A non-overlapping Schwarz algorithm for Navier-Stokes problem with DDFV discretization

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Context

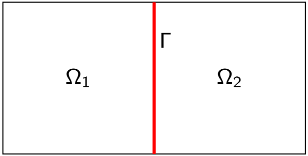
The Navier-Stokes problem

- Find $\mathbf{u} : \Omega_T \to \mathbb{R}^2$ and $\mathrm{p} : \Omega_T \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) = \mathbf{f} & \text{in} \quad \Omega_T = \Omega \times [0, T] \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T \end{cases}$$

with $T > 0$, $\mathbf{u} = 0$ on $\partial\Omega$, $\mathbf{u}_0 = \mathbf{u}_{init} \in (L^\infty(\Omega))^2$.
The stress tensor: $\sigma(\mathbf{u}, \mathrm{p}) = \dfrac{2}{\mathrm{Re}} \mathrm{D}\mathbf{u} - \mathrm{pId}$, with $\mathrm{D}\mathbf{u} = \dfrac{(\nabla \mathbf{u} + \nabla^t \mathbf{u})}{2}$.

- Domain decomposition:



$\Omega = \Omega_1 \cup \Omega_2$

The Navier-Stokes problem

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- Transmission conditions on Γ, for $j = 1, 2$, $i \neq j$:

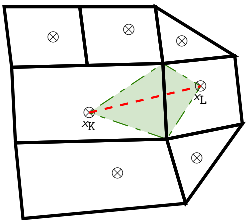
$$\sigma(\mathbf{u}_j^l, \mathrm{p}_j^l) \cdot \vec{\mathbf{n}}_j - \frac{1}{2}(\mathbf{u}_j^l \cdot \vec{\mathbf{n}}_j)(\mathbf{u}_j^l) + \lambda \mathbf{u}_j^l$$
$$= \sigma(\mathbf{u}_i^{l-1}, \mathrm{p}_i^{l-1}) \cdot \vec{\mathbf{n}}_i - \frac{1}{2}(\mathbf{u}_i^{l-1} \cdot \vec{\mathbf{n}}_i)(\mathbf{u}_i^{l-1}) + \lambda \mathbf{u}_i^{l-1}$$

$$\mathrm{div}(\mathbf{u}_j^l) + \alpha \mathrm{p}_j^l = -\mathrm{div}(\mathbf{u}_i^{l-1}) + \alpha \mathrm{p}_i^{l-1}$$

where $\vec{\mathbf{n}}_j$ is the outer normal to Ω_j, $\lambda, \alpha > 0$.

Discrete Duality Finite Volume method

DDFV meshes

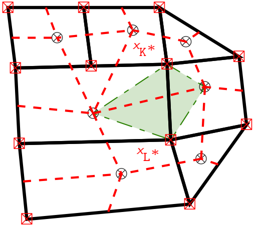


Primal mesh

$\mathfrak{M} \cup \partial\mathfrak{M}$

$\downarrow$

$\mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{K})_{K \in \mathfrak{M} \cup \partial\mathfrak{M}}$

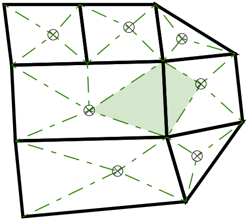


Dual mesh

$\mathfrak{M}^* \cup \partial\mathfrak{M}^*$

$\downarrow$

$\mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{K^*})_{K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*}$



Diamond mesh

$\mathfrak{D}$

$\downarrow$

$\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}$, $\mathrm{p}^{\mathfrak{D}}$

with $\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M} \cup \partial\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^* \cup \partial\mathfrak{M}^*})$ and $\mathfrak{T} = \mathfrak{M} \cup \partial\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$.

