École Polytechnique Fédérale de Lausanne

Finite time singularities in the Euler Equation

Semester Project Report

Supervisor: Prof. Maria Colombo
Tutor: Dr. Elio Marconi
Student: Giulia Mescolini

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1 Introduction

Global regularity of solutions to the incompressible Euler Equation is one of the main open problems in the study of partial differential equations. While the possibility of *blow-up* of solutions $u \in C^\infty(\mathbb{R}^3 \times [0,T))$ seems far to be solved, the works by Elgindi et al. led to the discovery of a solution in the class $C^{1,\alpha}(\mathbb{R}^3 \times [0,T))$ with a *blow-up* (we will use this term to mean that at a finite time the solution ceases to belong to $C^{1,\alpha}$).
In this work, we develop more in detail two examples sketched by Elgindi in [7], which involve equations with a structure similar to the Euler Equation and enable to get a glimpse of Elgindi's intuition. Then, we present the backbone of the construction of a solution with blow-up in finite time in [7], underlining the main similarities and difficulties with respect to the examples.

1.1 The Euler Equation

We recall the incompressible Euler equation governing the flow of an ideal fluid on $\mathbb{R}^3$, arising from Navier-Stokes equations by neglecting the *diffusive term*.
Denoting by $u : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ the fluid velocity and by $p : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ its pressure, the equations read as follows:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0 \\ \nabla \cdot u = 0 \\ u|_{t=0} = u_0. \end{cases} \tag{1}$$

The constraint $\nabla \cdot u = 0$ is known as *incompressibility constraint*, and expresses the property for which the volume of portions of fluid cannot be squeezed or expanded by the velocity field.
Moreover, for a smooth solution of Equation 1 with decay at infinity, it holds that

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u(x,t)|^2 dx = 0.$$

Proof. By multiplying the first equation in 1 by u and integrating over $\mathbb{R}^3$, one can get:

$$\int_{\mathbb{R}^3} \partial_t u u + (u \cdot \nabla)uu + \nabla p u = 0.$$

Note that $\int_{\mathbb{R}^3} \nabla p u = -\int_{\mathbb{R}^3} (\nabla \cdot u)p = 0$ and $\int_{\mathbb{R}^3} (u \cdot \nabla)uu = -\frac{1}{2}\int_{\mathbb{R}^3} (\nabla \cdot u)u^2 = 0$ by the incompressibility constraint. Hence, since u is smooth, we can get:

$$\int_{\mathbb{R}^3} \partial_t |u(x,t)|^2 dx = \frac{d}{dt} \int_{\mathbb{R}^3} |u(x,t)|^2 dx = 0.$$

□

1.2 The Vorticity Equation

Let us now introduce a physical quantity that plays a central role in the study of blow-ups: the fluid *vorticity*,

$$\omega = \nabla \times u,$$

which describes the spinning behavior of the fluid. Since $\nabla \cdot u = 0$, one has that $\nabla \times (\nabla \times u) = -\Delta u$.
Hence, u can be related to ω with the *Biot-Savart law* as

$$u = (-\Delta)^{-1}(\nabla \times \omega).$$

By applying the curl operator to Equation 1, we can derive an equation for the vorticity:

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u \\ \omega|_{t=0} = \omega_0. \end{cases} \tag{2}$$

Proof. We can obtain Equation 2 from $\nabla \times (\partial_t u) + \nabla \times ((u \cdot \nabla)u) + \nabla \times \nabla p = 0$ by observing that:

- $\nabla \times \nabla p = 0$ (curl of a gradient is zero);
- $\nabla \times ((u \cdot \nabla)u) = \nabla \times (\nabla(\frac{1}{2}u \cdot u) - u \times \omega) = -\nabla \times (u \times \omega) = -(\omega \cdot \nabla)u + (u \cdot \nabla)\omega$.

□

In the vorticity equation, we can identify two terms that are extremely relevant for singularity formation:

- $(\omega \cdot \nabla)u$, the *vortex stretching* term;
- $(u \cdot \nabla)\omega$, term describing *advection* of vorticity. It complicates the analysis, and we will try to set arguments in a framework where it is not an obstacle.

Let us consider for example the 2D case: there is no vortex stretching, as $\omega \perp u$, and global existence of 2D solutions to Equation 1 according to the *Beale-Kato-Majda* theorem, that we will present in subsection 1.3.
For the 3D scenario, instead, we will look for singularities in settings in which the depletion of vorticity due to advection is minimized.

1.3 Blow-up Criteria

Among the numerous works on the global regularity problem for the Euler Equation, we mainly refer to the blow-up criterion by Beale, Kato and Majda in [18].
In order to state this theorem in its *Lagrangian* formulation, we need to introduce the *integro-differential equations for particle trajectories* $X(\alpha, t)$, which are proved to be equivalent to the Euler equation when the flow is sufficiently smooth (see Proposition 2.23 in [18]):

$$\begin{cases} \frac{dX}{dt}(\alpha, t) = \int_{\mathbb{R}^2} K_2[X(\alpha, t) - X(\alpha', t)]\omega_0(\alpha')d\alpha' =: F(X(\alpha, t)) \\ X(\alpha, t)|_{t=0} = \alpha \end{cases} \qquad \text{(2D case)} \qquad (3a)$$

$$\begin{cases} \frac{dX}{dt}(\alpha, t) = \int_{\mathbb{R}^3} K_3[X(\alpha, t) - X(\alpha', t)]\nabla_\alpha X(\alpha', t)\omega_0(\alpha')d\alpha' =: F(X(\alpha, t)) \\ X(\alpha, t)|_{t=0} = \alpha \end{cases} \qquad \text{(3D case)} \qquad (3b)$$

where K_2, K_3 are kernels defined as

$$K_2(x) := \frac{1}{2\pi}\left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)^T, \qquad K_3(x) := \frac{1}{4\pi}\frac{x \times h}{|x|^3}.$$

The advantage in shifting to this formulation consists in the fact that Equation 3 are ODEs on an infinite-dimensional Banach space where the operator F is *bounded* (if the choice of the Banach space is suitable), while in Equation 1 the term $u \cdot \nabla$ is *unbounded* on standard Banach spaces.

Theorem 1. *Given a compactly supported initial vorticity $\omega_0 = \nabla \times u_0$ (with $\nabla \cdot u_0 = 0$) such that $||\omega_0||_\gamma < +\infty$ for[1] some $\gamma \in (0, 1)$, denoting by O_M the subset of γ-Hölder continuous functions with norm[2] smaller than M and such that $\inf_{\alpha \in \mathbb{R}^3} \det \nabla_\alpha X(\alpha) > \frac{1}{2}$ and indicating with $|\omega(\cdot, s)|_0$ the supremum norm at a fixed time s:*

(i) Suppose that $\forall T > 0\ \exists M_1 > 0$ such that $\int_0^T |\omega(\cdot, s)|_0 ds \leq M_1$.
Then, $\forall T > 0\ \exists M > 0$ such that the solution X lies in the space $C^1([0, T); O_M)$.

(ii) Suppose that $\forall M > 0\ \exists T(M)$, finite maximal time of existence of solutions $X \in C^1([0, T(M)); O_M)$, and that $\lim_{M \to \infty} T(M) = T^* < +\infty$.
Then, $\lim_{t \to T^*} \int_0^t |\omega(\cdot, s)|_0 ds = +\infty$.

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[1] Here, $|| \cdot ||_\gamma$ denotes the norm in the space of γ-Hölder continouous functions.

[2] We adopt the norm $|X|_{1,\gamma} := |X(0)| + |\nabla_\alpha X(0)|_0 + \sup_{\alpha \neq \alpha', \alpha, \alpha' \in \mathbb{R}^3} \frac{|X(\alpha) - X(\alpha')|}{|\alpha - \alpha'|}$, where $| \cdot |_0$ is the norm in C^0.

Hence, this theorem relates accumulation of vorticity to singularity formation. Note that, since in 2D vorticity is conserved along particle trajectories[3], its magnitude cannot become unbounded, so the hypothesis in (*i*) of Theorem 1 always holds.

1.4 Axi-symmetric Flows

A class of fluids whose properties have been studied, and that includes Elgindi's test case in [7], is the one of *axi-symmetric flows*.
A motivation for the choice of this setting is the fact that it represents an intermediate workspace, in terms of complexity, between the 2D and the 3D setting. One one hand, indeed, the 3D case is still too difficult to be analyzed and in 2D blow-ups are not possible.
We will see that, in the axi-symmetric case, there is vortex stretching, but at the same time there are only two coordinates to be considered, hence it could be regarded as a generalization of the easier 2D case.
Denoting by e_r, e_θ and e_3 the standard orthonormal unit vectors and defining the cylindrical coordinate system

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^T \quad e_\theta = \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right)^T \quad e_3 = (0,0,1)^T,$$

we say that a velocity field is *axi-symmetric* if it is independent from the θ coordinate, i.e.

$$u = u^r(r, x_3, t)e_r + u^\theta(r, x_3, t)e_\theta + u^3(r, x_3, t)e_3.$$

The velocity in the θ direction, u^θ, is called *swirl velocity*; for the class of *axi-symmetric flows without swirl* (for which $u^\theta = 0$), the vorticity has the form

$$\omega(r, x_3, t) = \omega^\theta e_\theta = (\partial_3 u^r - \partial_r u^3)e_\theta.$$

For this specific class of fluids, we can derive[4] from Equation 2 the *vortex-dynamics equation* by introducing the quanitity $\xi = \frac{\omega^\theta}{r}$. In particular, it states that ξ is conserved along particle trajectories, since it has a null material derivative (see its definition for the axi-symmetric case in Appendix A).

$$\frac{\tilde{D}}{Dt}\xi = 0. \tag{4}$$

If there is a swirl, the vortex dynamics equation is more complex, but still simpler than in the case without axial symmetry; by proceeding as in Chapter 2.3 of [18], we can derive the equations

$$\begin{cases} \frac{\tilde{D}}{Dt}(ru^\theta) = 0 \\ \frac{\tilde{D}}{Dt}\xi = -\frac{1}{r^4}\partial_3[(ru^\theta)^2] \\ \frac{\tilde{D}}{Dt}(ru^\theta)^2 = 0. \end{cases} \tag{5}$$

Note that in this case the quantity ξ changes depending on the swirl velocity and it is not conserved along trajectories. Since the spatial derivatives present in the material derivative $\frac{\tilde{D}}{Dt}$ do not include the coordinate θ, we have an interaction between the axial-plane velocities (u^r, u^3) and the swirl velocity.

