. I.2 DDFV mesh $\mathfrak{T}$.

Construction of the diamond mesh

The diamond mesh $\mathfrak{D}$ is made of quadrilaterals with disjoint interiors (thanks to Hp I.1.1), such that their principal diagonals are a primal edge $\sigma = \mathtt{K}|\mathtt{L} = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}]$ and the dual edge $\sigma^* = [x_\mathtt{K}, x_\mathtt{L}]$. Those quadrilaterals are called diamonds and they are denoted with $\mathtt{D}$ or $\mathtt{D}_{\sigma,\sigma^*}$. Thus a diamond is a quadrilateral with vertices $x_\mathtt{K}, x_\mathtt{L}, x_{\mathtt{K}^*}$ and $x_{\mathtt{L}^*}$ (see Fig. I.3).
We remark that diamonds are the union of two disjoints triangles $(x_\mathtt{K}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*})$ and $(x_\mathtt{L}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*})$ and that diamonds are not necessarily convex.
Moreover, if $\sigma \in \mathcal{E} \cap \partial\Omega$, the quadrilateral $\mathtt{D}_{\sigma,\sigma^*}$ degenerates into a triangle.

The set of all diamonds is denoted with $\mathfrak{D}$ and we have $\Omega = \bigcup_{\mathtt{D}\in\mathfrak{D}} \mathtt{D}$.
We distinguish the diamonds on the interior and of the boundary:

$$\mathfrak{D}_{ext} = \{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D},\ \text{such that } \sigma \subset \partial\Omega\}$$
$$\mathfrak{D}_{int} = \mathfrak{D}\backslash\mathfrak{D}_{ext}.$$

Remark I.1.3 *We have a bijection between the diamonds $D \in \mathfrak{D}$ and the edges $\mathcal{E}$ of the primal mesh; also between the diamonds $D \in \mathfrak{D}$ and the edges $\mathcal{E}^*$ of the dual mesh.*

For a volume $\mathtt{V} \in \mathfrak{M} \cup \partial\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ we define:

- $m_\mathtt{V}$ the measure of the cell $\mathtt{V}$,
- $\mathcal{E}_\mathtt{V}$ the set of edges of $\mathtt{V} \in \mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and the edge $\sigma = \mathtt{V}$ for $\mathtt{V} \in \partial\mathfrak{M}$,
- $\mathfrak{D}_\mathtt{V} = \{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}, \sigma \in \mathcal{E}_\mathtt{V}\}$,
- $\mathfrak{D}_\mathtt{V}^{int} = \{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_\mathtt{V} \cap \mathfrak{D}_{int}\}$, $\mathfrak{D}_\mathtt{V}^{ext} = \{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_\mathtt{V} \cap \mathfrak{D}_{ext}\}$,
- $d_\mathtt{V}$ the diameter of $\mathtt{V}$,
- $B_\mathtt{V} := B(x_\mathtt{V}, \rho_\mathtt{V}) \cap \partial\Omega \subset \mathtt{V}$ for $\mathtt{V} \in \partial\mathfrak{M} \cup \partial\mathfrak{M}^*$, $m_{B_\mathtt{V}}$ its length, $\rho_\mathtt{V}$ chosen to verify the inclusion.

For a diamond $\mathtt{D}_{\sigma,\sigma^*}$ whose vertices are $(x_\mathtt{K}, x_{\mathtt{K}^*}, x_\mathtt{L}, x_{\mathtt{L}^*})$, we denote:

- $x_\mathtt{D}$ the center of the diamond $\mathtt{D}$: $x_\mathtt{D} = \sigma \cap \sigma^*$,
- m_σ the length of the edge σ,

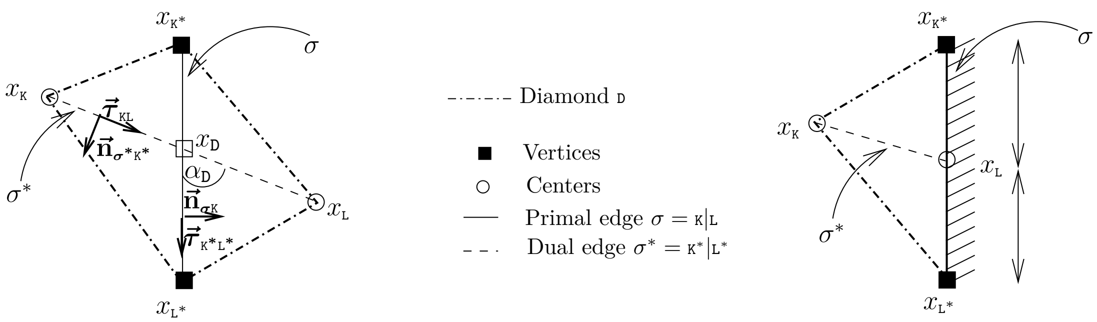


Fig. I.3 A diamond $D = D_{\sigma,\sigma^*}$, on the interior (left) and on the boundary (right).

- m_{σ^*} the length of σ^*,
- m_D the measure of the diamond D_{σ,σ^*},
- d_D the diameter of the diamond D_{σ,σ^*},
- α_D the angle between σ and σ^*.

We introduce for every diamond two orthonormal basis $(\vec{\tau}_{K^*L^*}, \vec{n}_{\sigma K})$ and $(\vec{n}_{\sigma^*K^*}, \vec{\tau}_{KL})$, where:

- $\vec{n}_{\sigma K}$ the unit normal to σ going out from K,
- $\vec{\tau}_{K^*L^*}$ the unit tangent vector to σ oriented from K^* to L^*,
- $\vec{n}_{\sigma^*K^*}$ the unit normal vector to σ^* going out from K^*,
- $\vec{\tau}_{KL}$ the unit tangent vector to σ^* oriented from K to L.

We denote for each diamond:

- its sides $\mathfrak{s}$ (for example $\mathfrak{s} = [x_K, x_{K^*}]$),
- $\mathcal{E}_D = \{\mathfrak{s}, \mathfrak{s} \subset \partial D \text{ and } \mathfrak{s} \not\subseteq \partial\Omega\}$ the set of all interior sides of the diamond,
- $m_{\mathfrak{s}}$ the length of $\mathfrak{s}$,
- $\vec{n}_{\mathfrak{s}D}$ the unit normal to $\mathfrak{s}$ going out from D,
- $\mathfrak{S} = \{\mathfrak{s} \in \mathcal{E}_D, \forall D \in \mathfrak{D}\}$ the set of interior edges of all diamond cells $D \in \mathfrak{D}$,
- $\mathfrak{S}_K = \{\mathfrak{s} \in \mathfrak{S}, \text{ such that } \mathfrak{s} \subset K\}$ and $\mathfrak{S}_{K^*} = \{\mathfrak{s} \in \mathfrak{S}, \text{ such that } \mathfrak{s} \subset K^*\}$.

Remark I.1.4 *Every diamond is star-shaped with respect to x_D.*

Remark I.1.5 *It can happen that dual cells can overlap; to avoid this inconvenient, we can either suppose that the diamonds are convexes or consider the barycentric dual mesh, obtained by joining the centers x_K of the primal control volumes to the middle point of the edges that have x_{K^*} as a vertex. Thanks to Hyp. I.1.1, barycentryc dual cells have disjoint interiors.*

Regularity of the mesh

Let size($\mathfrak{T}$) be the maximum of the diameters of the diamonds cells in $\mathfrak{D}$.
To measure the flattening of the triangles we denote with $\alpha_{\mathfrak{T}}$ the only real in $]0, \frac{\pi}{2}]$ such that $\sin(\alpha_{\mathfrak{T}}) := \min_{\mathtt{D}\in\mathfrak{D}} |\sin(\alpha_{\mathtt{D}})|$.
We introduce a positive number reg($\mathfrak{T}$) that measures the regularity of the mesh. It is defined as:

$$\mathrm{reg}(\mathfrak{T}) = \max\left(\frac{1}{\sin(\alpha_T)}, \mathcal{N}, \mathcal{N}^*, \max_{\mathtt{D}\in\mathfrak{D}} \max_{\mathfrak{s}\in\mathcal{E}_{\mathtt{D}}} \frac{d_{\mathtt{D}}}{m_{\mathfrak{s}}}, \max_{\mathtt{K}\in\mathfrak{M}} \frac{d_{\mathtt{K}}}{\sqrt{m_{\mathtt{K}}}}, \max_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \left(\frac{d_{\mathtt{K}^*}}{\sqrt{m_{\mathtt{K}^*}}}\right),\right.$$
$$\left.\max_{\mathtt{K}\in\mathfrak{M}} \max_{\mathtt{D}\in\mathfrak{D}_K} \left(\frac{d_{\mathtt{K}}}{d_{\mathtt{D}}}\right), \max_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \max_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} \left(\frac{d_{\mathtt{K}^*}}{d_{\mathtt{D}}}\right)\right), \tag{I.1}$$

where $\mathcal{N}$ and $\mathcal{N}^*$ are the maximum number of edges of each primal cell and the maximum number of edges incident to any vertex. The number reg($\mathfrak{T}$) should be uniformly bounded when size($\mathfrak{T}$) $\to 0$ for the convergence to hold.
From the definition of reg($\mathfrak{T}$), the following geometrical result holds: there exist two constants C_1 and C_2 depending on reg($\mathfrak{T}$) such that $\forall \mathtt{K} \in \mathfrak{M}, \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\forall \mathtt{D} \in \mathfrak{D}$ such that $\mathtt{D} \cap \mathtt{K} \neq 0$ and $\mathtt{D} \cap \mathtt{K}^* \neq 0$ we have:

$$C_1 m_{\mathtt{K}} \leq m_{\mathtt{D}} \leq C_2 m_{\mathtt{K}}, \qquad C_1 m_{\mathtt{K}^*} \leq m_{\mathtt{D}} \leq C_2 m_{\mathtt{K}^*},$$

and

$$C_1\, d_{\mathtt{K}} \leq d_{\mathtt{D}} \leq C_2\, d_{\mathtt{K}}, \qquad C_1\, d_{\mathtt{K}^*} \leq d_{\mathtt{D}} \leq C_2\, d_{\mathtt{K}^*}.$$

I.2 Approximation spaces and projections on DDFV meshes

The DDFV method for Stokes and incompressible Navier-Stokes problem uses staggered unknowns; this is a classical approach, see for instance [HW65].
We associate to each primal volume $\mathtt{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$ an unknown $\mathbf{u}_{\mathtt{K}} \in \mathbb{R}^2$ for the velocity, to every dual volume $\mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ an unknown $\mathbf{u}_{\mathtt{K}^*} \in \mathbb{R}^2$ for the velocity and to each diamond $\mathtt{D} \in \mathfrak{D}$ an unknown $\mathrm{p}^{\mathtt{D}} \in \mathbb{R}$ for the pressure. Those unknowns are collected in the families:

$$\mathbf{u}^{\mathfrak{T}} = ((\mathbf{u}_{\mathtt{K}})_{\mathtt{K}\in\mathfrak{M}\cup\partial\mathfrak{M}}, (\mathbf{u}_{\mathtt{K}^*})_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}) \in (\mathbb{R}^2)^{\mathfrak{T}} \quad \text{and} \quad \mathrm{p}^{\mathfrak{D}} = ((\mathrm{p}^{\mathtt{D}})_{\mathtt{D}\in\mathfrak{D}}) \in \mathbb{R}^{\mathfrak{D}}.$$

We approximatively have twice the number of velocity unknowns with respect to finite volume methods like TPFA (see [Dro14]), but we will show that this allows to build a complete approximation of the gradient (not only in the normal direction), which gives robustness to the method, since it does not demand any "admissibility constraint" on the mesh; this is not the case in TPFA methods, which require an orthogonality condition: this is a strong hypothesis since it excludes, for instance, non conformal meshes (such as locally refined meshes).

In all our works, we will deal with mixed boundary conditions; so we define two subsets of the boundary mesh, useful to take into account two different types of boundary conditions (see Fig. I.4):

$$\begin{aligned}
\partial\mathfrak{M}_i &= \{\mathtt{K} \in \partial\mathfrak{M} : x_{\mathtt{K}} \in \Gamma_i\}, \text{ for } i = 1, 2, \\
\partial\mathfrak{M}_1^* &= \{\mathtt{K}^* \in \partial\mathfrak{M}^* : x_{\mathtt{K}^*} \in \Gamma_1\}, \\
\partial\mathfrak{M}_2^* &= \{\mathtt{K}^* \in \partial\mathfrak{M}^* : x_{\mathtt{K}^*} \in \Gamma_2 \setminus \Gamma_1\}.
\end{aligned}$$

The conditions will change from a chapter to another, so Γ_1, Γ_2 will represent different types of boundaries: in Chap. II, we will impose Dirichlet and Neumann boundary conditions; in Chap. III, we will impose Dirichlet and open boundary conditions on the outflow. Finally, in Chap. IV, we will take into account transmission conditions on the interface between subdomains.

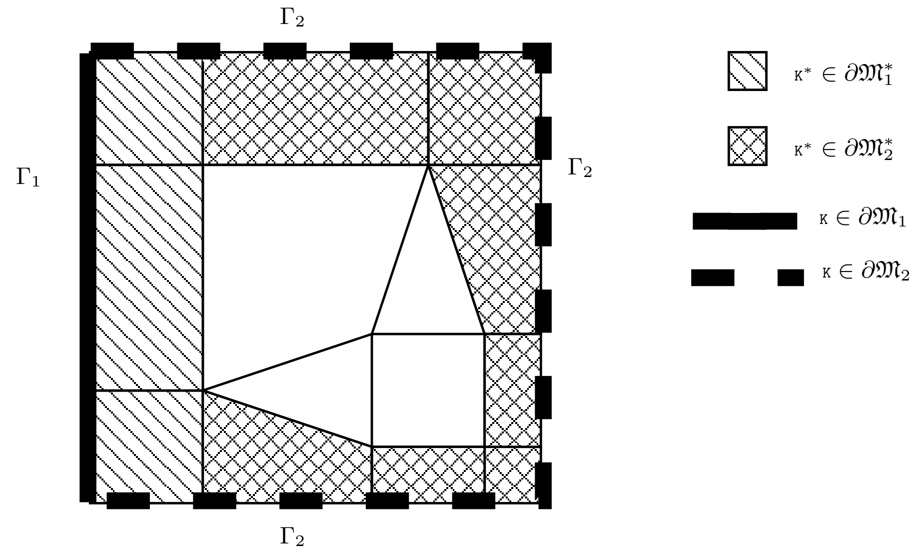


Fig. I.4 Domain with mixed boundary conditions

We define now two discrete average projections, for all functions $\mathbf{v}$ in $(H^1(\Omega))^2$:

- one on the interior:

$$\mathbb{P}_m^{\mathfrak{M}}\mathbf{v} = \left(\left(\frac{1}{m_{\mathtt{K}}}\int_{\mathtt{K}} \mathbf{v}(x)\mathrm{d}x\right)_{\mathtt{K}\in\mathfrak{M}}\right) \quad \mathbb{P}_m^{\mathfrak{M}^*}\mathbf{v} = \left(\left(\frac{1}{m_{\mathtt{K}^*}}\int_{\mathtt{K}^*} \mathbf{v}(x)\mathrm{d}x\right)_{\mathtt{K}^*\in\mathfrak{M}^*}\right),$$

- one on the boundary :

$$\mathbb{P}_m^{\partial\Omega}\mathbf{v} = \left(\left(\frac{1}{m_{B_{\mathtt{K}}}}\int_{B_{\mathtt{K}}} \mathbf{v}(x)\mathrm{d}x\right)_{\mathtt{K}\in\partial\mathfrak{M}}, \left(\frac{1}{m_{B_{\mathtt{K}^*}}}\int_{B_{\mathtt{K}^*}} \mathbf{v}(x)\mathrm{d}x\right)_{\mathtt{K}^*\in\partial\mathfrak{M}^*}\right).$$

We can collect them in a shortened notation:

$$\mathbb{P}_m^{\mathfrak{T}}\mathbf{v} = (\mathbb{P}_m^{\mathfrak{M}}\mathbf{v}, \mathbb{P}_m^{\mathfrak{M}^*}\mathbf{v}, \mathbb{P}_m^{\partial\Omega}\mathbf{v}), \quad \forall \mathbf{v} \in (H^1(\Omega))^2).$$

We introduce also a centered projection on the mesh $\mathfrak{T}$:

$$\mathbb{P}_c^{\mathfrak{T}}\mathbf{v} = ((\mathbf{v}(x_{\mathtt{K}}))_{\mathtt{K}\in(\mathfrak{M}\cup\partial\mathfrak{M})}, (\mathbf{v}(x_{\mathtt{K}^*}))_{\mathtt{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)}), \quad \forall \mathbf{v} \in (H^2(\Omega))^2, \tag{I.2}$$

and an average projection on the diamond mesh $\mathfrak{D}$:

$$\mathbb{P}_m^{\mathfrak{D}}\mathrm{q} = \left(\left(\frac{1}{m_{\mathtt{D}}}\int_{\mathtt{D}} \mathrm{q}(x)\mathrm{d}x\right)_{\mathtt{D}\in\mathfrak{D}}\right) \quad \forall \mathrm{q} \in H^1(\Omega).$$

We define two discrete subsets of $(\mathbb{R}^2)^{\mathfrak{T}}$, useful to take in account Dirichlet boundary conditions. In the following, we will denote by Γ_D the Dirichlet boundary (instead of Γ_1), and consequently

$\partial\mathfrak{M}_D, \partial\mathfrak{M}_D^*$ the corresponding primal and dual boundary mesh (instead of $\partial\mathfrak{M}_1, \partial\mathfrak{M}_1^*$).

$$\mathbb{E}_0^{\Gamma_D} = \{\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}, \text{s. t. } \forall \mathtt{K} \in \partial\mathfrak{M}_D,\ \mathbf{u}_{\mathtt{K}} = 0 \text{ and } \forall \mathtt{K}^* \in \partial\mathfrak{M}_D^*,\ \mathbf{u}_{\mathtt{K}^*} = 0\}.$$

$$\mathbb{E}_{m,\mathbf{g}}^{\Gamma_D} = \{\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}, \text{s. t. } \forall \mathtt{K} \in \partial\mathfrak{M}_D,\ \mathbf{u}_{\mathtt{K}} = (\mathbb{P}_m^{\partial\Omega}\mathbf{g})_{\mathtt{K}} \text{ and } \forall \mathtt{K}^* \in \partial\mathfrak{M}_D^*,\ \mathbf{u}_{\mathtt{K}^*} = (\mathbb{P}_m^{\partial\Omega}\mathbf{g})_{\mathtt{K}^*}\}.$$

We define also the projection $\mathfrak{P}_{m,\mathbf{g}}^D$ on the space $\mathbb{E}_{m,\mathbf{g}}^D$:

$$\begin{aligned}\mathfrak{P}_{m,\mathbf{g}}^{\Gamma_D} : (\mathbb{R}^2)^{\mathfrak{T}} &\longrightarrow \mathbb{E}_{m,\mathbf{g}}^{\Gamma_D} \\ \mathbf{u}^{\mathfrak{T}} &\longmapsto \left((\mathbf{u}_{\mathtt{K}})_{\mathtt{K}\in\mathfrak{M}\cup\partial\mathfrak{M}_2}, (\mathbb{P}_m^{\partial\Omega}\mathbf{g})_{\mathtt{K}\in\mathfrak{M}_D}, (\mathbf{u}_{\mathtt{K}^*})_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}_2^*}, (\mathbb{P}_m^{\partial\Omega}\mathbf{g})_{\mathtt{K}^*\in\partial\mathfrak{M}_D^*}\right).\end{aligned}$$

I.3 Discrete operators

In this section we define the discrete operators that are necessary to write and to analyse the DDFV schemes that we will build. We start by defining a *discrete gradient* and a *discrete divergence*. Those two operators are in "discrete duality" (this is what gives the name to the scheme) since we can prove a discrete Green formula (see Thm. I.5.1 below) that links them. For the proof we refer to [CVV99, Her00, DO05] and [Kre11a]. Then, we will define some other operators that will be useful later, such as a *discrete strain rate tensor*, a *discrete curl* and a *discrete rotational*.

Definition I.3.1 (Discrete gradient on $\mathfrak{D}$) *We define the discrete gradient of a vector field of $(\mathbb{R}^2)^{\mathfrak{T}}$ the operator*

$$\nabla^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})_{\mathtt{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$

such that for $\mathtt{D} \in \mathfrak{D}$:

$$\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{\sin(\alpha_{\mathtt{D}})}\left[\frac{\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}}}{m_{\sigma^*}} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}} + \frac{\mathbf{u}_{\mathtt{L}^*} - \mathbf{u}_{\mathtt{K}^*}}{m_\sigma} \otimes \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\right],$$

where $\otimes$ represents the tensor product.

We remark that the area of a diamond $\mathtt{D}$ is $m_{\mathtt{D}} = \frac{1}{2}m_\sigma m_{\sigma^*}\sin(\alpha_{\mathtt{D}})$. So we can rewrite the discrete gradient as:

$$\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_{\mathtt{D}}}\left[m_\sigma(\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}}) \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_{\sigma^*}(\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}^*}) \otimes \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\right].$$

Remark I.3.2 *The gradient $\nabla^{\mathfrak{D}}$, that is constant on each diamond $\mathtt{D} \in \mathfrak{D}$, is the "composition" of two directional derivatives. In fact, the gradient in the direction $\overrightarrow{x_{\mathtt{K}}, x_{\mathtt{L}}}$ can be approximated by $(\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}})$, and the one in the direction $\overrightarrow{x_{\mathtt{K}^*}, x_{\mathtt{L}^*}}$ by $(\mathbf{u}_{\mathtt{L}^*} - \mathbf{u}_{\mathtt{K}^*})$ (see Fig. I.3). Thanks to trigonometric formulas, we can combine those two and obtain a full approximation of the gradient on $\mathtt{D}$, given by Def. I.3.1 .*

Remark I.3.3 *For all $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$, the property $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = 0$ implies the existence of two constants $\boldsymbol{c}_0 \in \mathbb{R}^2$ and $\boldsymbol{c}_1 \in \mathbb{R}^2$ such that:*

$$\begin{aligned}\mathbf{u}_{\mathtt{K}} &= \boldsymbol{c}_0 \quad \forall \mathtt{K} \in (\mathfrak{M} \cup \partial\mathfrak{M}) \\ \mathbf{u}_{\mathtt{K}^*} &= \boldsymbol{c}_1 \quad \forall \mathtt{K}^* \in (\mathfrak{M}^* \cup \partial\mathfrak{M}^*).\end{aligned}$$

If, moreover, $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, we deduce $c_0 = c_1 = 0$ and finally $\mathbf{u}^{\mathfrak{T}} = 0$.

Definition I.3.4 (Discrete strain rate tensor on $\mathfrak{D}$) *We define the discrete strain rate tensor of a vector field in $(\mathbb{R}^2)^{\mathfrak{T}}$ as the operator*

$$\mathrm{D}^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})_{\mathtt{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$$

such that for $\mathtt{D} \in \mathfrak{D}$*:*

$$\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + {}^t(\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})}{2}. \tag{I.3}$$

To define a discrete divergence, we remark that for a regular vectorial function ξ, by applying Green's Formula we can write:

$$\int_{\mathtt{K}} \mathrm{div}(\xi(x))\mathrm{d}x = \sum_{\sigma\subset\partial\mathtt{K}} \int_{\sigma} \xi(s)\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s, \qquad \forall\mathtt{K}\in\mathfrak{M}. \tag{I.4}$$

By means of the discrete counterpart of (I.4), we can define our operator of discrete divergence.

Definition I.3.5 (Discrete divergence on $\mathfrak{T}$) *We define the discrete divergence of a discrete tensor field of $(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ as the operator*

$$\boldsymbol{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \boldsymbol{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}.$$

Let $\xi^{\mathfrak{D}} = (\xi^{\mathtt{D}})_{\mathtt{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, we set:

$$\boldsymbol{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} = (\boldsymbol{div}^{\mathfrak{M}}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\partial\mathfrak{M}}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\mathfrak{M}^*}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\partial\mathfrak{M}^*}\xi^{\mathfrak{D}}),$$

where we define $\boldsymbol{div}^{\mathfrak{M}}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathtt{K}}\xi^{\mathfrak{D}})_{\mathtt{K}\in\mathfrak{M}}$, $\boldsymbol{div}^{\partial\mathfrak{M}}\xi^{\mathfrak{D}} = 0$, $\boldsymbol{div}^{\mathfrak{M}^*}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}})_{\mathtt{K}\in\mathfrak{M}^*}$ and $\boldsymbol{div}^{\partial\mathfrak{M}^*}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}})_{\mathtt{K}^*\in\partial\mathfrak{M}^*}$ with:

$$\begin{aligned}
\mathbf{div}^{\mathtt{K}}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathtt{K}}} \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}} && \forall\mathtt{K}\in\mathfrak{M},\\
\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathtt{K}^*}} \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} && \forall\mathtt{K}^*\in\mathfrak{M}^*,\\
\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathtt{K}^*}} \Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}} \frac{m_\sigma}{2} \xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}} \Bigg) && \forall\mathtt{K}^*\in\partial\mathfrak{M}^*.
\end{aligned}$$

Definition I.3.6 (Discrete gradient on $\mathfrak{T}$) *We define the discrete gradient of a scalar field of $\mathbb{R}^{\mathfrak{D}}$ as the operator:*

$$\boldsymbol{\nabla}^{\mathfrak{T}} : \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \mapsto \boldsymbol{\nabla}^{\mathfrak{T}}\mathrm{p}^{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$$

with

$$\boldsymbol{\nabla}^{\mathfrak{T}}\mathrm{p}^{\mathfrak{D}} = \mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}} Id).$$

Similarly to the continuous setting, in which for a vectorial function $\mathbf{f} = (f_1, f_2)$ of two variables (x_1, x_2) the gradient and the divergence are defined by:

$$\nabla\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}, \qquad \mathrm{div}(\mathbf{f}) = \mathrm{Tr}(\nabla\mathbf{f}) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \tag{I.5}$$

and the curl and the rotational by:

$$\mathrm{curl}\,\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_2} & -\frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_2} & -\frac{\partial f_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{rot}(\mathbf{f}) = \mathrm{Tr}(\mathrm{curl}\,\mathbf{f}) = -\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}; \tag{I.6}$$

we can further define (on the diamond mesh $\mathfrak{D}$) a discrete divergence, a discrete curl and a discrete rotational.

Definition I.3.7 (Discrete divergence on $\mathfrak{D}$) *We define the discrete divergence of a vector field of* $(\mathbb{R}^2)^{\mathfrak{T}}$ *as the operator*

$$div^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (div^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})_{\mathtt{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$$

with

$$div^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = Tr(\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}), \quad \forall \mathtt{D} \in \mathfrak{D}.$$

Definition I.3.8 (Discrete curl on $\mathfrak{D}$) *We define the discrete curl of a vector field of* $(\mathbb{R}^2)^{\mathfrak{T}}$ *as the operator*

$$curl^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto curl^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$

such that for $\mathtt{D} \in \mathfrak{D}$*:*

$$curl^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_{\mathtt{D}}}\left[m_\sigma(\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}}) \otimes \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*} - m_{\sigma^*}(\mathbf{u}_{\mathtt{L}^*} - \mathbf{u}_{\mathtt{K}^*}) \otimes \vec{\boldsymbol{\tau}}_{\mathtt{KL}}\right].$$

Definition I.3.9 (Discrete rotational on $\mathfrak{D}$) *We define the discrete rotational of a vector field of* $(\mathbb{R}^2)^{\mathfrak{T}}$ *as the operator*

$$rot^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto rot^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{D}}$$

with

$$rot^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = -Tr(curl^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}), \quad \forall \mathtt{D} \in \mathfrak{D}.$$

I.4 Scalar products and norms

We define the *trace operators* on $(\mathbb{R}^2)^{\mathfrak{T}}$ and $\mathbb{R}^{\mathfrak{D}}$; see Fig. I.5 for the notations.

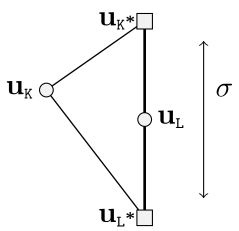


Fig. I.5 A boundary diamond, $\sigma \in \partial\mathfrak{M}$.

Let $\gamma^{\mathfrak{T}} : \mathbf{u}^{\mathfrak{T}} \mapsto \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) = (\gamma_\sigma(\mathbf{u}^{\mathfrak{T}}))_{\sigma\in\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\partial\mathfrak{M}}$, such that:

$$\gamma_\sigma(\mathbf{u}^{\mathfrak{T}}) = \frac{\mathbf{u}_{\mathtt{K}^*} + 2\mathbf{u}_{\mathtt{L}} + \mathbf{u}_{\mathtt{L}^*}}{4}, \qquad \forall \sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}] \in \partial\mathfrak{M}. \tag{I.7}$$

We can also define $\widetilde{\gamma}^{\mathfrak{T}} : \mathbf{u}^{\mathfrak{T}} \mapsto \widetilde{\gamma}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) = (\widetilde{\gamma}_\sigma(\mathbf{u}^{\mathfrak{T}}))_{\sigma\in\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\partial\mathfrak{M}}$, such that:

$$\widetilde{\gamma}_\sigma(\mathbf{u}^{\mathfrak{T}}) = \frac{\mathbf{u}_{\mathtt{K}^*} + 2\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}^*}}{4}, \qquad \forall \sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}] \in \partial\mathfrak{M}. \tag{I.8}$$

On the diamond mesh we define $\gamma^{\mathfrak{D}} : \Phi^{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}} \to (\Phi^{\mathtt{D}})_{\mathtt{D}\in\mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$, which is the operator of restriction to the boundary diamonds.

Now we define the scalar products on the approximation spaces:

$$
\begin{aligned}
&[[\mathbf{v}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = \frac{1}{2}\Big(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}\, \mathbf{u}_{\mathtt{K}} \cdot \mathbf{v}_{\mathtt{K}} + \sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathtt{K}^*}\, \mathbf{u}_{\mathtt{K}^*} \cdot \mathbf{v}_{\mathtt{K}^*}\Big) && \forall \mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}\\
&(\Phi^{\mathfrak{D}}, \mathbf{v}^{\partial\mathfrak{M}})_{\partial\Omega} = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \Phi^{\mathtt{D}} \cdot \mathbf{v}^{\sigma} && \forall \Phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}, \mathbf{v}^{\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\partial\mathfrak{M}}\\
&(\xi^{\mathfrak{D}} : \Phi^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}}(\xi^{\mathtt{D}} : \Phi^{\mathtt{D}}) && \forall \xi^{\mathfrak{D}}, \Phi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}\\
&(\mathrm{p}^{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}} \mathrm{p}^{\mathtt{D}} \mathrm{q}^{\mathtt{D}} && \forall \mathrm{p}^{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}},
\end{aligned}
$$

where $(\xi : \widetilde{\xi}) = \sum_{1\le i,j\le 2} \xi_{i,j}, \widetilde{\xi}_{i,j} = \mathrm{Tr}({}^t\xi\widetilde{\xi})$ for all $\xi, \widetilde{\xi} \in \mathcal{M}_2(\mathbb{R})$.
For all $p \ge 1$ we can define for any $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ and $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ the norms:

$$
\begin{aligned}
&\|\mathbf{u}^{\mathfrak{T}}\|_p = \left[\frac{1}{2}\Big(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}|\mathbf{u}_{\mathtt{K}}|^p + \sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^p\Big)\right]^{1/p} = \left[\frac{1}{2}\Big(\|\mathbf{u}^{\mathfrak{M}}\|_p^p + \|\mathbf{u}^{\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\|_p^p\Big)\right]^{1/p},\\
&\|\mathbf{v}^{\partial\mathfrak{M}}\|_{p,\partial\Omega} = \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\mathbf{v}^{\sigma}|^p\Big)^{1/p},\\
&|||\xi^{\mathfrak{D}}|||_p = \Big(\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}|\xi^{\mathtt{D}}|^p\Big)^{1/p},\\
&\|\mathbf{u}^{\mathfrak{T}}\|_{1,p} = \Big[\|\mathbf{u}^{\mathfrak{T}}\|_p^p + \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p^p\Big]^{1/p},\\
&\|\mathrm{p}^{\mathfrak{D}}\|_p = \Big(\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}|\mathrm{p}^{\mathtt{D}}|^p\Big)^{1/p}.
\end{aligned}
$$

Remark I.4.1 *Remark that, if we denote by $|||\cdot|||_{\mathcal{F}}$ the Frobenius norm $|||\xi|||_{\mathcal{F}}^2 = (\xi : \xi)$ for all matrices $\xi \in \mathcal{M}_2(\mathbb{R})$, the following holds:*

$$\left|\left|\left|\frac{\xi +^t \xi}{2}\right|\right|\right|_{\mathcal{F}} \le |||\xi|||_{\mathcal{F}}.$$

I.5 Green's formula

In [ABH07], [DO05] the discrete gradient and discrete divergence for a scalar-valued function are linked by a discrete Stokes formula. This is precisely the duality property that gives its name to the method.

Theorem I.5.1 ***Discrete Green's formula***
For all $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$, we have:

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\boldsymbol{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega},$$

where $\vec{\boldsymbol{n}}$ is the unitary outer normal.

The proof can be found in [Kre10, Thm IV.9].

I.6 Inf-sup stability

In this section, we first recall the Inf-sup stability property for the DDFV method: its definition and its main consequences. Then, since it has been proven to hold in the case of homogeneous Dirichlet boundary conditions, we will extend the result to the case of $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$. i.e. Dirichlet boundary conditions just on a fraction of the domain.

Inf-sup stability inequality (or LBB), in the continuous setting, is formulated as:

$$\inf_{\mathrm{p}\in L_0^2(\Omega)} \left(\sup_{\mathbf{v}\in(H_0^1(\Omega))^2} \frac{a(\mathbf{v},\mathrm{p})}{\|\mathbf{v}\|_{H^1}\|\mathrm{p}\|_{L^2}} \right) > 0$$

where $a(\mathbf{v},\mathrm{p}) = \displaystyle\int_\Omega \mathrm{p}(\mathrm{div}(\mathbf{v}))$ and $L_0^2(\Omega) = \{\mathrm{p} \in L^2(\Omega) : m(\mathrm{p}) = \frac{1}{|\Omega|}\int_\Omega \mathrm{p} = 0\}$. This inequality is equivalent to the existence of a continuous right-inverse of the divergence operator (see [GR11, BF12]).

This condition is related to the well-posedness of the Stokes problem coupled with homogeneous Dirichlet boundary conditions. In the DDFV framework, Inf-sup stability has been proven to hold unconditionally for conforming acute triangle meshes, non-conforming triangle meshes and chechkerboard meshes. For some conforming or non-conforming Cartesian meshes, it holds up to a single unstable pressure mode. Moreover, it has been proven numerically for many other families of meshes and it has still not been found a mesh that does not satisfy it. For more details, see [BKN15]).

Definition I.6.1 *A given DDFV mesh $\mathfrak{T}$ is said to satisfy the Inf-sup stability if the following condition holds:*

$$\beta_{\mathfrak{T}} := \inf_{\mathrm{p}^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}} \left(\sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0} \frac{a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2\|\mathrm{p}^{\mathfrak{D}} - m(\mathrm{p}^{\mathfrak{D}})\|_2} \right) > 0, \tag{I.9}$$

where $a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}}) = (\mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}$ and $m(\mathrm{p}^{\mathfrak{D}}) = \displaystyle\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}}$.
For a given family of meshes such that size$(\mathfrak{T}) \to 0$, the scheme is stable if and only if

$$\liminf_{size(\mathfrak{T})\to 0} \beta_{\mathfrak{T}} > 0.$$

We shall use the following two consequences of the Inf-sup condition (I.9):

- $\forall \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$:
$$\|\mathrm{p}^{\mathfrak{D}} - m(\mathrm{p}^{\mathfrak{D}})\|_2 \le \frac{1}{\beta_{\mathfrak{T}}} \sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0} \frac{(\mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}}{\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2}, \tag{I.10}$$

- For every $\mathrm{p}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ such that $m(\mathrm{p}^{\mathfrak{D}}) = 0$, there exists $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0$ such that:
$$\begin{aligned} \mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}) &= \mathrm{p}^{\mathfrak{D}} \\ \|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 &\le \frac{1}{\beta_{\mathfrak{T}}^2}\|\mathrm{p}^{\mathfrak{D}}\|_2. \end{aligned} \tag{I.11}$$

The former is a direct consequence of the definition. The latter is more subtle, in particular when considering the constant arising in estimate (I.11). It is worth detailing this issue.

The Inf-sup property is a crucial property of the continuous gradient/divergence operators, which enters into the analysis of the Stokes problem. In particular it is equivalent to the possibility to define a continuous right-inverse of the divergence operator defined from $H_0^1(\Omega)$ to $L^2(\Omega)$, as a consequence of the open mapping theorem [BF12, Sec. IV.3] and (I.11) is nothing but the discrete analog of this property.

For the numerical analysis, it is crucial to check whether or not the constants are uniform with respect to the mesh parameters; which is not completely clear when one uses such an abstract argument.

Therefore let us justify that the discrete divergence operator admits a right inverse, which, furthermore, satisfies a continuity estimate that depends only on the (possibly uniform) constant of the Inf-sup condition (I.9). To this end, we adapt the sketch of proof presented in [DPE12, Rem. 6.7].

Proposition I.6.2 *Let $\mathfrak{T}$ be a mesh that satisfies Inf-sup stability condition. Then, for every $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ with $m(\mathrm{p}^{\mathfrak{D}}) = 0$, there exists $\mathbf{w}^{\mathfrak{T}} \in \mathbb{E}_0$ such that:*

$$\begin{aligned} \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}) &= \mathrm{p}^{\mathfrak{D}} \\ \|\nabla^{\mathfrak{D}} \mathbf{w}^{\mathfrak{T}}\|_2 &\leq \frac{1}{\beta_{\mathfrak{T}}^2} \|\mathrm{p}^{\mathfrak{D}}\|_2, \end{aligned}$$

where $\beta_{\mathfrak{T}}$ is the Inf-sup constant defined in (I.9).

Proof As a warm-up we shall need a discrete analog of the Riesz isomorphism between H_0^1 and its dual space H^{-1}; the analogous discrete spaces are $\mathbb{E}_0$ and its dual $\mathbb{E}_0'$.

Let $l : \mathbb{E}_0 \to \mathbb{R}$ be a continuous linear form on $\mathbb{E}_0$, equipped with the norm induced by the scalar product $(\nabla^{\mathfrak{D}} \cdot : \nabla^{\mathfrak{D}} \cdot)_{\mathfrak{D}}$, i.e. $\forall \mathbf{v}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0$, $(\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}} : \nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} = \sum_{\mathrm{D} \in \mathfrak{D}} m_{\mathrm{D}} (\nabla^{\mathrm{D}} \mathbf{v}^{\mathfrak{T}} : \nabla^{\mathrm{D}} \mathbf{u}^{\mathfrak{T}})$.
By continuity, the linear form l satisfies:

$$|\langle l, \mathbf{v}^{\mathfrak{T}} \rangle_{\mathbb{E}_0', \mathbb{E}_0}| \leq C \|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2.$$

By Riesz representation theorem, there exists a unique $\mathcal{J}_l^{\mathfrak{T}} \in \mathbb{E}_0$ such that $\forall \mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0$:

$$\langle l, \mathbf{v}^{\mathfrak{T}} \rangle_{\mathbb{E}_0', \mathbb{E}_0} = (\nabla^{\mathfrak{D}} \mathcal{J}_l^{\mathfrak{T}} : \nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}})_{\mathfrak{D}}.$$

Moreover, by applying the previous relation, we have:

$$\|l\|_{\mathbb{E}_0'} = \sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0} \frac{|\langle l, \mathbf{v}^{\mathfrak{T}} \rangle_{\mathbb{E}_0', \mathbb{E}_0}|}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2} = \sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0} \frac{|(\nabla^{\mathfrak{D}} \mathcal{J}_l^{\mathfrak{T}}, \nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}})_{\mathfrak{D}}|}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2}.$$

Since $\mathcal{J}_l^{\mathfrak{T}} \in \mathbb{E}_0$ and by Cauchy-Schwarz, it holds:

$$\|l\|_{\mathbb{E}_0'} = \|\nabla^{\mathfrak{D}} \mathcal{J}_l^{\mathfrak{T}}\|_2.$$

We can then deduce that:

$$\langle l, \mathcal{J}_l^{\mathfrak{T}} \rangle_{\mathbb{E}_0', \mathbb{E}_0} = \|\nabla^{\mathfrak{D}} \mathcal{J}_l^{\mathfrak{T}}\|_2^2 = \|l\|_{\mathbb{E}_0'}^2. \tag{I.12}$$

Let $M^{\mathfrak{D}} = \{\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} : m(\mathrm{p}^{\mathfrak{D}}) = 0\}$. We consider now $\forall \mathrm{p}^{\mathfrak{D}} \in M^{\mathfrak{D}}$ and $\forall \mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0$:

$$a^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}}) = (\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}))_{\mathfrak{D}}.$$

Let $\mathrm{p}^{\mathfrak{D}} \in M^{\mathfrak{D}}$. We define the linear form $\mathcal{B}_{\mathrm{p}^{\mathfrak{D}}} : \mathbb{E}_0 \to \mathbb{R}$ on $\mathbb{E}_0$ as:

$$\langle \mathcal{B}_{\mathrm{p}^{\mathfrak{D}}}, \mathbf{v}^{\mathfrak{T}} \rangle_{\mathbb{E}_0', \mathbb{E}_0} = a^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}}). \tag{I.13}$$

The linear form $\mathcal{B}_{\mathrm{p}^{\mathfrak{D}}}$ on $\mathbb{E}_0$ is continuous, since it holds by Cauchy-Schwarz inequality:

$$|\langle \mathcal{B}_{\mathrm{p}^{\mathfrak{D}}}, \mathbf{v}^{\mathfrak{T}} \rangle_{\mathbb{E}_0',\mathbb{E}_0}| = |(\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}))_{\mathfrak{D}}| \leq \|\mathrm{p}^{\mathfrak{D}}\|_2 \|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2. \tag{I.14}$$

We can deduce the following properties on the norm of $\mathcal{B}_{\mathrm{p}^{\mathfrak{D}}}$:

- by (I.14):

$$\|\mathcal{B}_{\mathrm{p}^{\mathfrak{D}}}\|_{\mathbb{E}_0'} = \sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0} \frac{a^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2} = \sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0} \frac{(\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}))_{\mathfrak{D}}}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2} \leq \|\mathrm{p}^{\mathfrak{D}}\|_2 \tag{I.15}$$

- by (I.10), since we suppose that the mesh $\mathfrak{T}$ satisfies Inf-sup condition and that $m(\mathrm{p}^{\mathfrak{D}}) = 0$:

$$\|\mathcal{B}_{\mathrm{p}^{\mathfrak{D}}}\|_{\mathbb{E}_0'} = \sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0} \frac{a^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2} \geq \beta_{\mathfrak{T}} \|\mathrm{p}^{\mathfrak{D}}\|_2 \tag{I.16}$$

Consider now the following problem:

$$\begin{aligned} &\text{Find } \mathrm{q}^{\mathfrak{D}} \in M^{\mathfrak{D}} \text{ such that } \forall \mathrm{r}^{\mathfrak{D}} \in M^{\mathfrak{D}} : \\ &\underbrace{\langle \mathcal{B}_{\mathrm{r}^{\mathfrak{D}}}, \mathcal{J}^{\mathfrak{T}}_{\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}} \rangle_{\mathbb{E}_0',\mathbb{E}_0}}_{=:\widetilde{a}^{\mathfrak{T}}(\mathrm{r}^{\mathfrak{D}},\mathrm{q}^{\mathfrak{D}})} = (\mathrm{p}^{\mathfrak{D}}, \mathrm{r}^{\mathfrak{D}})_{\mathfrak{D}}. \end{aligned} \tag{I.17}$$

If we define $\widetilde{a}^{\mathfrak{T}}(\mathrm{r}^{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}}) := \langle \mathcal{B}_{\mathrm{r}^{\mathfrak{D}}}, \mathcal{J}^{\mathfrak{T}}_{\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}} \rangle_{\mathbb{E}_0',\mathbb{E}_0}$, we can show that $\widetilde{a}^{\mathfrak{T}}$ is coercive. In fact, if we apply (I.12):

$$\widetilde{a}^{\mathfrak{T}}(\mathrm{q}^{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}}) = \langle \mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}, \mathcal{J}^{\mathfrak{T}}_{\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}} \rangle_{\mathbb{E}_0',\mathbb{E}_0} = \|\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}\|^2_{\mathbb{E}_0'},$$

and by (I.16):

$$\widetilde{a}^{\mathfrak{T}}(\mathrm{q}^{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}}) = \|\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}\|^2_{\mathbb{E}_0'} \geq \beta^2_{\mathfrak{T}} \|\mathrm{q}^{\mathfrak{D}}\|^2_2. \tag{I.18}$$

The coercivity of $\widetilde{a}^{\mathfrak{T}}$ implies that problem (I.17) is well-posed. Let $\mathrm{q}^{\mathfrak{D}} \in M^{\mathfrak{D}}$ be the unique solution of (I.17); then:

$$\beta^2_{\mathfrak{T}} \|\mathrm{q}^{\mathfrak{D}}\|^2_2 \underbrace{\leq}_{\text{by (I.18)}} \widetilde{a}^{\mathfrak{T}}(\mathrm{q}^{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}}) \underbrace{=}_{\text{by (I.17)}} (\mathrm{p}^{\mathfrak{D}}, \mathrm{q}^{\mathfrak{D}}) \underbrace{\leq}_{\text{by Cauchy-Schwarz}} \|\mathrm{p}^{\mathfrak{D}}\|_2 \|\mathrm{q}^{\mathfrak{D}}\|_2,$$

which implies $\frac{1}{\beta^2_{\mathfrak{T}}} \|\mathrm{p}^{\mathfrak{D}}\|_2 \geq \|\mathrm{q}^{\mathfrak{D}}\|_2$. From (I.15) and (I.12), we deduce:

$$\frac{1}{\beta^2_{\mathfrak{T}}} \|\mathrm{p}^{\mathfrak{D}}\|_2 \geq \|\mathrm{q}^{\mathfrak{D}}\|_2 \geq \|\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}\|_{\mathbb{E}_0'} = \|\nabla^{\mathfrak{D}} \mathcal{J}^{\mathfrak{T}}_{\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}}\|_2. \tag{I.19}$$

If now we set $\mathbf{w}^{\mathfrak{T}} = \mathcal{J}^{\mathfrak{T}}_{\mathcal{B}_{\mathrm{q}^{\mathfrak{D}}}} \in \mathbb{E}_0$, we obtain

$$\|\nabla^{\mathfrak{D}} \mathbf{w}^{\mathfrak{T}}\|_2 \leq \frac{1}{\beta^2_{\mathfrak{T}}} \|\mathrm{p}^{\mathfrak{D}}\|_2$$

that is the estimate we want to prove. It remains to show that $\mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}) = \mathrm{p}^{\mathfrak{D}}$.

Let $\mathrm{r}^{\mathfrak{D}} \in M^{\mathfrak{D}}$; we can write:

$$(\mathrm{r}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} \underbrace{=}_{\text{by (I.13)}} \langle \mathcal{B}_{\mathrm{r}^{\mathfrak{D}}}, \mathbf{w}^{\mathfrak{T}} \rangle_{\mathbb{E}_0',\mathbb{E}_0} \underbrace{=}_{\text{by (I.17)}} (\mathrm{p}^{\mathfrak{D}}, \mathrm{r}^{\mathfrak{D}})_{\mathfrak{D}}$$

that implies

$$(\mathrm{r}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}} - \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} = 0 \quad \forall \mathrm{r}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}, m(\mathrm{r}^{\mathfrak{D}}) = 0. \tag{I.20}$$

If now we consider $\widetilde{r}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, we can decompose it in:

$$\begin{aligned}\widetilde{r}^{\mathfrak{D}} &= \underbrace{\widetilde{r}^{\mathfrak{D}} - \frac{m(\widetilde{r}^{\mathfrak{D}})}{|\Omega|}\mathbb{1}_{\mathfrak{D}}}_{:=\mathrm{r}^{\mathfrak{D}}} + \frac{m(\widetilde{r}^{\mathfrak{D}})}{|\Omega|}\mathbb{1}_{\mathfrak{D}} \\ &= \mathrm{r}^{\mathfrak{D}} + \frac{m(\widetilde{r}^{\mathfrak{D}})}{|\Omega|}\mathbb{1}_{\mathfrak{D}}.\end{aligned}$$

where $\mathbb{1}_{\mathfrak{D}} = (\mathbb{1}_{\mathtt{D}})_{\mathtt{D}\in\mathfrak{D}}$ and $\mathbb{1}_{\mathtt{D}}$ is the indicator function on $\mathtt{D}$. Remark that $m(\mathrm{r}^{\mathfrak{D}}) = 0$. Thus, if we compute:

$$(\widetilde{r}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}} - \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} = (\mathrm{r}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}} - \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} + \frac{m(\widetilde{r}^{\mathfrak{D}})}{|\Omega|}(\mathbb{1}_{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}} - \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} \tag{I.21}$$

By (I.20), we deduce $(\mathrm{r}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}} - \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} = 0$; remark that $(\mathbb{1}_{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathtt{D}\in DD} m_{\mathtt{D}}\mathrm{p}^{\mathfrak{D}} = m(\mathrm{p}^{\mathfrak{D}}) = 0$ and that by Green's formula (Thm. I.5.1) $(\mathbb{1}_{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} = [[\boldsymbol{\nabla}^{\mathfrak{T}}(\mathbb{1}_{\mathfrak{D}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} = 0$. So (I.21) gives:

$$(\widetilde{r}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}} - \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} = 0 \quad \forall \widetilde{r}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}},$$

that implies $\mathrm{p}^{\mathfrak{D}} = \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}})$. ■

The following result is an extension to Necas Lemma, [GR86, Corollary 2.4]. Instead of considering $v \in H_0^1(\Omega)$, with zero boundary data and $p \in L_0^2(\Omega)$ with zero mean, we take $(v, p) \in H_\Gamma^1(\Omega) \times L^2(\Omega)$, with $H_\Gamma^1 = \{v \in (H^1(\Omega))^2,\ v = 0 \text{ on } \Gamma_D\}$, with $|\Gamma_D| > 0$.

Lemma I.6.3 *Let $H_\Gamma^1 = \{v \in (H^1(\Omega))^2,\ v = 0 \text{ on } \Gamma_D\}$. Then, for every $p \in L^2(\Omega)$ there exists $v \in H_\Gamma^1$ and a constant $C > 0$ depending only on Ω, such that:*

$$\begin{aligned} &div(v) = p \\ &\|v\|_{H^1} \leq C\|p\|_2. \end{aligned} \tag{I.22}$$

Proof Consider $\omega \in H_\Gamma^1$ such that $\frac{1}{|\Omega|}m(\mathrm{div}(\omega)) = 1$, i.e. $\frac{1}{|\Omega|}\int_\Omega \mathrm{div}(\omega) = 1$. We can decompose any $p \in L^2(\Omega)$ into:

$$\begin{aligned} p &= \underbrace{p - \frac{1}{|\Omega|}m(p)\mathrm{div}(\omega)}_{:=\widetilde{p}} + \frac{1}{|\Omega|}m(p)\mathrm{div}(\omega) \\ &= \widetilde{p} + \frac{1}{|\Omega|}m(p)\mathrm{div}(\omega). \end{aligned}$$

Remark that $\widetilde{p} \in L_0^2(\Omega)$. By Necas Lemma, [[GR86], Corollary 2.4], there exists $\widetilde{v} \in (H_0^1(\Omega))^2$ such that

$$\begin{aligned} &\mathrm{div}(\widetilde{v}) = \widetilde{p} \\ &\|\widetilde{v}\|_{H^1} \leq C\|\widetilde{p}\|_{L^2}. \end{aligned}$$

If we set $v = \widetilde{v} + m(p)\omega$, we can observe that $v \in H^1_\Gamma(\Omega)$ and that

$$\begin{aligned}\mathrm{div}(v) &= \mathrm{div}(\widetilde{v}) + \frac{1}{|\Omega|} m(p)\mathrm{div}(\omega)\\ &= \widetilde{p} + \frac{1}{|\Omega|} m(p)\mathrm{div}(\omega) = p,\end{aligned}$$

that is the first property of (I.22).
We now look at $\|v\|_{H^1}$. By Minkowski inequality:

$$\begin{aligned}\|v\|_{H^1} &= \|\widetilde{v} + \frac{1}{|\Omega|} m(p)\omega\|_{H^1}\\ &\leq \|\widetilde{v}\|_{H^1} + \frac{|m(p)|}{|\Omega|}\|\omega\|_{H^1}.\end{aligned}$$

We apply Necas Lemma to $\|\widetilde{v}\|_{H^1}$ and Cauchy-Schwarz inequality to $|m(p)|\|\omega\|_{H^1}$:

$$\|v\|_{H^1} \leq C\|\widetilde{p}\|_{L^2} + \frac{1}{\sqrt{|\Omega|}}\|p\|_{L^2}\,\|\omega\|_{H^1}. \tag{I.23}$$

We now need to estimate $\|\widetilde{p}\|_{L^2}$. By definition of $\widetilde{p}$ and Minkowsky inequality, we can write:

$$\|\widetilde{p}\|_{L^2} \leq \|p\|_{L^2} + \frac{|m(p)|}{|\Omega|}\|\mathrm{div}(\omega)\|_{L^2}.$$

By Cauchy-Schwarz inequality, the fact that $\|\mathrm{div}(\omega)\|_{L^2} \leq \|\omega\|_{H^1}$ and the bound of $|m(\mathrm{p})|$ we deduce:

$$\|\widetilde{p}\|_{L^2} \leq \left(1 + \frac{1}{\sqrt{|\Omega|}}\|\omega\|_{H^1}\right)\|p\|_{L^2}.$$

Injecting this estimate in (I.23), we obtain:

$$\|v\|_{H^1} \leq \left[C\left(1 + \frac{1}{\sqrt{|\Omega|}}\|\omega\|_{H^1}\right) + \frac{1}{\sqrt{|\Omega|}}\|\omega\|_{H^1}\right]\|p\|_{L^2}$$

that, if we define $\widetilde{C} = C\left(1 + \frac{1}{\sqrt{|\Omega|}}\|\omega\|_{H^1}\right) + \frac{1}{\sqrt{|\Omega|}}\|\omega\|_{H^1}$, gives:

$$\|v\|_{H^1} \leq \widetilde{C}\|p\|_{L^2}.$$

This proves (I.22) of Lemma I.6.3. ∎

I.6.1 Extension to inhomogeneous Dirichlet boundary conditions

In this section we extend the results of [BKN15] to the case in which $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ instead of $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0$; this will come as a natural extension since $\mathbb{E}_0 \subset \mathbb{E}_0^{\Gamma_D}$.

Theorem I.6.4 *For a given DDFV mesh $\mathfrak{T}$ that satisfies Inf-sup stability (see Def.I.6.1), there exists $\widetilde{\beta}_{\mathfrak{T}}$ such that:*

$$\widetilde{\beta}_{\mathfrak{T}} := \inf_{\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}}\left(\sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}} \frac{a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2\|\mathrm{p}^{\mathfrak{D}} - m(\mathrm{p}^{\mathfrak{D}})\|_2}\right) > 0, \tag{I.24}$$

where $a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}}) = (\mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}$ *and* $m(\mathrm{p}^{\mathfrak{D}}) = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}}$.
Moreover, for a given family of meshes which are Inf-sup stable such that $size(\mathfrak{T}) \to 0$, *we have:*

$$\liminf_{size(\mathfrak{T})\to 0} \widetilde{\beta}_{\mathfrak{T}} > 0.$$

Proof To prove this result, it is necessary to refer on how in [BKN15] the authors prove that Inf-sup condition (of Def. I.6.1) holds for a given mesh $\mathfrak{T}$. We show how it is possible to extend the results of this proof to the case of $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$. To do that, we do not detail the computations (that can be found in [BKN15]), but we give an idea of the key points of the proof and we show how to deduce (I.24).

The strategy used in [BKN15] is to relate the value of $\beta_{\mathfrak{T}}$ to the eigenvalues of a suitable matrix. The reformulation of (I.9) as an eigenvalue problem reads:

$$\beta_{\mathfrak{T}} = \inf_{\substack{\mathrm{p}^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}\\ \langle M_{\mathfrak{T}}\mathrm{p}^{\mathfrak{D}},1\rangle=0}} \left(\sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0} \frac{\langle B_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}}\rangle}{\langle R_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}\rangle^{\frac12}\langle M_{\mathfrak{T}}\mathrm{p}^{\mathfrak{D}},\mathrm{p}^{\mathfrak{D}}\rangle^{\frac12}}\right), \tag{I.25}$$

where $R_{\mathfrak{T}}$ is the stiffness matrix, $B_{\mathfrak{T}}$ is the divergence matrix and $M_{\mathfrak{T}}$ the pressure mass matrix. Those matrices are built so that:

$$\begin{aligned}
\langle R_{\mathfrak{T}}\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}\rangle &= (\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}})_{\mathfrak{D}},\\
\langle B_{\mathfrak{T}}\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}}\rangle &= a^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}}) = (\mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}},\\
\langle M_{\mathfrak{T}}\mathrm{p}^{\mathfrak{D}},\mathrm{q}^{\mathfrak{D}}\rangle &= (\mathrm{p}^{\mathfrak{D}},\mathrm{q}^{\mathfrak{D}})_{\mathfrak{D}},
\end{aligned} \tag{I.26}$$

where we denote by $\langle\cdot,\cdot\rangle$ the Euclidian inner product on the spaces $\mathbb{R}^{2\mathcal{N}_{\mathfrak{T}}}$ and $\mathbb{R}^{\mathcal{N}_{\mathfrak{D}}}$, with $\mathcal{N}_{\mathfrak{T}} = \mathrm{Card}(\mathfrak{T})$, $\mathcal{N}_{\mathfrak{D}} = \mathrm{Card}(\mathfrak{D})$.
Our aim is to extend the reformulation (I.25) to the case of $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, thus we need to build different stiffness and divergence matrices that take into account boundary terms and maintain the same properties of (I.26).

The new stiffness matrix $\widetilde{R}_{\mathfrak{T}}$, modified only on $\partial\mathfrak{M}_N$ and $\partial\mathfrak{M}_N^*$, is built as:

$$\widetilde{R}_{\mathfrak{T}}\mathbf{u}^{\mathfrak{T}} = \begin{pmatrix}
\left(-\dfrac{m_{\mathtt{K}}}{2}\mathbf{div}^{\mathtt{K}}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})\right)_{\mathtt{K}\in\mathfrak{M}}\\
(\mathbf{u}_\sigma)_{\sigma\in\partial\mathfrak{M}_D}\\
\left(\dfrac{m_\sigma}{2}\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right)_{\sigma\in\partial\mathfrak{M}_N}\\
\left(-\dfrac{m_{\mathtt{K}^*}}{2}\mathbf{div}^{\mathtt{K}^*}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})\right)_{\mathtt{K}^*\in\mathfrak{M}^*}\\
(\mathbf{u}_{\mathtt{K}^*})_{\mathtt{K}^*\in\partial\mathfrak{M}_D^*}\\
\left(-\dfrac{m_{\mathtt{K}^*}}{2}\mathbf{div}^{\mathtt{K}^*}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}) + \sum\limits_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}}\dfrac{m_\sigma}{4}\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right)_{\mathtt{K}^*\in\partial\mathfrak{M}_N^*}
\end{pmatrix},$$

so that, if we take the product with a vector $\mathbf{v}^{\mathfrak{T}}$, we have:

$$\langle\widetilde{R}_{\mathfrak{T}}\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}\rangle = -[[\mathbf{div}^{\mathfrak{T}}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}),\mathbf{v}^{\mathfrak{T}}]]_{\mathfrak{T}} + \left(\gamma^{\mathfrak{D}}(\nabla^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}),\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\right)_{\partial\Omega},$$

that by Green's formula (Thm. I.5.1) implies $\langle \widetilde{R}_{\mathfrak{T}} \mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}} \rangle_{\mathfrak{T}} = (\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}} : \nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}})_{\mathfrak{D}}$ so that:

$$\langle \widetilde{R}_{\mathfrak{T}} \mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}} \rangle = \langle R_{\mathfrak{T}} \mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}} \rangle. \tag{I.27}$$

For what concerns the divergence matrix, we remark that $B_{\mathfrak{T}}$ is identical, while its transposed is modified into ${}^t\widetilde{B}_{\mathfrak{T}}$ in order to take into account the boundary terms on $\partial\mathfrak{M}_N$ and $\partial\mathfrak{M}_N^*$:

$$B_{\mathfrak{T}} \mathbf{u}^{\mathfrak{T}} = \left(-m_{\mathtt{D}} \mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}})_{\mathtt{D}\in\mathfrak{D}} \right) \quad \text{and} \quad {}^t\widetilde{B}_{\mathfrak{T}} \mathrm{p}^{\mathfrak{D}} = \frac{1}{2} \begin{pmatrix} \left(m_{\mathtt{K}} \nabla^{\mathtt{K}} \mathrm{p}^{\mathfrak{D}} \right)_{\mathtt{K}\in\mathfrak{M}} \\ (\,0\,)_{\sigma\in\partial\mathfrak{M}_D} \\ (\,-m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma\mathtt{K}}\,)_{\sigma\in\partial\mathfrak{M}_N} \\ \left(m_{\mathtt{K}^*} \nabla^{\mathtt{K}^*} \mathrm{p}^{\mathfrak{D}} \right)_{\mathtt{K}^*\in\mathfrak{M}^*} \\ (0)_{\mathtt{K}^*\in\partial\mathfrak{M}_D^*} \\ \left(m_{\mathtt{K}^*} \nabla^{\mathtt{K}^*} \mathrm{p}^{\mathfrak{D}} - \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}} \frac{m_\sigma}{2} \mathrm{p}^{\mathtt{D}} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} \right)_{\mathtt{K}^*\in\partial\mathfrak{M}_N^*} \end{pmatrix}.$$

So, if we take the product with a vector $\mathrm{p}^{\mathfrak{D}}$, we have:

$$\begin{aligned} \langle B_{\mathfrak{T}} \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}} \rangle &= -(\mathrm{div}^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}} \\ \langle {}^t\widetilde{B}_{\mathfrak{T}} \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}} \rangle &= [[\boldsymbol{\nabla}^{\mathfrak{T}} \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} - \left(\mathrm{p}^{\mathfrak{D}_{ext}} \vec{\mathbf{n}}_{\sigma\mathtt{K}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) \right)_{\partial\Omega}. \end{aligned}$$

Green's formula (Thm. I.5.1) gives $\langle {}^t\widetilde{B}_{\mathfrak{T}} \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}} \rangle = -(\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}}$, so that:

$$\langle B_{\mathfrak{T}} \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}} \rangle = \langle {}^t\widetilde{B}_{\mathfrak{T}} \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}} \rangle. \tag{I.28}$$

So, if we define the DDFV for the Stokes scheme as:

$$\text{Find } (\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_0^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}} \text{ such that:}$$

$$\begin{cases} \mathbf{div}^{\mathtt{K}}(-\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}_{\mathtt{K}} & \forall \mathtt{K} \in \mathfrak{M} \\ \mathbf{div}^{\mathtt{K}^*}(-\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}_{\mathtt{K}^*} & \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_N^* \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) = 0 & \\ (\nabla^{\mathtt{D}} \mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}} \mathrm{Id}) \vec{\mathbf{n}}_{\sigma\mathtt{K}} = \Phi_\sigma & \forall \mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N, \end{cases} \tag{I.29}$$

this is equivalent to

$$\begin{pmatrix} \widetilde{R}_{\mathfrak{T}} & {}^t\widetilde{B}_{\mathfrak{T}} \\ B_{\mathfrak{T}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}^{\mathfrak{T}} \\ \mathrm{p}^{\mathfrak{D}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{\mathfrak{M}} \\ 0 \\ \Phi_{\partial\mathfrak{M}_N} \\ \mathbf{f}_{\mathfrak{M}^*} \\ 0 \\ \mathbf{f}_{\partial\mathfrak{M}^*} \\ 0 \end{pmatrix}.$$

By (I.27)- (I.28) and the fact that $\mathbb{E}_0 \subset \mathbb{E}_0^{\Gamma_D}$, we can conclude:

$$\begin{aligned}\beta_{\mathfrak{T}} = & \inf_{\substack{p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}\\ \langle M_{\mathfrak{T}}p^{\mathfrak{D}},1\rangle=0}} \left(\sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0} \frac{\langle B_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}}, p^{\mathfrak{D}}\rangle}{\langle R_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}\rangle^{\frac{1}{2}}\langle M_{\mathfrak{T}}p^{\mathfrak{D}},p^{\mathfrak{D}}\rangle^{\frac{1}{2}}}\right)\\ = & \inf_{\substack{p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}\\ \langle M_{\mathfrak{T}}p^{\mathfrak{D}},1\rangle=0}} \left(\sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0} \frac{\langle B_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}}, p^{\mathfrak{D}}\rangle}{\langle \widetilde{R}_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}\rangle^{\frac{1}{2}}\langle M_{\mathfrak{T}}p^{\mathfrak{D}},p^{\mathfrak{D}}\rangle^{\frac{1}{2}}}\right)\\ \leq & \inf_{\substack{p^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}\\ \langle M_{\mathfrak{T}}p^{\mathfrak{D}},1\rangle=0}} \left(\sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0^{\Gamma_D}} \frac{\langle B_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}}, p^{\mathfrak{D}}\rangle}{\langle \widetilde{R}_{\mathfrak{T}}\mathbf{v}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}\rangle^{\frac{1}{2}}\langle M_{\mathfrak{T}}p^{\mathfrak{D}},p^{\mathfrak{D}}\rangle^{\frac{1}{2}}}\right) = \widetilde{\beta}_{\mathfrak{T}},\end{aligned}$$

so that every time that $\beta_{\mathfrak{T}}$ is proven (theoretically or numerically) to be positive, also $\widetilde{\beta}_{\mathfrak{T}} > 0$. This, in particular, implies that for a given family of meshes such that $\mathrm{size}(\mathfrak{T}) \to 0$, by Def.I.6.1:

$$\liminf_{\mathrm{size}(\mathfrak{T})\to 0} \widetilde{\beta}_{\mathfrak{T}} > \liminf_{\mathrm{size}(\mathfrak{T})\to 0} \beta_{\mathfrak{T}} > 0.$$

■

From (I.24), as from Def. I.6.1, we deduce the analogous of properties (I.10) and (I.11). In fact, the following holds:

- $\forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$:
$$\|p^{\mathfrak{D}} - m(p^{\mathfrak{D}})\|_2 \leq \frac{1}{\widetilde{\beta}_{\mathfrak{T}}^2} \sup_{\mathbf{v}^{\mathfrak{T}}\in\mathbb{E}_0^{\Gamma_D}} \frac{(\mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}, p^{\mathfrak{D}})_{\mathfrak{D}}}{\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2}, \tag{I.30}$$

- For every $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, there exists $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and there exists $h_0 > 0$ such that $\forall h = \mathrm{size}(\mathfrak{T}) \leq h_0$, there exists a constant C depending only on $\beta_{\mathfrak{T}}, \Omega, \mathrm{reg}(\mathfrak{T}), h_0$:
$$\begin{aligned}\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}) =& p^{\mathfrak{D}}\\ \|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \leq& C\|p^{\mathfrak{D}}\|_2.\end{aligned} \tag{I.31}$$

As for the case of (I.10), property (I.30) is a direct consequence of (I.24). For (I.31), the proof is the discrete analog of Lemma I.6.3, that we detail here.

Lemma I.6.5 *Let $\mathfrak{T}$ be a mesh that satisfies Inf-sup stability condition; we denote $h = size(\mathfrak{T})$. Then, for every $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, there exists $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$:*

$$\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}) = p^{\mathfrak{D}}$$

and there exists $h_0 > 0$ such that $\forall h \leq h_0$, there exists a constant C depending only on $\beta_{\mathfrak{T}}, \Omega, reg(\mathfrak{T}), h_0$ such that:

$$\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \leq C\|p^{\mathfrak{D}}\|_2.$$

Proof To prove this result, we adapt the proof of Lemma I.6.3 to the discrete case; in this proof, by means of a particular function, we reduce to the case of homogeneous Dirichlet in order to apply the result of Prop. I.6.2.

Let $\mathbf{v} \in H^1_{\Gamma_D}(\Omega)$ with $\frac{1}{|\Omega|}\int_\Omega \mathrm{div}(\mathbf{v}) = 1$. Let $\mathbf{v}^{\mathfrak{T}} = \mathbb{P}_m^{\mathfrak{T}}\mathbf{v}$.

By definition of the projection (see Sec. I.2), we have $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0$. Moreover, by [ABH07, Corollary 3.1], given a sequence of meshes $(\mathfrak{T}_n)_n$ such that $\mathrm{size}(\mathfrak{T}_n) \to 0$ as $n \to \infty$ and $\mathrm{reg}(\mathfrak{T}_n)$ is bounded, we have:

$$\mathbf{v}^{\mathfrak{T}_n} \to \mathbf{v} \quad \text{in } (L^2(\Omega))^2 \text{ as } n \to \infty,$$
$$\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}_n} \to \nabla\mathbf{v} \quad \text{in } (L^2(\Omega))^{2\times 2} \text{ as } n \to \infty.$$

Remark that $\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}_n}$ can be seen as the L^2 function $\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}_n} = \sum_{\mathtt{D}\in\mathfrak{D}} \nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}_n}\mathbb{1}_{\mathtt{D}}$; as a matter of fact, it implies $\forall\varphi \in (L^2(\Omega))^{2\times 2}$

$$\int_\Omega \left(\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}_n} : \varphi\right) \to \int_\Omega (\nabla\mathbf{v} : \varphi)\,.$$

Thanks to the relation between the gradient and the divergence operator (see (I.5)-(I.6)) and by choosing $\varphi = \frac{1}{|\Omega|}\begin{pmatrix}\mathbb{1}_\Omega & 0\\ 0 & 0\end{pmatrix}$ and $\varphi = \frac{1}{|\Omega|}\begin{pmatrix}0 & 0\\ 0 & \mathbb{1}_\Omega\end{pmatrix}$, we obtain:

$$\widetilde{m}^{\mathfrak{T}_n} = \frac{1}{|\Omega|}m(\operatorname{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}_n})) \to \frac{1}{|\Omega|}m(\operatorname{div}(\mathbf{v})) = 1 \text{ as } n \to \infty. \tag{I.32}$$

Define now $\mathbf{w}^{\mathfrak{T}} = \dfrac{1}{\widetilde{m}^{\mathfrak{T}}}\mathbf{v}^{\mathfrak{T}}$, where $\widetilde{m}^{\mathfrak{T}} := \dfrac{1}{|\Omega|}m(\operatorname{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}))$; it satisfies $\mathbf{w}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and $\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}} = \dfrac{1}{\widetilde{m}^{\mathfrak{T}}}\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}$. Moreover, by definition of $\widetilde{m}^{\mathfrak{T}}$,

$$m(\operatorname{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}})) = \frac{1}{\widetilde{m}^{\mathfrak{T}}}\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\operatorname{div}^{\mathtt{D}}(\mathbf{v}^{\mathfrak{T}}) = |\Omega|. \tag{I.33}$$

Its gradient is bounded by:

$$\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \le \frac{1}{\widetilde{m}^{\mathfrak{T}}}\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2$$

and by [ABH07, Lemma 3.4], there exists C that depends only on $\operatorname{reg}(\mathfrak{T})$ such that

$$\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \le \frac{C}{\widetilde{m}^{\mathfrak{T}}}\|\nabla\mathbf{v}\|_2.$$

By (I.32), there exists $h_0 > 0$ such that $\forall h \le h_0$ (where $h = \operatorname{size}(\mathfrak{T})$), $\widetilde{m}^{\mathfrak{T}} > \frac{1}{2}$, so:

$$\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \le 2C\|\nabla\mathbf{v}\|_2. \tag{I.34}$$

We can then decompose any $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ into:

$$\begin{aligned}\mathrm{p}^{\mathfrak{D}} &= \underbrace{\mathrm{p}^{\mathfrak{D}} - \frac{1}{|\Omega|}m(\mathrm{p}^{\mathfrak{D}})\operatorname{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}})}_{:=\widetilde{p}^{\mathfrak{D}}} + \frac{1}{|\Omega|}m(\mathrm{p}^{\mathfrak{D}})\operatorname{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}})\\ &= \widetilde{p}^{\mathfrak{D}} + \frac{1}{|\Omega|}m(\mathrm{p}^{\mathfrak{D}})\operatorname{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}).\end{aligned}$$

Remark that $\widetilde{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ with $m(\widetilde{p}^{\mathfrak{D}}) = 0$ by (I.33). By Prop. I.6.2, there exists $\widetilde{\mathbf{v}}^{\mathfrak{T}} \in \mathbb{E}_0$ such that

$$\begin{aligned}\operatorname{div}^{\mathfrak{D}}(\widetilde{\mathbf{v}}^{\mathfrak{T}}) &= \widetilde{\mathrm{p}}^{\mathfrak{D}}\\ \|\nabla^{\mathfrak{D}}\widetilde{\mathbf{v}}^{\mathfrak{T}}\|_2 &\le \frac{1}{\beta_{\mathfrak{T}}^2}\|\widetilde{\mathrm{p}}^{\mathfrak{D}}\|_2.\end{aligned} \tag{I.35}$$

If we set $\mathbf{v}^{\mathfrak{T}} = \widetilde{\mathbf{v}}^{\mathfrak{T}} + \frac{1}{|\Omega|}m(\mathrm{p}^{\mathfrak{D}})\mathbf{w}^{\mathfrak{T}}$, we can observe that $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and that

$$\operatorname{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}) = \operatorname{div}^{\mathfrak{D}}(\widetilde{\mathbf{v}}^{\mathfrak{T}}) + \frac{1}{|\Omega|}m(\mathrm{p}^{\mathfrak{D}})\operatorname{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}).$$

Using (I.35) and the decomposition of $\mathrm{p}^{\mathfrak{D}}$ we get:

$$\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}) = \widetilde{p}^{\mathfrak{D}} + \frac{1}{|\Omega|}m(\mathrm{p}^{\mathfrak{D}})\mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}) = \mathrm{p}^{\mathfrak{D}},$$

that is the first property of Lemma I.6.5 that we wanted to prove.
We now look at $\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2$. By Minkowski inequality:

$$\begin{aligned}\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 &= \|\nabla^{\mathfrak{D}}\widetilde{\mathbf{v}}^{\mathfrak{T}} + \frac{1}{|\Omega|}m(\mathrm{p}^{\mathfrak{D}})\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \\ &\leq \|\nabla^{\mathfrak{D}}\widetilde{\mathbf{v}}^{\mathfrak{T}}\|_2 + \frac{|m(\mathrm{p}^{\mathfrak{D}})|}{|\Omega|}\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2.\end{aligned}$$

By (I.35) and applying Cauchy-Schwarz inequality to $m(\mathrm{p}^{\mathfrak{D}}) = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}}$, by recalling that $|\Omega| = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}$, we obtain:

$$\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \leq \frac{1}{\beta_{\mathfrak{T}}^2}\|\widetilde{p}^{\mathfrak{D}}\|_2 + \frac{1}{\sqrt{|\Omega|}}\|\mathrm{p}^{\mathfrak{D}}\|_2\,\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2. \tag{I.36}$$

We now need to estimate $\|\widetilde{p}^{\mathfrak{D}}\|_2$. By definition of $\widetilde{p}^{\mathfrak{D}}$ and Minkowsky inequality, we can write:

$$\|\widetilde{p}^{\mathfrak{D}}\|_2 \leq \|\mathrm{p}^{\mathfrak{D}}\|_2 + \frac{1}{|\Omega|}|m(\mathrm{p}^{\mathfrak{D}})|\|\mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}})\|_2.$$

By Cauchy-Schwarz inequality, the fact that $\|\mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}})\|_2 \leq \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2$ and the bound of $|m(\mathrm{p}^{\mathfrak{D}})|$ we deduce:

$$\|\widetilde{p}^{\mathfrak{D}}\|_2 \leq \left(1 + \frac{1}{\sqrt{|\Omega|}}\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\right)\|\mathrm{p}^{\mathfrak{D}}\|_2.$$

Injecting this estimate in (I.36), we obtain:

$$\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \leq \left[\frac{1}{\beta_{\mathfrak{T}}^2}\left(1 + \frac{1}{\sqrt{|\Omega|}}\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\right) + \frac{1}{\sqrt{|\Omega|}}\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\right]\|\mathrm{p}^{\mathfrak{D}}\|_2$$

and by (I.34) it becomes:

$$\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \leq \left[\frac{1}{\beta_{\mathfrak{T}}^2}\left(1 + \frac{2C}{\sqrt{|\Omega|}}\|\nabla\mathbf{v}\|_2\right) + \frac{2C}{\sqrt{|\Omega|}}\|\nabla\mathbf{v}\|_2\right]\|\mathrm{p}^{\mathfrak{D}}\|_2$$

that, if we define $\widetilde{C} = \frac{1}{\beta_{\mathfrak{T}}^2}\left(1 + \frac{2C}{\sqrt{|\Omega|}}\|\nabla\mathbf{v}\|_2\right) + \frac{2C}{\sqrt{|\Omega|}}\|\nabla\mathbf{v}\|_2$, gives:

$$\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \leq \widetilde{C}\|\mathrm{p}^{\mathfrak{D}}\|_2.$$

This concludes the proof of Lemma I.6.5. ■

I.7 Stabilization of Brezzi-Pitkaranta

This stabilization term is inspired by the Brezzi-Pitkäranta method [BP84] in the finite element framework. This strategy has been previously used in the finite volume framework by [EHL06, EHL07]; in particular, in the DDFV framework, it was proposed by the author of [Kre11a] to add this term in the mass conservation equation, in order to deal with the lack of a uniform discrete Inf-sup condition for general meshes. Later, in [BKN15], Inf-sup condition was studied (as presented in Sec. I.6); it has been proven to hold for a large class of meshes but not for general meshes, thus sometimes it will be preferable to stabilize the equation of conservation of mass in order to avoid the Inf-sup hypothesis on the mesh.