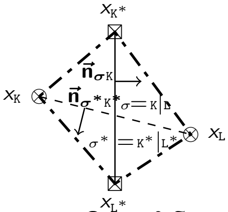
DDFV operators

- Discrete gradient operator $\nabla^{\mathfrak{D}} : (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ where

$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}(x_{\mathrm{L}} - x_{\mathrm{K}}) = \mathbf{u}_{\mathrm{L}} - \mathbf{u}_{\mathrm{K}},$$
$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}(x_{\mathrm{L}^*} - x_{\mathrm{K}^*}) = \mathbf{u}_{\mathrm{L}^*} - \mathbf{u}_{\mathrm{K}^*}.$$

[Krell '11]

$$\rightsquigarrow \mathrm{div}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \mathrm{Tr}(\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}})$$
$$\rightsquigarrow \mathrm{D}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \frac{\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} +^{t} (\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}})}{2}$$
$$\rightsquigarrow \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathrm{D}}) = \frac{2}{\mathrm{Re}}\mathrm{D}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{\mathrm{D}}\mathrm{Id}.$$



- Discrete divergence operator $\mathbf{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ where:

$$\mathbf{div}^{\mathrm{K}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathrm{K}}}\sum_{\sigma\subset\partial\mathrm{K}} m_\sigma \xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma\mathrm{K}}, \qquad \forall \mathrm{K} \in \mathfrak{M}$$

$$\mathbf{div}^{\mathrm{K}^*}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathrm{K}^*}}\sum_{\sigma^*\subset\partial\mathrm{K}^*} m_{\sigma^*} \xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*}, \qquad \forall \mathrm{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$$

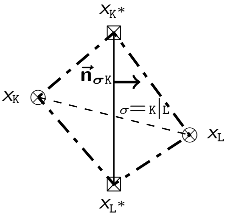
Theorem (Discrete duality property)

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\mathbf{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}$$

DDFV for Navier-Stokes : ($\mathcal{P}$)

At each time step we solve:

$$\begin{cases} m_{\mathrm{K}} \dfrac{\mathbf{u}_{\mathrm{K}} - \bar{\mathbf{u}}_{\mathrm{K}}}{\delta t} + \displaystyle\sum_{\sigma \subset \partial \mathrm{K}} m_\sigma \mathcal{F}_{\sigma\mathrm{K}} = m_{\mathrm{K}} \mathbf{f}_{\mathrm{K}} & \forall \mathrm{K} \in \mathfrak{M} \\ m_{\mathrm{K}^*} \dfrac{\mathbf{u}_{\mathrm{K}^*} - \bar{\mathbf{u}}_{\mathrm{K}^*}}{\delta t} + \displaystyle\sum_{\sigma^* \subset \partial \mathrm{K}^*} m_{\sigma^*} \mathcal{F}_{\sigma^*\mathrm{K}^*} = m_{\mathrm{K}^*} \mathbf{f}_{\mathrm{K}^*} & \forall \mathrm{K}^* \in \mathfrak{M}^* \\ \mathrm{div}^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}) = 0 & \forall \mathrm{D} \in \mathfrak{D} \end{cases}$$



with $\mathbf{u}^{\partial\mathfrak{M}} = \mathbf{u}^{\partial\mathfrak{M}^*} = 0$ and $\sum_{\mathrm{D}\in\mathfrak{D}} m_{\mathrm{D}} \mathrm{p}^{\mathrm{D}} = 0$.

The fluxes are a sum of a "diffusion" and a "convection" term:

$$m_\sigma \mathcal{F}_{\sigma\mathrm{K}} = m_\sigma (\mathcal{F}^d_{\sigma\mathrm{K}} + \mathcal{F}^c_{\sigma\mathrm{K}}) \approx \int_\sigma \sigma(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} + \int_\sigma (\mathbf{u} \cdot \vec{\mathbf{n}})\mathbf{u}$$

- The **diffusion fluxes**: $m_\sigma \mathcal{F}^d_{\sigma\mathrm{K}} = -m_\sigma \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathrm{K}}$.
- The **convection fluxes**, with $B : \mathbb{R} \to \mathbb{R}^+$:

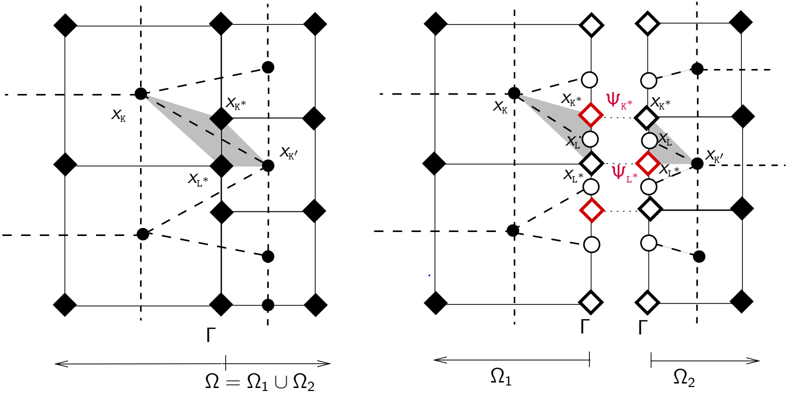
$$m_\sigma \mathcal{F}^c_{\sigma\mathrm{K}} = m_\sigma F_{\sigma\mathrm{K}} \left(\frac{\mathbf{u}_{\mathrm{K}} + \mathbf{u}_{\mathrm{L}}}{2} \right) + m_\sigma B\left(F_{\sigma\mathrm{K}}\right)(\mathbf{u}_{\mathrm{K}} - \mathbf{u}_{\mathrm{L}}),$$