1.5 The stream function

In the 2D case, being *u solenoidal*, i.e. divergence-free, we can introduce its *stream function* $\tilde{\psi}$.
In section 4 we will be set in the *axi-symmetric case*, hence let us directly develop the exposition for that framework. Note that this setting can be viewed as a 2D case, as there are only two free variables (r and z).
The stream function $\tilde{\psi}$ is defined in such a way that

$$\nabla^\perp \tilde{\psi} = u \iff u^r = \frac{1}{r}\partial_3\tilde{\psi}, \quad u^3 = -\frac{1}{r}\partial_r\tilde{\psi},$$

[3]Indeed, Equation 2 becomes simply $\frac{D\omega}{Dt} = 0$.
[4]See its derivation in Appendix B, section 1.

and since $\omega = \nabla \times u$, we have that

$$-\omega = -\partial_r u^3 + \partial_3 u^r = \partial_r \left(\frac{1}{r} \partial_r \tilde{\psi} \right) + \frac{1}{r} \partial_{33} \tilde{\psi}.$$

We can obtain a cleaner expression working with $\psi = \frac{1}{r}\tilde{\psi}$:

$$\omega = -\partial_r \left(\frac{1}{r} \partial_r (r\psi) \right) - \partial_{33}\psi = -\partial_{rr}\psi - \partial_{33}\psi - \frac{1}{r}\partial_r \psi + \frac{\psi}{r^2}. \tag{6}$$

1.6 Statement of the main theorem in [7] and comparison with literature

In this section, we will compare the result obtained in the main reference [7] with existing results in literature. Let us first state the theorem proved in [7]:

Theorem 2. *There exists an $\alpha > 0$ and a divergence-free and odd $u_0 \in C^{1,\alpha}(\mathbb{R}^3)$ with initial vorticity $|\omega_0(x)| \leq \frac{C}{|x|^{\alpha}+1}$ for some constant $C > 0$ so that the unique local odd solution to the Euler Equation belonging to the class $C^{1,\alpha}([0,1) \times \mathbb{R}^3)$ satisfies*

$$\lim_{t \to 1} \int_0^t |\omega(s)|_0 ds = +\infty.$$

The solutions built here have infinite energy, but this characteristics, that weakens the physical meaning of the solutions, has been dropped in a posterior work as explained in Appendix C, section 3.
There exist previous blow-up results for infinite energy solutions ([4, 12, 21]), but in that cases vorticity grows linearly at spatial infinity and the blow-up occurs on an infinite line or plane; by contrast, in Elgindi's solution, the vorticity is decaying (which ensures more physical concreteness) and the blow-up occurs at a single point.

2 Local model for the 3D axi-symmetric case with swirl

We now analyze a simple model in which the influence on blow-ups of the vorticity regularity and of spatial boundaries is highlighted. Those concepts are strictly related, as the presence of solid boundaries where the vorticity does not vanish could be seen as equivalent to jumps in the vorticity, leading to a regularity decay, so they both represent a factor in support of blow-ups.
The association between walls and jumps is due to the fact that we may not be able to extend the solution in a domain with a wall to a regular solution in the whole $\mathbb{R}^d$, but there may be a lack of regularity such as a jump while crossing the boundary of the original domain, hence theorems for smooth solutions could not be applied.

Note also that the model we will deal with here refers to a case of an axi-symmetric flow *with swirl*; this scenario seems to be a good candidate for the research of singularities, as in the case *without swirl*, instead, global existence of smooth solutions can be proved under mild assumptions (see chapter 4.3 in [18]).

2.1 The problem and its physical meaning

We are considering the 2D system:

$$\begin{cases} \partial_t \omega - (x_1 \lambda(t), -x_2 \lambda(t)) \cdot \nabla \omega = \partial_1 \rho \\ \partial_t \rho - (x_1 \lambda(t), -x_2 \lambda(t)) \cdot \nabla \rho = 0 \\ \lambda(t) = \int_{\mathbb{R}^2} \frac{y_1 y_2}{|y|^4} \omega(y,t) dy. \end{cases} \tag{7}$$

The analytical work on this model problem has been inspired by the paper by Luo and Hou [17], where the 3D axi-symmetric Euler equations in a periodic cylinder have been numerically studied and a class of potentially singular solutions from suitably chosen initial data has been discovered.
Although in numerical experiments the results have sometimes been misleading, since singularities may have

been created by numerical artefacts, in the paper [17] data have been checked against some major blow-up criteria (including Theorem 1), hence there is evidence for an actual blow-up of the solution.

If, to begin, we imagine to substitute the vector field $-(x_1\lambda(t), -x_2\lambda(t))$ in Equation 7 with the velocity field $u = (u_1, u_2)$, we can interpret more easily the physical phenomenon described by the equation (we will legitimate this substitution at the end of the section).
We now briefly sketch the path leading to the interpretation of Equation 7 as a model for the 3D axi-symmetric Euler equation *with swirl* away from the the axis of symmetry and near $(0, 0)$, taken to be the *hyperbolic stagnation* point. Indeed, $(0, 0)$ is an *hyperbolic* point (as can be visually assessed by Figure 1b) and a *stagnation* point as well since trajectories starting from it are points and remain still at the origin.

We start by remarking the similarity of the system 7, after the mentioned switch to u, with the 2D *Boussinesq equations* for vorticity ω and density ρ of fluids under the influence of gravitational forces (identifying the axis $x_1 = 0$ with the vertical one):

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \partial_1\rho \\ \partial_t\rho + (u\cdot\nabla)\rho = 0 \\ u = \nabla^{\perp}(-\Delta)^{-1}\omega \\ \omega(x,0) = \omega_0(x) \quad \rho(x,0) = \rho_0(x). \end{cases} \tag{8}$$

From Equation 8, we can get a model such as Equation 5 for axi-symmetric swirling flows making the following correspondances between the fluid variables (left) and the variables in the Boussinesq equation (right):

$$x_3 \leftrightarrow x_1 \quad r \leftrightarrow x_2 \quad \omega^\theta \leftrightarrow \omega \quad (ru^\theta)^2 \leftrightarrow \rho,$$

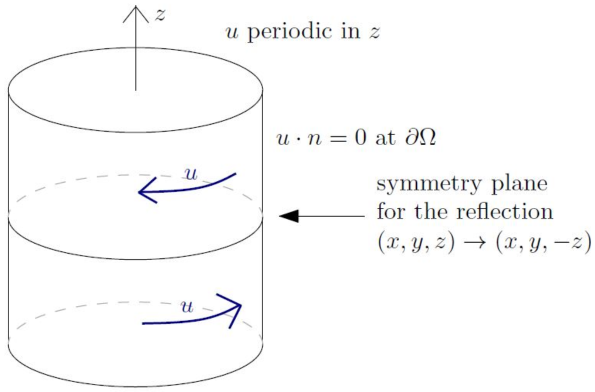
and evaluating the external variable coefficients at $r = 1$, which is consistent with the assumption we are using of being far from the axis of singularity $r = 0$. If we are close to $r = 0$, indeed, the dynamics of the solutions of the two systems may be different, due to the presence of factors of r.
Note also that $\partial_1\rho$ in the right-hand side of Equation 7 determines the presence of a *vortex stretching term*, which involves the swirl velocity u^θ given its relationship with ρ.

Finally, let us discuss the reason why u is substituted by $-(x_1\lambda(t), -x_2\lambda(t))$.
In the 1D case, for example, a monodimensional version of the Boussinesq problem in the domain $D = [0, 1]$ with Dirichlet boundary conditions has been studied by [2, 3], where the problem formulation reads as:

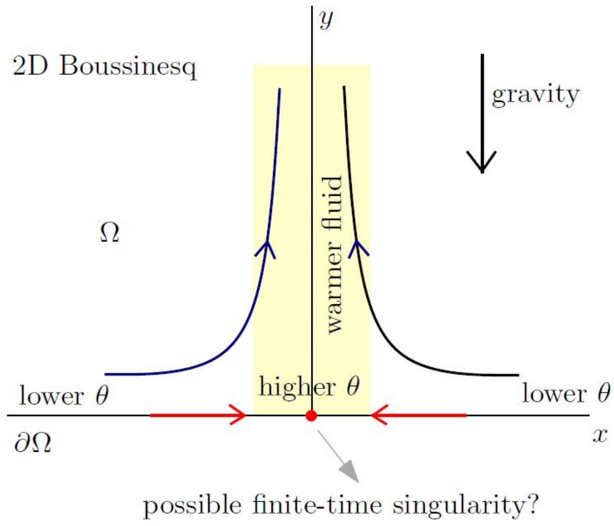
$$\begin{cases} \partial_t\omega - x\tilde{\lambda}(t)\omega = \partial_1\rho \\ \partial_t\rho - x\tilde{\lambda}(t)\rho = 0 \\ \tilde{\lambda}(t) = \int_x^1 \frac{\omega(y,t)}{y}dy. \end{cases} \tag{9}$$

In our case, we can proceed as done in the $1D$ case above and apply a simplified version of the *Biot-Savart* law to approximate u with the vector field $-(x_1\lambda(t), -x_2\lambda(t))$. The reasonableness of doing this, provided that we are inside a portion of disk close the origin $(0, 0)$, is widely discussed in [14], where it is proved that the non-negligible terms in $u = (u_1, u_2)$ are exactly $-(x_1\lambda(t), -x_2\lambda(t))$.
Note that the justification of the approximation presented in the paper [14] holds provided ω is odd in x_1, and this is the case in our example, where we will make the following hypothesis on parity of ω, ρ (inspired by [3] and by the numerical experiment in [17]):

ω *odd* in x_1 and x_2 (separately), ρ *odd* in x_2 and *even* in x_1. $(\star)$



(a) Setting of the numerical example by Huo and Luo



(b) 2D Boussinesq system

Figure 1
Visual representation of the current example (taken from [3]).