To define this Brezzi-Pitkäranta stabilization term in Stokes and Navier-Stokes problem we need a second order discrete operator, denoted by $\Delta^{\mathfrak{D}} : \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \mapsto \Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, defined as follows:

$$\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}} = \frac{1}{m_{\mathtt{D}}} \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathcal{E}_{\mathtt{D}}} \frac{\mathrm{d}_{\mathtt{D}}^2 + \mathrm{d}_{\mathtt{D}'}^2}{\mathrm{d}_{\mathtt{D}}^2}(\mathrm{p}^{\mathtt{D}'} - \mathrm{p}^{\mathtt{D}}), \quad \forall \mathtt{D} \in \mathfrak{D}.$$

Remark I.7.1 *We recall that $\mathcal{E}_{\mathtt{D}}$ contains just the interior edges of the diamonds. Actually, we never consider $\mathfrak{s} \in \partial\Omega$. This means that we imposed automatically an homogeneous Neumann condition on the boundary $\partial\Omega$ inside the operator $\Delta^{\mathfrak{D}}$.*

It resembles an approximation of the Laplace's operator, however it is consistent only under orthogonality condition (as in the case of *admissible* meshes, see [EGH00, Dro14]); that is not true in general for diamond meshes obtained from $\mathfrak{M}$.

In relation with this operator we define a semi-norm $|\cdot|_h$ on $\mathbb{R}^{\mathfrak{D}}$ that depends on the mesh:

$$|\mathrm{p}^{\mathfrak{D}}|_h^2 = \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathfrak{G}} (\mathrm{d}_{\mathtt{D}}^2 + \mathrm{d}_{\mathtt{D}'}^2)(\mathrm{p}^{\mathtt{D}'} - \mathrm{p}^{\mathtt{D}})^2, \quad \forall \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}. \tag{I.37}$$

Remark I.7.2 *By reorganizing the sum on the diamond edges $\mathfrak{s} \in \mathfrak{G}$, we have that for all $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$:*

$$\begin{aligned} -(d_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}p^{\mathfrak{D}}, p^{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathtt{D}\in\mathfrak{D}} p^{\mathtt{D}} \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathcal{E}_{\mathtt{D}}} (d_{\mathtt{D}}^2 + d_{\mathtt{D}'}^2)(p^{\mathtt{D}} - p^{\mathtt{D}'}) \\ &= \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathfrak{G}} (d_{\mathtt{D}}^2 + d_{\mathtt{D}'}^2)(p^{\mathtt{D}'} - p^{\mathtt{D}}) \\ &= |p^{\mathfrak{D}}|_h^2. \end{aligned}$$

The following lemma is an inverse Sobolev lemma, i.e. the seminorm $|\cdot|_h$ is bounded by the L^2 norm $\|\cdot\|_2$.

Lemma I.7.3 ([Kre10], Lemma IV.13) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on $reg(\mathfrak{T})$, such that $\forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ we have:*

$$|p^{\mathfrak{D}}|_h^2 \le C\|p^{\mathfrak{D}}\|_2.$$

I.8 Results on the strain rate tensor

In this section, we compare the strain rate tensor to the gradient: this will be useful in the estimates, in order to pass from a Laplace form of Stokes (or Navier-Stokes) problem to a Divergence form, and viceversa. The most important result of this section is the Korn inequality.

I.8.1 Bound for the strain rate tensor

This estimate comes straightforward from the definition of the operators; the goal is to bound the norm of the strain rate tensor with the one of the gradient.

Proposition I.8.1 *For all* $\mathbf{u}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$*, we have:*

$$\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2.$$

Proof Thanks to Rem. I.4.1, we have:

$$\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_{\mathcal{F}}^2 \leq \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_{\mathcal{F}}^2 = \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2.$$

■

I.8.2 Korn inequality

The proof of the discrete Korn inequality is inspired by the continuous version in [BS07]. In DDFV setting in the case of homogeneous Dirichlet boundary conditions, i.e. if $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0$, the theorem was proved in [Kre10]. In this case the proof relies on the definition of the operators and the constant of the estimate can be explicitly computed. By adding a part of the boundary with non-zero data, we introduce some difficulties and we are able to prove the result only by contradiction, just as in the continuous setting.

Theorem I.8.2 *(Korn's inequality) Let* $\mathfrak{T}$ *be a mesh that satisfies Inf-sup stability condition. Then there exists* $C > 0$*, that depends only on* $reg(\mathfrak{T})$*,*$\beta_{\mathfrak{T}}$*, such that :*

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \qquad \forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}.$$

In order to prove this result, it is necessary to first consider the case in which $\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}$ has zero mean.

Lemma I.8.3 *Let* $\mathfrak{T}$ *be a mesh that satisfies Inf-sup stability condition. Then* $\forall \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ *that satisfies* $m(rot^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} rot^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = 0$ *it holds:*

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq \frac{1}{\beta_{\mathfrak{T}}^2}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2,$$

where $\beta_{\mathfrak{T}}$ *is the Inf-sup constant defined in* (I.9)*.*

Proof (of Lemma I.8.3) Let $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ such that $m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}) = 0$. If we consider the function

$$\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \sum_{\mathtt{D}\in\mathfrak{D}} \mathrm{rot}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}\mathbf{1}_{\mathtt{D}},$$

this is an L^2 function with zero mean, by hypothesis. This means that, by infsup stability condition (I.11), $\exists \mathbf{w}^{\mathfrak{T}} \in \mathbb{E}_0$ such that:

$$\begin{aligned} \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}) =& \mathrm{rot}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) \\ \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \leq& \frac{1}{\beta_{\mathfrak{T}}^2}\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2. \end{aligned} \tag{I.38}$$

Moreover, if we define the matrix $\chi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have the following property:

$$\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \frac{1}{2}\mathrm{rot}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}})\chi \quad \forall \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{D}}.$$

Let us compute by replacing the value of $\mathrm{D}^{\mathfrak{D}}$ and by developing:

$$\begin{aligned}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}&-\mathrm{curl}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}}\\ &= (\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \frac{1}{2}\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\chi : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}}\\ &= \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 - (\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\chi : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}}.\end{aligned}$$

Since

$$(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} = 0,$$

and

$$\begin{aligned}(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\chi : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} &= (\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, -\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}},\\ (\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\chi : \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} &= (\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, -\mathrm{div}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}}.\end{aligned}$$

(pay attention that the product between matrices becomes a product between scalars), we then obtain:

$$\begin{aligned}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}&-\mathrm{curl}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}}\\ &= \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + 0 + \frac{1}{2}(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, \underbrace{-\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{div}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}}_{=0 \text{ by infsup}})_{\mathfrak{D}} \qquad (\mathrm{I}.39)\\ &= \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2.\end{aligned}$$

This means that, if we apply the Cauchy-Schwarz inequality and triangle inequality to (I.39), we deduce:

$$\begin{aligned}\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 &\leq \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\\ &\leq \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 + \|\mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2).\end{aligned}$$

By applying the definition of $\mathrm{curl}^{\mathfrak{D}}$ and (I.38) we get:

$$\begin{aligned}\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 &\leq \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 + \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2)\\ &\leq \frac{1}{\beta_{\mathfrak{T}}^2}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2.\end{aligned}$$

We conclude that:

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq \frac{1}{\beta_{\mathfrak{T}}^2}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2.$$

■

Thanks to this result, we give the proof of Korn's inequality in the general case.
Proof (of Theorem I.8.2) Let $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$. We define $\mathbf{z}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ as:

$$\mathbf{z}^{\mathfrak{T}} = \mathbf{u}^{\mathfrak{T}} + \frac{1}{2}m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})\mathbf{x}^{\mathfrak{T}},$$

where $\mathbf{x}^{\mathfrak{T}} = \mathbb{P}_c^{\mathfrak{T}}\begin{pmatrix} y \\ -x \end{pmatrix}$ is a vector that satisfies for all $\mathrm{D} \in \mathfrak{D}$: $\nabla^{\mathrm{D}}\mathbf{x}^{\mathfrak{T}} = \chi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathrm{D}^{\mathrm{D}}\mathbf{x}^{\mathfrak{T}} = 0$ and $\mathrm{rot}^{\mathrm{D}}\mathbf{x}^{\mathfrak{T}} = -2$.
As a consequence, we have that $m(\mathrm{rot}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}) = 0$ and $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \mathrm{D}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}$. By Lemma I.8.3 to $\mathbf{z}^{\mathfrak{T}}$:

$$\|\nabla^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2 \leq \frac{1}{\beta_{\mathfrak{T}}^2}\|\mathrm{D}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2. \qquad (\mathrm{I}.40)$$

If we compute $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}$, using the fact that $\nabla^{\mathrm{D}}\mathbf{x}^{\mathfrak{T}} = \chi$ we obtain:

$$\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \nabla^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}} - \frac{1}{2}m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})\chi,$$

from which we deduce

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \le C(\|\nabla^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2 + |m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})|),$$

where C depends on the size of the domain Ω. By (I.40)

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \le \bar{C}(\|\mathrm{D}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2 + |m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})|),$$

that by the definition of $\mathbf{z}^{\mathfrak{T}}$ becomes:

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \le \bar{C}(\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 + |m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})|). \tag{I.41}$$

It remains to prove that $\exists\widetilde{C} > 0$ such that:

$$|m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})| \le \widetilde{C}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \quad \forall\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}. \tag{I.42}$$

We prove this result by contradiction.
Let $(h_n)_{n\in\mathbb{N}}$ be a sequence such that $h_n \to 0$ as $n \to +\infty$, and let $(\mathfrak{T}_n)_n$ be a sequence of meshes such that size$(\mathfrak{T}_n) = h_n$ while reg$(\mathfrak{T}_n)$ is bounded. For every n, there exists a constant C_n such that:

$$|m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n})| \le C_n\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n}\|_2 \quad \forall\mathbf{u}^{\mathfrak{T}_n} \in \mathbb{E}_0^{\Gamma_D}, \tag{I.43}$$

with $C_n := \sup\limits_{\mathbf{u}^{\mathfrak{T}_n}\in\mathbb{E}_0^{\Gamma_D}} \dfrac{|m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n})|}{\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n}\|_2}$. Inequality (I.43) holds because of Thm. I.8.4 (proved below in Sec. I.8.3), that ensures that $\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n}\|_2$ is actually a norm.
Proving (I.42), it is equivalent to show that the bound (I.43) is a uniform bound. Thus we argue by contradiction, and we suppose that:

$$\forall k \in \mathbb{N},\ \exists n_k \text{ with } n_k \ge k \text{ such that } C_{n_k} \ge k,$$

that is

$$\forall k \in \mathbb{N},\ \exists\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}} \text{ such that } |m(\mathrm{rot}^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}})| \ge k\,\|\mathrm{D}^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}}\|_2 \quad \forall\widetilde{\mathbf{u}}^{\mathfrak{T}_n} \in \mathbb{E}_0^{\Gamma_D}.$$

Let $\mathbf{u}^{\mathfrak{T}_{n_k}} = \dfrac{\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}}}{m(\mathrm{rot}^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}})}$, so that:

$$m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}) = 1, \quad \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}\|_2 \le \frac{1}{k}. \tag{I.44}$$

From (I.41), we can deduce that $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}$ is bounded as $k \to +\infty$, since:

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}\|_2 \le C\Big(\frac{1}{k} + 1\Big).$$

We can thus apply the compactness result of [ABH07, Lemma 3.6], which implies the existence of $\mathbf{u} \in (H^1_D(\Omega))^2$ such that, up to a subsequence:

$$\begin{aligned}\mathbf{u}^{\mathfrak{T}_{n_k}} &\to \mathbf{u} \quad \text{in } (L^2(\Omega))^2 \text{ as } k \to \infty,\\ \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}} &\rightharpoonup \nabla\mathbf{u} \quad \text{in } (L^2(\Omega))^{2\times 2} \text{ as } k \to \infty.\end{aligned}$$

The weak convergence of $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}$, that can be seen as the L^2 function $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}} = \sum_{\mathtt{D}\in\mathfrak{D}} \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}\mathbf{1}_{\mathtt{D}}$, implies that $\forall\varphi \in (L^2(\Omega))^{2\times 2}$

$$\int_\Omega \left(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}} : \varphi\right) \to \int_\Omega (\nabla\mathbf{u} : \varphi)\,.$$

Thanks to the relation between the gradient and the rotational operator (see (I.5)-(I.6)) and by choosing $\varphi = \frac{1}{|\Omega|}\begin{pmatrix} 0 & 0 \\ \mathbf{1}_\Omega & 0 \end{pmatrix}$ and $\varphi = \frac{1}{|\Omega|}\begin{pmatrix} 0 & \mathbb{1}_\Omega \\ 0 & 0 \end{pmatrix}$, we obtain:

$$m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}) \to m(\mathrm{rot}\,\mathbf{u}).$$

From (I.44), we deduce $m(\mathrm{rot}\,\mathbf{u}) = 1$.

Moreover, the weak convergence of the gradient implies the weak convergence of the strain rate tensor $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}$, by definition of the operators; this, with (I.44), implies $\mathrm{D}\mathbf{u} = 0$, i.e. $\mathbf{u}$ is a rigid motion. The only rigid motion that satisfies $\mathbf{u}|_{\Gamma_D} = 0$ is $\mathbf{u} = 0$ since $\mathrm{meas}(\Gamma_D) > 0$ (see [BS07]). We have therefore a contradiction, so we proved (I.42) and Thm. I.8.2. ∎

I.8.3 Study of the kernel of $\mathrm{D}^{\mathfrak{D}}$

The following result is necessary in order to prove Korn's inequality. It will be useful even to show the wellposedness of the DDFV scheme in Sec. II.5.

Theorem I.8.4 *Let Ω be an open connected bounded polygonal domain of $\mathbb{R}^2$ and Γ_D be a part of the boundary such that $m(\Gamma_D) > 0$.*
Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies Inf-sup stability condition. Then $\forall\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ such that $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = 0$ we have $\mathbf{u}^{\mathfrak{T}} = 0$ in Ω.

Proof Since we are not able to give a general proof of this theorem for all meshes, we focus on all the ones that are unconditionally *Inf-sup* stable (see [BKN15]), since to prove Lemma I.8.3 we need this last hypothesis.
When studying those meshes, we observe a propagation phenomenon of the zero boundary data on Γ_D to the entire mesh.

In fact, it is important to remark that in DDFV meshes all boundary diamonds are triangles (see Fig. I.3). If we focus on one of those diamonds, the condition on Γ_D implies that the velocity is zero on the three vertices $\mathtt{L}, \mathtt{K}^*$ and $\mathtt{L}^*$:

$$\mathbf{u}_{\mathtt{L}} = \begin{pmatrix} \mathrm{u}^x_{\mathtt{L}} \\ \\ \mathrm{u}^y_{\mathtt{L}} \end{pmatrix} = 0, \mathbf{u}_{\mathtt{K}^*} = \begin{pmatrix} \mathrm{u}^x_{\mathtt{K}^*} \\ \\ \mathrm{u}^y_{\mathtt{K}^*} \end{pmatrix} = 0,$$

$$\mathbf{u}_{\mathtt{L}^*} = \begin{pmatrix} \mathrm{u}^x_{\mathtt{L}^*} \\ \\ \mathrm{u}^y_{\mathtt{L}^*} \end{pmatrix} = 0.$$

Since we are supposing $\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = 0$ for all $\mathtt{D} \in \mathfrak{D}$, this is true in particular for the boundary diamonds (the white ones in Fig. I.6). By the definition of the discrete strain rate tensor (I.3) we are led to

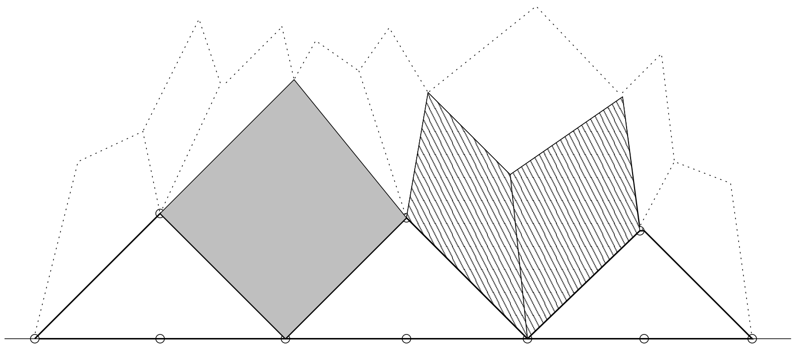


Fig. I.6 Possible configurations of diamonds adjacent to the boundary ones in Inf-sup stable meshes.

the following system:

$$\begin{cases} m_\sigma \, \mathrm{u}^x_{\mathtt{K}} \, \mathrm{n}^x_{\sigma,\mathtt{K}} = 0 \\ m_\sigma \, \mathrm{u}^y_{\mathtt{K}} \, \mathrm{n}^y_{\sigma,\mathtt{K}} = 0 \\ m_\sigma \, (\mathrm{u}^x_{\mathtt{K}} \, \mathrm{n}^y_{\sigma,\mathtt{K}} + \mathrm{u}^y_{\mathtt{K}} \, \mathrm{n}^x_{\sigma,\mathtt{K}}) = 0, \end{cases} \tag{I.45}$$

that implies $\mathbf{u}_{\mathtt{K}} = \begin{pmatrix} \mathrm{u}^x_{\mathtt{K}} \\ \\ \mathrm{u}^y_{\mathtt{K}} \end{pmatrix} = 0$, since the outer normal $\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \begin{pmatrix} \mathrm{n}^x_{\sigma,\mathtt{K}} \\ \\ \mathrm{n}^y_{\sigma,\mathtt{K}} \end{pmatrix}$ cannot be zero.

This means that for all diamonds in $\mathfrak{D}_{ext} \cap \Gamma_D$ the four components of the velocity, $\mathbf{u}_{\mathtt{K}}, \mathbf{u}_{\mathtt{L}}, \mathbf{u}_{\mathtt{K}^*}, \mathbf{u}_{\mathtt{L}^*}$, are zero.

We now look at the diamonds that are adjacent to ones on the boundary: for the meshes under consideration, we can distinguish two possible situations that we illustrate in Fig. I.6.

The first one is the case of the shaded diamond, for which the situation is equivalent to the one of boundary diamonds. In fact, we know that the velocity is zero on three of its vertices. So we can conclude, by solving a system similar to (I.45) deduced by $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = 0$, that even the last component of the velocity is zero on that diamond.

The second structure is described by the hatched diamonds. This is the case of two neighbors, that we will denote with $\mathtt{D}^1$,$\mathtt{D}^2$ which share a common vertex. Remark that on that vertex the velocity is zero and both diamonds have one more vertex with zero velocity. Thus we are considering a structure composed by 6 vertices, where the values of the velocity are zero on 3 among them.
In this case, we denote the normal vectors of $\mathtt{D}^1, \mathtt{D}^2$ with

$$\vec{\mathbf{n}}^i_{\sigma\mathtt{K}} = \begin{pmatrix} \mathrm{n}^{x,i}_{\sigma} \\ \\ \mathrm{n}^{y,i}_{\sigma} \end{pmatrix}, \quad \vec{\mathbf{n}}^i_{\sigma^*\mathtt{K}^*} = \begin{pmatrix} \mathrm{n}^{x,i}_{\sigma^*} \\ \\ \mathrm{n}^{y,i}_{\sigma^*} \end{pmatrix} \qquad \text{for } i = 1, 2,$$

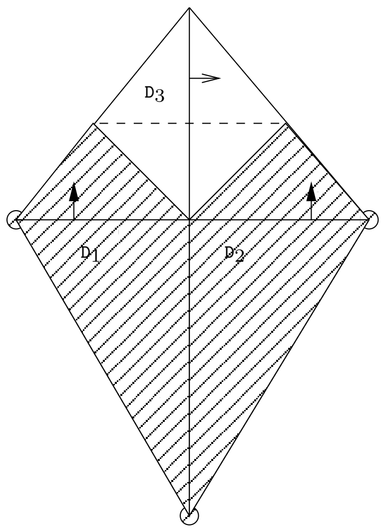


Fig. I.7 Degenerate case

and we write the system of equations equivalent to the conditions $\mathrm{D}^{\mathtt{D}^i}\mathbf{u}^{\mathfrak{T}} = 0$ for $i = 1, 2$.
The 6×6 matrix of that system has determinant

$$det = (\mathrm{n}^{x,2}_{\sigma^*}\,\mathrm{n}^{y,2}_{\sigma} - \mathrm{n}^{x,2}_{\sigma}\,\mathrm{n}^{y,2}_{\sigma^*})(\mathrm{n}^{x,1}_{\sigma^*}\,\mathrm{n}^{y,1}_{\sigma} - \mathrm{n}^{x,1}_{\sigma}\,\mathrm{n}^{y,1}_{\sigma^*})(\mathrm{n}^{x,1}_{\sigma}\,\mathrm{n}^{y,2}_{\sigma} - \mathrm{n}^{x,2}_{\sigma}\,\mathrm{n}^{y,1}_{\sigma}) \neq 0,$$

that is always different from zero, except in a degenerate case that we treat in the following section where the normals of the two diamonds are parallel. Thus the matrix is invertible, that implies that all the six components of the velocity on those two diamonds are zero: $\mathbf{u}^i_{\mathtt{K}} = \mathbf{u}^i_{\mathtt{L}} = \mathbf{u}^i_{\mathtt{K}^*} = \mathbf{u}^i_{\mathtt{L}^*} = 0$ for $i = 1, 2$.

Degenerate case: checkerboard mesh

This is a particular case of the second structure, in which the normal vectors of the two hatched diamonds are parallel. In order to have an invertible system to solve, it is necessary to consider a third diamond.
In particular, if we call $\mathtt{D}_1, \mathtt{D}_2$ the hatched diamonds and $\mathtt{D}_3$ the white one, we have for instance: $\vec{\mathbf{n}}^i_{\sigma\mathtt{K}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $i = 1, 2$ and $\vec{\mathbf{n}}^3_{\sigma\mathtt{K}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
If, as we did in the previous cases, we write the system of equations equivalent to $\mathrm{D}^{\mathtt{D}^i}\mathbf{u}^{\mathfrak{T}} = 0$, but this time for $i = 1, 2, 3$, we get again an invertible system, this time of size 8×8. As before, we find that all the components of the velocity are zero on the three diamonds.
By proceeding step by step, we can prove that the velocity $\mathbf{u}^{\mathfrak{T}}$ is zero on the entire domain Ω.

■

Remark I.8.5 *Since the study of the kernel of* $\mathrm{D}^{\mathfrak{D}}$ *is related to the mesh geometry, there is no general proof for all meshes. We only focused on meshes that satisfy Inf-sup condition because in the study of Navier-Stokes problem we are already in this setting, due to Thm. I.8.3. The technique, though, can be potentially extended to all mesh geometries considering one mesh at a time; for instance, it is valid also on Cartesian meshes, which are Inf-sup stable up to a single pressure mode.*

I.9 Trace inequalities

Given a vector $\mathbf{u}^{\mathfrak{T}} = ((\mathbf{u}_{\mathrm{K}})_{\mathrm{K}\in\mathfrak{M}\cup\partial\mathfrak{M}}, (\mathbf{u}_{\mathrm{K}^*})_{\mathrm{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*})$ defined on a DDFV mesh $\mathfrak{T}$, we associate the approximate solution on the boundary in two different ways:

$$\widetilde{\phi}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) = \frac{1}{2}\sum_{\mathrm{K}\in\mathfrak{M}} \mathbf{u}_{\mathrm{K}} \mathbf{1}_{\bar{\mathrm{K}}\cap\partial\Omega} + \frac{1}{2}\sum_{\mathrm{K}^*\in\partial\mathfrak{M}^*} \mathbf{u}_{\mathrm{K}^*} \mathbf{1}_{\bar{\mathrm{K}}^*\cap\partial\Omega},$$

$$\phi^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) = \frac{1}{2}\sum_{\mathrm{L}\in\partial\mathfrak{M}} \mathbf{u}_{\mathrm{L}} \mathbf{1}_{\mathrm{L}} + \frac{1}{2}\sum_{\mathrm{K}^*\in\partial\mathfrak{M}^*} \mathbf{u}_{\mathrm{K}^*} \mathbf{1}_{\bar{\mathrm{K}}^*\cap\partial\Omega}.$$

With this definition, we use simultaneously the values on the primal mesh and the values on the dual mesh. The difference between the two traces can be explained if we look at a diamond D on the boundary, illustrated in Fig. I.8: $\widetilde{\phi}^{\mathfrak{T}}$ averages values at the interior and on the boundary of the mesh, i.e. $\mathbf{u}_{\mathrm{K}}, \mathbf{u}_{\mathrm{K}^*}, \mathbf{u}_{\mathrm{L}^*}$, while $\phi^{\mathfrak{T}}$ takes values just on the boundary, i.e. $\mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{K}^*}, \mathbf{u}_{\mathrm{L}^*}$.

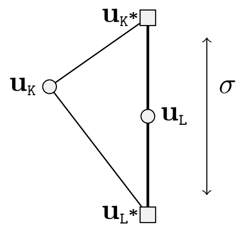


Fig. I.8 A boundary diamond, $\sigma \in \partial\mathfrak{M}$.

We can also consider two different reconstructions based either on the primal values or the dual values:

$$\widetilde{\phi}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathrm{K}\in\mathfrak{M}} \mathbf{u}_{\mathrm{K}} \mathbf{1}_{\bar{\mathrm{K}}\cap\partial\Omega} \quad \text{or} \quad \phi^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathrm{L}\in\partial\mathfrak{M}} \mathbf{u}_{\mathrm{L}} \mathbf{1}_{\mathrm{L}}$$

$$\widetilde{\phi}^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}}) = \phi^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathrm{K}^*\in\partial\mathfrak{M}^*} \mathbf{u}_{\mathrm{K}^*} \mathbf{1}_{\bar{\mathrm{K}}^*\cap\partial\Omega}(\mathrm{x}).$$

With respect to the traces defined in Sec.I.4, they satisfy

$$\|\widetilde{\gamma}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} \leq \|\widetilde{\phi}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega}$$
$$\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} \leq \|\widetilde{\phi}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega}.$$

We point out that, if we consider the object we want to estimate, we have for both cases (by Minkowski's inequality):

$$\|\widetilde{\phi}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} \leq \|\widetilde{\phi}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} + \|\widetilde{\phi}^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega},$$
$$\|\phi^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} \leq \|\phi^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} + \|\phi^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega}.$$

Before proving the trace theorem, we introduce a discrete Poincaré inequality, proved in [ABH07] for scalar fields and L^2 norm; here we need the one proved in [BCCHF15] for vector fields.

Theorem I.9.1 (Discrete Poincaré inequality, [BCCHF15], Thm. 11) *Let Ω be an open connected bounded polygonal domain of $\mathbb{R}^2$ and Γ_D be a part of the boundary such that $m(\Gamma_D) > 0$. Let $\mathfrak{T}$ be a DDFV mesh associated to Ω.*

- *If $1 \leq p < 2$, let $1 \leq q \leq p^*$*

- *If $p \geq 2$, let $1 \leq q < \infty$.*

There exists a constant $C > 0$, depending only on p,q, Γ_D and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$:

$$\|\mathbf{u}^{\mathfrak{T}}\|_q \leq \frac{C}{\sin(\alpha_{\mathfrak{T}})^{\frac{1}{p}} \; reg(\mathfrak{T})^{\frac{p-1}{p}}} \; \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_p.$$

Theorem I.9.2 (Trace inequality) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $p > 1$ there exists a constant $C > 0$, depending only on p, $\sin(\alpha_{\mathfrak{T}})$, $reg(\mathfrak{T})$ and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and $\forall s \geq 1$:*

$$\|\widetilde{\phi}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{s,\partial\Omega}^s \leq C \|\mathbf{u}^{\mathfrak{T}}\|_{1,p} \|\mathbf{u}^{\mathfrak{T}}\|_{\frac{p(s-1)}{p-1}}^{s-1}. \tag{I.46}$$

The computations of the proof are similar to those present in [EGH00] and [CHKM15]. In [EGH00], the proof is given for finite volume methods; in [CHKM15], the proof is given for DDFV method but in the case of L^1 norm. Our proof has been adapted to the vectorial case and to general L^s, L^p norms.

Proof

Boundary properties: By compactness of $\partial\Omega$, there exists a finite number of open hyper-rectangles $\{R_i, i = 1 \dots N\}$, and normalized vectors of $\mathbb{R}^2$, $\{\eta_i, i = 1, \dots, N\}$, such that:

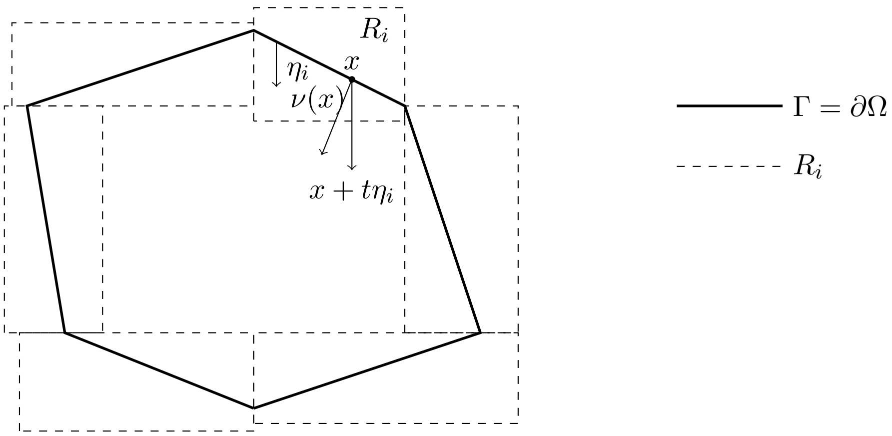


Fig. I.9 Properties of the boundary $\partial\Omega$.

$$\begin{cases} \partial\Omega \subset \bigcup_{i=1}^N R_i \\ \\ (\eta_i, \overrightarrow{\nu}(x)) \geq \lambda > 0 \quad \forall x \in R_i \cap \partial\Omega, i \in \{1 \dots N\} \\ \\ \{x + t\eta_i, x \in R_i \cap \partial\Omega, t \in \mathbb{R}^+\} \cap R_i \subset \Omega, \end{cases}$$

where λ is a strictly positive number and $\overrightarrow{\nu}(x)$ is the normal vector to $\partial\Omega$ at x, inward to Ω (see Figure 1). Let $\{\lambda_i, i = 1 \dots N\}$ be a family of functions such that $\sum_{i=1}^{N} \lambda_i(x) = 1$, for all $x \in \partial\Omega$, $\lambda_i \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^+)$ and $\lambda_i = 0$ outside of R_i, for all $i = 1 \dots N$. Let $\partial\Omega_i = R_i \cap \partial\Omega$; we shall prove that there exists $C_i > 0$ depending only on $\lambda, \mathrm{reg}(\mathfrak{T})$ and λ_i such that

$$\int_{\partial\Omega_i} \lambda_i(x) |\widetilde{\phi}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})(x)|^s \mathrm{d}x + \int_{\partial\Omega_i} \lambda_i(x) |\widetilde{\phi}^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}})(x)|^s \mathrm{d}x \leq C_i \|\mathbf{u}^{\mathfrak{T}}\|_{1,p} \|\mathbf{u}^{\mathfrak{T}}\|_{\frac{p(s-1)}{p-1}}^{s-1}.$$

Then it will be sufficient to define $C := \sum_{i=1}^N C_i$ to get (I.46). We study separately the two terms.

On the primal mesh: We introduce the functions to determine the successive neighbours of a cell $\mathbf{u}_{\mathtt{K}}$. Consider $x, y \in \Omega$, then:

$$\text{for } \sigma \in \mathcal{E} \quad \Psi_\sigma(x,y) := \begin{cases} 1 & \text{if } [x,y] \cap \sigma \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } \mathtt{K} \in \mathfrak{M} \quad \Psi_{\mathtt{K}}(x,y) := \begin{cases} 1 & \text{if } [x,y] \cap \mathtt{K} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

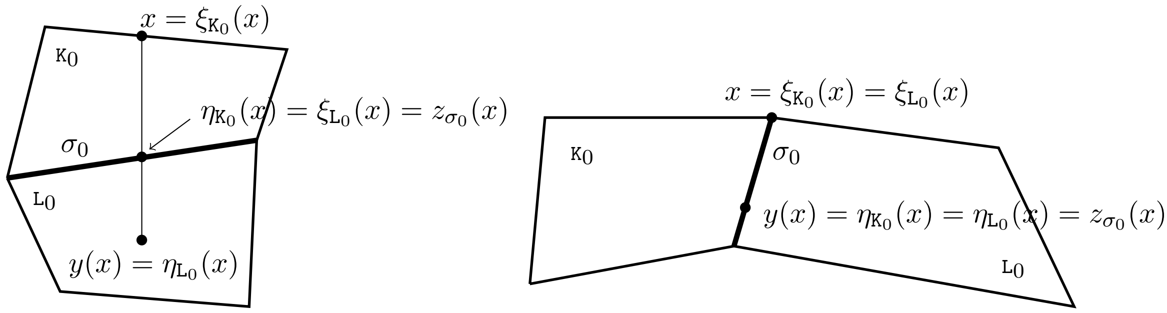


Fig. I.10 (Left) $[x, y(x)] \cap \sigma_0$ is reduced to a point $z_{\sigma_0}(x)$. (Right) $[x, y(x)] \cap \sigma_0$ is the segment $[x, y(x)]$.

Now, we fix $i \in \{1 \ldots N\}$ and $x \in \partial\Omega_i$.
Then there exists a unique $t > 0$ such that $x + t\eta_i = y(x) \in \partial R_i$. Then, for $\sigma \in \mathcal{E}$, if $[x, y(x)] \cap \sigma \neq \emptyset$, then it is:

- either a point: $z_\sigma(x) := [x, y(x)] \cap \sigma$
- either a segment: $[a(x), b(x)] := [x, y(x)] \cap \sigma$ and let $z_\sigma(x) := b(x)$.

For $\mathtt{K} \in \mathfrak{M}$, if $[x, y(x)] \cap \mathtt{K} \neq \emptyset$ we have:

$$[\xi_{\mathtt{K}}(x), \eta_{\mathtt{K}}(x)] := [x, y(x)] \cap \mathtt{K}.$$

Let us fix $x \in \mathtt{K}_0$, with $\mathtt{K}_0 \in \mathfrak{M}$ such that $y(x) \in \mathtt{L}_0$, $\sigma_0 = \mathtt{K}_0|\mathtt{L}_0$. We distinguish the following two cases:

1. For the left case (see Fig. I.10):

$$\begin{aligned} \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s &= \big(\lambda_i(\xi_{\mathtt{K}_0}(x)) - \lambda_i(\eta_{\mathtt{K}_0}(x))\big)|\mathbf{u}_{\mathtt{K}_0}|^s \\ &\quad + \big(\lambda_i(\xi_{\mathtt{L}_0}(x)) - \lambda_i(\eta_{\mathtt{L}_0}(x))\big)|\mathbf{u}_{\mathtt{L}_0}|^s \\ &\quad + \lambda_i(z_{\sigma_0}(x))\big(|\mathbf{u}_{\mathtt{K}_0}|^s - |\mathbf{u}_{\mathtt{L}_0}|^s\big), \end{aligned}$$

2. for the right case (see Fig. I.10):

$$\lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s = \big(\lambda_i(\xi_{\mathtt{K}_0}(x)) - \lambda_i(\eta_{\mathtt{K}_0}(x))\big)|\mathbf{u}_{\mathtt{K}_0}|^s.$$

In both cases:

$$\lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \leq \sum_{\mathtt{D}\in\mathfrak{D}} \Psi_\sigma(x,y(x))\lambda_i(z_\sigma(x))\big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\big| \\ + \sum_{\mathtt{K}\in\mathfrak{M}} \Psi_{\mathtt{K}}(x,y(x))|\lambda_i(\xi_{\mathtt{K}}(x)) - \lambda_i(\eta_{\mathtt{K}}(x))||\mathbf{u}_{\mathtt{K}}|^s,$$

that we can write as

$$\lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \leq A(x) + B(x),$$

by defining

$$A(x) := \sum_{\mathtt{D}\in\mathfrak{D}} \Psi_\sigma(x,y(x))\lambda_i(z_\sigma(x))\big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\big| \\ B(x) := \sum_{\mathtt{K}\in\mathfrak{M}} \Psi_{\mathtt{K}}(x,y(x))|\lambda_i(\xi_{\mathtt{K}}(x)) - \lambda_i(\eta_{\mathtt{K}}(x))||\mathbf{u}_{\mathtt{K}}|^s.$$

We proceed by estimating separately the two terms.

<u>Estimate of A:</u>
Since λ_i is bounded, we get:

$$A(x) \leq \|\lambda_i\|_\infty \sum_{\mathtt{D}\in\mathfrak{D}} \Psi_\sigma(x,y(x))\Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big|;$$

We now use the following estimate (with $c_\sigma = |(\eta_i, \overrightarrow{\nu}_\sigma(x))|$)

$$\int_{\partial\Omega_i} \Psi_\sigma(x,y(x))\mathrm{d}x \leq \frac{c_\sigma}{\lambda} m_\sigma,$$

that is proved in [EGH00], to conclude:

$$A = \int_{\partial\Omega_i} A(x)\mathrm{d}x \leq \|\lambda_i\|_\infty \sum_{\mathtt{D}\in\mathfrak{D}} \Big(\int_{\partial\Omega_i} \Psi_\sigma(x,y(x))\mathrm{d}x\Big)\Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big| \\ \leq C_i \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma \Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big|,$$

where in the 3rd inequality we used [Kre10, Lemma I.19].
Now, as in [BCCHF15], we use the inequality:

$$\Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big| \leq s(|\mathbf{u}_{\mathtt{K}}|^{s-1} + |\mathbf{u}_{\mathtt{L}}|^{s-1})|\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}|,$$

that leads to:

$$\sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma \Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big| \leq s \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma(|\mathbf{u}_{\mathtt{K}}|^{s-1} + |\mathbf{u}_{\mathtt{L}}|^{s-1})|\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}| \\ \leq C \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma m_{\sigma^*}(|\mathbf{u}_{\mathtt{K}}|^{s-1} + |\mathbf{u}_{\mathtt{L}}|^{s-1})\Big|\frac{\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}}{m_{\sigma^*}}\Big|,$$

that by integration by parts and Hölder gives:

$$\sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma \Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big| \leq C\Big(\sum_{\mathtt{K}\in\mathfrak{M}}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma m_{\sigma^*}|\mathbf{u}_{\mathtt{K}}|^{\frac{(s-1)p}{p-1}}\Big)^{\frac{p-1}{p}} \Big(\sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma m_{\sigma^*}\Big|\frac{\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}}{m_{\sigma^*}}\Big|^p\Big)^{\frac{1}{p}}.$$

By regularity hypothesis on the mesh and the definition of the discrete gradient we can write:

$$A \leq \frac{C}{\sin(\alpha_{\mathfrak{T}})^{\frac{1}{p}} \ \mathrm{reg}(\mathfrak{T})^{\frac{p-1}{p}}} \ \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \ \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p.$$

Estimate of B:
Since λ_i is C^∞, we have, by Taylor's formula:

$$B(x) \leq \|\nabla\lambda_i\|_\infty \sum_{K\in\mathfrak{M}} \Psi_{\mathtt{K}}(x,y(x))|\xi_{\mathtt{K}}(x)-\eta_{\mathtt{K}}(x)||\mathbf{u}_{\mathtt{K}}|^s,$$

and thanks to the inequality that can be found in [EGH00, Lemma 3.10]

$$\int_{\partial\Omega_i} \Psi_{\mathtt{K}}(x,y(x))|\xi_{\mathtt{K}}(x)-\eta_{\mathtt{K}}(x)|\mathrm{d}x \leq \frac{m_{\mathtt{K}}}{\lambda},$$

we can conclude:

$$\begin{aligned} B = \int_{\partial\Omega_i} B(x) &\leq \|\nabla\lambda_i\|_\infty \sum_{\mathtt{K}\in\mathfrak{M}} \Big(\int_{\partial\Omega_i} \Psi_{\mathtt{K}}(x,y(x))|\xi_{\mathtt{K}}(x)-\eta_{\mathtt{K}}(x)|\Big)|\mathbf{u}_{\mathtt{K}}|^s \\ &\leq C_i \sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}|\mathbf{u}_{\mathtt{K}}|^s. \end{aligned}$$

Thus

$$B \leq C\|\mathbf{u}^{\mathfrak{T}}\|_s^s.$$

Putting together the terms, we find:

$$\int_{\partial\Omega_i} \lambda_i(x)|\widetilde{\phi}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s \leq C_i \left(\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \ \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p + \|\mathbf{u}^{\mathfrak{T}}\|_s^s\right).$$

By proceeding as in the proof of [BCCHF15, Lemma 1], we use interpolation between L^p spaces and we write:

$$\|\mathbf{u}^{\mathfrak{T}}\|_s^s \leq \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \ \|\mathbf{u}^{\mathfrak{T}}\|_p,$$

that leads to

$$\int_{\partial\Omega_i} \lambda_i(x)|\widetilde{\phi}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s \leq C_i\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \ \|\mathbf{u}^{\mathfrak{T}}\|_{1,p},$$

that proves our theorem.

On the dual mesh: the computations are exactly the same, exchanging $\mathtt{K}$ with $\mathtt{K}^*$ and σ in σ^* . ■

Corollary I.9.3 (Second trace inequality) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C>0$, depending only on p,q, $\sin(\alpha_{\mathfrak{T}})$, reg($\mathfrak{T}$) and Ω such that $\forall\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and for all $s\geq 1$, $p>1$:*

$$\|\phi^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|^s_{s,\partial\Omega} \leq C\|\mathbf{u}^{\mathfrak{T}}\|_{1,p}\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{p(s-1)}{p-1}}.$$

Proof The proof is almost the same as Thm. I.9.2.
What changes is just that we now fix $x\in\mathtt{L}$, $\mathtt{L}\in\partial\mathfrak{M}$ and $\mathtt{K}_0\in\mathfrak{M}$ such that $\mathtt{L}\subset\mathtt{K}_0$, $y(x)\in\mathtt{K}_0$, $\sigma_0=\mathtt{K}_0|\mathtt{L}$ (see Fig. I.11).

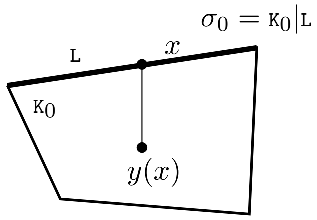


Fig. I.11 $\mathtt{L}$ on the boundary $\partial\mathfrak{M}$ and $\mathtt{K}_0 \in \mathfrak{M}$ such that $\mathtt{L} \subset \mathtt{K}_0$, $y(x) \in \mathtt{K}_0$, $\sigma_0 = \mathtt{K}_0|\mathtt{L}$.

The term that we want to study now is $\lambda_i(x)|\mathbf{u}_{\mathtt{L}}|^s$, since we are focusing on the boundary. It can be written as:

$$\lambda_i(x)|\mathbf{u}_{\mathtt{L}}|^s = \lambda_i(x)(|\mathbf{u}_{\mathtt{L}}|^s - |\mathbf{u}_{\mathtt{K}_0}|^s) + \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s, \tag{I.47}$$

that can be estimated by:

$$\begin{aligned}\lambda_i(x)|\mathbf{u}_{\mathtt{L}}|^s &\leq \lambda_i(x)\Big||\mathbf{u}_{\mathtt{L}}|^s - |\mathbf{u}_{\mathtt{K}_0}|^s\Big|\mathbf{1}_{\mathtt{L}}(x) + \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \\ &:= A_b(x) + \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s.\end{aligned}$$

<u>Estimate of A_b:</u>
Since λ is bounded, we have:

$$A_b = \int_{\partial\Omega_i} A_b(x) \leq \|\lambda_i\|_\infty \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma \Big||\mathbf{u}_{\mathtt{L}}|^s - |\mathbf{u}_{\mathtt{K}}|^s\Big|.$$

We can proceed exactly as in the proof of Thm I.9.2 for A, so we get:

$$A_b \leq \frac{C}{\sin(\alpha_{\mathfrak{T}})^{\frac{1}{p}} \ \mathrm{reg}(\mathfrak{T})^{\frac{p-1}{p}}} \ \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \ \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p.$$

Putting together all the terms, we find:

$$\left(\int_{\partial\Omega_i} \lambda_i(x)|\phi^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s\right) \leq A_b + \left(\int_{\partial\Omega_i} \lambda_i(x)|\widetilde{\phi}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s\right).$$

Thanks to the previous theorem, we conclude:

$$\int_{\partial\Omega_i} \lambda_i(x)|\phi^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s \leq C_i\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \ \|\mathbf{u}^{\mathfrak{T}}\|_{1,p}$$

that proofs our statement.

On the dual mesh: the computations are the same as the previous theorem. ■

Corollary I.9.4 (L^2 norm) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$ that depends only on Ω and reg($\mathfrak{T}$) such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$:*

$$\|\phi^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{2,\partial\Omega} \leq C\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2.$$

Proof It is a direct consequence of Corollary I.9.3 with $s = p = 2$ and Poincaré inequality (Thm. I.9.1). ■

I.10 Properties of discrete operators

We give now some results on discrete operators and projections, that will be useful in the error estimate proof of Chapter II. All the proofs can be found in [Kre10].

Lemma I.10.1 ([Kre10], Lemma IV.14) *For $d = 2, 3$. Let K be an open non empty polygonal convex set of $\mathbb{R}^d$ such that, for some $\alpha > 0$, there exists a ball of radius α diam(K) contained in K. Let E be an affine hyperplan of $\mathbb{R}^d$ and σ an open non empty set of E contained in $\partial K \cap E$. Then there exists a constant $C > 0$, that depends only on α, such that for all $\boldsymbol{v} \in (H^1(K))^d$:*

$$\left|\frac{1}{m_\sigma}\int_\sigma \boldsymbol{v}(s)\,ds\right|^2 \leq \frac{C\,diam(\mathrm{K})}{m_\sigma}\int_K |||\nabla \boldsymbol{v}(s)|||_{\mathcal{F}}^2 ds + \frac{C}{diam(\Omega)m_\sigma}\int_K |\boldsymbol{v}(s)|^2 ds.$$

Properties of the projection $\mathbb{P}_c^{\mathfrak{T}}$

Lemma I.10.2 ([Kre10], Lemma IV.16) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg$(\mathfrak{T})$, such that for all functions $\boldsymbol{v} \in (H^2(\Omega))^2$, we have:*

$$\|\nabla \boldsymbol{v} - \nabla^{\mathfrak{D}}\mathbb{P}_c^{\mathfrak{T}}\boldsymbol{v}\|_2 \leq C size(\mathfrak{T})\|\nabla \boldsymbol{v}\|_{H^1}.$$

Corollary I.10.3 ([Kre10], Coro IV.17) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg$(\mathfrak{T})$, such that for all functions $\boldsymbol{v} \in (H^2(\Omega))^2$, we have:*

$$\|\nabla^{\mathfrak{D}}\mathbb{P}_c^{\mathfrak{T}}\boldsymbol{v}\|_2 \leq C\|\nabla \boldsymbol{v}\|_{H^1}.$$

Corollary I.10.4 ([Kre10], Coro IV.18) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg$(\mathfrak{T})$, such that for all functions $\boldsymbol{v} \in (H^2(\Omega))^2$ that verifies* $\mathrm{div}(v) = 0$, *we have:*

$$\|div^{\mathfrak{D}}\mathbb{P}_c^{\mathfrak{T}}\boldsymbol{v}\|_2 \leq C size(\mathfrak{T})\|\nabla \boldsymbol{v}\|_{H^1}.$$

Properties of the projection $\mathbb{P}_m^{\mathfrak{T}}$

Lemma I.10.5 ([Kre10], Lemma IV.19) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg$(\mathfrak{T})$, such that for all functions $\boldsymbol{v} \in (H^1(\Omega))^2$, we have:*

$$\|\nabla^{\mathfrak{D}}\mathbb{P}_m^{\mathfrak{T}}\boldsymbol{v}\|_2 \leq C\|\nabla \boldsymbol{v}\|_2 \quad \textit{and} \quad \|\boldsymbol{v} - \mathbb{P}_m^{\mathfrak{T}}\boldsymbol{v}\|_2 \leq C size(\mathfrak{T})\|\nabla \boldsymbol{v}\|_2.$$

Error between the projections $\mathbb{P}_c^{\mathfrak{T}}$ and $\mathfrak{P}_{m,g}^{\Gamma_D}\mathbb{P}_c^{\mathfrak{T}}$:

Lemma I.10.6 ([Kre10], Lemma IV.20) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg$(\mathfrak{T})$, such that for all functions $\boldsymbol{v} \in (H^1(\Omega))^2$, we have:*

$$\|\nabla^{\mathfrak{D}}\mathbb{P}_c^{\mathfrak{T}}\boldsymbol{v} - \nabla^{\mathfrak{D}}\mathfrak{P}_{m,g}^{\Gamma_D}\mathbb{P}_c^{\mathfrak{T}}\boldsymbol{u}\ \|_2 \leq C size(\mathfrak{T})\|\boldsymbol{v}\|_{H^2}.$$

Lemma I.10.7 ([Kre10], Lemma IV.22) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg$(\mathfrak{T})$, such that for all functions $\boldsymbol{v} \in (H^2(\Omega))^2$, we have:*

$$\|\boldsymbol{v} - \mathbb{P}_c^{\mathfrak{T}}\boldsymbol{v}\|_2 \leq C size(\mathfrak{T})\|\nabla \boldsymbol{v}\|_{H^1} \quad \textit{and} \quad \|\boldsymbol{v} - \mathfrak{P}_{m,g}^{D}\mathbb{P}_c^{\mathfrak{T}}\boldsymbol{v}\|_2 \leq C size(\mathfrak{T})\|\nabla \boldsymbol{v}\|_{H^1}.$$

Properties of the projection on diamonds $\mathbb{P}_m^{\mathfrak{D}}$

Lemma I.10.8 ([Kre10], Lemma IV.23) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg($\mathfrak{T}$), such that for all functions $p \in H^1(\Omega)$, we have:*

$$\sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathfrak{S}} (\mathbb{P}_m^{\mathtt{D}'}p - \mathbb{P}_m^{\mathtt{D}}p)^2 \le C\|\nabla p\|_2^2.$$

Lemma I.10.9 ([Kre10], Lemma IV.24) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg($\mathfrak{T}$), such that for all functions $p \in H^1(\Omega)$, we have:*

$$\|\mathbb{P}_m^{\mathfrak{D}}p - p\|_2 \le C size(\mathfrak{T})\|\nabla p\|_2.$$

I.11 Basic inequalities

Here we recall some basic inequalities that we will need in the following chapters.

Lemma I.11.1 (Young's inequality) *Let a, b, c be three non negative numbers. Let p_1, p_2 and p_3 be positive real numbers such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Then, we have:*

$$abc \le \frac{C_1}{p_1}\, a^{p_1} + \frac{C_2}{p_2}\, b^{p_2} + \frac{1}{p_3\, C_1 C_2}\, c^{p_3},$$

for some positive constants C_1, C_2,

We adapted the proof of Grönwall's lemma, Lemma 16.I.6 in [Sch01], to obtain the following:

Lemma I.11.2 (Discrete Grönwall's lemma) *If a sequence $(a_n)_n$, $n = 0 \dots N$, satisfies*

$$a_0 \le A, \qquad a_n \le A + B\delta t \sum_{i=0}^{n-1} a_i \quad \forall n \in 1, \dots N, \delta t = \frac{T}{N},$$

where A and B are two positive constants independent of δt, then

$$\max_{n=1\dots N} a_n \le Ae^{BT}.$$

Lemma I.11.3 (Hölder's inequality) *Let $p, q, r \in (1, +\infty)$ with $1/p + 1/q + 1/r = 1$. For every $(x_1, \dots x_n)$, $(y_1, \dots y_n), (z_1, \dots z_n) \in \mathbb{R}^n$ it holds*

$$\sum_{i=1}^n |x_i\, y_i\, z_i| \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n |y_i|^q\right)^{1/q} \left(\sum_{i=1}^n |z_i|^r\right)^{1/r}.$$

Chapter II

Stokes problem with mixed Dirichlet-Neumann boundary conditions

Contents

A condensed version of this chapter has been published in [GKL17]; further details are given here, such as the proofs the error estimate for the DDFV scheme, the extension to the Divergence form and the study of the unstabilized scheme.

The goal of this chapter is to approximate with DDFV method the solution of the following Stokes problem:

$$\begin{cases} -\Delta \mathbf{u} + \nabla \mathrm{p} = \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_D, \\ (\nabla \mathbf{u} - \mathrm{pId})\,\vec{\mathbf{n}} = \Phi & \text{on } \Gamma_N, \end{cases} \tag{II.1}$$

where the unknowns are the velocity $\mathbf{u} : \Omega \to \mathbb{R}^2$ and the pressure $\mathrm{p} : \Omega \to \mathbb{R}$. The data are $\mathbf{f} \in (L^2(\Omega))^2$, $\Phi, \mathbf{g} \in (H^{\frac{1}{2}}(\partial\Omega))^2$ and $\vec{\mathbf{n}}$ is the unitary outer normal. Ω is an open bounded polygonal domain of $\mathbb{R}^2$, with $\partial\Omega = \Gamma_D \cup \Gamma_N$, where $|\Gamma_D| > 0$ is the fraction of the boundary with Dirichlet boundary conditions and $\Gamma_N \neq \emptyset$ the one with Neumann boundary conditions.

In the previous works of [Del07], [Kre11a] and [BKN15], DDFV method was studied for Stokes problem in the case of homogeneous Dirichlet boundary conditions. In the case of [Del07], well-posedness of the scheme was proved only for conformal triangle meshes, conformal and non conformal square meshes, in the case of unstabilized mass equation. This result was then improved in [Kre11a] by adding a stabilization term to the equation of conservation of mass that led to prove existence and uniqueness of the solution on general meshes. Successively, since it was observed that very accurate approximations could be computed even without stabilization on general meshes, in [BKN15] Boyer, Krell and Nabet worked on the Inf-sup stability condition (Def. I.6.1) for the unstabilized scheme. This condition relies on the wellposedness of the scheme; it holds unconditionally for certain meshes (e.g. conforming acute triangle meshes) or, with some restrictions, for specific mesh geometries (see Sec. I.6 for more details).

The work of this chapter aims at extending the theory known for the Stokes problem to the case of Neumann boundary conditions on a fraction of the boundary, namely the 4^{th} equation in (II.1).

Outline. This chapter is organized as follows. In Sec. II.1 we define a DDFV discretization of the Stokes problem (II.1); we choose to stabilize the mass conservation equation through two parameters, $\mu, \beta \geq 0$. We prove the wellposedness of this scheme under the hypothesis $\beta + \mu > 0$ in Sec. II.2. In Sec. II.3 we prove some error estimates, by showing a first rough error estimate only for the velocity (when $\mu > 0$) followed by a stability result (true under the hypothesis $\beta > 0$) that leads to an optimal error estimate for the velocity and the pressure. This result is tested numerically in Sec. II.4, by also showing the influence of the parameters in the convergence. In Sec. II.5 we extend the obtained results to the Divergence form of Stokes problem (II.28), thanks to the results of Sec. I.8. In Sec. II.6, we consider the case of a DDFV scheme for the Stokes problem (II.1) without stabilization on the mass conservation equation, with "weak" boundary conditions.

II.1 DDFV scheme

We recall that DDFV scheme uses staggered unknowns. We approximate the velocity on the centers and vertices of the primal mesh (i.e. on $\mathfrak{T} = \mathfrak{M} \cup \partial\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$) and the pressure on the diamond mesh (i.e. $\mathfrak{D}$).
As introduced in Sec. I.2 and as illustrated in Fig. II.1, the boundary meshes will be denoted by:

$$\begin{aligned}
\partial\mathfrak{M}_D &= \{\kappa \in \partial\mathfrak{M} : x_\kappa \in \Gamma_D\},\\
\partial\mathfrak{M}_N &= \{\kappa \in \partial\mathfrak{M} : x_\kappa \in \Gamma_N\},\\
\partial\mathfrak{M}_D^* &= \{\kappa^* \in \partial\mathfrak{M}^* : x_{\kappa^*} \in \Gamma_D\},\\
\partial\mathfrak{M}_N^* &= \{\kappa^* \in \partial\mathfrak{M}^* : x_{\kappa^*} \in \Gamma_N \setminus \Gamma_D\}.
\end{aligned}$$

To obtain our scheme, we integrate the momentum equation over all $\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}_N^*$. We impose Dirichlet boundary conditions on $\partial\mathfrak{M}_D \cup \partial\mathfrak{M}_D^*$ and Neumann boundary conditions on $\partial\mathfrak{M}_N$. The equation of conservation of mass is directly approximated on the diamond mesh equation over $\mathfrak{D}$, and it is stabilized through two parameters $\beta \geq 0$, associated to a stabilization of Brezzi-Pitkäranta (see Sec. I.7) and $\mu \geq 0$, associated to a linear stabilization.

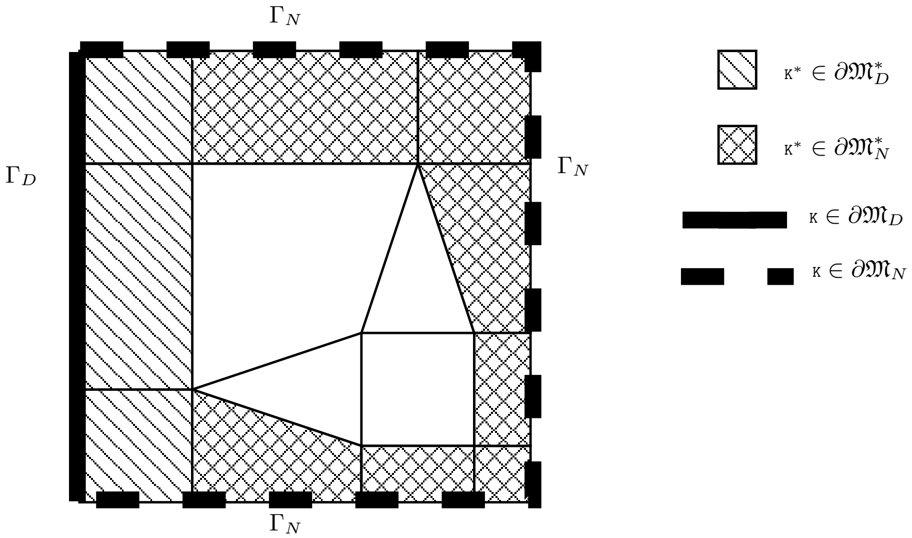


Fig. II.1 Domain with mixed boundary conditions

The scheme is the following:

$$\text{Find } \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}^{\Gamma_D}_{m,\mathbf{g}} \text{ and } \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that}$$

$$\begin{cases} \mathbf{div}^{\kappa}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\kappa} & \forall \kappa \in \mathfrak{M} \\ \mathbf{div}^{\kappa^*}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\kappa^*} & \forall \kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*_N \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) + \mu \mathrm{size}(\mathfrak{T})\mathrm{p}^{\mathfrak{D}} - \beta \mathrm{d}^2_{\mathfrak{D}}\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}} = 0 & \\ (\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathrm{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\kappa} = \Phi_\sigma & \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N, \end{cases} \qquad (\mathcal{P}_{\beta\mu})$$

where we denote by $\mathbf{f}_{\kappa}, \mathbf{g}_{\kappa}$ (resp. $\mathbf{f}_{\kappa^*}, \mathbf{g}_{\kappa^*}$) the mean-value of the source term $\mathbf{f}$ and of the Dirichlet data $\mathbf{g}$ on $\kappa \in \mathfrak{M}$ (resp. on $\kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$) and Φ_σ the mean-value of the Neumann data on $\sigma \in \Gamma_N$:

$$\mathbf{f}_{\kappa} = \frac{1}{m_{\kappa}} \int_{\kappa} \mathbf{f}(x)\mathrm{d}x, \qquad \mathbf{f}_{\kappa^*} = \frac{1}{m_{\kappa^*}} \int_{\kappa^*} \mathbf{f}(x)\mathrm{d}x,$$
$$\mathbf{g}_{\kappa} = \frac{1}{m_{\kappa}} \int_{\kappa} \mathbf{g}(x)\mathrm{d}x, \qquad \mathbf{g}_{\kappa^*} = \frac{1}{m_{\kappa^*}} \int_{\kappa^*} \mathbf{g}(x)\mathrm{d}x,$$
$$\Phi_\sigma = \frac{1}{m_\sigma} \int_\sigma \Phi(x)\mathrm{d}x.$$

Moreover, we denote by $\mathbf{g}_\sigma = \gamma^\sigma(\mathbf{g}^{\mathfrak{T}})$, $\sigma \in \partial\mathfrak{M}$. Remark that, as the mesh becomes finer, the stabilization terms vanish (we recall that $\mathrm{d}^2_{\mathfrak{D}}$ is the diameter of a diamond, thus it depends on $size(\mathfrak{T})$).

We define now the bilinear form associated to the scheme $(\mathcal{P}_{\beta\mu})$.

Definition II.1.1 (Bilinear form) *For all* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}), (\widetilde{\mathbf{u}}^{\mathfrak{T}}, \widetilde{p}^{\mathfrak{D}}) \in \mathbb{E}^{\Gamma_D}_{m,\boldsymbol{g}} \times \mathbb{R}^{\mathfrak{D}}$*, the bilinear form associated to* $(\mathcal{P}_{\beta\mu})$ *is:*

$$\begin{aligned} B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \widetilde{\mathbf{u}}^{\mathfrak{T}}, \widetilde{p}^{\mathfrak{D}}) := [[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}), \widetilde{\mathbf{u}}^{\mathfrak{T}}]]_{\mathfrak{T}} \\ + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) + \mu size(\mathfrak{T})\mathrm{p}^{\mathfrak{D}} - \beta d^2_{\mathfrak{D}}\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}}, \widetilde{p}^{\mathfrak{D}})_{\mathfrak{D}}. \end{aligned} \quad \text{(II.2)}$$

II.2 Wellposedness of the scheme

Before showing that there exists a unique solution to $(\mathcal{P}_{\beta\mu})$, we prove the following a priori estimate:

Proposition II.2.1 (A priori estimate) *Let* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_{m,\boldsymbol{g}}^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$ *be a solution of* $(\mathcal{P}_{\beta\mu})$*. Then:*

$$\begin{aligned}
&\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \mu size(\mathfrak{T})\|\mathrm{p}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 \\
&\quad \leq \left|[[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}\right| + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma|\boldsymbol{g}_\sigma|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma\left|\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}} Id\right|^2\right)^{\frac{1}{2}} \\
&\qquad + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\Phi_\sigma|^2\right)^{\frac{1}{2}} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma\left|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})\right|^2\right)^{\frac{1}{2}}. \qquad \text{(II.3)}
\end{aligned}$$

Proof We consider the bilinear form (II.2) associated to the scheme:

$$\begin{aligned}
B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = [[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}), \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} \\
+ (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) + \mu\mathrm{size}(\mathfrak{T})\mathrm{p}^{\mathfrak{D}} - \beta \mathrm{d}_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}.
\end{aligned}$$

On one hand, if we apply Green's formula (Thm. I.5.1) to the first term, by taking into account the boundary terms, we get:

$$\begin{aligned}
B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) &= \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 - (\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}))_{\mathfrak{D}} \\
&\quad + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot (-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}}\mathrm{Id})\,\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}} \\
&\quad + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) + \mu\mathrm{size}(\mathfrak{T})\mathrm{p}^{\mathfrak{D}} - \beta \mathrm{d}_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}.
\end{aligned}$$

The terms $(\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}))_{\mathfrak{D}}$ simplify and we apply Remark I.7.2 to the term $-\beta(\mathrm{d}_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}$. The boundary diamonds $\mathfrak{D}_{ext}$ can be split between $\mathfrak{D}_{ext} \cap \Gamma_D$ and $\mathfrak{D}_{ext} \cap \Gamma_N$, so by applying Dirichlet and Neumann boundary conditions we obtain:

$$\begin{aligned}
B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) &= \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \mu\mathrm{size}(\mathfrak{T})\|\mathrm{p}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 \\
&\quad + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma\mathbf{g}_\sigma \cdot (-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}}\mathrm{Id})\,\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}} - \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \Phi_\sigma. \qquad \text{(II.4)}
\end{aligned}$$

On the other hand, since $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ is a solution to $(\mathcal{P}_{\beta\mu})$, we have:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = [[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}. \qquad \text{(II.5)}$$

By putting together (II.4) and (II.5) :

$$\begin{aligned}
&\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \mu\mathrm{size}(\mathfrak{T})\|\mathrm{p}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 \\
&\quad \leq [[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} + \left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma\mathbf{g}_\sigma \cdot (\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}}\mathrm{Id})\,\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}}\right| + \left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \Phi_\sigma\right|.
\end{aligned}$$

We then apply Cauchy-Schwarz inequality to get our result (II.3). ■

We can now prove the well-posedness of the scheme, that is a direct consequence to the *a priori* estimate of Prop. II.2.1.

Theorem II.2.2 (Well-posedness of the scheme) *Let* $\mathfrak{T}$ *a DDFV mesh associated to* Ω *and* $\beta + \mu > 0$ *. Then the stabilized scheme* $(\mathcal{P}_{\beta\mu})$ *has a unique solution* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_{m,\boldsymbol{g}}^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$*.*

Proof By linearity, it is sufficient to prove that if $\mathbf{f}^{\mathfrak{T}} = 0$, $\mathbf{g}^{\partial\mathfrak{M}} = 0$, $\mathbf{g}^{\partial\mathfrak{M}^*} = 0$ and $\Phi_\sigma = 0$, then $\mathbf{u}^{\mathfrak{T}} = 0$ and $\mathrm{p}^{\mathfrak{D}} = 0$. Directly from (II.3), we deduce:

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \mu \mathrm{size}(\mathfrak{T})\|\mathrm{p}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 \le 0.$$

This implies $\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 = 0$: from Remark I.3.3 and since $\mathbf{g}^{\partial\mathfrak{M}} = 0$, $\mathbf{g}^{\partial\mathfrak{M}^*} = 0$, we obtain $\mathbf{u}^{\mathfrak{T}} = 0$. Moreover, if $\mu > 0$, then $\|\mathrm{p}^{\mathfrak{D}}\|_2^2 = 0$ that implies $\mathrm{p}^{\mathfrak{D}} = 0$; otherwise, we have $\beta > 0$, from which we can deduce $|\mathrm{p}^{\mathfrak{D}}|_h^2 = 0$ that leads to $\mathrm{p}^{\mathfrak{D}}$ constant. Thanks to Neumann boundary condition (the 4^{th} equation in $(\mathcal{P}_{\beta\mu})$), since $\mathbf{u}^{\mathfrak{T}} = 0$, and $\Phi_\sigma = 0$ we get that $\mathrm{p}^{\mathfrak{D}} = 0$. ■

Remark II.2.3 *Without the stabilization term, we can still prove the wellposedness of the scheme. In fact, if the velocity $\mathbf{u}^{\mathfrak{T}} = 0$, the momentum equation and the Neumann boundary condition become:*

$$\begin{cases} \mathbf{div}^{\mathrm{K}}\left(\mathrm{p}^{\mathfrak{D}}\mathrm{Id}\right) = 0 & \forall \mathrm{K} \in \mathfrak{M} \\ \mathbf{div}^{\mathrm{K}^*}\left(\mathrm{p}^{\mathfrak{D}}\mathrm{Id}\right) = 0 & \forall \mathrm{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_N^* \\ (\mathrm{p}^{\mathrm{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathrm{K}} = 0 & \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N. \end{cases} \tag{II.6}$$

Our goal is to show that $\mathrm{p}^{\mathfrak{D}} = 0$. For every $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, thanks to Green's formula (Thm. I.5.1), we can write:

$$\left(\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}}\right)_{\mathfrak{D}} = -\left[\left[\mathbf{v}^{\mathfrak{T}}, \mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}\mathrm{Id})\right]\right]_{\mathfrak{T}} + \sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \cdot (\mathrm{p}^{\mathrm{D}}\mathrm{Id})\,\vec{\mathbf{n}}_{\sigma\mathrm{K}}. \tag{II.7}$$

By definition of the scalar products (see Sec. I.4), by (II.6) *and by the fact that $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, we get that $\left[\left[\mathbf{v}^{\mathfrak{T}}, \mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}\mathrm{Id})\right]\right]_{\mathfrak{T}} = 0$ and $\sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \cdot (\mathrm{p}^{\mathrm{D}}\mathrm{Id})\,\vec{\mathbf{n}}_{\sigma\mathrm{K}} = 0$. Thus* (II.7) *becomes:*

$$\left(\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}}\right)_{\mathfrak{D}} = 0. \tag{II.8}$$

Assuming that the mesh $\mathfrak{T}$ satisfies Inf-sup stability condition (see Sec. I.6), inequality (I.30) *is verified; since* (II.8) *holds for any $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, the supremum in the right hand side of* (I.30) *vanishes so we can deduce that the pressure $\mathrm{p}^{\mathfrak{D}}$ is constant. We can then conclude thanks to Neumann boundary conditions as in the previous proof.*

II.3 Error estimates for the DDFV scheme

In this section, we prove error estimates for the scheme $(\mathcal{P}_{\beta\mu})$.
First, we show an error estimate of order 0.5 just for the velocity and its gradient, under the hypothesis $\mu > 0$. We then improve this result if $\beta > 0$, by showing an estimate of order 1 for the velocity, its gradient and the pressure, thanks to a stability study of the scheme.

II.3.1 Rough error estimate

Since we are working with mixed boundary conditions of the type Dirichlet/ Neumann, i.e. $\Gamma_N \neq \emptyset$, we need to suppose more regularity for the exact solution $\mathbf{u}$ in order to get a better error estimate with respect to the homogeneous Dirichlet case of [Kre11a].

Thus, we define the space of regularity of the solution as follows:

$$(W^{2,\infty}(\mathfrak{D}))^2 = \left\{ \mathbf{u} \in (W^{1,\infty}(\Omega))^2 \ \ s.t. \ \ \mathbf{u}|_{\mathtt{D}} \in (W^{2,\infty}(\mathtt{D}))^2, \quad \forall \mathtt{D} \in \mathfrak{D} \right\}.$$

$$W^{1,\infty}(\mathfrak{D}) = \left\{ \mathrm{p} \in L^{\infty}(\Omega) \ \ s.t. \ \ \mathrm{p}|_{\mathtt{D}} \in W^{1,\infty}(\mathtt{D}), \quad \forall \mathtt{D} \in \mathfrak{D} \right\}.$$

We will now prove an error estimate for the error of order 0.5 in the L^2 norm of the velocity field and of its gradient.

Theorem II.3.1 *Let* $(\mathbf{u}, p) \in (W^{2,\infty}(\mathfrak{D}))^2 \times W^{1,\infty}(\mathfrak{D})$ *be the solution of* (II.1) *and* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}^{\Gamma_D}_{m,g} \times \mathbb{R}^{\mathfrak{D}}$ *be the solution of the problem* $(\mathcal{P}_{\beta\mu})$*. Suppose that* $\mu > 0$*. Then there exists a constant* $C > 0$ *that depends on* $reg(\mathfrak{T}), \mu, \|\mathbf{u}\|_{W^{2,\infty}}$ *and* $\|p\|_{W^{1,\infty}}$ *such that*

$$\|\mathbf{u} - \mathbf{u}^{\mathfrak{T}}\|_2 + \|\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2 \leq C size(\mathfrak{T})^{\frac{1}{2}}.$$

Proof Let $\mathbf{e}^{\mathfrak{T}} = (\mathfrak{P}^D_{m,g} \mathbb{P}^{\mathfrak{T}}_c \mathbf{u}) - \mathbf{u}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$ be the error for the velocity field and $\mathrm{e}^{\mathfrak{D}} = \mathbb{P}^{\mathfrak{D}}_m \mathrm{p} - \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ the error for the pressure field.

The proof is divided into four parts. In Step 1, we start by defining the problem satisfied by the errors. Then, in Step 2, we give a first estimate of the bilinear form associated to the scheme, followed by estimates of the consistency errors in Step 3. We will conclude in Step 4 by gathering all the estimates together.

II.3.1.1 Step 1 : Error scheme

We look for the equations satisfied by $(\mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathfrak{T}} \times \mathbb{R}^{\mathfrak{D}}$.

Thanks to (II.1) and $(\mathcal{P}_{\beta\mu})$, we can write $\forall \mathtt{K} \in \mathfrak{M}$:

$$\begin{cases} \mathbf{div}^{\mathtt{K}}(-\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}} Id) = \mathbf{f}_{\mathtt{K}}, \\ -\dfrac{1}{m_{\mathtt{K}}} \displaystyle\int_{\mathtt{K}} \mathrm{div}(\nabla \mathbf{u}(x)) \mathrm{d}x + \dfrac{1}{m_{\mathtt{K}}} \displaystyle\int_{\mathtt{K}} \nabla \mathrm{p}(x) \mathrm{d}x = \mathbf{f}_{\mathtt{K}}. \end{cases}$$

Thus, we deduce:

$$m_{\mathtt{K}} \mathbf{div}^{\mathtt{K}}(-\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}} + \mathrm{e}^{\mathfrak{D}} \mathrm{Id}) = m_{\mathtt{K}} \mathbf{div}^{\mathtt{K}}(-\nabla^{\mathtt{D}} \mathfrak{P}^D_{m,g} \mathbb{P}^{\mathfrak{T}}_c \mathbf{u} + \mathbb{P}^{\mathfrak{D}}_m \mathrm{p}\, \mathrm{Id}) + \int_{\mathtt{K}} \mathrm{div}(\nabla \mathbf{u}(x)) \mathrm{d}x - \int_{\mathtt{K}} \nabla \mathrm{p}(x) \mathrm{d}x.$$

By Def. I.3.5 of the discrete divergence and Green's formula (Thm. I.5.1), we obtain for all $\mathtt{K} \in \mathfrak{M}$:

- $$m_{\mathtt{K}} \mathbf{div}^{\mathtt{K}}(-\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}} + \mathrm{e}^{\mathfrak{D}} \mathrm{Id}) = \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}} \int_{\sigma} (\nabla \mathbf{u} - \nabla^{\mathtt{D}} \mathfrak{P}^D_{m,g} \mathbb{P}^{\mathfrak{T}}_c \mathbf{u}) \vec{\mathbf{n}}_{\sigma\mathtt{K}} \mathrm{d}s + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathtt{K}}} \int_{\sigma} (\mathbb{P}^D_m \mathrm{p} - \mathrm{p}(s)) \vec{\mathbf{n}}_{\sigma\mathtt{K}} \mathrm{d}s.$$

We define the *consistency errors*

$$\begin{aligned} R^{\mathbf{u}}_{\mathfrak{D}}(z) &= \nabla \mathbf{u}(z) - \nabla^{\mathtt{D}} \mathfrak{P}^{\mathrm{D}}_{m,g} \mathbb{P}^{\mathfrak{T}}_c \mathbf{u}, \qquad && \text{for } z \in \mathtt{D},\ \mathtt{D} \in \mathfrak{D}, \\ R^{p}_{\mathfrak{D}}(z) &= \mathbb{P}^D_m \mathrm{p} - \mathrm{p}(z), && \text{for } z \in \mathtt{D},\ \mathtt{D} \in \mathfrak{D}. \end{aligned}$$

so that

$$\bullet m_{\mathtt{K}}\mathbf{div}^{\mathtt{K}}(-\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}+\mathrm{e}^{\mathfrak{D}}\mathrm{Id}) = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}}\int_{\sigma} R^{\mathbf{u}}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}}\int_{\sigma} R^{p}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s.$$

In the same way, for all $\mathtt{K}^*\in\mathfrak{M}*$:

$$\bullet m_{\mathtt{K}^*}\mathbf{div}^{\mathtt{K}^*}(-\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}+\mathrm{e}^{\mathfrak{D}}\mathrm{Id}) = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}}\int_{\sigma^*} R^{\mathbf{u}}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{d}s + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}}\int_{\sigma^*} R^{p}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{d}s.$$

For the boundary primal and dual cells, we consider only $\partial\mathfrak{M}^*_N$ and $\partial\mathfrak{M}_N$ since we impose strong Dirichlet boundary conditions on $\partial\mathfrak{M}^*_D$ and $\partial\mathfrak{M}_D$, i.e. $\mathbf{u}_{\mathtt{K}^*}=0\ \forall\mathtt{K}^*\in\partial\mathfrak{M}^*_D$, $\mathbf{u}_{\mathtt{K}}=0\ \forall\mathtt{K}\in\partial\mathfrak{M}_D$.