Theorem

The scheme $\mathcal{P}$ is well-posed.

DDFV on composite meshes

DDFV on composite meshes



DDFV meshes.

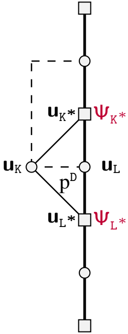
DDFV scheme for the subdomain problem

We define the DDFV discretization for the transmission conditions, to which we refer by

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j}, \mathrm{p}_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}}, \mathbf{h}_{\mathfrak{T}_j}, g_{\mathfrak{D}_j}) = 0$$

the following system:

$$\begin{cases} m_{\mathrm{K}} \dfrac{\mathbf{u}_{\mathrm{K}} - \bar{\mathbf{u}}_{\mathrm{K}}}{\delta t} + \displaystyle\sum_{\sigma \subset \partial \mathrm{K}} m_\sigma \widetilde{\mathcal{F}}_{\sigma \mathrm{K}} = m_{\mathrm{K}} \mathbf{f}_{\mathrm{K}} & \forall \mathrm{K} \in \mathfrak{M}_j \\ m_{\mathrm{K}^*} \dfrac{\mathbf{u}_{\mathrm{K}^*} - \bar{\mathbf{u}}_{\mathrm{K}^*}}{\delta t} + \displaystyle\sum_{\sigma^* \subset \partial \mathrm{K}^*} m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^* \mathrm{K}^*} = m_{\mathrm{K}^*} \mathbf{f}_{\mathrm{K}^*} & \forall \mathrm{K}^* \in \mathfrak{M}_j^* \\ m_{\mathrm{K}^*} \dfrac{\mathbf{u}_{\mathrm{K}^*} - \bar{\mathbf{u}}_{\mathrm{K}^*}}{\delta t} + \displaystyle\sum_{\sigma^* \subset \partial \mathrm{K}^*} m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^* \mathrm{K}^*} + m_{\sigma \mathrm{K}^*} \Psi_{\mathrm{K}^*} = m_{\mathrm{K}^*} \mathbf{f}_{\mathrm{K}^*} & \forall \mathrm{K}^* \in \partial\mathfrak{M}_{j,\Gamma}^* \\ \mathrm{div}^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}) = 0 & \forall \mathrm{D} \in \mathfrak{D}_j \setminus \mathfrak{D}_j^{\Gamma} \end{cases}$$



with $\mathbf{u}^{\partial\mathfrak{M}_{j,D}} = 0$ and $\mathbf{u}^{\partial\mathfrak{M}^*_{j,D}} = 0$, plus the *transmission conditions* on Γ.

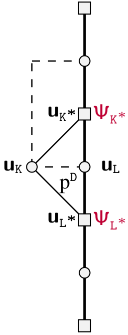
Transmission conditions

- Transmission conditions on Γ at the continuous level:

$$\sigma(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} - \frac{1}{2}(\mathbf{u} \cdot \vec{\mathbf{n}})\mathbf{u} + \lambda \mathbf{u} = \mathbf{h}$$
$$\mathrm{div}(\mathbf{u}) + \alpha \mathrm{p} = \mathbf{g}$$