2.2 Regularity of solutions

In this setting, we will prove that there is development of singularity in finite time in the case where $\omega, \partial_1\rho$ are smooth, but the domain has a boundary on which the solution is not vanishing.
More precisely, the statement to be proved is the following:

Proposition 1. *For solutions with the parity above described (⋆),*

(i) if $\omega_0, \partial_1\rho_0 \in C_c^2(\mathbb{R}^2)$, the unique local solution to Equation 7 is global.

(ii) if $\omega_0, \partial_1\rho_0 \in C_c^\infty(\mathbb{R}_+^2)$ the unique solution to Equation 7 develops a singularity in finite time.

Proof. The proof follows by solving the equation (for simplicity of exposition, in the case $\omega_0 = 0$, but the result can be easily extended as done in Appendix B, section 2).
Let us define the quantity:

$$\mu(t) = \exp\left(\int_0^t \lambda(s)ds\right).$$

We recognize that both PDEs in Equation 7 are of the form:

$$\partial_t\psi + \mathbf{b}\cdot\nabla\psi = f, \tag{10}$$

with a given initial value ψ_0. Hence, their solution will be of the form

$$\psi(\Phi(x,t),t) = \psi_0(x) + \int_0^t f(\Phi(x,s),s)ds,$$

where $\Phi(x,t)$ denotes the *flux*[5] of the vector field $\mathbf{b}$. In our case, we note that the forcing term in the equation for ω is $\partial_1\rho$, hence it is convenient to derive first an equation for $\partial_1\rho$, by deriving in x_1 the equation for ρ:

$$\partial_1(\partial_t\rho - (x_1\lambda(t), -x_2\lambda(t))\cdot\nabla\rho) = 0$$

$$\partial_t(\partial_1\rho) + (-\lambda(t), 0)\cdot(\partial_1\rho, \partial_2\rho) - (x_1\lambda(t), -x_2\lambda(t))\cdot\nabla\partial_1\rho = 0$$

[5]The *flux* of a vector field $\mathbf{b}$ is defined as follows: $\Phi : [0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ such that $\begin{cases}\partial_t\Phi(\Phi(x,t),t) = \mathbf{b}(\Phi(x,t),t)\\ \Phi(x,0) = x\end{cases}$.

$$\partial_t(\partial_1\rho) - (x_1\lambda(t), -x_2\lambda(t)) \cdot \nabla\partial_1\rho = \lambda(t)\partial_1\rho.$$

Noting that

$$\frac{d}{dt}\partial_1\rho(\Phi(x,t),t) = [\partial_t(\partial_1\rho) - (x_1\lambda(t), -x_2\lambda(t)) \cdot \nabla(\partial_1\rho)](\Phi(x,t),t),$$

we can conclude that

$$\partial_1\rho(\Phi(x,t),t) = \partial_1\rho_0(x)\exp\left(\int_0^t \lambda(s)ds\right) = \partial_1\rho_0(x)\mu(t).$$

The vector field $\mathbf{b}$ of Equation 10 is in this case $(-x_1\lambda(t), x_2\lambda(t))$, and its flux Φ satisfies

$$\partial_t\Phi(x_1,x_2,t) = (-x_1\lambda(t), x_2\lambda(t)),$$

which implies

$$\Phi\left(\mu(t)x_1, \frac{x_2}{\mu(t)}\right) = (x_1, x_2).$$

Now, we can get that

$$\partial_1\rho(x_1,x_2,t) = \mu(t)\partial_1\rho_0\left(\mu(t)x_1, \frac{x_2}{\mu(t)}\right).$$

Having obtained this identity, we can move on to the equation for ω; by referring to the structure of solutions of Equation 10 and recalling that in our case the forcing term is $\partial_1\rho$, we can easily obtain

$$\omega(x_1,x_2,t) = \partial_1\rho_0\left(\mu(t)x_1, \frac{x_2}{\mu(t)}\right)\int_0^t \mu(s)ds.$$

Observing that $\dot\mu(t) = \mu(t)\lambda(t)$ and that data satisfy the parity assumptions $(\star)$, one can get that:

$$\frac{\dot\mu(t)}{\mu(t)} = \left(\int_0^t \mu(s)ds\right)4\int_0^\infty\int_0^\infty \frac{y_1y_2}{|y|^4}\partial_1\rho_0\left(\mu(t)y_1, \frac{y_2}{\mu(t)}\right)dy_1dy_2. \tag{11}$$

Let us now analyze separately the two cases:

(i) **Case without singularity.** *$\omega_0, \partial_1\rho_0 \in C^2$, compactly supported on $\mathbb{R}^2$.*
First of all, we note that necessarily

$$|\partial_1\rho_0(x_1,x_2)| \le |x_1x_2|D(x_1,x_2), \tag{12}$$

for x_1, x_2 small and D uniformly bounded and compactly supported function.
To prove this, we observe that $\partial_1\rho_0$ is odd in x_1 (being ρ_0 even in x_1) and in x_2 (being ρ odd in x_2); hence, $\partial_1\rho_0(x_1,0) = \partial_1\rho_0(0,x_2) = 0 \quad \forall x_1,x_2$.
By performing a Taylor expansion of $\partial_1\rho_0$ around the origin, we get that $\partial_1\rho_0(x_1,x_2) = Cx_1x_2 + o(|x|^2)$ and we obtain the desired result.
Note that we can even extend this result to a ball wider than the domain of validity of the Taylor expansion, as near to the axis (remaining still in the support of ρ_0), we have estimates such as:

$$|\partial_1\rho_0| \le Cx_1, \qquad |\partial_1\rho_0| \le Cx_2.$$

Therefore, let us now suppose that $\partial_1\rho_0 \neq 0$ for $x_1 \le A, x_2 \le B$. By plugging the estimate in 12 into Equation 11, we get

$$\begin{aligned}\left|\frac{\dot\mu(t)}{\mu(t)}\right| &\le C\int_0^t \mu(s)ds\int_0^\infty\int_0^{\frac{A}{\mu(t)}} \frac{(y_1y_2)^2}{|y|^4}dy_1dy_2 \overset{\clubsuit}{=} C\int_0^t \mu(s)ds\int_0^{\frac{A}{\mu(t)}}\int_0^\infty \frac{(y_1y_2)^2}{(y_1+y_2)^2}dy_2dy_1\\ &\le C\int_0^t \mu(s)ds\int_0^{\frac{A}{\mu(t)}} y_1^2\int_0^\infty \frac{y_2^2}{(y_1^2+y_2^2)^2}dy_2dy_1 = \int_0^{\frac{A}{\mu(t)}} \frac{\pi}{4}y_1dy_1 \le \frac{C}{\mu(t)^2}\int_0^t \mu(s)ds.\end{aligned} \tag{13}$$

Note that ♣ equality follows from the application of Tonelli's theorem, being the integrand non-negative and measurable.
Now we are left to prove that Equation 13 implies that $\mu(t)$ is bounded for all finite times; to do so, let us introduce the auxiliary function

$$\tilde{\mu}(t) := \max_{r\in[0,t]} \mu(r),$$

which satisfies $\dot{\tilde{\mu}}(t) = \dot{\mu}(t)$ whenever $\tilde{\mu}$ and μ coincide, and $\dot{\tilde{\mu}}(t) = 0$ elsewhere, so $\dot{\tilde{\mu}}(t) \le \dot{\mu}(t)$.
Note also that a blow-up of μ implies a blow-up of $\tilde{\mu}$. Hence, we have that

$$\dot{\tilde{\mu}}(t) \le \frac{C}{\tilde{\mu}(t)} \int_0^t \mu(s)ds \le \frac{C}{\tilde{\mu}(t)} t\tilde{\mu}(t) \le Ct,$$

so $\tilde{\mu}$ grows at most quadratically, and by its definition it cannot be overcome by μ. Therefore, μ is bounded at each finite time.
Note that by the study of μ one can draw conclusions on ω: if ω could become infinite in finite time, then also μ would have a singularity, and this would be in contradiction with the proof above.

(ii) **Case with singularity.** $\omega_0, \partial_1\rho_0 \in C^\infty(\mathbb{R}^2_+)$, *with* ρ *not vanishing on* $x_2 = 0$.
In this case, we take $\partial_1\rho_0(x_1, x_2)$ to be smooth and odd in x_1 of the form

$$\begin{cases} \partial_1\rho_0(x_1, x_2) = x_1 \text{ in } [0,1]^2 \\ \partial_1\rho_0(x_1, x_2) = 0 \text{ in } \mathbb{R}^2 \setminus [0,2]^2 \\ \partial_1\rho_0(x_1, x_2) \text{ non-negative in } [0,2]^2 \setminus [0,1]^2. \end{cases} \tag{14}$$

Starting again from Equation 11, we have that

$$\begin{aligned} \frac{\dot{\mu}(t)}{\mu(t)} &= 4\left(\int_0^t \mu(s)ds\right)\int_0^\infty \int_0^\infty \frac{y_1y_2}{|y|^4}\partial_1\rho_0\left(\mu(t)y_1, \frac{y_2}{\mu(t)}\right)dy_1dy_2 \ge 4\left(\int_0^t \mu(s)ds\right)\mu(t)\int_0^{\mu(t)}\int_0^{\frac{1}{\mu(t)}} \frac{y_1^2y_2}{(y_1^2+y_2^2)^2}dy_1dy_2 \\ &\overset{♣}{=} 4\left(\int_0^t \mu(s)ds\right)\mu(t)\int_0^{\frac{1}{\mu(t)}} y_1^2 \int_0^{\mu(t)} \frac{y_2}{(y_1^2+y_2^2)^2}dy_2dy_1 = \frac{4}{2}\left(\int_0^t \mu(s)ds\right)\mu(t)\int_0^{\frac{1}{\mu(t)}} y_1^2 \frac{\mu(t)^2}{y_1^2(y_1^2+\mu(t)^2)}dy_1 \\ &= 2\left(\int_0^t \mu(s)ds\right)\mu(t)\int_0^{\frac{1}{\mu(t)}} y_1^2\left(\frac{1}{y_1^2} - \frac{1}{y_1^2+\mu(t)^2}\right)dy_1 = 2\left(\int_0^t \mu(s)ds\right)\mu(t)\left[\frac{1}{\mu(t)} - \left| y_1 - \mu(t)\arctan\left(\frac{y_1}{\mu(t)}\right)\right|_0^{\frac{1}{\mu(t)}}\right] \\ &= 2\left(\int_0^t \mu(s)ds\right)\mu(t)^2 \arctan\left(\frac{1}{\mu(t)^2}\right) \overset{♦}{\ge} \frac{\pi}{2}\left(\int_0^t \mu(s)ds\right). \end{aligned}$$