For all $\mathtt{K}^*\in\partial\mathfrak{M}^*_N$:

$$\begin{aligned}\bullet\, m_{\mathtt{K}^*}\mathbf{div}^{\mathtt{K}^*}(-\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}+\mathrm{e}^{\mathfrak{D}}\mathrm{Id}) &= \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}}\int_{\sigma^*} R^{\mathbf{u}}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{d}s + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}}\int_{\sigma^*} R^{p}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{d}s\\ &+ \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}\cap\mathfrak{D}_{ext}}\int_{x_{\mathtt{K}^*}}^{x_\sigma} R^{\mathbf{u}}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}\cap\mathfrak{D}_{ext}}\int_{x_{\mathtt{K}^*}}^{x_\sigma} R^{p}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s.\end{aligned}$$

Finally, for all $\sigma\in\partial\mathfrak{M}_N$:

$$\bullet(\nabla^{\mathtt{D}}\mathbf{e}^{\mathfrak{T}}-\mathbf{e}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}} = -\frac{1}{m_\sigma}\int_{\sigma} R^{\mathbf{u}}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s - \frac{1}{m_\sigma}\int_{\sigma} R^{p}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s.$$

We can finally write in a compact way the system satisfied by the error:

$$\text{Find } \mathbf{e}^{\mathfrak{T}}\in\mathbb{E}_0^{\Gamma_D} \text{ and } \mathrm{e}^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}} \text{ such that:}$$

$$\begin{cases} \mathbf{div}^{\mathtt{K}}(-\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}+\mathrm{e}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{R}^{\mathtt{K}} & \forall\mathtt{K}\in\mathfrak{M}\\ \mathbf{div}^{\mathtt{K}^*}(-\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}+\mathrm{e}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{R}^{\mathtt{K}^*} & \forall\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*_N\\ \mathrm{div}^{\mathfrak{D}}(\mathbf{e}^{\mathfrak{T}})+\mu\mathrm{size}(\mathfrak{T})\mathrm{e}^{\mathfrak{D}}-\beta d_{\mathtt{D}}^2\Delta^{\mathfrak{D}}\mathrm{e}^{\mathfrak{D}} = \mathbf{R}^{\mathfrak{D}} & \\ (\nabla^{\mathtt{D}}\mathbf{e}^{\mathfrak{T}}-\mathbf{e}^{\mathfrak{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}} = -(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}+\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}) & \forall\forall\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N \end{cases}$$

where $\mathbf{R}^{\mathfrak{T}}=\Big((\mathbf{R}^{\mathtt{K}})_{\mathtt{K}\in\mathfrak{M}},(\mathbf{R}^{\mathtt{K}^*})_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*_N}\Big)$ and $\mathbf{R}^{\mathfrak{D}}=\Big(\mathbf{R}^{\mathtt{D}}\Big)_{\mathtt{D}\in\mathfrak{D}}$ with:

$$\begin{aligned}
\mathbf{R}^{\mathtt{K}} &= \frac{1}{m_{\mathtt{K}}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \frac{1}{m_{\mathtt{K}}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}, && \forall\mathtt{K}\in\mathfrak{M},\\
\mathbf{R}^{\mathtt{K}^*} &= \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*} + \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*}, && \forall\mathtt{K}^*\in\mathfrak{M}^*,\\
\mathbf{R}^{\mathtt{K}^*} &= \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}(\mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*}+\mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*}) + \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} \frac{m_\sigma}{2}(\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}+\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}), && \forall\mathtt{K}^*\in\partial\mathfrak{M}^*_N,\\
\mathbf{R}^{\mathtt{D}} &= \mathrm{div}^{\mathtt{D}}(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_c\mathbf{u}) + \mu\mathrm{size}(\mathfrak{T})\mathbb{P}^{\mathrm{D}}_m\mathrm{p} - \beta d_{\mathtt{D}}^2\Delta^{\mathtt{D}}\mathbb{P}^{\mathfrak{D}}_m\mathrm{p}, && \forall\mathtt{D}\in\mathfrak{D}.
\end{aligned}$$

We define for $i \in \{\mathbf{u}, \mathrm{p}\}$:

$$\begin{aligned}
&\mathbf{R}^i_{\sigma\mathtt{K}} = -\mathbf{R}^i_{\sigma\mathtt{L}} = \frac{1}{m_\sigma}\int_\sigma R^i_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s,\\
&\mathbf{R}^i_{\sigma^*\mathtt{K}^*} = -\mathbf{R}^i_{\sigma^*\mathtt{L}^*} = \frac{1}{m_{\sigma^*}}\int_{\sigma^*} R^i_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{d}s,\\
&\mathbf{R}^i_{\mathtt{K}^*\mathtt{L}} = \frac{2}{m_\sigma}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{D}}} R^i_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s,\\
&\mathbf{R}^i_{\sigma} = |\mathbf{R}^i_{\sigma\mathtt{K}}| = |\mathbf{R}^i_{\sigma\mathtt{L}}|,\\
&\mathbf{R}^i_{\sigma^*} = |\mathbf{R}^i_{\sigma^*\mathtt{K}^*}| = |\mathbf{R}^i_{\sigma^*\mathtt{L}^*}|.
\end{aligned}$$

We denote the L^2 norms of consistency error as follows:

$$\|\mathbf{R}^i_\sigma\|_2^2 = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}}|\mathbf{R}^i_\sigma|^2 \quad \text{and} \quad \|\mathbf{R}^i_{\sigma^*}\|_2^2 = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}}|\mathbf{R}^i_{\sigma^*}|^2, \quad \text{for } i \in \{\mathbf{u}, \mathrm{p}\}.$$

We also remark that for $\sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}] \subset \partial\Omega$:

$$\mathbf{R}^i_{\sigma\mathtt{K}} = \frac{1}{2}\mathbf{R}^i_{\mathtt{K}^*\mathtt{L}} + \frac{1}{2}\mathbf{R}^i_{\mathtt{L}^*\mathtt{L}}, \quad \text{for } i \in \{\mathbf{u}, \mathrm{p}\}.$$

II.3.1.2 Step 2 : Estimate of $B(\mathbf{e}^{\mathfrak{T}}, \mathbf{e}^{\mathfrak{D}}; \mathbf{e}^{\mathfrak{T}}, \mathbf{e}^{\mathfrak{D}})$

We can now start with the estimate. Thanks to the Def. (II.2) of B, we have:

$$B(\mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}; \mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}) = [[\mathbf{R}^{\mathfrak{T}}, \mathbf{e}^{\mathfrak{T}}]]_{\mathfrak{T}} + (\mathbf{R}^{\mathfrak{D}}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}}.$$

We note $I := [[\mathbf{R}^{\mathfrak{T}}, \mathbf{e}^{\mathfrak{T}}]]_{\mathfrak{T}}$ and $T := (\mathbf{R}^{\mathfrak{D}}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}}$.

Estimate of $I = [[\mathbf{R}^{\mathfrak{T}}, \mathbf{e}^{\mathfrak{T}}]]_{\mathfrak{T}}$:
By definition, I is:

$$\begin{aligned}
I = \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}})\cdot\mathbf{e}_{\mathtt{K}} + \frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}(\mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*} + \mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*})\cdot\mathbf{e}_{\mathtt{K}^*}\\
+ \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} \frac{m_\sigma}{2}(\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}} + \mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}})\cdot\mathbf{e}_{\mathtt{K}^*}.
\end{aligned}$$

If we reorganize the sum on diamonds, we get:

$$\begin{aligned}
I = &\frac{1}{2}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_\sigma(\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}})\cdot(\mathbf{e}_{\mathtt{K}} - \mathbf{e}_{\mathtt{L}}) + \frac{1}{2}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\sigma^*}(\mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*} + \mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*})\cdot(\mathbf{e}_{\mathtt{K}^*} - \mathbf{e}_{\mathtt{L}^*})\\
&+ \frac{1}{2}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma(\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}})\cdot\mathbf{e}_{\mathtt{L}} + \frac{1}{2}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}(\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}})\cdot\mathbf{e}_{\mathtt{K}^*}\\
&+ \frac{1}{2}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}(\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}})\cdot\mathbf{e}_{\mathtt{L}^*}\\
:= &I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Remark that the boundary terms depend only on the values on Γ_N; in fact, since we impose Dirichlet boundary conditions, if $\mathtt{D} \in \mathfrak{D}_{ext} \cap \Gamma_D$ we have $\mathbf{e}_{\mathtt{L}} = \mathbf{e}_{\mathtt{K}^*} = \mathbf{e}_{\mathtt{L}^*} = 0$.

We estimate separately the terms.

► Firstly from Def. (I.3.1) of the discrete gradient and secondly by Cauchy-Schwarz inequality, we have:

$$\begin{aligned} I_1 &= \frac{1}{2} \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_\sigma (\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}) \cdot (\mathbf{e}_{\mathtt{K}} - \mathbf{e}_{\mathtt{L}}) \\ &= - \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}} \frac{m_{\mathtt{D}}}{\sin(\alpha_D)} (\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}) \cdot ((\nabla^{\mathtt{D}} \mathbf{e}^{\mathfrak{T}}) \cdot \vec{\boldsymbol{\tau}}_{\mathtt{KL}}) \\ &\leq C(\mathrm{reg}(\mathfrak{T})) \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2 \left(\|\mathbf{R}^{\mathrm{p}}_{\sigma}\|_2 + \|\mathbf{R}^{\mathbf{u}}_{\sigma}\|_2 \right). \end{aligned}$$

► As the previous term, from Def. (I.3.1) of the discrete gradient and then by Cauchy-Schwarz inequality, we have:

$$\begin{aligned} I_2 &= \frac{1}{2} \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_\sigma (\mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*} + \mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*}) \cdot (\mathbf{e}_{\mathtt{K}^*} - \mathbf{e}_{\mathtt{L}^*}) \\ &= - \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}} \frac{m_{\mathtt{D}}}{\sin(\alpha_D)} (\mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*} + \mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*}) \cdot ((\nabla^{\mathtt{D}} \mathbf{e}^{\mathfrak{T}}) \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}) \\ &\leq C(\mathrm{reg}(\mathfrak{T})) \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2 \left(\|\mathbf{R}^{\mathrm{p}}_{\sigma^*}\|_2 + \|\mathbf{R}^{\mathbf{u}}_{\sigma^*}\|_2 \right). \end{aligned}$$

► By applying Cauchy-Schwarz inequality and by the definitions of traces (and their norms) of Sec. I.4, we have:

$$\begin{aligned} I_3 &= \frac{1}{2} \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma (\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}) \cdot \mathbf{e}_{\mathtt{L}} \\ &\leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma |\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2 \right)^{\frac{1}{2}} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma |\mathbf{e}_{\mathtt{L}}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma |\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2 \right)^{\frac{1}{2}} \|\gamma^\sigma(\mathbf{e}^{\mathfrak{T}})\|_{2,\partial\Omega}. \end{aligned}$$

By applying Thm. I.9.4, we can write:

$$I_3 \leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma |\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2 \right)^{\frac{1}{2}} \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2.$$

► In the same way, by applying Cauchy-Schwarz inequality and by the definitions of traces (and their norms) of Sec. I.4, we have:

$$\begin{aligned} I_4 &= \frac{1}{2} \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} (\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}) \cdot \mathbf{e}_{\mathtt{K}^*} \\ &\leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{e}_{\mathtt{K}^*}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \|\gamma^\sigma(\mathbf{e}^{\mathfrak{T}})\|_{2,\partial\Omega}. \end{aligned}$$

By applying Thm. I.9.4, we can write:

$$I_4 \leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2.$$

► Finally, as the previous two estimates, we have:

$$\begin{aligned} I_5 &= \frac{1}{2} \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} (\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}) \cdot \mathbf{e}_{\mathtt{L}^*} \\ &\leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{e}_{\mathtt{L}^*}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \|\gamma^\sigma(\mathbf{e}^{\mathfrak{T}})\|_{2,\partial\Omega}. \end{aligned}$$

By applying Thm. I.9.4, we can write:

$$I_5 \leq \frac{1}{2} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}} + \mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2.$$

Putting all together, we get the following estimate for I:

$$\begin{aligned} |I| \leq\; & C(\mathrm{reg}(\mathfrak{T})) \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2 \left(\|\mathbf{R}^{\mathrm{p}}_{\sigma}\|_2 + \|\mathbf{R}^{\mathbf{u}}_{\sigma}\|_2 + \|\mathbf{R}^{\mathrm{p}}_{\sigma^*}\|_2 + \|\mathbf{R}^{\mathbf{u}}_{\sigma^*}\|_2\right) \\ &+ C(\mathrm{reg}(\mathfrak{T})) \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2 \left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \left(\frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \right] \\ &+ C(\mathrm{reg}(\mathfrak{T})) \|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathfrak{T}}\|_2 \left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \left(\frac{m_\sigma}{2} |\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}}|^2 \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Estimate of $T = (\mathbf{R}^{\mathfrak{D}}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}}$:

We remark that $T = (\mathrm{div}^{\mathfrak{D}}(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u}), \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}} + (\mu\, \mathrm{size}(\mathfrak{T})\, \mathbb{P}^{\mathfrak{D}}_{m}\mathrm{p}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}} - (\beta d^2_{\mathfrak{D}} \Delta^{\mathfrak{D}} \mathbb{P}^{\mathfrak{D}}_{m}\mathrm{p}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}}$, so by adding and subtracting $\mathrm{div}^{\mathfrak{D}}(\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u})$ and by Minkowski inequality we have:

$$\|\mathrm{div}^{\mathfrak{D}}(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u})\|_2 \leq \|\nabla^{\mathfrak{D}}(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u} - \mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u})\|_2 + \|\mathrm{div}^{\mathfrak{D}}(\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u})\|_2.$$

From Corollary I.10.4 and Lemma I.10.6 we get:

$$\|\mathrm{div}^{\mathfrak{D}}(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u})\|_2 \leq C\mathrm{size}(\mathfrak{T})\|\mathbf{u}\|_{H^2}.$$

Cauchy-Schwarz inequality on the previous estimate gives:

$$(\mathrm{div}^{\mathfrak{D}}(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u}), \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}} \leq C\mathrm{size}(\mathfrak{T})\|\mathbf{u}\|_{H^2}\|\mathrm{e}^{\mathfrak{D}}\|_2.$$

By reorganizing the sum on $\mathfrak{s} \in \mathfrak{S}$ in the term $T_1 := -(\beta d_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} \mathbb{P}_m^{\mathfrak{D}} \mathrm{p}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}}$ we have, as in Rem. I.7.2:

$$T_1 = -\beta \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} \mathrm{e}^{\mathfrak{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \mathbb{P}_m^{\mathfrak{D}} \mathrm{p} = \beta \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathfrak{S}} (d_{\mathtt{D}}^2 + d_{\mathtt{D}'}^2)(\mathbb{P}_m^{\mathtt{D}'}\mathrm{p} - \mathbb{P}_m^{\mathtt{D}}\mathrm{p})(\mathbf{e}^{\mathtt{D}'} - \mathbf{e}^{\mathtt{D}}).$$

Cauchy-Schwarz inequality and the definition of the semi-norm $|\cdot|_h$ (see (I.37)) give:

$$\begin{aligned}|T_1| &\leq \left(\sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathfrak{S}} (d_{\mathtt{D}}^2 + d_{\mathtt{D}'}^2)(\mathbb{P}_m^{\mathtt{D}'}\mathrm{p} - \mathbb{P}_m^{\mathtt{D}}\mathrm{p})^2\right)^{\frac{1}{2}} \left(\sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathfrak{S}} (d_{\mathtt{D}}^2 + d_{\mathtt{D}'}^2)(\mathbf{e}^{\mathtt{D}'} - \mathbf{e}^{\mathtt{D}})^2\right)^{\frac{1}{2}} \\ &\leq 2\mathrm{size}(\mathfrak{T})\beta|\mathrm{e}^{\mathfrak{D}}|_h \left(\sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathfrak{S}} (\mathbb{P}_m^{\mathtt{D}'}\mathrm{p} - \mathbb{P}_m^{\mathtt{D}}\mathrm{p})^2\right)^{\frac{1}{2}}.\end{aligned}$$

Lemma I.7.3 and Lemma I.10.8 lead to:

$$|T_1| \leq C\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2\|\nabla \mathrm{p}\|_2.$$

Then, Cauchy-Schwarz inequality and Lemma I.10.9 give:

$$(\mu\, \mathrm{size}(\mathfrak{T})\, \mathbb{P}_m^{\mathfrak{D}}\mathrm{p}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}} \leq C\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2\|\mathrm{p}\|_{H^1}.$$

We recall that $T = (\mathrm{div}^{\mathfrak{D}}(\mathfrak{P}_{m,g}^{\mathrm{D}} \mathbb{P}_c^{\mathfrak{T}}\mathbf{u}), \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}} + (\mu\, \mathrm{size}(\mathfrak{T})\, \mathbb{P}_m^{\mathfrak{D}}\mathrm{p}, \mathrm{e}^{\mathfrak{D}})_{\mathfrak{D}} + T_1$, so we can write:

$$|T| \leq C\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2(\|\mathbf{u}\|_{H^2} + \|\mathrm{p}\|_{H^1}).$$

<u>Estimate of $B(\mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}; \mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}})$:</u>

By gathering all the estimates on I and T we get:

$$\begin{aligned}|B(\mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}; \mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}})| \leq\ & |I| + |T| \\ \leq\ & C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2 \left(\|\mathbf{R}_{\sigma}^{\mathrm{p}}\|_2 + \|\mathbf{R}_{\sigma}^{\mathbf{u}}\|_2 + \|\mathbf{R}_{\sigma^*}^{\mathrm{p}}\|_2 + \|\mathbf{R}_{\sigma^*}^{\mathbf{u}}\|_2\right) \\ &+ C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2 \Bigg[\Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \frac{m_\sigma}{2}|\mathbf{R}_{\mathtt{K}^*\mathtt{L}}^{\mathbf{u}}|^2\Bigg)^{\frac{1}{2}} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \left(\frac{m_\sigma}{2}|\mathbf{R}_{\mathtt{K}^*\mathtt{L}}^{\mathrm{p}}|^2\right)^{\frac{1}{2}}\Bigg] \\ &+ C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2 \Bigg[\Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \frac{m_\sigma}{2}|\mathbf{R}_{\mathtt{L}^*\mathtt{L}}^{\mathbf{u}}|^2\Bigg)^{\frac{1}{2}} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \left(\frac{m_\sigma}{2}|\mathbf{R}_{\mathtt{L}^*\mathtt{L}}^{\mathrm{p}}|^2\right)^{\frac{1}{2}}\Bigg] \\ &+ C\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2(\|\mathbf{u}\|_{H^2} + \|\mathrm{p}\|_{H^1}).\end{aligned} \tag{II.9}$$

As in Prop. II.2.1, equivalently to (II.4) the following equality holds:

$$B(\mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}, \mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}) = \|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 + \mu\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{e}^{\mathfrak{D}}|_h^2 - \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \gamma^\sigma(\mathbf{e}^{\mathfrak{T}}) \cdot (\mathbf{R}_{\sigma\mathtt{K}}^{\mathbf{u}} + \mathbf{R}_{\sigma\mathtt{K}}^{\mathrm{p}}),$$

from which we deduce:

$$\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 + \mu\text{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{e}^{\mathfrak{D}}|_h^2 \leq |B(\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}},\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}})| + \left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma\gamma^\sigma(\mathbf{e}^{\mathfrak{T}})\cdot(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}})\right|.$$

By applying Cauchy-Schwarz inequality we obtain:

$$\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 + \mu\text{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{e}^{\mathfrak{D}}|_h^2$$
$$\leq |B(\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}},\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}})| + \left(\sum_{D_{\sigma,\sigma*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}})^2\right)^{\frac{1}{2}} \left(\sum_{D_{\sigma,\sigma*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma(\gamma^\sigma(\mathbf{e}^{\mathfrak{T}}))^2\right)^{\frac{1}{2}}.$$

Thanks to Thm. I.9.4 and Minkowsky inequality, we can write:

$$\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 + \mu\text{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{e}^{\mathfrak{D}}|_h^2$$
$$\leq |B(\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}},\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}})| + C\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2\right)^{\frac{1}{2}} + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2\right)^{\frac{1}{2}}\right]\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2.$$

By replacing the estimates obtained in (II.9), we get:

$$\begin{aligned}
\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 &+ \mu\text{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{e}^{\mathfrak{D}}|_h^2 \\
&\leq C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\,(\|\mathbf{R}^{\mathrm{p}}_\sigma\|_2 + \|\mathbf{R}^{\mathbf{u}}_\sigma\|_2 + \|\mathbf{R}^{\mathrm{p}}_{\sigma^*}\|_2 + \|\mathbf{R}^{\mathbf{u}}_{\sigma^*}\|_2) \\
&+ C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2\right)^{\frac{1}{2}} + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2\right)^{\frac{1}{2}}\right] \\
&+ C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2\right)^{\frac{1}{2}} + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} \frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}}|^2\right)^{\frac{1}{2}}\right] \\
&+ C\text{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2(\|\mathbf{u}\|_{H^2} + \|\mathrm{p}\|_{H^1}) \\
&+ C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2\right)^{\frac{1}{2}} + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2\right)^{\frac{1}{2}}\right].
\end{aligned}$$

It remains to show the estimates for the consistency errors.

II.3.1.3 Step 3 : Consistency errors

First, we remark that the consistency error for the velocity can be decomposed into three contributions $R^{\mathbf{u},\eta}_{\mathfrak{D}}$, $R^{\mathbf{u},D\mathbf{u}}_{\mathfrak{D}}$ and $R^{\mathbf{u},bd}_{\mathfrak{D}}$ that come from, respectively, the error due to the flux approximation, to the gradient approximation and the boundary data approximation:

$$R^{\mathbf{u}}_{\mathfrak{D}}(z) = R^{\mathbf{u},\eta}_{\mathfrak{D}}(z) + R^{\mathbf{u},D\mathbf{u}}_{\mathfrak{D}} + R^{\mathbf{u},bd}_{\mathfrak{D}},$$

where, for $z \in \mathtt{D}$,

$$R^{\mathbf{u},\eta}_{\mathfrak{D}}(z) = \nabla \mathbf{u}(z) - \frac{1}{m_{\mathtt{D}}} \int_{\mathtt{D}} \nabla \mathbf{u}(x) \mathrm{d}x,$$
$$R^{\mathbf{u},D\mathbf{u}}_{\mathfrak{D}} = \frac{1}{m_{\mathtt{D}}} \int_{\mathtt{D}} (\nabla \mathbf{u}(x) - \nabla^{\mathtt{D}} \mathbb{P}^{\mathfrak{T}}_{c} \mathbf{u}) \mathrm{d}x,$$
$$R^{\mathbf{u},bd}_{\mathfrak{D}} = \nabla^{\mathtt{D}} \mathbb{P}^{\mathfrak{T}}_{c} \mathbf{u} - \nabla^{\mathtt{D}} \mathfrak{P}^{D}_{m,g} \mathbb{P}^{\mathfrak{T}}_{c} \mathbf{u}.$$

We now estimate all the consistency errors, by stating every result in the form of a lemma. For the first two lemmas, we just give the statement, since they have been proven in [Kre10]; we detail only the proofs of the new results.

Lemma II.3.2 ([Kre10], Lemma V.8) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg($\mathfrak{T}$), such that for all $\mathbf{u} \in (H^2(\Omega))^2$*

$$\|\boldsymbol{R}^{\mathbf{u}}_{\sigma}\|_2 + \|\boldsymbol{R}^{\mathbf{u}}_{\sigma^*}\|_2 \le C size(\mathfrak{T}) \|\nabla \mathbf{u}\|_{H^1}.$$

Lemma II.3.3 ([Kre10], Lemma V.10) *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg($\mathfrak{T}$), such that for all $\mathbf{u} \in H^1(\Omega)$:*

$$\|\boldsymbol{R}^{p}_{\sigma}\|_2 + \|\boldsymbol{R}^{p}_{\sigma^*}\|_2 \le C size(\mathfrak{T}) \|\nabla p\|_2.$$

The following result estimates boundary terms on Γ_N.

Lemma II.3.4 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω.*
There exists a constant $C > 0$, that depends only on reg($\mathfrak{T}$) and $m(\Gamma_N)$, such that for all $\mathbf{u} \in (W^{2,\infty}(\mathfrak{D}))^2$:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\boldsymbol{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2 + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\boldsymbol{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2 \le C size(\mathfrak{T})^2 \|\mathbf{u}\|^2_{W^{2,\infty}(\mathfrak{D})}$$

Proof By definition, we can write:

$$\begin{aligned}
\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}} &= \frac{2}{m_\sigma} \int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}} R^{\mathbf{u}}_{\mathfrak{D}}(s) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} \mathrm{d}s \\
&= \frac{2}{m_\sigma} \int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}} (R^{\mathbf{u},\eta}_{\mathfrak{D}}(s) + R^{u,Du}_{\mathfrak{D}} + R^{\mathbf{u},bd}_{\mathfrak{D}}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} \mathrm{d}s.
\end{aligned}$$

Now, it's important to notice that $R^{\mathbf{u},bd}_{\mathfrak{D}} = 0 \quad \forall \mathtt{D} \in \mathfrak{D}_{int} \cup (\mathfrak{D}_{ext} \cap \Gamma_N)$. So we are left only with the terms depending on $R^{\mathbf{u},\eta}_{\mathfrak{D}}(s), R^{\mathbf{u},D\mathbf{u}}_{\mathfrak{D}}$.
By applying Jensen's inequality and convexity we get:

$$\begin{aligned}
\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2 &= \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} \left| \frac{2}{m_\sigma} \int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}} (R^{\mathbf{u},\eta}_{\mathfrak{D}}(s) + R^{\mathbf{u},D\mathbf{u}}_{\mathfrak{D}}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} \mathrm{d}s \right|^2 \\
&\le \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}} |R^{\mathbf{u},\eta}_{\mathfrak{D}}(s) + R^{\mathbf{u},D\mathbf{u}}_{\mathfrak{D}}|^2 \mathrm{d}s \\
&\le 2 \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}} |R^{\mathbf{u},\eta}_{\mathfrak{D}}(s)|^2 \mathrm{d}s + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} \frac{m_\sigma}{2} |R^{\mathbf{u},D\mathbf{u}}_{\mathfrak{D}}|^2 \Big).
\end{aligned}$$

We estimate separately the terms.

► By Jensen's inequality:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}}|R_{\mathfrak{D}}^{\mathbf{u},\eta}(s)|^2\mathrm{d}s=\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}}\left|\frac{1}{m_{\mathtt{D}}}\int_{\mathtt{D}}(\nabla\mathbf{u}(s)-\nabla\mathbf{u}(z))\mathrm{d}z\right|^2\mathrm{d}s$$
$$\leq\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}}\frac{1}{m_{\mathtt{D}}}\int_{\mathtt{D}}|\nabla\mathbf{u}(s)-\nabla\mathbf{u}(z)|^2\mathrm{d}z\,\mathrm{d}s.$$

By [Kre10, Lemma I.12] and the fact that $\mathrm{diam}(\widehat{\mathtt{D}\cup[x_{\mathtt{K}^*},x_{\mathtt{L}}]})\leq C\mathrm{d}_{\mathtt{D}}$

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}}|R_{\mathfrak{D}}^{\mathbf{u},\eta}(s)|^2\mathrm{d}s\leq\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{Cd_{\mathtt{D}}^3}{m_{\mathtt{D}}}\int_{\widehat{\mathtt{D}}}|\nabla^2\mathbf{u}(s)|^2\mathrm{d}s,$$

and since we are on the boundary, $\mathtt{D}$ is a triangle (see Fig. I.3): so $\widehat{\mathtt{D}}=\mathtt{D}$. This implies:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}}|R_{\mathfrak{D}}^{\mathbf{u},\eta}(s)|^2\mathrm{d}s\leq C\mathrm{size}(\mathfrak{T})\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{\mathtt{D}}|\nabla^2\mathbf{u}(s)|^2\mathrm{d}s.$$

By the regularity of $\mathbf{u}$ and the fact that

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}m_{\mathtt{D}}\leq\frac{1}{2}\mathrm{size}(\mathfrak{T})m(\Gamma_N),\tag{II.10}$$

we conclude:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}}|R_{\mathfrak{D}}^{\mathbf{u},\eta}(s)|^2\mathrm{d}s\leq C\mathrm{size}(\mathfrak{T})^2m(\Gamma_N)\|\mathbf{u}\|^2_{W^{2,\infty}(\mathfrak{D})}.$$

► By Jensen's inequality:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|R_{\mathfrak{D}}^{\mathbf{u},D\mathbf{u}}|^2=\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}\left|\frac{1}{m_{\mathtt{D}}}\int_{\mathtt{D}}(\nabla\mathbf{u}(z)-\nabla^{\mathtt{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})\mathrm{d}z\right|^2$$
$$\leq\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2\,m_{\mathtt{D}}}\int_{\mathtt{D}}|\nabla\mathbf{u}(z)-\nabla^{\mathtt{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}|^2\mathrm{d}z.$$

By [Kre10, Lemma I.13]:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|R_{\mathfrak{D}}^{\mathbf{u},D\mathbf{u}}|^2\leq\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{Cd_{\mathtt{D}}^2m_\sigma}{m_{\mathtt{D}}}\int_{\widehat{\mathtt{D}}}|\nabla^2\mathbf{u}(z)|^2\mathrm{d}z$$

and since we are on the boundary, $\mathtt{D}$ is a triangle: so $\widehat{\mathtt{D}}=\mathtt{D}$. This implies:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|R_{\mathfrak{D}}^{\mathbf{u},D\mathbf{u}}|^2\leq C\mathrm{size}(\mathfrak{T})\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{\mathtt{D}}|\nabla^2\mathbf{u}(z)|^2\mathrm{d}z.\tag{II.11}$$

By the regularity of $\mathbf{u}$ and (II.10) : we conclude:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|R_{\mathfrak{D}}^{\mathbf{u},D\mathbf{u}}|^2\leq C\mathrm{size}(\mathfrak{T})^2m(\Gamma_N)\|\mathbf{u}\|^2_{W^{2,\infty}(\mathfrak{D})}.$$

We can conclude by putting all the terms together.

We proceed in the same way to estimate the term $\displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2$. ■

Remark II.3.5 *Remark that we obtained an estimate of order size*$(\mathfrak{T})^2$ *thanks to the regularity* $\mathbf{u}\in(W^{2,\infty}(\mathfrak{D}))^2$*; otherwise, from* (II.11)*,we would have obtained an estimate of order size*$(\mathfrak{T})$ *if* $\mathbf{u}\in(H^2(\mathfrak{D}))^2$.

The following result is obtained by replacing in the previous proof $[x_{\mathtt{K}^*},x_{\mathtt{L}}]$ by σ.

Lemma II.3.6 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C>0$, that depends only on reg$(\mathfrak{T})$, such that for all $\mathbf{u}\in(W^{2,\infty}(\mathfrak{D}))^2$:*

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\boldsymbol{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2\le C size(\mathfrak{T})^2\|\mathbf{u}\|^2_{W^{2,\infty}(\mathfrak{D})}.$$

Lemma II.3.7 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C>0$, that depends only on reg$(\mathfrak{T})$, such that for all $p\in W^{1,\infty}(\mathfrak{D})$:*

$$\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\boldsymbol{R}^{p}_{\mathtt{K}^*\mathtt{L}}|^2+\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\boldsymbol{R}^{p}_{\mathtt{L}^*\mathtt{L}}|^2\right)\le C size(\mathfrak{T})^2\|p\|^2_{W^{1,\infty}(\mathfrak{D})}.$$

Proof By definition, we can write:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2=\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}\left|\frac{2}{m_\sigma}\int_{x_{\mathtt{K}^*}}^{x_{\mathtt{L}}}R^{\mathrm{p}}_{\mathfrak{D}}(s)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\mathrm{d}s\right|^2.$$

By Lemma I.10.1:

$$\begin{aligned}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2\le&\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}C\,h_{\mathtt{D}}\int_{\mathtt{D}}|||\nabla(R^{\mathrm{p}}_{\mathfrak{D}}(z)\vec{\mathbf{n}}_{\sigma\mathtt{K}}|||^2_{\mathcal{F}}\mathrm{d}z\\&+\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{C}{h_{\mathtt{D}}}\int_{\mathtt{D}}|R^{\mathrm{p}}_{\mathfrak{D}}(z)\vec{\mathbf{n}}_{\sigma\mathtt{K}}|^2\mathrm{d}z.\end{aligned}$$

We apply now the definition of $R^{\mathrm{p}}_{\mathfrak{D}}(z)$ to get:

$$\begin{aligned}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2\le&\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}C\,h_{\mathtt{D}}\int_{\mathtt{D}}|\nabla\mathrm{p}(z)|^2\mathrm{d}z\\&+\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{C}{h_{\mathtt{D}}}\int_{\mathtt{D}}|\mathbb{P}^D_m\mathrm{p}-\mathrm{p}(z)|^2\mathrm{d}z.\end{aligned}$$

We estimate separately the terms.

► By regularity of p and (II.10) we have:

$$\begin{aligned}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}C\,h_{\mathtt{D}}\int_{\mathtt{D}}|\nabla\mathrm{p}(z)|^2\mathrm{d}z&\le C\mathrm{size}(\mathfrak{T})\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{\mathtt{D}}|\nabla\mathrm{p}(z)|^2\mathrm{d}z\\&\le C\mathrm{size}(\mathfrak{T})^2m(\Gamma_N)\|\mathrm{p}\|^2_{W^{1,\infty}(\mathfrak{D})}.\end{aligned}$$

► By Lemma I.10.9 and the regularity of p:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{C}{h_{\mathtt{D}}}\int_{\mathtt{D}}|\mathbb{P}_m^D\mathrm{p}-\mathrm{p}(z)|^2\mathrm{d}z\leq\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{C}{h_{\mathtt{D}}}\mathrm{size}(\mathfrak{T})^2\int_{\mathtt{D}}|\nabla\mathrm{p}(z)|^2\mathrm{d}z.$$

Again by regularity of p, we have:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{C}{h_{\mathtt{D}}}\int_{\mathtt{D}}|\mathbb{P}_m^D\mathrm{p}-\mathrm{p}(z)|^2\mathrm{d}z\leq C\mathrm{size}(\mathfrak{T})^2\|\mathrm{p}\|^2_{W^{1,\infty}(\mathfrak{D})}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_{\mathtt{D}}}{h_{\mathtt{D}}}.$$

Finally, since $\frac{m_{\mathtt{D}}}{h_{\mathtt{D}}}\leq\frac{1}{2}m_\sigma$, we conclude:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{C}{h_{\mathtt{D}}}\int_{\mathtt{D}}|\mathbb{P}_m^D\mathrm{p}-\mathrm{p}(z)|^2\mathrm{d}z\leq C\,\mathrm{m}(\Gamma_N)\mathrm{size}(\mathfrak{T})^2\|\mathrm{p}\|^2_{W^{1,\infty}(\mathfrak{D})}.$$

We can conclude by putting all the terms together.

We proceed in the same way to estimate the term $\displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}}|^2$. ■

The following result is obtained by replacing in the previous proof $[x_{\mathtt{K}^*},x_{\mathtt{L}}]$ by σ.

Lemma II.3.8 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C>0$, that depends only on reg($\mathfrak{T}$), such that for all $p\in W^{1,\infty}(\mathfrak{D})$:*

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}m_\sigma|\boldsymbol{R}^p_{\sigma\mathtt{K}}|^2\leq C\,size(\mathfrak{T})^2\|p\|^2_{W^{1,\infty}(\mathfrak{D})}.$$

II.3.1.4 Step 4 : Conclusion

We recall that $\mathbf{e}^{\mathfrak{T}}=(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_c\mathbf{u})-\mathbf{u}^{\mathfrak{T}}\in((\mathbb{R}^2)^{\mathfrak{T}}$ and $e^{\mathfrak{D}}=\mathbb{P}^{\mathfrak{D}}_m\mathrm{p}-\mathrm{p}^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}}$ and that we obtained the following inequality:

$$\begin{aligned}
&\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2+\mu\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2^2+\beta|\mathrm{e}^{\mathfrak{D}}|_h^2\\
&\qquad\leq C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\left(\|\mathbf{R}^{\mathrm{p}}_\sigma\|_2+\|\mathbf{R}^{\mathbf{u}}_\sigma\|_2+\|\mathbf{R}^{\mathrm{p}}_{\sigma^*}\|_2+\|\mathbf{R}^{\mathbf{u}}_{\sigma^*}\|_2\right)\\
&\qquad+C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\Bigg[\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2\Big)^{\frac{1}{2}}+\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2\Big)^{\frac{1}{2}}\Bigg]\\
&\qquad+C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\Bigg[\Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2\Bigg)^{\frac{1}{2}}+\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}}|^2\Big)^{\frac{1}{2}}\Bigg]\\
&\qquad+C\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2(\|\mathbf{u}\|_{H^2}+\|\mathrm{p}\|_{H^1})\\
&\qquad+C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2\Bigg[\Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2\Bigg)^{\frac{1}{2}}+\Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2\Bigg)^{\frac{1}{2}}\Bigg].
\end{aligned}$$

Lemmas II.3.2, II.3.3, II.3.4, II.3.6, II.3.7, II.3.8 and the fact that $\beta|\mathrm{e}^{\mathfrak{D}}|_h^2\geq0$ imply:

$$\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2+\mu\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2^2\leq C\mathrm{size}(\mathfrak{T})\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2+C\mathrm{size}(\mathfrak{T})\|\mathrm{e}^{\mathfrak{D}}\|_2.$$

We apply Young's inequality to obtain:

$$\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 \leq C\text{size}(\mathfrak{T}) \quad \text{and} \quad \|\mathrm{e}^{\mathfrak{D}}\|_2^2 \leq C. \tag{II.12}$$

Estimate of $\|\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2$:
By using discrete Poincaré inequality, we find:

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2 &\leq \|\mathbf{u}-\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 + \|\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2\\ &\leq \|\mathbf{u}-\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 + C\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2.\end{aligned}$$

By [Kre10, Lemma I.18] , we have:

$$\|\mathbf{u}-\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2^2 \leq \|\mathbf{u}-\mathbb{P}_c^{\tau}\mathbf{u}\|_2^2 + \|u-\mathbb{P}_m^{\tau}\mathbf{u}\|_2^2.$$

and if we apply Lemma I.10.7, it gives:

$$\|\mathbf{u}-\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2^2 \leq C\text{size}(\mathfrak{T}).$$

So we have:

$$\|\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2 \leq C(\text{size}(\mathfrak{T}) + \|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2). \tag{II.13}$$

Thus, if we apply estimate (II.12) we finally obtain:

$$\|\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\text{size}(\mathfrak{T})^{\frac{1}{2}}.$$

Estimate of $\|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2$:
We can decompose it in:

$$\begin{aligned}\|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 &\leq \|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 + \|\nabla\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}-\nabla^{\mathfrak{D}}\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 + \|\nabla\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2\\ &\leq \|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 + \|\nabla\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}-\nabla^{\mathfrak{D}}\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 + \|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2.\end{aligned}$$

By Lemma I.10.2 and Lemma I.10.6, we have:

$$\|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 + \|\nabla\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}-\nabla^{\mathfrak{D}}\mathfrak{P}_{m,g}^{\mathrm{D}}\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2 \leq Csize(\mathfrak{T}),$$

so that :

$$\|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\text{size}(\mathfrak{T}) + \|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2. \tag{II.14}$$

By estimate II.12, we conclude:

$$\|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\text{size}(\mathfrak{T})^{\frac{1}{2}}.$$

■

II.3.2 Stability of the DDFV scheme

In order to improve the error estimate, we prove the stability result for the scheme $(\mathcal{P}_{\beta\mu})$. This result has been proven in the case of homogeneous Dirichlet boundary conditions in [Kre10], here the difficulty relies in taking into account the boundary Γ_N.

Theorem II.3.9 *Suppose that $\beta > 0$. There exist two constants $C_1, C_2 > 0$, depending only on Ω, β and reg(τ), such that for every couple $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_0^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$ with*

$$(-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}} Id)\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \Phi_\sigma \quad \forall\forall_{\mathtt{D}_{\sigma,\sigma^*}} \in \mathfrak{D}_{ext} \cap \Gamma_N,$$

there exist $\widetilde{\mathbf{u}}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and $\widetilde{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}}\|_2^2 + \|\widetilde{p}^{\mathfrak{D}}\|_2^2 \le C_1(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \|\mathrm{p}^{\mathfrak{D}}\|_2^2), \tag{II.15}$$

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \|\mathrm{p}^{\mathfrak{D}}\|_2^2 \le C_2\left(B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathfrak{T}}, \widetilde{p}^{\mathfrak{D}}) + \left|\sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \Phi_\sigma \gamma^\sigma(\widetilde{\mathbf{u}}^\tau)\right| + \|\Phi_\sigma\|_2^2\right). \tag{II.16}$$

Proof Let $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_0^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$ with

$$(-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \Phi_\sigma \quad \forall_{\mathtt{D}_{\sigma,\sigma^*}} \in \mathfrak{D}_{ext} \cap \Gamma_N.$$

The proof consists into building explicitely $(\widetilde{\mathbf{u}}^\tau, \widetilde{\mathrm{p}}^{\mathfrak{D}}) \in \mathbb{E}_0^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$ such that relations (II.15)-(II.16) are true.

▶ **Step 1:** As in the proof of Prop. II.2.1, we apply discrete Green's formula (Thm. I.5.1) to the bilinear form B, defined in (II.2). We obtain:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}; \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \mu\mathrm{size}(\mathfrak{T})\|\mathrm{p}^{\mathfrak{D}}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 - \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \Phi_\sigma \cdot \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}). \tag{II.17}$$

▶ **Step 2:** We apply Lemma I.6.3 to the function $\mathrm{p}^{\mathfrak{D}} = \sum_{\mathtt{D}\in\mathfrak{D}} \mathrm{p}^{\mathtt{D}}\mathbb{1}_{\mathtt{D}} \in L^2(\Omega)$. So there exists $C > 0$ depending only on Ω, $\mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^2$ such that (mind the sign):

$$\begin{aligned} &\mathrm{div}(\mathbf{v}) = -\mathrm{p}^{\mathfrak{D}} \\ &\|\mathbf{v}\|_{H^1} \le C\|\mathrm{p}^{\mathfrak{D}}\|_2. \end{aligned} \tag{II.18}$$

We set $\mathbf{v}^{\mathfrak{T}} = \mathbb{P}_m^\tau \mathbf{v}$. In particular, $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$.
We apply Lemma I.10.5 to obtain:

$$\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \le C\|\mathbf{v}\|_{H^1} \le C\|\mathrm{p}^{\mathfrak{D}}\|_2. \tag{II.19}$$

▶ **Step 3:** Green's formula (Thm. I.5.1) implies:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}}, 0) = (\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}})_{\mathfrak{D}} - (\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}})_{\mathfrak{D}} - \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \Phi_\sigma \cdot \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}).$$

We pass the boundary term on the left hand side. Then, by applying Cauchy-Scwharz inequality and adding and subtracting the term $\sum_{\mathtt{D}\in\mathfrak{D}} \int_{\mathtt{D}} \mathrm{p}^{\mathtt{D}}(\mathrm{div}(\mathbf{v}(z))\mathrm{d}z$, we end up with:

$$\begin{aligned} \mathrm{B}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}}, 0) + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \Phi_\sigma \cdot \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) &\ge -\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \\ &- \sum_{\mathtt{D}\in\mathfrak{D}} \int_{\mathtt{D}} \mathrm{p}^{\mathtt{D}}(\mathrm{div}(\mathbf{v}(z))\mathrm{d}z - \sum_{\mathtt{D}\in\mathfrak{D}} \int_{\mathtt{D}} \mathrm{p}^{\mathtt{D}}(\mathrm{div}^{\mathtt{D}}(\mathbf{v}^{\mathfrak{T}}) - \mathrm{div}(\mathbf{v}(z)))\mathrm{d}z. \end{aligned}$$

Thanks to (II.18), we get:

$$\mathrm{B}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}}, 0) + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma \cdot \Phi_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \geq -C \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2 \|\mathrm{p}^{\mathfrak{D}}\|_2$$
$$+ \|\mathrm{p}^{\mathfrak{D}}\|_2^2 - \sum_{\mathtt{D} \in \mathfrak{D}} \int_{\mathtt{D}} \mathrm{p}^{\mathtt{D}}(\mathrm{div}^{\mathtt{D}}(\mathbf{v}^{\mathfrak{T}}) - \mathrm{div}(\mathbf{v}(z)))\mathrm{d}z.$$

We need the following estimate:

$$\sum_{\mathtt{D} \in \mathfrak{D}} \int_{\mathtt{D}} \mathrm{p}^{\mathtt{D}}(\mathrm{div}^{\mathtt{D}}(\mathbf{v}^{\mathfrak{T}}) - \mathrm{div}(\mathbf{v}(z)))\mathrm{d}z \leq C(|\mathrm{p}^{\mathfrak{D}}|_h + \|\Phi_\sigma\|_2 + \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2)\|\mathbf{v}\|_{H^1}.$$

Let us assume temporary that this statement holds (it will be proven in Lemma II.3.10 below), then we can write:

$$\mathrm{B}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}}, 0) + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma \Phi_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \geq -C \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2 \|\mathrm{p}^{\mathfrak{D}}\|_2$$
$$+ \|\mathrm{p}^{\mathfrak{D}}\|_2^2 - C(|\mathrm{p}^{\mathfrak{D}}|_h + \|\Phi_\sigma\|_2 + \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2)\|\mathbf{v}\|_{H^1}.$$

Using (II.18), we get:

$$\mathrm{B}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}}, 0) + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma \Phi_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \geq -C \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2 \|\mathrm{p}^{\mathfrak{D}}\|_2$$
$$+ \|\mathrm{p}^{\mathfrak{D}}\|_2^2 - C(|\mathrm{p}^{\mathfrak{D}}|_h + \|\Phi_\sigma\|_2 + \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2)\|\mathrm{p}^{\mathfrak{D}}\|_2.$$

Thanks to Young's inequality (Lemma I.11.1), we get the existence of four constants $C_1, C_2, C_3, C_4 > 0$ that depend only on Ω and reg($\mathfrak{T}$), such that:

$$\mathrm{B}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}; \mathbf{v}^{\mathfrak{T}}, 0) + \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma \Phi_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \geq$$
$$C_1 \|\mathrm{p}^{\mathfrak{D}}\|_2^2 - C_2 \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2^2 - C_3 |\mathrm{p}^{\mathfrak{D}}|_h^2 - C_4 \|\Phi_\sigma\|_2^2. \quad \text{(II.20)}$$

Step 3: The bilinearity of B, inequality (II.20) and (II.17) give, for all $\xi > 0$:

$$\mathrm{B}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}; \mathbf{u}^{\mathfrak{T}} + \xi \mathbf{v}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) + \left| \sum_{\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N} m_\sigma \Phi_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}} + \xi \mathbf{v}^{\mathfrak{T}}) \right| + \xi C_4 \|\Phi_\sigma\|_2 \geq$$
$$(1 - \xi C_2) \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_2^2 + \xi C_1 \|\mathrm{p}^{\mathfrak{D}}\|_2^2 + (\beta - \xi C_3) |\mathrm{p}^{\mathfrak{D}}|_h^2.$$

We choose the value of $\xi > 0$ sufficiently small (that depends only on C_2, β and C_3) such that all the constants in front of the norms are strictly positive. In this way we find (II.16).
To recover inequality (II.15), it's sufficient to consider $\widetilde{\mathbf{u}}^{\mathfrak{T}} = \mathbf{u}^{\mathfrak{T}} + \xi \mathbf{v}^{\mathfrak{T}}$ and $\widetilde{\mathrm{p}}^{\mathfrak{D}} = \mathrm{p}^{\mathfrak{D}}$; since $\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2 \leq C \|\mathrm{p}^{\mathfrak{D}}\|_2$, we can conclude. ■

The following lemma is an extension of the result proved in [[Kre10], Lemma IV.25]. The difference is in the space in which the function $\mathbf{v}$ belongs: we consider the case in which $\mathbf{v} \in H^1_{\Gamma_D}(\Omega)$ instead of $\mathbf{v} \in H^1_0(\Omega)$.

Lemma II.3.10 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, that depends only on reg($\mathfrak{T}$), such that for all $\boldsymbol{v} \in (H^1_{\Gamma_D}(\Omega))^2$ and for every couple $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_0^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$*

with

$$(-\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathrm{D}} Id)\vec{\mathbf{n}}_{\sigma\mathrm{K}} = \Phi_\sigma \quad \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N, \tag{II.21}$$

we have:

$$\sum_{\mathrm{D}\in\mathfrak{D}} \int_{\mathrm{D}} \mathrm{p}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}(\mathbf{v}^{\mathfrak{T}}) - div(\boldsymbol{v}(z)))dz \le C(|\mathrm{p}^{\mathfrak{D}}|_h + \|\Phi_\sigma\|_2 + \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2)\|\boldsymbol{v}\|_{H^1},$$

with $\mathbf{v}^{\mathfrak{T}} = \mathbb{P}_m^{\mathcal{T}}\boldsymbol{v}$ *is the average projection of* $\mathbf{v}$ *on the mesh* $\mathfrak{T}$*, defined in Sec. I.2.*

Proof The set $\mathfrak{D}$ can be split into interior and exterior diamonds: $\mathfrak{D} = \mathfrak{D}_{int} \cup \mathfrak{D}_{ext}$. From [[Kre10], Lemma IV.25] we have:

$$\sum_{\mathrm{D}\in\mathfrak{D}_{int}} \int_{\mathrm{D}} \mathrm{p}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}(\mathbf{v}^{\mathfrak{T}}) - \mathrm{div}(\mathbf{v}(z)))\mathrm{d}z \le C|\mathrm{p}^{\mathfrak{D}}|_h\|\mathbf{v}\|_{H^1}.$$

So, we focus on diamonds $\mathrm{D} \in \mathfrak{D}_{ext}$. In particular, the only terms that are non zero are on $\mathrm{D} \in \mathfrak{D}_{ext} \cap \Gamma_N$, since $\mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^2$. Thanks to the divergence formula (I.3.5) we can write:

$$\begin{aligned}\int_{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}(\mathbf{v}^{\mathfrak{T}}) - \mathrm{div}(\mathbf{v}(z)))\mathrm{d}z = &\sum_{\mathfrak{s}=[x_{\mathrm{K}},x_{\mathrm{K}^*}]\in\mathcal{E}_{\mathrm{D}}} m_{\mathfrak{s}}\frac{1}{m_{\mathfrak{s}}}\int_{\mathfrak{s}}\left(\frac{\mathbf{v}_{\mathrm{K}}+\mathbf{v}_{\mathrm{K}^*}}{2} - \mathbf{v}(z)\right)\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}}\mathrm{d}z\\ &+\int_{x_{\mathrm{K}^*}}^{x_{\mathrm{L}}}\frac{m_\sigma}{2}\frac{2}{m_\sigma}\left(\frac{\mathbf{v}_{\mathrm{L}}+\mathbf{v}_{\mathrm{K}^*}}{2} - \mathbf{v}(z)\right)\cdot\vec{\mathbf{n}}_{\sigma\mathrm{K}}\mathrm{d}z\\ &+\int_{x_{\mathrm{L}}}^{x_{\mathrm{L}^*}}\frac{m_\sigma}{2}\frac{2}{m_\sigma}\left(\frac{\mathbf{v}_{\mathrm{L}^*}+\mathbf{v}_{\mathrm{L}}}{2} - \mathbf{v}(z)\right)\cdot\vec{\mathbf{n}}_{\sigma\mathrm{K}}\mathrm{d}z.\end{aligned} \tag{II.22}$$

We define:

$$\begin{aligned}R^{\mathfrak{s}}_{\mathrm{div}}(\mathbf{v}) &= \frac{1}{m_{\mathfrak{s}}}\int_{\mathfrak{s}}\left(\frac{\mathbf{v}_{\mathrm{K}}+\mathbf{v}_{\mathrm{K}^*}}{2} - \mathbf{v}(z)\right)\mathrm{d}z\\ R^{\mathrm{K}^*}_{\mathrm{div}}(\mathbf{v}) &= \frac{2}{m_\sigma}\int_{x_{\mathrm{K}^*}}^{x_{\mathrm{L}}}\left(\frac{\mathbf{v}_{\mathrm{L}}+\mathbf{v}_{\mathrm{K}^*}}{2} - \mathbf{v}(z)\right)\mathrm{d}z\\ R^{\mathrm{L}^*}_{\mathrm{div}}(\mathbf{v}) &= \frac{2}{m_\sigma}\int_{x_{\mathrm{L}}}^{x_{\mathrm{L}^*}}\left(\frac{\mathbf{v}_{\mathrm{L}^*}+\mathbf{v}_{\mathrm{L}}}{2} - \mathbf{v}(z)\right)\mathrm{d}z.\end{aligned}$$

So, if in (II.22) we multiply by p^{D} and we sum over $\mathfrak{D}_{ext} \cap \Gamma_N$:

$$\begin{aligned}\sum_{\mathrm{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}\int_{\mathrm{D}}\mathrm{p}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}(\mathbf{v}^{\mathfrak{T}}) - \mathrm{div}(\mathbf{v}(z)))\mathrm{d}z = &\sum_{\mathrm{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}\mathrm{p}^{\mathrm{D}}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathrm{D}}} m_{\mathfrak{s}}R^{\mathfrak{s}}_{\mathrm{div}}(\mathbf{v})\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}}\\ &+\sum_{D\in\mathfrak{D}_{ext}\cap\Gamma_N}\mathrm{p}^{\mathrm{D}}\frac{m_\sigma}{2}R^{\mathrm{K}^*}_{\mathrm{div}}(\mathbf{v})\cdot\vec{\mathbf{n}}_{\sigma\mathrm{K}}\\ &+\sum_{D\in\mathfrak{D}_{ext}\cap\Gamma_N}\mathrm{p}^{\mathrm{D}}\frac{m_\sigma}{2}R^{\mathrm{L}^*}_{\mathrm{div}}(\mathbf{v})\cdot\vec{\mathbf{n}}_{\sigma\mathrm{K}}.\end{aligned}$$

The first term $\displaystyle\sum_{\mathrm{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}\mathrm{p}^{\mathrm{D}}\sum_{\mathfrak{s}\in\mathcal{E}_{\mathrm{D}}} m_{\mathfrak{s}}R^{\mathfrak{s}}_{\mathrm{div}}(\mathbf{v})\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}}$ contains only interior edges $\mathfrak{s}$ of the diamonds, so we refer to the proof [[Kre10], Lemma IV.25].

We consider then the other two terms. In particular, we prove the result just for $R^{\mathrm{K}^*}_{\mathrm{div}}(\mathbf{v})$, since it is equivalent for $R^{\mathrm{L}^*}_{\mathrm{div}}(\mathbf{v})$.
We recall that $m_{\mathrm{D}} = \frac{1}{2}m_\sigma m_{\sigma^*}\sin(\alpha)$. We apply Neumann boundary conditions (II.21) and Cauchy-

Schwarz inequality to obtain:

$$\begin{aligned}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N} \mathrm{p}^{\mathtt{D}}\frac{m_\sigma}{2}R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}} &\leq C\Big(\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_{\mathtt{D}}|\Phi_\sigma|^2\Big)^{\frac{1}{2}}\Big(\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}|R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})|^2\Big)^{\frac{1}{2}}\\
&+C\Big(\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_{\mathtt{D}}|\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}|^2\Big)^{\frac{1}{2}}\Big(\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}|R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})|^2\Big)^{\frac{1}{2}}\\
&\leq C(\|\Phi_\sigma\|_2)\Big(\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}|R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})|^2\Big)^{\frac{1}{2}}\\
&+C(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2)\Big(\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}|R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

To conclude, we need to prove that $\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}|R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})|^2\leq C\|\mathbf{v}\|_{H^1}$. Notice that, if we call the segment $[x_{\mathtt{K}^*},x_{\mathtt{L}}]=\sigma_1$ and if we define

$$\mathbf{v}_{\sigma_1}:=\frac{1}{m_{\sigma_1}}\int_{\sigma_1}\mathbf{v}(y)\mathrm{d}y,$$

we can write by the definition of $R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})$:

$$|R_{\mathrm{div}}^{\mathtt{K}^*}(\mathbf{v})|^2\leq 2\left(|\mathbf{v}_{\mathtt{K}^*}-\mathbf{v}_{\sigma_1}|^2+2|\mathbf{v}_{\mathtt{L}}-\mathbf{v}_{\sigma_1}|^2\right).$$

If we denote $\mathbf{v}=\begin{pmatrix}\mathtt{v}^1\\ \mathtt{v}^2\end{pmatrix}$, we can work componentwise. For $i=1,2$, each term on the right hand side can be controlled by:

$$|\mathtt{v}^i_{\mathtt{K}^*}-\mathtt{v}^i_{\sigma_1}|^2\leq 2\left|\mathtt{v}^i_{\mathtt{K}^*}-\frac{1}{m_{\mathtt{K}^*}}\int_{\mathtt{K}^*}\mathtt{v}^i(x)\mathrm{d}x\right|^2+\left|\frac{1}{m_{\mathtt{K}^*}}\int_{\mathtt{K}^*}\mathtt{v}^i(x)\mathrm{d}x-\mathtt{v}^i_{\sigma_1}\right|^2.$$

By defintion, both $\mathtt{v}^i_{\mathtt{K}^*}$ and $\mathtt{v}^i_{\sigma_1}$ are averages of $\mathtt{v}^i$ on a segment and $\frac{1}{m_{\mathtt{K}^*}}\int_{\mathtt{K}^*}\mathtt{v}^i(x)\mathrm{d}x$ is an average over a bounded polygonal domain. We can thus apply [[ABH07], Lemma 3.4] and deduce:

$$|\mathtt{v}^i_{\mathtt{K}^*}-\mathtt{v}^i_{\sigma_1}|^2\leq C\int_{\widehat{\mathtt{K}^*}}|\nabla\mathtt{v}^i(z)|^2\mathrm{d}z.$$

We obtain for $i=1,2$:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_N}|R_{\mathrm{div}}^{\mathtt{K}^*}(\mathtt{v}^i)|^2\leq C\int_\Omega|\nabla\mathtt{v}^i(z)|^2\mathrm{d}z.$$

that by summing over i, gives us the desired result. ■

II.3.3 Optimal error estimate

We improve the result of Thm. II.3.1 by applying the stability result of Thm. II.3.9; remark that to apply this result it is necessary to suppose $\beta>0$.

Theorem II.3.11 *We suppose that the solution of* (II.1) *satisfies* $(\mathbf{u},p)\in(W^{2,\infty}(\mathfrak{D}))^2\times W^{1,\infty}(\mathfrak{D})$. *Let* $\beta>0$ *and* $(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})$ *be the solution of* $(\mathcal{P}_{\beta\mu})$. *Then there exists a constant* $C>0$ *that depends* *on* $reg(\mathfrak{T}), \mu, \beta, \|\mathbf{u}\|_{W^{2,\infty}}$ *and* $\|p\|_{W^{1,\infty}}$ *such that*

$$\|\mathbf{u} - \mathbf{u}^{\mathfrak{T}}\|_2 + \|\nabla\mathbf{u} - \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C size(\mathfrak{T}),$$
$$\|p - \mathrm{p}^{\mathfrak{D}}\|_2 \leq C size(\mathfrak{T}).$$

Proof We proceed exactly as in Thm. II.3.1 by defining the errors $\mathbf{e}^{\mathfrak{T}} = (\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u}) - \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$, the error for the velocity field, and $e^{\mathfrak{D}} = \mathbb{P}^{\mathfrak{D}}_{m}\mathrm{p} - \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, the error for the pressure field. We recall that they satisfy the following system:

Find $\mathbf{e}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and $\mathrm{e}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\begin{cases} \mathbf{div}^{\mathtt{K}}(-\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}} + \mathrm{e}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{R}^{\mathtt{K}} & \forall \mathtt{K} \in \mathfrak{M} \\ \mathbf{div}^{\mathtt{K}^*}(-\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}} + \mathrm{e}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{R}^{\mathtt{K}^*} & \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*_N \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{e}^{\mathfrak{T}}) + \mu\mathrm{size}(\mathfrak{T})\mathrm{e}^{\mathfrak{D}} - \beta d_{\mathtt{D}}^2\Delta^{\mathfrak{D}}\mathrm{e}^{\mathfrak{D}} = \mathbf{R}^{\mathfrak{D}} & \\ (-\nabla^{\mathtt{D}}\mathbf{e}^{\mathfrak{T}} + \mathrm{e}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}} & \forall \mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N \end{cases}$$

where $\mathbf{R}^{\mathfrak{T}} = \Big((\mathbf{R}^{\mathtt{K}})_{\mathtt{K}\in\mathfrak{M}}, (\mathbf{R}^{\mathtt{K}^*})_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*_N}\Big)$ and $\mathbf{R}^{\mathfrak{D}} = \Big(\mathbf{R}^{\mathtt{D}}\Big)_{\mathtt{D}\in\mathfrak{D}}$ with:

$$\mathbf{R}^{\mathtt{K}} = \frac{1}{m_{\mathtt{K}}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \frac{1}{m_{\mathtt{K}}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}, \qquad \forall \mathtt{K} \in \mathfrak{M},$$
$$\mathbf{R}^{\mathtt{K}^*} = \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*} + \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*}, \qquad \forall \mathtt{K}^* \in \mathfrak{M}^*,$$
$$\mathbf{R}^{\mathtt{K}^*} = \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} (\mathbf{R}^{\mathbf{u}}_{\sigma^*\mathtt{K}^*} + \mathbf{R}^{\mathrm{p}}_{\sigma^*\mathtt{K}^*}) + \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} \frac{m_\sigma}{2}(\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}} + \mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}), \quad \forall \mathtt{K}^* \in \partial\mathfrak{M}^*_N,$$
$$\mathbf{R}^{\mathtt{D}} = \mathrm{div}^{\mathtt{D}}(\mathfrak{P}^{\mathrm{D}}_{m,g}\mathbb{P}^{\mathfrak{T}}_{c}\mathbf{u}) + \mu\mathrm{size}(\mathfrak{T})\mathbb{P}^{\mathtt{D}}_{m}\mathrm{p} - \beta d_{\mathtt{D}}^2\Delta^{\mathtt{D}}\mathbb{P}^{\mathfrak{D}}_{m}\mathrm{p}, \qquad \forall \mathtt{D} \in \mathfrak{D}.$$

Thm. II.3.9 implies that there exist $\widetilde{\mathbf{e}}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ and $\widetilde{\mathrm{e}}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{e}}^{\mathfrak{T}}\|_2^2 + \|\widetilde{\mathrm{e}}^{\mathfrak{D}}\|_2^2 \leq C(\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 + \|\mathrm{e}^{\mathfrak{D}}\|_2^2), \tag{II.23}$$

and

$$\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 + \|\mathrm{e}^{\mathfrak{D}}\|_2^2 \leq C\mathrm{B}(\mathbf{e}^{\mathfrak{T}}, \mathrm{e}^{\mathfrak{D}}; \widetilde{\mathbf{e}}^{\mathfrak{T}}, \widetilde{\mathrm{e}}^{\mathfrak{D}})$$
$$+ C\Bigg|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}})\cdot\gamma^\sigma(\widetilde{\mathbf{e}}^{\mathfrak{T}})\Bigg| + C\|(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}} + \mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}})\|^2_{2,\Gamma_N}.$$

Thus, by applying Cauchy-Schwarz inequality and trace theorem (Thm. I.9.4), we can write:

$$\begin{aligned}
\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2+\|\mathrm{e}^{\mathfrak{D}}\|_2^2 &\leq CB(\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}};\widetilde{\mathbf{e}}^{\mathfrak{T}},\widetilde{\mathrm{e}}^{\mathfrak{D}})\\
&+C\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}+\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}})^2\right)^{\frac12}\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma(\gamma^\sigma(\widetilde{\mathbf{e}}^{\mathfrak{T}}))^2\right)^{\frac12}\\
&+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2\\
&\leq CB(\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}};\widetilde{\mathbf{e}}^{\mathfrak{T}},\widetilde{\mathrm{e}}^{\mathfrak{D}})+C\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma(\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}+\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}})^2\right)^{\frac12}\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{e}}^{\mathfrak{T}}\|_2\\
&+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2.
\end{aligned}$$

Thanks to the definition of B we have $B(\mathbf{e}^{\mathfrak{T}},\mathrm{e}^{\mathfrak{D}};\widetilde{\mathbf{e}}^{\mathfrak{T}},\widetilde{\mathrm{e}}^{\mathfrak{D}})=[[\mathbf{R}^{\mathfrak{T}},\widetilde{\mathbf{e}}^{\mathfrak{T}}]]_{\mathcal{T}}+(\mathrm{R}^{\mathfrak{D}},\widetilde{\mathrm{e}}^{\mathfrak{D}})_{\mathfrak{D}}=:\mathrm{I}+\mathrm{T}$ and by proceeding as in the proof of Thm. II.3.1 we get:

$$\begin{aligned}
\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2+\|\mathrm{e}^{\mathfrak{D}}\|_2^2 &\leq C\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{e}}^{\mathfrak{T}}\|_2\left(\|\mathbf{R}^{\mathrm{p}}_{\sigma}\|_2+\|\mathbf{R}^{\mathbf{u}}_{\sigma}\|_2+\|\mathbf{R}^{\mathrm{p}}_{\sigma^*}\|_2+\|\mathbf{R}^{\mathbf{u}}_{\sigma^*}\|_2\right)\\
&+C\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{e}}^{\mathfrak{T}}\|_2\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2\right)^{\frac12}+\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2\right)^{\frac12}\right]\\
&+C\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{e}}^{\mathfrak{T}}\|_2\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2\right)^{\frac12}+\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}}|^2\right)^{\frac12}\right]\\
&+C\mathrm{size}(\mathfrak{T})\|\widetilde{\mathrm{e}}^{\mathfrak{D}}\|_2(\|\mathbf{u}\|_{H^2}+\|\mathrm{p}\|_{H^1})\\
&+C\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{e}}^{\mathfrak{T}}\|_2\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2\right)^{\frac12}+\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2\right)^{\frac12}\right]\\
&+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2.
\end{aligned}$$

Now we use relation (II.23) to get:

$$\begin{aligned}
&\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2+\|\mathrm{e}^{\mathfrak{D}}\|_2^2\\
&\quad\leq C(\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2+\|\mathrm{e}^{\mathfrak{D}}\|_2)\left(\|\mathbf{R}^{\mathrm{p}}_{\sigma}\|_2+\|\mathbf{R}^{\mathbf{u}}_{\sigma}\|_2+\|\mathbf{R}^{\mathrm{p}}_{\sigma^*}\|_2+\|\mathbf{R}^{\mathbf{u}}_{\sigma^*}\|_2\right)\\
&\quad+C(\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2+\|\mathrm{e}^{\mathfrak{D}}\|_2)\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{K}^*\mathtt{L}}|^2\right)^{\frac12}+\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{K}^*\mathtt{L}}|^2\right)^{\frac12}\right]\\
&\quad+C(\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2+\|\mathrm{e}^{\mathfrak{D}}\|_2)\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathbf{u}}_{\mathtt{L}^*\mathtt{L}}|^2\right)^{\frac12}+\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N}\frac{m_\sigma}{2}|\mathbf{R}^{\mathrm{p}}_{\mathtt{L}^*\mathtt{L}}|^2\right)^{\frac12}\right]\\
&\quad+C\mathrm{size}(\mathfrak{T})(\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2+\|\mathrm{e}^{\mathfrak{D}}\|_2)(\|\mathbf{u}\|_{H^2}+\|\mathrm{p}\|_{H^1})\\
&\quad+C(\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2+\|\mathrm{e}^{\mathfrak{D}}\|_2)\left[\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2\right)^{\frac12}+\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2\right)^{\frac12}\right]\\
&\quad+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathbf{u}}_{\sigma\mathtt{K}}|^2+C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma|\mathbf{R}^{\mathrm{p}}_{\sigma\mathtt{K}}|^2.
\end{aligned}$$

Lemmas II.3.2, II.3.3, II.3.4, II.3.6, II.3.7, II.3.8 imply:

$$\|\nabla^{\mathfrak{D}}\mathbf{e}^{\mathfrak{T}}\|_2^2 + \|\mathrm{e}^{\mathfrak{D}}\|_2^2 \leq C\mathrm{size}(\mathfrak{T})^2. \tag{II.24}$$

<u>Estimate of $\|\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2$ and $\|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2$:</u>

If we apply the estimate (II.24) (different from (II.12) in Thm. II.3.1) to (II.13) and (II.14), we obtain:

$$\|\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\mathrm{size}(\mathfrak{T}).$$

$$\|\nabla\mathbf{u}-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\mathrm{size}(\mathfrak{T}).$$

<u>Estimate of $\|\mathrm{p}-\mathrm{p}^{\mathfrak{D}}\|_2$:</u>
We have:

$$\|\mathrm{p}-\mathrm{p}^{\mathfrak{D}}\|_2 \leq \|\mathrm{p}-\mathbb{P}_m^{\mathfrak{D}}\mathrm{p}\|_2 + \|\mathbb{P}_m^{\mathfrak{D}}\mathrm{p}-\mathrm{p}^{\mathfrak{D}}\|_2.$$

We conclude thanks to Lemma I.10.9 and the estimate (II.24). ■

II.4 Numerical results

We validate the scheme $(\mathcal{P}_{\beta\mu})$ by showing some numerical experiments. The computational domain is $\Omega=[0,1]^2$. The configuration of the boundaries is illustrated in Fig. II.2.

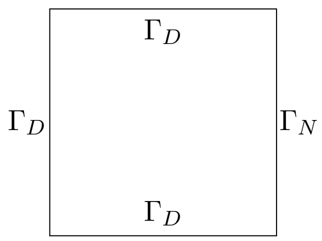


Fig. II.2 $\Omega=[0,1]^2$, Dirichlet boundary conditions on Γ_D , Neumann boundary conditions on Γ_N.

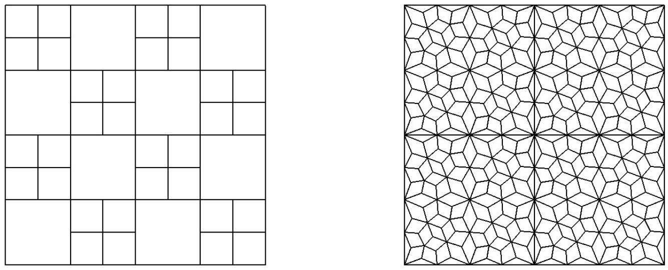
We study the error in the case of unstabilized and stabilized mass equation (i.e. with a linear stabilization, $\mu>0$, or Brezzi-Pitkaranta type stabilization, $\beta>0$). In the following discussion, we show how we obtain the expected convergence rates and how the stabilization terms do not influence the result.
For those tests we give the expression of the exact solution $(\mathbf{u},p)$, from which we deduce the source term $\mathbf{f}$, the Dirichlet boundary condition $\mathbf{g}$ and the Neumann boundary condition Φ for which $(\mathbf{u},\mathrm{p})$ is solution of (II.1). We will compare the L^2-norm of the error for the velocity (denoted Ervel), the velocity gradient (Ergradvel) and the pressure (Erpre). In particular we denote:

$$\mathrm{Ergradvel}=\frac{\|\nabla^{\mathfrak{D}}(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2}{\|\nabla^{\mathfrak{D}}(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})\|_2},\quad \mathrm{Erpre}=\frac{\|\mathbb{P}_c^{\mathfrak{D}}\mathrm{p}-\mathrm{p}^{\mathfrak{D}}\|_2}{\|\mathbb{P}_c^{\mathfrak{D}}\mathrm{p}\|_2},\quad \mathrm{Ervel}=\frac{\|\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}-\mathbf{u}^{\mathfrak{T}}\|_2}{\|\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\|_2}, \tag{II.25}$$

where $\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}$ and $\mathbb{P}_c^{\mathfrak{D}}\mathrm{p}$ are the centered projections of $\mathbf{u}$ and p.

On Tables III.1 -II.6 we give the number of primal cells (NbCell) and the convergence rates (Ratio). We remark that, to discuss the error estimates, a family of meshes (Fig. II.3) is obtained by refining successively and uniformly the original mesh.

Fig. II.3 Family of meshes. On the *left*: non conformal square mesh. On the *right*: quadrangle-triangle mesh.



Green-Taylor vortices: In this test case, the exact solution is given by:

$$\mathbf{u}(x,y) = \begin{pmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y), \\ -\frac{1}{2}\cos(2\pi x)\sin(2\pi y) \end{pmatrix} \quad \mathrm{p}(x,y) = \frac{1}{8}\cos(2\pi x)\sin(2\pi y). \tag{II.26}$$

In this example, we show the results obtained using the non conformal square mesh of Fig. II.3; many other meshes were tested and the geometry of the mesh did not influence the accuracy of the approximation. As we can see in Tables II.1, II.2, II.3, we observe super convergence in L^2 norm of the velocity; instead, for the H^1 norm of the velocity and for the L^2 norm of the pressure we get exactly what was expected from Thm. II.3.11. As we mentioned before, an important remark is that the order of convergence does not change whether or not a stabilization is present and this has been observed in all the tests; it is sufficient to compare Tables II.1, II.2, II.3. This underlines the fact that the stabilization term is just a useful tool for the proofs of Theorems II.2.2 and II.3.11, but in practice it does not affect the results. Moreover, we tested the unstabilized scheme on other meshes for which we are not able to prove well-posedness because of their geometry and we numerically observed good behaviour. Remark also that the mesh in this example is non conformal.