- Discrete transmission conditions :

$$-\widetilde{\mathcal{F}}_{\sigma\mathrm{K}} + \frac{1}{2} F_{\sigma\mathrm{K}} \mathbf{u}_{\mathrm{L}} + \lambda \mathbf{u}_{\mathrm{L}} = \mathbf{h}_{\mathrm{L}} \qquad \forall \sigma \in \partial\mathfrak{M}_{j,\Gamma}$$
$$-\Psi_{\mathrm{K}^*} + \frac{1}{2}(\bar{\mathbf{u}}_{\mathrm{K}^*} \cdot \vec{\mathbf{n}}_{\sigma\mathrm{K}})\, \mathbf{u}_{\mathrm{K}^*} + \lambda \mathbf{u}_{\mathrm{K}^*} = \mathbf{h}_{\mathrm{K}^*} \qquad \forall \mathrm{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma}$$
$$\mathrm{div}^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}) + \alpha \mathrm{p}^{\mathrm{D}} = g_{\mathrm{D}} \qquad \forall \mathrm{D} \in \mathfrak{D}_j^{\Gamma}$$



with $\lambda, \alpha > 0$ and the flux:

$$m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathrm{K}} = \underbrace{-m_\sigma \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathrm{K}}}_{m_\sigma \mathcal{F}^d_{\sigma\mathrm{K}}} \underbrace{+ m_\sigma F_{\sigma\mathrm{K}} \left(\frac{\mathbf{u}_{\mathrm{K}} + \mathbf{u}_{\mathrm{L}}}{2}\right) + m_\sigma \widetilde{B}\,(F_{\sigma\mathrm{K}})\,(\mathbf{u}_{\mathrm{K}} - \mathbf{u}_{\mathrm{L}})}_{m_\sigma \widetilde{\mathcal{F}}^c_{\sigma\mathrm{K}}},$$

Theorem

The scheme $\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j}, p_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}}, \mathbf{h}_{\mathfrak{T}_j}, g_{\mathfrak{D}_j}) = 0$ *is well-posed.*

Schwarz algorithm

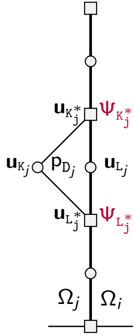
Iterative domain decomposition solver

1. Choose $\mathbf{h}^0_{\mathfrak{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$ and $g^0_{\mathfrak{D}_j} \in \mathbb{R}^{\mathfrak{D}_j}$.
2. Compute $(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}}$ solution to

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}^l_{\mathfrak{T}_j}, \mathrm{p}^l_{\mathfrak{D}_j}, \Psi^l_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}_j}, \mathbf{h}^{l-1}_{\mathfrak{T}_j}, g^{l-1}_{\mathfrak{D}^\Gamma_j}) = 0. \qquad (\mathcal{S}_1)$$

3. Compute the new values of $\mathbf{h}^l_{\mathfrak{T}_J}$ and of $g^l_{\mathfrak{D}^\Gamma_j}$ by:

$$\begin{aligned}
\mathbf{h}^l_{\mathtt{L_j}} &= \widetilde{\mathcal{F}}^l_{\sigma\mathtt{K}_i} - \frac{1}{2} F_{\sigma\mathtt{K}_i}\mathbf{u}^l_{\mathtt{L_i}} + {\color{red}\lambda}\, \mathbf{u}^l_{\mathtt{L_i}}, && \forall \mathtt{L_j} \in \partial\mathfrak{M}_{j,\Gamma} \\
\mathbf{h}^l_{\mathtt{K}^*_\mathtt{j}} &= \Psi^l_{\mathtt{K}^*_\mathtt{i}} - \frac{1}{2}(\bar{\mathbf{u}}_{\mathtt{K}^*_\mathtt{i}} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}})\, \mathbf{u}^l_{\mathtt{K}^*_\mathtt{i}} + {\color{red}\lambda}\, \mathbf{u}^l_{\mathtt{K}^*_\mathtt{i}}, && \forall \mathtt{K}^*_\mathtt{j} \in \partial\mathfrak{M}^*_{j,\Gamma} \qquad (\mathcal{S}_2) \\
g^l_{\mathtt{D}_j} &= \frac{1}{m_{\mathtt{D}_j}} \left(-m_{\mathtt{D}_i}\, \mathrm{div}^{\mathtt{D}_i}(\mathbf{u}^l_{\mathfrak{T}_i}) + {\color{red}\alpha}\, m_{\mathtt{D}_i}\, \mathrm{p}^l_{\mathtt{D}_i} \right), && \forall \mathtt{D}_j \in \mathfrak{D}^\Gamma_j
\end{aligned}$$



Theorem

The solution of the DDFV Schwarz algorithm converges when $l \to \infty$ the solution of the classical DDFV scheme $(\mathcal{P})$ on Ω .