Equality ♣ is obtained using again Tonelli's theorem to exchange the order of integration, while ♦ holds as long as $\mu(t) \ge 1$.
Hence, we work with an inequality of the form

$$\dot{\mu}(t) \ge c\mu(t)\int_0^t \mu(s)ds, \tag{15}$$

for some constant $c > 0$.
To prove that μ becomes infinite in finite time, we will use a strategy based on *comparison*.
In particular, we aim at finding a function f with blow-up at time t^* such that $f(t) \le \mu(t)\ \forall t \in (0, t^*)$, to conclude that μ has a blow-up in finite time as well.
Let us consider the following solution with blow-up at $t = 1$:

$$f(t) = \frac{4}{(1-t)^2}.$$

For $t \geq 1/2$, this solution is such that[6]

$$\dot{f}(t) \leq f(t) \int_0^t f(s)ds.$$

Lemma 1. *Let $\psi : [0, \bar{t}) \to \mathbb{R}_+$, $g : [0, \bar{t}) \to \mathbb{R}_+$ continuous and differentiable, blowing up at the same time instant $\bar{t}$ by simplicity of exposition, be such that:*

$$\dot{g}(t) \leq g(t) \left[a + \int_0^t g(s)ds \right], \qquad \dot{\psi}(t) \geq \psi(t) \left[b + \int_0^t \psi(s)ds \right], \qquad g(0) < \psi(0),\ a \leq b$$

Then, $g(t) \leq \psi(t) \quad \forall t \in (0, \bar{t})$.

Proof of Lemma 1. Let us consider the set of time instants for which $g(t) < \psi(t)$. By hypothesis, it contains $t = 0$, and let us suppose $[0, t^*)$ to be the largest interval for which $g(t) < \psi(t)$.
We want to prove that this interval is actually $[0, \bar{t})$.
Suppose that $t^* < \bar{t}$ and $g(t^*) = \psi(t^*)$. Then,

$$\dot{g}(t^*) \leq g(t) \left[a + \int_0^{t^*} g(s)ds \right] < \psi(t^*) \left[b + \int_0^{t^*} \psi(s)ds \right] \leq \dot{\psi}(t^*)$$

as $g(t) < \psi(t)\ \forall t \in [0, t^*)$. Note that now we can detect a contradiction, since we know that

$$(\psi - g)(t) > 0 \quad \forall t \in [0, t^*),$$

$$(\psi - g)(t^*) = 0.$$

Hence, having $(\dot{\psi} - \dot{g})(t^*) > 0$ leads to a contradiction and enables us to conclude that t^* is the blow-up instant. ◻

In our case, note that $f(1/2) = 16$ and suppose that $\exists t_M : \mu(t) > 16$ and $\int_0^{t_M} \mu(s)ds \geq 2\ \forall t \geq t_M$. We are interested in the case $t_M \geq 1/2$, else the presence of the blow-up is trivial to prove. The existence of such a t_M is reasonable, as we know from the condition $\mu(0) = 1$ and Equation 15 that $\mu(t) \geq 1$, hence $\dot{\mu}(t) \geq \mu(t)t$, so μ at least goes to $+\infty$ in infinite time, but cannot remain bounded.
We will apply Lemma 1 using two translations of the above defined functions:

$$\psi(t) := \mu(t + t_M) \text{ and } g(t) := f\left(t + \frac{1}{2}\right).$$

We have that

$$\dot{\psi}(t) \geq \psi(t) \left[\int_0^{t_M} \mu(s)ds + \int_0^t \psi(s)ds \right] \text{ and } \dot{g}(t) \leq g(t) \left[\int_0^{\frac{1}{2}} f(s)ds + \int_0^t g(s)ds \right],$$

by imposing the inequalities obtained for μ and f evaluated, respectively, in $t + t_M$ and $t + \frac{1}{2}$, and noting that $\int_{t_M}^{t+t_M} \mu(s)ds = \int_0^t \psi(s)ds$ with a change of variables.
Note also that

$$a := \int_0^{\frac{1}{2}} f(s)ds = 2 \leq b := \int_0^{t_M} \mu(s)ds, \qquad \psi(0) \geq g(0) \text{ by the choice of } t_M.$$

Hence, by applying Lemma 1, we conclude that

$$\psi(t) \geq g(t),$$

and since there is a blow-up for g, there will be a blow-up for ψ as well.

[6]Indeed, $\dot{f}(t) = \frac{8}{(1-t)^3}$ and $\int_0^t f(s)ds = \frac{4t}{(1-t)}$. By comparing the quantities $\frac{8}{(1-t)^3}$ and $\frac{16t}{(1-t)^3}$ we get the result.

This concludes the proof of the theorem. □

Remark 1. *From the explicit calculation performed, we can also conclude that if ρ_0 vanishes to order y^α, with α sufficiently small, there will be a singularity in finite time in $\mathbb{R}^2$ as well.*

This remark highlights the importance of regularity in blow-up creation, even in the setting when the domain does not have boundaries. Indeed, in [7], the singularity is created for the class of Hölder functions $C^{1,\alpha}(\mathbb{R}^3)$, with $\alpha > 0$ small.

Proof. Let us suppose to have ρ_0 behaving like x_1^α close to the axis $x_1 = 0$, and vanishing outside the rectangle $[0, A] \times [0, B]$.
Its derivative close to the axis $x_1 = 0$ would be $\partial_1\rho_0 = \alpha x_1^{\alpha-1}$, and by plugging $\partial_1\rho_0$ into Equation 11, we would have to compute the integral

$$\int_0^B \int_0^A \frac{y_1^\alpha y_2}{|y|^4} dy_1 dy_2 = \frac{1}{2} \int_0^A y_1^\alpha \left(\frac{1}{y_1^2} - \frac{1}{y_1^2 + \mu(t)^2} \right) dy_1.$$

Note that, as $y_1 \to 0$, the first integral of the sum does not converge, as it behaves like $\frac{1}{y_1^{2-\alpha}}$ with $2 - \alpha > 1$ whenever $\alpha < 1$.
The same happens if we suppose that $\partial_1\rho_0$ behaves like $x_2^{\alpha-1}$ close to the axis $x_2 = 0$. □

3 Stable singularity formation in the simplest setting

We now analyze a simple model in which the same method developed in [7] for the Euler Equation can be witnessed.
In particular way, we start by analyzing a model which presents a singular solution such as:

$$\partial_t f = f^2, \tag{16}$$

and to ideally extend the singularity to a perturbed version of the problem:

$$\partial_t f = f^2 + \varepsilon \mathcal{N}(f), \tag{17}$$

where $\varepsilon > 0$ is a small constant and $\mathcal{N}(f)$ is a *quadratic non-linearity* with total degree 0, which means that:

- $\mathcal{N}(af) = a^2\mathcal{N}(f) \quad \forall a \in \mathbb{R}$;
- $\mathcal{N}(f(a\cdot)) = \mathcal{N}(f)(a\cdot) \quad \forall a \in \mathbb{R}$.

To continue the singularity to the perturbed problem, two strategies can be developed:

1. the first involves a *fixed point* approach; even if it is more straightforward, it requires stronger assumptions on $\mathcal{N}$ which are missing in the Euler Equation setting.
2. the second idea, to whom Elgindi refers as *compactness method*, is more general; essentially, it is based on constructing an auxiliary problem for which our solution is a stationary solution, and then apply standard results in the ODE theory.

The discussion on the *fixed point approach* can be found in the Appendix C (section 2); here, we now focus on the *compactness method*.

3.1 The compactness method

While looking for *self-similar* solutions to Equation 16 of the form

$$f(x,t) = \frac{1}{1-t} F\left(\frac{x}{1-t}\right),$$

we have to solve the equation

$$F + z\partial_z F = F^2, \tag{18}$$

where z is the self-similar variable $z = \frac{x}{1-t}$.
$F \equiv 1$ is clearly a solution, which has a blow-up at every point x; however, it is *unstable*, in the sense that perturbing the problem we may fall into a very different singular behaviour, such as a scenario with a blow-up only at one point. By contrast, a *stable* singular solution is

$$F_0(z) = \frac{1}{1+z}.$$

For the perturbed problem in Equation 17 with $\varepsilon > 0$, we search for a self-similar solution with two free parameters[7] (μ, λ) of the form

$$f(x,t) = \frac{1}{1-(1+\mu)t} F_\varepsilon\left(\frac{x}{(1-(1+\mu)t)^{1+\lambda}}\right),$$

where $F_\varepsilon = F_0 + g$, and μ, λ and g are of the order of ε.
By inserting the above f into Equation 17, we get

$$(1+\mu)F_\varepsilon + (1+\lambda)(1+\mu)z\partial_z F_\varepsilon = F_\varepsilon^2 + \varepsilon \mathcal{N}(F_\varepsilon),$$

and since we are considering F_ε as a perturbation of F_0, it is legitimate to substitute it with $F_0 + g$ and obtain

$$\begin{aligned} \mathcal{L}(g) &:= g + z\partial_z g - \frac{2}{1+z} g \\ &= -\mu F_0 - (\mu + \lambda + \lambda\mu) z\partial_z F_0 - \mu g - (\mu+\lambda+\lambda\mu) z\partial_z g + \varepsilon\mathcal{N}(F_0 + g) + g^2 =: \mathrm{RHS}, \end{aligned} \tag{19}$$

where we isolated the operator $\mathcal{L} = \mathrm{Id} + z\partial_z - \frac{2}{1+z}\mathrm{Id}$, that can be seen as a linearization of $F_0 + z\partial_z F_0 = F_0^2$ around F_0, as

$$\begin{aligned} (g + F_0) + z\partial_z(g+F_0) &= (g+F_0)^2 \\ g + \tfrac{1}{1+z} + z\partial_z g - \tfrac{z}{(1+z)^2} &= g^2 + \tfrac{2g}{1+z} + \tfrac{1}{(1+z)^2} \\ g + z\partial_z g - \tfrac{2g}{1+z} &= g^2. \end{aligned}$$