Table II.1 Green-Taylor vortices on the non conformal square mesh of Fig. III.3, with $\mu = 0, \beta = 0$.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
64	6.693E-02	-	9.762E-02	-	1.179E+00	-
208	1.665E-02	2.00	4.485E-02	1.12	5.621E-01	1.07
736	4.173E-03	1.99	2.167E-02	1.05	2.770E-01	1.02
2752	1.045E-03	1.99	1.068E-02	1.02	1.380E-01	1.00
10624	2.615E-04	**1.99**	5.304E-03	**1.01**	6.895E-02	**1.00**

Table II.2 Green-Taylor vortices on the non conformal square mesh of Fig. III.3, with $\mu = 10^{-2}, \beta = 0$.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
64	6.695E-02	-	9.769E-02	-	1.175E+00	-
208	1.665E-02	2.00	4.487E-02	1.12	5.612E-01	1.06
736	4.173E-03	1.99	2.167E-02	1.05	2.767E-01	1.02
2752	1.045E-03	1.99	1.068E-02	1.02	1.379E-01	1.00
10624	2.614E-04	**1.99**	5.305E-03	**1.00**	6.894E-02	**1.00**

Table II.3 Green-Taylor vortices on the non conformal square mesh of Fig. III.3, with $\mu = 0, \beta = 10^{-2}$.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
64	6.970E-02	-	1.080E-01	-	6.979E-01	-
208	1.719E-02	2.01	5.031E-02	1.10	3.189E-01	1.13
736	4.305E-03	1.99	2.447E-02	1.04	1.528E-01	1.06
2752	1.079E-03	1.99	1.210E-02	1.01	7.498E-02	1.02
10624	2.700E-04	**1.99**	6.021E-03	**1.00**	3.717E-02	**1.01**

Polynomial solutions: The exact solution is given by

$$\mathbf{u}(x, y) = \begin{pmatrix} 2000(x^4 - 2x^3 + x^2)(2y^2 - 3y^2 + y), \\ -2000(y^4 - 2y^3 + y^2)(2x^3 - 3x^2 + x) \end{pmatrix} \quad \mathrm{p}(x, y) = x^2 + y^2 - 1. \tag{II.27}$$

In this example we use the quadrangle mesh on the right of Fig. II.3. Remark that, for this mesh, the well-posedness of the unstabilized scheme has not been proven. However, we numerically observe that the scheme is invertible and, in Tables II.4, II.5, II.6, we observe (as in the previous test case) super convergence in L^2 norm of the velocity and the expected rate for the gradient of the velocity and for the pressure. The order of convergence does not change if we work with or without stabilization. As in the previous case, we tested our schemes on different general meshes, and every time we got good results.

Table II.4 Polynomial solutions on the quadrangle-triangle mesh of Fig. III.3, with $\mu = 0, \beta = 0$.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
400	5.081E-02	-	6.309E-02	-	5.450E+00	-
1536	1.284E-02	1.98	2.796E-02	1.17	2.643E+00	1.04
6016	3.225E-03	1.99	1.346E-02	1.05	1.307E+00	1.01
23808	8.078E-04	1.99	6.660E-03	1.01	6.517E-01	1.00
94720	2.022E-04	**1.99**	3.320E-03	**1.00**	3.256E-01	**1.00**

Table II.5 Polynomial solutions on the quadrangle-triangle mesh of Fig. III.3, with $\mu = 10^{-2}, \beta = 0$.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
400	5.080E-02	-	6.312E-02	-	5.443E+00	-
1536	1.284E-02	1.98	2.797E-02	1.17	2.641E+00	1.04
6016	3.224E-03	1.99	1.346E-02	1.05	1.307E+00	1.01
23808	8.079E-04	1.99	6.660E-03	1.01	6.516E-01	1.00
94720	2.022E-04	**1.99**	3.320E-03	**1.00**	3.256E-01	**1.00**

II.5 Extension to the Divergence form

In this section, we would like to discuss the extension of the work done for the Laplace form of the Stokes problem (II.1) to the Divergence form:

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) = \mathbf{f} & \text{in} \quad \Omega, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on} \quad \Gamma_D, \\ \sigma(\mathbf{u}, \mathrm{p})\vec{\mathbf{n}} = \Phi & \text{on} \quad \Gamma_N, \end{cases} \tag{II.28}$$

Table II.6 Polynomial solutions on the quadrangle-triangle mesh of Fig. III.3, with $\mu = 0, \beta = 10^{-2}$.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
400	4.580E-02	-	7.500E-02	-	3.045E+00	-
1536	1.152E-02	1.99	3.436E-02	1.12	1.434E+00	1.08
6016	2.887E-03	1.99	1.673E-02	1.03	7.051E-01	1.02
23808	7.230E-04	1.99	8.302E-03	1.01	3.510E-01	1.00
94720	1.809E-04	**1.99**	4.142E-03	**1.00**	1.753E-01	**1.00**

where the unknowns are the velocity $\mathbf{u} : \Omega \to \mathbb{R}^2$ and the pressure $\mathrm{p} : \Omega \to \mathbb{R}$. The data are $\mathbf{f} \in (L^2(\Omega))^2$, $\Phi, \mathbf{g} \in (H^{\frac{1}{2}}(\partial\Omega))^2$ and $\vec{\mathbf{n}}$ is the unitary outer normal. The stress tensor is $\sigma(\mathbf{u}, \mathrm{p}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$, with $\mathrm{Re} > 0$. In particular, the strain rate tensor is defined by the symmetric part of the velocity gradient $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} +^t \nabla\mathbf{u})$.
We will consider an open bounded polygonal domain Ω of $\mathbb{R}^2$ with $\partial\Omega = \Gamma_D \cup \Gamma_N$, where $\Gamma_D \neq \emptyset$ is the fraction of domain with Dirichlet boundary conditions, $\Gamma_N \neq \emptyset$ is the fraction is the one with Neumann boundary conditions.

The DDFV discretization of (II.28) and the properties of the resulting scheme come as a natural extension of the results for $(\mathcal{P}_{\beta\mu})$. This is due to the results proved in Sec. I.8, that relate the discrete strain rate tensor with the discrete gradient.

To obtain our scheme, we integrate the momentum equation over all $\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*_N$. We impose Dirichlet boundary conditions on $\partial\mathfrak{M}_D \cup \partial\mathfrak{M}^*{}_D$ and Neumann boundary conditions on $\partial\mathfrak{M}_N$ (see Fig. II.1). The incompressibility constraint is directly approximated on the diamond mesh $\mathfrak{D}$. We remark that, since Korn's inequality (Thm. I.8.2) is proved under the assumption that the mesh satisfies inf-sup stability (see Sec. I.6), it is not interesting to stabilize this equation (as we did for $(\mathcal{P}_{\beta\mu})$).

The scheme reads:

$$\text{Find } \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}^{\Gamma_D}_{m,\mathbf{g}} \text{ and } \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that:}$$

$$\begin{cases} -\mathbf{div}^{\mathrm{K}}\left(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\right) = \mathbf{f}_{\mathrm{K}} & \forall \mathrm{K} \in \mathfrak{M} \\ -\mathbf{div}^{\mathrm{K}^*}\left(\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\right) = \mathbf{f}_{\mathrm{K}^*} & \forall \mathrm{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*_N \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) = 0 & \\ \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathrm{K}} = \Phi_\sigma & \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N. \end{cases} \tag{$\mathcal{D}$}$$

with the discrete stress tensor defined by $\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathfrak{D}}\mathrm{Id}$ and where we denote as before by $\mathbf{f}_{\mathrm{K}}, \mathbf{g}_{\mathrm{K}}$ (resp. $\mathbf{f}_{\mathrm{K}^*}, \mathbf{g}_{\mathrm{K}^*}$) the mean-value of the source term $\mathbf{f}$ and of the Dirichlet data $\mathbf{g}$ on $\mathrm{K} \in \mathfrak{M}$ (resp. on $\mathrm{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*_\Gamma$) and Φ_σ the mean-value of the Neumann data on $\sigma \in \Gamma_N$:

$$\mathbf{f}_{\mathrm{K}} = \frac{1}{m_{\mathrm{K}}}\int_{\mathrm{K}} \mathbf{f}(x)\mathrm{d}x, \qquad \mathbf{f}_{\mathrm{K}^*} = \frac{1}{m_{\mathrm{K}^*}}\int_{\mathrm{K}^*} \mathbf{f}(x)\mathrm{d}x,$$

$$\mathbf{g}_{\mathrm{K}} = \frac{1}{m_{\mathrm{K}}}\int_{\mathrm{K}} \mathbf{g}(x)\mathrm{d}x, \qquad \mathbf{g}_{\mathrm{K}^*} = \frac{1}{m_{\mathrm{K}^*}}\int_{\mathrm{K}^*} \mathbf{g}(x)\mathrm{d}x,$$

$$\Phi_\sigma = \frac{1}{m_\sigma}\int_\sigma \Phi(x)\mathrm{d}x.$$

Moreover, we denote by $\mathbf{g}_\sigma = \gamma^\sigma(\mathbf{g}^{\mathfrak{T}})$.

II.5.1 Well-posedness of the scheme

In the following proposition, we show an a priori estimate to the solution of $(\mathcal{D})$ before proving that the problem is well-posed:

Proposition II.5.1 (A priori estimate) *Let* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_{m,\boldsymbol{g}}^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$ *be a solution of* $(\mathcal{D})$*. Then:*

$$\begin{aligned}\frac{2}{Re}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq &\left|[[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}\right| \\ &+ \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma |\boldsymbol{g}_\sigma|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma \left|\frac{2}{Re}\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}} Id\right|^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma |\Phi_\sigma|^2\right)^{\frac{1}{2}} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \left|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})\right|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{II.29}$$

Proof We define the bilinear form associated to the scheme:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = \left[\left[\mathbf{div}^{\mathfrak{T}}\left((-\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}\right), \mathbf{u}^{\mathfrak{T}}\right]\right]_{\mathfrak{T}} + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}.$$

If we apply Green's formula (Thm. I.5.1), with the remark that, since $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}$ is symmetric and $\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) = 0$, we have $[[\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = [[\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}$, we get:

$$\begin{aligned}B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = &\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 - (\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}))_{\mathfrak{D}} \\ &+ \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \left(-\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathtt{K}} + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}.\end{aligned}$$

The terms $(\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}))_{\mathfrak{D}}$ simplify. The boundary diamonds $\mathfrak{D}_{ext}$ can be split between $\mathfrak{D}_{ext} \cap \Gamma_D$ and $\mathfrak{D}_{ext} \cap \Gamma_N$; by applying Dirichlet and Neumann boundary conditions, we obtain:

$$\begin{aligned}B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = &\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma \mathbf{g}_\sigma \cdot \left(-\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathtt{K}} \\ &- \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \Phi_\sigma.\end{aligned} \tag{II.30}$$

On the other hand, since $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ is a solution to $(\mathcal{D})$, we have:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = [[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}. \tag{II.31}$$

By putting together (II.30) and (II.31) :

$$\begin{aligned}\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq &[[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} \\ &+ \left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma \mathbf{g}_\sigma \cdot \left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right| + \left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \Phi_\sigma\right|.\end{aligned}$$

We then apply Cauchy-Schwarz inequality to get our result (II.29). ■

In the following result, we prove that there exists a unique solution to $(\mathcal{D})$.

Theorem II.5.2 (Well-posedness of the scheme) *Let $\mathfrak{T}$ a DDFV mesh associated to Ω that satisfies Inf-sup stability (Def. I.6.1) . Then the scheme $(\mathcal{D})$ has a unique solution $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_{m,\boldsymbol{g}}^{\Gamma_D} \times \mathbb{R}^{\mathfrak{D}}$.*

Proof By linearity, it is sufficient to prove that if $\mathbf{f}^{\mathfrak{T}} = 0$, $\mathbf{g}^{\partial\mathfrak{M}} = 0$,$\mathbf{g}^{\partial\mathfrak{M}^*} = 0$ and $\Phi_\sigma = 0$, then $\mathbf{u}^{\mathfrak{T}} = 0$ and $\mathrm{p}^{\mathfrak{D}} = 0$. Directly from (II.29), we deduce:

$$\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq 0.$$

that implies $\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 = 0$. Thanks to Thm. I.8.4, we deduce $\mathbf{u}^{\mathfrak{T}} = 0$. Thus, the momentum equation and the Neumann boundary condition become:

$$\begin{cases} \mathbf{div}^{\mathrm{K}}\left(\mathrm{p}^{\mathfrak{D}}\mathrm{Id}\right) = 0 & \forall \mathrm{K} \in \mathfrak{M} \\ \mathbf{div}^{\mathrm{K}^*}\left(\mathrm{p}^{\mathfrak{D}}\mathrm{Id}\right) = 0 & \forall \mathrm{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_N^* \\ (\mathrm{p}^{\mathrm{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathrm{K}} = 0 & \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N. \end{cases} \tag{II.32}$$

Our goal is now to show that $\mathrm{p}^{\mathfrak{D}} = 0$. For every $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, thanks to Green's formula (Thm. I.5.1), we can write:

$$\left(\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}}\right)_{\mathfrak{D}} = -\left[\left[\mathbf{v}^{\mathfrak{T}}, \mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}\mathrm{Id})\right]\right]_{\mathfrak{T}} + \sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \cdot (\mathrm{p}^{\mathrm{D}}\mathrm{Id})\,\vec{\mathbf{n}}_{\sigma\mathrm{K}}. \tag{II.33}$$

By definition of the scalar products (see Sec. I.4), by (II.32) and by the fact that $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, we get that $\left[\left[\mathbf{v}^{\mathfrak{T}}, \mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}\mathrm{Id})\right]\right]_{\mathfrak{T}} = 0$ and $\sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \cdot (\mathrm{p}^{\mathrm{D}}\mathrm{Id})\,\vec{\mathbf{n}}_{\sigma\mathrm{K}} = 0$. Thus (II.33) becomes:

$$\left(\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}}\right)_{\mathfrak{D}} = 0. \tag{II.34}$$

We now go back to inequality (I.30) ensured by Inf-sup stability: since (II.34) holds for any $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$, the supremum in the right hand side of (I.30) vanishes and we can deduce that $\mathrm{p}^{\mathfrak{D}}$ is constant. Thanks to Neumann boundary condition, i.e. $\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathrm{D}}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathrm{K}} = 0$, and $\mathbf{u}^{\mathfrak{T}} = 0$, we get that $\mathrm{p}^{\mathfrak{D}} = 0$. ■

Remark II.5.3 (Error estimates) *For what concerns the error estimates for the scheme $(\mathcal{D})$, we will not detail the computations here. The proof of an optimal error estimate of order 1 for the velocity, its gradient and the pressure comes straightforwardly from the one in Sec. II.3, Thm. II.3.11.*

First, remark that it is sufficient to apply the results of Sec. I.8 in order to pass from $\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2$ to $\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2$ and viceversa when necessary.

Moreover, the hypothesis $\beta > 0$ in the optimal error estimate of Thm. II.3.11 can be eliminated if we suppose Inf-Sup stability condition. It is necessary to suppose $\beta > 0$ in order to apply the stability result of Thm. II.3.9; under Inf-Sup stability condition and with $\beta = 0 = \mu$, the proof remains valid. In fact, if we look at Step 1 with $\beta = 0 = \mu$, instead of (II.17) we get:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}; \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 - \sum_{\mathrm{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \Phi_\sigma \cdot \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}).$$

Step 2 is replaced by Lemma I.6.5, which directly ensures $\forall \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ *the existence of* $\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}$ *such that* $\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{\mathfrak{T}}) = -\mathrm{p}^{\mathfrak{D}}$ *and* $\|\nabla^{\mathfrak{D}}\mathbf{v}^{\mathfrak{T}}\|_2 \leq C\|\mathrm{p}^{\mathfrak{D}}\|_2$ *; Step 3 is then obtained straightforwardly.*

II.5.2 Numerical results

We validate the scheme $(\mathcal{D})$ by showing a few numerical experiments. The setting of the simulations is the same as the one introduced in Sec. II.4.
The computational domain is $\Omega = [0,1]^2$. The configuration of the boundaries is illustrated in Fig. II.4.

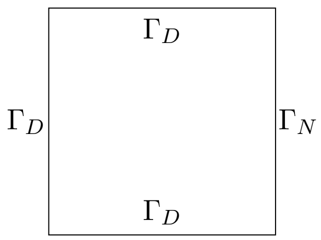
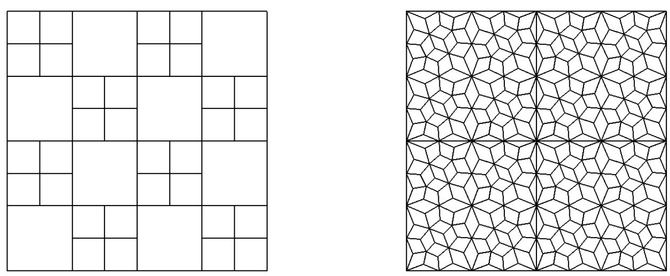


Fig. II.4 $\Omega = [0,1]^2$, Dirichlet boundary conditions on Γ_D , Neumann boundary conditions on Γ_N.

For those tests we give the expression of the exact solution ($\mathbf{u}$,p), from which we deduce the source term $\mathbf{f}$, the Dirichlet boundary condition $\mathbf{g}$ and the Neumann boundary condition Φ for which ($\mathbf{u}$,p) is solution of (II.1). We will compare the L^2-norm of the error obtained with the DDFV scheme for the velocity (denoted Ervel), the velocity gradient (Ergradvel) and the pressure (Erpre); see (II.25) for the definition of the norms. On the Tables II.7, II.14 we give the number of primal cells (NbCell) and the convergence rates (Ratio).

The difference with respect to Sec. II.4 is that now the scheme is not stabilized; so, instead of studying the influence of the parameters, we present tests on different mesh geometries to show how this does not affect the accuracy of the scheme. With respect to the previous section, we add the meshes of Fig. II.6. Moreover, for some of the meshes that we tested, Inf-sup inequality (Def. I.6.1) has not been proven, but we observe numerically that it holds even in those cases.
The main difficulty with respect to the implementation of the scheme $(\mathcal{P}_{\beta\mu})$ is that here we deal with the discretization of the term $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}$.

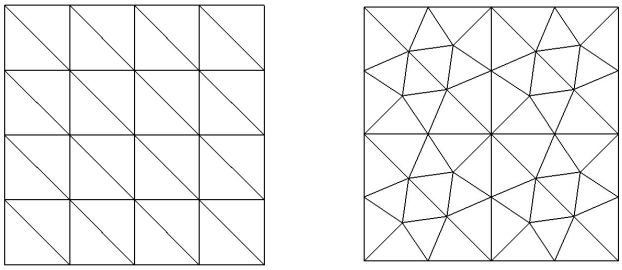
Fig. II.5 Two families of quadrangular meshes.



We remark that, to discuss the error estimates, a family of meshes is obtained by refining successively and uniformly the original mesh.

In Tables II.7,II.8,II.9, II.10 we show the results for the solution of Green-Taylor vortices (II.26); in Tables II.11,II.12,II.13, II.14 we show the results for the polynomial solutions (II.27). In all the

Fig. II.6 Two family of triangular meshes.



cases, we observe super convergence in L^2 norm of the velocity and the expected order 1 for the H^1 norm of the velocity and for the L^2 norm of the pressure (as proved in Thm. II.3.11) ; we also remark that the mesh geometry does not influence the results, even for meshes that do not satisfy Inf-sup stability (Def. I.6.1), like the quadrangular mesh of Fig. II.5.

Table II.7 Green-Taylor vortices on the *left* triangular mesh of Fig. II.6.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
72	1.122E-02	-	3.669E-02	-	5.226E-01	-
256	2.729E-03	2.04	1.572E-02	1.22	2.179E-01	1.26
960	7.113E-04	1.94	7.355E-03	1.09	1.150E-01	0.92
3712	1.860E-04	1.93	3.534E-03	1.05	5.915E-02	0.96
14592	4.772E-05	**1.96**	1.728E-03	**1.03**	2.992E-02	**0.98**

Table II.8 Green-Taylor vortices on the *right* triangular mesh of Fig. II.6.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
48	4.048E-02	-	6.850E-02	-	1.394E+00	-
160	1.001E-02	2.01	2.270E-02	1.59	3.486E-01	2.00
576	2.489E-03	2.00	7.783E-03	1.54	8.732E-02	1.99
2176	6.207E-04	2.00	2.712E-03	1.52	2.184E-02	1.99
8448	1.550E-04	**2.00**	9.519E-04	**1.51**	5.459E-03	**2.00**

Table II.9 Green-Taylor vortices on the *left* quadrangular mesh of Fig. II.5.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
64	3.292E-02	-	9.172E-02	-	1.878E+00	-
208	7.827E-03	2.07	4.423E-02	1.05	6.809E-01	1.46
736	1.932E-03	2.01	2.163E-02	1.03	3.002E-01	1.18
2752	4.801E-04	2.00	1.069E-02	1.01	1.432E-01	1.06
10624	1.197E-04	**2.00**	5.308E-03	**1.00**	7.021E-02	**1.03**

Table II.10 Green-Taylor vortices on the *right* quadrangular mesh of Fig. II.5.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
400	3.658E-03	-	2.591E-02	-	6.480E-01	-
1536	9.242E-04	1.98	1.280E-02	1.01	3.206E-01	1.01
6016	2.326E-04	1.99	6.376E-03	1.00	1.595E-01	1.00
23808	5.833E-05	1.99	3.184E-03	1.00	7.957E-02	1.00
94720	1.460E-05	**1.99**	1.591E-03	**1.00**	3.974E-02	**1.00**

Table II.11 Polynomial solutions on the *left* triangular mesh of Fig. II.6.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
72	7.574E-02	-	1.266E-01	-	7.616E+00	-
256	1.507E-02	2.32	4.247E-02	1.57	2.494E+00	1.61
960	4.019E-03	1.90	1.757E-02	1.27	1.048E+00	1.25
3712	1.085E-03	1.89	7.782E-03	1.17	4.984E-01	1.07
14592	2.836E-04	**1.93**	3.626E-03	**1.10**	2.471E-01	**1.01**

Table II.12 Polynomial solutions on the *right* triangular mesh of Fig. II.6.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
48	1.381E-01	-	2.041E-01	-	1.991E+01	-
160	3.697E-02	1.90	6.771E-02	1.59	5.614E+00	1.82
576	9.553E-03	1.95	2.333E-02	1.53	1.465E+00	1.93
2176	2.417E-03	1.98	8.172E-03	1.51	3.709E-01	1.98
8448	6.072E-04	**1.99**	2.879E-03	**1.50**	9.305E-02	**1.99**

Table II.13 Polynomial solutions on the *left* quandrangular mesh of Fig. II.5.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
64	1.669E-01	-	2.739E-01	-	3.525E+01	-
208	4.027E-02	2.05	1.138E-01	1.26	1.017E+01	1.79
736	8.839E-02	2.01	4.964E-02	1.19	3.528E+00	1.52
2752	4.419E-02	2.00	2.273E-02	1.12	1.428E+00	1.30
10624	2.210E-02	**2.00**	1.080E-02	**1.07**	6.372E-01	**1.16**

Table II.14 Polynomial solutions on the *right* quandrangular mesh of Fig. II.5

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
400	2.468E-02	-	5.906E-02	-	5.974E+00	-
1536	6.451E-03	1.93	2.741E-02	1.10	2.726E+00	1.13
6016	1.639E-03	1.97	1.339E-02	1.03	1.319E+00	1.04
23808	4.117E-04	1.99	6.651E-03	1.01	6.534E-01	1.01
94720	1.031E-04	**1.99**	3.319E-03	**1.00**	3.259E-01	**1.00**

II.6 Unstabilized scheme: weak boundary conditions

Here we discuss another extension to the results proved for the scheme $(\mathcal{P}_{\beta\mu})$. If we do not stabilize the incompressibility constraint, the problem remains well-posed under the hypothesis that the mesh satifies Inf-sup inequality (see Sec. I.6). We recall that this inequality has been proven for a large class of meshes, but in the simplest case of conformal square meshes it is valid up to an unstable mode for the pressure.

We want to show that there is a way of avoiding this inconvenient on simple mesh geometries: it is to impose boundary conditions in a "weak sense". This means that, instead of imposing the conditions on $\partial\mathfrak{M}_D \cup \partial\mathfrak{M}_D^* \cup \partial\mathfrak{M}_N$, they will only be imposed on $\partial\mathfrak{M}_D \cup \partial\mathfrak{M}_N$.
The scheme is thus obtained by integrating the momentum equation over $\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and the incompressibility constraint over $\mathfrak{D}$. It reads:

$$\text{Find } \mathbf{u}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}} \text{ and } \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} :$$

$$\begin{cases} \mathbf{div}^{\kappa}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\kappa} & \forall \kappa \in \mathfrak{M}, \\ \mathbf{div}^{\kappa^*}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\kappa^*} & \forall \kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) = 0, & \\ (\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathrm{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\kappa} = \Phi_\sigma & \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_N, \\ \gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}}) = \mathbf{g}_\sigma & \forall \mathrm{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_D \\ \sum\limits_{\kappa\in\mathfrak{M}} m_{\kappa}\mathbf{u}_{\kappa} - \sum\limits_{\kappa^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\kappa^*}\mathbf{u}_{\kappa^*} = 0, & \end{cases} \tag{$\mathcal{P}_w$}$$

where we denote as before by $\mathbf{f}_{\kappa}$ (resp. $\mathbf{f}_{\kappa^*}$) the mean-value of the source term $\mathbf{f}$ on $\kappa \in \mathfrak{M}$ (resp. on $\kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_\Gamma^*$), $\mathbf{g}_\sigma, \Phi_\sigma$ the mean-value of the Dirichlet and Neumann data respectively for $\sigma \in \Gamma_D, \sigma \in \Gamma_N$:

$$\mathbf{f}_{\kappa} = \frac{1}{m_{\kappa}}\int_{\kappa}\mathbf{f}(x)\mathrm{d}x, \qquad \mathbf{f}_{\kappa^*} = \frac{1}{m_{\kappa^*}}\int_{\kappa^*}\mathbf{f}(x)\mathrm{d}x,$$
$$\mathbf{g}_\sigma = \frac{1}{m_\sigma}\int_\sigma \mathbf{g}(x)\mathrm{d}x, \qquad \Phi_\sigma = \frac{1}{m_\sigma}\int_\sigma \Phi(x)\mathrm{d}x.$$

Linear dependence of the equations.
By imposing weak boundary conditions, the equations are no more linearly independent. This is why we add the relation:

$$\sum_{\kappa\in\mathfrak{M}} m_{\kappa}\mathbf{u}_{\kappa} - \sum_{\kappa^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\kappa^*}\mathbf{u}_{\kappa^*} = 0.$$

To show the linear dependence, set firstly $\psi^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ such that

$$\begin{aligned} \psi_{\kappa} &= \vec{\mathbf{e}_1} := (1,0)^t, & \forall \kappa \in \mathfrak{M} \cup \partial\mathfrak{M}, \\ \psi_{\kappa^*} &= \vec{\mathbf{0}}, & \forall \kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*, \end{aligned}$$

that implies $\nabla^{\mathfrak{D}}\psi^{\mathfrak{T}} = 0$ and $\gamma^{\mathfrak{T}}(\psi) = (\frac{1}{2}, 0)^t$. We have:

- on one hand, thanks to the scheme $(\mathcal{P}_w)$:

$$2\left[\left[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}), \psi^{\mathfrak{T}}\right]\right]_{\mathfrak{T}} = \sum_{\kappa\in\mathfrak{M}} m_{\kappa}\mathbf{f}_{\kappa} \cdot \vec{\mathbf{e}_1},$$

that thanks to the definition of the source term leads to

$$2\left[\left[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathfrak{D}}\mathrm{Id}),\psi^{\mathfrak{T}}\right]\right]_{\mathfrak{T}}=\int_{\Omega}\mathbf{f}(x)\cdot\vec{\mathbf{e}_1}\mathrm{d}x.$$

- on the other hand, Green's formula (I.5.1) gives:

$$2\left[\left[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathfrak{D}}\mathrm{Id}),\psi^{\mathfrak{T}}\right]\right]_{\mathfrak{T}}= \\ 2\left(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}-\mathrm{p}^{\mathfrak{D}}\mathrm{Id}:\nabla^{\mathfrak{D}}\psi^{\mathfrak{T}}\right)_{\mathfrak{D}}-2\left(\gamma^{\mathfrak{D}}\left((\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}-\mathrm{p}^{\mathfrak{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right),\gamma^{\mathfrak{T}}(\psi^{\mathfrak{T}})\right)_{\partial\Omega},$$

that since $\nabla^{\mathfrak{D}}\Psi^{\mathfrak{T}}=0$ and $\gamma^{\mathfrak{T}}(\psi)=(\frac{1}{2},0)^t$ is equivalent to:

$$2\left[\left[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathfrak{D}}\mathrm{Id}),\psi^{\mathfrak{T}}\right]\right]_{\mathfrak{T}}=\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}m_\sigma\left((-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right)\cdot\vec{\mathbf{e}}_1.$$

Putting all together, we get:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}m_\sigma\left((-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right)\cdot\vec{\mathbf{e}}_1=\int_{\Omega}\mathbf{f}(x)\cdot\vec{\mathbf{e}}_1\mathrm{d}x. \tag{II.35}$$

We can repeat the same computation by choosing $\psi^{\mathfrak{T}}\in\left(\mathbb{R}^2\right)^{\mathfrak{T}}$ such that

$$\begin{aligned}\psi_{\mathtt{K}}&=\vec{\mathbf{e}_2}:=(0,1)^t, &&\forall\mathtt{K}\in\mathfrak{M}\cup\partial\mathfrak{M},\\ \psi_{\mathtt{K}^*}&=\vec{\mathbf{0}}, &&\forall\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*,\end{aligned}$$

that implies $\nabla^{\mathfrak{D}}\psi^{\mathfrak{T}}=0$ and $\gamma^{\tau}(\psi)=(0,\frac{1}{2})^t$. Now we obtain:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}m_\sigma\left((-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right)\cdot\vec{\mathbf{e}}_2=\int_{\Omega}\mathbf{f}(x)\cdot\vec{\mathbf{e}}_2\mathrm{d}x. \tag{II.36}$$

Relations (II.35)-(II.36) imply then:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}m_\sigma(-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}}=\int_{\Omega}\mathbf{f}(x)\mathrm{d}x. \tag{II.37}$$

By proceeding in the same way, we choose first $\psi^{\mathfrak{T}}\in\left(\mathbb{R}^2\right)^{\mathfrak{T}}$ such that

$$\begin{aligned}\psi_{\mathtt{K}}&=\vec{\mathbf{0}}:=(1,0)^t, &&\forall\mathtt{K}\in\mathfrak{M}\cup\partial\mathfrak{M},\\ \psi_{\mathtt{K}^*}&=\vec{\mathbf{e}_1}, &&\forall\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*,\end{aligned}$$

that implies $\nabla^{\mathfrak{D}}\psi^{\mathfrak{T}}=0$ and $\gamma^{\mathfrak{T}}(\psi)=(\frac{1}{2},0)^t$, and next $\psi^{\mathfrak{T}}\in\left(\mathbb{R}^2\right)^{\mathfrak{T}}$ such that

$$\begin{aligned}\psi_{\mathtt{K}}&=\vec{\mathbf{0}}:=(1,0)^t, &&\forall\mathtt{K}\in\mathfrak{M}\cup\partial\mathfrak{M},\\ \psi_{\mathtt{K}^*}&=\vec{\mathbf{e}_2}, &&\forall\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*,\end{aligned}$$

that implies $\nabla^{\mathfrak{D}}\psi^{\mathfrak{T}}=0$ and $\gamma^{\mathfrak{T}}(\psi)=(0,\frac{1}{2})^t$. Those choices lead to:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}m_\sigma(-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathtt{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}}=\int_{\Omega}\mathbf{f}(x)\mathrm{d}x. \tag{II.38}$$

We observe that (II.37) and (II.38) give the same result; this shows that the equations on $\mathfrak{T}$ are not linearly independent.

II.6.1 Well-posedness of the scheme

Equivalently to Sec. II.2, we start by proving an a priori estimate:

Proposition II.6.1 (A priori estimate) *Let* $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in (\mathbb{R}^2)^{\mathfrak{T}} \times \mathbb{R}^{\mathfrak{D}}$ *be a solution of* $(\mathcal{P}_w)$*. Then:*

$$
\begin{aligned}
\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq & \left|[[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}\right| \\
& + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma |\boldsymbol{g}_\sigma|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma \left|\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}} Id\right|^2\right)^{\frac{1}{2}} \\
& + \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma |\Phi_\sigma|^2\right)^{\frac{1}{2}} \left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \left|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})\right|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{II.39}
$$

Proof We can associate the following bilinear form to the scheme:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = [[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}}\mathrm{Id}), \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}.$$

If we apply Green's formula (Thm. I.5.1), by taking into account the boundary terms, we get:

$$
\begin{aligned}
B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = & \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 - (\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}))_{\mathfrak{D}} \\
& + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot (-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}}\mathrm{Id})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} + (\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}), \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}.
\end{aligned}
$$

The terms $(\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}))_{\mathfrak{D}}$ simplify. The boundary diamonds $\mathfrak{D}_{ext}$ can be split between $\mathfrak{D}_{ext} \cap \Gamma_D$ and $\mathfrak{D}_{ext} \cap \Gamma_N$; by applying Dirichlet and Neumann boundary conditions, we obtain:

$$
B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma g^\sigma \cdot (-\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathtt{D}}\mathrm{Id})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} - \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \Phi_\sigma. \tag{II.40}
$$

On the other hand, since $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ is a solution to $(\mathcal{P}_w)$, we have:

$$B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = [[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}. \tag{II.41}$$

By putting together (II.40) and (II.41) :

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq [[\mathbf{f}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} + \left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_D} m_\sigma \mathbf{g}_\sigma \cdot (\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}}\mathrm{Id})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}}\right| + \left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}\cap\Gamma_N} m_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \Phi_\sigma\right|.$$

We then apply Cauchy-Schwarz inequality to get our result (II.39). ■

We can now prove the well-posed character of our scheme.

Theorem II.6.2 (Well-posedness of the scheme) *The scheme DDFV* $(\mathcal{P}_w)$ *has a unique solution* $(\mathbf{u}^{\mathfrak{T}}, p^{\mathfrak{D}}) \in (\mathbb{R}^2)^{\mathfrak{T}} \times \mathbb{R}^{\mathfrak{D}}$ *on conformal triangle meshes, conformal and non-conformal square meshes.*

Proof By linearity, it is sufficient to prove that if $\mathbf{f}^{\mathfrak{T}} = 0$, $\mathbf{g}_\sigma = 0$ and $\Phi_\sigma = 0$, then $\mathbf{u}^{\mathfrak{T}} = 0$ and $\mathrm{p}^{\mathfrak{D}} = 0$. Directly from (II.39), we deduce:

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq 0.$$

This implies $\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 = 0$: from Remark I.3.3 and since $\mathbf{g}_\sigma = 0$, we obtain $\mathbf{c}_0 + \mathbf{c}_1 = 0$. Since we impose $\sum_{\mathrm{K}\in\mathfrak{M}} m_{\mathrm{K}}\mathbf{u}_{\mathrm{K}} - \sum_{\mathrm{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathrm{K}^*}\mathbf{u}_{\mathrm{K}^*} = 0$ and by hypothesis $\sum_{\mathrm{K}\in\mathfrak{M}} m_{\mathrm{K}} = \sum_{\mathrm{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathrm{K}^*}$, we deduce $\mathbf{c}_0 = \mathbf{c}_1$ so we can conclude $\mathbf{u}^{\mathfrak{T}} = 0$.

The momentum equation becomes:

$$\mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = 0.$$

Here is the point where the structure of the mesh is crucial. In fact, we can prove that $\mathrm{p}^{\mathfrak{D}} = 0$ if we are on a conformal triangle mesh, conformal or non conformal square mesh.
Here, we illustrate the case of conformal triangle mesh: this is an adaptation of the proof that can be found in [Del07], done in the case of Dirichlet boundary conditions. We also refer to [Del07] for the case of conformal/non-conformal squares.

If we consider a volume K of the primal mesh and we enumerate its edges with $\alpha = 1, 2, 3$, by using the definition of discrete divergence we can write:

$$0 = \mathbf{div}^{\mathrm{K}}(\mathrm{p}^{\mathfrak{D}}\mathrm{Id}) = \frac{1}{m_{\mathrm{K}}}(|\sigma_1|\mathrm{p}_1\mathrm{Id}\vec{\mathbf{n}}_{\sigma_1\mathrm{K}} + |\sigma_2|\mathrm{p}_2\mathrm{Id}\vec{\mathbf{n}}_{\sigma_2,\mathrm{K}} + |\sigma_3|\mathrm{p}_3\mathrm{Id}\vec{\mathbf{n}}_{\sigma_3\mathrm{K}}),$$

with $|\sigma_1|\vec{\mathbf{n}}_{\sigma_1\mathrm{K}} + |\sigma_2|\vec{\mathbf{n}}_{\sigma_2\mathrm{K}} + |\sigma_3|\vec{\mathbf{n}}_{\sigma_3\mathrm{K}} = 0$, so:

$$(\mathrm{p}_1\mathrm{Id} - \mathrm{p}_3\mathrm{Id})|\sigma_1|\vec{\mathbf{n}}_{\sigma_1\mathrm{K}} + (\mathrm{p}_2\mathrm{Id} - \mathrm{p}_3\mathrm{Id})|\sigma_2|\vec{\mathbf{n}}_{\sigma_2\mathrm{K}} = 0,$$

and since $\vec{\mathbf{n}}_{\sigma_1\mathrm{K}}$ and $\vec{\mathbf{n}}_{\sigma_2\mathrm{K}}$ are not co-linear, it implies $\mathrm{p}_1 = \mathrm{p}_2 = \mathrm{p}_3$.
If we proceed in the same way for an element K' neighbour of K, we obtain $\mathrm{p}'_1 = \mathrm{p}'_2 = \mathrm{p}'_3$ and since $\mathrm{p}_1 = \mathrm{p}'_1$ we deduce $\mathrm{p}_1 = \mathrm{p}_2 = \mathrm{p}_3 = \mathrm{p}'_2 = \mathrm{p}'_3$.
If we make the same reasoning for all the others triangles, we get that all the p_j are equal to the same constant. Thanks to Neumann boundary conditions, we deduce that this constant is equal to 0, i.e. $\mathrm{p}^{\mathrm{D}} = 0$ $\forall \mathrm{D} \in \mathfrak{D}$. ■

II.6.2 Numerical results

We validate the scheme $(\mathcal{P}_w)$ through some numerical experiments. We recall that the goal of designing this scheme is to show that, since Inf-sup inequality (Def. I.6.1) in the simplest case of conformal square meshes has been proven to be valid up to an unstable mode for the pressure, we can avoid this inconvenient by imposing boundary conditions in a "weak sense", instead of stabilizing the equation of conservation of mass.
The computational domain is $\Omega = [0, 1]^2$. The configuration of the boundaries is illustrated in Fig. II.7.
As in Sections II.4, II.5.2, we give the expression of the exact solution ($\mathbf{u}$,p), from which we deduce the source term $\mathbf{f}$, the Dirichlet boundary condition $\mathbf{g}$ and the Neumann boundary condition Φ for which ($\mathbf{u}$,p) is solution of (II.1). We will compare the L^2-norm of the error obtained with the DDFV scheme for the velocity (denoted Ervel), the velocity gradient (Ergradvel) and the pressure (Erpre). On the two Tables III.1, II.4 we give the number of primal cells (NbCell) and

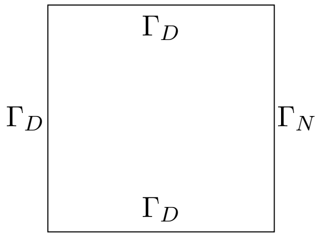


Fig. II.7 $\Omega = [0,1]^2$, Dirichlet boundary conditions on Γ_D , Neumann boundary conditions on Γ_N.

the convergence rates (Ratio).
We remark that, to discuss the error estimates, a family of meshes is obtained by refining successively and uniformly the original mesh.
In Tables II.15, II.17 we show the results for the solution of Green-Taylor vortices (II.26); in Tables II.16,II.18 we show the results for the polynomial solutions (II.27).

In Tables II.15,II.16 we show how on a cartesian mesh we obtain super convergence in L^2 norm of the velocity and the expected order 1 for the H^1 norm of the velocity and for the L^2 norm of the pressure.

Table II.15 Green-Taylor vortices on a cartesian mesh with weak boundary conditions.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
12	2.980E-01	-	4.242E-01	-	3.429E+00	-
32	9.168E-02	1.70	1.361E-01	1.640	1.789E+00	0.93
96	2.567E-02	1.83	5.273E-02	1.36	7.571E-01	1.24
320	7.050E-03	1.86	2.320E-02	1.18	3.546E-01	1.09
1152	1.915E-03	**1.88**	1.094E-02	**1.08**	1.733E-01	**1.03**

Table II.16 Polynomial solutions on a cartesian mesh with weak boundary conditions.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
12	1.122E+00	-	1.456E+00	-	3.771E+01	-
32	2.396E-01	2.22	3.632E-01	2.00	1.243E+01	1.60
96	6.708E-02	1.83	1.320E-01	1.46	5.541E+00	1.16
320	1.899E-02	1.82	5.312E-02	1.31	2.578E+00	1.10
1152	5.265E-03	**1.85**	2.283E-02	**1.22**	1.262E+00	**1.03**

In Tables II.17,II.18, we numerically tested the scheme on the non conformal square mesh of Fig. II.5 and we obtained the same results as for the cartesian case.

Table II.17 Green-Taylor vortices on the *left* quadrangular mesh of Fig. II.5 with weak boundary conditions.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
64	5.851E-02	-	1.211E-01	-	1.746E+00	-
208	1.680E-02	1.80	5.576E-02	1.11	8.070E-01	1.11
736	4.707E-03	1.83	2.704E-02	1.04	3.901E-01	1.05
2752	1.300E-03	1.85	1.336E-02	1.01	1.925E-01	1.02
10624	3.548E-04	**1.87**	6.651E-03	**1.00**	9.572E-02	**1.00**

Table II.18 Polynomial solutions on the *left* quadrangular mesh of Fig. II.5 with weak boundary conditions.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
64	1.851E-01	-	2.829E-01	-	1.234E+01	-
208	5.208E-02	1.82	1.227E-01	1.20	6.087E+00	1.01
736	1.430E-02	1.86	5.592E-02	1.13	3.006E+00	1.01
2752	3.907E-03	1.87	2.645E-02	1.08	1.496E+00	1.00
10624	1.061E-03	**1.88**	1.282E-02	**1.04**	7.473E-01	**1.00**

Conclusions

In this chapter, we first presented a stabilized DDFV scheme $(\mathcal{P}_{\beta\mu})$ for the Stokes problem with mixed Dirichlet/Neumann boundary conditions and we proved its wellposedness on general meshes. We obtained an error estimate in the L^2 norm of order 1 for the velocity, its gradient and the pressure. Numerically, we observed a super convergence in the L^2 norm of the velocity and the expected convergences for the gradient of the velocity and for the pressure; moreover, we remarked that the order of convergence is not influenced by the presence of the parameters of stabilization. Second, we extended the results to the divergence form of Stokes problem, by obtaining the scheme $(\mathcal{D})$; we proved that the same results proven for $(\mathcal{P}_{\beta\mu})$ are valid for $(\mathcal{D})$, thanks to the results of Sec. I.8 (such as Korn inequality) and we numerically tested the scheme. At last, we wrote a non-stabilized DDFV scheme for for the Stokes problem, with "weak" boundary conditions. We proved its well-posedness in the case of conformal triangle meshes, conformal and non-conformal square meshes and we tested numerically the convergence, that turns out to have the same rates as the previous cases.

Chapter III

Navier Stokes problem with outflow boundary conditions

Contents

Most of the content of this chapter appeared in [GKL19].

The problem we are interested in is the computation of a flow whose velocity is prescribed at one part of the boundary and it flows freely on the other one. In this framework, we are often required to truncate the physical domain to obtain a reduced computational domain, either because we want to save computational ressources or because the physical domain is unbounded. We illustrate this setting in Fig. III.1.

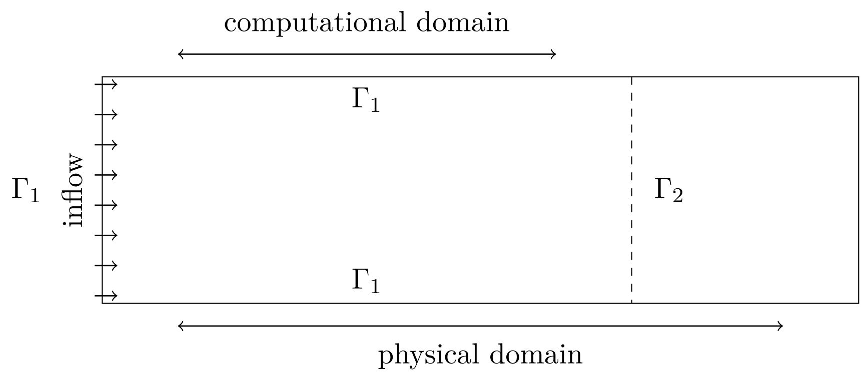


Fig. III.1 Domain and notations.

The aim of this chapter is to design and analyze a finite volume approximation of the 2D unsteady incompressible Navier-Stokes problem:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u},\mathrm{p})) = 0; & \text{in} \quad \Omega_T = \Omega\times[0,T] \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T, \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1\times(0,T), \\ \sigma(\mathbf{u},\mathrm{p})\,\vec{\mathbf{n}} + \dfrac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{ref}) = \sigma_{ref}\,\vec{\mathbf{n}} & \text{on} \quad \Gamma_2\times(0,T), \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega \end{cases} \tag{III.1}$$

with $0 < T < \infty$, Ω an open connected bounded polygonal domain of $\mathbb{R}^2$, whose boundary is $\partial\Omega = \Gamma_1\cup\Gamma_2$ and whose outward unit normal is $\vec{\mathbf{n}}$, $\mathbf{u}_{init}\in(L^\infty(\Omega))^2$, $\mathbf{g}_1\in(H^{\frac{1}{2}}(\Gamma_{1,T}))^2$ and where $\mathbf{u}:\Omega_T\to\mathbb{R}^2$ is the velocity, $\mathrm{p}:\Omega_T\to\mathbb{R}$ is the pressure and $\sigma(\mathbf{u},\mathrm{p}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$ is the stress tensor, with $\mathrm{Re}>0$. In particular, the strain rate tensor is defined by the symmetric part of the velocity gradient $\mathrm{D}\mathbf{u} = \dfrac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$.

On the physical part of the boundary Γ_1 we impose Dirichlet boundary conditions. On the "non-physical" part, Γ_2, we impose the artificial boundary condition

$$\sigma(\mathbf{u},\mathrm{p})\,\vec{\mathbf{n}} + \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{ref}) = \sigma_{ref}\,\vec{\mathbf{n}} \tag{III.2}$$

that was first introduced in [BF94] and then further studied in [BF96] and [BF12]. We use the notation $(a)^- = -\min(a,0)$. In order to build it, we need to choose some *reference flow* $\mathbf{u}_{ref}$, which is any $\mathbf{u}_{ref}\in(H^1(\Omega))^2$ such that $\mathbf{u}_{ref} = \mathbf{g}_1$ on Γ_1, chosen so as to be a reasonable approximation of the expected flow near Γ_2, and a *reference stress tensor* σ_{ref} such that $\sigma_{ref}\,\vec{\mathbf{n}}\in(H^{-\frac{1}{2}}(\Omega))^2$. This nonlinear condition is physically meaningful: if the flow is outward, we impose the constraint coming from the selected reference flow; if it is inward, we need to control the increase of energy, so we add a term that is quadratic with respect to velocity. Other techniques to model artificial boundaries have been studied during the years. For instance, in [HS89] an artificial boundary condition is designed for the Navier-Stokes equations under the hypothesis of small viscosity. The method consists into the approximation of the transparent boundary conditions, since they are non-local. The technique was then generalized to parabolic perturbations of hyperbolic systems in [Hal91] and to compressible flows in [Tou97]. We choose to work with the condition (III.2) of [BF94] since it is defined locally and it does not add hypothesis on the viscosity. It has been derived by a particular weak formulation of Navier-Stokes equation that ensures an energy estimate: we would like to reproduce the same property at a discrete level with the DDFV formalism.

The analysis of problem (III.1) is done in [BF96] and [BF07] from the continuous point of view and simulations are performed in [BF94] by the use of Finite Differences schemes in the case of Cartesian meshes. Within the framework of DDFV methods, we are able to reproduce those simulations by extending to the case of general meshes and we also offer a complete analysis of the discrete problem, perspective that was never addressed in the literature.

Outline. This chapter is organized as follows. In Section III.1 we show how approximate the nonlinear convection term. In Section III.2, we introduce the DDFV scheme for the Navier-Stokes problem (III.1) and we prove its well-posedness in Section III.3 (see Theorem (III.3.1)). In Section

III.4 we show an estimate of the convection term. In Section III.5, we prove a discrete energy estimate. Finally, in Section III.6, theoretical results are illustrated with numerical simulations.

III.1 Approximation of the nonlinear convection term

As in [Kre10, Kre11b], we construct a bilinear form $\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})$ as an approximation of $\int_{\Omega}(\mathbf{u}\cdot\nabla)\mathbf{v}$. The form introduced in [Kre10, Kre11b] is built in order to take into account homogeneous Dirichlet boundary conditions, so we need to modify it in order to handle the outflow condition (III.2).

To obtain the approximation of the convection term, we need to integrate the equation over the primal and dual mesh; we approximate $\int_{\mathtt{K}}(\mathbf{u}\cdot\nabla)\mathbf{v}$ when $\mathtt{K}\in\mathfrak{M}$ with $m_{\mathtt{K}}\,\mathbf{b}^{\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})$.
We remark that for $\mathbf{u}$ and $\mathbf{v}$ smooth functions:

$$\int_{\mathtt{K}}(\mathbf{u}\cdot\nabla)\mathbf{v} = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}}\int_{\sigma}(\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}})\mathbf{v}, \quad \forall\mathtt{K}\in\mathfrak{M}.$$

Such as for the Dirichlet case [Kre11b], we look for an approximation of the fluxes: $\int_{\sigma}(\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}) \rightsquigarrow F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})$. For the interior edges $\sigma\in\mathcal{E}_{int}$, we obtain them by calculating the fluxes on the sides $\mathfrak{s}$ of diamonds (see Fig. III.2). In fact, we remark that by integrating the solenoidal constraint on the semi-diamond $\mathtt{D}_{\mathtt{K}}$ of vertices $x_{\mathtt{K}}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}$ we have:

$$0 = \int_{\mathtt{D}_{\mathtt{K}}}\mathrm{div}(\mathbf{u}^{\mathfrak{T}})\mathrm{d}x = \int_{\sigma}\mathbf{u}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}} + \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}\int_{\mathfrak{s}}\mathbf{u}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}\mathrm{d}s.$$

The integral $\int_{\mathfrak{s}}\mathbf{u}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}\mathrm{d}s$ is approximated by $G_{\mathfrak{s},\mathtt{D}} = m_{\mathfrak{s}}\dfrac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}$, for $\mathfrak{s}=[x_{\mathtt{K}}, x_{\mathtt{K}^*}]=\mathtt{D}|\mathtt{D}'$, $\mathfrak{s}\in\mathcal{E}_{\mathtt{D}}$. Remark that those fluxes on the sides $\mathfrak{s}$ of diamonds are the same that we obtain by integrating the solenoidal constraint on each diamond $\mathtt{D}\in\mathfrak{D}$:

$$\int_{\mathtt{D}}\mathrm{div}(\mathbf{u}^{\mathfrak{T}})\mathrm{d}x = \sum_{\mathfrak{s}\in\partial\mathtt{D}}\int_{\mathfrak{s}}\mathbf{u}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}\mathrm{d}s,$$

that at a discrete level is written as:

$$m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathcal{E}_{\mathtt{D}}} G_{\mathfrak{s},\mathtt{D}}. \tag{III.3}$$

Remark III.1.1 *If* $\mathtt{D}\in\mathfrak{D}_{ext}$ *(see Fig. III.2),* $m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}})$ *can be rewritten as:*

$$m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathcal{E}_{\mathtt{D}}} G_{\mathfrak{s},\mathtt{D}} + m_{\sigma}\gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}.$$

For what concerns the boundary edges $\sigma\in\partial\Omega$, we replace $\mathbf{u}$ by its trace $\gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}})$. So we impose:

$$F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} G_{\mathfrak{s},\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) & \text{if } \sigma\in\mathcal{E}_{int} \\ \\ m_{\sigma}\gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}} & \text{if } \sigma\in\partial\Omega \end{cases} \tag{III.4}$$

and with an equivalent argument, we define for the dual edges:

$$F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}^*}\cap\mathcal{E}_{\mathtt{D}}} G_{\mathfrak{s},\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) & \text{if } \mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*,\ \sigma^*\cap\partial\Omega=\emptyset, \\ \\ -G_{\mathfrak{s},\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \dfrac{1}{2}F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) & \text{if } \mathtt{K}^*\in\partial\mathfrak{M}^*,\ \sigma^*\cap\partial\Omega\neq\emptyset, \text{i.e. } \mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}. \end{cases} \tag{III.5}$$

As illustrated in the *right* of Fig. III.2, for $\mathtt{K}^*\in\partial\mathfrak{M}^*$ such that $\sigma^*\cap\partial\Omega\neq\emptyset$, σ corresponds to the boundary edge $[x_{\mathtt{K}^*}, x_{\mathtt{L}^*}]$ in the diamond $\mathtt{D}_{\sigma,\sigma^*}$.

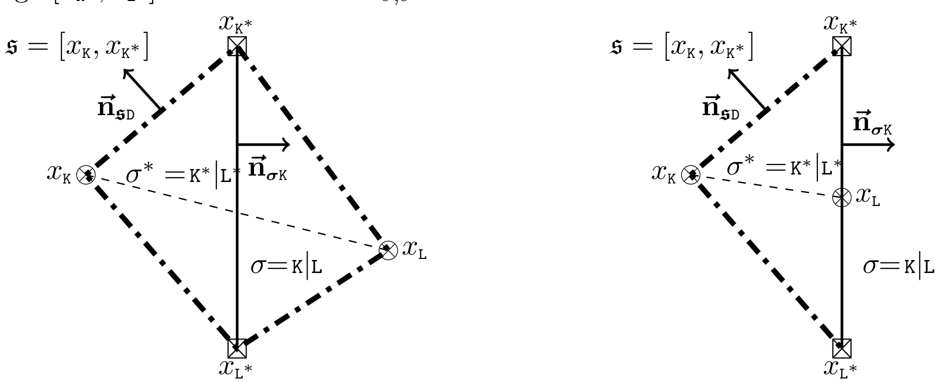


Fig. III.2 *Left:* A diamond $\mathtt{D}=\mathtt{D}_{\sigma,\sigma^*}$ with $\sigma\subset\mathcal{E}_{int}$. *Right:* A diamond $\mathtt{D}=\mathtt{D}_{\sigma,\sigma^*}$ with $\sigma\in\partial\Omega$.

Remark that thanks to the solenoidal constraint $\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}})=0$ we have conservativity of the fluxes $F_{\sigma\mathtt{K}}$ and $F_{\sigma^*\mathtt{K}^*}$:

$$\begin{aligned} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) &= -F_{\sigma\mathtt{L}}(\mathbf{u}^{\mathfrak{T}}), \qquad \forall\sigma=\mathtt{K}|\mathtt{L} \\ F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) &= -F_{\sigma^*\mathtt{L}^*}(\mathbf{u}^{\mathfrak{T}}), \quad \forall\sigma^*=\mathtt{K}^*|\mathtt{L}^*. \end{aligned} \tag{III.6}$$

Unlike in [Kre10, Kre11b], we do not stabilize the solenoidal constraint thus we do not need to add a stabilization term in the flux $G_{\mathfrak{s},\mathtt{D}}$; in fact, you can remark the link between the solenoidal constraint and $G_{\mathfrak{s},\mathtt{D}}$ in (III.3).

We define the bilinear form on the primal mesh as:

$$\boxed{m_{\mathtt{K}}\mathbf{b}^{\mathtt{K}}(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}) = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{int}_{\mathtt{K}}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_{\sigma} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\gamma^{\sigma}(\mathbf{v}^{\mathfrak{T}}) \qquad \forall\mathtt{K}\in\mathfrak{M}}$$

where

$$\mathbf{v}^+_{\sigma} = \begin{cases} \mathbf{v}_{\mathtt{K}} & \text{if } F_{\sigma\mathtt{K}}\geq 0 \\ \mathbf{v}_{\mathtt{L}} & \text{otherwise} \end{cases} \qquad \forall\sigma\in\mathcal{E}_{int},$$

and on the dual mesh as:

$$\boxed{\begin{aligned} m_{\mathtt{K}^*}\mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}) &= \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_{\sigma_*} && \forall\mathtt{K}^*\in\mathfrak{M}^* \\ m_{\mathtt{K}^*}\mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}) &= \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_{\sigma_*} + \frac{1}{2}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\gamma^{\sigma}(\mathbf{v}^{\mathfrak{T}}) && \forall\mathtt{K}^*\in\partial\mathfrak{M}^* \end{aligned}}$$

where

$$\mathbf{v}^+_{\sigma_*} = \begin{cases} \mathbf{v}_{\mathtt{K}^*} & \text{if } F_{\sigma^*\mathtt{K}^*}\geq 0 \\ \mathbf{v}_{\mathtt{L}^*} & \text{otherwise} \end{cases} \qquad \forall\sigma^*\in\mathcal{E}^*.$$

We choose to do upwinding on the interior diamonds because we started from the analysis done in [Kre10] and [Kre11b]. In the case of Dirichlet boundary conditions in [Kre10, Kre11b], it is necessary to upwind in order to get well-posedness of the scheme and an energy estimate, since it is the key to prove an inequality of the type $[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}),\mathbf{v}^{\mathfrak{T}}]]_{\mathfrak{T}} \geq 0$. In our case, in the weak formulation the convection term is skew-symmetrized, and we will see that upwind and centered discretizations lead to the same scheme, see Remark III.2.2.

Proposition III.1.2 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}}) \in (\mathbb{R}^2)^{\mathfrak{T}} \times \mathbb{R}^{\mathfrak{D}}$, we have:*

$$\begin{aligned}
&\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) = 0 && \forall \mathtt{K}\in\mathfrak{M}\\
&\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) = 0 && \forall \mathtt{K}^*\in\mathfrak{M}^*\\
&\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) = -\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}} \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}} && \forall \mathtt{K}^*\in\partial\mathfrak{M}^*
\end{aligned}$$

Proof For $\mathtt{K}\in\mathfrak{M}$, we distinguish two cases, depending on Def. (III.4).

• If $\mathtt{K}\cap\partial\Omega=\emptyset$, that means that $\forall \mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}$ $\sigma\in\mathcal{E}_{int}$, by reorganizing the sum on the sides $\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}$ belonging to the primal cell $\mathtt{K}$, we obtain:

$$-\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}}\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} = -\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}} m_{\mathfrak{s}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot(\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}+\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}'}) = 0 \tag{III.7}$$

since $\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} = -\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}'}$, where $\mathtt{D}$ and $\mathtt{D}'$ denote the two neighbor diamonds which share the edge $\mathfrak{s}$, of vertices $x_{\mathtt{K}}, x_{\mathtt{K}^*}$.

•If $\mathtt{K}\cap\partial\Omega\neq\emptyset$, we remark that, thanks to Rem. III.1.1 and since $\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}})=0$, for $\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}^{ext}$ we have:

$$m_\sigma\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} G_{\mathfrak{s},\mathtt{D}}.$$

So we get:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) = -\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}}\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} = 0$$

where we applied (III.7). We deduce that $\displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) = 0$ for all $\mathtt{K}\in\mathfrak{M}$.

The proof is similar for $\displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) = 0$ if $\mathtt{K}^*\in\mathfrak{M}^*$.

We now focus on the case in which $\mathtt{K}^*\in\partial\mathfrak{M}^*$; by Def. (III.5) of $F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})$:

$$-\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}}\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}^*}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} - \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}}\frac{1}{2}F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) = 0 - \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}}\frac{1}{2}F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}).$$

where the first sum is zero thanks to a similar argument to (III.7).

We deduce that $\displaystyle\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) = -\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}}\frac{1}{2}F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})$ for all $\mathtt{K}^*\in\partial\mathfrak{M}^*$. ■

III.2 DDFV scheme

Let $N\in\mathbb{N}^*$. We note $\delta t = \frac{T}{N}$ and $t_n = n\delta t$ for $n\in\{0,\dots,N\}$. To obtain the DDFV scheme, we choose to use an implicit Euler time discretization, except for the nonlinear term, which is

linearized by using a semi-implicit approximation.
We look for $\mathbf{u}^{\mathfrak{T},[0,T]} = (\mathbf{u}^n)_{n\in\{0,\dots N\}} \in (\mathbb{E}_{\mathbf{g}_1}^{\Gamma_1})^{N+1}$ and $\mathrm{p}^{\mathfrak{D},[0,T]} = (\mathrm{p}^n)_{n\in\{0,\dots N\}} \in (\mathbb{R}^{\mathfrak{D}})^{N+1}$, that we initialize with:

$$\mathbf{u}^0 = \mathbb{P}_c^{\mathfrak{T}} \mathbf{u}_0 \in \mathbb{E}_{\mathbf{g}_1}^{\Gamma_1} \tag{III.8}$$

where $\mathbb{P}_c^{\mathfrak{T}}$ is the centered projection defined in (I.2). We would like to write the system (III.1) in our setting.

For what concerns the **momentum equation**, we start by finding the discrete equivalent of the variational formulation of the problem. For the continuous problem, as presented in [BF12], the velocity $\mathbf{u}$ satisfies:

$$\begin{aligned}\int_\Omega \partial_t \mathbf{u}\cdot\Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\Psi) + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u} \\ = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})^+(\mathbf{u}\cdot\Psi) + \frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}_{ref}\cdot\Psi) + \int_{\Gamma_2} (\sigma_{ref}\vec{\mathbf{n}})\cdot\Psi,\end{aligned} \tag{III.9}$$

where Ψ is a test function in the space

$$V = \{\Psi \in (H^1(\Omega))^2,\ \Psi|_{\Gamma_1} = 0,\ \mathrm{div}(\Psi) = 0\}.$$

This weak formulation (III.9) can be rewritten in the DDFV framework (with the operators introduced in section I.1) as:

$$\begin{aligned}[[\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} &+ \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1} : \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} \\ &+ \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} \\ &= -\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma\mathtt{K}}(\mathbf{u}^n))^+\,\gamma^\sigma(\mathbf{u}^{n+1})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}) \\ &\quad + \frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma\mathtt{K}}(\mathbf{u}^n))^-\gamma^\sigma(\mathbf{u}_{ref})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}) \\ &\quad + \sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma_{\mathrm{ref}}^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}),\end{aligned} \tag{III.10}$$

where $\Psi^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ is a test function in the discrete space that satisfies similar properties compared to the continuous test function Ψ:

$$\Psi^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}, \quad \mathrm{div}^{\mathfrak{D}}(\Psi^{\mathfrak{T}}) = 0. \tag{III.11}$$

To simplify the computations, even though it is not necessary, as in the continuous case (see [BF12]), the reference flow $(\mathbf{u}_{ref}^{\mathfrak{T}}, \mathrm{p}_{ref}^{\mathfrak{D}}) \in \mathbb{E}_{\mathbf{g}_1}^{\Gamma_1} \times \mathbb{R}^{\mathfrak{D}}$ is supposed to be a solution of the under-determined steady Stokes problem:

$$\begin{cases} -\mathrm{div}^{\mathfrak{M}}\Big(\dfrac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}(\mathbf{u}_{ref}^{\mathfrak{T}}) - \mathrm{p}_{ref}^{\mathfrak{D}}\mathrm{Id}\Big) = 0, \\ -\mathrm{div}^{\mathfrak{M}^*\cup\mathfrak{M}_2^*}\Big(\dfrac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}(\mathbf{u}_{ref}^{\mathfrak{T}}) - \mathrm{p}_{ref}^{\mathfrak{D}}\mathrm{Id}\Big) = 0, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}_{ref}^{\mathfrak{T}}) = 0. \end{cases} \tag{III.12}$$

We also need to define a reference stress tensor $\sigma^{\mathtt{D}}_{\mathrm{ref}}$ on the outflow part of the boundary; we choose:

$$\sigma^{\mathtt{D}}_{\mathrm{ref}}\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}, \mathrm{p}^{\mathfrak{D}}_{ref})\vec{\mathbf{n}}_{\sigma\mathtt{K}} \quad \forall \sigma \in \partial\mathfrak{M}_2 \tag{III.13}$$

where the discrete stress tensor is defined by $\sigma^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}, \mathrm{p}^{\mathfrak{D}}_{ref}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}) - \mathrm{p}^{\mathfrak{D}}_{ref}\mathrm{Id}.$

From this formulation, we design our DDFV scheme. We project (III.10) on the mesh. For instance, to obtain the equation on the primal mesh, that is $\forall \mathtt{K} \in \mathfrak{M}$, we consider a primal cell $\mathtt{K}_0 \in \mathfrak{M}$ and we build $\Psi^{\mathfrak{T}}$ such that $\Psi_{\mathtt{K}_0} = 1$, $\Psi_{\mathtt{K}'} = 0$ for all $\mathtt{K}' \in \mathfrak{M}, \mathtt{K}' \neq \mathtt{K}_0$ and $\Psi_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*} = 0$; we then replace $\Psi^{\mathfrak{T}}$ in (III.10). We proceed in the same way for the dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*_2$ and the boundary mesh $\partial\mathfrak{M}_2$. The solenoidal constraint is projected on the diamond mesh $\mathfrak{D}$.

Remark III.2.1 *When projecting* (III.10) *on the boundary dual mesh, i.e. when considering* $\Psi^{\mathfrak{T}}$ *such that for a* $\mathtt{K}^*_0 \in \partial\mathfrak{M}^*_2$*,* $\Psi_{\mathtt{K}^*_0} = 1$*,* $\Psi_{\mathtt{K}^{*\prime}} = 0$ *for all* $\mathtt{K}^{*\prime} \in \mathfrak{M}, \mathtt{K}^{*\prime} \neq \mathtt{K}^*_0$ *and* $\Psi_{\mathfrak{M}\cup\partial\mathfrak{M}} = 0$*, remark that the term* $-\dfrac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} = -\dfrac{1}{2}\sum\limits_{\mathtt{K}\in\mathfrak{M}} \mathbf{b}^{\mathtt{K}}(\mathbf{u}^n, \Psi^{\mathfrak{T}})\mathbf{u}^{n+1}_{\mathtt{K}} - \dfrac{1}{2}\sum\limits_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^n, \Psi^{\mathfrak{T}})\mathbf{u}^{n+1}_{\mathtt{K}^*}$ *has also contributions that comes from the trace term* $\gamma^\sigma(\Psi^{\mathfrak{T}})$ *in* $\mathbf{b}^{\mathtt{K}}(\mathbf{u}^n, \Psi^{\mathfrak{T}})$*, which then cancel by replacing terms coming from* $(\mathcal{N}_3)$*. This is why the equations for* $\mathtt{K}^* \in \mathfrak{M}^*$ *and* $\mathtt{K}^* \in \partial\mathfrak{M}^*_2$ *can be written in the compact form* $(\mathcal{N}_2)$*.*

The resulting DDFV is the following:

- For all $\mathtt{K} \in \mathfrak{M}$:

$$m_{\mathtt{K}}\frac{\mathbf{u}^{n+1}_{\mathtt{K}} - \mathbf{u}^n_{\mathtt{K}}}{\delta t} - m_{\mathtt{K}}\mathbf{div}^{\mathtt{K}}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})) + \frac{1}{2}m_{\mathtt{K}}\mathbf{b}^{\mathtt{K}}(\mathbf{u}^n, \mathbf{u}^{n+1}) - \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^{int}_{\mathtt{K}}}\left(F^+_{\sigma\mathtt{K}}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{K}} - F^-_{\sigma\mathtt{L}}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{L}}\right) = 0; \tag{$\mathcal{N}_1$}$$

- For all $\mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*_2$:

$$m_{\mathtt{K}^*}\frac{\mathbf{u}^{n+1}_{\mathtt{K}^*} - \mathbf{u}^n_{\mathtt{K}^*}}{\delta t} - m_{\mathtt{K}^*}\mathbf{div}^{\mathtt{K}^*}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})) + \frac{1}{2}m_{\mathtt{K}^*}\mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^n, \mathbf{u}^{n+1}) - \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}\left(F^+_{\sigma^*\mathtt{K}^*}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{K}^*} - F^-_{\sigma^*\mathtt{L}^*}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{L}^*}\right) = 0; \tag{$\mathcal{N}_2$}$$

- For all $\mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_2$:

$$m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1}))\vec{\mathbf{n}}_{\sigma\mathtt{L}} - \frac{1}{4}F_{\sigma\mathtt{L}}(\mathbf{u}^n)\,(\mathbf{u}^{n+1}_{\mathtt{K}} - \mathbf{u}^{n+1}_{\mathtt{L}}) = -\frac{1}{2}(F_{\sigma\mathtt{L}}(\mathbf{u}^n))^-(\gamma^\sigma(\mathbf{u}^{n+1}) - \gamma^\sigma(\mathbf{u}_{ref})) + m_\sigma(\sigma^{\mathtt{D}}_{\mathrm{ref}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}); \tag{$\mathcal{N}_3$}$$

- For all $\mathtt{D} \in \mathfrak{D}$:

$$\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{n+1}) = 0. \tag{$\mathcal{N}_4$}$$

Remark III.2.2 *If we consider the definition of* $\mathbf{b}^{\mathtt{K}}(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}})$ *on the interior diamonds* $\mathfrak{D}_{\mathtt{K}}^{int}$, *we remark that it can be written as a centered discretization plus a diffusion term:*

$$m_{\mathtt{K}}\mathbf{b}^{\mathtt{K}}(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}) = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}^{int}} \left(F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\left(\frac{\mathbf{v}_{\mathtt{K}}+\mathbf{v}_{\mathtt{L}}}{2}\right) + B_{\sigma\mathtt{K}}(\mathbf{v}_{\mathtt{K}}-\mathbf{v}_{\mathtt{L}})\right) + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}^{ext}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\gamma^{\sigma}(\mathbf{v}^{\mathfrak{T}}),$$

where $B_{\sigma\mathtt{K}}$ *is defined by a function* B *of* $F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})$. *In particular, this formulation allows to generalize the results: if* $B_{\sigma\mathtt{K}}=0$ *we get a centered approximation, if* $B_{\sigma\mathtt{K}}=\frac{1}{2}|F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})|$ *it is an upwind scheme (this kind of generalization will be useful in Chap. IV).*

If we consider (III.10) *and we project it on* $\mathtt{K}\in\mathfrak{M}$ *with this new definition of convection, we get:*

$$m_{\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}^{n+1}-\mathbf{u}_{\mathtt{K}}^{n}}{\delta t} - m_{\mathtt{K}}\mathbf{div}^{\mathtt{K}}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1},p^{n+1})) + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^{int}} F_{\sigma\mathtt{K}}(\mathbf{u}^{n})\,\frac{\mathbf{u}_{\mathtt{L}}^{n+1}}{2}$$
$$+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^{int}}(B_{\sigma\mathtt{L}}(\mathbf{u}^{n})-B_{\sigma\mathtt{K}}(\mathbf{u}^{n}))\mathbf{u}_{\mathtt{L}}^{n+1} + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^{ext}} F_{\sigma\mathtt{K}}(\mathbf{u}^{n})\left(\gamma^{\sigma}(\mathbf{u}^{n+1})\right)=0.$$

Since $\forall\mathtt{K}\in\mathfrak{M}$, $\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}F_{\sigma\mathtt{K}}=0$ *by Prop. III.1.2, we can add the term* $\frac{1}{4}\mathbf{u}_{\mathtt{K}}^{n+1}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}F_{\sigma\mathtt{K}}$ *to the last expression and find:*

$$m_{\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}^{n+1}-\mathbf{u}_{\mathtt{K}}^{n}}{\delta t} - m_{\mathtt{K}}\mathbf{div}^{\mathtt{K}}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1},p^{n+1})) + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^{int}} F_{\sigma\mathtt{K}}(\mathbf{u}^{n})\,\frac{\mathbf{u}_{\mathtt{K}}^{n+1}+\mathbf{u}_{\mathtt{L}}^{n+1}}{2}$$
$$+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^{int}}(B_{\sigma\mathtt{L}}(\mathbf{u}^{n})-B_{\sigma\mathtt{K}}(\mathbf{u}^{n}))\mathbf{u}_{\mathtt{L}}^{n+1} + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^{ext}} F_{\sigma\mathtt{K}}(\mathbf{u}^{n})\underbrace{\left(\frac{1}{2}\mathbf{u}_{\mathtt{K}}^{n+1}+\gamma^{\sigma}(\mathbf{u}^{n+1})\right)}_{=\text{trace term on }\mathtt{D}}=0. \quad \text{(III.14)}$$

We observe that for both the centered and the upwind schemes, we have $B_{\sigma\mathtt{L}}=B_{\sigma\mathtt{K}}$, *and the schemes are thus equivalent to a centered discretization on the interior cells. This property is due to the skew-symmetrization of the convection term.*

For a dual cell $\mathtt{K}^*\in\partial\mathfrak{M}_2^*$, *the equation is similar to* (III.14), *except for the "trace term on* $\mathtt{D}$*", that becomes* $\dfrac{1}{2}\mathbf{u}_{\mathtt{K}^*}^{n+1}+\gamma^{\sigma}(\mathbf{u}^{n+1})$. *For an interior dual cell* $\mathtt{K}^*\in\mathfrak{M}^*$, *this trace term is zero.*

III.3 Well-posedness

We now prove the existence and uniqueness of the solution of our DDFV scheme.
The well-posedness result relies on a uniform discrete *Inf-sup condition*. We could have add a stabilization term to the equation of conservation of mass to generalize the result to general meshes, as done in [Kre11a, GKL17] for Stokes and in [Kre11b] for Navier-Stokes, but since our proof for Korn's inequality (that is crucial to prove the energy estimate) requires the hypothesis of Inf-sup stability, we decided not to stabilize the equation. This hypothesis does not add a lot of restriction on the choice of the mesh; see Sec. I.6 for more details.

III.3.1 Existence and uniqueness

Theorem III.3.1 *(Well-posedness) Let* $\mathfrak{T}$ *be a DDFV mesh associated to* Ω *that satisfies the Inf-sup stability condition. The scheme* (III.8), $(\mathcal{N}_1)$-$(\mathcal{N}_4)$ *has a unique solution* $(\mathbf{u}^{\mathfrak{T},[0,T]},p^{\mathfrak{D},[0,T]})\in(\mathbb{E}_{\boldsymbol{g}_1}^{\Gamma_1})^{N+1}\times(\mathbb{R}^{\mathfrak{D}})^{N+1}$.