Conclusions and Perspectives

Conclusions:

- Designed a DDFV scheme for Navier-Stokes.
- Designed a DDFV scheme for the subdomain problem.
- Constructed a Schwarz algorithm.
- Proved the convergence of the Schwarz algorithm.

Perspectives:

- Numerical simulations.
- Optimize the parameters λ, α.
- Study the overlapping case.

M.J. Gander, L. Halpern, F. Hubert, S. Krell. *Optimized Schwarz Methods for Anisotropic Diffusion with DDFV discretizations*. submitted, 2019.

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Grazie per l'attenzione!

Nonlinear convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ (1/2)

We observe:

$$\int_{\mathrm{K}} (\bar{\mathbf{u}}^{\mathfrak{T}}\cdot\nabla)\mathbf{u}^{\mathfrak{T}} = \sum_{\sigma\subset\partial\mathrm{K}} \int_{\sigma} (\bar{\mathbf{u}}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\sigma\mathrm{K}})\mathbf{u}^{\mathfrak{T}} \quad \forall \mathrm{K}\in\mathfrak{M}$$

We define the fluxes:

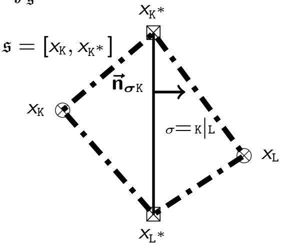
$$\int_{\sigma} (\bar{\mathbf{u}}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\sigma\mathrm{K}}) \rightsquigarrow F_{\sigma\mathrm{K}}(\bar{\mathbf{u}}^{\mathfrak{T}})$$

We impose: $F_{\sigma\mathrm{K}}(\bar{\mathbf{u}}^{\mathfrak{T}}) = -\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathrm{K}}\cap\mathcal{E}_{\mathrm{D}}} G_{\mathfrak{s},\mathrm{D}}(\bar{\mathbf{u}}^{\mathfrak{T}})$ where

$$G_{\mathfrak{s},\mathrm{D}}(\bar{\mathbf{u}}^{\mathfrak{T}}) = m_{\mathfrak{s}}\frac{\mathbf{u}^n_{\mathrm{K}}+\mathbf{u}^n_{\mathrm{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}} \rightsquigarrow \int_{\mathfrak{s}} \bar{\mathbf{u}}^{\mathfrak{T}}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}},$$

We have conservativity:

$$F_{\sigma\mathrm{K}} = -F_{\sigma\mathrm{L}}, \quad \forall\sigma = \mathrm{K}|\mathrm{L}$$



Nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ (2/2)

$$\int_{\mathrm{K}} (\mathbf{u} \cdot \nabla)\mathbf{u} \rightsquigarrow \sum_{\sigma \subset \partial \mathrm{K}} \int_{\sigma} (\bar{\mathbf{u}}^{\mathfrak{T}} \cdot \vec{\mathbf{n}}_{\sigma\mathrm{K}})\mathbf{u}^{\mathfrak{T}}$$

So we define $\forall \mathrm{K} \in \mathfrak{M}$:

$$\boxed{m_\sigma \mathcal{F}^c_{\sigma\mathrm{K}} = m_\sigma F_{\sigma\mathrm{K}}(\bar{\mathbf{u}}^{\mathfrak{T}}) \frac{\mathbf{u}_{\mathrm{K}} + \mathbf{u}_{\mathrm{L}}}{2} + \frac{m_\sigma^2}{2\mathrm{Re}\, m_{\mathrm{D}}} B\left(\frac{2\mathrm{Re}\, m_{\mathrm{D}}}{m_\sigma^2}\, m_\sigma F_{\sigma\mathrm{K}}(\bar{\mathbf{u}}^{\mathfrak{T}})\right)(\mathbf{u}_{\mathrm{K}} - \mathbf{u}_{\mathrm{L}}).}$$

If $B(s) = 0$ we get a centered approximation.
If $B(s) = \frac{1}{2}|s|$ we get an upwind scheme.

Finally we get:

$$\int_{\mathrm{K}} (\mathbf{u} \cdot \nabla)\mathbf{u} \rightsquigarrow \sum_{\sigma \subset \partial \mathrm{K}} m_\sigma \mathcal{F}^c_{\sigma\mathrm{K}}$$

Back