3.1.1 Coercivity of $\mathcal{L}$

We will now decide in which space we should set our argument.
We are interested in cases in which the operator $\mathcal{L}$ is *positive* (even if not *invertible*), hence we want to avoid the possibility of obtaining situations where $(\mathcal{L}(g), g) < 0$.
Taking as space L^2, this may happen for example with a g supported in an small interval such as $[0,1]$, because when z is small the term $\frac{-2g}{1+z}$ of the operator prevails over g, determining negativity of $\mathcal{L}(g)$. Those kind of situations are linked with the presence of mass at $z = 0$, hence let us set into a weighted L^2 space with a strong weight near $z = 0$, such as

$$L^2_w := \left\{ g : \int g^2 w < \infty \right\} \text{ with } w(z) = \frac{(1+z)^4}{z^4},$$

where mass at $z = 0$ is penalized to such an extent that we require at least quadratically vanishing of the function for $z \to 0$; in this space, we can easily prove that $\mathcal{L}$ is a coercive operator, satisfying

$$(\mathcal{L}(g), g)_{L^2_w} \geq c|g|^2_{L^2_w}. \tag{20}$$

[7]Their role will be crucial for imposing two constraints, as we will see in few steps.

Proof of coercivity on L^2_w. Note that

$$(z\partial_z g, g)_{L^2_w} = \int z\partial_z g g \frac{(1+z)^4}{z^4} = -\frac{1}{2}\int g^2 \partial_z \frac{(1+z)^4}{z^3} = -\frac{1}{2}\int g^2 \frac{(z+1)^3(z-3)}{z^4},$$

$$\left(-\frac{2g}{1+z}, g\right)_{L^2_w} = \int \frac{-2(1+z)^3}{z^4} g^2.$$

Hence, we get coercivity since

$$(\mathcal{L}(g), g)_{L^2_w} = \left(g + z\partial_z g - \frac{2g}{1+z}, g\right)_{L^2_w} = |g|^2_{L^2_w} - \frac{1}{2}\int g^2 \frac{(1+z)^4}{z^4} = \frac{1}{2}|g|^2_{L^2_w}.$$

□

For this simple abstract problem, the space L^2_w would be enough for the coercivity of the linearized operator. However, for the Euler case we will have to consider a more complicated operator, and a different weighted Sobolev space will be needed, as we will show in section 4.

Let us now prove that the coercivity holds on a properly weighted H^2 space as well. Let us introduce the inner product

$$(f, g)_X \coloneqq (f, g)_{L^2_w} + c_1(z\partial_z f, z\partial_z g)_{L^2_w} + c_2((z\partial_z)^2 f, (z\partial_z)^2 g)_{L^2_w}.$$

where $c_1, c_2 > 0$ must be chosen suitably.

Proof of coercivity on the weighted H^2.

$$(\mathcal{L}(g), g)_X = \underbrace{(g, g)_X}_{\text{(I)}} + \underbrace{(z\partial_z g, g)_X}_{\text{(II)}} \underbrace{-\left(\frac{2}{1+z} g, g\right)_X}_{\text{(III)}}.$$

Let us deal with the three terms separately:

$$\text{(I)} = |g|^2_X;$$

$$\begin{aligned}
\text{(II)} &= \int (z\partial_z g) g \frac{(1+z)^4}{z^4} + c_1 \int z(\partial_z(z\partial_z g) z\partial_z g \frac{(1+z)^4}{z^4} + c_2 \int z\partial_z[z\partial_z(z\partial_z g)] z\partial_z(z\partial_z g) \frac{(1+z)^4}{z^4} \\
&= -\frac{1}{2}\int g^2 \partial_z\left[\frac{(1+z)^4}{z^3}\right] - \frac{c_1}{2}\int (z\partial_z g)^2 \partial_z \left[\frac{(1+z)^4}{z^3}\right] - \frac{c_2}{2}\int [(z\partial_z)(z\partial_z g)]^2 \partial_z \left[\frac{(1+z)^4}{z^3}\right] \\
&= -\frac{1}{2}\int g^2 \frac{(1+z)^3(z-3)}{z^4} - \frac{c_1}{2}\int (z\partial_z g)^2 \frac{(1+z)^3(z-3)}{z^4} - \frac{c_2}{2}\int [z\partial_z(z\partial_z) g]^2 \frac{(1+z)^3(z-3)}{z^4};
\end{aligned}$$

$$\begin{aligned}
\text{(III)} &= -\int \frac{2g^2}{(1+z)} \frac{(1+z)^4}{z^4} - c_1 \int \frac{-2zg}{(1+z)^2}(z\partial_z g)\frac{(1+z)^4}{z^4} - c_1 \int \frac{2z}{1+z}\partial_z g (z\partial_z g) \frac{(1+z)^4}{z^4} \\
&\quad - c_2 \int z\partial_z \left(\frac{-2zg}{(1+z)^2} + \frac{2z\partial_z g}{1+z}\right) z\partial_z(z\partial_z g) \frac{(1+z)^4}{z^4} \\
&= \int \frac{-2(1+z)^3}{z^4} g^2 + c_1 \int \frac{2(1+z)^2}{z^2} g\partial_z g - c_1 \int (z\partial_z g)^2 \frac{2(1+z)^3}{z^4} \\
&\quad \underbrace{-c_2 \int g\,(2(z-1))\, z\partial_z(z\partial_z g) \frac{(1+z)}{z^3} + 4c_2 \int (z\partial_z g) z\partial_z(z\partial_z g) \frac{(1+z)^2}{z^3}}_{\clubsuit} - 2c_2 \int z\partial_z(z\partial_z g) z\partial_z(z\partial_z g) \frac{(1+z)^3}{z^4},
\end{aligned}$$

where

$$\begin{aligned}
\clubsuit &= c_2 \int \frac{2(z-1)(z+1)}{z^3}(z\partial_z g)^2 + c_2 \int g\partial_z g \frac{4}{z^2} + 2c_2 \int (z\partial_z g)^2 \frac{2(1+z)}{z^3} \\
&= c_2 \int \frac{2(z-1)(z+1)}{z^3}(z\partial_z g)^2 + \frac{c_2}{2}\int g^2 \frac{8}{z^3} + 2c_2 \int (z\partial_z g)^2 \frac{2(1+z)}{z^3}.
\end{aligned}$$

Let us now study (II) + (III).
By grouping the terms with g^2, $(z\partial_z g)^2$ and $(z\partial_z)^2 g$ from the above calculations for (II) and (III), we obtain:

$$\int g^2\left[\frac{-(1+z)^3(z-3)}{2z^4}-\frac{2(1+z)^3}{z^4}+c_2\frac{4}{z^3}+c_1\frac{4(z+1)}{z^3}\right]\geq -\frac{1}{2}\int g^2\frac{(1+z)^4}{z^4};$$

$$\int (z\partial_z g)^2\left[-c_1\frac{(1+z)^3(z-3)}{2z^4}-c_1\frac{2(z+1)^3}{z^4}+c_2\frac{(z-1)(z+1)}{z^3}\right]\geq -\frac{1}{2}c_1\int (z\partial_z g)^2\frac{(1+z)^4}{z^4}+c_2\int (z\partial_z g)^2\frac{(z-1)(z+1)}{z^3};$$

$$\int ((z\partial_z)^2 g)^2\left[-c_2\frac{(1+z)^3(z-3)}{2z^4}-c_2\frac{2(1+z)^3}{z^4}\right]\geq -\frac{1}{2}c_2\int ((z\partial_z)^2 g)^2\frac{(1+z)^4}{z^4}.$$

By putting all the addends together, we have that

$$\text{(I)}+\text{(II)}+\text{(III)}\ \geq |g|_X^2-\frac{1}{2}|g|_X^2+c_2\int (z\partial_z g)^2\frac{(z-1)(z+1)}{z^3}\geq \frac{1}{2}|g|_X^2-\hat{c}\int (z\partial_z g)^2\frac{(1+z)^4}{z^4}\geq\left(\frac{1}{2}-\hat{c}\right)|g|_X^2.$$

To have coercivity, it is necessary that we can find a $\hat{c}<\frac{1}{2}$. Note that this is possible since:

$$c_2\frac{(z-1)(z+1)}{z^3}\geq -\hat{c}\frac{(1+z)^4}{z^4}\quad \forall z\geq 0,$$

provided that $\hat{c}\geq \frac{c_2}{10}$. Hence, by choosing in the definition $c_2<5$, a coercivity estimate can be obtained:

$$(\mathcal{L}(g),g)_X=(g,g)_{L^2_w}+(z\partial_z g,z\partial_z g)_{L^2_w}+((z\partial_z)^2 g,(z\partial_z)^2 g)_{L^2_w}\geq C(g,g)_X.$$

This proves the coercivity of the operator $\mathcal{L}$ on the weighted Sobolev space. □

3.1.2 The non-linear operator $\mathcal{N}$

Let us now discuss the nonlinear operator $\mathcal{N}$.
The theorem that validates the *fixed point approach* relies upon the *boundedness* of the operator $\mathcal{N}$, while we treat in this section cases in which the operator is not bounded, but it satisfies an estimate such as

$$|(\mathcal{N}(f),f)|_X\leq c|f|_X^2, \tag{21}$$

for some constant $c>0$. Although this is the main hypothesis, we will add other hypothesis in the following proofs, inspired by the properties of the operator examined by Elgindi in [7].
The role of the scaling parameters μ and λ comes into play to impose that RHS in Equation 19 belongs to the space in which Equation 19 can be solved, which turns into imposing that RHS vanishes quadratically at $z=0$. We can determine them, under the assumption that g too vanishes quadratically at $z=0$, by imposing:

$$\text{RHS}(0)=-\mu F_0(0)+\varepsilon\mathcal{N}(F_0+g)(0)=-\mu+\varepsilon\mathcal{N}(F_0+g)(0)=0,$$

$$\text{RHS}'(0)=-\mu\partial_z F_0(0)-(\mu+\lambda+\lambda\mu)\partial_z F_0(0)+\varepsilon\partial_z(\mathcal{N}(F_0+g))(0)=\mu+(\mu+\lambda+\lambda\mu)+\varepsilon\partial_z(\mathcal{N}(F_0+g))(0).$$

Hence, we end up with

$$\begin{cases}\mu=\varepsilon\mathcal{N}(F_0+g)(0)\\ \lambda=\frac{-1}{1+\mu}\left(2\mu+\varepsilon\partial_z(\mathcal{N}(F_0+g))(0)\right).\end{cases}\tag{22}$$

With this choice of μ,λ, which implies that the parameters' order is ε, we can prove that

$$(\text{RHS},g)_X\leq C\varepsilon(|g|_x+|g|_X^2)+|g|_X^3,$$

for some constant $C>0$.