Proof The scheme issued from the equations ($\mathcal{N}_1$)-($\mathcal{N}_4$) is a linear system $Av = b$ with A square matrix at each time step. We want to show that A is injective, thus we study the kernel of the matrix. Let $v = (\mathbf{u}^{n+1}, \mathrm{p}^{n+1}) \in \mathbb{E}_{\mathbf{g}_1}^{\Gamma_1} \times \mathbb{R}^{\mathfrak{D}}$ be in $ker(A)$: we then obtain the system $Av = 0$. If we multiply this relation by a test function $\Psi^{\mathfrak{T}}$ that satisfies (III.18), this is equivalent to consider the discrete variational formulation (III.10) in the form:

$$\begin{aligned}\frac{1}{\delta t}[[\mathbf{u}^{n+1}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} \\ + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} \\ = -\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma\mathtt{K}}(\mathbf{u}^n))^+ \gamma^\sigma(\mathbf{u}^{n+1}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}).\end{aligned}$$

The choice $\Psi^{\mathfrak{T}} = \mathbf{u}^{n+1}$ leads to:

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 + \underbrace{\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{u}^n))^+ |\gamma^\sigma(\mathbf{u}^{n+1})|^2}_{\geq 0} = 0,$$

that implies

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 \leq 0,$$

from which we deduce that $\mathbf{u}^{n+1} = 0$.
To conclude the proof, we need to show that p^{n+1} is equal to zero too. Since $\mathbf{u}^{n+1} = 0$, the momentum equation and the outflow boundary condition become:

$$\begin{cases} \mathbf{div}^{\mathtt{K}}\left(\mathrm{p}^{n+1}\mathrm{Id}\right) = 0 & \forall \mathtt{K} \in \mathfrak{M} \\ \mathbf{div}^{\mathtt{K}^*}\left(\mathrm{p}^{n+1}\mathrm{Id}\right) = 0 & \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_2^* \\ (\mathrm{p}^{n+1}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathtt{K}} = 0 & \forall \mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext} \cap \Gamma_2. \end{cases} \tag{III.15}$$

For every $\mathbf{v}^{n+1} \in \mathbb{E}_0^{\Gamma_D}$, thanks to Green's formula (Thm. I.5.1), we can write:

$$\left(\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{n+1}), \mathrm{p}^{n+1}\right)_{\mathfrak{D}} = -\left[\left[\mathbf{v}^{n+1}, \mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{n+1}\mathrm{Id})\right]\right]_{\mathfrak{T}} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{v}^{n+1}) \cdot \left(\mathrm{p}^{n+1}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathtt{K}}. \tag{III.16}$$

By definition of the scalar products (see Sec. I.4), by (III.15) and by the fact that $\mathbf{v}^{n+1} \in \mathbb{E}_0^{\Gamma_D}$, we get that $\left[\left[\mathbf{v}^{n+1}, \mathbf{div}^{\mathfrak{T}}(\mathrm{p}^{n+1}\mathrm{Id})\right]\right]_{\mathfrak{T}} = 0$ and $\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{v}^{n+1}) \cdot \left(\mathrm{p}^{n+1}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathtt{K}} = 0$. Thus (III.16) becomes:

$$\left(\mathrm{div}^{\mathfrak{D}}(\mathbf{v}^{n+1}), \mathrm{p}^{n+1}\right)_{\mathfrak{D}} = 0. \tag{III.17}$$

We now go back to inequality (I.30) ensured by Inf-sup stability: since (III.17) holds for any $\mathbf{v}^{n+1} \in \mathbb{E}_0^{\Gamma_D}$, the supremum in the right hand side of (I.30) vanishes and we can deduce that p^{n+1} is constant. Then, the condition ($\mathcal{N}_3$) on $\partial\mathfrak{M}_O$ implies that $\mathrm{p}^{n+1} = 0$ on the boundary, since we recall that $m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1}))\vec{\mathbf{n}}_{\sigma\mathtt{L}} = m_\sigma\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathtt{D}}(\mathbf{u}^{n+1}) - \mathrm{p}^{n+1}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathtt{L}}$, that $\mathbf{u}_{ref}^{\mathfrak{T}} = \sigma_{ref}^{\mathfrak{D}} = 0$ because we are studying $ker(A)$ and that $\mathbf{u}^{n+1} = 0$.
Thus, by putting together the fact that p^{n+1} is constant and it is zero on the boundary, we have $\mathrm{p}^{n+1} = 0$ in all the domain. ∎

Remark III.3.2 *Supposing Inf-sup condition is not that restrictive; just in the case of Cartesian meshes the stability is proved up to a checkerboard mode for the pressure, but thanks to the boundary conditions that we impose even in this case we can deduce that* $p^{n+1} = 0$*. Moreover, lots of numerical tests have been done and it still has not been found another mesh that does not satisfy the condition, see [BKN15].*

III.4 Property of the convection term

We need to prove the following estimate in order to establish a discrete energy estimate:

Proposition III.4.1 *Let* $\mathfrak{T}$ *be a DDFV mesh associated to* Ω*. For all* $(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}, \mathbf{w}^{\mathfrak{T}}) \in \mathbb{E}^{\Gamma_1}_{\boldsymbol{g}_1} \times \mathbb{E}^{\Gamma_1}_{\boldsymbol{g}_1} \times \mathbb{E}^{\Gamma_1}_{\boldsymbol{g}_1}$*, there exists a constant* $C > 0$ *that depends only on* Ω *and reg*$(\mathfrak{T})$ *such that:*

$$\begin{aligned}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} \leq & C \left(\|\mathbf{u}^{\mathfrak{T}}\|_3 + \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{3,\partial\Omega}\right) \|\mathbf{v}^{\mathfrak{T}}\|_6 \|\nabla^{\mathfrak{D}} \mathbf{w}^{\mathfrak{T}}\|_2 \\ & + C \,\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega} \|\gamma^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega} \|\widetilde{\gamma}^{\mathfrak{T}}(\mathbf{w}^{\mathfrak{T}})\|_{4,\partial\Omega}.\end{aligned}$$

Proof By the definition of the scalar product $[[\cdot,\cdot]]_{\mathfrak{T}}$ and of the convection term:

$$\begin{aligned}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} &= \frac{1}{2}\left(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}} \mathbf{w}_{\mathtt{K}} \cdot \mathbf{b}^{\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}) + \sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathtt{K}^*} \mathbf{w}_{\mathtt{K}^*} \cdot \mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})\right) \\ &= \frac{1}{2}\Bigg(\sum_{\mathtt{K}\in\mathfrak{M}} \mathbf{w}_{\mathtt{K}} \cdot \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{int}_{\mathtt{K}}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}^{+}_{\sigma} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) \gamma^{\sigma}(\mathbf{v}^{\mathfrak{T}})\Big) \\ &+ \sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \mathbf{w}_{\mathtt{K}^*} \cdot \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}^{+}_{\sigma^*} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) \frac{\gamma^{\sigma}(\mathbf{v}^{\mathfrak{T}})}{2}\Big)\Bigg).\end{aligned}$$

If we reorganize the sum over diamonds, since the fluxes are conservative (see (IV.6)), we get:

$$\begin{aligned}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} = \frac{1}{2}\Bigg(&\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}^{+}_{\sigma} \cdot (\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}}) \\ &+ 2 \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) \gamma^{\sigma}(\mathbf{v}^{\mathfrak{T}}) \cdot \widetilde{\gamma}^{\sigma}(\mathbf{w}^{\mathfrak{T}}) + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}^{+}_{\sigma^*} \cdot (\mathbf{w}_{\mathtt{K}^*} - \mathbf{w}_{\mathtt{L}^*})\Bigg) \\ &:= \frac{1}{2}(\mathrm{T}_1 + 2\mathrm{T}_2 + \mathrm{T}_3).\end{aligned}$$

<u>Estimate of T_1:</u>

By the definition of $\mathbf{v}^{+}_{\sigma}$, we have:

$$\begin{aligned}|\mathrm{T}_1| &= \Big|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}^{+}_{\sigma} \cdot (\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}})\Big| \\ &= \Big|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} \left(F^{+}_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\, \mathbf{v}_{\mathtt{K}} - F^{-}_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\, \mathbf{v}_{\mathtt{L}}\right) \cdot (\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}})\Big| \\ &\leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} |F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})||\mathbf{v}_{\mathtt{K}} + \mathbf{v}_{\mathtt{L}}||\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}}|.\end{aligned}$$

If we look at the flux $F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})$, we remark that $\forall\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^{int}$:

$$|F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})| = \Big| -\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}\Big| \le Cm_\sigma\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}\Big|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\Big|,$$

where C depends on $\mathrm{reg}(\mathfrak{T})$ (see (I.1)). We use this result in the estimate of T_1 to obtain:

$$|\mathrm{T}_1| \le C\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*}|\mathbf{v}_{\mathtt{K}}+\mathbf{v}_{\mathtt{L}}|\Big|\frac{\mathbf{w}_{\mathtt{K}}-\mathbf{w}_{\mathtt{L}}}{m_{\sigma^*}}\Big|\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}\Big|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\Big|.$$

We apply Hölder's inequality with $p=6$, $q=2$, $r=3$:

$$\begin{aligned}|\mathrm{T}_1| \le C\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*}|\mathbf{v}_{\mathtt{K}}+\mathbf{v}_{\mathtt{L}}|^6\Big)^{1/6}&\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*}\Big|\frac{\mathbf{w}_{\mathtt{K}}-\mathbf{w}_{\mathtt{L}}}{m_{\sigma^*}}\Big|^2\Big)^{1/2}\\ &\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*}\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}\Big|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\Big|^3\Big)^{1/3},\end{aligned}$$

and thanks to the definition (I.3.1) of the gradient operator and (I.1), we can write:

$$\begin{aligned}|\mathrm{T}_1| \le C\Big(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}|\mathbf{v}_{\mathtt{K}}|^6\Big)^{1/6}&\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}}|\nabla^{\mathtt{D}}\mathbf{w}^{\mathfrak{T}}|^2\Big)^{1/2}\\ &\cdot\Big(\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}|\mathbf{u}_{\mathtt{K}}|^3+\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*} m_{\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^3\Big)^{1/3}\\ &\le C\|\mathbf{v}^{\mathfrak{T}}\|_6\ \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\ \|\mathbf{u}^{\mathfrak{T}}\|_3.\end{aligned}$$

<u>Estimate of T_2:</u>

For what concerns boundary terms, the definition of fluxes changes (see (III.4)). Thus T_2 can be estimated by:

$$|\mathrm{T}_2| = \Big|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})\cdot\widetilde{\gamma}^\sigma(\mathbf{w}^{\mathfrak{T}})\Big| \le \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})||\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})||\widetilde{\gamma}^\sigma(\mathbf{w}^{\mathfrak{T}})|.$$

By applying Hölder's inequality with $p=\frac{8}{3}$, $q=\frac{8}{3}$, $r=4$ we get

$$\begin{aligned}|\mathrm{T}_2| \le \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|^{8/3}\Big)^{3/8}&\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})|^{8/3}\Big)^{3/8}\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\widetilde{\gamma}^\sigma(\mathbf{w}^{\mathfrak{T}})|^4\Big)^{1/4}\\ &\le \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega}\ \|\gamma^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega}\ \|\widetilde{\gamma}^{\mathfrak{T}}(\mathbf{w}^{\mathfrak{T}})\|_{4,\partial\Omega}.\end{aligned}$$

<u>Estimate of T_3:</u>

As we did for T_1, by the definition of $\mathbf{v}_{\sigma^*}^+$, we have:

$$|\mathrm{T}_3| \le \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} |F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})||\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}||\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}|.$$

By the definition of the flux (see (III.5)), this term can be split into two contributions:

$$|\mathrm{T}_3| \leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} |F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})||\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}||\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}| \\ + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} |F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})||\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}||\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}| = \mathrm{T}_3^1+\mathrm{T}_3^2.$$

For what concerns the estimate of T_3^1, the definition of the flux $F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})$ is the same as the one of $F_{\sigma\mathtt{K}}$ when $\sigma\in\mathcal{E}_{int}$.
Thus we can proceed as for the estimate of T_1 and we get:

$$\mathrm{T}_3^1 \leq C\,\|\mathbf{v}^{\mathfrak{T}}\|_6\ \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\ \|\mathbf{u}^{\mathfrak{T}}\|_3.$$

For the term T_3^2, the definition of the flux changes and we can estimate it by:

$$|F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})| = \Big| - \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} - \frac{1}{2}m_\sigma\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}\Big| \\ \leq Cm_\sigma\left(\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}\left|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\right| + |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|\right).$$

In this case, we can write:

$$|\mathrm{T}_3^2| \leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2|\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}|\left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right|\left(\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}\left|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\right| + |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|\right).$$

We split the right hand side into two terms. The first one is estimated exactly as T_1:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2|\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}|\left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right|\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}\left|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\right| \leq C\,\|\mathbf{v}^{\mathfrak{T}}\|_6\ \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\ \|\mathbf{u}^{\mathfrak{T}}\|_3.$$

For the second one, we apply Hölder's inequality with $p=6$, $q=2$, $r=3$ and we obtain:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2|\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}|\left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right||\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})| \\ \leq C\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2|\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}|^6\right)^{\frac{1}{6}}\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2\left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right|^2\right)^{\frac{1}{2}}\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|^3\right)^{\frac{1}{3}} \\ \leq C\left(\sum_{\mathtt{K}\in\mathfrak{M}^*} m_{\mathtt{K}^*}|\mathbf{v}_{\mathtt{K}^*}|^6\right)^{\frac{1}{6}}\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}}|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}|^2\right)^{\frac{1}{2}}\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|^3\right)^{\frac{1}{3}} \\ \leq C\|\mathbf{v}^{\mathfrak{T}}\|_6\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{3,\partial\Omega}.$$

By collecting the estimates we find the announced result:

$$\mathrm{T}_3 = \mathrm{T}_3^1+\mathrm{T}_3^2 \leq C\,\|\mathbf{v}^{\mathfrak{T}}\|_6\ \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\ (\|\mathbf{u}^{\mathfrak{T}}\|_3 + \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{3,\partial\Omega}).$$

■

III.5 Discrete energy estimate

The open boundary condition (III.2) that we study is derived from a weak formulation of the Navier-Stokes equation that ensures an energy estimate, presented in [BF96]. In this section we prove a discrete version of the energy estimate.
In order to do so, we will need to consider the variational formulation (III.10) and select the solution as a test function. Since the solution $\mathbf{u}^{\mathfrak{T},[0,T]}$ is not zero on the Dirichlet boundary Γ_1, it does not satisfy the hypothesis (III.18). We decompose it as $\mathbf{u}^{\mathfrak{T},[0,T]} = \mathbf{v}^{\mathfrak{T},[0,T]} + \mathbf{u}^{\mathfrak{T}}_{ref}$ and, thanks to the definition of $\mathbf{u}^{\mathfrak{T}}_{ref}$ (see (III.12)) , $\mathbf{v}^{\mathfrak{T},[0,T]}$ will be a good candidate to be our test function.

Theorem III.5.1 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies Inf-sup stability condition. Let $(\mathbf{u}^{\mathfrak{T},[0,T]}, p^{\mathfrak{D},[0,T]}) \in (\mathbb{E}^{\Gamma_1}_{\boldsymbol{g}_1})^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$ be the solution of the DDFV scheme* (III.8)*,* $(\mathcal{N}_1)$-$(\mathcal{N}_4)$ *, where $\mathbf{u}^{\mathfrak{T},[0,T]} = \mathbf{v}^{\mathfrak{T},[0,T]} + \mathbf{u}^{\mathfrak{T}}_{ref}$.*
For $N > 1$, there exists a constant $C > 0$, depending on $\Omega, reg(\mathfrak{T}), \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{u}_0, Re$ and T such that:

$$\sum_{j=0}^{N-1} \|\mathbf{v}^{j+1} - \mathbf{v}^j\|_2^2 \le C, \quad \|\mathbf{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1} \delta t \frac{1}{Re} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^{j+1}\|_2^2 \le C, \quad \delta t \frac{1}{Re} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1} \delta t \sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^j + \mathbf{u}^{\mathfrak{T}}_{ref}))^+ \left|\gamma^\sigma(\mathbf{v}^{j+1})\right|^2 \le C.$$

Proof The first step to obtain the energy inequality consists in rewriting the variational formulation (III.10) for the unknown $\mathbf{v}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^{\mathfrak{T}}_{ref}$. Let $\Psi^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ be a test function such that

$$\Psi^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_D}, \quad \mathrm{div}^{\mathfrak{D}}(\Psi^{\mathfrak{T}}) = 0. \tag{III.18}$$

We recall that $(\mathbf{u}^{\mathfrak{T}}_{ref}, \mathrm{p}^{\mathfrak{D}}_{ref})$ is a solution of the steady Stokes problem (III.12), so in particular

$$0 = -\left[\left[\mathbf{div}^{\mathfrak{T}}\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}) - \mathrm{p}^{\mathfrak{D}}_{ref}\mathrm{Id}\right), \Psi^{\mathfrak{T}}\right]\right]_{\mathfrak{T}};$$

by Green formula (Thm. I.5.1) we have:

$$\begin{aligned} 0 = -\left[\left[\mathbf{div}^{\mathfrak{T}}\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}) - \mathrm{p}^{\mathfrak{D}}_{ref}\mathrm{Id}\right), \Psi^{\mathfrak{T}}\right]\right]_{\mathfrak{T}} \\ = \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}_{ref}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} - \sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}, \mathrm{p}^{\mathfrak{D}}_{ref})\vec{\mathbf{n}}_{\sigma\mathtt{K}}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}). \end{aligned}$$

Then, since $\sigma^{\mathfrak{D}}_{\mathrm{ref}}$ is given by (III.13), the following holds:

$$0 = \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}_{ref}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} - \sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma^{\mathtt{D}}_{\mathrm{ref}}\vec{\mathbf{n}}_{\sigma\mathtt{K}}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}).$$

This implies that (III.10), for the unknown $\mathbf{v}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^{\mathfrak{T}}_{ref}$, becomes:

$$\begin{aligned}
&[[\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\delta t}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} \\
&\quad + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{v}^{n+1} + \mathbf{u}^{\mathfrak{T}}_{ref}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}, \Psi^{\mathfrak{T}}), \mathbf{v}^{n+1} + \mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}} \\
&\qquad + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}))^+ \gamma^\sigma(\mathbf{v}^{n+1} + \mathbf{u}^{\mathfrak{T}}_{ref}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}) \\
&\qquad\qquad = -\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} F_{\sigma\mathtt{K}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref})^- \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}).
\end{aligned}$$

The second step consists in selecting $\Psi^{\mathfrak{T}} = (\mathbf{v}^{n+1} + \mathbf{u}^{\mathfrak{T}}_{ref}) - \mathbf{u}^{\mathfrak{T}}_{ref}$ as a test function. If we define:

$$E := [[\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\delta t}, \mathbf{v}^{n+1}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}))^+ \left|\gamma^\sigma(\mathbf{v}^{n+1})\right|^2,$$

it follows that:

$$\begin{aligned}
E \leq &\left|\frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{v}^{n+1}), \mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{u}^{\mathfrak{T}}_{ref}), \mathbf{v}^{n+1}]]_{\mathfrak{T}}\right| \\
&\qquad + \left|\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} F_{\sigma\mathtt{K}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref})^- \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref}) \cdot \gamma^\sigma(\mathbf{v}^{n+1})\right|.
\end{aligned}$$

We apply Proposition III.4.1 to the convection terms $[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{v}^{n+1}), \mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}}$ and $[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}, \mathbf{u}^{\mathfrak{T}}_{ref}), \mathbf{v}^{n+1}]]_{\mathfrak{T}}$; for what concerns the boundary term, thanks to the definition of $F_{\sigma\mathtt{K}}$ for $\sigma \in \partial\Omega$, we have:

$$\left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} F_{\sigma\mathtt{K}}\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref}) \cdot \gamma^\sigma(\mathbf{v}^{n+1})\right| \leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma |\gamma^\sigma(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref})||\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref})||\gamma^\sigma(\mathbf{v}^{n+1})|$$

and by applying Hölder's inequality with $p = \frac{8}{3}$, $q = \frac{8}{3}$, $r = 4$ we get:

$$\left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} F_{\sigma\mathtt{K}}\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref}) \cdot \gamma^\sigma(\mathbf{v}^{n+1})\right| \leq C\|\gamma^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{v}^{n+1})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}_{ref})\|_{4,\partial\Omega}.$$

Thus we are led to:

$$\begin{aligned}
E \leq C&\Big(\|\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}\|_3 + \|\gamma^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref})\|_{3,\partial\Omega}\Big) \\
&\qquad \cdot \Big(\|\mathbf{v}^{n+1}\|_6 \ \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}_{ref}\|_2 + \|\mathbf{u}^{\mathfrak{T}}_{ref}\|_6 \ \|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2\Big) \\
&\qquad\quad + C\|\gamma^{\mathfrak{T}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{v}^{n+1})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}_{ref})\|_{4,\partial\Omega}.
\end{aligned}$$

By Sobolev inequalities of [BCCHF15, Theorem 9], we bound $\|\mathbf{v}^n\|_3$ from above by $C\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathbf{v}^n\|_2^{\frac{2}{3}}$ and $\|\mathbf{v}^{n+1}\|_6$ by $C\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^{\frac{2}{3}}\|\mathbf{v}^{n+1}\|_2^{\frac{1}{3}}$. Moreover, thanks to the trace theorem (Thm. I.9.2) and to [BCCHF15, Theorem 9], we dominate $\|\gamma^{\mathfrak{T}}(\mathbf{v}^{n+1})\|_{\frac{8}{3},\partial\Omega}$ and $\|\gamma^{\mathfrak{T}}(\mathbf{v}^n)\|_{3,\partial\Omega}$ by $C\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^{\frac{5}{8}}\|\mathbf{v}^{n+1}\|_2^{\frac{3}{8}}$ and $C\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathbf{v}^n\|_2^{\frac{1}{3}}$.

We then apply the discrete Poincaré inequality, Theorem I.9.1, to get rid of the norms of $\mathbf{v}^{n+1}$. Finally we recall that $\mathbf{u}^{\mathfrak{T}}_{ref}$ is a fixed reference steady flow. Hence there exists a constant $C > 0$ that depends only on Ω, $\mathrm{reg}(\mathfrak{T})$ and $\mathbf{u}^{\mathfrak{T}}_{ref}$ such that:

$$E \leq C\Big(2\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathbf{v}^n\|_2^{\frac{2}{3}}\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + 2\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathbf{v}^n\|_2^{\frac{1}{3}}\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 \\ + 5\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + \|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{5}{8}}\|\mathbf{v}^n\|_2^{\frac{3}{8}}\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2\Big).$$

We control the norm of the gradients with the norms of $\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}$ and $\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n$ thanks to Korn's inequality, Theorem I.8.2:

$$E \leq C\Big(2\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + 2\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 \\ + 5\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + \|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{5}{8}}\|\mathbf{v}^n\|_2^{\frac{3}{8}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2\Big).$$

Hence, by suitable use of Young's inequality (Lemma I.11.1) we end up with:

$$[[\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\delta t}, \mathbf{v}^{n+1}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}))^+ \left|\gamma^\sigma(\mathbf{v}^{n+1})\right|^2 \\ \leq 16\mathrm{Re}^2C^3\|\mathbf{v}^n\|_2^2 + \frac{1}{2\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^2 + \frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2 + \frac{25}{2}\mathrm{Re}C^2$$

We combine $\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2$ with the left hand side, we multiply this relation by δt and we apply

$$2[[\mathbf{v}^{n+1}-\mathbf{v}^n, \mathbf{v}^{n+1}]]_{\mathfrak{T}} = \|\mathbf{v}^{n+1}-\mathbf{v}^n\|_2^2 + \|\mathbf{v}^{n+1}\|_2^2 - \|\mathbf{v}^n\|_2^2.$$

We obtain:

$$\|\mathbf{v}^{n+1}-\mathbf{v}^n\|_2^2 + \|\mathbf{v}^{n+1}\|_2^2 + \delta t\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2 + \delta t\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}))^+ \left|\gamma^\sigma(\mathbf{v}^{n+1})\right|^2 \\ \leq \|\mathbf{v}^n\|_2^2 + 32\mathrm{Re}^2C^3\delta t\|\mathbf{v}^n\|_2^2 + \frac{1}{\mathrm{Re}}\delta t\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^2 + 25\mathrm{Re}C^2\delta t.$$

We sum over $n = 0\ldots m-1$ with $m \in \{1\ldots N\}$ to obtain:

$$\sum_{n=0}^{m-1}\|\mathbf{v}^{n+1}-\mathbf{v}^n\|_2^2 + \|\mathbf{v}^m\|_2^2 + \sum_{n=0}^{m-1}\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2 \\ + \delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^m\|_2^2 + \sum_{n=0}^{m-1}\delta t\sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^n + \mathbf{u}^{\mathfrak{T}}_{ref}))^+ |\gamma^\sigma(\mathbf{v}^{n+1})|^2 \\ \leq \|\mathbf{v}^0\|_2^2 + \frac{1}{\mathrm{Re}}\delta t\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^0\|_2^2 + 25\mathrm{Re}C^2T + 32\mathrm{Re}^2C^3\delta t\sum_{n=0}^{m-1}\|\mathbf{v}^n\|_2^2. \quad \text{(III.19)}$$

We can now apply Grönwall's lemma (Lemma I.11.2), with:

$$a_0 := \|\mathbf{v}^0\|_2^2 + \frac{1}{\mathrm{Re}}\delta t\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^0\|_2^2$$

$$a_m := \sum_{j=0}^{m-1} \|\mathbf{v}^{j+1} - \mathbf{v}^j\|_2^2 + \|\mathbf{v}^m\|_2^2 + \sum_{j=0}^{m-1} \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^{j+1}\|_2^2$$
$$+ \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^m\|_2^2 + \sum_{j=0}^{m-1} \delta t \sum_{\mathtt{D} \in \mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^j + \mathbf{u}_{ref}^{\mathfrak{T}}))^+ \left|\gamma^\sigma(\mathbf{v}^{j+1})\right|^2$$

for $m = \{1 \dots N\}$. In fact, we deduce from (III.19) that

$$a_m \leq \underbrace{a_0 + 25\mathrm{Re}C^2 T}_{:=A} + \underbrace{32\mathrm{Re}^2 C^3}_{:=B} \delta t \sum_{i=0}^{m-1} a_i,$$

that implies:

$$\max_{m=1\dots N} a_m \leq A e^{BT}.$$

This proves our initial statement, since we can choose $m = N$ and write:

$$\sum_{j=0}^{N-1} \|\mathbf{v}^{j+1} - \mathbf{v}^j\|_2^2 + \|\mathbf{v}^N\|_2^2 + \sum_{j=0}^{N-1} \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^{j+1}\|_2^2$$
$$+ \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}^N\|_2^2 + \sum_{j=0}^{N-1} \delta t \sum_{\mathtt{D} \in \mathfrak{D}_{ext}} (F_{\sigma\mathtt{K}}(\mathbf{v}^j + \mathbf{u}_{ref}^{\mathfrak{T}})^+ \left|\gamma^\sigma(\mathbf{v}^{j+1})\right|^2 \leq C(T).$$

■

III.6 Numerical results

We validate the scheme through a series of numerical experiments. First, we study numerically the consistency properties of the scheme. Second, we reproduce the simulations of a flow in a channel presented in [BF94] and [Joh04].

III.6.1 Convergence results

Test case 1. The computational domain is $\Omega = [0,1]^2$, whose boundary is divided into $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We impose Dirichlet boundary conditions on Γ_1, composed by the two horizontal boundaries and the left vertical one. The open boundary condition (III.2) is imposed on Γ_2, the right vertical boundary. We set the viscosity to 1.
For the tests we give the expression of the exact solution ($\mathbf{u}$,p), from which we deduce a source term $\mathbf{f}$ for the momentum equation and the Dirichlet boundary condition g_1. As a reference flow $(\mathbf{u}_{ref}^{\mathfrak{T}}, \mathrm{p}_{ref}^{\mathfrak{D}})$, we consider the projection of the exact solution on Γ_2. We will compare the $L^\infty(L^2)$-norm of the error (difference between a centered projection of the exact solution and the approximated solution obtained with DDFV scheme) for the velocity (denoted Ervel), the $L^2(L^2)$-norm of the error for the velocity gradient (Ergradvel) and the pressure (Erpre). In particular we denote:

$$\mathrm{Ergradvel} = \frac{\left(\sum_{n=0}^{N} \delta t \|\nabla^{\mathfrak{D}} (\mathbb{P}_c^{\mathfrak{T}} \mathbf{u})^n - \nabla^{\mathfrak{D}} \mathbf{u}^n\|_2^2\right)^{1/2}}{\left(\sum_{n=0}^{N} \delta t \|\nabla^{\mathfrak{D}} \left(\mathbb{P}_c^{\mathfrak{T}} \mathbf{u}\right)^n \|_2^2\right)^{1/2}}$$

$$\text{Erpre} = \frac{\left(\sum_{n=0}^{N} \delta t \|(\mathbb{P}_c^{\mathfrak{D}}\mathrm{p})^n - \mathrm{p}^n\|_2^2\right)^{1/2}}{\left(\sum_{n=0}^{N} \delta t \| \left(\mathbb{P}_c^{\mathfrak{D}}\mathrm{p}\right)^n \|_2^2\right)^{1/2}}, \quad \text{Ervel} = \frac{\max_{n=0\ldots N} \| (\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})^n - \mathbf{u}^n\|_2}{\max_{n=1\ldots N} \| (\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})^n \|_2},$$

where $(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})^n$ and $(\mathbb{P}_c^{\mathfrak{D}}\mathrm{p})^n$ are the centered projections of $\mathbf{u}$ and p at the time step $t_n = n\delta t$.
On Table III.1 we give the number of primal cells (NbCell) and the convergence rates (Ratio). We remark that, to discuss the error estimates, a family of meshes (Fig. III.3) is obtained by refining successively and uniformly the original mesh. The exact solution is:

$$\mathbf{u}(x,y) = \begin{pmatrix} -2\pi\cos(\pi x)\sin(2\pi y)\exp(-5\eta t\pi^2), \\ \pi\sin(\pi x)\cos(2\pi y)\exp(-5\eta t\pi^2) \end{pmatrix},$$
$$\mathrm{p}(x,y) = -\frac{\pi^2}{4}(4\cos(2\pi x) + \cos(4\pi y))\exp(-10t\eta\pi^2).$$

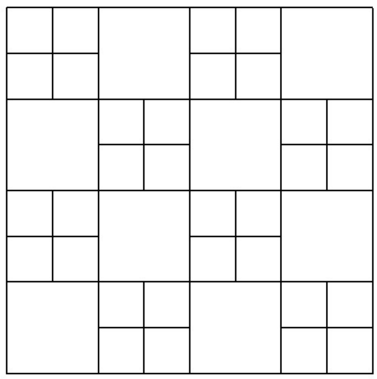


Fig. III.3 Non conformal square mesh.

The final time is T = 0.03 and we set $\delta t = 3 \times 10^{-5}$. As we can see in Table III.1, we observe super convergence in $L^\infty(L^2)$ norm of the velocity, that is a classical result for Finite Volume methods. For what concerns the gradient of the velocity and the pressure, we remark that the non-conformity of the mesh does not influence the good convergence of the method. We get a first order accuracy on the velocity gradient, and an order of 1.5 for the pressure, that is better than what we expected.

Table III.1 Test case 1 on the non conformal square mesh Fig. III.3.

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
64	1.424E-01	-	1.612E-01	-	6.127E+00	-
208	4.095E-02	1.80	7.316E-02	1.14	1.725E+00	1.83
736	1.019E-02	2.00	3.489E-02	1.07	5.836E-01	1.56
2752	2.559E-03	1.99	1.710E-02	1.03	1.947E-01	1.58
10624	6.493E-04	**1.98**	8.474E-03	**1.01**	6.189E-02	**1.65**

We tested many other meshes and the results do not change. The geometry of the mesh does not influence the accuracy of the approximation.

III.6.2 Simulations of a flow in a pipe

Figure III.4 describes the situation we are dealing with: we consider Ω, a connected bounded polygonal domain of $\mathbb{R}^2$, whose boundary $\partial\Omega$ is split into Γ_0, Γ_1 and Γ_2 and whose outer normal is denoted by $\vec{\mathbf{n}}$. We add a cylindrical obstacle inside Ω. The Dirichlet part of the boundary is composed by Γ_0 and Γ_1: on the physical boundary Γ_0 we impose no slip boundary conditions (i.e. the velocity is set to zero) and on the inflow boundary Γ_1 the velocity is prescribed. On the *artificial* boundary Γ_2, that we wish to set as close as possible to the obstacle, we impose the nonlinear boundary condition (III.2).

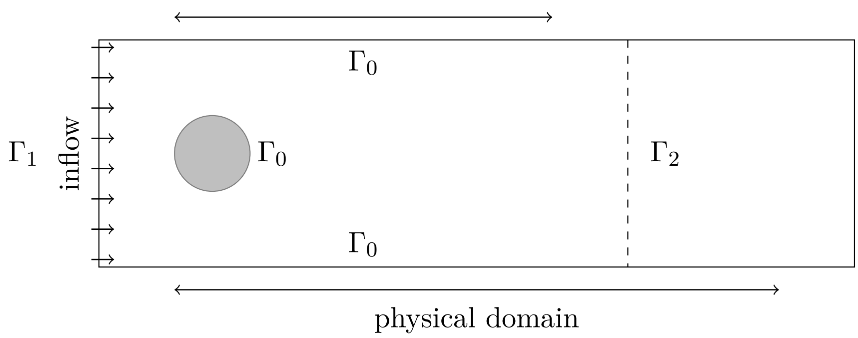


Fig. III.4 Domain and notations.

We reproduce two different test cases, proposed in [BF94] and in [Joh04]. In both cases, the simulations are performed on a triangular mesh, generated by GMSH, that is locally refined around the cylinder.

Test case 2. We show that by adding an artificial boundary, thanks to condition (III.2), we do not introduce any perturbation to the flow. For this purpose, we consider an original domain that we cut into two smaller domains and we draw the streamlines of the respective solution.
We consider the symmetric domain $\Omega = [0, 5] \times [0, 1]$ with a cylindrical obstacle of diameter $L = 0.4$. The smaller domains are obtained by cutting at the horizontal axis first in $x = 3$, then in $x = 1.5$. The mesh for Ω is composed by 12118 cells, and we pass to $\Omega' = [0, 3] \times [0, 1]$ with 8636 cells and to $\Omega'' = [0, 1.5] \times [0, 1]$ with 6534 cells. The time step is $\delta t = 0.035$. The inflow on Γ_1 is:

$$\mathbf{g}_1 = (6y(1 - y), 0).$$

Since our first simulations are performed with $\mathrm{Re} = 250$, it makes sense to set as reference flow a Poiseuille flow. Therefore we choose $\mathbf{u}_{ref} = \mathbf{g}_1$, $\mathrm{p}_{ref} = 0$ and $\sigma_{ref}(\mathbf{u}, \mathrm{p})\,\vec{\mathbf{n}} = (0, \frac{6}{\mathrm{Re}}(1 - 2y))$. As initial condition, we impose $\mathbf{u}_{init} = \mathbf{g}_1$ and the final time is $T = 3.5$. If $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_2)$, the stream

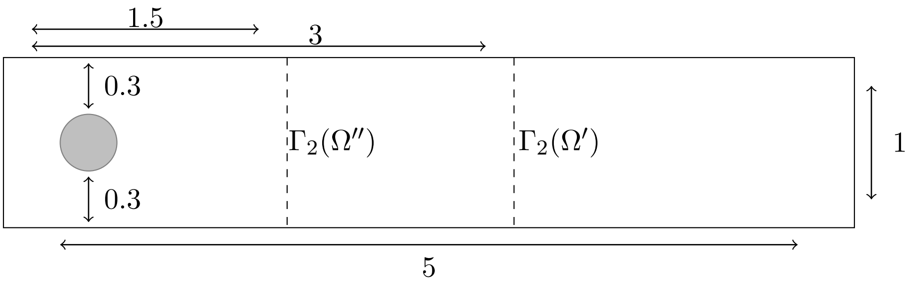


Fig. III.5 Domains $\Omega = [0,5]\times[0,1]$, $\Omega' = [0,3]\times[0,1]$ and $\Omega'' = [0,1.5]\times[0,1]$ for the test case 2.

function Ψ is defined in the continuous setting as the solution of

$$\frac{\partial \Psi}{\partial y} = \mathrm{u}_1, \quad \frac{\partial \Psi}{\partial x} = -\mathrm{u}_2,$$

in particular it has to satisfy the following system:

$$\begin{cases} \Delta\Psi = \mathrm{rot}(\mathbf{u}) & \text{in } \Omega \\ \nabla\Psi\cdot\vec{\mathbf{n}} = \mathbf{u}\cdot\vec{\tau} & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{rot}(\mathbf{u}) = -\dfrac{\partial \mathrm{u}_1}{\partial x} + \dfrac{\partial \mathrm{u}_1}{\partial y}$, $\vec{\mathbf{n}}$ is the outward unit normal to the domain and $\vec{\tau}$ is the unitary tangent to the boundary. In the DDFV setting, given the discrete solution $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathfrak{T}} \times \mathbb{R}^{\mathfrak{D}}$, we look for $\Psi^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ solution of:

$$\begin{cases} \mathrm{div}^{\mathfrak{D}}\boldsymbol{\nabla}^{\mathfrak{T}}\Psi^{\mathfrak{D}} = \mathrm{rot}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) \\ \nabla^{\mathtt{K}}\Psi^{\mathtt{D}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*} \quad \forall \mathtt{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}. \end{cases}$$

We observe by the streamlines in Fig. III.6 (at T=1.5) and Fig. III.7 (at T=3.5), for which we point out that the scale is the same in all the sub-figures, that we can cut close to the obstacle without adding any perturbation to the whole flow. The recirculations are well located and there is no spurious vortices. Clearly, the closer we cut, the more we loose in precision in the cells right before the artificial boundary. This is due to the artificiality of the conditions and to the choice of the reference flow. But in any case, the boundary can cut the recirculation right in the middle without affecting the whole flow.
The choice of the *reference flow* is crucial. In [BF94], it is proposed to use a Poiseuille flow as we reproduce in our numerical tests of Fig. III.6 and Fig. III.7. In [Bru00], since to write down the variational formulation (III.9) the reference flow is assumed to be the solution of a steady Stokes problem with $\mathbf{u} = \mathbf{g}_1$ on Γ_1, the author chooses the flow at infinity: $\mathbf{u}_{ref} = \mathbf{u}_\infty$, $\sigma_{ref} = \sigma_\infty$. Nevertheless, when the flow is chaotic or turbulent such a reference flow does not give a good equivalent of the traction. Thus for higher Reynold's numbers, such as Re $= 1000$ as in Fig. III.8 ($T = 1.5$) Fig. III.9 ($T = 3.5$), other techniques can be envisaged for the choice of the reference flow; for example, it looks reasonable to choose a reference flow that changes with time.

We might think that a good approximation of the solution at the boundary Γ_2 is the solution computed at the previous time step (or even just before the boundary at the same time step), but actually we numerically observed that these techniques lead to strong instabilities.
Remark that by replacing $\mathbf{u}^{\mathfrak{T}}_{ref}$ with $\mathbf{u}^n$, the energy stability is no longer guaranteed. Therefore, in

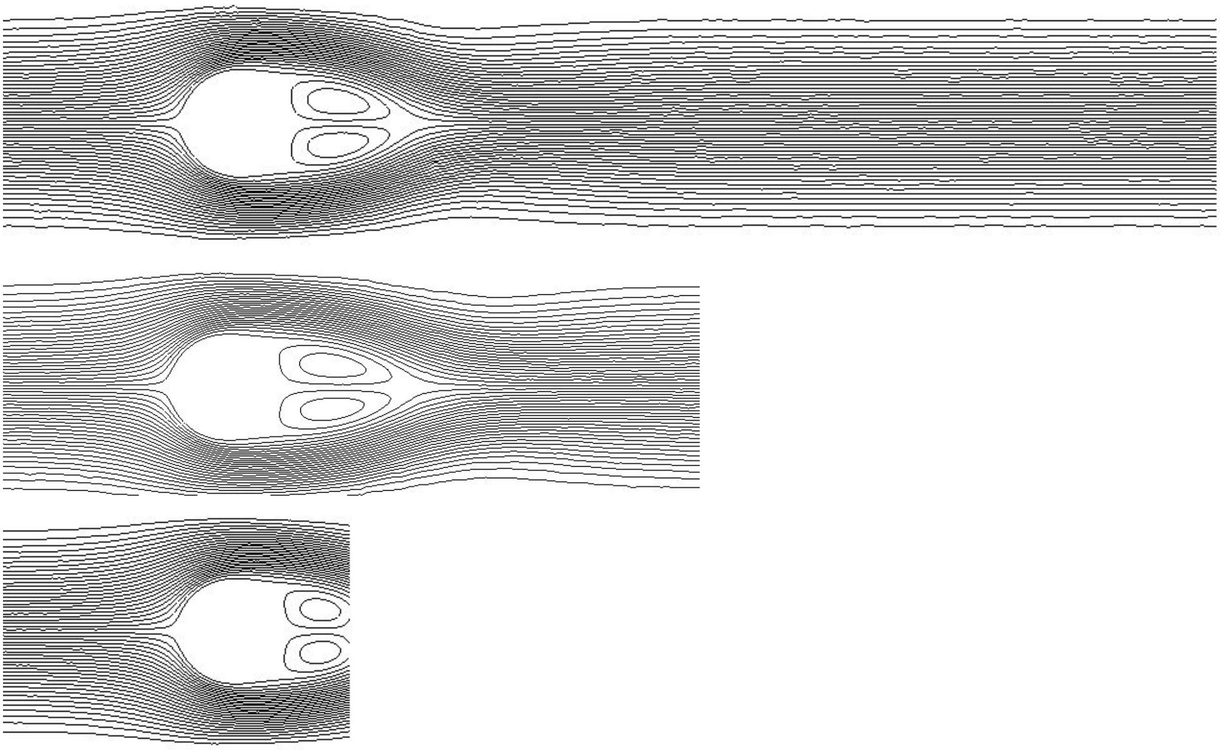


Fig. III.6 Streamline of Test case 2 at $T = 1.5$, Re $= 250$, $\delta t = 0.035$. On the top: $\Omega = [0, 5] \times [0, 1]$, NbCell$= 12118$. In the middle: $\Omega' = [0, 3] \times [0, 1]$, NbCell=8636. On the bottom: $\Omega'' = [0, 1.5] \times [0, 1]$, NbCell=6534.

order to avoid numerical instabilities, for the simulations of Fig. III.8, Fig. III.9 we do not update the value of $\mathbf{u}_{ref}^{\mathfrak{T}}$ at each time step, but each 400 iterations. This choice is arbitrary and we cannot provide a generalization to all the values of Reynold's number. A way to overcome this difficulty could be to compute the flow on a strictly larger domain (with respect to the smaller one) with a less refined mesh, and then take as reference flow the trace of the solution on Γ_2. Anyway, those results lead us to observe that even for higher Reynold's number and with a consistent choice of $\mathbf{u}_{ref}^{\mathfrak{T}}$, the conditions behave well: the recirculations are still well located and there are no spurious vortices.

Test case 3. This test is inspired from the benchmark of [ST96] and we precisely use the detailed results in [Joh04]. In both [ST96], [Joh04] the drag and lift coefficients of the flow past an obstacle are computed from simulations on a long domain, by imposing Dirichlet boundary conditions. Our idea is to measure the quality of the DDFV solution we obtain on the shorter domain with outflow boundary conditions.
The benchmark is defined with dimensional equations, so we adopt the same framework. References [ST96] and [Joh04] consider a long channel $\Omega = [0, 2.2\text{m}] \times [0, 0.41\text{m}]$, that we cut at $x = 0.6$m, with a cylindrical obstacle S whose center is in $(0.2\text{m}, 0.2\text{m})$. We perform the computation of the drag and lift coefficients by working on the smaller domain $\Omega' = [0, 0.6\text{m}] \times [0, 0.41\text{m}]$, with the outflow boundary condition (III.2) on Γ_2 (at $x = 0.6$m) and Dirichlet on the other boundaries. The triangular mesh that we considered on Ω', obtained with GMSH, has 8020 cells and it is locally refined around the cylinder.

The viscosity of the fluid is set to $\eta = 10^{-3}\text{m}^2\text{s}^{-1}$ and the final time is $T = 8$s. The time-dependent

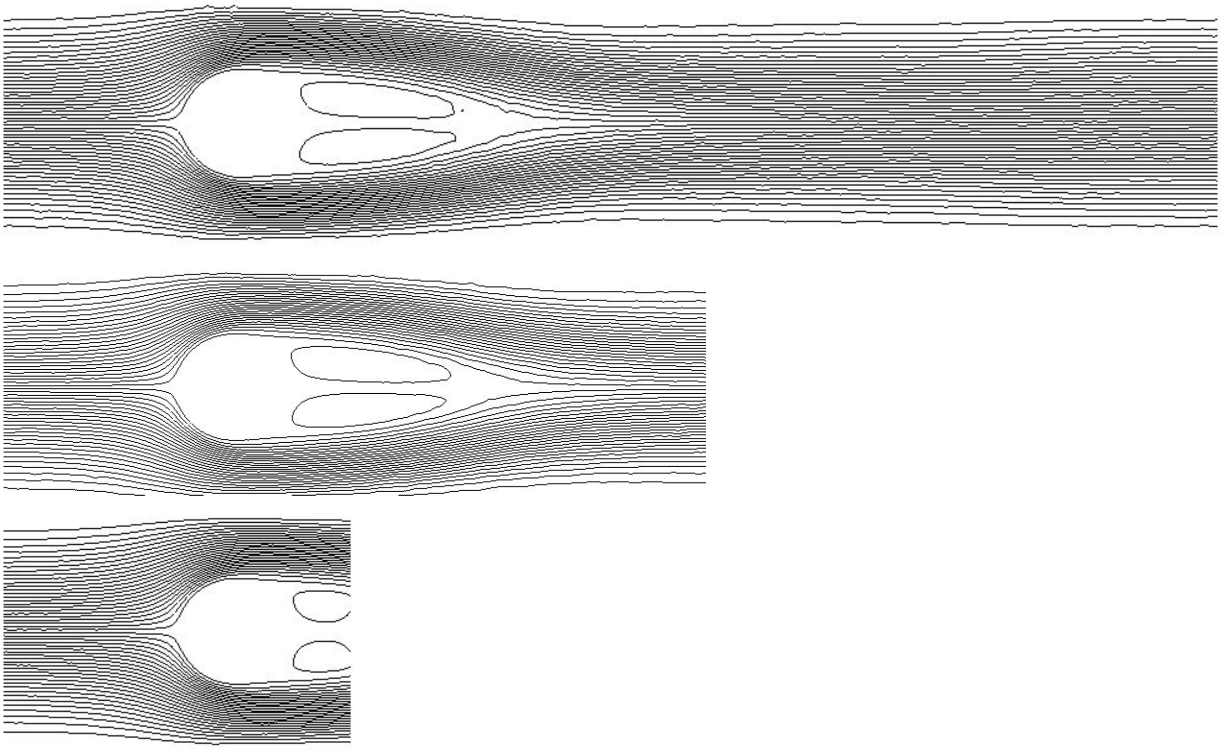


Fig. III.7 Streamline of Test case 2 at $T = 3.5$, Re $= 250$, $\delta t = 0.035$. On the top: $\Omega = [0,5] \times [0,1]$, NbCell= 12118. In the middle: $\Omega' = [0,3] \times [0,1]$, NbCell=8636. On the bottom: $\Omega'' = [0,1.5] \times [0,1]$, NbCell=6534.

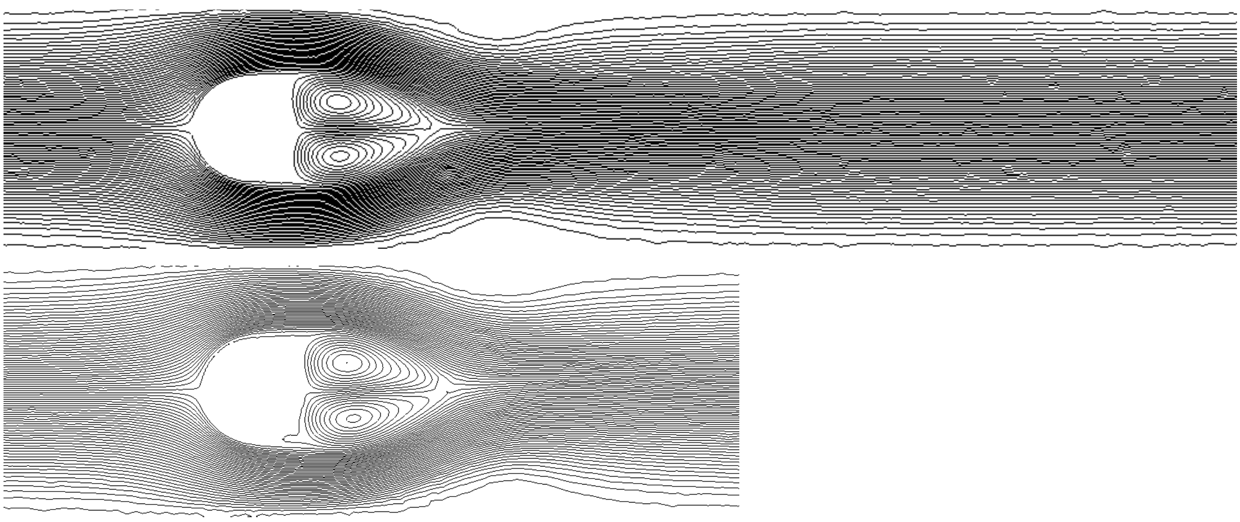


Fig. III.8 Streamline of Test case 2 at $T = 1.5$, Re $= 1000$, $\delta t = 0.035$. On the top: $\Omega = [0,5] \times [0,1]$, NbCell= 12118. On the bottom: $\Omega' = [0,3] \times [0,1]$, NbCell=8636.

inflow on Γ_1 is:

$$\mathbf{g}_1 = 0.41^{-2} \sin(\pi t/8)(6y(0.41 - y), 0),$$

and as a reference flow on Γ_2 we choose $\mathbf{u}_{ref} = \mathbf{g}_1$, $\mathrm{p}_{ref} = 0$ and $\sigma_{ref}\,\vec{\mathbf{n}} = \sigma(\mathbf{u}_{ref}, 0)\,\vec{\mathbf{n}}$, where $\vec{\mathbf{n}}$ is the outer normal to Ω. The initial condition is $\mathbf{u}_{init} = (0,0)$. The density of the fluid is given by $\rho = 1\mathrm{kgm}^{-3}$, and the reference velocity is $\bar{U} = 1\mathrm{ms}^{-1}$ (note that the maximum velocity is $\frac{3}{2}\bar{U}$).

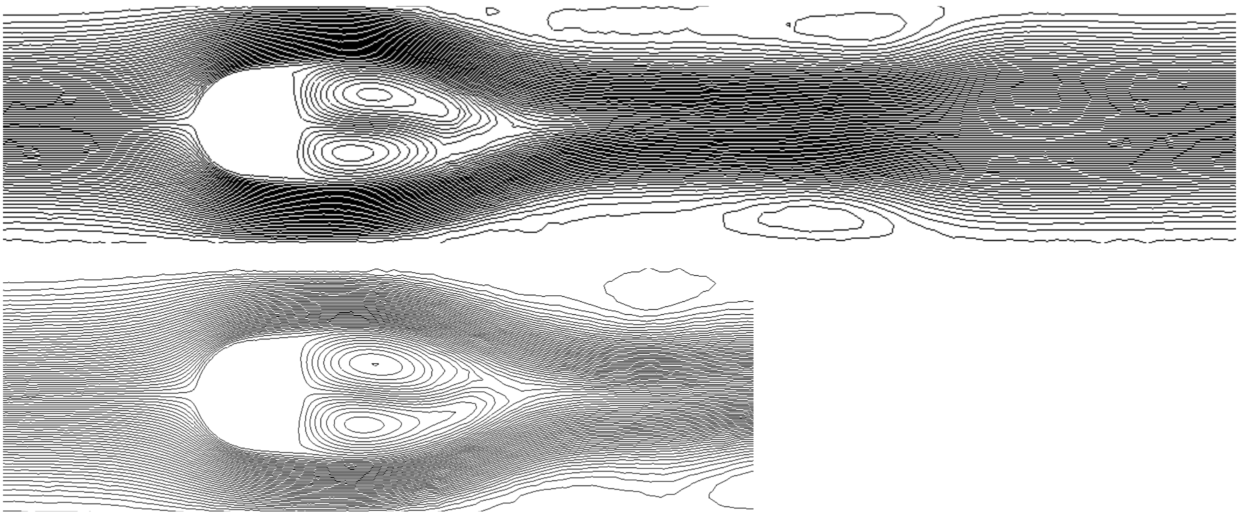


Fig. III.9 Streamline of Test case 2 at $T = 3.5$, Re $= 1000$, $\delta t = 0.035$. On the top: $\Omega = [0,5] \times [0,1]$, NbCell= 12118. On the bottom: $\Omega' = [0,3] \times [0,1]$, NbCell=8636.

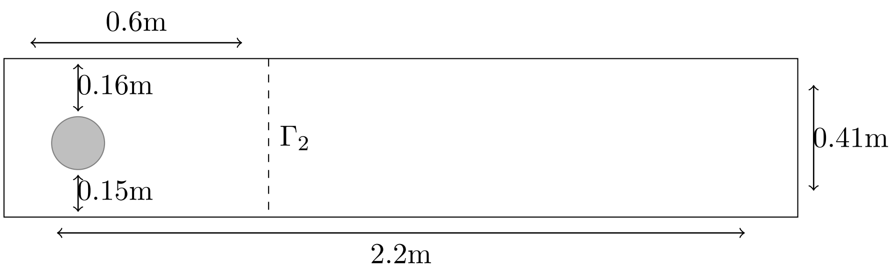


Fig. III.10 Domains $\Omega = [0, 2.2] \times [0, 0.41]$ and $\Omega' = [0, 0.6] \times [0, 0.41]$ for the test case 3.

The diameter of the cylinder is $L = 0.1$m, so that the Reynold's number is $0 \leq \mathrm{Re}(t) \leq 100$. We define the drag coefficient $c_d(t)$ and the lift coefficient $c_l(t)$ as:

$$\begin{aligned} c_d(t) &= \frac{2}{\rho L \bar{U}^2} \int_S \left(\rho\eta \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_y - p(t) n_x \right), \\ c_l(t) &= -\frac{2}{\rho L \bar{U}^2} \int_S \left(\rho\eta \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_x + p(t) n_y \right), \end{aligned}$$

where here $\vec{\mathbf{n}}_S = (n_x, n_y)$ is the normal vector on S directing into Ω , $\mathbf{t}_S = (n_y, -n_x)$ the tangential vector and $\mathbf{u}_{t_S}$ the tangential velocity.
The corresponding formula in the DDFV setting is:

$$\begin{aligned} c_d^n &= \frac{2}{\rho L \bar{U}^2} \sum_{\mathtt{D} \in \mathfrak{D}_{ext} \cap S} m_\sigma \left(\rho\eta \nabla^{\mathtt{D}} (\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}\ n_y - \mathrm{p}^n n_x \right), \\ c_l^n &= -\frac{2}{\rho L \bar{U}^2} \sum_{\mathtt{D} \in \mathfrak{D}_{ext} \cap S} m_\sigma \left(\rho\eta \nabla^{\mathtt{D}} (\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}\ n_x + \mathrm{p}^n n_y \right), \end{aligned}$$

where:

$$\nabla^{\mathtt{D}} (\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} = \frac{m_\sigma}{2 m_{\mathtt{D}}} (\mathbf{u}_{\mathtt{L}}^n - \mathbf{u}_{\mathtt{K}}^n) \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*} + \frac{m_{\sigma^*}}{2 m_{\mathtt{D}}} (\mathbf{u}_{\mathtt{L}^*}^n - \mathbf{u}_{\mathtt{K}^*}^n) \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*\mathtt{L}^*}\ \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}.$$

We study the evolution of the coefficients in Fig. III.11 and their maximum value in Table III.2, defined as:

$$c_{d,max} = \max_{n\in\{0\dots N\}} c_d^n, \qquad c_{l,max} = \max_{n\in\{0\dots N\}} c_l^n.$$

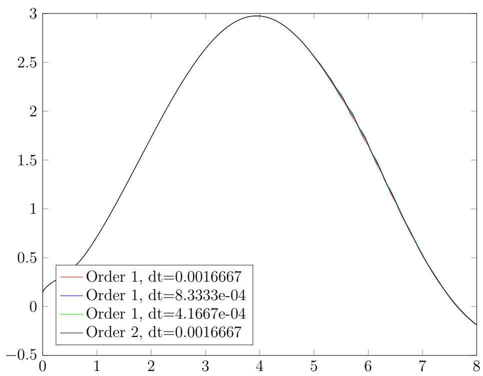
The results shown in Table III.2 and in Fig. III.11 prove that the boundary conditions are robust and the solution we find is quantitatively correct. The small difference in the coefficients, with respect to the reference values, is due to the different kind of condition on the boundary and to the level of refinement of the mesh. We work with approximately 24000 unknowns, for all velocity components and pressure, with respect to the approximately 500 000 unknowns used to compute the reference coefficients. Even if our grid is coarser, we still obtain a good approximation.
In Figure III.11 we can also observe how the time step and the choice of the scheme influences the result for the lift coefficient: for the reference values, [Joh04] considers a time step $\delta t = 0.00125s$ with a second order scheme in time. Our scheme is first order in time: firstly, we test this scheme with a decreasing time step, starting with $\delta t = 0.0016667$. Secondly, we implement a second order backward difference formula in time to see if the approximation improves. The first iteration of the scheme remains unchanged, and for $n \in \{1, \dots N\}$ the variational formulation (III.10) becomes:

$$\begin{aligned}
&[[\frac{1}{\delta t}\left(\frac{3}{2}\mathbf{u}^{n+1} - 2\mathbf{u}^n + \frac{1}{2}\mathbf{u}^{n-1}\right), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} \\
&\qquad + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(2\mathbf{u}^n - \mathbf{u}^{n-1}, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} \\
&\qquad = -\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma\mathtt{K}}(2\mathbf{u}^n - \mathbf{u}^{n-1}))^+ \gamma^\sigma(\mathbf{u}^{n+1}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}) \\
&\qquad\quad + \frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma\mathtt{K}}(2\mathbf{u}^n - \mathbf{u}^{n-1}))^- \gamma^\sigma(\mathbf{u}_{ref}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}) \\
&\qquad\qquad + \sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma^{\mathfrak{D}}_{\mathrm{ref}}\vec{\mathbf{n}}_{\sigma\mathtt{K}}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}).
\end{aligned}$$

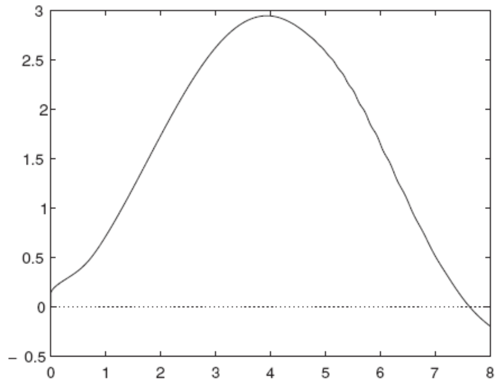
We observe in Fig. III.11 that this technique actually improves the quality of the approximation of the lift coefficient.

	DDFV	**Reference**
$\mathbf{c_{d,max}}$	**2.9**754	**2.9**509
$\mathbf{c_{l,max}}$	**0.4**4902	**0.4**7795

Table III.2 Comparison between the values of $c_{d,max}, c_{l,max}$ obtained with DDFV scheme (left) and the reference values of [Joh04] (right).

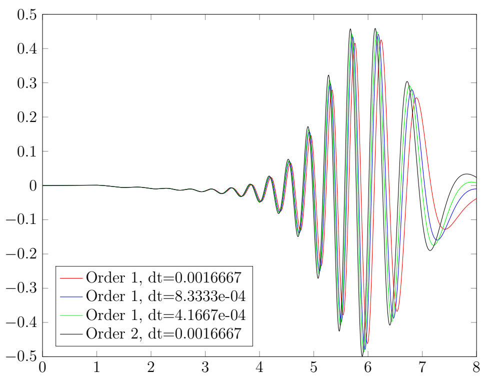


(a) DDFV.

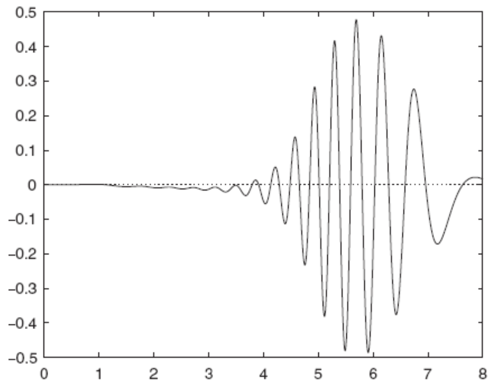


(b) Reference values.

Evolution of the drag coefficient c_d^n



(c) DDFV.



(d) Reference values.

Evolution of the lift coefficient c_l^n

Fig. III.11 Comparison between the evolution of c_d^n, c_l^n on the time interval $[0, 8]$ obtained with DDFV scheme (left) and the reference values of [Joh04] (right). We plot the results for the scheme of order 1 in time, with respect to different time steps, and for the scheme of order 2.

Conclusions and perspectives

In this chapter, we proposed a DDFV scheme for the Navier–Stokes problem with outflow boundary conditions. The DDFV scheme is proved to be well-posed and it satisfies a discrete energy estimate. We numerically showed the good convergence of the scheme and we performed numerical tests that establish the accuracy of this condition. In particular, we remarked the influence in the simulations of the *reference flow* that appears in the outflow boundary condition: it is supposed to be a steady flow, but when the Reynold's number increases and the flow it is no more laminar, this choice is not the optimal one. We would like to investigate this point in order to find the best technique to choose it. Moreover, these results are proved in the case of a constant viscosity, but they could be extended to the case of variable viscosity, by starting from the works of [BF07, Kre11b].

Chapter IV

Non-overlapping DDFV Schwarz algorithm for Navier-Stokes problem

Contents

The aim of this chapter is to develop a DDFV version of a non-overlapping iterative Schwarz algorithm for the incompressible Navier-Stokes problem:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u},\mathrm{p})) = 0 & \text{in} \quad \Omega\times[0,T], \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega\times[0,T], \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega\times[0,T], \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega, \end{cases} \tag{IV.1}$$

where Ω is an open connected bounded polygonal domain of $\mathbb{R}^2$, $\mathbf{u}_{init} \in (L^\infty(\Omega))^2$, and where $\mathbf{u} : \Omega\times[0,T] \to \mathbb{R}^2$ is the velocity, $\mathrm{p} : \Omega\times[0,T] \to \mathbb{R}$ is the pressure and $\sigma(\mathbf{u},\mathrm{p}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$ is the stress tensor, with $\mathrm{Re} > 0$. In particular, the strain rate tensor is defined by the symmetric part of the velocity gradient $\mathrm{D}\mathbf{u} = \dfrac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$.

Non-overlapping Schwarz algorithm enters the class of domain decomposition methods, in which a domain is decomposed into smaller subdomains. The main advantage is that the problems on the subdomains are independent, which makes these methods suitable for parallel computing and thus interesting for high performance computing perspectives. The classical Schwarz algorithm,

proposed in 1870 by H.A. Schwarz for the Laplace's problem, is an iterative method that consists in transmitting the solution, or its normal derivative, from a subdomain to the other, in order to deal with complex domains. This method converges only if the subdomains overlap. Moreover, this convergence becomes slower as the overlap between the subdomains is smaller.
In non-overlapping Schwarz algorithms, the subdomains intersect only on their interface and in order to obtain convergence, different transmission conditions on the interfaces have been investigated. It was shown in 1990 by P.L.Lions [Lio90] that, with Fourier (i.e. Robin) transmission conditions, Schwarz algorithm for the Laplace operator converges even without overlap. This method has been adapted to the discrete case for many problems of isotropic diffusion, [AJNM02, CHH04, GJMN05], for advection-diffusion-reaction problems, [GH07, HH14] and for anisotropic diffusion in a DDFV discretization, [BHK10b, GHHK18].

Our goal is thus to decompose the domain Ω of problem (IV.1) into smaller subdomains, solve the Navier-Stokes problem on those subdomains by imposing some transmission conditions on the interfaces, and recover by an iterative Schwarz algorithm the discrete solution of (IV.1) on Ω. We work with an unsteady problem; we decide to apply this iterative algorithm at each time iteration.

We refer as a starting point of this study to the works of [GHHK18] and [HH14]: they both build a non-overlapping Schwarz algorithm in a finite volume framework with Fourier-like transmission conditions between subdomains; the first considers the case of anisotropic diffusion with a DDFV discretization, the latter considers a problem of advection-diffusion-reaction in a TPFA discretization. In our case, when switching to the Navier-Stokes equations (IV.1) in a DDFV framework, the difficulty mainly consists into designing the same type of transmission conditions, by taking into account both the non-linear convection terms and the incompressibility constraint.

This is why we choose to impose the following: when we decompose the domain Ω into two (or more) smaller subdomains $\Omega = \Omega_1 \cup \Omega_2$, the Schwarz algorithm defines a sequence of solutions $\mathbf{u}_j^l$ of the Navier-Stokes problem in Ω_j, where the transmission conditions on the interface between the subdomains (denoted by Γ) for $(j,i) = (1,2)$ or $(2,1)$, are defined by:

$$\begin{aligned} \sigma(\mathbf{u}_j^l, \mathrm{p}_j^l) \cdot \vec{\mathbf{n}}_j - \frac{1}{2}(\mathbf{u}_j^l \cdot \vec{\mathbf{n}}_j)(\mathbf{u}_j^l) + \lambda \mathbf{u}_j^l &= \sigma(\mathbf{u}_i^{l-1}, \mathrm{p}_i^{l-1}) \cdot \vec{\mathbf{n}}_i - \frac{1}{2}(\mathbf{u}_i^{l-1} \cdot \vec{\mathbf{n}}_i)(\mathbf{u}_i^{l-1}) + \lambda \mathbf{u}_i^{l-1}, \\ \operatorname{div}(\mathbf{u}_j^l) + \alpha \mathrm{p}_j^l &= -\operatorname{div}(\mathbf{u}_i^{l-1}) + \alpha \mathrm{p}_i^{l-1}, \end{aligned} \tag{IV.2}$$

where $\vec{\mathbf{n}}_j$ is the outer normal to Ω_j and l is the iteration of the Schwarz algorithm. The first condition, which depends on λ, is inspired by the classical Fourier condition, which linearly combines of the values of a the unknown and the values of its derivative; here, also the convection is included. The second, which depends on α, combines the divergence of the velocity with the pressure; it will be useful to conserve the incompressibility constraint at the convergence of the algorithm. This is the first time, to our knowledge, that this kind of condition appears.

Another key point of our study is the discretization of the convection terms, inspired by [HH14]: we decide to approximate this terms by a centered discretization plus a diffusive perturbation, which depends on a general function B; this function will play an important role in the convergence of the algorithm. We refer to the so called "B-schemes", which first appeared in [CHD11] when designing finite volume schemes for non-coercive elliptic problems with Neumann boundary conditions.

Outline. This chapter is organized as follows. In Sec. IV.1, we define a discretization of the Navier-Stokes problem on the entire domain Ω by using B-schemes for the discretization of the nonlinear convection terms; we prove that the scheme, to which we will refer as $(\mathcal{P})$, is well-posed. In particular, $(\mathcal{P})$ is the limit scheme towards which the solution of the iterative Schwarz algorithm should converge. In Sec.IV.2, we introduce the composite meshes, i.e. the meshes on the subdomains, and we build a scheme for the subdomain problem with transmission boundary conditions, that are the discrete version of (IV.2). In the same section, we introduce then the DDFV Schwarz algorithm. The convergence of that algorithm is proven in Sec. IV.3, and we show that it actually converges to a modified version of $(\mathcal{P})$, that we name $(\widetilde{\mathcal{P}})$. The difference between $(\mathcal{P})$ and $(\widetilde{\mathcal{P}})$ is the choice of the function B that defines the convection terms on the interface. In Sec. IV.4, we prove how to recover the convergence towards $(\mathcal{P})$, by proposing an alternative Schwarz algorithm. Finally, in Sec. IV.5 we illustrate the theoretical results by means of some numerical simulations; in particular, we show and compare the convergence of the Schwarz algorithms built, by underlying the influence of the parameters λ, α of (IV.2).

IV.1 DDFV scheme for the Navier-Stokes problem on Ω

In this section, we propose a DDFV discretization for the Navier-Stokes problem with Dirichlet boundary conditions on the entire domain Ω. Such a scheme for this problem was already studied in [Kre10], with the choice of an upwind discretization to treat the convection term. Our aim is to generalize this result to B-schemes; in those schemes, the convection term is approximated by a centered discretization plus a diffusive perturbation, which depends on a general function B. This is inspired from the work of [HH14], which handles scalar advection-diffusion-reaction equation with a classic finite volume discretization.

Just a remark, all along this chapter $\mathtt{D}_{\sigma,\sigma^*}$ will be denoted by $\mathtt{D}$, to simplify the notations.

IV.1.1 The scheme $\mathcal{P}$

Let $N \in \mathbb{N}^*$ and $0 < T < \infty$. We note $\delta t = \frac{T}{N}$ and $t_n = n\delta t$ for $n \in \{0, \dots, N\}$. To design the DDFV scheme, we choose to use an implicit Euler time discretization, except for the nonlinear term, which is linearized by using a semi-implicit approximation. This is why we need $\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^n) - \beta d_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} \mathrm{p}^n = 0$ at each time step $n \in \{0, \dots N\}$, even at the initial time step. Remark that the parameter β is relied to a Brezzi-Pitkäranta stabilization for the mass conservation equation; see Sec. I.7.

We look for $\mathbf{u}^{\mathfrak{T},[0,T]} = (\mathbf{u}^n)_{n\in\{0,\dots N\}} \in \left(\mathbb{E}_0\right)^{N+1}$ and $\mathrm{p}^{\mathfrak{D},[0,T]} = (\mathrm{p}^n)_{n\in\{0,\dots N\}} \in (\mathbb{R}^{\mathfrak{D}})^{N+1}$, that we initialize with:

$$\begin{aligned}
&\mathbf{u}^0 = \mathbb{P}_c^{\mathfrak{T}} \mathbf{u}_0 \in \mathbb{E}_0, \\
&\mathrm{p}^0 \in \mathbb{R}^{\mathfrak{D}} \text{such that } \Delta^{\mathfrak{D}} \mathrm{p}^0 = \frac{1}{\beta d_{\mathfrak{D}}^2} \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^0) \text{ with } \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} \mathrm{p}_{\mathtt{D}}^0 = 0.
\end{aligned}$$

The vector p^0 is well defined since it is solution of a square system, whose matrix is invertible. With those choices of $(\mathbf{u}^0, \mathrm{p}^0)$ we guarantee the property $\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^n) - \beta d_{\mathtt{D}} \Delta^{\mathfrak{D}} \mathrm{p}^n = 0$ at the initial time step.

From now on, to simplify the notations we will denote $(\mathbf{u}^{n+1}, \mathrm{p}^{n+1})$ with $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ and $(\mathbf{u}^n, \mathrm{p}^n)$ with $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$ that at each time step are known.

To obtain our scheme, we integrate the momentum equation over all $\mathfrak{M} \cup \mathfrak{M}^*$ and we impose Dirichlet boundary conditions on $\partial\mathfrak{M} \cup \partial\mathfrak{M}^*$. The equation of conservation of mass is directly approximated on the diamond mesh equation over $\mathfrak{D}$, and it is stabilized through a parameter $\beta > 0$ with a Brezzi-Pitkäranta stabilization (see Sec. I.7).

Given $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$, satisfying $\mathrm{div}^{\mathfrak{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta d_{\mathtt{D}}^2 \Delta^{\mathfrak{D}} \bar{\mathrm{p}}^{\mathfrak{D}} = 0$, we look for $\mathbf{u}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$ and $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\begin{cases} m_{\mathtt{K}} \dfrac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \displaystyle\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_{\mathtt{K}} \mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} & \forall \mathtt{K} \in \mathfrak{M} \\ m_{\mathtt{K}^*} \dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \displaystyle\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^* \in \mathfrak{M}^* \\ \mathbf{u}^{\partial\mathfrak{M}} = 0 \\ \mathbf{u}^{\partial\mathfrak{M}^*} = 0 \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta d_{\mathtt{D}}^2 \Delta^{\mathfrak{D}} \mathrm{p}^{\mathfrak{D}} = 0 \\ \displaystyle\sum_{\mathtt{D} \in \mathfrak{D}} m_{\mathtt{D}} \mathrm{p}^{\mathtt{D}} = 0, \end{cases} \tag{$\mathcal{P}$}$$

with $\beta > 0$.
The fluxes are defined as a sum of a "diffusion" term and a "convection" term:

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_\sigma (\mathcal{F}^d_{\sigma\mathtt{K}} + \mathcal{F}^c_{\sigma\mathtt{K}}), \tag{IV.3}$$

$$m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\sigma^*} (\mathcal{F}^d_{\sigma^*\mathtt{K}^*} + \mathcal{F}^c_{\sigma^*\mathtt{K}^*}). \tag{IV.4}$$

The **diffusion fluxes** are defined, as in Chapter III, as:

$$\begin{aligned} m_\sigma \mathcal{F}^d_{\sigma\mathtt{K}} &= - m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} \\ &= - m_\sigma \left(\frac{2}{\mathrm{Re}} \mathrm{D}^{\mathtt{D}} \mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}} \mathrm{Id} \right) \vec{\mathbf{n}}_{\sigma\mathtt{K}}, \end{aligned}$$

$$\begin{aligned} m_{\sigma^*} \mathcal{F}^d_{\sigma^*\mathtt{K}^*} &= - m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \\ &= - msige \left(\frac{2}{\mathrm{Re}} \mathrm{D}^{\mathtt{D}} \mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathtt{D}} \mathrm{Id} \right) \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}, \end{aligned}$$

where the stress tensor is approximated through the operators defined in Sec. I.3.

The **convection fluxes** are defined as:

$$\begin{aligned} m_\sigma \mathcal{F}^c_{\sigma\mathtt{K}} =& m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}}}{2} \right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} B \left(\frac{2 m_{\mathtt{D}} \mathrm{Re}}{m_\sigma} F_{\sigma\mathtt{K}} \right) (\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}), \\ m_{\sigma^*} \mathcal{F}^c_{\sigma^*\mathtt{K}^*} =& m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathtt{D}}} B \left(\frac{2 m_{\mathtt{D}} \mathrm{Re}}{m_{\sigma^*}} F_{\sigma^*\mathtt{K}^*} \right) (\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}). \end{aligned}$$

They are the sum of a centered discretization and a diffusive perturbation, which depends on the function B. This function B embeds the different schemes that we want to work with: for instance, if we want a centered scheme, we choose $B(s) = 0$ and for an upwind scheme, we set $B(s) = \frac{1}{2}|s|$. We denote $B\left(\frac{2m_{\mathtt{D}}\mathrm{Re}}{m_\sigma} F_{\sigma\mathtt{K}}\right)$ with $B_{\sigma\mathtt{K}}$ and $B\left(\frac{2m_{\mathtt{D}}\mathrm{Re}}{m_{\sigma^*}} F_{\sigma^*\mathtt{K}^*}\right)$ with $B_{\sigma^*\mathtt{K}^*}$. In the particular case of the upwind discretization, $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$ are scalars, but we will see that those coefficients can be generalized to matrices and to more general expressions. The total fluxes then become:

$$\boxed{\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} &= -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}}\left(\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{L}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}\, m_{\mathtt{D}}} B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}) \\ m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} &= -m_{\sigma^*}\sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*} F_{\sigma^*\mathtt{K}^*}\left(\frac{\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}}{2}\right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re}\, m_{\mathtt{D}}} B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) \end{aligned}} \quad \text{(IV.5)}$$

The definition of $F_{\sigma\mathtt{K}}$, $F_{\sigma^*\mathtt{K}^*}$ comes straightforward by Sec. III.1 and [Kre10, Kre11b]. They are an approximation of the fluxes: $\int_\sigma (\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}) \rightsquigarrow F_{\sigma\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})$ and $\int_{\sigma^*}(\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}) \rightsquigarrow F_{\sigma^*\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})$.
We obtain them by calculating the fluxes on the sides $\mathfrak{s}$ of diamonds for the interior edges (see Fig. IV.1). In fact, we remark that by integrating the solenoidal constraint on the semi-diamond $\mathtt{D}_{\mathtt{K}}$ of vertices $x_{\mathtt{K}}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}$ we have:

$$0 = \int_{\mathtt{D}_{\mathtt{K}}} \mathrm{div}(\bar{\mathbf{u}}^{\mathfrak{T}})\mathrm{d}x = \int_\sigma \bar{\mathbf{u}}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}} + \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} \int_{\mathfrak{s}} \bar{\mathbf{u}}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}\mathrm{d}s.$$

Those fluxes on the sides $\mathfrak{s}$ of diamonds are the ones that we obtain by integrating the solenoidal constraint on each diamond $\mathtt{D}\in\mathfrak{D}$:

$$\int_{\mathtt{D}} \mathrm{div}(\bar{\mathbf{u}}^{\mathfrak{T}})\mathrm{d}x = \sum_{\mathfrak{s}\in\partial\mathtt{D}} \int_{\mathfrak{s}} \bar{\mathbf{u}}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}\mathrm{d}s,$$

that at a discrete level is written, by adding a stabilization, as:

$$m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}}\bar{\mathrm{p}}^{\mathfrak{D}} = \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}},$$

so that $\int_{\mathfrak{s}} \bar{\mathbf{u}}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}\mathrm{d}s$ is approximated by $m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} = m_{\mathfrak{s}}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}}+\bar{\mathbf{u}}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} - \beta(d_{\mathtt{D}}+d_{\mathtt{D}'})(\mathrm{p}^{\mathtt{D}} - \mathrm{p}^{\mathtt{D}'})$ for $\mathfrak{s} = [x_{\mathtt{K}}, x_{\mathtt{K}^*}] = \mathtt{D}|\mathtt{D}'$, $\mathfrak{s}\in\mathcal{E}_{\mathtt{D}}$.