Proof. Let us study separately the various terms in

$(\mathrm{RHS}, g)_X = -(\mu F_0, g)_X - ((\mu+\lambda+\lambda\mu)z\partial_z F_0, g)_X - (\mu g, g)_X - ((\mu+\lambda+\lambda\mu)z\partial_z g, g)_X + \varepsilon(\mathcal{N}(F_0+g), g)_X + (g^2, g)_X.$

We recall that, by Equation 22, the parameters μ, λ are of order ε; hence, for the sake of simplicity, we will study all the scalar products omitting the scaling parameters and multiply the estimates obtained by ε at the end.

Terms bounded by $|g|_X$:
Note that the above choice of μ, λ implies that there are no terms vanishing less than linearly in the RHS at $z = 0$. Hence, the linear combination of the terms F_0 and $z\partial_z F_0$ gives rise to a term $h := -\mu F_0 - (\mu + \lambda + \lambda\mu)z\partial_z F_0$ vanishing quadratically at $z = 0$, as it happens for the Euler case in [7]. Thanks to this property, $h \in X$ and its norm could be bounded by $C\varepsilon$ thanks to the presence of the parameters μ, λ in the definition of h.
It is now possible to bound $(-\mu F_0 - (\mu + \lambda + \lambda\mu)z\partial_z F_0, g)_X = (h, g)_X$:

$$(h, g)_X \leq |h|_X |g|_X \leq C\varepsilon |g|_X.$$

Note that if one considered the terms individually, it would not be possible to proceed with *Cauchy-Schwartz* inequality as above, because, for example, F_0 does not belong to X.

Terms bounded by $|g|_X^2$:

$$(g, g)_X = |g|_X^2, \tag{23}$$

and by (II) of the above coercivity estimate, we recall that

$$\begin{aligned}(z\partial_z g, g)_X &= -\frac{1}{2}\int g^2 \frac{(1+z)^3(z-3)}{z^4} - \frac{c_1}{2}\int (z\partial_z g)^2 \frac{(1+z)^3(z-3)}{z^4} - \frac{c_2}{2}\int [z\partial_z(z\partial_z)g]^2 \frac{(1+z)^3(z-3)}{z^4} \\ &\leq C(g,g)_X \\ &\leq C|g|_X^2.\end{aligned} \tag{24}$$

Term bounded by $|g|_X^3$:

$$(g^2, g)_X \leq \|g\|_{L_\infty}(g, g)_X \leq C|g|_X^3 \tag{25}$$

by using the fact that we can control $L^\infty-$ norm with $X-$ norm. Indeed, we have that

$$\begin{aligned}|g(t)|^2 &\overset{\heartsuit}{=} |g(t) - g(0)|^2 \leq \left|\int_0^t \partial_z g(z)dz\right|^2 \leq \left(\int_0^t |z\partial_z g(z)| \frac{(1+z)^2}{z^2}\frac{z}{(1+z)^2}dz\right)^2 \\ &\leq \int_0^t |z\partial_z g(z)|^2 \frac{(1+z)^4}{z^4}dz \underbrace{\int_0^t \frac{z^2}{(1+z)^4}dz}_{=\frac{t^3}{3(1+t)^3}\leq\frac{1}{3} \quad \forall t\geq 0} \leq \frac{1}{3}|z\partial_z g|^2_{L^2_w} \leq C|g|_X^2 \quad \forall t \geq 0,\end{aligned} \tag{26}$$

where $\heartsuit$ holds because $g(0) = 0$, since g is taken in the space where mass at 0 is penalized. From Equation 26, we have that $\|g\|_{L^\infty} := \sup_{t\geq 0}|g(t)| \leq C|g|_X$.

Term with $\mathcal{N}$:
We are only left to bound the term $(\mathcal{N}(F_0 + g), g)_X$. As sufficient information on the operator $\mathcal{N}$ is not available, we can refer to what happens in Elgindi's test case, and add the additional hypothesis that $\varepsilon(\mathcal{N}(F_0 + g), g)_X$ can be bounded by $C\varepsilon(|g|_x + |g|_X^2) + |g|_X^3$.
In particular, we will introduce in section 4 Elgindi's RHS, and note that the part which behaves like the term we are analyzing is the one denoted by $\mathcal{N}(F)$, where $F = F_0+g$ (see Appendix A, section 1 for its precise expression).

Having bounded all the terms in RHS, the desired bound has been obtained. □

Combining this result with the coercivity of the operator $\mathcal{L}$ on X, and with the assumption that g is small enough (coherent with the fact that it represents a small perturbation), we obtain the *a-priori* estimate

$$c|g|_X^2 \leq (\mathcal{L}(g), g)_X = (\mathrm{RHS}, g)_X \leq C\varepsilon(|g|_x + |g|_X^2) + |g|_X^3 \quad \Rightarrow \quad |g|_X \leq C\varepsilon.$$

3.1.3 EXISTENCE AND UNIQUENESS OF A SOLUTION

Having obtained the *a-priori* estimate, we are left to discuss existence and uniqueness of our problem, which consists in solving for g an equation of the form

$$\mathcal{L}(g) = f + \varepsilon\mathcal{N}(g). \tag{27}$$

Let us now add a fake time variable τ to transform the problem into an ODE:

$$\begin{cases} \partial_\tau g + \mathcal{L}(g) = f + \varepsilon\mathcal{N}(g) \\ g|_{\tau=0} = 0. \end{cases} \tag{28}$$

Assumptions 20 and 21 enable us to prove the following results on g:

1. $|g|_X \leq C$;
2. $|\partial_\tau g|_X \leq e^{-c\tau}$.

Proof. 1. Multiplying Equation 28 by g, we can obtain

$$\tfrac{1}{2}\partial_\tau |g|_X^2 + (\mathcal{L}(g), g)_X = (f, g)_X + \varepsilon(\mathcal{N}(g), g).$$

By inserting the estimates on $\mathcal{L}$ and $\mathcal{N}$, the following inequality can be derived

$$\tfrac{1}{2}\partial_\tau |g|_X^2 + c|g|_X^2 \leq (f, g)_X + \varepsilon|g|_X^2.$$

By using *Young's inequality*, we get $|(f, g)_X| \leq |f|_X|g|_X \leq \frac{|f|_X^2}{2c} + \frac{c|g|_X^2}{2}$. Hence,

$$\tfrac{1}{2}\partial_\tau |g|_X^2 \leq \frac{|f|_X^2}{2c} - \frac{c|g|_X^2}{2} + \varepsilon|g|_X^2 = \frac{|f|_X^2}{2c} + \left(\varepsilon - \frac{c}{2}\right)|g|_X^2.$$

We are interested in cases in which $\frac{|f|_X^2}{2c} + \left(\varepsilon - \frac{c}{2}\right)|g|_X^2 \leq 0$; indeed, in this setting, $\partial_\tau |g|_X^2$ would be non-positive, hence $|g|_X^2$ would not be able to grow over a certain threshold, let us stay $\tilde{c}^2$.
We can prove that $\exists \bar{\varepsilon}, \tilde{c}$ such that $\bar{\varepsilon}\tilde{c}^2 = \frac{|f|_X^2}{2c}$ and $\frac{c}{2}\tilde{c}^2 \geq \frac{|f|_X^2}{c}$, by taking

$$\tilde{c}^2 = \frac{2|f|_X^2}{c^2} \text{ and } \bar{\varepsilon} = \frac{|f|_X^2}{2c\tilde{c}^2}.$$

2. Here, instead, we derive Equation 28 with respect to τ, and noting that f is independent from that variable, we get

$$\partial_{\tau\tau} g + \partial_\tau \mathcal{L}(g) = \varepsilon\partial_\tau \mathcal{N}(f).$$

Let us now multiply by $\partial_\tau g$. We get that

$$\tfrac{1}{2}\partial_\tau |\partial_\tau g|^2 + (\partial_\tau \mathcal{L}(g), \partial_\tau g)_X = \varepsilon(\partial_\tau \mathcal{N}(g), \partial_\tau g)_X.$$

Note that $\mathcal{L}$ is linear, so it commutes with the derivative, and by coercivity of $\mathcal{L}$ we get that:

$$\tfrac{1}{2}\partial_\tau |\partial_\tau g|^2 + c|\partial_\tau g|^2 \leq \varepsilon(\partial_\tau \mathcal{N}(g), \partial_\tau g)_X.$$

We now suppose that the right-hand side can be bounded by

$$\varepsilon(\partial_\tau \mathcal{N}(g), \partial_\tau g)_X \leq C(\varepsilon)|\partial_\tau g|_X^2.$$

This assumption is motivated by the specific expression of the operator $\mathcal{N}$ and the properties that it satisfies in the case of the Euler Equation, that will be dealt with in section 4. Hence, we obtain that

$$\tfrac{1}{2}\partial_\tau |\partial_\tau g|^2 \leq \tilde{c}|\partial_\tau g|^2,$$

which leads to an exponential decay of $|\partial_\tau g|$:

$$|\partial_\tau g| \leq \bar{C}e^{-(\tilde{c}/2)\tau}.$$

□

Thanks to the above bound, we can prove the existence of stationary solutions to Equation 28.
Indeed, for $\tau \to \infty$, we get that the growth/reduction of the τ-dependent g is null, and this solution to the auxiliary problem is a solution to the original one in Equation 27, too.