Remark IV.1.1 *If* $\mathtt{D}\in\mathfrak{D}_{ext}$ *(see Fig. IV.1),* $m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}})$ *can be rewritten as:*

$$m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}}\bar{p}^{\mathfrak{D}} = \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}'\in\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} + m_\sigma \gamma^\sigma(\bar{\mathbf{u}}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}.$$

We thus impose:
► For the primal edges:

$$m_\sigma F_{\sigma\mathtt{K}} = -\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}}.$$

► For the dual edges:

$$m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}^*}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} & \text{if } \mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*,\ \sigma^*\cap\partial\Omega=\emptyset \\ -m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} - \dfrac{1}{2} m_{\partial\Omega\cap\partial\mathtt{K}^*} H_{\mathtt{K}^*} & \text{if } \mathtt{K}^*\in\partial\mathfrak{M}^*,\ \sigma^*\cap\partial\Omega\neq\emptyset, \text{i.e. } \mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}^{ext} \end{cases}$$

where $m_{\partial\Omega\cap\partial\mathtt{K}^*}$ indicates the measure of the intersection between $\partial\mathtt{K}^* \cap \partial\Omega$ and:

$$\begin{aligned} m_{\mathfrak{s}} G_{\mathfrak{s},\mathtt{D}} &= m_{\mathfrak{s}} \frac{\bar{\mathbf{u}}_{\mathtt{K}} + \bar{\mathbf{u}}_{\mathtt{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} - \beta(d_{\mathtt{D}} + d_{\mathtt{D}'})(\mathrm{p}^{\mathtt{D}} - \mathrm{p}^{\mathtt{D}'}), && \forall \mathfrak{s} = [x_{\mathtt{K}}, x_{\mathtt{K}^*}] = \mathtt{D}|\mathtt{D}', \\ m_{\partial\Omega\cap\partial\mathtt{K}^*} H_{\mathtt{K}^*} &= \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}} m_{\sigma\cap\partial\mathtt{K}^*} \bar{\mathbf{u}}_{\mathtt{K}^*} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}, && \forall \mathtt{K}^* \in \partial\mathfrak{M}^*. \end{aligned}$$

The difference with respect to the fluxes of Sec. III.1 is their definition on the boundary edges; this is due to type of boundary conditions that we consider. Remark that at this stage, since we are imposing (for simplicity) homogeneous Dirichlet boundary conditions on $\partial\Omega$, those contributions will not be taken into account.

Remark that if $m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \bar{\mathrm{p}}^{\mathfrak{D}} = 0$ we have conservativity of the fluxes $F_{\sigma\mathtt{K}}$ and

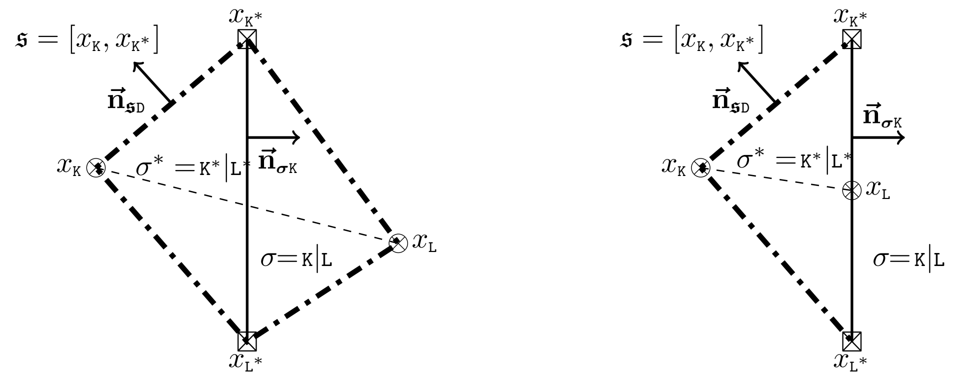


Fig. IV.1 *Left:* A diamond $\mathtt{D} = \mathtt{D}_{\sigma,\sigma^*}$ with $\sigma \subset \mathcal{E}_{int}$. *Right:* A diamond $\mathtt{D} = \mathtt{D}_{\sigma,\sigma^*}$ with $\sigma \in \partial\Omega$.

$F_{\sigma^*\mathtt{K}^*}$, that is:

$$F_{\sigma\mathtt{K}} = -F_{\sigma\mathtt{L}}, \quad \forall \sigma = \mathtt{K}|\mathtt{L} \quad \text{and} \quad F_{\sigma^*\mathtt{K}^*} = -F_{\sigma^*\mathtt{L}^*}, \quad \forall \sigma^* = \mathtt{K}^*|\mathtt{L}^*. \tag{IV.6}$$

Proposition IV.1.2 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$, $\beta \in \mathbb{R}^*$ we have:*

$$\begin{aligned} &\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}} = 0 && \forall \mathtt{K} \in \mathfrak{M} \\ &\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} = 0 && \forall \mathtt{K}^* \in \mathfrak{M}^* \\ &\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} = -m_{\partial\Omega\cap\partial\mathtt{K}^*} H_{\mathtt{K}^*} && \forall \mathtt{K}^* \in \partial\mathfrak{M}^* \end{aligned}$$

Proof For the interior mesh, we proceed as in [Kre10].
If $\mathtt{K} \in \mathfrak{M}$, by reorganizing the sum on the sides $\mathfrak{s} \in \mathfrak{S}_{\mathtt{K}}$ belonging to the primal cell $\mathtt{K}$, we obtain:

$$-\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} = -\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}} m_{\mathfrak{s}} \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \cdot (\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} + \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}'}) = 0 \tag{IV.7}$$

since $\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} = -\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}'}$, where $\mathtt{D}$ and $\mathtt{D}'$ denote the two neighbor diamonds which share the edge $\mathfrak{s}$, of vertices $x_{\mathtt{K}}, x_{\mathtt{K}^*}$. In the same way,

$$-\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}}(d_{\mathtt{D}}^2+d_{\mathtt{D}'}^2)(\mathrm{p}^{\mathtt{D}'}-\mathrm{p}^{\mathtt{D}}) = -\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}}(d_{\mathtt{D}}^2+d_{\mathtt{D}'}^2)(\mathrm{p}^{\mathtt{D}'}-\mathrm{p}^{\mathtt{D}}+\mathrm{p}^{\mathtt{D}}-\mathrm{p}^{\mathtt{D}'}) = 0. \tag{IV.8}$$

We deduce that $\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}} = 0$. The proof is similar for $\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*} = 0$ if $\mathtt{K}^*\in\mathfrak{M}^*$.

We now focus on the case in which $\mathtt{K}^*\in\partial\mathfrak{M}^*$; by definition of $m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}$:

$$-\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}^*}\cap\mathcal{E}_{\mathtt{D}}}\left\{m_{\mathfrak{s}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}+(d_{\mathtt{D}}^2+d_{\mathtt{D}'}^2)(\mathrm{p}^{\mathtt{D}'}-\mathrm{p}^{\mathtt{D}})\right\} - \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}\cap\partial\Omega}\frac{1}{2}m_{\partial\Omega\cap\partial\mathtt{K}^*}H_{\mathtt{K}^*}$$
$$= 0 - m_{\partial\Omega\cap\partial\mathtt{K}^*}H_{\mathtt{K}^*}.$$

where the first sum is zero thanks to (IV.7),(IV.8), and for the second term we use the fact that each vertex $\mathtt{K}^*$ is shared by two boundary diamonds. ■

IV.1.2 Wellposedness of problem $(\mathcal{P})$

The following theorem states the wellposedness of the scheme $(\mathcal{P})$. This is quite a classical result for a discretization of Navier-Stokes problem with Dirichlet boundary conditions, if we consider a centered or an upwind scheme (see for instance [Kre10] for the DDFV setting); what is crucial is to understand the properties that need to be satisfied by $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$ in order to have wellposedness, with the aim to extend later to more general coefficients.

Theorem IV.1.3 *Under the hypothesis*

$$\begin{aligned} B_{\sigma\mathtt{K}} &= B_{\sigma\mathtt{L}}, & B_{\sigma\mathtt{K}} &\geq 0\\ B_{\sigma^*\mathtt{K}^*} &= B_{\sigma^*\mathtt{L}^*}, & B_{\sigma^*\mathtt{K}^*} &\geq 0\end{aligned} \tag{$\mathcal{H}p$}$$

problem $(\mathcal{P})$ is well-posed.

Remark IV.1.4 *In the more general case in which $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$ are matrices, instead of $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}\geq 0$, we ask $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$ to be semi-definite positive; the proof does not change, except for the notations.*

Proof The scheme $(\mathcal{P})$ is a linear system in $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}\times\mathbb{R}^{\mathfrak{D}}$. Let us denote by N the dimension of $\left(\mathbb{R}^2\right)^{\mathfrak{T}}\times\mathbb{R}^{\mathfrak{D}}$. Then $(\mathcal{P})$ can be written, with $\mathbf{g}^{\partial\mathfrak{M}} = \mathbf{g}^{\partial\mathfrak{M}^*} = q^{\mathfrak{D}} = \phi = 0$, as

$$\left\{\begin{aligned} m_{\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}} &= m_{\mathtt{K}}\mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}}\frac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} && \forall\mathtt{K}\in\mathfrak{M}\\ m_{\mathtt{K}^*}\frac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} &= m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} && \forall\mathtt{K}^*\in\mathfrak{M}^*\\ \mathbf{u}^{\partial\mathfrak{M}} &= \mathbf{g}^{\partial\mathfrak{M}}\\ \mathbf{u}^{\partial\mathfrak{M}^*} &= \mathbf{g}^{\partial\mathfrak{M}^*}\\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta d_{\mathtt{D}}^2\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}} &= q^{\mathfrak{D}}\\ \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}} &= \phi\end{aligned}\right. \tag{$\mathcal{P}$}$$

This is a linear system $Av = b$ with a rectangular matrix $A \in \mathcal{M}_{N+1,N}(\mathbb{R})$, $v \in \mathbb{R}^N$ and $b \in \mathbb{R}^{N+1}$. Let X be the following set:

$$X = \left\{ (\mathbf{f}^{\mathfrak{M}}, \mathbf{f}^{\mathfrak{M}^*}, \mathbf{g}^{\partial\mathfrak{M}}, \mathbf{g}^{\partial\mathfrak{M}^*}, q^{\mathfrak{D}}, \phi) \in \mathbb{R}^{N+1}, \sum_{\mathtt{D}\in\mathfrak{D}_{ext}} m_\sigma \gamma^\sigma(\mathbf{g}^{\mathfrak{T}}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} q^{\mathfrak{D}} \right\},$$

then $dim(X) = N$. We have that ${}^t(\mathbf{f}^{\mathfrak{M}}, \mathbf{f}^{\mathfrak{M}^*}, 0, 0, 0)$ belongs to X and that $\mathrm{Im}(A) \subset X$ since we have a relation between the solenoidal constraint and the Dirichlet boundary conditions thanks to Green's formula (Thm. I.5.1). If we show that the matrix is injective, we conclude that $dim(\mathrm{Im}(A)) = N$ and that $\mathrm{Im}(A) = X$. So we study the kernel of the matrix A, that is equivalent to show that if $\mathbf{f}^{\mathfrak{M}} = \mathbf{f}^{\mathfrak{M}^*} = 0$, then $\mathbf{u}^{\mathfrak{T}} = 0$ and $\mathrm{p}^{\mathfrak{D}} = 0$.

We multiply the equations on the primal and dual mesh of $(\mathcal{P})$ by $\mathbf{u}^{\mathfrak{T}}$ and we sum over all the control volumes:

$$\frac{1}{2}\left[\frac{1}{\delta t}\left(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}} |\mathbf{u}_{\mathtt{K}}|^2 + \sum_{\mathtt{K}\in\mathfrak{M}^*} m_{\mathtt{K}^*} |\mathbf{u}_{\mathtt{K}^*}|^2\right) + \sum_{\mathtt{K}\in\mathfrak{M}} \mathbf{u}_{\mathtt{K}} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} + \sum_{\mathtt{K}^*\in\mathfrak{M}^*} \mathbf{u}_{\mathtt{K}^*} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*}\right] = 0.$$

By definition of the scalar products we have $\frac{1}{2}\left[\frac{1}{\delta t}\left(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}} |\mathbf{u}_{\mathtt{K}}|^2 + \sum_{\mathtt{K}\in\mathfrak{M}^*} m_{\mathtt{K}^*} |\mathbf{u}_{\mathtt{K}^*}|^2\right)\right] = \frac{1}{\delta t}\|\mathbf{u}^{\mathfrak{T}}\|^2$ and, by replacing the definition of the fluxes, we have:

$$\begin{aligned}\frac{1}{\delta t}\|\mathbf{u}^{\mathfrak{T}}\|^2 &- \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}} \mathbf{u}_{\mathtt{K}} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma\mathtt{K}} - \frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*} \mathbf{u}_{\mathtt{K}^*} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \\ &+ \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}} \mathbf{u}_{\mathtt{K}} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}} \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}}}{2} + \frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*} \mathbf{u}_{\mathtt{K}^*} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \\ + \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}} \mathbf{u}_{\mathtt{K}} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}) &+ \frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*} \mathbf{u}_{\mathtt{K}^*} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathtt{D}}} B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) = 0. \quad \text{(IV.9)}\end{aligned}$$

We can consider separately the terms. By replacing the definition of the divergence operator (Def. (I.3.5)) and then by appying Green's formula (Thm. I.5.1) for $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0$, we have:

- $-\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}} \mathbf{u}_{\mathtt{K}} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma\mathtt{K}} - \frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*} \mathbf{u}_{\mathtt{K}^*} \cdot \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}$

$$= -\left[\left[\mathbf{div}^{\mathfrak{T}}\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathfrak{D}}\right), \mathbf{u}^{\mathfrak{T}}\right]\right]_{\mathfrak{T}} = \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 - (\mathrm{p}^{\mathfrak{D}}, \mathrm{div}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} = \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2,$$

where for the last equality we use that $\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta \mathrm{d}_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}} = 0$ and we apply Remark I.7.2 to the term $-\beta(\mathrm{d}_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}$. .

For all the convection terms, we pass to a sum over diamonds recalling that $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0$, so we do not have boundary terms.

For the centered part, we apply Prop. IV.1.2, to conclude that:

$$\bullet\ \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{L}}}{2}+\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}\frac{\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}}{2}$$
$$=\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma F_{\sigma\mathtt{K}}(|\mathbf{u}_{\mathtt{K}}|^2-|\mathbf{u}_{\mathtt{L}}|^2)+\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}(|\mathbf{u}_{\mathtt{K}^*}|^2-|\mathbf{u}_{\mathtt{L}^*}|^2)$$
$$=\frac{1}{4}\sum_{\mathtt{K}\in\mathfrak{M}}|\mathbf{u}_{\mathtt{K}}|^2\underbrace{\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}}}_{=0}+\frac{1}{4}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}|\mathbf{u}_{\mathtt{K}^*}|^2\underbrace{\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_{\sigma^*}F_{\sigma^*\mathtt{K}^*}}_{=0}=0.$$

For the terms that depend on $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$, we apply hypothesis $(\mathcal{H}p)$ to conclude that:

$$\bullet\ \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})$$
$$=\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})^2+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})^2\geq 0.$$

Putting all together, (IV.9) becomes:

$$\frac{1}{\delta t}||\mathbf{u}^{\mathfrak{T}}||_2^2+\frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}||_2^2+\beta|\mathrm{p}^{\mathfrak{D}}|_h^2\leq 0,$$

from which we deduce that $\mathbf{u}^{\mathfrak{T}}=0$ and $\mathrm{p}^{\mathfrak{D}}$ is a constant (we recall that $\beta>0$). Since $\mathrm{p}^{\mathfrak{D}}$ verifies $\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}}=0$, we have $\mathrm{p}^{\mathfrak{D}}=0$. ■

IV.2 DDFV scheme for the subdomain problem on Ω_j

In this section, we define a discretization for the subdomain problem on Ω_j, for $j=1,2$. We present the study for two subdomains for simplicity, but it could be extended to a generic number of adjacent subdomains. As in Sec. IV.1, the nonlinear convection term will be approximated through B-schemes; we will see that the coefficients $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$ play an important role in the convergence of the Schwarz algorithm.
We start by defining the scheme, denoted by $(\mathcal{P}_j)$, and the related Schwarz algorithm for the domain decomposition; then we prove some a priori estimates in order to show the wellposedness of $(\mathcal{P}_j)$ at the end of the section.

IV.2.1 DDFV on composite meshes

On each subdomain Ω_j of Ω, $j=1,2$, we want to solve a Navier-Stokes system with mixed boundary conditions. On a fraction of the boundary (the one that intersects $\partial\Omega$) we impose Dirichlet boundary conditions; on the remaining part (the interface Γ between the two subdomains) we impose the discretized version of the transmission conditions (IV.2), that depend on the two parameters λ, α.

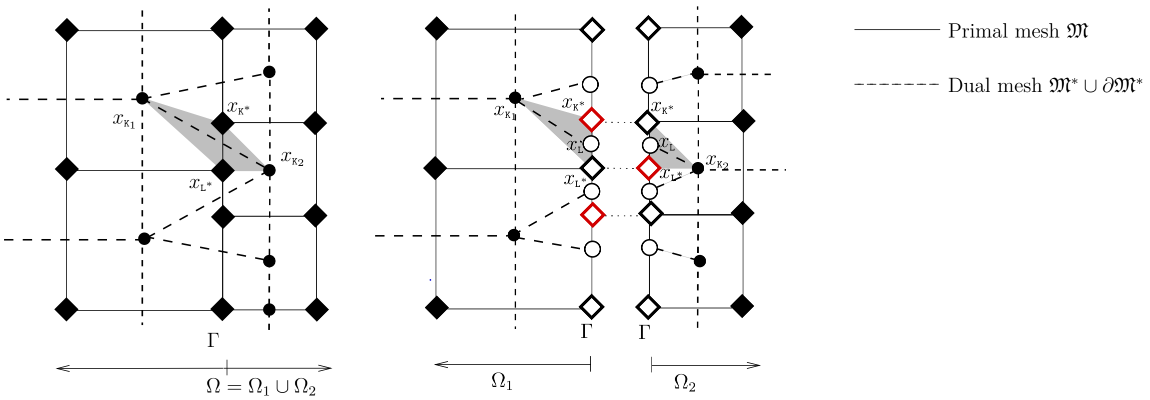


Fig. IV.2 DDFV meshes.

Description of the meshes

For each subdomain Ω_j of Ω, $j = 1, 2$, we consider a DDFV mesh $\mathfrak{T}_j = (\mathfrak{M}_j \cup \partial\mathfrak{M}_j, \mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_j)$ and the associated diamond mesh $\mathfrak{D}_j$. We remark that, thanks to DDFV scheme, we can work with non conformal meshes, and in particular the two subdomains can be meshed differently. Letting Γ be the interface between the two subdomains, we denote by:

- ⋄ the diamond cells intersecting Γ : $\mathfrak{D}^\Gamma_j := \{\mathtt{D} \in \mathfrak{D}_j, \mathtt{D} \cap \Gamma \neq \emptyset\}$;
- ⋄ the boundary primal cells intersecting Γ : $\partial\mathfrak{M}_{j,\Gamma} := \{\mathtt{K} \in \partial\mathfrak{M}_j, \mathtt{K} \cap \Gamma \neq \emptyset\}$;
- ⋄ the boundary dual cells intersecting Γ : $\partial\mathfrak{M}^*_{j,\Gamma} := \{\mathtt{K}^* \in \partial\mathfrak{M}^*_j, \mathtt{K}^* \cap \Gamma \neq \emptyset\}$;
- ⋄ the boundary primal cells intersecting $\partial\Omega$: $\partial\mathfrak{M}_{j,D} := \{\mathtt{K} \in \partial\mathfrak{M}_j, \mathtt{K} \cap \partial\Omega \neq \emptyset\}$;
- ⋄ the boundary dual cells intersecting $\partial\Omega$: $\partial\mathfrak{M}^*_{j,D} := \{\mathtt{K}^* \in \partial\mathfrak{M}^*_j, \mathtt{K}^* \cap \partial\Omega \neq \emptyset\}$;

see Fig. IV.2 for an example.

Definition IV.2.1 (Composite mesh) *We say that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are compatible, if the following conditions are satisfied:*

1. *the two meshes share the same vertices on Γ. This, in particular, implies that the two meshes have the same degenerate volumes on Γ, i.e. $\partial\mathfrak{M}_{1,\Gamma} = \partial\mathfrak{M}_{2,\Gamma}$.*
2. *The center $x_{\mathtt{L}}$ of the degenerate volumes of the interface $\mathtt{L} = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}] \in \partial\mathfrak{M}_{1,\Gamma} = \partial\mathfrak{M}_{2,\Gamma}$ is the intersection between $(x_{\mathtt{K}^*}, x_{\mathtt{L}^*})$ and $(x_{\mathtt{K}_1}, x_{\mathtt{K}_2})$, where $\mathtt{K}_1 \in \mathfrak{M}_1$ and $\mathtt{K}_2 \in \mathfrak{M}_2$ are the two primal cells such that $\mathtt{L} \in \partial\mathtt{K}_1$ and $\mathtt{L} \in \partial\mathtt{K}_2$ (see Fig. IV.2).*

Consider the composite mesh of Fig. IV.2; remark that:

- a diamond $\mathtt{D}$, of vertices $x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_{\mathtt{K}_2}$ that intersects Γ in the domain Ω can be written as the union of diamonds $\mathtt{D}_1$, of vertices $x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}}$, and $\mathtt{D}_2$, of vertices $x_{\mathtt{K}_2}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}}$, respectively in Ω_1, Ω_2. Moreover, on the subdomain meshes we have the additional unknowns on $x_{\mathtt{L}}$ on Γ with respect to the mesh on Ω;
- equivalently, a volume $\mathtt{K}^*$ that intersects Γ in Ω is the union of $\mathtt{K}^*_1, \mathtt{K}^*_2$ in Ω_1, Ω_2. In particular, an edge $\sigma^* = [x_{\mathtt{K}_1}, x_{\mathtt{K}_2}]$ can be split into $\sigma^* = \sigma^*_1 \cup \sigma^*_2 = [x_{\mathtt{K}_1}, x_{\mathtt{L}}] \cup [x_{\mathtt{L}}, x_{\mathtt{K}_2}]$;
- an edge $\sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}]$ on the interface Γ is shared by all the meshes.

We add some fluxes unknowns $\Psi_{\mathfrak{T}_j}$ on each dual cell that intersect Γ , i.e. $\forall \mathtt{K}_\mathtt{j}^* \in \partial\mathfrak{M}_{j,\Gamma}^*$. Those unknowns approximate the dual fluxes $\mathcal{F}_{\sigma^*\mathtt{K}^*}$ on the interface; in particular, for a diamond $\mathtt{D} \in \mathfrak{D}_j^\Gamma$, the unknowns are illustrated in Fig. IV.3.

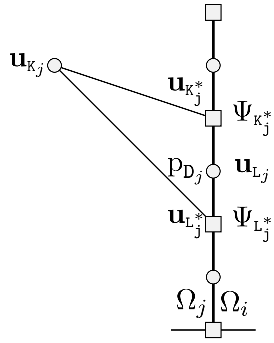


Fig. IV.3 The unknowns on a diamond on the interface for the subdomain Ω_j.

To obtain our scheme, we integrate the momentum equation over $\mathfrak{M}_j \cup \mathfrak{M}_j^* \cup \partial\mathfrak{M}_{j,\Gamma}^*$, we impose Dirichlet boundary conditions on $\partial\mathfrak{M}_{j,D} \cup \partial\mathfrak{M}_{j,D}^*$ and transmission conditions on $\partial\mathfrak{M}_{j,\Gamma} \cup \partial\mathfrak{M}_{j,\Gamma}^*$ (depending on λ). The equation of conservation of mass is directly approximated on the diamond mesh equation over $\mathfrak{D}_j$, and it is stabilized through a parameter $\beta > 0$ with a Brezzi-Pitkäranta stabilization (see Sec. I.7); for the diamonds in $\mathfrak{D}_j^\Gamma$ a transmission term is added, controlled by the parameter α.

We then define the DDFV discretization for the transmission conditions the following system:

Find $(\mathbf{u}_{\mathfrak{T}_j}, \mathrm{p}_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \partial\mathfrak{M}_{j,\Gamma}^*$ such that

$$
\left\{
\begin{aligned}
&m_\mathtt{K} \frac{\mathbf{u}_\mathtt{K}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_\mathtt{K}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_\mathtt{K}\mathbf{f}_\mathtt{K} + m_\mathtt{K} \frac{\bar{\mathbf{u}}_\mathtt{K}}{\delta t} && \forall \mathtt{K} \in \mathfrak{M}_j \\
&m_{\mathtt{K}^*} \frac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} + m_\mathtt{K} \frac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} && \forall \mathtt{K}^* \in \mathfrak{M}_j^* \\
&m_{\mathtt{K}^*} \frac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} + m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi_{\mathtt{K}^*} = m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} + m_\mathtt{K} \frac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} && \forall \mathtt{K}^* \in \partial\mathfrak{M}_{j,\Gamma}^* \\
&-\mathcal{F}_{\sigma\mathtt{K}} + \frac{1}{2} F_{\sigma\mathtt{K}}\mathbf{u}_\mathtt{L} + \lambda\mathbf{u}_\mathtt{L} = \mathbf{h}_\mathtt{L} && \forall \sigma \in \partial\mathfrak{M}_{j,\Gamma} \\
&-\Psi_{\mathtt{K}^*} + \frac{1}{2} H_{\mathtt{K}^*}\, \mathbf{u}_{\mathtt{K}^*} + \lambda\mathbf{u}_{\mathtt{K}^*} = \mathbf{h}_{\mathtt{K}^*} && \forall \mathtt{K}^* \in \partial\mathfrak{M}_{j,\Gamma}^* \\
&\mathbf{u}^{\partial\mathfrak{M}_{j,D}} = 0 \\
&\mathbf{u}^{\partial\mathfrak{M}_{j,D}^*} = 0 \\
&m_\mathtt{D}\mathrm{div}^\mathtt{D}(\mathbf{u}^\mathfrak{T}) - \beta m_\mathtt{D} d_\mathtt{D}^2 \Delta^\mathtt{D}\mathrm{p}^\mathfrak{D} = 0 && \forall \mathtt{D} \in \mathfrak{D}_j \setminus \mathfrak{D}_j^\Gamma \\
&m_\mathtt{D}\mathrm{div}^\mathtt{D}(\mathbf{u}^\mathfrak{T}) - \beta m_\mathtt{D} d_\mathtt{D}^2 \Delta^\mathtt{D}\mathrm{p}^\mathfrak{D} + \alpha m_\mathtt{D}\mathrm{p}^\mathtt{D} = g_\mathtt{D} && \forall \mathtt{D} \in \mathfrak{D}_j^\Gamma,
\end{aligned}
\right. \tag{$\mathcal{P}_j$}
$$

with $\lambda, \beta, \alpha > 0$ and $\bar{\mathbf{u}}_{\mathfrak{T}_j}$ the solution computed at the previous time step $t_{n-1} = (n-1)\delta t$ for $n \in \{1, \dots N-1\}$.
We will refer to the system $(\mathcal{P}_j)$ in the more compact form:

$$
\mathcal{L}_{\Omega_j,\Gamma}^{\mathfrak{T}_j,\mu}(\mathbf{u}_{\mathfrak{T}_j}, \mathrm{p}_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}, \mathbf{f}_\mathfrak{T}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}_{\mathfrak{T}_j}, g_{\mathfrak{D}_j}) = 0. \tag{IV.10}
$$

Remark IV.2.2 *When we impose transmission conditions in Schwarz algorithm, the term on the boundary that we want to approximate is* $\int_\sigma \left(\sigma(\mathbf{u},p)\cdot\vec{\mathbf{n}} - \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\right)$*; this comes from the anti-symmetrization of the convection term.*
Formally, at the continuous level, if φ *is a test function in* $V = \{\varphi \in (H^1(\Omega))^2,\ \Psi|_{\Gamma_D} = 0,\ div(\varphi) = 0\}$*, the variational formulation of* (IV.1) *reads:*

$$\int_\Omega \partial_t \mathbf{u}\cdot\varphi + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi - \int_\Omega \operatorname{div}(\sigma(\mathbf{u},p))\varphi = 0. \tag{IV.11}$$

The convection term can be written as

$$\begin{aligned}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi &= \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi\\ &= \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\varphi\cdot\mathbf{u} + \int_{\partial\Omega}\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u},\end{aligned}$$

by integration by parts. If we inject this result in (IV.11) *and we integrate by parts also the diffusion terms, we end up with:*

$$\int_\Omega \partial_t\mathbf{u}\cdot\varphi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\varphi\cdot\mathbf{u} + \int_\Omega \sigma(\mathbf{u},p) : \nabla\varphi - \int_{\partial\Omega}\left(\sigma(\mathbf{u},p)\vec{\mathbf{n}} - \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\right)\cdot\varphi = 0.$$

This is the reason why, when working with transmission conditions, we want to impose a condition on $\sigma(\mathbf{u},p)\vec{\mathbf{n}} - \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}$*, that contains just half of the convection. Remark then that the numerical flux* $\mathcal{F}_{\sigma\mathtt{K}}$ *is constructed to approximate the term*

$$\mathcal{F}_{\sigma\mathtt{K}} \approx \int_\sigma \left(-\sigma(\mathbf{u},p)\vec{\mathbf{n}} + (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\right).$$

This is why in the approximation it gives:

$$\sigma(\mathbf{u},p)\vec{\mathbf{n}} - \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u} = \sigma(\mathbf{u},p)\vec{\mathbf{n}} - (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u} + \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u} \approx -\mathcal{F}_{\sigma\mathtt{K}} + \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}}.$$

IV.2.2 DDFV Schwarz algorithm

Let $N \in \mathbb{N}^*$. We note $\delta t = \frac{T}{N}$ and $t_n = n\delta t$ for $n \in \{0,\dots N\}$.

At each time step t_n we apply the following parallel DDFV Schwarz algorithm: for arbitrary initial guesses $\mathbf{h}^0_{\mathfrak{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$ and $g^0_{\mathfrak{D}_j} \in \mathbb{R}^{\mathfrak{D}_j}$, at each iteration $l = 1,2,\dots$ and $i,j,\in\{1,2\}$, $j \neq i$ the algorithm performs two steps:

1. Compute $(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}) \in \mathbb{R}^{\mathfrak{T}_j}\times\mathbb{R}^{\mathfrak{D}_j}\times\mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}}$ solution to

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}_j}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}^{l-1}_{\mathfrak{T}_j}, g^{l-1}_{\mathfrak{D}^\Gamma_j}) = 0. \tag{$\mathcal{S}_1$}$$

2. Compute the new values of $\mathbf{h}^l_{\mathfrak{T}_J}$ and of $g^l_{\mathfrak{D}^\Gamma_j}$ by:

$$\begin{aligned}
\mathbf{h}^l_{\mathtt{L_j}} &= \mathcal{F}^l_{\sigma\mathtt{K}_i} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{u}^l_{\mathtt{L_i}} + \lambda\mathbf{u}^l_{\mathtt{L_i}}, && \forall \mathtt{L}_j = \mathtt{L}_i \in \partial\mathfrak{M}_{j,\Gamma}\\
\mathbf{h}^l_{\mathtt{K}^*_\mathtt{j}} &= \Psi^l_{\mathtt{K}^*_\mathtt{i}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{u}^l_{\mathtt{K}^*_\mathtt{i}} + \lambda\mathbf{u}^l_{\mathtt{K}^*_\mathtt{i}}, && \forall \mathtt{K}^*_\mathtt{j} \in \partial\mathfrak{M}^*_{j,\Gamma} \text{ such that } x_{\mathtt{K}^*_\mathtt{j}} = x_{\mathtt{K}^*_\mathtt{i}}\\
g^l_{\mathtt{D}_j} &= -\left(m_{\mathtt{D}_i}\operatorname{div}^{\mathtt{D}_i}(\mathbf{u}^l_{\mathfrak{T}_i}) - \beta m_{\mathtt{D}_i} d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\mathrm{p}^l_{\mathfrak{D}_i}\right) + \alpha m_{\mathtt{D}_i}\mathrm{p}^l_{\mathtt{D}_i}, && \forall \mathtt{D}_j \in \mathfrak{D}^\Gamma_j \text{ such that } x_{\mathtt{D}_j} = x_{\mathtt{D}_i}
\end{aligned} \tag{$\mathcal{S}_2$}$$

To prove that the algorithm is well-posed, we need the following result:

Theorem IV.2.3 (A priori estimate on $(\mathcal{P}_j)$**)** *The scheme* $(\mathcal{P}_j)$ *satisfies the following:*

$$\begin{aligned}\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|_2^2 &+ \frac{2}{Re}\|D^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j}\|_2^2 - (p^{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j})_{\mathfrak{D}_j}\\
&+ \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^{\Gamma}} m_\sigma(\mathcal{F}_{\sigma\mathtt{K}} - \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}})\cdot\mathbf{u}_{\mathtt{L}} + \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}_{j,\Gamma}^*} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Psi_{\mathtt{K}^*} - \frac{1}{2}H_{\mathtt{K}^*}\mathbf{u}_{\mathtt{K}^*})\cdot\mathbf{u}_{\mathtt{K}^*}\\
&+ \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2Rem_{\mathtt{D}}}B_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}|^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2Rem_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}|^2 = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}.\end{aligned} \quad \text{(IV.12)}$$

Proof We multiply the equations on the primal and dual mesh of $(\mathcal{P}_j)$ by $\mathbf{u}_{\mathfrak{T}_j}$ and we sum over all the control volumes:

$$\begin{aligned}\frac{1}{2}\Bigg[\frac{1}{\delta t}\Bigg(\sum_{\mathtt{K}\in\mathfrak{M}_j} m_{\mathtt{K}}|\mathbf{u}_{\mathtt{K}}|^2 + \sum_{\mathtt{K}\in\mathfrak{M}_j^*\cup\partial\mathfrak{M}_j^*} m_{\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2\Bigg) + \sum_{\mathtt{K}^*\in\partial\mathfrak{M}_{j,\Gamma}^*} m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi_{\mathtt{K}^*}\cdot\mathbf{u}_{\mathtt{K}^*}\\
+ \sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}} + \sum_{\mathtt{K}^*\in\mathfrak{M}_j^*\cup\partial\mathfrak{M}_j^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*}\Bigg] = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}.\end{aligned} \quad \text{(IV.13)}$$

By definition of the scalar products we have $\frac{1}{2}\Bigg[\frac{1}{\delta t}\Bigg(\sum_{\mathtt{K}\in\mathfrak{M}_j} m_{\mathtt{K}}|\mathbf{u}_{\mathtt{K}}|^2 + \sum_{\mathtt{K}\in\mathfrak{M}_j^*\cup\partial\mathfrak{M}_j^*} m_{\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2\Bigg)\Bigg] = \frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|^2$ and, by rewriting the fluxes as a sum of the diffusive and convective contribution we have:

$$\begin{aligned}\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|^2 + \frac{1}{2}\Bigg[\sum_{\mathtt{K}^*\in\partial\mathfrak{M}_{j,\Gamma}^*} m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi_{\mathtt{K}^*}\cdot\mathbf{u}_{\mathtt{K}^*} + \sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}}^d + \sum_{\mathtt{K}^*\in\mathfrak{M}_j^*\cup\partial\mathfrak{M}_j^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*}^d\\
+ \sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}}^c + \sum_{\mathtt{K}^*\in\mathfrak{M}_j^*\cup\partial\mathfrak{M}_j^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*}^c\Bigg] = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}.\end{aligned}$$

We consider separately the two contributions. For the **diffusion terms**, we have, by the definition of the divergence operator (Def. I.3.5):

$$\begin{aligned}\frac{1}{2}\Bigg[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}}^d + \sum_{\mathtt{K}^*\in\mathfrak{M}_j^*\cup\partial\mathfrak{M}_j^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*}^d\Bigg]\\
= -\Bigg[\Bigg[\mathrm{div}^{\mathfrak{T}_j}\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}_{\mathfrak{T}_j} - \mathrm{p}^{\mathfrak{D}_j}\mathrm{Id}\right), \mathbf{u}_{\mathfrak{T}_j}\Bigg]\Bigg]_{\mathfrak{T}_j} - \frac{1}{4}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}_{j,\Gamma}^*}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}^{ext}} m_\sigma\mathcal{F}_{\sigma\mathtt{K}}^d.\end{aligned}$$

We can now apply Green's formula to the RHS, and remark that $\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} \mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}} = \sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot(\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*})$. We thus find:

$$\frac{1}{2}\left[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^d_{\sigma^*\mathtt{K}^*}\right]$$
$$=\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}^{\mathfrak{T}}\|_2^2-(\mathrm{p}^{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}_j}+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})-\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot\frac{\mathbf{u}_{\mathtt{K}^*}+\mathbf{u}_{\mathtt{L}^*}}{4}.$$

By the definition of the trace operator, we obtain:

$$\frac{1}{2}\left[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^d_{\sigma^*\mathtt{K}^*}\right]$$
$$=\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}^{\mathfrak{T}}\|_2^2-(\mathrm{p}^{\mathfrak{D}_j},\mathrm{div}^{\mathfrak{D}_j}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}_j}+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}. \quad \text{(IV.14)}$$

For the **convection terms**:

$$\frac{1}{2}\left[\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}+\sum_{\mathtt{K}^*\in\mathfrak{M}^*_j\cup\partial\mathfrak{M}^*_j}\mathbf{u}_{\mathtt{K}^*}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}^c_{\sigma^*\mathtt{K}^*}\right]:=\frac{1}{2}(T_1+T_2)$$

We estimate the term T_1; we first integrate by parts:

$$T_1=\sum_{\mathtt{K}\in\mathfrak{M}_j}\mathbf{u}_{\mathtt{K}}\cdot\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}$$
$$=\sum_{\mathtt{D}\in\mathfrak{D}_j} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}.$$

We replace the definition of $\mathcal{F}^c_{\sigma\mathtt{K}}$ for all $\mathtt{D}\in\mathfrak{D}_j$:

$$T_1=\sum_{\mathtt{D}\in\mathfrak{D}_j} m_\sigma F_{\sigma\mathtt{K}}\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{L}}}{2}\cdot(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}}|^2+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}$$
$$=\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j} m_\sigma F_{\sigma\mathtt{K}}(|\mathbf{u}_{\mathtt{K}}|^2-|\mathbf{u}_{\mathtt{L}}|^2)+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}}|^2+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}.$$

Passing to the sum over primal cells $\mathtt{K}$ for the first term and applying Prop. IV.1.2 we get:

$$T_1=\frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}_j}|\mathbf{u}_{\mathtt{K}}|^2\underbrace{\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma F_{\sigma\mathtt{K}}}_{=0}-\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma F_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{L}}|^2+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}}|^2+\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}\cdot\mathbf{u}_{\mathtt{L}}.$$

It can be rewritten as:

$$T_1=\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma(\mathcal{F}^c_{\sigma\mathtt{K}}-\frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}})\cdot\mathbf{u}_{\mathtt{L}}+\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}}|^2 \quad \text{(IV.15)}$$

We estimate the term T_2; we first integrate by parts:

$$T_2 = \sum_{\kappa^* \in \mathfrak{M}_j^* \cup \partial\mathfrak{M}_j^*} \mathbf{u}_{\kappa^*} \cdot \sum_{\mathrm{D}\in\mathfrak{D}_{\kappa^*}} m_{\sigma^*} \mathcal{F}^c_{\sigma^*\kappa^*}$$
$$= \sum_{\mathrm{D}\in\mathfrak{D}_j} m_{\sigma^*} \mathcal{F}^c_{\sigma^*\kappa^*} \cdot (\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}).$$

We replace the definition of $\mathcal{F}^c_{\sigma\kappa}$ for all $\mathrm{D} \in \mathfrak{D}_j$:

$$T_2 = \sum_{\mathrm{D}\in\mathfrak{D}_j} m_{\sigma^*} F_{\sigma^*\kappa^*} \frac{\mathbf{u}_{\kappa^*} + \mathbf{u}_{\mathrm{L}^*}}{2} \cdot (\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}) + \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma^*\kappa^*} |\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}|^2$$
$$= \frac{1}{2} \sum_{\mathrm{D}\in\mathfrak{D}_j} m_{\sigma^*} F_{\sigma^*\kappa^*} (|\mathbf{u}_{\kappa^*}|^2 - |\mathbf{u}_{\mathrm{L}^*}|^2) + \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma^*\kappa^*} |\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}|^2.$$

Passing to the sum over dual cells κ^* for the first term we get:

$$T_2 = \frac{1}{2} \sum_{\kappa^* \in \mathfrak{M}_j^* \cup \partial\mathfrak{M}_j^*} |\mathbf{u}_{\kappa^*}|^2 \sum_{\mathrm{D}\in\mathfrak{D}_{\kappa^*}} m_{\sigma^*} F_{\sigma^*\kappa^*} + \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma^*\kappa^*} |\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}|^2. \tag{IV.16}$$

From the definition of $F_{\sigma^*\kappa^*}$ and by Prop. IV.1.2 we have that $\sum_{\mathrm{D}\in\mathfrak{D}_{\kappa^*}} m_{\sigma^*} F_{\sigma^*\kappa^*} = 0$ for all $\kappa^* \in \mathfrak{M}_j^*$ and $\sum_{\mathrm{D}\in\mathfrak{D}_{\kappa^*}} m_{\sigma^*} F_{\sigma^*\kappa^*} = -m_{\partial\Omega\cap\partial\kappa^*} H_{\kappa^*}$ for all $\kappa^* \in \partial\mathfrak{M}_{j,\Gamma}^*$, that gives:

$$T_2 = -\frac{1}{2} \sum_{\kappa^* \in \partial\mathfrak{M}_{j,\Gamma}^*} m_{\partial\Omega\cap\partial\kappa^*} H_{\kappa^*} |\mathbf{u}_{\kappa^*}|^2 + \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma^*\kappa^*} |\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}|^2.$$

If we put the estimates (IV.14),(IV.15) and (IV.16) together, we find:

$$\begin{aligned}
&\frac{1}{\delta t} \|\mathbf{u}_{\mathfrak{T}_j}\|^2 + \frac{2}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}_j} \mathbf{u}_{\mathfrak{T}_j}\|_2^2 - (\mathrm{p}^{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j} \mathbf{u}_{\mathfrak{T}_j})_{\mathfrak{D}_j} \\
&+ \frac{1}{2} \sum_{\mathrm{D}\in\mathfrak{D}_j^\Gamma} m_\sigma (\mathcal{F}^d_{\sigma\kappa} + \mathcal{F}^c_{\sigma\kappa} - \frac{1}{2} F_{\sigma\kappa} \mathbf{u}_{\mathrm{L}}) \cdot \mathbf{u}_{\mathrm{L}} + \frac{1}{2} \sum_{\kappa^* \in \partial\mathfrak{M}_{j,\Gamma}^*} m_{\partial\Omega\cap\partial\kappa^*} \Psi_{\kappa^*} \cdot \mathbf{u}_{\kappa^*} - \frac{1}{4} \sum_{\kappa^* \in \partial\mathfrak{M}_{j,\Gamma}^*} m_{\partial\Omega\cap\partial\kappa^*} H_{\kappa^*} |\mathbf{u}_{\kappa^*}|^2 \\
&\quad + \frac{1}{2} \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma\kappa} |\mathbf{u}_\kappa - \mathbf{u}_{\mathrm{L}}|^2 + \frac{1}{2} \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma^*\kappa^*} |\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}|^2 = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j},
\end{aligned}$$

that, since $\mathcal{F}^d_{\sigma\kappa} + \mathcal{F}^c_{\sigma\kappa} = \mathcal{F}_{\sigma\kappa}$, leads to our result:

$$\begin{aligned}
&\frac{1}{\delta t} \|\mathbf{u}_{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}_j} \mathbf{u}_{\mathfrak{T}_j}\|_2^2 - (\mathrm{p}^{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j} \mathbf{u}_{\mathfrak{T}_j})_{\mathfrak{D}_j} \\
&\quad + \frac{1}{2} \sum_{\mathrm{D}\in\mathfrak{D}_j^\Gamma} m_\sigma (\mathcal{F}_{\sigma\kappa} - \frac{1}{2} F_{\sigma\kappa} \mathbf{u}_{\mathrm{L}}) \cdot \mathbf{u}_{\mathrm{L}} + \frac{1}{2} \sum_{\kappa^* \in \partial\mathfrak{M}_{j,\Gamma}^*} m_{\partial\Omega\cap\partial\kappa^*} (\Psi_{\kappa^*} - \frac{1}{2} H_{\kappa^*} \mathbf{u}_{\kappa^*}) \cdot \mathbf{u}_{\kappa^*} \\
&\qquad + \frac{1}{2} \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma\kappa} |\mathbf{u}_\kappa - \mathbf{u}_{\mathrm{L}}|^2 + \frac{1}{2} \sum_{\mathrm{D}\in\mathfrak{D}_j} \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathrm{D}}} B_{\sigma^*\kappa^*} |\mathbf{u}_{\kappa^*} - \mathbf{u}_{\mathrm{L}^*}|^2 = [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j}.
\end{aligned}$$

■

Thanks to Thm. IV.2.3, we are able to deduce the following:

Theorem IV.2.4 (Wellposedness of the DDFV subdomain problem) *Under the hypothesis $(\mathcal{H}p)$ and $\lambda, \beta, \alpha > 0$, the problem $(\mathcal{P}_j)$ is well-posed.*

Proof By linearity we can prove that if $\mathbf{f}_{\mathfrak{T}_j} = 0 = \mathbf{h}^{\mathfrak{T}_j} = g^{\mathfrak{D}_j}$, then $\mathbf{u}_{\mathfrak{T}_j} = 0 = \Psi^{\mathfrak{T}_j}$ and $\mathrm{p}^{\mathfrak{D}_j} = 0$. Starting from the estimate (IV.12) of Thm. IV.2.3, we apply:

- the transmission conditions on the sums over $\mathfrak{D}_j^\Gamma$ and $\partial\mathfrak{M}^*_{j,\Gamma}$:

$$-\mathcal{F}_{\sigma\mathtt{K}} + \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}} + \lambda\mathbf{u}_{\mathtt{L}} = \mathbf{h}_{\mathtt{L}} \quad \forall\sigma \in \partial\mathfrak{M}_{j,\Gamma},$$

$$-\Psi_{\mathtt{K}^*} + \frac{1}{2}H_{\mathtt{K}^*}\,\mathbf{u}_{\mathtt{K}^*} + \lambda\mathbf{u}_{\mathtt{K}^*} = \mathbf{h}_{\mathtt{K}^*} \quad \forall\mathtt{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma},$$

- the conditions on the equation of conservation of mass:

$$\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}}\mathrm{p}^{\mathfrak{D}} = 0 \quad \forall\mathtt{D} \in \mathfrak{D}_j \setminus \mathfrak{D}_j^\Gamma,$$

$$\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}}\mathrm{p}^{\mathfrak{D}} + \alpha m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}} = g_{\mathtt{D}} \quad \forall\mathtt{D} \in \mathfrak{D}_j^\Gamma.$$

This implies:

$$\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{u}^{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}^{\mathfrak{T}_j}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 \\
&\qquad + \alpha\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}}|\mathrm{p}^{\mathtt{D}}|^2 + \frac{\lambda}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathbf{u}_{\mathtt{L}}|^2 + \frac{\lambda}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2 \\
&\quad + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}|^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}|^2 \\
&= [[\mathbf{f}_{\mathfrak{T}_j}, \mathbf{u}_{\mathfrak{T}_j}]]_{\mathfrak{T}_j} + (\mathrm{p}^{\mathfrak{D}_j}, g^{\mathfrak{D}_j})_{\mathfrak{D}_j^\Gamma} + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma\mathbf{h}_{\mathtt{L}}\cdot\mathbf{u}_{\mathtt{L}} + \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}\mathbf{h}_{\mathtt{K}^*}\cdot\mathbf{u}_{\mathtt{K}^*}. \qquad \text{(IV.17)}
\end{aligned}$$

If now we impose $\mathbf{f}_{\mathfrak{T}_j} = 0 = \mathbf{h}^{\mathfrak{T}_j} = g^{\mathfrak{D}_j}$ in (IV.17), we have:

$$\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{u}_{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}^{\mathfrak{T}}\|_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 \\
&\qquad \underbrace{+\alpha\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}}|\mathrm{p}^{\mathtt{D}}|^2 + \frac{\lambda}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathbf{u}_{\mathtt{L}}|^2 + \frac{\lambda}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^2}_{\geq 0} \\
&\qquad\qquad + \underbrace{\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma\mathtt{K}}|\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}|^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m_{\sigma^*}^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma^*\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}|^2}_{\geq 0} = 0,
\end{aligned}$$

that leads to:

$$\frac{1}{\delta t}||\mathbf{u}_{\mathfrak{T}_j}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}_j}\mathbf{u}^{\mathfrak{T}}||_2^2 + \beta|\mathrm{p}^{\mathfrak{D}}|_h^2 \leq 0,$$

from which we deduce that $\mathbf{u}_{\mathfrak{T}_j} = 0$ and $\mathrm{p}^{\mathfrak{D}_j}$ is a constant (we recall that $\beta > 0$). Thanks to the transmission conditions on $\mathfrak{D}_j^\Gamma$, since $\alpha > 0$ and $\mathbf{u}_{\mathfrak{T}_j} = 0$, we obtain $\mathrm{p}^{\mathfrak{D}_j} = 0$. Finally, thanks to the transmission condition on $\partial\mathfrak{M}^*_{j,\Gamma}$ and $\mathbf{u}_{\mathfrak{T}_j} = 0$, we also have $\Psi^{\mathfrak{T}_j} = 0$. ■

IV.3 Convergence analysis of the DDFV Schwarz algorithm

Before studying the convergence of the DDFV Schwarz algorithm towards the solution of the Navier-Stokes problem defined on Ω, we recall some remarks that may be useful to understand the analysis. Consider the composite mesh of Fig. IV.2; remark that:

- a diamond $\mathtt{D}$ that intersects Γ in the domain Ω can be written as the union of diamonds $\mathtt{D}_1, \mathtt{D}_2$ respectively in Ω_1, Ω_2. Moreover, on the subdomain meshes we have the additional unknowns $\mathrm{u}_{\mathtt{L}}$ on Γ with respect to the mesh on Ω;
- equivalently, a volume $\mathtt{K}^*$ that intersects Γ in Ω is the union of $\mathtt{K}_1^*, \mathtt{K}_2^*$ in Ω_1, Ω_2. In particular, an edge $\sigma^* = [x_{\mathtt{K}_1}, x_{\mathtt{K}_2}]$ can be split into $\sigma^* = \sigma_1^* \cup \sigma_2^* = [x_{\mathtt{K}_1}, x_{\mathtt{L}}] \cup [x_{\mathtt{L}}, x_{\mathtt{K}_2}]$;
- an edge $\sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}]$ on the interface Γ is shared by all the meshes.

Those characterizations of the meshes imply that, at the limit, the fluxes need to satisfy the following:

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = -m_\sigma \mathcal{F}_{\sigma\mathtt{K}_2}, \quad \forall \mathtt{D} \in \mathfrak{D}^\Gamma \tag{IV.18}$$

$$m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\sigma_1^*} \mathcal{F}_{\sigma_1^*\mathtt{K}^*} + m_{\sigma_2^*} \mathcal{F}_{\sigma_2^*\mathtt{K}^*}, \quad \forall \sigma^* = \sigma_1^* \cup \sigma_2^*,\ \mathtt{K}^* \in \partial\mathfrak{M}_\Gamma^*. \tag{IV.19}$$

In order to obtain this properties, it will become necessary to modify the fluxes on the interface, either for the limit or for the subdomain problem. For this reason, the convergence will be studied in two steps.

The first step will be to identify the limit of Schwarz algorithm defined in Sec. IV.2.2. We will show that this limit is still a DDFV scheme for the problem (IV.1), but with modified fluxes on Γ. We will then prove convergence to this limit scheme, to which we will refer as $(\widetilde{\mathcal{P}})$.

Secondly, in Sec. IV.4, we will show that it is possible to modify the fluxes of the Schwarz algorithm $(\mathcal{S}_1)$ in order to converge exactly to $(\mathcal{P})$.

IV.3.1 The limit problem $(\widetilde{\mathcal{P}})$

The following scheme is another DDFV approximation of (IV.1), on the domain Ω; what changes with respect to $(\mathcal{P})$ are the fluxes on the interface Γ.

To obtain our scheme, we will not detail the steps here since we proceed as in Sec. IV.1.
Given $(\bar{\mathbf{u}}^{\mathfrak{T}}, \bar{\mathrm{p}}^{\mathfrak{D}})$, satisfying $\mathrm{div}^{\mathfrak{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) - \beta d_{\mathtt{D}}^2 \Delta^{\mathfrak{D}} \bar{\mathrm{p}}^{\mathfrak{D}} = 0$, we look for $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ and $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\left\{\begin{array}{rll} m_{\mathtt{K}} \dfrac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \displaystyle\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}\setminus\mathfrak{D}_{\mathtt{K}}^\Gamma} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}^\Gamma} m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} &= m_{\mathtt{K}} \mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} & \forall \mathtt{K} \in \mathfrak{M} \\ m_{\mathtt{K}^*} \dfrac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \displaystyle\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}\setminus\mathfrak{D}_{\mathtt{K}^*}^\Gamma} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}^\Gamma} m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} &= m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} & \forall \mathtt{K}^* \in \mathfrak{M}^* \\ \mathbf{u}^{\partial\mathfrak{M}} &= 0 & \\ \mathbf{u}^{\partial\mathfrak{M}^*} &= 0 & \\ m_{\mathtt{D}} \mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d_{\mathtt{D}}^2 \Delta^{\mathtt{D}} \mathrm{p}^{\mathfrak{D}} &= 0 & \forall \mathtt{D} \in \mathfrak{D} \\ \displaystyle\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} \mathrm{p}^{\mathtt{D}} &= 0, & \end{array}\right. \tag{$\widetilde{\mathcal{P}}$}$$

with $\beta > 0$. At the interior of the domain, the fluxes coincide with the fluxes in $(\mathcal{P})$, see (IV.5). On the interface, they are defined as:

$$\boxed{\begin{aligned} m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} &= -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} \widetilde{B}_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}), \\ m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} &= -m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2}\right) + \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathtt{D}}} \widetilde{B}_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}), \end{aligned}}$$

where $\widetilde{B}_{\sigma\mathtt{K}}$ and $\widetilde{B}_{\sigma^*\mathtt{K}^*}$ are defined in the next section in Prop. IV.3.4 and Prop. IV.3.5.

IV.3.1.1 Definition of $\widetilde{B}_{\sigma\mathtt{K}}$ and $\widetilde{B}_{\sigma^*\mathtt{K}^*}$

We first give the following definition of some matrices that will be useful in the following; the index i, j relating to the subdomain will be specified if the matrix is not symmetric with respect to the subdomains.

Definition IV.3.1 *We define for $i = 1, 2$, $\sigma \in \partial\mathfrak{M}_{i,\Gamma}$ and $P = Id + \vec{\mathbf{n}}_{\sigma\mathtt{K}} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}}$ the matrix:*

$$A_i = \frac{m_\sigma^2}{2Re m_{\mathtt{D}_i}}(P + B_{\sigma\mathtt{K}_i} Id),$$

where we recall that $B_{\sigma\mathtt{K}_i} = B\left(\frac{2m_{\mathtt{D}_i} Re}{m_\sigma} F_{\sigma\mathtt{K}_i}\right)$ and we define:

$$A = A_1 + A_2.$$

Remark IV.3.2 *The matrix $A = A_1 + A_2$ is symmetric and definite positive, thus invertible, since it is the sum of two symmetric and definite positive matrices.*
In fact, thanks to the definition of P, if $\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \begin{pmatrix} x \\ y \end{pmatrix}$, we have:

$$A_i = \begin{pmatrix} 1 + B_{\sigma\mathtt{K}_i} + x^2 & xy \\ xy & 1 + B_{\sigma\mathtt{K}_i} + y^2 \end{pmatrix},$$

which is symmetric and for any $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ it holds:

$$\langle A_i v, v\rangle = (1 + B_{\sigma\mathtt{K}_i})(v_1^2 + v_2^2) + (xv_1 + yv_2)^2 \geq 0 \quad \textit{and} \quad \langle A_i v, v\rangle = 0 \iff v = 0,$$

thanks to Hypothesis $(\mathcal{H}p)$, which implies $B_{\sigma\mathtt{K}_i} \geq 0$.
For $i, j = 1, 2$, $i \neq j$, since A_i and A_j are polynomial in P, the following properties hold:

$$A_i A_j = A_j A_i,$$
$$A_j A^{-1} = A^{-1} A_j,$$

since from Hyp. $(\mathcal{H}p)$ we have $B_{\sigma\mathtt{K}_i} = B_{\sigma\mathtt{K}_j}$ for $\sigma \in \partial\mathfrak{M}_{i,\Gamma}$.

The fluxes $\widetilde{\mathcal{F}}_{\sigma\mathtt{K}}, \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*}$ are constructed in order to satisfy the properties (IV.18)-(IV.19) defined in the introduction of Sec. IV.3.
It is important to recall that $(\widetilde{\mathcal{P}})$ is a scheme defined on the mesh $\mathfrak{T}$ on Ω; in particular, this means that there are no additional unknowns $\mathbf{u}_{\mathtt{L}}$ on the interface Γ, see Fig. IV.2. The following results apply for a general diamond:

Proposition IV.3.3 *Let $\mathtt{D} \in \mathfrak{D}_\Gamma$ be a diamond and let $\mathtt{D}_1, \mathtt{D}_2$ be the two semi-diamonds such that $\mathtt{D} = \mathtt{D}_1 \cup \mathtt{D}_2$, see Fig. IV.4 . We denote by $(x_{\mathtt{K}}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}})$ its vertices and by $(x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_\sigma)$, $(x_{\mathtt{K}_2}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_\sigma)$ the vertices of $\mathtt{D}_1$ and $\mathtt{D}_2$. Then, there exists a unique $\mathbf{u}_{\boldsymbol{\sigma}}$ that, for $\sigma = \mathtt{K}_1|\mathtt{K}_2$, verifies*

$$\mathcal{F}_{\sigma\mathtt{K}_1} = -\mathcal{F}_{\sigma\mathtt{K}_2} \tag{IV.20}$$

given by:

$$\mathbf{u}_{\boldsymbol{\sigma}} = A^{-1}\left[A_1\mathbf{u}_{\mathtt{K}_1} + A_2\mathbf{u}_{\mathtt{K}_2} + \frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2})\right]. \tag{IV.21}$$

where A, A_1 and A_2 are given in Def. IV.3.1.

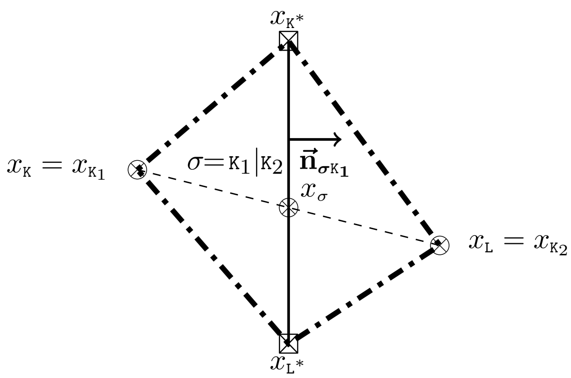


Fig. IV.4 A diamond $\mathtt{D}$, of vertices $x_{\mathtt{K}}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_{\mathtt{L}}$ as a union of two semi-diamonds: $\mathtt{D}_1$ of vertices $x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_\sigma, x_{\mathtt{L}^*}$ and $\mathtt{D}_2$ of vertices $x_{\mathtt{K}_2}, x_{\mathtt{K}^*}, x_\sigma, x_{\mathtt{L}^*}$. In particular, $\sigma_1^* = [x_{\mathtt{K}_1}, x_{\mathtt{L}}]$ and $\sigma_2^* = [x_{\mathtt{L}}, x_{\mathtt{K}_2}]$.

Proof The condition (IV.20) is a linear equation in $\mathbf{u}_{\boldsymbol{\sigma}}$. In fact, $\mathcal{F}_{\sigma\mathtt{K}_1}$ is a flux on $\mathtt{D}_1$, of vertices $x_{\mathtt{K}_1}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_\sigma$, and $\mathcal{F}_{\sigma\mathtt{K}_2}$ is a flux on $\mathtt{D}_2$, of vertices $x_{\mathtt{K}_2}, x_{\mathtt{K}^*}, x_{\mathtt{L}^*}, x_\sigma$. If we insert the definitions of the fluxes, (IV.20) becomes:

$$\begin{aligned} m_\sigma\mathcal{F}_{\sigma\mathtt{K}_1} &= -m_\sigma\sigma^{\mathtt{D}_1}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1} + m_\sigma F_{\sigma\mathtt{K}_1}\left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\boldsymbol{\sigma}}}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}B_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\boldsymbol{\sigma}}) \\ &= -\mathcal{F}_{\sigma\mathtt{K}_2} = m_\sigma\sigma^{\mathtt{D}_2}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_2} - m_\sigma F_{\sigma\mathtt{K}_2}\left(\frac{\mathbf{u}_{\mathtt{K}_2} + \mathbf{u}_{\boldsymbol{\sigma}}}{2}\right) - \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}B_{\sigma\mathtt{K}_2}(\mathbf{u}_{\mathtt{K}_2} - \mathbf{u}_{\boldsymbol{\sigma}}) \end{aligned} \tag{IV.22}$$

The strain rate tensors can be written by using the matrix P as:

$$\begin{aligned} &- m_\sigma\sigma^{\mathtt{D}_1}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1} \\ &= \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}P(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\boldsymbol{\sigma}}) + \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\otimes\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma\mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}, \end{aligned} \tag{IV.23}$$

$$\begin{aligned} &m_\sigma\sigma^{\mathtt{D}_2}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_2} \\ &\quad = -\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}P(\mathbf{u}_{\mathtt{K}_2} - \mathbf{u}_{\boldsymbol{\sigma}}) - \frac{m_\sigma m_{\sigma_2^*}}{2\mathrm{Re}m_{\mathtt{D}_2}}(\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_2}\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\otimes\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_2})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) - m_\sigma\mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_2}. \end{aligned} \tag{IV.24}$$

If we replace (IV.23), (IV.24) into (IV.22), since $\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1} = -\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_2}$ and $\dfrac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}} = \dfrac{1}{\sin(\alpha_{\mathtt{D}})} = \dfrac{m_\sigma m_{\sigma_2^*}}{2\mathrm{Re}m_{\mathtt{D}_2}}$, the contributions of the pressure $\mathrm{p}^{\mathtt{D}}$ and of the velocity $\mathbf{u}_{\mathtt{K}^*}, \mathbf{u}_{\mathtt{L}^*}$ on the vertices cancel.

So (IV.22) becomes:

$$\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}P(\mathbf{u}_{\mathtt{K}_1}-\mathbf{u}_{\boldsymbol{\sigma}})+m_\sigma F_{\sigma\mathtt{K}_1}\left(\frac{\mathbf{u}_{\mathtt{K}_1}+\mathbf{u}_{\boldsymbol{\sigma}}}{2}\right)+\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}B_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1}-\mathbf{u}_{\boldsymbol{\sigma}})=$$
$$-\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}P(\mathbf{u}_{\mathtt{K}_2}-\mathbf{u}_{\boldsymbol{\sigma}})-m_\sigma F_{\sigma\mathtt{K}_2}\left(\frac{\mathbf{u}_{\mathtt{K}_2}+\mathbf{u}_{\boldsymbol{\sigma}}}{2}\right)-\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}B_{\sigma\mathtt{K}_2}(\mathbf{u}_{\mathtt{K}_2}-\mathbf{u}_{\boldsymbol{\sigma}}).$$

We group the terms in $\mathbf{u}_{\boldsymbol{\sigma}}$ thanks to $F_{\sigma\mathtt{K}_1}=-F_{\sigma\mathtt{K}_2}$, to obtain:

$$\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}\left(P+B_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\mathbf{u}_{\mathtt{K}_1}+\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}\left(P+B_{\sigma\mathtt{K}_2}\mathrm{Id}\right)\mathbf{u}_{\mathtt{K}_2}+\frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1}-\mathbf{u}_{\mathtt{K}_2})=$$
$$\left(\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_1}}\left(P+B_{\sigma\mathtt{K}_1}\mathrm{Id}\right)+\frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_2}}\left(P+B_{\sigma\mathtt{K}_2}\mathrm{Id}\right)\right)\mathbf{u}_{\boldsymbol{\sigma}}. \tag{IV.25}$$

By Def. IV.3.1, (IV.25) becomes:

$$A_1\mathbf{u}_{\mathtt{K}_1}+A_2\mathbf{u}_{\mathtt{K}_2}+\frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1}-\mathbf{u}_{\mathtt{K}_2})=A\mathbf{u}_{\boldsymbol{\sigma}}. \tag{IV.26}$$

It is sufficient to show that this expression is injective; if $(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})$ are equal to zero, we need to deduce that $\mathbf{u}_{\boldsymbol{\sigma}}$ is zero. This is true because, if $(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})$ are zero, this means in particular $\mathbf{u}_{\mathtt{K}_1}=\mathbf{u}_{\mathtt{K}_2}=0$; so condition (IV.26) becomes:

$$A\mathbf{u}_{\boldsymbol{\sigma}}=0.$$

Since the matrix A is definite positive, by Rem. IV.3.2, we deduce $\mathbf{u}_{\mathtt{L}}=0$. ■

The following proposition is a way to obtain property (IV.18), by modifying the fluxes on the interface:

Proposition IV.3.4 *Let* $\mathtt{D}$ *be a diamond and let* $\mathtt{D}_1,\mathtt{D}_2$ *be the two semi-diamonds such that* $\mathtt{D}=\mathtt{D}_1\cup\mathtt{D}_2$*, see Fig. IV.4 . Then there exists a unique flux* $\widetilde{\mathcal{F}}_{\sigma\mathtt{K}}$ *on* $\sigma=\mathtt{K}_1|\mathtt{K}_2$ *such that*

$$m_\sigma\widetilde{\mathcal{F}}_{\sigma\mathtt{K}}=m_\sigma\mathcal{F}_{\sigma\mathtt{K}_1}=-m_\sigma\mathcal{F}_{\sigma\mathtt{K}_2}, \tag{IV.27}$$

given by

$$\widetilde{B}_{\sigma\mathtt{K}}=\frac{2Rem_{\mathtt{D}}}{m_\sigma^2}\left(A_1A_2+\left(\frac{1}{2}m_\sigma F_{\sigma\mathtt{K}}\right)^2 Id\right)A^{-1}-P, \tag{IV.28}$$

$$m_\sigma\widetilde{\mathcal{F}}_{\sigma\mathtt{K}}=-m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathtt{K}}+m_\sigma F_{\sigma\mathtt{K}}\left(\frac{\mathbf{u}_{\mathtt{K}_1}+\mathbf{u}_{\mathtt{K}_2}}{2}\right)+\frac{m_\sigma^2}{2Rem_{\mathtt{D}}}\widetilde{B}_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}_1}-\mathbf{u}_{\mathtt{K}_2}). \tag{IV.29}$$

Proof We consider $\mathcal{F}_{\sigma\mathtt{K}_1}$ and we refer to Fig. IV.4: we recall that it is a flux on the semi-diamond $\mathtt{D}_1$ of vertices $x_{\mathtt{K}_1},x_{\mathtt{K}^*},x_{\mathtt{L}^*},x_\sigma$. Thanks to (IV.23), it can be written as:

$$m_\sigma\mathcal{F}_{\sigma\mathtt{K}_1}=A_1(\mathbf{u}_{\mathtt{K}_1}-\mathbf{u}_{\boldsymbol{\sigma}})+m_\sigma F_{\sigma\mathtt{K}_1}\left(\frac{\mathbf{u}_{\mathtt{K}_1}+\mathbf{u}_{\boldsymbol{\sigma}}}{2}\right)$$
$$+\frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re}m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\sigma\mathtt{K}_1}\cdot\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\mathrm{Id}+\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\otimes\vec{\mathbf{n}}_{\sigma\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*}-\mathbf{u}_{\mathtt{L}^*})+m_\sigma\mathrm{p}^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}_1}.$$

By grouping the terms in $\mathbf{u}_{\mathtt{K}_1}$ and $\mathbf{u}_{\boldsymbol{\sigma}}$ in $m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1}$ and injecting the definition (IV.21) of $\mathbf{u}_{\boldsymbol{\sigma}}$, that ensures (IV.20), i.e. $m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = -m_\sigma \mathcal{F}_{\sigma\mathtt{K}_2}$

$$\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\left(A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\mathbf{u}_{\mathtt{K}_1} + \left(-A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right) A^{-1}\left[A_1 \mathbf{u}_{\mathtt{K}_1} + A_2 \mathbf{u}_{\mathtt{K}_2} + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2})\right] \\ &+ \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re} m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}\cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}. \end{aligned}$$

By regrouping the terms in $\mathbf{u}_{\mathtt{K}_1}$ and $\mathbf{u}_{\mathtt{K}_2}$ we have:

$$\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\left[\left(A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right) + \left(-A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right) A^{-1}\left(A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\right]\mathbf{u}_{\mathtt{K}_1} \\ &+ \left(-A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right) A^{-1}\left(A_2 - \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\right)\mathbf{u}_{\mathtt{K}_2} \\ &+ \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re} m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}\cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}. \end{aligned}$$

As by Rem. IV.3.2, the matrices A and A_i commute, for $i = 1, 2$, we can write:

$$\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\left(A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\left[\overbrace{A - A_1}^{=A_2} + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right] A^{-1}\,\mathbf{u}_{\mathtt{K}_1} \\ &+ \left(-A_1 + \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right)\left(A_2 - \frac{1}{2} m_\sigma F_{\sigma\mathtt{K}_1}\mathrm{Id}\right) A^{-1}\,\mathbf{u}_{\mathtt{K}_2} \\ &+ \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re} m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}\cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}. \end{aligned}$$

We develop the computations and we find:

$$\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\left[\left(A_1 A_2 + \left(\frac{1}{2} m_\sigma F_{\sigma\mathtt{K}}\right)^2 \mathrm{Id}\right) A^{-1}\right](\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) \\ &+ \frac{m_\sigma m_{\sigma_1^*}}{2\mathrm{Re} m_{\mathtt{D}_1}}(\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}\cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1} + m_\sigma F_{\sigma\mathtt{K}_1}\left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\mathtt{K}_2}}{2}\right). \end{aligned}$$

If we define:

$$\widetilde{B}_{\sigma\mathtt{K}} = \frac{2\mathrm{Re} m_{\mathtt{D}}}{m_\sigma^2}\left(A_1 A_2 + \left(\frac{1}{2} m_\sigma F_{\sigma\mathtt{K}}\right)^2 \mathrm{Id}\right) A^{-1} - P,$$

we get:

$$\frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}}(P + \widetilde{B}_{\sigma\mathtt{K}}) = \left(A_1 A_2 + \left(\frac{1}{2} m_\sigma F_{\sigma\mathtt{K}}\right)^2 \mathrm{Id}\right) A^{-1},$$

and since $m_\sigma F_{\sigma\mathtt{K}} = m_\sigma F_{\sigma\mathtt{K}_1} = -m_\sigma F_{\sigma\mathtt{K}_2}$, $\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}} = \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}_1}$ and $\frac{m_{\sigma_1^*}}{m_{\mathtt{D}_1}} = \frac{m_{\sigma^*}}{m_{\mathtt{D}}}$ (see Fig. IV.4), we end up with:

$$\begin{aligned} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = &\frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}}\left(P + \widetilde{B}_{\sigma\mathtt{K}}\right)(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) \\ &+ \frac{m_\sigma m_{\sigma^*}}{2\mathrm{Re} m_{\mathtt{D}}}(\vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}}\cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\mathrm{Id} + \vec{\mathbf{n}}_{\boldsymbol{\sigma}^*\mathtt{K}^*}\otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}})(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) + m_\sigma \mathrm{p}^{\mathtt{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma}\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}}\left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\mathtt{K}_2}}{2}\right). \end{aligned}$$

We remark that now the expression of $m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1}$ depends only on the unknowns $\mathbf{u}_{\mathtt{K}_1}, \mathbf{u}_{\mathtt{K}_2}, \mathbf{u}_{\mathtt{K}^*}, \mathbf{u}_{\mathtt{L}^*}$; so it is a flux defined on the entire diamond $\mathtt{D}$ (see Fig. IV.4). It can be rewritten as:

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}_1} = -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}_1} + \mathbf{u}_{\mathtt{K}_2}}{2} \right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} \widetilde{B}_{\sigma\mathtt{K}}(\mathbf{u}_{\mathtt{K}_1} - \mathbf{u}_{\mathtt{K}_2}) := m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}}.$$

so that we find (IV.29). ■

The following proposition is a way to obtain property IV.19:

Proposition IV.3.5 *Let $\mathtt{D}$ be a diamond and let $\mathtt{D}_1, \mathtt{D}_2$ be the two semi-diamonds such that $\mathtt{D} = \mathtt{D}_1 \cup \mathtt{D}_2$, see Fig. IV.4 . Then, for $\mathtt{K}^* \in \partial\mathfrak{M}^*_\Gamma$, there exists a unique flux $\widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*}$ on $\sigma^* = \sigma_1^* \cup \sigma_2^* = [x_{\mathtt{K}_1}, x_{\mathtt{L}}] \cup [x_{\mathtt{L}}, x_{\mathtt{K}_2}]$ such that*

$$m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = m_{\sigma_1^*} \mathcal{F}_{\sigma_1^*\mathtt{K}^*} + m_{\sigma_2^*} \mathcal{F}_{\sigma_2^*\mathtt{K}^*}, \tag{IV.30}$$

given by

$$\widetilde{B}_{\sigma^*\mathtt{K}^*} = \frac{m_{\sigma_1^*}}{m_{\sigma^*}} B_{\sigma_1^*\mathtt{K}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}} B_{\sigma_2^*\mathtt{K}^*}. \tag{IV.31}$$

$$\widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = -m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) + \frac{m_{\sigma^*}^2}{2Rem_{\mathtt{D}}} \widetilde{B}_{\sigma^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}) \tag{IV.32}$$

Proof This is a direct consequence of the computation of (IV.30). By definition,

$$m_{\sigma_1^*} \mathcal{F}_{\sigma_1^*\mathtt{K}^*} = -m_{\sigma_1^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_1^*} F_{\sigma^*\mathtt{K}_1^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) + \frac{m_{\sigma_1^*}^2}{2\mathrm{Re} m_{\mathtt{D}_1}} B_{\sigma_1^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}),$$

$$m_{\sigma_2^*} \mathcal{F}_{\sigma_2^*\mathtt{K}^*} = -m_{\sigma_2^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_2^*} F_{\sigma^*\mathtt{K}_2^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) + \frac{m_{\sigma_2^*}^2}{2\mathrm{Re} m_{\mathtt{D}_2}} B_{\sigma_2^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}).$$

Since $\sigma^* = \sigma_1^* \cup \sigma_2^*$, we have $m_{\sigma^*} = m_{\sigma_1^*} + m_{\sigma_2^*}$ and $\frac{m_{\sigma_2^*}}{m_{\mathtt{D}_2}} = \frac{m_{\sigma_1^*}}{m_{\mathtt{D}_1}}$; by definition, it holds $m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} = m_{\sigma_1^*} F_{\sigma^*\mathtt{K}_1^*} + m_{\sigma_2^*} F_{\sigma^*\mathtt{K}_2^*}$. So if we take the sum of the two fluxes, we get:

$$\begin{aligned} m_{\sigma_1^*} \mathcal{F}_{\sigma_1^*\mathtt{K}^*} + m_{\sigma_2^*} \mathcal{F}_{\sigma_2^*\mathtt{K}^*} = -m_{\sigma^*} \sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*} F_{\sigma^*\mathtt{K}^*} \left(\frac{\mathbf{u}_{\mathtt{K}^*} + \mathbf{u}_{\mathtt{L}^*}}{2} \right) \\ + \frac{m_{\sigma^*}^2}{2\mathrm{Re} m_{\mathtt{D}_1}} \left[\frac{m_{\sigma_1^*}}{m_{\sigma^*}} B_{\sigma_1^*\mathtt{K}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}} B_{\sigma_2^*\mathtt{K}^*} \right] (\mathbf{u}_{\mathtt{K}^*} - \mathbf{u}_{\mathtt{L}^*}). \end{aligned}$$

By defining $\widetilde{B}_{\sigma^*\mathtt{K}^*} := \frac{m_{\sigma_1^*}}{m_{\sigma^*}} B_{\sigma_1^*\mathtt{K}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}} B_{\sigma_2^*\mathtt{K}^*}$, we obtain (IV.32). ■

Remark IV.3.6 *Remark that, by construction, $(\widetilde{\mathcal{P}})$ is the limit of the Schwarz algorithm defined in Sec. IV.2.2 if and only if* (IV.28) *and* (IV.31) *hold. This is equivalent to say that $\widetilde{B}_{\sigma\mathtt{K}}$ is defined as a function F of $(B_{\sigma\mathtt{K}_1}, B_{\sigma\mathtt{K}_2})$ and $\widetilde{B}_{\sigma^*\mathtt{K}^*}$ defined as a function G of $(B_{\sigma_1^*\mathtt{K}^*}, B_{\sigma_2^*\mathtt{K}^*})$:*

$$\begin{aligned} \widetilde{B}_{\sigma\mathtt{K}} &= F(B_{\sigma\mathtt{K}_1}, B_{\sigma\mathtt{K}_2}), \\ \widetilde{B}_{\sigma^*\mathtt{K}^*} &= G(B_{\sigma_1^*\mathtt{K}^*}, B_{\sigma_2^*\mathtt{K}^*}). \end{aligned} \tag{IV.33}$$

IV.3.1.2 Wellposedness of the limit problem $(\widetilde{\mathcal{P}})$

In the following theorem, we prove that the solution of the limit problem $(\widetilde{\mathcal{P}})$ exists and it is unique. This will rely just on the expression of the new fluxes $\widetilde{\mathcal{F}}_{\sigma\mathtt{K}}, \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*}$.

Theorem IV.3.7 *Under the hypothesis ($\mathcal{H}p$) for $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$, problem ($\widetilde{\mathcal{P}}$) is well-posed.*

Proof By Thm. IV.1.3, we need to verify that hypothesis ($\mathcal{H}p$) holds; since we are supposing it for $B_{\sigma\mathtt{K}}, B_{\sigma^*\mathtt{K}^*}$, we just need to verify it for $\widetilde{B}_{\sigma\mathtt{K}}, \widetilde{B}_{\sigma^*\mathtt{K}^*}$ (the modified fluxes on the interface). As a direct consequence of (IV.28) and (IV.31), we have:

$$\widetilde{B}_{\sigma\mathtt{K}} = \widetilde{B}_{\sigma\mathtt{L}},$$
$$\widetilde{B}_{\sigma^*\mathtt{K}^*} = \widetilde{B}_{\sigma^*\mathtt{L}^*}.$$

In fact, if we consider a diamond on the interface between the two subdomains Ω_1, Ω_2, it can be seen as the one in Fig. IV.4.
For $\mathtt{K} = \mathtt{K}_1$ and $\mathtt{L} = \mathtt{K}_2$, we have:

$$\widetilde{B}_{\sigma\mathtt{K}} = \widetilde{B}_{\sigma\mathtt{K}_1} = \frac{2\mathrm{Re}m_{\mathtt{D}}}{m_\sigma^2}\left(A_1A_2 + \left(\frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_1}\right)^2 \mathrm{Id}\right)A^{-1} - P,$$

$$\widetilde{B}_{\sigma\mathtt{L}} = \widetilde{B}_{\sigma\mathtt{K}_2} = \frac{2\mathrm{Re}m_{\mathtt{D}}}{m_\sigma^2}\left(A_2A_1 + \left(\frac{1}{2}m_\sigma F_{\sigma\mathtt{K}_2}\right)^2 \mathrm{Id}\right)A^{-1} - P,$$

We remark that A, P do not depend on the index of the subdomain; moreover, $m_\sigma F_{\sigma\mathtt{K}_1} = -m_\sigma F_{\sigma\mathtt{K}_2}$, so that $(m_\sigma F_{\sigma\mathtt{K}_1})^2 = (m_\sigma F_{\sigma\mathtt{K}_2})^2$ and $A_1A_2 = A_2A_1$ from Rem. IV.3.2. So we can conclude $\widetilde{B}_{\sigma\mathtt{K}} = \widetilde{B}_{\sigma\mathtt{L}}$.

For the dual flux, we have:

$$\widetilde{B}_{\sigma^*\mathtt{K}^*} = \frac{m_{\sigma_1^*}}{m_{\sigma^*}}B_{\sigma_1^*\mathtt{K}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}}B_{\sigma_2^*\mathtt{K}^*},$$
$$\widetilde{B}_{\sigma^*\mathtt{L}^*} = \frac{m_{\sigma_1^*}}{m_{\sigma^*}}B_{\sigma_1^*\mathtt{L}^*} + \frac{m_{\sigma_2^*}}{m_{\sigma^*}}B_{\sigma_2^*\mathtt{L}^*}.$$

Thanks to Hp. ($\mathcal{H}p$), we have $B_{\sigma_1^*\mathtt{K}^*} = B_{\sigma_1^*\mathtt{L}^*}$ and $B_{\sigma_2^*\mathtt{K}^*} = B_{\sigma_2^*\mathtt{L}^*}$. So we get $\widetilde{B}_{\sigma^*\mathtt{K}^*} = \widetilde{B}_{\sigma^*\mathtt{L}^*}$.

We now have to prove that $\widetilde{B}_{\sigma\mathtt{K}}, \widetilde{B}_{\sigma^*\mathtt{K}^*}$ are semi-definite positive.

If $\vec{\mathbf{n}}_{\sigma\mathtt{K}} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $P = \mathrm{Id} + \vec{\mathbf{n}}_{\sigma\mathtt{K}} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}} = \begin{pmatrix} 1+x^2 & xy \\ xy & 1+y^2 \end{pmatrix}$; if we insert the definition (IV.28) of $\widetilde{B}_{\sigma\mathtt{K}}$, by defining two constants:

$$\mathrm{den} = 4m_\sigma^2(2 + 3B_{\sigma\mathtt{K}} + B_{\sigma\mathtt{K}}^2),$$

and

$$a = (m_{\mathtt{D}}\mathrm{Re}F_{\sigma\mathtt{K}})^2(1 + B_{\sigma\mathtt{K}}) + 8m_\sigma^2 B_{\sigma\mathtt{K}} + 12m_\sigma^2 B_{\sigma\mathtt{K}}^2 + 4m_\sigma^2 B_{\sigma\mathtt{K}}^3,$$

we have

$$\widetilde{B}_{\sigma\mathtt{K}} = \frac{1}{\mathrm{den}}\left[a\mathrm{Id} + (m_{\mathtt{D}}\mathrm{Re}F_{\sigma\mathtt{K}})^2\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}\right].$$

Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$; then:

$$\begin{aligned}\langle \widetilde{B}_{\sigma\mathtt{K}}v, v\rangle &= \frac{1}{\mathrm{den}}a\langle v, v\rangle + \frac{(m_{\mathtt{D}}\mathrm{Re}F_{\sigma\mathtt{K}})^2}{\mathrm{den}}\left(y^2v_1^2 - 2xyv_1v_2 + x^2v_2^2\right)\\ &= \frac{1}{\mathrm{den}}a\|v\|^2 + \frac{(m_{\mathtt{D}}\mathrm{Re}F_{\sigma\mathtt{K}})^2}{\mathrm{den}}(yv_1 - xv_2)^2 \geq 0,\end{aligned}$$

and

$$\langle \widetilde{B}_{\sigma\kappa} v, v\rangle = 0 \iff v = 0,$$

thanks to hypothesis $(\mathcal{H}p)$ on $B_{\sigma\kappa}$, that ensures $a \geq 0$ and den ≥ 0. So $\widetilde{B}_{\sigma\kappa}$ is semi-definite positive.

For what concerns the dual flux, by (IV.31) we obtain directly that $\widetilde{B}_{\sigma^*\kappa^*}$ is semi-definite positive since it is the sum of two semi-definite positive matrices $B_{\sigma_1^*\kappa^*}$ and $B_{\sigma_2^*\kappa^*}$ (by $(\mathcal{H}p)$). ■

IV.3.1.3 Identification of the limit

In order to prove the convergence of the Schwarz algorithm towards the solution of $(\widetilde{\mathcal{P}})$, it is necessary to project this solution, that is defined on Ω, on the subdomains Ω_j, $j = 1, 2$.

Theorem IV.3.8 *Let $\mathfrak{T}$ be the composite mesh $\mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2$ and $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ be the solution of the DDFV scheme $(\widetilde{\mathcal{P}})$ on the domain Ω. For $j \in \{1, 2\}$, there exists a projection $(\mathbf{u}^{\infty}_{\mathfrak{T}_j}, p^{\infty}_{\mathfrak{D}_j}, \Psi^{\infty}_{\mathfrak{T}_j}, h^{\infty}_{\mathfrak{T}_j}, g^{\infty}_{\mathfrak{D}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}} \times \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j}$ of $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$, such that:*

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}^{\infty}_{\mathfrak{T}_j}, p^{\infty}_{\mathfrak{D}_j}, \Psi^{\infty}_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}_j}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}^{\infty}_{\mathfrak{T}_j}, g^{\infty}_{\mathfrak{D}_j}) = 0. \qquad (\mathcal{P}^{\infty})$$

Proof On the primal cells $\mathfrak{M}_j \cup \partial\mathfrak{M}_{j,D}$ and on the dual cells $\mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_{j,D} \cup \partial\mathfrak{M}^*_{j,\Gamma}$ we can simply define the values of $\mathbf{u}^{\infty}_{\mathfrak{T}_j}$ as the values of $\mathbf{u}^{\mathfrak{T}}$:

- for all $\kappa \in \mathfrak{M}_j$ and $\kappa^* \in \mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_{j,\Gamma}$, we set $\mathbf{u}^{\infty}_{\kappa_j} = \mathbf{u}_{\kappa}$ and $\mathbf{u}^{\infty}_{\kappa^*_j} = \mathbf{u}_{\kappa^*}$,
- for all $\kappa \in \partial\mathfrak{M}_{j,D}$ and $\kappa^* \in \partial\mathfrak{M}^*_{j,D}$, we set $\mathbf{u}^{\infty}_{\kappa_j} = 0$ and $\mathbf{u}^{\infty}_{\kappa^*_j} = 0$.
- for all $D \in \mathfrak{D}_j$ such that $x_D \notin \Gamma$, we set $\mathrm{p}^{\infty}_{D_j} = \mathrm{p}^{D}$.
- for all $D \in \mathfrak{D}_j$ such that $x_D \in \Gamma$, $D_j \in \mathfrak{D}^{\Gamma}_j$ and $D_i \in \mathfrak{D}^{\Gamma}_i$, we set $\mathrm{p}^{\infty}_{D_j} = \mathrm{p}^{\infty}_{D_i} = \mathrm{p}^{D}$.

We then need to introduce new unknowns near the boundary Γ:

- for all $L \in \partial\mathfrak{M}_{j,\Gamma}$, we impose (see Prop. IV.3.3):

$$\mathbf{u}^{\infty}_{L} = \mathbf{u}^{\infty}_{L_j} = \mathbf{u}^{\infty}_{L_i} = A^{-1}\left[A_j\mathbf{u}_{\kappa_j} + A_i\mathbf{u}_{\kappa_i} + \frac{1}{2}m_\sigma F_{\sigma\kappa_1}(\mathbf{u}_{\kappa_j} - \mathbf{u}_{\kappa_i})\right]. \qquad \text{(IV.34)}$$

- for all $\kappa^* \in \mathfrak{M}^*$ such that $x_{\kappa^*} \in \Gamma$, $\kappa^* = \kappa^*_j \cup \kappa^*_i$ with $\kappa^*_j \in \partial\mathfrak{M}^*_{j,\Gamma}$, we impose:

$$\Psi^{\infty}_{\kappa^*_j} = -\Psi^{\infty}_{\kappa^*_i} = -\frac{m_{\kappa^*_j}}{m_{\partial\Omega\cap\partial\kappa^*}}\frac{\mathbf{u}^{\infty}_{\kappa^*_j}}{\delta t} - \frac{1}{m_{\partial\Omega\cap\partial\kappa^*}}\sum_{D\in\mathfrak{D}_{\kappa^*_j}}\mathcal{F}^{\infty}_{\sigma^*_j\kappa^*} + \frac{m_{\kappa^*_j}}{m_{\partial\Omega\cap\partial\kappa^*}}\mathbf{f}_{\kappa^*_j} + \frac{m_{\kappa^*_j}}{m_{\partial\Omega\cap\partial\kappa^*}}\frac{\bar{\mathbf{u}}_{\kappa^*_j}}{\delta t}. \qquad \text{(IV.35)}$$

- for all $L = L_j \in \partial\mathfrak{M}_{j,\Gamma}$ and for all $\kappa^* \in \mathfrak{M}^*$ such that $x_{\kappa^*} \in \Gamma$, $\kappa^* = \kappa^*_j \cup \kappa^*_i$ with $\kappa^*_j \in \partial\mathfrak{M}^*_{j,\Gamma}$, $\kappa^*_i \in \partial\mathfrak{M}^*_{i,\Gamma}$we impose:

$$\mathbf{h}^{\infty}_{L_j} = \mathcal{F}^{\infty}_{\sigma\kappa_i} - \frac{1}{2}F_{\sigma\kappa_i}\mathbf{u}^{\infty}_{L} + \lambda\mathbf{u}^{\infty}_{L},$$
$$\mathbf{h}^{\infty}_{\kappa^*_j} = \Psi^{\infty}_{\kappa^*_i} - \frac{1}{2}H_{\kappa^*_i}\,\mathbf{u}^{\infty}_{\kappa^*_i} + \lambda\mathbf{u}^{\infty}_{\kappa^*_i}.$$

- for all $D \in \mathfrak{D}$ such that $x_D \in \Gamma$, $D_j \in \mathfrak{D}^{\Gamma}_j$ and $D_i \in \mathfrak{D}^{\Gamma}_i$, we set

$$g^{\infty}_{D_j} = -\left(m_{D_i}\mathrm{div}^{D_i}(\mathbf{u}^{\infty}_{\mathfrak{T}_i}) - \beta m_{D_i} d^2_{D_i}\Delta^{D_i}\mathrm{p}^{\infty}_{D_i}\right) + \alpha m_{D_i}\mathrm{p}^{\infty}_{D_i}.$$

Consequence on the equations.
We now show that from a solution $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ of the DDFV scheme $(\widetilde{\mathcal{P}})$ we built a solution to $(\mathcal{P}^{\infty})$.

- $\forall \mathtt{K} \in \mathfrak{M}$, $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ satisfies:

$$m_{\mathtt{K}} \frac{\mathbf{u}_{\mathtt{K}}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}} \setminus \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = m_{\mathtt{K}} \mathbf{f}_{\mathtt{K}} + m_{\mathtt{K}} \frac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t}.$$

If we look at the composite mesh (see Fig. IV.2), we can remark that the primal cells $\mathtt{K} \in \mathfrak{M}$ correspond to $\mathtt{K_j} \in \mathfrak{M}_j$ (or to $\mathtt{K_i} \in \mathfrak{M}_i$). This implies that $m_{\mathtt{K}} = m_{\mathtt{K_j}}$, $m_{\mathtt{K}} \mathbf{f}_{\mathtt{K}} = \int_{\mathtt{K}} \mathbf{f}(x) \mathrm{d}x = m_{\mathtt{K_j}} \mathbf{f}_{\mathtt{K}_j}$ and $m_{\mathtt{K}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K}}}{\delta t} = m_{\mathtt{K_j}} \dfrac{\bar{\mathbf{u}}_{\mathtt{K_j}}}{\delta t}$.
Moreover, for a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}} \setminus \mathfrak{D}_{\mathtt{K}}^{\Gamma}$, remark that the limit unknowns $\mathbf{u}_{\mathtt{K}_j}^{\infty}, \mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^{\infty}, \mathrm{p}_{\mathtt{D}_j}^{\infty}$ on $\mathfrak{T}_j$ for $j = 1, 2$ coincide with $\mathbf{u}_{\mathtt{K}}, \mathbf{u}_{\mathtt{K}^*}, \mathrm{p}^{\mathtt{D}}$ on $\mathfrak{T}$; so if

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty} = -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}_j}^{\infty}, \mathrm{p}_{\mathfrak{D}_j}^{\infty})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}_j} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}_j}^{\infty} + \mathbf{u}_{\mathtt{L}_j}^{\infty}}{2} \right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} B_{\sigma\mathtt{K}} (\mathbf{u}_{\mathtt{K}_j}^{\infty} - \mathbf{u}_{\mathtt{L}_j}^{\infty}),$$

we have:

$$\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}} \setminus \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}} = \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K_j}} \setminus \mathfrak{D}_{\mathtt{K}_j}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty}.$$

For a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{\Gamma}$, if

$$m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty} = -m_\sigma \sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}_j}^{\infty}, \mathrm{p}_{\mathfrak{D}_j}^{\infty})\, \vec{\mathbf{n}}_{\sigma\mathtt{K}_j} + m_\sigma F_{\sigma\mathtt{K}} \left(\frac{\mathbf{u}_{\mathtt{K}_j}^{\infty} + \mathbf{u}_{\mathtt{L}}^{\infty}}{2} \right) + \frac{m_\sigma^2}{2\mathrm{Re} m_{\mathtt{D}}} B_{\sigma\mathtt{K}_j} (\mathbf{u}_{\mathtt{K}_j}^{\infty} - \mathbf{u}_{\mathtt{L}}^{\infty}),$$

thanks to the choice (IV.34) of $\mathbf{u}_{\mathtt{L}}^{\infty}$ for all $\mathtt{L} \in \partial\mathfrak{M}_{j,\Gamma}$ and thanks to Prop. IV.3.4, we have

$$m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty},$$

that implies:

$$\sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{\Gamma}} m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathtt{K}} = \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}_j}^{\Gamma}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty}.$$

So in the end $(\mathbf{u}_{\mathfrak{T}_j}^{\infty}, \mathrm{p}_{\mathfrak{D}_j}^{\infty}, \Psi_{\mathfrak{T}_j}^{\infty})$ satisfies:

$$\boxed{m_{\mathtt{K_j}} \frac{\mathbf{u}_{\mathtt{K}_j}^{\infty}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K_j}}} m_\sigma \mathcal{F}_{\sigma\mathtt{K}_j}^{\infty} = m_{\mathtt{K_j}} \mathbf{f}_{\mathtt{K_j}} + m_{\mathtt{K_j}} \frac{\bar{\mathbf{u}}_{\mathtt{K_j}}}{\delta t}, \qquad \forall \mathtt{K_j} \in \mathfrak{M}_j.} \tag{IV.36}$$

- $\forall \mathtt{K}^* \in \mathfrak{M}^*$, $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ satisfies:

$$m_{\mathtt{K}^*} \frac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*} \setminus \mathfrak{D}_{\mathtt{K}^*}^{\Gamma}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}^{\Gamma}} m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \frac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t}. \tag{IV.37}$$

We need to distinguish two cases.

1. If $\partial\mathtt{K}^* \cap \Gamma = \emptyset$, equation (IV.37) reduces to:

$$m_{\mathtt{K}^*} \frac{\mathbf{u}_{\mathtt{K}^*}}{\delta t} + \sum_{\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathtt{K}^*} = m_{\mathtt{K}^*} \mathbf{f}_{\mathtt{K}^*} + m_{\mathtt{K}^*} \frac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t},$$

and the cells $\mathtt{K}^* \in \mathfrak{M}^*$ correspond to $\mathtt{K}_\mathtt{j}^* \in \mathfrak{M}_j^*$ (or to $\mathtt{K}_\mathtt{i} \in \mathfrak{M}_i^*$). This implies that $m_{\mathtt{K}^*} = m_{\mathtt{K}_\mathtt{j}^*}$, $m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} = \int_{\mathtt{K}^*} \mathbf{f}(x)\mathrm{d}x = m_{\mathtt{K}_\mathtt{j}^*}\mathbf{f}_{\mathtt{K}_\mathtt{j}^*}$, $m_{\mathtt{K}^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} = m_{\mathtt{K}_\mathtt{j}^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{j}^*}}{\delta t}$ and $m_{\sigma^*} = m_{\sigma_j^*}$.
Moreover, for a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*} \setminus \mathfrak{D}_{\mathtt{K}^*}^\Gamma$, (that is the case here since we are supposing $\partial\mathtt{K}^* \cap \Gamma = \emptyset$) remark that the limit unknowns $\mathbf{u}_{\mathtt{K}_j}^\infty, \mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^\infty, \mathrm{p}_{\mathtt{D}_j}^\infty$ on $\mathfrak{T}_j$ for $j = 1,2$ coincide with $\mathbf{u}_\mathtt{K}, \mathbf{u}_{\mathtt{K}^*}, \mathrm{p}^\mathtt{D}$ on $\mathfrak{T}$. So if

$$m_{\sigma^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^\infty = -m_{\sigma_j^*}\sigma^{\mathtt{D}_j}(\mathbf{u}_{\mathfrak{T}_j}^\infty, \mathrm{p}_{\mathfrak{D}_j}^\infty)\,\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_j^*}F_{\sigma_j^*\mathtt{K}^*}\left(\frac{\mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^\infty + \mathbf{u}_{\mathtt{L}_\mathtt{j}^*}^\infty}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_j}}B_{\sigma_j^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^\infty - \mathbf{u}_{\mathtt{L}_\mathtt{j}^*}^\infty),$$

we have:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{j}^*}} m_{\sigma^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^\infty.$$

So $(\mathbf{u}_{\mathfrak{T}_j}^\infty, \mathrm{p}_{\mathfrak{D}_j}^\infty, \Psi_{\mathfrak{T}_j}^\infty)$ satisfies on the interior dual mesh:

$$\boxed{m_{\mathtt{K}_\mathtt{j}^*}\frac{\mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^\infty}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{j}^*}} m_\sigma\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^\infty = m_{\mathtt{K}_\mathtt{j}}\mathbf{f}_{\mathtt{K}_\mathtt{j}^*} + m_{\mathtt{K}_\mathtt{j}^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{j}^*}}{\delta t}, \qquad \forall \mathtt{K}_\mathtt{j}^* \in \mathfrak{M}_j^*.} \tag{IV.38}$$

2. If $\partial\mathtt{K}^* \cap \Gamma \neq \emptyset$, the cell $\mathtt{K}^*$ can be written as the union of $\mathtt{K}_\mathtt{j} \in \partial\mathfrak{M}_{j,\Gamma}^*$ and $\mathtt{K}_\mathtt{i} \in \partial\mathfrak{M}_{i,\Gamma}^*$. This implies that $m_{\mathtt{K}^*} = m_{\mathtt{K}_\mathtt{j}^*} + m_{\mathtt{K}_\mathtt{i}^*}$, $m_{\sigma^*} = m_{\sigma_j^*} + m_{\sigma_i^*}$, $m_{\mathtt{K}^*}\mathbf{f}_{\mathtt{K}^*} = \int_{\mathtt{K}^*}\mathbf{f}(x)\mathrm{d}x = m_{\mathtt{K}_\mathtt{j}^*}\mathbf{f}_{\mathtt{K}_\mathtt{j}^*} + m_{\mathtt{K}_\mathtt{i}^*}\mathbf{f}_{\mathtt{K}_\mathtt{i}^*}$ and $m_{\mathtt{K}^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}^*}}{\delta t} = m_{\mathtt{K}_\mathtt{j}^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{j}^*}}{\delta t} + m_{\mathtt{K}_\mathtt{i}^*}\dfrac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{i}^*}}{\delta t}$.
Moreover, for a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*} \setminus \mathfrak{D}_{\mathtt{K}^*}^\Gamma$, remark that the limit unknowns $\mathbf{u}_{\mathtt{K}_j}^\infty, \mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^\infty, \mathrm{p}_{\mathtt{D}_j}^\infty$ on $\mathfrak{T}_j$ for $j = 1,2$ coincide with $\mathbf{u}_\mathtt{K}, \mathbf{u}_{\mathtt{K}^*}, \mathrm{p}^\mathtt{D}$ on $\mathfrak{T}$. So if

$$m_{\sigma^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^\infty = -m_{\sigma^*}\sigma^{\mathtt{D}}(\mathbf{u}_{\mathfrak{T}_j}^\infty, \mathrm{p}_{\mathfrak{D}_j}^\infty)\,\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma^*}F_{\sigma_j^*\mathtt{K}^*}\left(\frac{\mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^\infty + \mathbf{u}_{\mathtt{L}_\mathtt{j}^*}^\infty}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}}}B_{\sigma_j^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}_\mathtt{j}^*}^\infty - \mathbf{u}_{\mathtt{L}_\mathtt{j}^*}^\infty),$$

and

$$m_{\sigma^*}\mathcal{F}_{\sigma_i^*\mathtt{K}^*}^\infty = -m_{\sigma_i^*}\sigma^{\mathtt{D}_j}(\mathbf{u}_{\mathfrak{T}_i}^\infty, \mathrm{p}_{\mathfrak{D}_i}^\infty)\,\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + m_{\sigma_i^*}F_{\sigma_i^*\mathtt{K}^*}\left(\frac{\mathbf{u}_{\mathtt{K}_\mathtt{i}^*}^\infty + \mathbf{u}_{\mathtt{L}_\mathtt{i}^*}^\infty}{2}\right) + \frac{m_\sigma^2}{2\mathrm{Re}m_{\mathtt{D}_i}}B_{\sigma_i^*\mathtt{K}^*}(\mathbf{u}_{\mathtt{K}_\mathtt{i}^*}^\infty - \mathbf{u}_{\mathtt{L}_\mathtt{i}^*}^\infty),$$

we have:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}\setminus\mathfrak{D}_{\mathtt{K}^*}^\Gamma} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{j}^*}\setminus\mathfrak{D}_{\mathtt{K}_\mathtt{j}^*}^\Gamma} m_{\sigma_j^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^\infty + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{i}^*}\setminus\mathfrak{D}_{\mathtt{K}_\mathtt{i}^*}^\Gamma} m_{\sigma_i^*}\mathcal{F}_{\sigma_i^*\mathtt{K}^*}^\infty.$$

For a diamond $\mathtt{D} \in \mathfrak{D}_{\mathtt{K}^*}^\Gamma$, thanks to (IV.30), we have

$$m_{\sigma^*}\widetilde{\mathcal{F}}_{\sigma^*\mathtt{K}^*} = m_{\sigma_j^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^\infty + m_{\sigma_i^*}\mathcal{F}_{\sigma_i^*\mathtt{K}^*}^\infty,$$

that implies:

$$\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}^\Gamma} m_{\sigma^*}\mathcal{F}_{\sigma^*\mathtt{K}^*} = \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{j}^*}^\Gamma} m_{\sigma_j^*}\mathcal{F}_{\sigma_j^*\mathtt{K}^*}^\infty + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{i}^*}^\Gamma} m_{\sigma_i^*}\mathcal{F}_{\sigma_i^*\mathtt{K}^*}^\infty.$$

We deduce from (IV.37):

$$m_{\mathtt{K}_\mathtt{j}^*}\frac{\mathbf{u}^\infty_{\mathtt{K}_\mathtt{j}^*}}{\delta t}+m_{\mathtt{K}_\mathtt{i}^*}\frac{\mathbf{u}^\infty_{\mathtt{K}_\mathtt{i}^*}}{\delta t}+\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{j}^*}} m_{\sigma_j^*}\mathcal{F}^\infty_{\sigma_j^*\mathtt{K}^*}+\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{i}^*}} m_{\sigma_i^*}\mathcal{F}^\infty_{\sigma_i^*\mathtt{K}^*}$$
$$=m_{\mathtt{K}_\mathtt{j}^*}\mathbf{f}_{\mathtt{K}_\mathtt{j}^*}+m_{\mathtt{K}_\mathtt{i}^*}\mathbf{f}_{\mathtt{K}_\mathtt{i}^*}+m_{\mathtt{K}_\mathtt{j}^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{j}^*}}{\delta t}+m_{\mathtt{K}_\mathtt{i}^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{i}^*}}{\delta t}.$$

By definition, $\Psi^\infty_{\mathtt{K}_\mathtt{i}^*}$ satisfies:

$$\Psi^\infty_{\mathtt{K}_\mathtt{i}^*}=-\Psi^\infty_{\mathtt{K}_\mathtt{j}^*}=-\frac{m_{\mathtt{K}_\mathtt{i}^*}}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\frac{\mathbf{u}^\infty_{\mathtt{K}_\mathtt{i}^*}}{\delta t}-\frac{1}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{i}^*}} m_{\sigma_i^*}\mathcal{F}^\infty_{\sigma_i^*\mathtt{K}^*}+\frac{m_{\mathtt{K}_\mathtt{i}^*}}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\mathbf{f}_{\mathtt{K}_\mathtt{i}^*}+\frac{m_{\mathtt{K}_\mathtt{i}^*}}{m_{\partial\Omega\cap\partial\mathtt{K}^*}}\frac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{i}^*}}{\delta t},$$

so $(\mathbf{u}^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}_j},\Psi^\infty_{\mathfrak{T}_j})$ satisfies on the boundary dual mesh:

$$\boxed{m_{\mathtt{K}_\mathtt{j}^*}\frac{\mathbf{u}^\infty_{\mathtt{K}_\mathtt{j}^*}}{\delta t}+\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}_\mathtt{j}^*}} m_{\sigma_j^*}\mathcal{F}^\infty_{\sigma_j^*\mathtt{K}^*}+m_{\partial\Omega\cap\partial\mathtt{K}^*}\Psi^\infty_{\mathtt{K}_\mathtt{j}^*}=m_{\mathtt{K}_\mathtt{j}^*}\mathbf{f}_{\mathtt{K}_\mathtt{j}^*}+m_{\mathtt{K}_\mathtt{j}^*}\frac{\bar{\mathbf{u}}_{\mathtt{K}_\mathtt{j}^*}}{\delta t},\quad \forall\mathtt{K}_\mathtt{j}^*\in\partial\mathfrak{M}^*_{j,\Gamma}.}\tag{IV.39}$$

- $\forall\mathtt{K}\in\mathfrak{M}$, with $\partial\mathtt{K}\cap\Gamma\neq\emptyset$, if we look at the composite mesh, the diamond $\mathtt{D}\in\mathfrak{D}^\Gamma_\mathtt{K}$ can be written as the union of $\mathtt{D}_j\in\mathfrak{D}^\Gamma_{\mathtt{K}_j}$ and $\mathtt{D}_i\in\mathfrak{D}^\Gamma_{\mathtt{K}_i}$. By definition, we have $F_{\sigma\mathtt{K}_j}=-F_{\sigma\mathtt{K}_i}$; moreover, thanks to the choice (IV.34) of $\mathbf{u}^\infty_\mathtt{L}$ for all $\mathtt{L}\in\partial\mathfrak{M}_{j,\Gamma}$ and thanks to Prop. IV.3.4, we have $m_\sigma\mathcal{F}^\infty_{\sigma\mathtt{K}_j}=-m_\sigma\mathcal{F}^\infty_{\sigma\mathtt{K}_i}$.
From the definition of $\mathbf{h}^\infty_{\mathfrak{T}_i}$, we get the relation:

$$\mathbf{h}^\infty_\mathtt{L}=\mathcal{F}^\infty_{\sigma\mathtt{K}_i}-\frac12F_{\sigma\mathtt{K}_i}\mathbf{u}^\infty_\mathtt{L}+\lambda\mathbf{u}^\infty_\mathtt{L}=-\mathcal{F}^\infty_{\sigma\mathtt{K}_j}+\frac12F_{\sigma\mathtt{K}_j}\mathbf{u}^\infty_\mathtt{L}+\lambda\mathbf{u}^\infty_\mathtt{L}.$$

So $(\mathbf{u}^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}_j},\Psi^\infty_{\mathfrak{T}_j})$ satisfies:

$$\boxed{\mathcal{F}^\infty_{\sigma\mathtt{K}_i}-\frac12F_{\sigma\mathtt{K}_i}\mathbf{u}^\infty_\mathtt{L}+\lambda\mathbf{u}^\infty_\mathtt{L}=-\mathcal{F}^\infty_{\sigma\mathtt{K}_j}+\frac12F_{\sigma\mathtt{K}_j}\mathbf{u}^\infty_\mathtt{L}+\lambda\mathbf{u}^\infty_\mathtt{L}.}\tag{IV.40}$$

- $\forall\mathtt{K}^*\in\mathfrak{M}^*$, with $\partial\mathtt{K}^*\cap\Gamma\neq\emptyset$, the cell $\mathtt{K}^*$ can be written as the union of $\mathtt{K}_\mathtt{j}\in\partial\mathfrak{M}^*_{j,\Gamma}$ and $\mathtt{K}_\mathtt{i}\in\partial\mathfrak{M}^*_{i,\Gamma}$. By definition, we have $H_{\mathtt{K}_\mathtt{j}^*}=-H_{\mathtt{K}_\mathtt{i}^*}$ and $\Psi^\infty_{\mathtt{K}_\mathtt{i}^*}=-\Psi^\infty_{\mathtt{K}_\mathtt{j}^*}$. This leads, from the definition of $\mathbf{h}^\infty_{\mathfrak{T}_i}$, to the relation:

$$\mathbf{h}^\infty_{\mathtt{K}^*j}=\Psi^\infty_{\mathtt{K}_\mathtt{i}^*}-\frac12H_{\mathtt{K}_\mathtt{i}^*}\mathbf{u}^\infty_{\mathtt{K}_\mathtt{i}^*}+\lambda\mathbf{u}^\infty_{\mathtt{K}_\mathtt{i}^*}=-\Psi^\infty_{\mathtt{K}_\mathtt{j}^*}+\frac12H_{\mathtt{K}_\mathtt{j}^*}\mathbf{u}^\infty_{\mathtt{K}_\mathtt{j}^*}+\lambda\mathbf{u}^\infty_{\mathtt{K}_\mathtt{j}^*}.$$

So $(\mathbf{u}^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}_j},\Psi^\infty_{\mathfrak{T}_j})$ satisfies:

$$\boxed{\Psi^\infty_{\mathtt{K}_\mathtt{i}^*}-\frac12H_{\mathtt{K}_\mathtt{i}^*}\mathbf{u}^\infty_{\mathtt{K}_\mathtt{i}^*}+\lambda\mathbf{u}^\infty_{\mathtt{K}_\mathtt{i}^*}=-\Psi^\infty_{\mathtt{K}_\mathtt{j}^*}+\frac12H_{\mathtt{K}_\mathtt{j}^*}\mathbf{u}^\infty_{\mathtt{K}_\mathtt{j}^*}+\lambda\mathbf{u}^\infty_{\mathtt{K}_\mathtt{j}^*}.}\tag{IV.41}$$

- for all $\mathtt{D}\in\mathfrak{D}$, $(\mathbf{u}^\mathfrak{T},\mathrm{p}^\mathfrak{D})$ satisfies:

$$m_\mathtt{D}\mathrm{div}^\mathtt{D}(\mathbf{u}^\mathfrak{T})-\beta m_\mathtt{D}d_\mathtt{D}^2\Delta^\mathtt{D}\mathrm{p}^\mathtt{D}=0,\qquad\forall\mathtt{D}\in\mathfrak{D}.\tag{IV.42}$$

We need to distinguish two cases:

1. If $\mathtt{D}\cap\Gamma=\emptyset$, the diamond $\mathtt{D}$ coincides with a diamond $\mathtt{D}_j\in\mathfrak{D}_j$ (or with a diamond $\mathtt{D}_i\in\mathfrak{D}_i$). For a diamond $\mathtt{D}\in\mathfrak{D}\setminus\mathfrak{D}^\Gamma$, remark that the limit unknowns $\mathbf{u}^\infty_{\mathtt{K}_j},\mathbf{u}^\infty_{\mathtt{K}^*_j},\mathrm{p}^\infty_{\mathtt{D}_j}$ on $\mathfrak{T}_j$ for $j=1,2$ coincide with $\mathbf{u}_{\mathtt{K}},\mathbf{u}_{\mathtt{K}^*},\mathrm{p}^{\mathtt{D}}$ on $\mathfrak{T}$. Thus we can directly deduce that $(\mathbf{u}^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}_j},\Psi^\infty_{\mathfrak{T}_j})$ satisfies $\forall\mathtt{D}_j\in\mathfrak{D}_j\setminus\mathfrak{D}^\Gamma_j$:

$$\boxed{m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j})-\beta m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\mathrm{p}^\infty_{\mathtt{D}_j}=0.} \tag{IV.43}$$

2. If $\mathtt{D}\cap\Gamma\neq\emptyset$, the diamond $\mathtt{D}$ can be written as the union of $\mathtt{D}_j\in\mathfrak{D}^\Gamma_j$ and $\mathtt{D}_i\in\mathfrak{D}^\Gamma_i$. This implies that the divergence can be split as : $m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}})=m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j})+m_{\mathtt{D}_i}\mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^\infty_{\mathfrak{T}_i})$. From (IV.42), the choice of unknowns $\mathrm{p}^\infty_{\mathfrak{D}}$ and from the definition of $g^\infty_{\mathtt{D}_j}$ we obtain:

$$\begin{aligned} g^\infty_{\mathtt{D}_j} &= -\left(m_{\mathtt{D}_i}\mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^\infty_{\mathfrak{T}_i})-\beta m_{\mathtt{D}_i}d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\mathrm{p}^\infty_{\mathtt{D}_i}\right)+\alpha m_{\mathtt{D}_i}\mathrm{p}^\infty_{\mathtt{D}_i}\\ &= \left(m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j})-\beta m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\mathrm{p}^\infty_{\mathtt{D}_j}\right)+\alpha m_{\mathtt{D}_j}\mathrm{p}^\infty_{\mathtt{D}_j}, \end{aligned}$$

that implies for $(\mathbf{u}^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}_j},\Psi^\infty_{\mathfrak{T}_j})$ that $\forall\mathtt{D}_j\in\mathfrak{D}^\Gamma_j$:

$$\boxed{\begin{aligned} &-(m_{\mathtt{D}_i}\mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^\infty_{\mathfrak{T}_i})-\beta m_{\mathtt{D}_i}d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\mathrm{p}^\infty_{\mathtt{D}_i})+\alpha m_{\mathtt{D}_i}\mathrm{p}^\infty_{\mathtt{D}_i}=\\ &\qquad(m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{u}^\infty_{\mathfrak{T}_j})-\beta m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\mathrm{p}^\infty_{\mathtt{D}_j})+\alpha m_{\mathtt{D}_j}\mathrm{p}^\infty_{\mathtt{D}_j}\end{aligned}} \tag{IV.44}$$

To recapitulate, (IV.36), (IV.38), (IV.39), (IV.40), (IV.41),(IV.43), (IV.44) show that $(\mathbf{u}^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}_j},\Psi^\infty_{\mathfrak{T}_j})$ is a solution to $(\mathcal{P}^\infty)$. ■

IV.3.1.4 Convergence of the DDFV Schwarz algorithm towards $(\widetilde{\mathcal{P}})$

Theorem IV.3.9 (Convergence of the discrete Schwarz algorithm) *Under the hypothesis that $m_{\sigma^*}=2m_{\sigma^*_i}=2m_{\sigma^*_j}$ for $i,j=1,2$, $i\neq j$, the iterates of the Schwarz algorithm $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ converge as l tends to infinity to the solution of the DDFV scheme $(\widetilde{\mathcal{P}})$ (up to a constant for the pressure).*

Proof The iterates of $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ satisfy:

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}^l_{\mathfrak{T}_j},\mathrm{p}^l_{\mathtt{D}_j},\Psi^l_{\mathfrak{T}_j},\mathbf{f}_{\mathfrak{T}},\bar{\mathbf{u}}_{\mathfrak{T}_j},\mathbf{h}^{l-1}_{\mathfrak{T}_j},g^{l-1}_{\mathfrak{D}_j})=0,$$

and $(\mathbf{u}^\infty_{\mathfrak{T}_j},\Psi^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}})$, constructed from the solution of $(\widetilde{\mathcal{P}})$ is solution to:

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}^\infty_{\mathfrak{T}_j},\mathrm{p}^\infty_{\mathfrak{D}_j},\Psi^\infty_{\mathfrak{T}_j},\mathbf{f}_{\mathfrak{T}_j},\bar{\mathbf{u}}_{\mathfrak{T}_j},\mathbf{h}^\infty_{\mathfrak{T}_j},g^\infty_{\mathfrak{D}_j})=0.$$

We define the errors

$$\begin{aligned} \mathbf{e}^l_{\mathfrak{T}_j}&=\mathbf{u}^l_{\mathfrak{T}_j}-\mathbf{u}^\infty_{\mathfrak{T}_j},\\ \Phi^l_{\mathfrak{T}_j}&=\Psi^l_{\mathfrak{T}_j}-\Psi^\infty_{\mathfrak{T}_j},\\ \Pi^l_{\mathfrak{D}_j}&=\mathrm{p}^l_{\mathfrak{D}_j}-\mathrm{p}^\infty_{\mathfrak{D}_j}. \end{aligned} \tag{IV.45}$$

By linearity, they satisfy:

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{e}^l_{\mathfrak{T}_j},\Pi^l_{\mathfrak{D}_j},\Phi^l_{\mathfrak{T}_j},0,0,\mathbf{H}^{l-1}_{\mathfrak{T}_j},\mathbf{G}^{l-1}_{\mathfrak{D}_j})=0, \tag{IV.46}$$

with

$$\begin{aligned}
&\mathbf{H}^{l-1}_{\mathtt{L}_j} = \mathcal{F}^{l-1}_{\sigma\mathtt{K}_i} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{e}^{l-1}_{\mathtt{L_i}} + \lambda\mathbf{e}^{l-1}_{\mathtt{L_i}}, && \forall \mathtt{L}_j = \mathtt{L}_i \in \partial\mathfrak{M}_{j,\Gamma}\\
&\mathbf{H}^{l-1}_{\mathtt{K}^*_\mathtt{j}} = \Phi^{l-1}_{\mathtt{K}^*_\mathtt{i}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{i}}\,\mathbf{e}^{l-1}_{\mathtt{K}^*_\mathtt{i}} + \lambda\mathbf{e}^{l-1}_{\mathtt{K}^*_\mathtt{i}}, && \forall \mathtt{K}^*_\mathtt{j} \in \partial\mathfrak{M}^*_{j,\Gamma} \text{ such that } x_{\mathtt{K}^*_\mathtt{j}} = x_{\mathtt{K}^*_\mathtt{i}}\\
&\mathbf{G}^{l-1}_{\mathtt{D}_j} = -\big(m_{\mathtt{D}_i}\mathrm{div}^{\mathtt{D}_i}(\mathbf{e}^{l-1}_{\mathfrak{T}_i}) - \beta m_{\mathtt{D}_i} d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\Pi^{l-1}_{\mathtt{D}_i}\big) + \alpha m_{\mathtt{D}_i}\Pi^{l-1}_{\mathtt{D}_i} && \forall \mathtt{D}_j \in \mathfrak{D}^\Gamma_j \text{ such that } x_{\mathtt{D}_j} = x_{\mathtt{D}_i}.
\end{aligned}$$

To prove the convergence of the iterates of Schwarz algorithm, it is sufficient to prove the convergence to 0 of the solution of (IV.46). In the expanded form, (IV.46) is written as:

$$\left\{\begin{aligned}
m_\mathtt{K}\frac{\mathbf{e}^l_{\mathtt{K}_j}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_\mathtt{K}} m_\sigma \mathcal{F}^l_{\sigma\mathtt{K}_j} &= 0 && \forall \mathtt{K_j}\in\mathfrak{M}_j\\
m_{\mathtt{K}^*}\frac{\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}^l_{\sigma^*_j\mathtt{K}^*} &= 0 && \forall \mathtt{K}^*_\mathtt{j}\in\mathfrak{M}^*_j\\
m_{\mathtt{K}^*}\frac{\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}}}{\delta t} + \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} \mathcal{F}^l_{\sigma^*_j\mathtt{K}^*} + m_{\partial\Omega\cap\partial\mathtt{K}^*}\Phi^l_{\mathtt{K}^*_\mathtt{j}} &= 0 && \forall \mathtt{K}^*_\mathtt{j}\in\partial\mathfrak{M}^*_{j,\Gamma}\\
-\mathcal{F}^l_{\sigma\mathtt{K}_j} + \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L_j}} + \lambda\mathbf{e}^l_{\mathtt{L_j}} &= \mathbf{H}^{l-1}_{\mathtt{L}_j} && \forall\sigma\in\partial\mathfrak{M}_{j,\Gamma}\\
-\Phi^l_{\mathtt{K}^*_\mathtt{j}} + \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\,\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}} + \lambda\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}} &= \mathbf{H}^{l-1}_{\mathtt{K}^*_\mathtt{j}} && \forall \mathtt{K}^*_\mathtt{j}\in\partial\mathfrak{M}^*_{j,\Gamma}\\
\mathbf{e}^{\partial\mathfrak{M}_{j,D}} &= 0\\
\mathbf{e}^{\partial\mathfrak{M}^*_{j,D}} &= 0\\
m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j} &= 0 && \forall \mathtt{D}_j\in\mathfrak{D}_j\setminus\mathfrak{D}^\Gamma_j\\
m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j} + \alpha m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j} &= \mathbf{G}^{l-1}_{\mathtt{D}_j} && \forall \mathtt{D}_j\in\mathfrak{D}^\Gamma_j.
\end{aligned}\right.$$

Thanks to the hypothesis $m_{\sigma^*} = 2m_{\sigma^*_i} = 2m_{\sigma^*_j}$, we have $m_{\mathtt{D}_i} = m_{\mathtt{D}_j}$; so, in the equation on $\mathtt{D}_j\in\mathfrak{D}^\Gamma_j$, we can simplify the measures and it becomes:

$$\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j} + \alpha\,\Pi^l_{\mathtt{D}_j} = -\big(\mathrm{div}^{\mathtt{D}_i}(\mathbf{e}^{l-1}_{\mathfrak{T}_i}) - \beta d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\Pi^{l-1}_{\mathtt{D}_i}\big) + \alpha\Pi^{l-1}_{\mathtt{D}_i}.$$

We multiply the equations by $\mathbf{e}^l_{\mathfrak{T}_j}$ and we sum over all the control volumes, as in the proof of Thm. IV.2.3. We obtain, analogously to (IV.12), the following:

$$\begin{aligned}
\frac{1}{\delta t}\|\mathbf{e}^l_{\mathfrak{T}_j}\|^2_2 &+ \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{e}^l_{\mathfrak{T}_j}\|^2_2 - (\Pi^l_{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j}\\
&+ \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L_j}})\cdot\mathbf{e}^l_{\mathtt{L_j}} + \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_\mathtt{j}} - \frac{1}{2}H_{\mathtt{K}^*_\mathtt{j}}\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}})\cdot\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}}\\
&+ \underbrace{\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m^2_\sigma}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma\mathtt{K}}|\mathbf{e}^l_{\mathtt{K}_j} - \mathbf{e}^l_{\mathtt{L_j}}|^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_j}\frac{m^2_{\sigma^*}}{2\mathrm{Re}m_\mathtt{D}}B_{\sigma^*\mathtt{K}^*}|\mathbf{e}^l_{\mathtt{K}^*_\mathtt{j}} - \mathbf{e}^l_{\mathtt{L}^*j}|^2}_{\geq 0} = 0. \qquad \text{(IV.47)}
\end{aligned}$$

By the equations on $\mathfrak{D}^\Gamma_j$, we can split the scalar product into interior diamonds $\mathfrak{D}\setminus\mathfrak{D}^\Gamma_j$ and boundary diamonds $\mathfrak{D}^\Gamma_j$:

$$-(\Pi^l_{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j} = -\sum_{\mathtt{D}_j\in\mathfrak{D}_j\setminus\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j});$$

for the diamonds $\mathtt{D}_j \in \mathfrak{D}_j \setminus \mathfrak{D}_j^\Gamma$ we apply the equation of conservation of mass, for the diamonds $\mathtt{D}_j \in \mathfrak{D}_j^\Gamma$ we add and subtract the term $\sum_{\mathtt{D}_j\in\mathfrak{D}_j} m_{\mathtt{D}_j}\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma}\cdot\Pi^l_{\mathfrak{D}_j^\Gamma}$:

$$
\begin{aligned}
-\,(\Pi^l_{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j} = -\beta \sum_{\mathtt{D}_j\in\mathfrak{D}_j\setminus\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j} d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma}\cdot\Pi^l_{\mathfrak{D}_j^\Gamma} - \beta \sum_{\mathtt{D}_j\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j} d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma}\cdot\Pi^l_{\mathfrak{D}_j^\Gamma}\\
- \sum_{\mathtt{D}_j\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\left(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma}\right).
\end{aligned}
$$

We apply Rem. I.7.2 to the term $-\beta\sum_{\mathtt{D}_j\in\mathfrak{D}_j} m_{\mathtt{D}_j} d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma}\cdot\Pi^l_{\mathfrak{D}_j^\Gamma} = -\beta\left(d^2_{\mathfrak{D}_j}\Delta^{\mathfrak{D}_j}\Pi^l_{\mathfrak{D}_j}, \Pi^l_{\mathfrak{D}_j}\right)$; we then multiply and divide $\sum_{\mathtt{D}_j\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}\Pi^l_{\mathtt{D}_j}\left(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma}\right)$ by α to finally obtain:

$$
-(\Pi^l_{\mathfrak{D}_j}, \mathrm{div}^{\mathfrak{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}))_{\mathfrak{D}_j} = \beta\,|\Pi^l_{\mathfrak{D}_j}|^2_h - \frac{1}{\alpha}\sum_{\mathtt{D}_j\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma}).
$$

So (IV.47) becomes:

$$
\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2 + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2 + \beta\,|\Pi^l_{\mathfrak{D}_j}|^2_h - \frac{1}{\alpha}\sum_{\mathtt{D}_j\in\mathfrak{D}_j^\Gamma} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}_j^\Gamma})\\
&+\frac{1}{2\lambda}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\lambda\mathbf{e}^l_{\mathtt{L}_j} + \frac{1}{2\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_j} - \frac{1}{2}H_{\mathtt{K}^*_j}\mathbf{e}^l_{\mathtt{K}^*_j})\cdot\lambda\mathbf{e}^l_{\mathtt{K}^*_j} \le 0, \quad \text{(IV.48)}
\end{aligned}
$$

where we multiplied and divided by $\lambda > 0$ the terms on the second line.
We start by considering $\dfrac{1}{2\lambda}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\lambda\mathbf{e}^l_{\mathtt{L}_j}$. By applying now the equality $-ab = \dfrac{1}{4}((-a+b)^2 - (a+b)^2)$ we can write:

- $$
\begin{aligned}
\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\lambda\mathbf{e}^l_{\mathtt{L}_j} = \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j} + \lambda\mathbf{e}^l_{\mathtt{L}_j}|^2\\
-\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\underbrace{-\mathcal{F}^l_{\sigma\mathtt{K}_j} + \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j} + \lambda\mathbf{e}^l_{\mathtt{L}_j}}_{=\mathbf{H}^{l-1}_{\mathtt{L}_j}}|^2,
\end{aligned}
$$

So thanks to transmission conditions it becomes:

- $$
\begin{aligned}
\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma(\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j})\cdot\lambda\mathbf{e}^l_{\mathtt{L}_j} = \frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathcal{F}^l_{\sigma\mathtt{K}_j} - \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_j} + \lambda\mathbf{e}^l_{\mathtt{L}_j}|^2\\
-\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}_j^\Gamma} m_\sigma|\mathcal{F}^{l-1}_{\sigma\mathtt{K}_i} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{e}^{l-1}_{\mathtt{L}_i} + \lambda\mathbf{e}^{l-1}_{\mathtt{L}_i}|^2,
\end{aligned}
$$

Equivalently for $\frac{1}{2\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_{\mathtt{j}}}-\frac{1}{2}H_{\mathtt{K}^*_{\mathtt{j}}}\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}})\cdot\lambda\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}}$, we obtain:

- $$\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}(\Phi^l_{\mathtt{K}^*_{\mathtt{j}}}-\frac{1}{2}H_{\mathtt{K}^*_{\mathtt{j}}}\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}})\cdot\lambda\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}}=\frac{1}{4}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^l_{\mathtt{K}^*_{\mathtt{j}}}-\frac{1}{2}H_{\mathtt{K}^*_{\mathtt{j}}}\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}}+\lambda\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}}|^2$$
$$-\frac{1}{4}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^{l-1}_{\mathtt{K}^*_{\mathtt{i}}}-\frac{1}{2}H_{\mathtt{K}^*_{\mathtt{j}}}\mathbf{e}^{l-1}_{\mathtt{K}^*_{\mathtt{i}}}+\lambda\mathbf{e}^{l-1}_{\mathtt{K}^*_{\mathtt{i}}}|^2.$$

If now we consider $m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j})$, thanks to the equality $-ab=\frac{1}{4}((-a+b)^2-(a+b)^2)$ we can write:

- $$-\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j})=\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}-\alpha\,\Pi^l_{\mathtt{D}_j}|^2$$
$$-\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}|\underbrace{\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}+\alpha\,\Pi^l_{\mathtt{D}_j}}_{=\frac{m_{\mathtt{D}_i}}{m_{\mathtt{D}_j}}\mathbf{G}^{l-1}_{\mathtt{D}_j}}|^2,$$

that under the hypothesis $m_{\sigma^*}=2m_{\sigma^*_i}=2m_{\sigma^*_j}$ which implies $m_{\mathtt{D}_i}=m_{\mathtt{D}_j}$ becomes:

- $$-\sum_{\mathtt{D}_j\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}\alpha\,\Pi^l_{\mathtt{D}_j}(\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j})=\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_j}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}-\alpha\,\Pi^l_{\mathtt{D}_j}|^2$$
$$-\frac{1}{4}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}_i}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^{l-1}_{\mathfrak{T}_i})-\beta d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\Pi^{l-1}_{\mathfrak{D}^\Gamma_i}-\alpha\,\Pi^{l-1}_{\mathtt{D}_i}|^2.$$

Replacing those results into (IV.48), we have:

$$\begin{aligned}
&\frac{1}{\delta t}\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2+\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}_j}\mathbf{e}^l_{\mathfrak{T}_j}\|_2^2+\beta\,|\Pi^l_{\mathfrak{D}_j}|_h^2\\
&+\frac{1}{4\alpha}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^l_{\mathfrak{T}_j})-\beta d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\Pi^l_{\mathfrak{D}^\Gamma_j}-\alpha\,\Pi^l_{\mathtt{D}_j}|^2-\frac{1}{4\alpha}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_{\mathtt{D}}|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^{l-1}_{\mathfrak{T}_i})-\beta d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\Pi^{l-1}_{\mathfrak{D}^\Gamma_i}-\alpha\,\Pi^{l-1}_{\mathtt{D}_i}|^2\\
&\qquad+\frac{1}{8\lambda}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma|\mathcal{F}^l_{\sigma\mathtt{K}_j}-\frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{e}^l_{\mathtt{L}_{\mathtt{j}}}+\lambda\mathbf{e}^l_{\mathtt{L}_{\mathtt{j}}}|^2-\frac{1}{8\lambda}\sum_{\mathtt{D}\in\mathfrak{D}^\Gamma_j} m_\sigma|\mathcal{F}^{l-1}_{\sigma\mathtt{K}_i}-\frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{e}^{l-1}_{\mathtt{L}_{\mathtt{i}}}+\lambda\mathbf{e}^{l-1}_{\mathtt{L}_{\mathtt{i}}}|^2\\
&+\frac{1}{8\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^l_{\mathtt{K}^*_{\mathtt{j}}}-\frac{1}{2}H_{\mathtt{K}^*_{\mathtt{j}}}\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}}+\lambda\mathbf{e}^l_{\mathtt{K}^*_{\mathtt{j}}}|^2-\frac{1}{8\lambda}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}} m_{\partial\Omega\cap\partial\mathtt{K}^*}|\Phi^{l-1}_{\mathtt{K}^*_{\mathtt{i}}}-\frac{1}{2}H_{\mathtt{K}^*_{\mathtt{j}}}\mathbf{e}^{l-1}_{\mathtt{K}^*_{\mathtt{i}}}+\lambda\mathbf{e}^{l-1}_{\mathtt{K}^*_{\mathtt{i}}}|^2\le 0.
\end{aligned}$$

Summing over $l = 0, \dots l_{max}$ and $j = 1, 2$ we obtain:

$$\sum_{l=0}^{l_{max}} \sum_{j=1,2} \frac{1}{\delta t} \|\mathbf{e}^{l}_{\mathfrak{T}_j}\|_2^2 + \sum_{l=0}^{l_{max}} \sum_{j=1,2} \frac{2}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}_j} \mathbf{e}^{l}_{\mathfrak{T}_j}\|_2^2 + \sum_{l=0}^{l_{max}} \sum_{j=1,2} \beta \, |\Pi^{l}_{\mathfrak{D}_j}|_h^2$$
$$+ \frac{1}{4\alpha} \sum_{j=1,2} \|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^{l_{max}}_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j} \Delta^{\mathtt{D}_j} \Pi^{l_{max}}_{\mathfrak{D}^{\Gamma}_j} - \alpha\, \Pi^{l_{max}}_{\mathtt{D}_j}\|^2_{\mathfrak{D}^{\Gamma}_j} + \frac{1}{8\lambda} \sum_{j=1,2} \|\mathcal{F}^{l_{max}}_{\sigma\mathtt{K}_j} - \frac{1}{2} F_{\sigma\mathtt{K}_j} \mathbf{e}^{l_{max}}_{\mathtt{L}_j} + \lambda \mathbf{e}^{l_{max}}_{\mathtt{L}_j}\|^2_{\partial\mathfrak{M}_{j,\Gamma}}$$
$$+ \frac{1}{8\lambda} \sum_{j=1,2} \|m_{\partial\Omega\cap\partial\mathtt{K}^*} \Phi^{l_{max}}_{\mathtt{K}^*_j} - \frac{1}{2} H_{\mathtt{K}^*_j} \mathbf{e}^{l_{max}}_{\mathtt{K}^*_j} + \lambda \mathbf{e}^{l_{max}}_{\mathtt{K}^*_j}\|^2_{\partial\mathfrak{M}^*_{j,\Gamma}} \leq \frac{1}{4\mu} \sum_{j=1,2} \|\mathrm{div}^{\mathtt{D}_j}(\mathbf{e}^{0}_{\mathfrak{T}_j}) - \beta d^2_{\mathtt{D}_j} \Delta^{\mathtt{D}_j} \Pi^{0}_{\mathfrak{D}^{\Gamma}_j} - \alpha\, \Pi^{0}_{\mathtt{D}_j}\|^2_{\mathfrak{D}^{\Gamma}_j}$$
$$+ \frac{1}{8\lambda} \sum_{j=1,2} \|\mathcal{F}^{0}_{\sigma\mathtt{K}_j} - \frac{1}{2} F_{\sigma\mathtt{K}_j} \mathbf{e}^{0}_{\mathtt{L}_j} + \lambda \mathbf{e}^{0}_{\mathtt{L}_j}\|^2_{\partial\mathfrak{M}_{j,\Gamma}} + \frac{1}{8\lambda} \sum_{j=1,2} \|m_{\partial\Omega\cap\partial\mathtt{K}^*} \Phi^{0}_{\mathtt{K}^*_j} - \frac{1}{2} H_{\mathtt{K}^*_j} \mathbf{e}^{0}_{\mathtt{K}^*_j} + \lambda \mathbf{e}^{0}_{\mathtt{K}^*_j}\|^2_{\partial\mathfrak{M}^*_{j,\Gamma}}$$

that shows how the total energy stays bounded as the iteration index l_{max} goes to infinity; the series $\sum_{l=0}^{l_{max}} \sum_{j=1,2} \frac{1}{\delta t} \|\mathbf{e}^{l}_{\mathfrak{T}_j}\|_2^2$ and $\sum_{l=0}^{l_{max}} \sum_{j=1,2} \beta \, |\Pi^{l}_{\mathfrak{D}_j}|_h^2$ converge, so their general term tends to zero, that implies the convergence to zero of the errors $\|\mathbf{e}^{l}_{\mathfrak{T}_j}\|_2^2$, $|\Pi^{l}_{\mathfrak{D}_j}|_h^2$, defined in (IV.45). Thus the algorithm converges.

The limit is the solution of problem $(\widetilde{\mathcal{P}})$, that is problem $(\mathcal{P})$ with an appropriate choice of the flux on Γ; in fact, we can deduce that, as l_{max} goes to infinity:

- $\|\mathbf{e}^{l}_{\mathfrak{T}_j}\|_2^2$ tends to zero implies:

$$\mathbf{u}^{l}_{\mathfrak{T}_j} \to \mathbf{u}^{\infty}_{\mathfrak{T}_j} \quad \text{for } j = 1, 2.$$

- $|\Pi^{l}_{\mathfrak{D}_j}|_h^2$ tends to zero implies:

$$\mathrm{p}^{l}_{\mathfrak{D}_j} \to \mathrm{p}^{\infty}_{\mathfrak{D}_j} + \mathrm{const}(\Omega_j) \quad \text{for } j = 1, 2.$$

Thus the pressure converges up to a constant that depends on the subdomain; we show in the following remark (Rem. IV.3.10) that in some cases we are able to determine $\mathrm{cost}(\Omega_j)$.

■

Remark IV.3.10 *We can determine the constant const(Ω_j) if we suppose that the mesh satisfies Inf-sup inequality (Def. I.6.1). In fact, this implies that:*

$$\|\Pi^{l}_{\mathfrak{D}_j} - m(\Pi^{l}_{\mathfrak{D}_j})\|_2 \leq \|\boldsymbol{e}^{l}_{\mathfrak{T}_j}\|_2^2 \to 0.$$

From which we deduce that:

$$p^{l}_{\mathfrak{D}_j} - m(p^{l}_{\mathfrak{D}_j}) \to p^{\infty}_{\mathfrak{D}_j} - m(p^{\infty}_{\mathfrak{D}_j}) \quad \textit{for } j = 1, 2.$$

Remark IV.3.11 *Numerically, we observed that $(p^{l}_{\mathfrak{D}_j} - const(\Omega_j)) \to p^{\infty}_{\mathfrak{D}_j}$ and that $(\Psi^{l}_{\mathfrak{T}_j} - const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}}) \to \Psi^{\infty}_{\mathfrak{T}_j}$.*

If $(\mathbf{u}_{\mathfrak{T}_j}, p_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j})$ is solution to $(\mathcal{P}_j)$

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j}, p_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \mathbf{h}_{\mathfrak{T}_j}, g_{\mathfrak{D}_j}) = 0,$$

then $(\mathbf{U}_{\mathfrak{T}_j}, \mathrm{P}_{\mathfrak{D}_j}, \Phi_{\mathfrak{T}_j}) := (\mathbf{u}_{\mathfrak{T}_j}, p_{\mathfrak{D}_j} - const(\Omega_j), \Psi_{\mathfrak{T}_j} - const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}})$ *satisfies:*

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j}, p_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}}, \bar{\mathbf{u}}_{\mathfrak{T}_j}, \widetilde{\mathbf{h}}_{\mathfrak{T}_j}, \widetilde{g}_{\mathfrak{D}_j}) = 0,$$

with $\widetilde{\mathbf{h}}_{\mathtt{L}_j} = \mathbf{h}_{\mathtt{L}_j} - const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}}$ *for all* $\mathtt{L}_j \in \partial\mathfrak{M}_{j,\Gamma}$, $\widetilde{\mathbf{h}}_{\mathtt{K}^*_j} = \mathbf{h}_{\mathtt{K}^*_j} - const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}}$ *for all* $\mathtt{K}^*_j \in \partial\mathfrak{M}^*_{j,\Gamma}$ *and* $\widetilde{g}_{\mathtt{D}_j} = g_{\mathtt{D}_j} + \alpha m_{\mathtt{D}} const(\Omega_j)$*. In fact:*

- $(\mathbf{U}_{\mathfrak{T}_j}, \mathrm{P}_{\mathfrak{D}_j}, \Phi_{\mathfrak{T}_j})$ *satisfies the same equations on* $\mathtt{K} \in \mathfrak{M}_j, \mathtt{K}^* \in \mathfrak{M}^*_j \cup \partial\mathfrak{M}^*_j$ *for the momentum. It is sufficient to remark that:*

$$\begin{aligned} \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}} &= 0 && \forall \mathtt{K} \in \mathfrak{M}_j, \\ \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} const(\Omega_j)\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} &= 0 && \forall \mathtt{K}^* \in \mathfrak{M}^*_j, \\ \sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*} const(\Omega_j)\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} &= -m_{\partial\Omega\cap\partial\mathtt{K}^*} const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}} && \forall \mathtt{K}^* \in \partial\mathfrak{M}^*_j; \end{aligned}$$

  where we recall that $m_{\partial\Omega\cap\partial\mathtt{K}^*}$ *is the intersection between* $\partial\mathtt{K}^* \cap \partial\Omega$.

- $(\mathbf{U}_{\mathfrak{T}_j}, \mathrm{P}_{\mathfrak{D}_j}, \Phi_{\mathfrak{T}_j})$ *the same equation on* $\mathtt{D} \in \mathfrak{D} \setminus \mathfrak{D}^\Gamma_j$ *since* $\Delta^{\mathfrak{D}}(\mathrm{p}^{\mathfrak{D}} - const(\Omega_j)) = \Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}}$ *by definition of the operator (see Sec. I.7);*

- *on* $\sigma \in \partial\mathfrak{M}_{j,\Gamma}$*:*

$$-\mathcal{F}_{\sigma\mathtt{K}} + \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}} + \lambda\mathbf{u}_{\mathtt{L}} = \mathbf{h}_{\mathtt{L}} - const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}};$$

- *on* $\mathtt{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma}$*:*

$$-\Psi_{\mathtt{K}^*} + \frac{1}{2}H_{\mathtt{K}^*}\mathbf{u}_{\mathtt{K}^*} + \lambda\mathbf{u}_{\mathtt{K}^*} = \mathbf{h}_{\mathtt{K}^*} - const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}};$$

- *on* $\mathtt{D} \in \mathfrak{D}^\Gamma_j$*:*

$$m_{\mathtt{D}}\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta m_{\mathtt{D}} d^2_{\mathtt{D}}\Delta^{\mathtt{D}}\mathrm{p}^{\mathfrak{D}} + \alpha m_{\mathtt{D}}\mathrm{p}^{\mathtt{D}} = g_{\mathtt{D}} + const(\Omega_j)\vec{\mathbf{n}}_{\sigma\mathtt{K}}.$$

IV.4 Second DDFV Schwarz algorithm

We now investigate whether is it possible to construct a discrete Schwarz algorithm with modified fluxes that converges to the solution of $(\mathcal{P})$.
We show that this is possible if we suppose an asymmetric discretization of our problem (IV.1), in the sense that we need to consider an upwind discretization of the convection term on the primal mesh and a centered scheme on the dual mesh, that corresponds to the choice $B_{\sigma\mathtt{K}}(s) = \frac{1}{2}|s|$ and $B_{\sigma^*\mathtt{K}^*}(s) = 0$ in $(\mathcal{P})$.
This comes from the fact that in the first DDFV approximation, we can prove convergence if and only if

- (IV.28) holds, i.e.:

$$\widetilde{B}_{\sigma\mathtt{K}} = \frac{2\mathrm{Re}m_{\mathtt{D}}}{m_\sigma^2}\left(A_1A_2 + \left(\frac{1}{2}m_\sigma F_{\sigma\mathtt{K}}\right)^2 \mathrm{Id}\right)A^{-1} - P,$$

  that, by Rem. IV.3.6, can be rewritten as $\widetilde{B}_{\sigma\mathtt{K}} = F(B_{\sigma\mathtt{K}_1}, B_{\sigma\mathtt{K}_2})$.

- (IV.31) holds, i.e.:

$$\widetilde{B}_{\sigma^*\mathtt{K}^*} = \frac{m_{\sigma^*_1}}{m_{\sigma^*}}B_{\sigma^*_1\mathtt{K}^*} + \frac{m_{\sigma^*_2}}{m_{\sigma^*}}B_{\sigma^*_2\mathtt{K}^*},$$

that, by Rem. IV.3.6, can be rewritten as $\widetilde{B}_{\sigma^*\kappa^*} = G(B_{\sigma_1^*\kappa^*}, B_{\sigma_2^*\kappa^*})$.

In fact, those two relations lead to the fundamental properties (IV.29) and (IV.32).
The idea for the second Schwarz algorithm, since we would like it to converge to the solution of $(\mathcal{P})$ (whose fluxes depends only on $B_{\sigma\kappa}, B_{\sigma^*\kappa^*}$), is to invert (IV.28) and (IV.31), that will lead to invert the functions F, G of Rem. IV.3.6.
The advantage of this second DDFV approximation, with respect to the convergence of $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ towards $(\widetilde{\mathcal{P}})$ described in Sec. IV.3, is that here the limit solution does not have a different definition of the fluxes on the interface Γ.

Theorem IV.4.1 *Let $(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})$ be a solution of $(\mathcal{P})$ for convective fluxes defined by a constant upwind flux $B_{\sigma\kappa}(s) = \frac{1}{2}|s|$ for all $\sigma \in \mathcal{E}$, and by the centered flux $B_{\sigma^*\kappa^*}(s) = 0$ for all $\sigma^* \in \mathcal{E}^*$. Define $(\bar{S})$ the Schwarz algorithm where*

- *On the primal mesh, the new discrete convective fluxes are defined as:*

$$\begin{cases} B_{\sigma\kappa}(s)Id & \text{if } \sigma \notin \mathcal{E}_\Gamma \\ \bar{B}_{\sigma\kappa}(s) & \text{if } \sigma \in \mathcal{E}_\Gamma \end{cases}$$

with:

$$\bar{B}_{\sigma\kappa}(s) = \frac{1}{2} Q \begin{pmatrix} |s| - 2 + 2\sqrt{1+|s|} & 0 \\ 0 & |s| - 1 + \sqrt{1+2|s|} \end{pmatrix} Q^{-1}, \tag{IV.49}$$

and $Q = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$, where $\vec{\mathbf{n}}_{\sigma\kappa} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the outer normal to the interface Γ.

- *On the dual mesh, $B_{\sigma^*\kappa^*}(s) = 0$*

Under the hypothesis that $m_{\sigma^*} = 2m_{\sigma_j^*} = 2m_{\sigma_i^*}$, for $j, i = 1, 2$, $j \neq i$, $(\mathcal{P})$ is the limit of the Schwarz algorithm $(\bar{S})$.

Proof Recall that $(\widetilde{\mathcal{P}})$ is the limit of $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ if and only if (IV.28) and (IV.31) hold.

We start by considering (IV.28), that we recall here.:

$$\widetilde{B}_{\sigma\kappa} = \frac{2\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2} \left(A_1 A_2 + \left(\frac{1}{2} m_\sigma F_{\sigma\kappa} \right)^2 \mathrm{Id} \right) A^{-1} - P, \tag{IV.50}$$

that is a condition on the fluxes on $\sigma \in \mathcal{E}_\Gamma$.
The assumption $m_{\sigma^*} = 2m_{\sigma_j^*} = 2m_{\sigma_i^*}$ implies that $m_{\mathrm{D}_1} = m_{\mathrm{D}_2} = 2m_{\mathrm{D}}$ and $B_{\sigma\kappa_1} = B_{\sigma\kappa_2} = B_{\sigma\kappa}$. This means that

$$A_1 = A_2 = \frac{m_\sigma^2}{\mathrm{Re}m_{\mathrm{D}}}(P + B_{\sigma\kappa}\mathrm{Id})$$

and

$$A = A_1 + A_2 = \frac{2m_\sigma^2}{\mathrm{Re}m_{\mathrm{D}}}(P + B_{\sigma\kappa}\mathrm{Id}) = 2A_1 = 2A_2.$$

Moreover, $A^{-1} = \dfrac{\mathrm{Re}m_{\mathrm{D}}}{2m_\sigma^2}(P + B_{\sigma\kappa}\mathrm{Id})^{-1}$. By replacing these remarks in (IV.50), we obtain:

$$\widetilde{B}_{\sigma\kappa} = \frac{2\mathrm{Re}m_{\mathrm{D}}}{m_\sigma^2} \left(\frac{1}{4} A^2 + \left(\frac{1}{2} m_\sigma F_{\sigma\kappa} \right)^2 \mathrm{Id} \right) A^{-1} - P.$$

By developing the computation, we get:

$$\widetilde{B}_{\sigma\mathrm{K}} = \left(\frac{\mathrm{Re} m_{\mathrm{D}}}{2m_\sigma^2}\right) A + \frac{2\mathrm{Re} m_{\mathrm{D}}}{m_\sigma^2}\left(\frac{1}{2} m_\sigma F_{\sigma\mathrm{K}}\right)^2 A^{-1} - P.$$

By replacing the definition of A and A^{-1}, it leads to:

$$\widetilde{B}_{\sigma\mathrm{K}} = P + B_{\sigma\mathrm{K}}\mathrm{Id} + \left(\frac{\mathrm{Re} m_{\mathrm{D}}}{m_\sigma^2}\right)^2 \left(\frac{1}{2} m_\sigma F_{\sigma\mathrm{K}}\right)^2 (P + B_{\sigma\mathrm{K}}\mathrm{Id})^{-1} - P.$$

Then, if $s = \dfrac{\mathrm{Re} m_{\mathrm{D}}}{m_\sigma} F_{\sigma\mathrm{K}}$, we have $\left(\dfrac{\mathrm{Re} m_{\mathrm{D}}}{m_\sigma^2}\right)^2 \left(\dfrac{1}{2} m_\sigma F_{\sigma\mathrm{K}}\right)^2 = \dfrac{1}{4}s^2$, so we end up with:

$$\widetilde{B}_{\sigma\mathrm{K}} = B_{\sigma\mathrm{K}}\mathrm{Id} + \frac{1}{4}s^2 (P + B_{\sigma\mathrm{K}}\mathrm{Id})^{-1}.$$

If we make explicit the dependences of $\widetilde{B}_{\sigma\mathrm{K}}, B_{\sigma\mathrm{K}}$ as a function of s, since $B_{\sigma\mathrm{K}}$ is a function of $B_{\sigma\mathrm{K}}(\frac{m_{\mathrm{D}}\mathrm{Re}}{m_\sigma} F_{\sigma\mathrm{K}})$ and $\widetilde{B}_{\sigma\mathrm{K}}$ a function of $\widetilde{B}_{\sigma\mathrm{K}}(\frac{2m_{\mathrm{D}}\mathrm{Re}}{m_\sigma} F_{\sigma\mathrm{K}})$, (IV.50) finally becomes:

$$\widetilde{B}_{\sigma\mathrm{K}}(2s) = B_{\sigma\mathrm{K}}(s)\mathrm{Id} + \frac{1}{4}s^2 \left(P + B_{\sigma\mathrm{K}}(s)\mathrm{Id}\right)^{-1}, \quad \text{for } l = 1, 2.$$

We can rewrite this condition, like in Rem. IV.3.6, as:

$$\widetilde{B}_{\sigma\mathrm{K}} = F(B_{\sigma\mathrm{K}}).$$

This relation implies that the Schwarz algorithm $(\mathcal{S}_1)$-$(\mathcal{S}_2)$, whose convection fluxes depend on $B_{\sigma\mathrm{K}}$, converges towards the solution of $(\widetilde{\mathcal{P}})$, whose convection fluxes depend on $\widetilde{B}_{\sigma\mathrm{K}}$ for $\sigma \in \mathcal{E}_\Gamma$.

We want to build a new Schwarz algorithm $(\bar{S})$ that converges toward $(\mathcal{P})$, whose fluxes are defined by $B_{\sigma\mathrm{K}}$; so we need to build $\bar{B}_{\sigma\mathrm{K}}$ such that:

$$B_{\sigma\mathrm{K}} = F(\bar{B}_{\sigma\mathrm{K}}),$$

where $B_{\sigma\mathrm{K}}$ can be a full matrix. In our case, since our goal is to converge towards the fluxes that define an upwind scheme, i.e. defined by $B(s) = \frac{1}{2}|s|$, $B_{\sigma\mathrm{K}}$ is actually a diagonal matrix, that will be denoted by $B_{\sigma\mathrm{K}}\mathrm{Id}$ to distinguish it by a full matrix.