4 Link with the Euler Equation

In this section, we deepen into the link between the examples proposed in Sections 2 and 3 and the method built for the Euler Equation in [7].

4.1 Towards the fundamental model

First of all, we will see how the so-called *fundamental model*, which plays the role of Equation 16, is obtained. The path towards this model starts from the standard vorticity equation, which, in the case of axi-symmetric 3D Euler equations with *vanishing swirl*, reads as

$$\partial_t \omega + u \cdot \nabla \omega = \partial_t \omega + u^r \partial_r \omega + u^3 \partial_3 \omega = \tfrac{1}{r} u^r \omega. \tag{29}$$

After introducing the *stream function*, we can add to our system Equation 6 as well:

$$\begin{cases} \partial_t \omega + u_r \partial_r \omega + u_3 \partial_3 \omega = \frac{1}{r} u_r \omega \\ \omega = -\partial_{rr}\psi - \partial_{33}\psi - \frac{1}{r}\partial_r \psi + \frac{\psi}{r^2} \\ u^r = \partial_3 \psi \quad u^3 = -\frac{1}{r}\psi - \partial_r \psi. \end{cases} \tag{30}$$

The system is set on the spatial domain $\Omega = \{(r, x_3) \in [0, +\infty) \times [0, +\infty)\}$; this choice is due to the fact that we decide to explore what happens in the half-plane $r \geq 0$, under the condition that ω is *odd* in x_3, which was the case of setting leading to singularity in Section 2.
We solve the problem with homogeneous[8] Dirichlet boundary conditions for ψ on $r = 0$ and $x_3 = 0$, and we impose that $\omega = 0$ on $r = 0$; the class of solutions in which we are looking for u is $C^{1,\alpha}$ with a small $\alpha \in (0, 1/3]$.

4.1.1 Passing to a form of polar coordinates

We define $\rho = \sqrt{r^2 + x_3^2}$, $R = \rho^\alpha$ and $\theta = \arctan(x_3/r)$. In the new coordinates, the relevant functions become

$$\Omega(R, \theta) = \omega(r, x_3), \qquad \rho^2 \Psi(R, \theta) = \psi(r, x_3).$$

After the change of variables in Equation 30, and writing explicitly the relationship between Ω and Ψ, we obtain the following equations:

$$\begin{cases} \partial_t \Omega + (-3\Psi - \alpha R \partial_R \Psi)\partial_\theta \Omega + (\partial_\theta \Psi - \tan(\theta)\Psi)\alpha R \partial_R \Omega = \frac{1}{\cos(\theta)}(2\sin(\theta)\Psi + \alpha \sin(\theta) R \partial_R \Psi + \cos(\theta)\partial_\theta \Psi)\Omega \\ -\alpha^2 R^2 \partial_{RR}\Psi - \alpha(5+\alpha) R \partial_R \Psi - \partial_{\theta\theta}\Psi + \partial_\theta(\tan(\theta)\Psi) - 6\Psi = \Omega \\ \Psi(R, 0) = \Psi(R, \pi/2) = 0. \end{cases} \tag{31}$$

In particular way, we exploit the new formulation of axi-symmetric 3D Euler to see the emergence of a blowing-up part with the addition of a perturbation.
Let us now focus on the second equation of Equation 31; one can prove (by expanding Ψ in Fourier series) that, if Ω is orthogonal[9] to $\cos^2(\theta)\sin(\theta)$ the unique L^2 solution with Dirichlet boundary conditions on $[0, \infty) \times [0, \pi/2]$ can be controlled by the L^2-norm of Ω, with an estimate of the form

$$\left| \partial_\theta \left(\frac{\Psi}{\cos(\theta)} \right) \right|_{L^2} + |\partial_{\theta\theta}\Psi|_{L^2} + \alpha^2 |R^2 \partial_{RR}\Psi|_{L^2} \leq 100 |\Omega|_{L^2}.$$

[8] After all, it is important to determine ψ only up to an additive constant, since u and ω depend only on its derivatives.
[9] In the sense of the L^2 scalar product.

Under the same assumption on the form of Ω and the requirement that α is small enough, one can prove that the control of the norms holds with the norm of a weighted Sobolev space $\mathcal{H}^k$, too (see Proposition 7.8 of [7]).

With a general Ω belonging to a weighted Sobolev space instead, the estimate that holds is

$$\alpha^2|R^2\partial_{RR}\Psi|_{\mathcal{H}^k} + \left|\partial_{\theta\theta}\left(\Psi - \frac{1}{4\alpha}\sin(2\theta)L_{12}(\Omega)\right)\right|_{\mathcal{H}^k} \leq C|\Omega|_{\mathcal{H}^k},$$

where the non-local operator L_{12} consists of

$$L_{12}(\Omega) = \int_R^{+\infty}\int_0^{\frac{\pi}{2}} \Omega(s,\theta)\frac{3\sin(\theta)\cos^2(\theta)}{s}dsd\theta.$$

Therefore, the solution to the second equation of Equation 31 is of the form

$$\Psi = \frac{1}{4\alpha}\sin(2\theta)L_{12}(\Omega) + \text{lower order terms which can be bounded.} \tag{32}$$

4.1.2 The fundamental model

The fundamental model arises from Equation 31 by making the following reductions:

- **Neglecting terms vanishing quadratically at $R = 0$ and containing a factor of α**, which is justified by the fact that we will eventually obtain *stable* self-similar blow-ups under this kind of perturbations.
- **Dropping the transport term**; as there is evidence for a regularizing effect of transport, one should look for singularities in a scenario in which transport is negligible.
  After the previous observation, and after having inserted the expression of Ψ in Equation 32, indeed, the first equation of Equation 31 becomes

  $$\partial_t\Omega - \frac{3}{2\alpha}\sin(2\theta)L_{12}(\Omega)\partial_\theta\Omega + L_{12}(\Omega)(\cos(2\theta) - \sin^2(\theta))R\partial_R\Omega = \frac{1}{\alpha}L_{12}(\Omega)\Omega, \tag{33}$$

  and if one considers Ω behaving like $R(\sin(\theta)\cos^2(\theta))^{\alpha/3}$ near $R = 0$, we can verify that the second and the third member of the left-hand-side annihilate themselves to leading order:

  Proof.

$$\begin{aligned}
&-\frac{3}{2\alpha}\sin(2\theta)L_{12}(\Omega)\partial_\theta\Omega + L_{12}(\Omega)(\cos(2\theta) - \sin^2(\theta))R\partial_R\Omega \\
&= -\frac{3}{2\alpha}\sin(2\theta)L_{12}(\Omega)R\frac{\alpha}{3}(\sin(\theta)\cos^2(\theta))^{\alpha/3-1}[\cos^3(\theta) - 2\sin^2(\theta)\cos(\theta)] + L_{12}(\Omega)(\cos(2\theta) - \sin^2(\theta))R(\sin(\theta)\cos^2(\theta))^{\alpha/3} \\
&= RL_{12}(\Omega)(\sin(\theta)\cos^2(\theta))^{\alpha/3-1}\left[-\frac{1}{2}\sin(2\theta)(\cos^3(\theta) - 2\sin^2(\theta)\cos(\theta)) + (\cos(2\theta) - \sin^2(\theta))\sin(\theta)\cos^2(\theta)\right] \\
&= RL_{12}(\Omega)(\sin(\theta)\cos^2(\theta))^{\alpha/3-1}\underbrace{\left[-\cos^4(\theta)\sin(\theta) + 2\sin^3(\theta)\cos^2(\theta) + \cos^4(\theta)\sin(\theta) - \sin^3(\theta)\cos^2(\theta) - \sin^3(\theta)\cos^2(\theta)\right]}_{=0}
\end{aligned}$$

  □

  Moreover, looking for a solution of the form $\Omega(R,t,\theta) = (\sin(\theta)\cos^2(\theta))^{\alpha/3}\Omega_*(R,t)$, we have that

  $$|\sin(2\theta)\partial_\theta(\sin(\theta)\cos^2(\theta))^{\alpha/3}| \leq 2\alpha(\sin(\theta)\cos^2(\theta))^{\alpha/3}.$$

  Proof.

$$\begin{aligned}
|\sin(2\theta)\partial_\theta(\sin(\theta)\cos^2(\theta))^{\alpha/3}| &= \left|\sin(2\theta)\frac{\alpha}{3}(\sin(\theta)\cos^2(\theta))^{\alpha/3-1}[\cos^3(\theta) - 2\sin^2(\theta)\cos(\theta)]\right| \\
&= \left|\frac{2\alpha}{3}(\sin(\theta)\cos^2(\theta))^{\alpha/3-1}\sin(\theta)\cos^2(\theta)[\cos^2(\theta) - 2\sin^2(\theta)]\right| \\
&= \left|\frac{2\alpha}{3}(\sin(\theta)\cos^2(\theta))^{\alpha/3}\underbrace{(\cos^2(\theta) - 2\sin^2(\theta))}_{\leq 3}\right| \leq 2\alpha|sin(\theta)\cos^2(\theta)|^{\alpha/3}
\end{aligned}$$

□

For the above considerations, and since we are considering small αs, the transport term becomes negligible with respect to $\frac{1}{\alpha}L_{12}(\Omega)\Omega$.

The above considerations are needed essentially to split the *Biot-Savart* Law linking u and ω into a singular and a regular part, which, as observed by T.Tao[10], makes the non-locality present in

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\omega(y) \times (x-y)}{|x-y|^3} dy$$

simpler. This leads to the possibility to decompose u into a singular (u_S) and regular part (u_R):

$$u = \frac{1}{\alpha}u_S + u_R. \tag{34}$$

With the above reductions on Equation 31, we can then finally obtain the *fundamental model*

$$\partial_t \Omega = \frac{1}{\alpha}L_{12}(\Omega)\Omega. \tag{35}$$

This equation possesses simple self-similar blow-up solutions of the form

$$\Omega(R, \theta, t) = \Gamma(\theta)\frac{1}{1-t}F\left(\frac{R}{1-t}\right) \tag{36}$$

with $F(z) = \frac{2z}{(1+z)^2}$, having the following properties:

1. their dependence on θ can be freely chosen; we choose to work with $\Gamma(\theta) = (\sin(\theta)\cos^2(\theta))^{\alpha/3}$ to neglect the transport terms;
2. they are of order α;
3. they are stable with respect to perturbations vanishing quadratically at $R = 0$.