Thus we need to invert the function F defined above to find the new coefficients $\bar{B}_{\sigma\mathrm{K}}$. The inverse of F does not exist for every $B_{\sigma\mathrm{K}}$. Given s and $B_{\sigma\mathrm{K}}(2s)$, we have a second-degree equation for $\bar{B}_{\sigma\mathrm{K}}(s)$:

$$\bar{B}_{\sigma\mathrm{K}}(s)^2 + \bar{B}_{\sigma\mathrm{K}}(s)\underbrace{\left(P - B_{\sigma\mathrm{K}}(2s)\mathrm{Id}\right)}_{T} + \underbrace{\frac{1}{4}s^2\mathrm{Id} - P\,B_{\sigma\mathrm{K}}(2s)\mathrm{Id}}_{V} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

that is:

$$\bar{B}_{\sigma\mathrm{K}}(s)^2 + \bar{B}_{\sigma\mathrm{K}}(s)T + V = 0.$$

Since the matrices T, V are symmetric and they commute (because they are polynomials on P), they can be diagonalized using the same basis of eigenvectors. If $Q = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$, Q orthogonal matrix, we can write:

$$T = Q\,\widetilde{T}\,Q^{-1}, \qquad V = Q\,\widetilde{V}\,Q^{-1},$$

with $\widetilde{T}$ and $\widetilde{V}$ diagonal matrices, whose expressions are:

$$\widetilde{T} = \begin{pmatrix} 2 - B_{\sigma\mathtt{K}} & 0 \\ 0 & 1 - B_{\sigma\mathtt{K}} \end{pmatrix}, \quad \widetilde{V} = \begin{pmatrix} \frac{1}{4}s^2 - 2B_{\sigma\mathtt{K}} & 0 \\ 0 & \frac{1}{4}s^2 - B_{\sigma\mathtt{K}} \end{pmatrix}.$$

We then look for $\bar{B}_{\sigma\mathtt{K}}(s)$ of the form $\bar{B}_{\sigma\mathtt{K}}(s) = Q\,\widetilde{M}\,Q^{-1}$, with $\widetilde{M}$ diagonal matrix such that:

$$\widetilde{M}^2 + \widetilde{M}\,\widetilde{T} + \widetilde{V} = 0.$$

Since we are supposing $B_{\sigma\mathtt{K}}(s) = \frac{1}{2}|s|$, the solution is given by

$$\widetilde{M} = \frac{1}{2}\begin{pmatrix} |s| - 2 + 2\sqrt{1+|s|} & 0 \\ 0 & |s| - 1 + \sqrt{1+2|s|} \end{pmatrix},$$

that leads to our result (IV.49).

For what concerns property (IV.31), we would like to define a unique $B_{\sigma^*\mathtt{K}^*}(s^*)$ for $\sigma^* \in \mathcal{E}^*$ in the limit scheme $(\mathcal{P})$.
With the assumption $m_{\sigma^*} = 2m_{\sigma_1^*} = 2m_{\sigma_2^*}$, we can define $s^* = \dfrac{\mathrm{Re}\, m_{\mathtt{D}}}{m_{\sigma^*}} F_{\sigma^*\mathtt{K}^*}$ and $s_j^* = \dfrac{\mathrm{Re}\, m_{\mathtt{D}_j}}{m_{\sigma_j^*}} F_{\sigma_j^*\mathtt{K}^*}$ for $j = 1, 2$: remark that there is no relation between the s_j^*. The only property that is satisfied is $s^* = s_j^* + s_i^*$, since $m_{\sigma_j^*} F_{\sigma_j^*\mathtt{K}^*} + m_{\sigma_i^*} F_{\sigma_i^*\mathtt{K}^*} = m_{\sigma^*} F_{\sigma^*\mathtt{K}^*}$. This leads to the new expression for (IV.31):

$$\widetilde{B}_{\sigma^*\mathtt{K}^*}(s_j^* + s_i^*) = \frac{1}{2}\left(B_{\sigma^*\mathtt{K}^*}(s_j^*) + B_{\sigma^*\mathtt{K}^*}(s_i^*)\right).$$

This is true only if $B_{\sigma^*\mathtt{K}^*} = \widetilde{B}_{\sigma^*\mathtt{K}^*} = 0$; in this way, even property (IV.32) is verified. So the dual flux for the algorithm $(\bar{S})$ and for the limit $(\mathcal{P})$ correspond to a centered discretization of the convection flux on the dual mesh.

The Schwarz algorithm $(\bar{S})$ is well posed, since $(\mathcal{H}p)$ is verified by his fluxes, and it converges towards $(\mathcal{P})$ with the choice of $B_{\sigma\mathtt{K}}(s) = \frac{1}{2}|s|$ for all $\sigma \in \mathcal{E}$ and $B_{\sigma^*\mathtt{K}^*}(s) = 0$ for all $\sigma^* \in \mathcal{E}$. ■

IV.5 Numerical results

In this section, the objectives are the following:

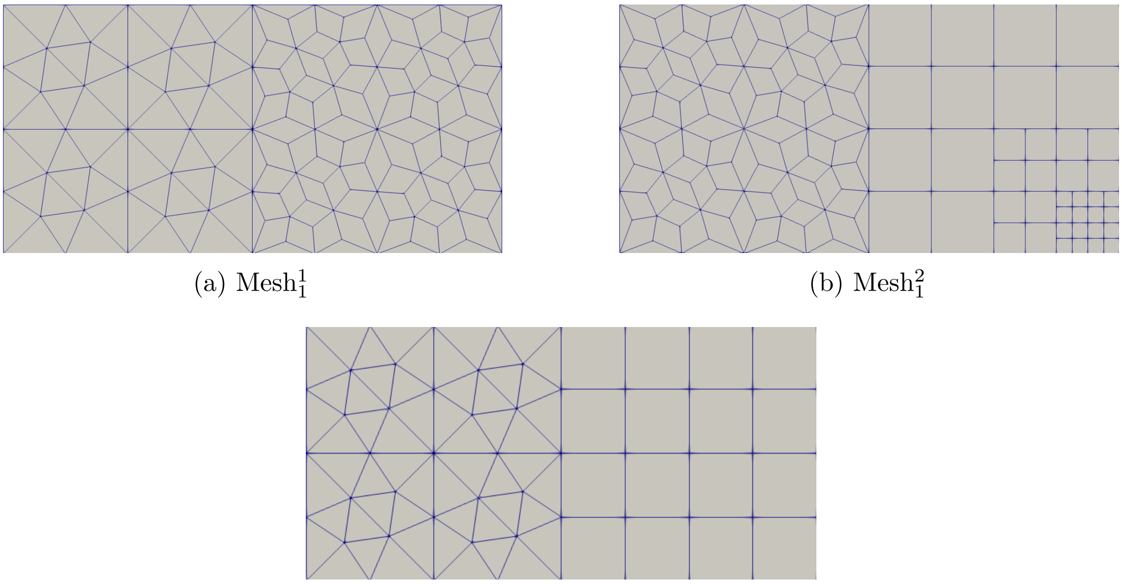
- showing and comparing the convergence properties of the Schwarz algorithms $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ (presented in Sec. IV.2.2)) and $(\bar{\mathcal{S}})$ (presented in Sec. IV.4);
- studying the influence of the parameters λ, α, β of (IV.2) in the convergence.

We recall that the difference between the two algorithms relies in the definition of the fluxes at the interface; the first one converges towards the solution of $(\widetilde{\mathcal{P}})$ (see Thm. IV.3.9), the second one towards the solution of $(\mathcal{P})$ (see Thm. IV.4.1).

We will refer to $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ as "**first Schwarz algorithm**", and to $(\bar{\mathcal{S}})$ as "**second Schwarz algorithm**". For the first Schwarz algorithm, in all the following test cases, we will consider an upwind discretization of the convection flux, i.e. we fix the function $B = \frac{1}{2}|s|$.

We recall that the domain decomposition algorithm is an iterative algorithm that is employed at each time step; this, in particular, implies that at each iteration of the Schwarz algorithm we solve a steady problem. In the tests we present, a time step is fixed and the iterative algorithm is applied; we choose to fix the first one, with $\delta t = 10^{-4}$, so every test presented in this section will be done in the time interval $[0, \delta t]$.

In all the test cases, the domain $\Omega = [-1, 1] \times [0, 1]$ will be divided into two subdomains $\Omega = \Omega_1 \cup \Omega_2$. The meshes we will consider are illustrated, in their first level of refinement, in Fig. IV.5.



(a) Mesh_1^1 (b) Mesh_1^2

(c) Mesh_1^3.

Fig. IV.5 Coarse level of refinement of the composite meshes on Ω, Mesh_1^k.

The sub-index in the name of the mesh (see Fig. IV.5) denotes the level of refinement, i.e. Mesh_1^k represents the coarse mesh of a family of refined meshes $(\mathrm{Mesh}_m^k)_m$. More precisely, Mesh_m^k is obtained by dividing by two all the edges of Mesh_{m-1}^k.

We consider the following exact solutions to (IV.1):

Test 1:

$$\begin{aligned} \mathbf{u}(x, y) &= \begin{pmatrix} -2\pi \cos(\pi x) \sin(2\pi y) \exp(-5\eta t \pi^2), \\ \pi \sin(\pi x) \cos(2\pi y) \exp(-5\eta t \pi^2) \end{pmatrix}, \\ \mathrm{p}(x, y) &= -\frac{\pi^2}{4}(4\cos(2\pi x) + \cos(4\pi y)) \exp(-10 t \eta \pi^2). \end{aligned} \tag{IV.51}$$

Test 2:

$$\begin{aligned} \mathbf{u}(x, y) &= \begin{pmatrix} \sin(2\pi x) \cos(2\pi y) \exp(-2\eta t), \\ -\cos(2\pi x) \sin(2\pi y) \exp(-2\eta t) \end{pmatrix}, \\ \mathrm{p}(x, y) &= -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y)) \exp(-4\eta t). \end{aligned} \tag{IV.52}$$

The algorithms, in all the following simulations, are initialized with initial random guesses $\mathbf{h}^0_{\mathfrak{I}_j}$ and $g^0_{\mathfrak{D}_j}$ for $j = 1, 2$.

As a stopping criterion, we impose:

$$\max\left(\|\mathbf{e}^l_{\mathfrak{T}_j}\|_2, \|\Pi^l_{\mathfrak{D}_j}\|_2\right) < 10^{-6},$$

where the errors are defined in (IV.45).

IV.5.1 Error on the interface

In this first test case, we consider the first Schwarz algorithm; our goal is to point out that the error computed with respect to the solution of $(\widetilde{\mathcal{P}})$, along the iterations of the algorithm, stays localized at the interface between the two subdomains.
The domain Ω is meshed with Mesh_5^3, we fix the parameters $\lambda = 100, \alpha = 1, \beta = 10^{-2}$.
In Fig. IV.6 we represent the error of the velocity on the entire domain at the initialization on the primal and dual mesh; the initialization assigns random values, and the initial error is 100 for both primal and dual mesh.
As we pass to the 1^{st} iteration, we observe in Fig. IV.7 how it immediately locates on the interface between the subdomains; it decreases, passing from 100 to 1.9 on the primal mesh and to 6.9 on the dual mesh. Already at the 10^{th} iteration we see in Fig. IV.8 how it has diminished, staying localized on the interface, passing from 1.9 to 0.52 on the primal mesh and from to 6.9 to 0.05 on the dual mesh.

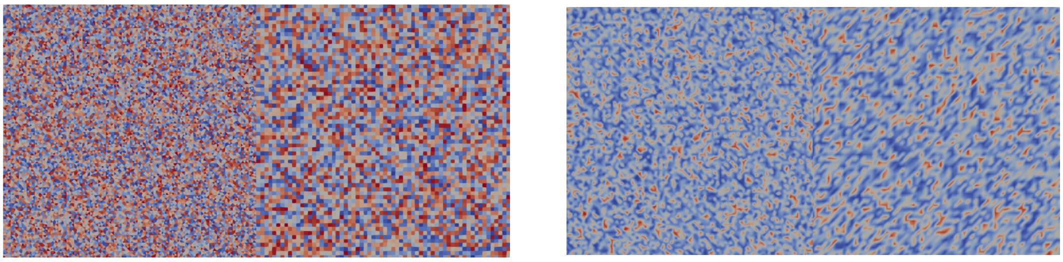


Fig. IV.6 Error $\mathbf{u}^0_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}$ at the initialization: $\|\mathbf{u}^0_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}\|_\infty = 100$. *Left:* Primal mesh. *Right:* Dual mesh.

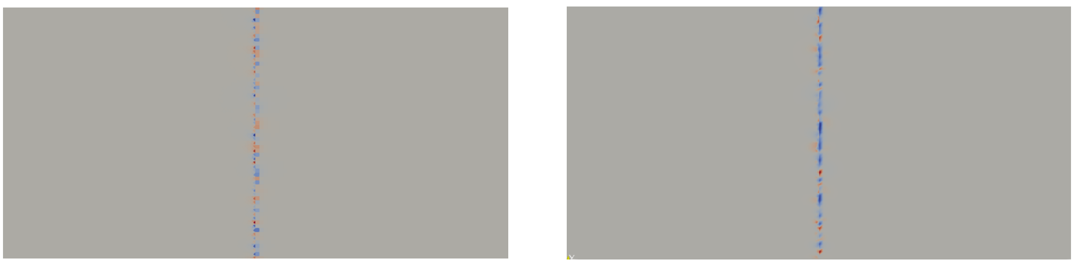
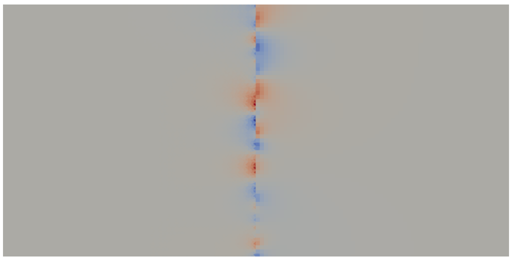


Fig. IV.7 Error $\mathbf{u}^0_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}$ at the 1^{st} iteration. *Left:* Primal mesh, $\|\mathbf{u}^1_{\mathfrak{M}} - \mathbf{u}_{\mathfrak{M}}\|_\infty = 1.9$. *Right:* Dual mesh, $\|\mathbf{u}^1_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*} - \mathbf{u}_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\|_\infty = 6.9$.



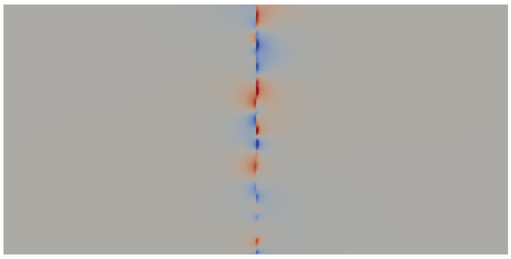


Fig. IV.8 Error $\mathbf{u}^0_{\mathfrak{T}} - \mathbf{u}_{\mathfrak{T}}$ at the 10^{th} iteration. *Left:* Primal mesh, $\|\mathbf{u}^2_{\mathfrak{M}} - \mathbf{u}_{\mathfrak{M}}\|_\infty = 0.52$. *Right:* Dual mesh, $\|\mathbf{u}^2_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*} - \mathbf{u}_{\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\|_\infty = 0.05$.

IV.5.2 Study of the parameters

In this section our goal is to study the influence of the parameters λ, α, β on the convergence of the first and second Schwarz algorithms.
We recall that β is associated to the Brezzi-Pitkaranta stabilization (see Sec. I.7), present in the mass conservation equation, while the parameters λ and α are associated the transmission conditions between subdomains, that we recall here, for $j, i \in 1, 2, j \neq i$:

- for all $\mathtt{L}_j = \mathtt{L}_i \in \partial\mathfrak{M}_{j,\Gamma}$:

$$-\mathcal{F}^l_{\sigma\mathtt{K}_j} + \frac{1}{2}F_{\sigma\mathtt{K}_j}\mathbf{u}^l_{\mathtt{L}_j} + \lambda\mathbf{u}^l_{\mathtt{L}_j} = \mathcal{F}^{l-1}_{\sigma\mathtt{K}_i} - \frac{1}{2}F_{\sigma\mathtt{K}_i}\mathbf{u}^{l-1}_{\mathtt{L}_i} + \lambda\mathbf{u}^{l-1}_{\mathtt{L}_i};$$

- for all $\mathtt{K}^*_j \in \partial\mathfrak{M}^*_{j,\Gamma}$ such that $x_{\mathtt{K}^*_j} = x_{\mathtt{K}^*_i}$:

$$-\Psi^l_{\mathtt{K}^*_j} + \frac{1}{2}H_{\mathtt{K}^*_j}\mathbf{u}^l_{\mathtt{K}^*_j} + \lambda\mathbf{u}^l_{\mathtt{K}^*_j} = \Psi^{l-1}_{\mathtt{K}^*_i} - \frac{1}{2}H_{\mathtt{K}^*_i}\mathbf{u}^{l-1}_{\mathtt{K}^*_i} + \lambda\mathbf{u}^{l-1}_{\mathtt{K}^*_i};$$

- for all $\mathtt{D}_j \in \mathfrak{D}^\Gamma_j$ such that $x_{\mathtt{D}_j} = x_{\mathtt{D}_i}$:

$$m_{\mathtt{D}_j}\mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^l_{\mathfrak{T}_j}) - \beta m_{\mathtt{D}_j}d^2_{\mathtt{D}_j}\Delta^{\mathtt{D}_j}\mathrm{p}^l_{\mathfrak{D}_j} + \alpha m_{\mathtt{D}_j}\mathrm{p}^l_{\mathtt{D}_j} = \\ -\left(m_{\mathtt{D}_i}\mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^{l-1}_{\mathfrak{T}_i}) - \beta m_{\mathtt{D}_i}d^2_{\mathtt{D}_i}\Delta^{\mathtt{D}_i}\mathrm{p}^{l-1}_{\mathfrak{D}_i}\right) + \alpha m_{\mathtt{D}_i}\mathrm{p}^{l-1}_{\mathtt{D}_i}.$$

Comparison between first and second Schwarz algorithm and parameters optimization. In those numerical tests, our goal is to compare the convergence between the first and the second Schwarz algorithm and to see the influence of λ and α; to do so, in each test case we fix one of the two parameters and we let the remaining vary. Here, the value of β associated to the stabilization is set to 10^{-2}; we will discuss its value in the next section. In Fig. IV.9-IV.11 we represent on the x-axis the number of iterations, on the y-axis the error.

We start by considering the first Schwarz algorithm; we can observe in Fig. IV.9 the convergence of the algorithm to the solution of Test 1 on Mesh^1_1.
In particular, on the *left* of Fig. IV.9, α is fixed to 1, and we observe how, as λ increases, the number of iterations necessary to converge decreases until $\lambda = 200$; passed this critical value, the number of iterations starts to increase again. In fact, for $\lambda = 10$ we need 650 iterations to reach an error of 10^{-5}, for $\lambda = 200$, we need 98 iterations and for $\lambda = 600$ we need 232 iterations. This suggests that for $\alpha = 1$, $\lambda = 200$ is a good choice to have a better convergence. On the *right* of Fig. IV.9, we set $\lambda = 100$ and we let α vary: we observe the same kind of behavior as the one of λ. If α is small, i.e. $\alpha = 0.01$, the algorithm converges in more than 1000 iterations; when α increases to 0.25, the number of iterations decreases to 97. Then, when α becomes bigger, such as $\alpha = 10$,

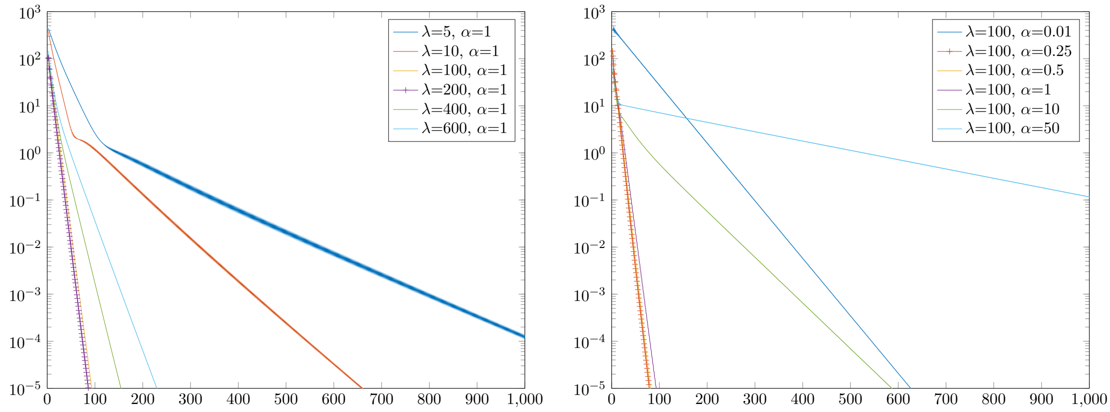


Fig. IV.9 Test 1, Mesh_1^1, first Schwarz algorithm. *Left:* optimization of λ, with $\alpha = 1$. *Right:* optimization of α, with $\lambda = 100$.

we need around 600 iterations to converge.

We consider now the second Schwarz algorithm on same test case, i.e. Test 1 on Mesh_1^1. We show its convergence in Fig. IV.10. This indicates that for $\lambda = 100$, $\alpha = 0.25$ is a good choice to have a better convergence.

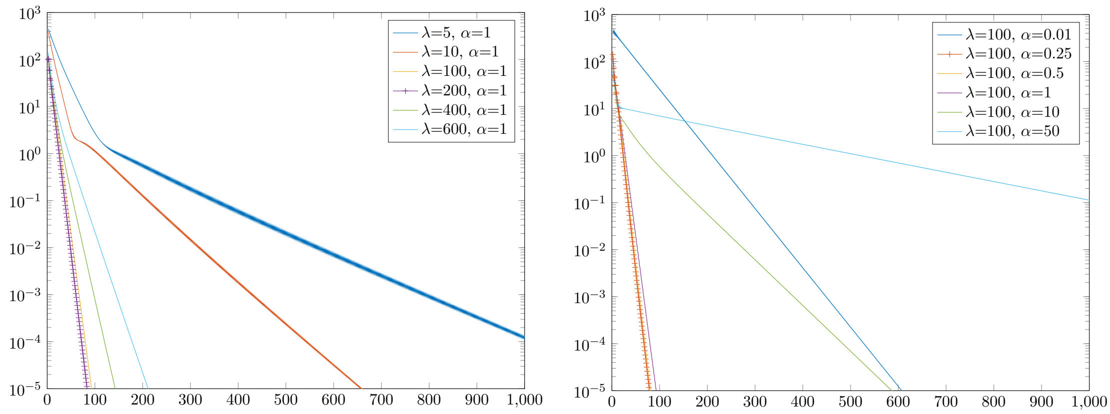


Fig. IV.10 Test 1, Mesh_1^1, second Schwarz algorithm.*Left:* optimization of λ, with $\alpha = 1$. *Right:* optimization of α, with $\lambda = 100$.

We remark that the second Schwarz algorithm behaves similarly to the first one, if we compare Fig. IV.10 and Fig. IV.9 ; thus, both algorithms converge and the speed of convergence is influenced by the choice of λ and α. The parameters have the same behavior and the number of iterations necessary to the convergence is almost identical between the two algorithms; this is why from now on we will focus just on the first one.

We consider now a different test case, i.e. Test 2 on Mesh_1^2 for the first Schwarz algorithm.

We observe in Fig. IV.11 that we still have the same kind of behavior for the parameters λ, α; but we point out that the optimal value of the parameters depends on the mesh and on the test case. In fact, if we compare the optimal α in Fig. IV.10 and in Fig. IV.11, we remark that $\alpha = 0.25$ for the first case and $\alpha = 0.5$ for the second case.

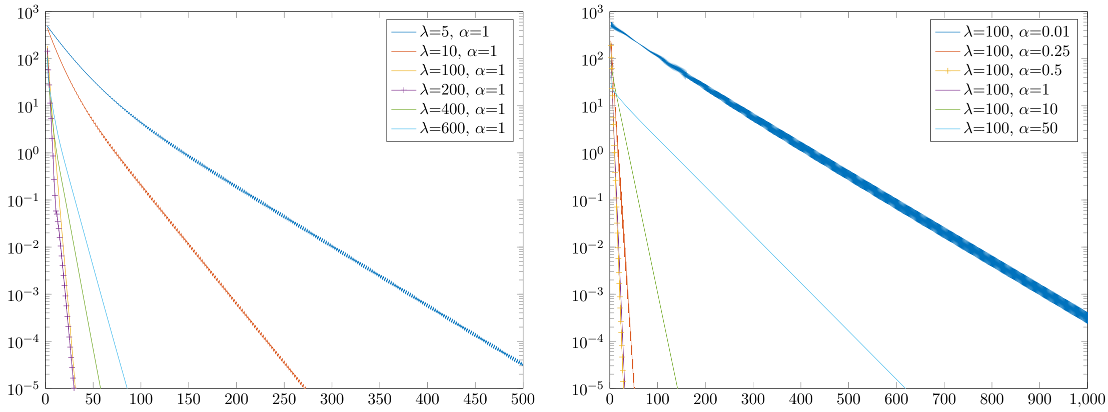


Fig. IV.11 Test 2, Mesh_1^2, first Schwarz algorithm. *Left:* optimization of λ, with $\alpha = 1$. *Right:* optimization of α, with $\lambda = 100$.

Influence of the mesh and of the stabilization. In the first test case of Fig. IV.12, our goal is to show how the level of refinement of the mesh can influence the choice of the optimal parameter; we consider Test 1 on the family $(\mathrm{Mesh}_m^1)_m$, $m = 1, 2, 3, 4$. As before, we fix one parameter at the time (λ or α) and we let the other vary; we represent on the x-axis the value of the parameter that changes, on the y-axis the number of iterations required to obtain an error of order 10^{-5}.

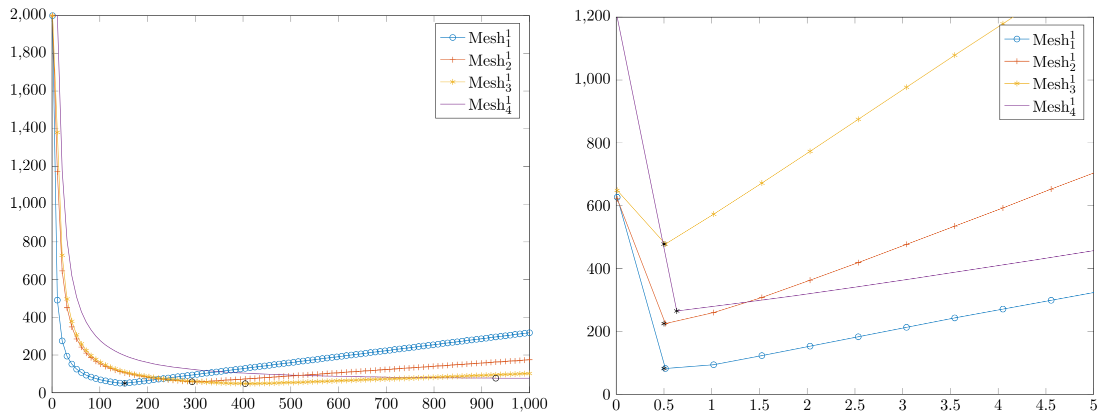


Fig. IV.12 Test 1, $(\mathrm{Mesh}_m^1)_m$, $m = 1, 2, 3, 4$. *Left:* optimization of λ to obtain an error of order 10^{-5}, with $\alpha = 1, \beta = 10^{-1}$. *Right:* optimization of α to obtain an error of order 10^{-5}, with $\lambda = 100, \beta = 10^{-1}$.

Table IV.1 Test 1 on $(\mathrm{Mesh}_m^1)_m$, $m = 1, 2, 3$. *First line:* Optimal value of λ for $\alpha = 1, \beta = 10^{-1}$. *Second line:* Optimal value of α for $\lambda = 100, \beta = 10^{-1}$.

	$\mathbf{Mesh}_1^1$	$\mathbf{Mesh}_2^1$	$\mathbf{Mesh}_3^1$	$\mathbf{Mesh}_4^1$
λ	152.36	293.36	404.63	929.36
α	0.5	0.5	0.5	0.6

As illustrated in Fig. IV.12 and summarized in Tab. IV.1, we observe different results for the two parameters; the mesh refinement has an impact on λ but not really on α. The mesh size h is divided by two at each level of refinement, and we see that it has an influence on the value of λ; unfortunately, we can not conclude by defining a relation between the two.

In Fig. IV.13 (*left*) and Tab. IV.2 we want to confirm the results obtained for λ on Fig. IV.12 (*left*) and Tab. IV.1, by considering the same test case (Test 1) on a different family of meshes, $(\text{Mesh}^3_m)_m$, $m = 1, 2, 3, 4$.

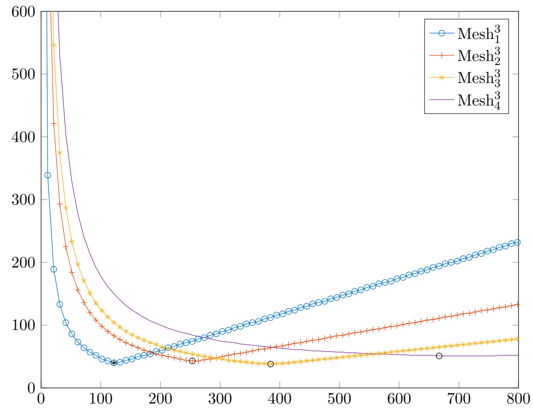


Table IV.2 Test 1 on $(\text{Mesh}^3_m)_m$, $m = 1, 2, 3, 4$. Optimal value of λ for $\alpha = 1, \beta = 10^{-1}$.

	Mesh$^3_1$	**Mesh$^3_2$**	**Mesh$^3_3$**	**Mesh$^3_4$**
λ	122	253.27	384.45	667.51

Fig. IV.13 Test 1, $(\text{Mesh}^3_m)_m$, $m = 1, 2, 3$. *Left:* optimization of λ to obtain an error of order 10^{-5}, with $\alpha = 1, \beta = 10^{-1}$. *Right:* Summary table of the optimal values of λ.

As before, λ is influenced by the mesh discretization step but we can not conclude by defining a relation between the two; moreover, we remark that its optimal values change with respect to Tab. IV.1, due to the different meshes.

In Fig. IV.14 and in Tab. IV.3 we want to point out the influence of the parameter β, associated to the Brezzi-Pitkäranta stabilization. We see how the choice of this parameter affects the convergence of the algorithm and how it affects the optimal value of λ: we pass from 818 iterations with $\lambda = 436.81$ (for $\beta = 10^{-4}$) to 40 iterations with $\lambda = 122$ (for $\beta = 10^{-1}$). There is then an optimal choice even for this parameter.

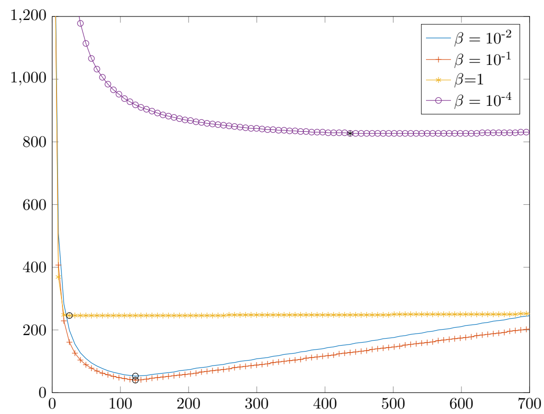


Table IV.3 Test 1 on Mesh$^3_1$. Optimal value of λ and the number of iterations for different values of β and for $\alpha = 1$.

β	10^{-4}	10^{-2}	10^{-1}	1
λ	436.81	122	122	25.2
# iter	818	53	40	246

Fig. IV.14 Test 1, Mesh$^3_1$. *Left:* optimization of λ with different values of β on Mesh$^3_1$; $\alpha = 1$. *Right:* Summary table of the optimal values of λ.

As last simulation, on Fig. IV.15 and Tab. IV.4 we compare the optimal values of λ for Test 1 on different meshes. We see that even the choice of the mesh influences the optimal choice of the parameter: for a cartesian mesh, $\lambda = 105.91$ while for Mesh$^2_1$ $\lambda = 154.3$.

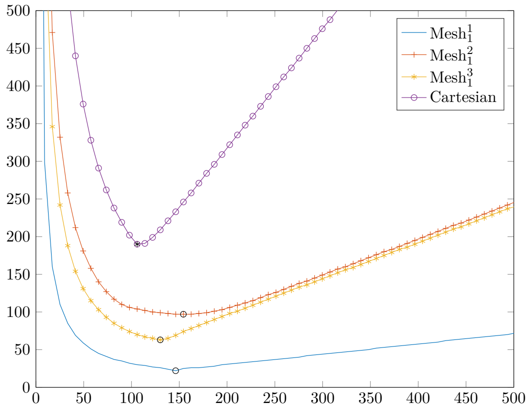


Table IV.4 Test 1. Optimal value of λ for $\alpha = 1, \beta = 10^{-1}$ on different meshes.

	Mesh$_1^1$	**Mesh$_1^2$**	**Mesh$_1^3$**	**Cartesian**
λ	146.2	154.3	130.1	105.91

Fig. IV.15 *Left:* Test1, optimization of λ for different meshes to obtain an error of order 10^{-6}, $\alpha = 1$ and $\beta = 10^{-1}$. *Right:* Summary table of the optimal values of λ.

Conclusions and perspectives

In this chapter, we proposed two non-overlapping DDFV Schwarz algorithm for Navier-Stokes problem. We started by defining a discretization $(\mathcal{P})$ of the Navier-Stokes problem on the entire domain Ω by means of B-schemes for the discretization of the nonlinear convection terms; we proved the well-posedness of this scheme, which is the limit scheme towards which the solution of the iterative Schwarz algorithm should converge. We then built a scheme for the subdomain problem with transmission boundary conditions and we introduced the DDFV Schwarz algorithm, to which we refer as "first Schwarz algorithm". We showed in Thm. IV.3.9 that it converges to a modified version of $(\mathcal{P})$, that we named $(\widetilde{\mathcal{P}})$. The difference between $(\mathcal{P})$ and $(\widetilde{\mathcal{P}})$ is the choice of the function B that defines the convection terms on the interface. We then built a "second Schwarz algorithm" and we recovered the convergence towards $(\mathcal{P})$ in Thm. IV.4.1.
We then numerically tested the two algorithms; in particular, we focused on the influence of the parameters λ, α of the transmission conditions, by observing the presence of an optimal value for both parameters in order to have a better convergence. Moreover, we remarked that the choice of the mesh, of the test case and of the parameter β linked to the stabilization of the mass conservation equation influences the optimal value of λ and α.
We are working on other numerical simulations; we would like to test the convergence on meshes that have non-conformal edges on the interface, to reproduce and compare the numerical results of Chapter III and also to investigate more fully the choice of the optimal parameters.

Chapter V

Curved interface reconstruction for 2D compressible multi-material flows

Contents

This chapter presents an independent work issued from a CEMRACS project (in 2018); it is a joint work with Igor Chollet, Théo Corot, Laurent Dumas, Philippe Hoch and Thomas Leroy and it has been submitted to ESAIM: proceedings and surveys.

A curved interface reconstruction procedure is presented here in the case of a 2D compressible flow made of two or more materials. Built with a dynamic programming procedure already introduced in [DGJM17], the curve interface is continuous and volume preserving in each cell. It is applied here to general test cases with non cartesian grids as well as triple point configurations.

V.1 Introduction

Interface reconstruction (IR) methods are encountered in numerical simulation of multi-material or multi-fluid flows. In a Volume of Fluid (VOF) approach in a case of two materials, denote C the volume fraction of material 1 encountered in volume V:

$$C = \frac{1}{V} \int_V \chi(x, y, z) dxdydz \qquad (1),$$

where χ denotes the indicator function of material 1:

$$\chi(x, y, z) = \begin{cases} 1 & \text{if } (x, y, z) \in \text{ material 1}, \\ 0 & \text{if } (x, y, z) \in \text{ material 2}. \end{cases}$$

The objective of IR methods is to define a geometric interface separating material 1 and material 2 with the following properties:

- P1: volume fractions conservation,
- P2: continuity of the interface,
- P3: robustness,
- P4: low or moderate computational cost.

The first IR method for volume tracking that has been introduced in 1982 is due to D.L. Youngs [You82]. This method consists in assuming that the interface for each mixed cell (that is such $0 < C < 1$) is made of a segment joining two of its edges. The normal of this segment is colinear to the gradient of the volume fraction ∇C and its position is obtained by assuming an exact conservation of partial volumes. With such construction, it is clear that the conservation property P1 is fulfilled by contrast with the continuity property P2. The two other desired properties, robustness and low cost, are also satisfied with this method.

There exists many variants to Youngs method, for instance an order 2 reconstruction ([RK98]) or an extension to more than two materials ([SGFL09]). Some correction terms for the normal computation have also been proposed to reduce undesirable effects and to smooth the interface ([GDSS05]). The two references [RK98] and [Rud97] give various examples of applications of Youngs IR method. Even though Young's method is still largely used up to now because of its simplicity and robustness, it suffers from the non continuity of the interface.

Recently, in [DGJM17], a new reconstruction method which ensures continuity of the interface and preserves volume fractions has been introduced. This new interface reconstruction method, called DPIR (*Dynamic Programming Interface Reconstruction*), is introduced in the next section and will be used as a starting point for the presented work. It consists of two main steps. First, minimize a suitable energy functional which gives a continuous linear interface. Secondly, add a control point in each cell to find the correct volume fractions. This last step is usually made by searching the point in the normal direction of the interface, in the line passing through the center of this one.

In this paper there are three main goals. First, the DPIR method is extended for curved interfaces (section V.3). It is of interest in particular in the case of curved meshes, where an exact reconstruction of the interface is expected. In order to be a real candidate for being used in multi-material hydrodynamic simulation using ALE remap methods, the DPIR method must be able to deal with distored meshes. Although the principle of the method remains unchanged (one minimization step, one correction step), several improvements are proposed (section V.4), in particular to deal with strongly distorded cells and small volume fraction issues. Finally, this work ends with a generalization of the method for three materials (section V.5). Interface reconstruction for multi-material simulations is a complicated issue, and a comparison of several existing methods can be found in [KGSS10]. The proposed method applies the DPIR method for all the materials *without choosing any material ordering* and a suitable average is applied to obtain the final interfaces. The method is tested on two test cases with triple point configuration on cartesian meshes, giving encouraging results for futur unstructured meshes cases.

V.2 Interface reconstruction with DPIR

DPIR method deals with interface reconstruction as a minimization problem of the sum of volume fraction errors. It relies on the minimization of the functional

$$J(y) = \sum_{(i,j)\in\{1,\ldots,N_x\}\times\{1,\ldots,N_y\}} |vol_{y,i,j} - vol_{i,j}|^2 \tag{2}$$

where $t \mapsto y(t)$ is the associated interface curve, $vol_{i,j}$ denotes the targeted volume fraction in cell (i, j) and $vol_{y,i,j}$ the volume fraction obtained with the curve y.

This method is splitted in two steps, first dynamic programming is used to minimize the cost function J. Then, a correction is made on the curve y obtained in each cell to recover the targeted volume fraction. DPIR algorithm is decribed below, for more information we invite the reader to consult [DGJM17].

V.2.1 First step: minimization of J with dynamic programming

During the first step, the interface curve $t \mapsto y(t)$ is assumed to be piecewise linear in each cell (see Fig. V.1). The minimization problem consists in finding a finite number of points $(M_i)_{0\mathrm{L}^*i\mathrm{L}^*N}$, located on the edges of mixed cells, such that $M_0 = M_N$. More precisely, the possible locations of points M_i are obtained after finding the so-called *internal* and *external curves* that will bound the interface curve (see Fig. V.1). The internal and external curves are also polygonal curves with nodes located at the mesh nodes and are obtained by a simple search algorithm among mixed cells. We denote by $vol(M_i, M_{i+1})$ the volume fraction in the cell computed bewteen the segment M_iM_{i+1} and the internal curve. Once these curves are found, a dynamic programming procedure is applied to find the piecewise closed linear curve that minimize the cost function J at a computational cost of $\mathcal{O}(NL^2)$ where L is the discretization number of each cell edges.

Note that a penalty term of the form $p(y) = \sum_{i=0}^{N-1} \lambda||M_{i+1} - M_i||$ can be added to the cost function J in order to reduce the interface length and to avoid wave effects.

Dynamic programming is a very efficient tool to minimize J in the meaning that it has a low cost (property P4) and is robust (property P3). This step gives a first approximation of the interface which is continuous (property P2). However, the interface obtained does not satisfy the volume conservation (property P1).

V.2.2 Second step: local correction of volume fractions

The goal of the second step is to correct volume fractions in order to recover Property P1. In the original DPIR algorithm described in [DGJM17], a control point is added in each mixed cell. This point is located on the perpendicular bisector of the interface segment and placed in order to have an exact volume conservation.

The complete method including the previous two steps, called DPIR (Dynamic Programming Interface Reconstruction) can then be summarized by:

DPIR Algorithm*: for a given distribution of volume fractions on a 2D cartesian grid:*

- **Initialization***: define the internal and external curves that will bound the interface.*

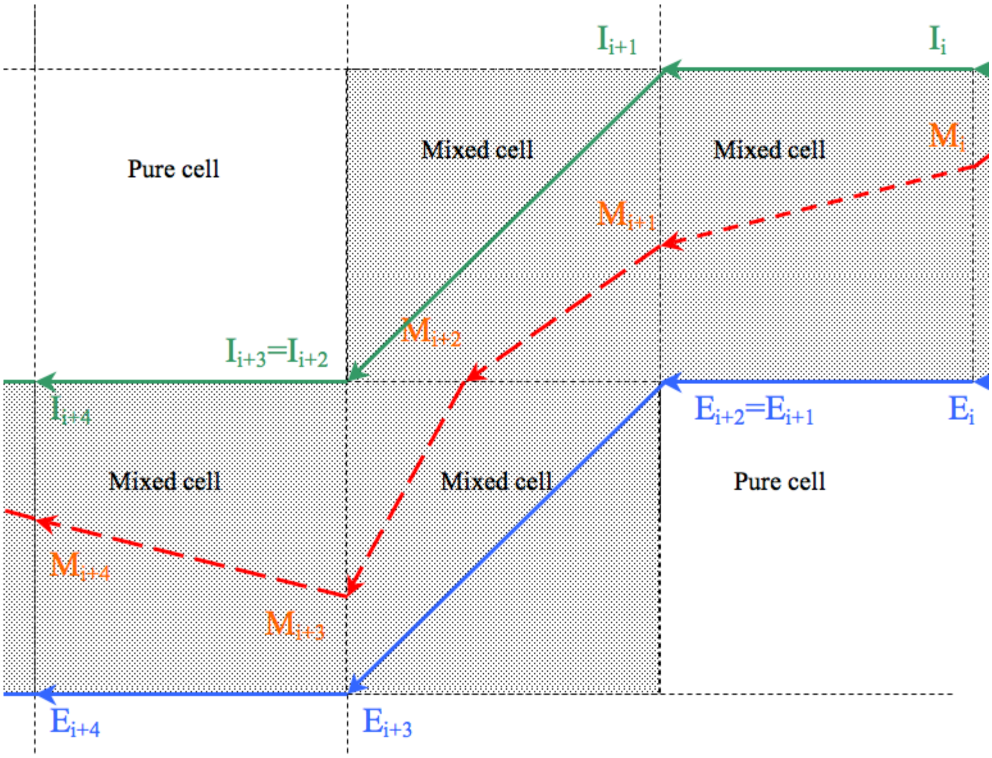


Fig. V.1 The interface curve (dotted line) with the internal (points I_i) and external (points E_i) curves.

- **Step 1 (global step)**: *minimize the cost functional J (2) with dynamic programming.*
- **Step 2 (local step)**: *add a control point in each cell to have an exact conservation of volume fractions.*

In the next sections, we present some extensions of this method. We extend it to curved interfaces in Section V.3, describes tools to make it more robust in Section V.4 and extend it to three materials in Section V.5.

V.3 DPIR extension to curved interfaces reconstruction

In order to obtain a curved interface, more suited to some cases (circle reconstruction for instance), rational quadratic Bezier curves are introduced in the local correction phase of DPIR.

A second order rational Bezier curve is a parametric curve defined by three control points P_0, P_1 and P_2. Here, P_1 will play the role of the control point introduced in the correction step. A weight $\omega \in [0, +\infty]$ is associated to this point (Fig. V.2).

$$\mathbf{M}^{\omega}(q) = \begin{pmatrix} x(q) \\ y(q) \end{pmatrix} = \frac{P_0(1-q)^2 + 2\omega q(1-q)P_1 + q^2 P_2}{(1-q)^2 + 2\omega q(1-q) + q^2}, \qquad q \in [0,1]. \tag{V.1}$$

The area $A(\mathbf{M}^{\omega}(q), P_0, P_1, P_2)$ under a Bezier curve (see Fig. V.2 on Left) can be computed with

$$A(\mathbf{M}^{\omega}(q), P_0, P_1, P_2) = f(\omega) \cdot A(P_0, P_1, P_2),$$

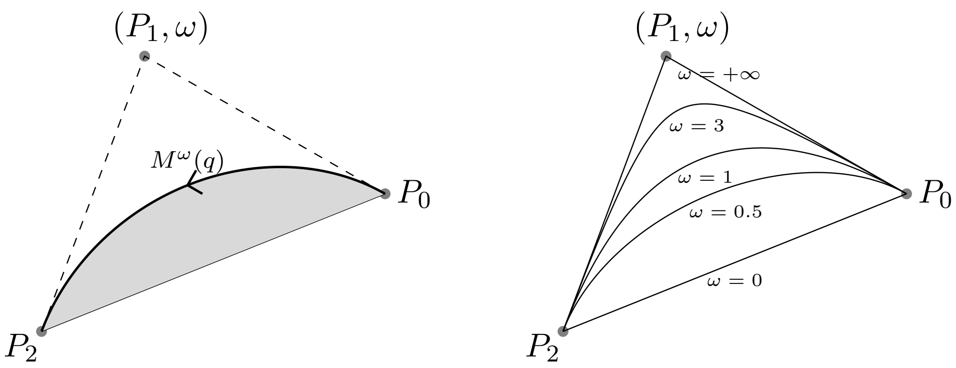


Fig. V.2 A second order rational Bezier curve (area under Bezier curve and straight segment $[P_0, P_2]$ and shape curve evolution with respect to weight parameter ω) of parameterization (V.1).

where $A(P_0, P_1, P_2)$ is the area of the triangle $\widehat{P_0 P_1 P_2}$ and

$$f(\omega) = \begin{cases} 0 & \text{if } \omega = 0 \\ \dfrac{2\omega}{1-\omega^2}\left(\dfrac{1}{1-\omega^2}\arctan\left(\sqrt{\dfrac{1-\omega}{1+\omega}}\right) - \dfrac{\omega}{2}\right) & \text{if } \omega \in (0,1) \\ \dfrac{2}{3} & \text{if } \omega = 1 \\ \dfrac{\omega}{\omega^2-1}\left(\omega + \dfrac{1}{\sqrt{1-\omega^2}}\ln\left(\omega - \sqrt{\omega^2-1}\right)\right) & \text{if } \omega > 1 \end{cases}$$

In this paper, the value of ω will be fixed to 1 if no other specification is made.

Let M_i, $i = 0, \ldots N$ be the coordinates of the points obtained after the first step of DPIR. Let P_i be the control point associated with the piece of interface $[M_i, M_{i+1}]$. The position of P_i is defined in order to preserve the volume fraction defined by the associated rational quadratic Bezier curve (M_i, P_i, M_{i+1}). We use a dichotomy to find the position of P_i. Let us apply this algorithm to the reconstruction of a circle. First we consider a circle of radius 2 on a coarse mesh. We compare the results obtained with Youngs method, the original DPIR and our extension with $\omega = 0.2$. Plots on Fig. V.3 show that the use of such curved parameterization can greatly improve results.

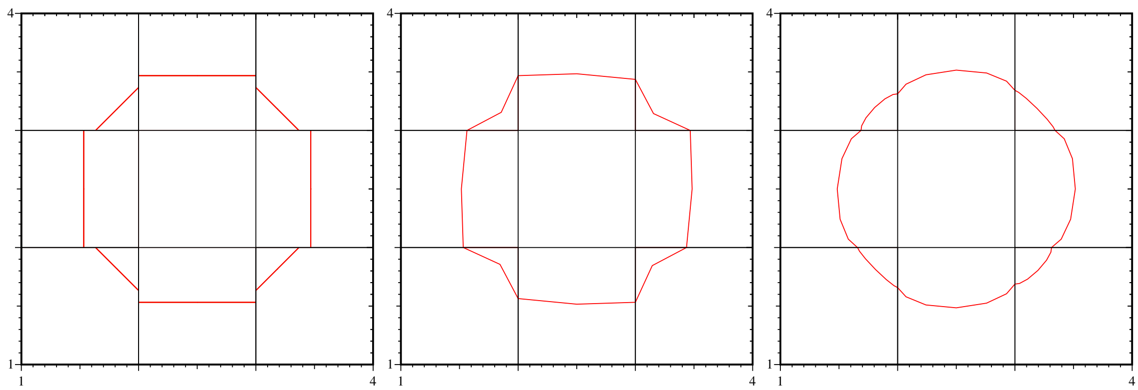


Fig. V.3 Circle reconstruction with Youngs method (left), DPIR (middle) and DPIR using curved interfaces (right).

Then we apply our method to a refined mesh (Fig. V.4). Here again, we can see improvements are made on results with curved interface.

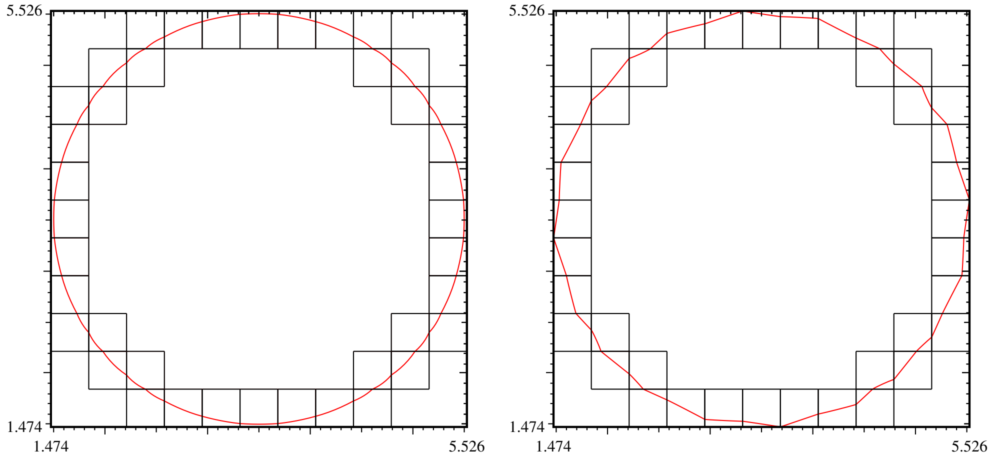


Fig. V.4 Reconstruction of a circle with DPIR with and without bezier curves.

Even if these good results are encouraging, we want a method that can work on general meshes. This method has shown some lack of robustness and it is what we want to investigate in the following section.

V.4 Robustness improvement of DPIR

To illustrate this robustness problem let us consider the same test as above on an unstructured mesh (Fig. V.5). In some cases the points on the edges M_i and M_{i+1} obtained after the first step of DPIR can be to close to the same node leading to a spike when the algorithm tries to balance the volume fraction. It has been observed, on some cases, that the algorithm can not balance the volume since the interface ends up outside the cell.

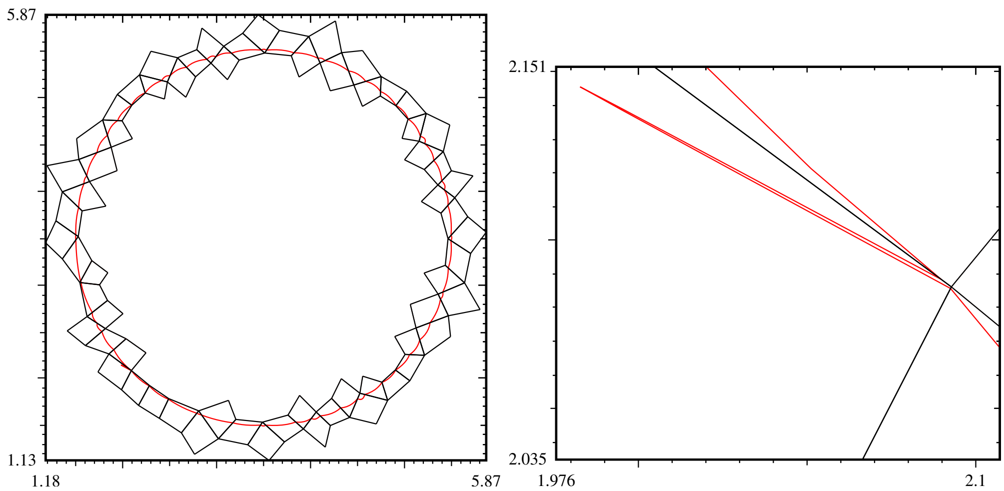


Fig. V.5 *On the left:* Reconstruction of a circle with the initial DPIR algorithm. *On the right:* a zoom on the reconstruction; the interface (in red) degenerates near a corner.

In order to tackle this problem and improve the robustness we describe three improvements of the method.

V.4.1 A new search direction for the control point

A first way to reinforce DPIR robustness is to move the control point in the direction to the cell center (Fig. V.6) instead of the perpendicular bisector. This ensures to move in a direction where there is more space available to apply the correction. Consequently it makes the correction step more robust.

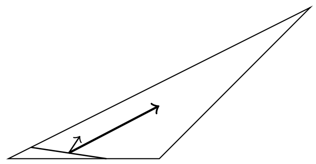


Fig. V.6 Search direction of the control point: perpendicular bisector vs center of the cell.

V.4.2 A new discretization of the cell edges

A second idea is to change, in the first step of DPIR, the discretization of the segments crossed by the interface: instead of considering uniform discretization, we use Chebyshev points in order to obtain a finer discretization around the corners than in the middle of those segments and avoid to end up with two points M_i M_{i+1} too close to each other. It permits in particular to reduce the number of points on each edge without altering the reconstruction quality.

V.4.3 A new penalty term

It may happen for some cells that the cost function term $vol(M_i, M_{i+1})$ and the penalty term $\|M_{i+1} - M_i\|$ have very different scales. In order to remedy to this problem, a second penalization term is added to the cost functional in the first step of DPIR. It is defined as:

$$\widetilde{p} = \sum_{i=0}^{N-1} \frac{|\text{vol}_{\text{target}} - \text{vol}(M_i, M_{i+1})|}{\text{vol}(M_i, M_{i+1})},$$

that leads to the following minimization problem:

$$\min_{M_0,\dots M_N} \sum_{i=0}^{N-1} |\text{vol}(M_i, M_{i+1}) - \text{vol}_{\text{target}}|^2 + \lambda\|M_{i+1} - M_i\| + \widetilde{p}.$$

If the volume fraction between M_i and M_{i+1} is too small with respect to the correction, the second penalization $\widetilde{p}$ becomes big and those points are not chosen by the minimization.

V.4.4 Numerical results

First let us apply the corrections illustrated in Sec. V.4.2-,V.4.3 in to the reconstruction of the circle on an unstructured grid. Fig. V.7 shows that the problem visible on Fig. V.5 has been solved: by looking at the zoom on the right hand side, we see that the interface does not degenerate anymore. In fact, the corrections do not let the interface pass too close to the corners of the mesh.

We also applied this method with the three improvements to the reconstruction of a square (Fig. V.8) and to the reconstruction of a J (Fig. V.9) on unperturbed and perturbed cartesian meshes.

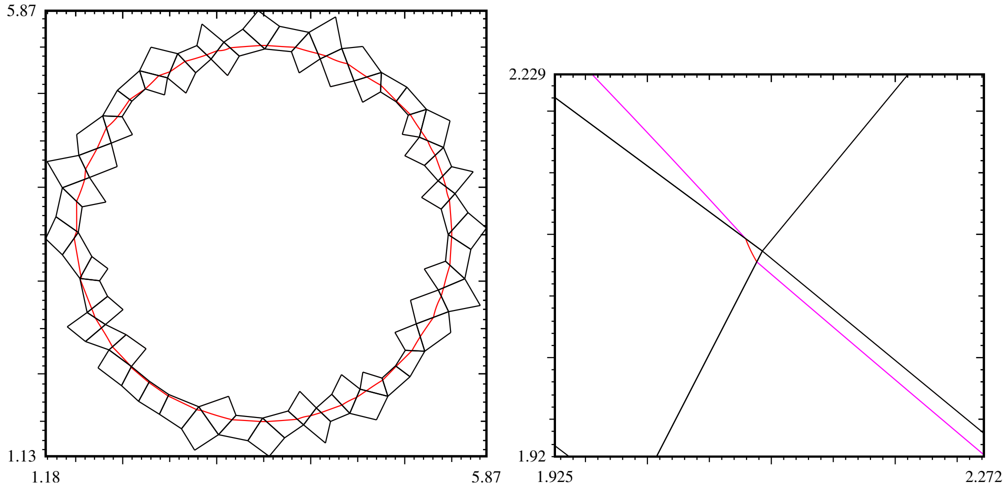


Fig. V.7 Reconstruction of a circle.

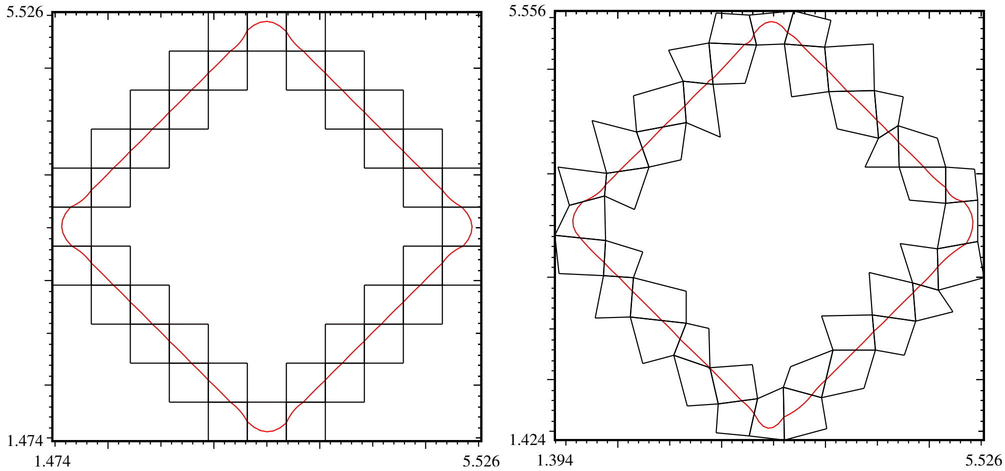


Fig. V.8 Reconstruction of a square on a distorted cartesian mesh

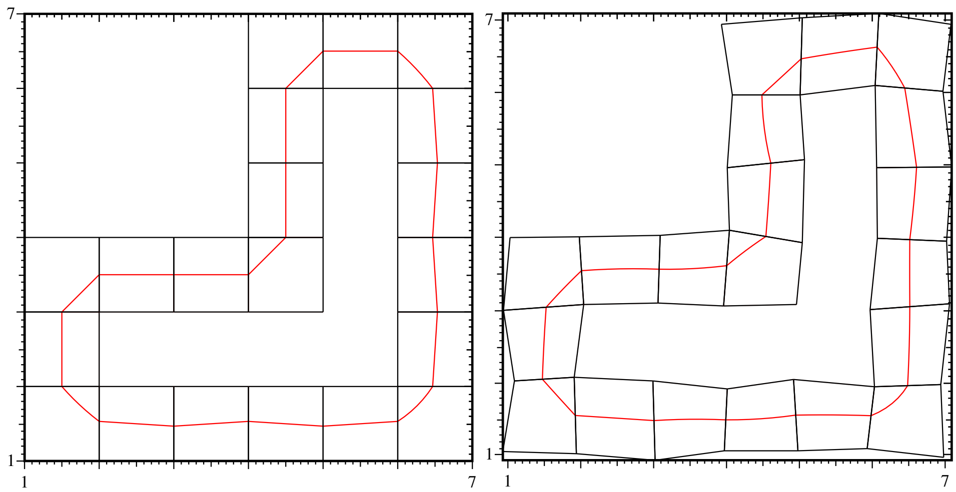


Fig. V.9 Reconstruction of a J on a distorted cartesian mesh

V.5 DPIR extension to triple point reconstruction

In this section, we describe an extension of DPIR to three or more materials. One of the main issues of this extension is the ability of the scheme to take into account triple points. First we describe different cases that can occur when considering more than two materials. Then we explain how to extend the algorithm and treat triple points. Then we apply it to three and four materials cases.

V.5.1 Classification of triple cells

A cell with exactly two strictly positive volume fractions is called a *double cell*. The treatment of double cells will the same as above. However the algorithm described above can not handle more than one interface point per edges. When an edge is crossed twice by the interface our method is not able to treat it. This means that DPIR can not handle filaments.

A *triple cell* contains exactly three materials. Two main cases can happen with this kind of cells: it contains a triple point or not. If the cell contains a triple point, you can quickly end up with the filament issue (see Fig. V.10b). Since we are not able to deal with this problem with two materials we will only give some perspectives on this matter at the end. If the cell does not contain a triple point, then materials are aligned in the cell (see Fig. V.10c).

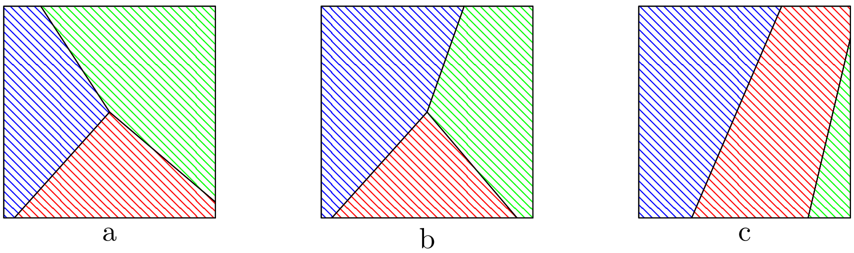


Fig. V.10 Types of triple cells.

We extend DPIR to three materials with a one-against-all approach. First we reconstruct three interfaces which are then merged in order to recover the final interface.

V.5.2 The new algorithm for interface reconstruction

The method described below is only suited for the first case of Fig. V.10a. Indeed, as explained above, we do not treat the filaments.

The algorithm is still divided in two steps : a first step leading to an interface prediction by running three Dynamic Programming algorithms, and a second step for a local correction of all mixed cells.

V.5.2.1 First step: one-against-all approach

1. Consider each material against all the others. Run the dynamic programming step of DPIR on all those materials. This step results in three interfaces (see Fig. V.11).

2. Every double cell is composed of edges with exactly 2 or 0 interface points (from outer and inner materials). Average those two points and fix the new resulting point as the final interface point (see Fig. V.12).

3. Every triple cell containing a triple point has three edges crossed by two interfaces. Glue the interfaces as previously and compute the temporary triple point as the barycenter of the triangle induced by those three new points (see Fig. V.12).

V.5.2.2 Second step: mixed cells correction

Each mixed cell is divided in different sub-cells whose number depends on its type (two for double cells and three for triple cells). We use a different correction algorithm depending on the type of mixed cell. If the mixed cell is a double cell, we simply use the standard correction step of DPIR in order to correct the partial volumes.

In the case of a triple cell containing a triple point, the objective is to obtain an interface regardless of the order of materials in the correction step. We describe here an algorithm that realizes such a correction.

- Compute signs of correction (decide if a material needs to increase or decrease its volume by evaluating the sign of the difference between the current volume in the sub-cell and the targeted volume)
- Correct the material that has a correction sign different of the two others by moving the triple point (considered here as a control point) in the direction given by the leading direction (that is the mean between the two correction directions of sub-cells that have the same correction sign)
- Correct the partial interface between the two others materials using the standard DPIR correction step (adding a control point on the sub-segment). The result of this step is illustrated on Fig. V.13.

Let us point out that as soon as you can treat the filament issue, the case of a cell with three materials aligned can be treated the same way as the cell with two materials.

V.5.2.3 Complexity

The complexity of the algorithm is equal to M times the complexity of DPIR, where M is the number of materials. As all one-against-all DPIR executions are independent, they can be executed in parallel. The correction step only involves very local corrections, that are independent. There are two different options :

- from the edge point of view: average contributions of computed temporary interface on each edge (that can be done in parallel) and correct the volume into each cell (possibly in parallel);
- from the cell point of view: average all interface contributions on each edge of the cell crossed by an interface and correct the volume into each cell (possibly in parallel).

Each local correction involves a (small) dichotomy search optimization that is $\mathcal{O}(log_2(|C|/\epsilon))$, with ϵ the requested precision of this optimization and $|C|$ the diameter of the largest mixed cell of the mesh. If we denote by T the number of mixed cells, the total complexity of the correction step is then $\mathcal{O}(Tlog_2(|C|/\epsilon))$ (triple cells only require 2 dichotomies).

V.5.3 A example of DPIR reconstruction on a triple point configuration

The new DPIR algorithm is applied to the interface reconstruction of a classical triple point test case where materials 1 and 2 are located inside a half sphere and material 3 outside this sphere. The results are displayed on figures V.11, V.12 and V.13.

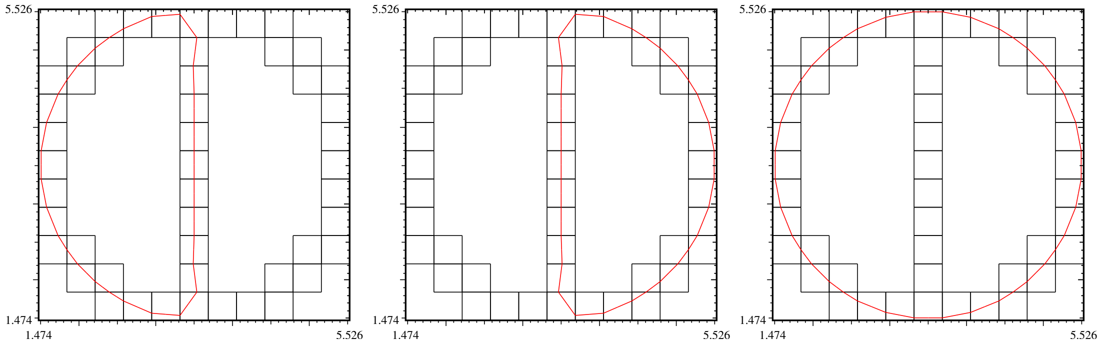


Fig. V.11 A triple point configuration: three interfaces obtained after the Dynamic programming execution, introduced in section V.5.2.1, point A).

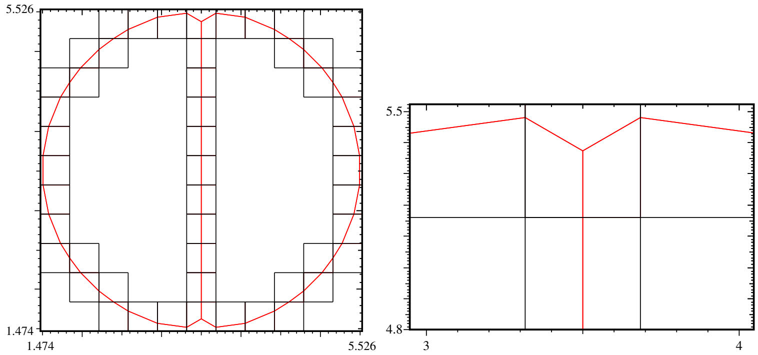


Fig. V.12 A triple point configuration: result after the first step, see section V.5.2.1, points B) and C).

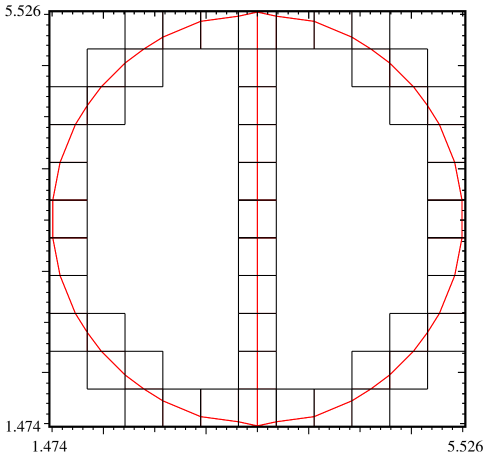


Fig. V.13 A triple point configuration: final result after the second step, see section V.5.2.2.

This new algorithm has also been successfully applied to a case with four materials (see Fig. V.14).

V.5.4 Perspectives on the filament issue

A filament is a portion of the interface of a given material that crosses twice the same edge (Fig. V.15). When it appears, one has to detect it before applying the algorithm. Indeed, DPIR can not handle this problem because it looks for one unique interface point per edge.

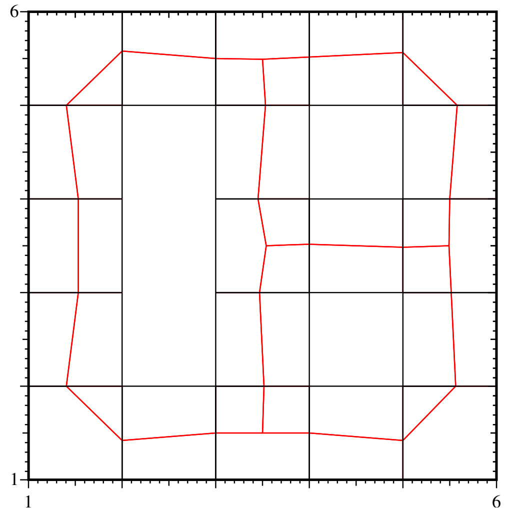


Fig. V.14 Interface reconstruction in a case with 4 materials.

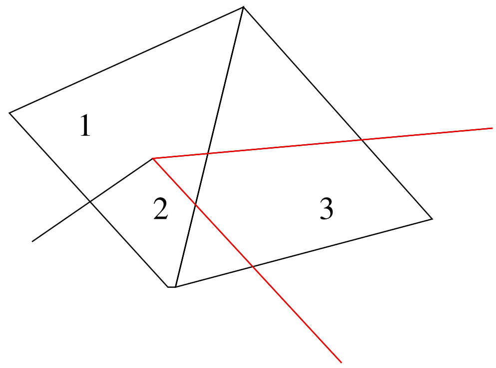


Fig. V.15 Example of filament. The numbers 1,2,3 indicate the three different materials.

To deal with this problem, we propose to first detect cells that contain filaments. The edges that should contain two interfaces points are then tagged and splitted into two sub-edges. At the stage methods described above can be applied since you are now looking for one unique interface point per edge (or sub-edge).

V.6 Conclusion and perspectives

A new and extended version of DPIR algorithm has been developed that preserves its major advantages, namely the continuity of the interface and the exact conservation of volumes. First, we have extended it to unstructured grids and curved interfaces using Bezier curves. We also described solutions to make it more robust at almost no additional cost.

In a second part, we extended DPIR to three or more materials. In this paper, we chose to use a one-against-all approach even if other methods could be considered in the future.

We are currently working on coupling this method to an ALE scheme in order to use the reconstructed interface with DPIR in anti-diffusive methods and curvature computation to deal with surface tension.

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Résumé: L'objectif de cette thèse est d'étudier et développer des schémas numérique du type volume finis pour des problèmes provenant de la mécanique des fluides, notamment le problème de Stokes et Navier-Stokes. Les schémas choisis sont du type dualité discrète, dénotés DDFV; cette méthode travaille sur des grilles décalées, où les inconnus de vitesse sont placés aux centres des volumes de contrôle et aux sommets du maillage, et les inconnus de pression aux arêtes du maillage. Ce type de construction a deux avantages principaux: elle permet de considérer des maillages généraux (qui ne vérifient pas nécessairement la condition d'orthogonalité classique des maillages volumes finis) et de reconstruire à niveau discret les propriétés de dualité des opérateurs différentiels continus. On commence par l'étude de la discrétisation du problème de Stokes avec des conditions aux bords mixtes de type Dirichlet/Neumann; le caractère bien posé de ce problème est strictement lié à l'inégalité Inf-sup, qui doit être vérifiée. Dans le cadre DDFV, cette inégalité a été prouvée pour des maillages particuliers; on peut éviter cette hypothèse, en ajoutant des termes de stabilisation dans l'équation de conservation de masse. Dans un premier temps, on étudie un schéma stabilisé pour le problème de Stokes en forme de Laplace, en montrant son caractère bien posé, des estimations d'erreur et des tests numériques. On étudie ensuite le même problème en forme divergence, où le tenseur des contraintes remplace le gradient; ici, on suppose que l'inégalité Inf-sup est vérifiée, et on écrit un schéma bien posé suivi des tests numériques. On considère ensuite le problème de Navier-Stokes incompressible. Initialement, on étude ce problème couplé avec des conditions aux bords « ouvertes » en sortie ; ce type de conditions apparaissent lors qu'on veut introduire une frontière artificielle, qui peut arriver pour des raisons de coût de calcul ou physiques. On écrit un schéma bien posé et des estimations d'énergie, validés par des simulations numériques. Deuxièmement, on s'intéresse à la méthode de décomposition de domaines sans recouvrement pour le problème de Navier-Stokes incompressible, en écrivant un algorithme de Schwarz discret. On discrétise le problème avec un schéma de type Euler semi-implicite en temps, et à chaque itération on applique l'algorithme de Schwarz au système linéaire résultant. Nous montrons également la convergence de cet algorithme et nous terminons par des expériences numériques. Cette thèse se termine par un cinquième chapitre issu d'une collaboration lors du CEMRACS 2019, où le but est d'étendre DPIR (une technique récente pour la reconstruction d'interfaces entre deux matériaux) au cas d'interfaces courbes et de trois matériaux. Des simulations numériques montrent les résultats.

Mots clefs : mécanique des fluides, volumes finis, DDFV, Stokes, Navier-Stokes, conditions aux bords mixtes, algorithmes de Schwarz, décompostion de domaine, reconstruction d'interfaces

Abstract: The goal of this thesis is to study and develop numerical schemes of finite volume type for problems arising in fluid mechanics, namely Stokes and Navier-Stokes problems. The schemes we choosed are of discrete duality type, denoted by DDFV; this method works on staggered grids, where the velocity unknowns are located at the centers of control volumes and at the vertices of the mesh, and the pressure unknowns are on the edges of the mesh. This kind of construction has two main advantages: it allows to consider general meshes (that do not necessarily verify the classical orthogonality condition required by finite volume meshes) and to reconstruct and mimic at the discrete level the dual properties of the continuous differential operators. We start by the study of the discretization of Stokes problem with mixed boundary conditions of Dirichlet/Neumann type; the well-posed character of this problem is strictly relied to Inf-sup inequality, that has to be verified. In the DDFV setting, this inequality has been proven for particular meshes; we can avoid this hypothesis, by adding some stabilization terms in the equation of conservation of mass. In the first place, we study a stabilized scheme for Stokes problem in Laplace form, by showing its well-posedness, some error estimates and numerical tests. We study the same problem in divergence form, where the strain rate tensor replaces the gradient; here, we suppose that the Inf-sup inequality is verified, and we design a well-posed scheme followed by some numerical tests. We consider then the incompressible Navier-Stokes problem. At first, we study this problem coupled with « open » boundary conditions on the outflow; this kind of conditions arises when an artificial boundary is introduced, to save computational ressources or for physical reasons. We write a well-posed scheme and some energy estimates, validated by numerical simulations. Secondly, we address the domain decomposition method without overlap for the incompressible Navier-Stokes problem, by writing a discrete Schwarz algorithm. We discretize the problem with a semi-implicit Euler scheme in time, and at each time iteration we apply Schwarz algorithm to the resulting linear system. We show the convergence of this algorithm and we end by some numerical experiments. This thesis ends with a last chapter concerning the work done during CEMRACS 2019, where the goal is to extend DPIR (a recent technique for interface reconstruction between two materials) to the case of curved interfaces and of three materials. Some numerical simulations show the results.

Key words : fluid mechanics, finite volumes, DDFV, Stokes, Navier-Stokes, mixed boundary conditions, Schwarz algorithm, domain decomposition, interface reconstruction