Note that thanks to 1. and 3. we are allowed to neglect the transport terms, while 2. and 3. enable us to to neglect the rest of the terms of the original formulation in Equation 31.
Referring to the decomposition of u sketched in Equation 34, with the choices made we can ensure that u_R is small and can be considered as a tiny perturbation of the fundamental model. Indeed, the vorticity equation Equation 2 can be rewritten as

$$\partial_t\omega + u_S \cdot \nabla\omega = \nabla u_S \cdot \omega + \alpha\mathcal{N}(u_R, \omega),$$

where the operator $\mathcal{N}$ satisfies suitable bounds which will be introduced later.

Remark 2. *Note that from Equation 36 we can see that the solution constructed by Elgindi is in the class $C^{1,\alpha/3}$. Indeed, we say that $u \in C^{1,\alpha}$ when the vorticity $\omega \in C^\alpha$, and, as Ω behaves in ρ like $R = \rho^\alpha$ for small ρ and in θ like $\Gamma(\theta) = (\sin(\theta)\cos^2(\theta))^{\alpha/3}$, we have that the solution is in the class $C^{1,\alpha/3}$.*

Rescaling time by a factor α, and switching from Ω to f to match the notation with section 3, we can find the formulation of the fundamental model discussed in Chapter 4 of [7]

$$\begin{cases} \partial_t f(\rho,\theta,t) = f(\rho,\theta,t)L_{12}(f)(\rho,t) \\ L_{12}(f) = \int_r^{+\infty}\int_0^{\frac{\pi}{2}} f(\rho,\theta)\frac{3\sin(\theta)\cos^2(\theta)}{\rho}d\rho d\theta. \end{cases} \tag{37}$$

This equation represents a more complicated version of the example in section 3; the main difficulty consists in the fact that the right-hand-side is *non-local*, but the steps to follow to reach the solution to the perturbed model are analogous to the case previously described.

[10] https://terrytao.wordpress.com/tag/tarek-elgindi/

In particular way, Elgindi applies a *modulation* strategy similar to the one discussed for the simpler case: starting from a profile with blow-up F_0, he searches for a self-similar solution depending on the parameters μ, λ and writes the equations in self-similar variables, then he fixes μ, λ to ensure that there exists a small perturbation g so that $F = F_0 + g$ solves the problem.
Before analyzing how to conclude the construction of a singularity for the Euler Equation, we shall discuss the operators that come into play in this case and the suitable spaces in which the argument should be set.

4.2 The weighted Sobolev space and the linearized operator

Throughout this section, we will use the same notation adopted in [7]. The core expressions will be defined within the section, while the marginal ones are moved to Appendix A.

Let us first define the operator

$$\mathcal{L}_\Gamma(f) := \underbrace{f + \partial_z f - \frac{2f}{1+z}}_{\text{(I)}} \underbrace{- \frac{2z\Gamma(\theta)}{c_*(1+z)^2} L_{12}(f)}_{\text{(II)}},$$

which is the sum of (I), the linearized operator of section 3 (proved to be coercive on on L^2 and on the weighted L^2_w with weight $w = \frac{(1+z)^4}{z^4}$) and an extra term (II).
A coercivity result can be obtained for the operator $\mathcal{L}_\Gamma$: indeed, we can prove, similarly to what has been done in section 3, that:

$$(\mathcal{L}_\Gamma(f), f)_{L^2_w} \geq \frac{1}{4}|f|^2_{L^2_w}.$$

Having defined the above operator, we obtain that the problem arising from the linearization of Equation 37 around $F_0 = \frac{\alpha\Gamma(\theta)}{c_*}\frac{2z}{(1+z)^2}$ (where $c_* = \int_0^{\pi/2}\Gamma(\theta)K(\theta)d\theta$) reads as

$$\mathcal{L}_\Gamma(g) - \frac{3}{2\alpha} L_{12}(F_0)\sin(2\theta)\partial_\theta g = -\mu F_0 - (\mu + \lambda + \mu\lambda)z\partial_z F_0 + \mathcal{N}_0 + \mathcal{N} + \mathcal{N}_*. \tag{38}$$

The definition of the $\mathcal{N}$ operators can be found in Appendix A; here, we focus on the left-hand-side of the equation. It suggests indeed that for the Euler problem, seen as a perturbation of the fundamental model, it is necessary to study the coercivity of $\mathcal{L}_\Gamma$ with an extra transport term in the angular direction.
For this reason, we introduce another operator, given by a slight variation of the left-hand side of Equation 38:

$$\mathcal{L}^T_\Gamma(g) = \mathcal{L}_\Gamma(g) - \mathbb{P}\left(\frac{3}{1+z}\sin(2\theta)\partial_\theta g\right),$$

where the projector $\mathbb{P}$ is defined as

$$\mathbb{P}(f)(z,\theta) = f(z,\theta) - \frac{\Gamma(\theta)}{c_*}\frac{2z^2}{(1+z)^3}L_{12}(f)(0).$$

The only difference between $\mathcal{L}^T_\Gamma$ and the term in the left-hand side of Equation 38 is the presence of a projector applied to $\frac{3}{1+z}\sin(2\theta)\partial_\theta g = L_{12}(F_0)\sin(2\theta)\partial_\theta g$, used to ensure that $L_{12}(\mathcal{L}^T_\Gamma(g))(0) = 0$. The reason why this condition is necessary will emerge in the final part of the argument, because it is exploited when looking for *a priori* estimates, so we postpone this discussion to the end of this chapter.
Applying the projector ensures that property, because, by Lemma 5.2 in [7], we already have that

$$L_{12}(\mathcal{L}_\Gamma(g))(0) = \mathcal{L}(L_{12}(g))(0) = L_{12}(g)(0) - 2L_{12}(g)(0) = -L_{12}(g)(0) \overset{11}{=} 0,$$

and we would like to add to it a term such that $L_{12}(\cdot)(0) = 0$, but we have no hint on whether

$$L_{12}\left(\frac{3}{2\alpha}L_{12}(F_0)\sin(2\theta)\partial_\theta g\right)(0) = 0$$

[11]This is a ground hypothesis, as some required estimates proved in [7] hold provided that $L_{12}(g)(0) = 0$.

or not. In this sense, we benefit from the projector, because the $\mathbb{P}$ introduced has the property that $L_{12}(\mathbb{P}(f))(0) = 0 \quad \forall f$. Therefore, we can introduce the extra transport term in the angular direction without losing the property $L_{12}(\mathcal{L}_\Gamma^T(g))(0) = 0$.

Clearly, to make the operator $\mathcal{L}_\Gamma^T$ appear in Equation 38 we need to add a correction term, and the new version reads as

$$\mathcal{L}_\Gamma^T(g) = +\frac{\Gamma(\theta)}{c_*}\frac{2z^2}{(1+z)^3}L_{12}\left(\frac{3}{1+z}\sin(2\theta)\partial_\theta g\right)(0) - \mu F_0 - (\mu+\lambda+\mu\lambda)z\partial_z F_0 + \mathcal{N}_0 + \mathcal{N} + \mathcal{N}_*. \tag{39}$$

To obtain a coercivity result on the operator $\mathcal{L}_\Gamma^T$, it is necessary to transfer the argument in a space with both *radial* and *angular* weights, since the latter enables us to hide the effect of the angular transport term.
Introducing the angular weight

$$\bar{w} = \frac{1}{\sin^\eta(2\theta)}, \qquad \eta = \frac{99}{100},$$

we can define a norm in the weighted Sobolev space of interest, denoted as $\mathcal{H}^k$ (in particular, we will use $\mathcal{H}^4$):

$$|f|_{\mathcal{H}^k}^2 := \sum_{i=0}^{k}\left|(R\partial_R)^i f\frac{(1+z)^2}{z^2\sin^{\eta/2}(2\theta)}\right|_{L^2}^2 + \sum_{0\le i+j\le k, i\ge 1}\left|(R\partial_R)^j(\sin(2\theta)\partial_\theta)^i f\frac{(1+z)^2}{z^2\sin^{\eta/2}(2\theta)}\right|_{L^2}^2.$$

The coercivity estimate holds for $\mathcal{L}_\Gamma^T(g)$ reads as

$$(\mathcal{L}_\Gamma^T(g), g)_{\mathcal{H}^k} \ge c_k |g|_{\mathcal{H}^k}^2 \quad \forall f \in \mathcal{H}^k \tag{40}$$

for some $c_k > 0$.

4.3 Construction of the solution

The scaling parameter λ is chosen in such a way that all terms vanishing linearly at $z = 0$ are annihilated, which leads to

$$\lambda = \frac{-2\mu}{\mu+1}.$$

The choice of μ is given by the constraint that the right-hand side of Equation 39 satisfies $L_{12}(\mathrm{RHS})(0) = 0$. By fixing μ in such a way that $\bar{\mu} := -L_{12}\left(\frac{3}{1+z}\sin(2\theta)\partial_\theta g\right)(0) + 2\alpha\mu = L_{12}(\mathcal{N}(0)) + L_{12}(\mathcal{N})(0)$, we can verify that $L_{12}(\mathrm{RHS})(0) = 0$.

Proof. With simple computations starting from the definition of L_{12}, we obtain

$$L_{12}(F_0) = \frac{2\alpha}{1+z}, \qquad L_{12}(z\partial_z F_0) = \frac{-2\alpha z}{(1+z)^2}, \qquad L_{12}\left(\frac{\Gamma(\theta)}{c_*}\frac{2z^2}{(1+z)^3}\right) = \frac{2z+1}{(1+z)^2},$$

hence $L_{12}(F_0)(0) = 2\alpha$, $L_{12}(z\partial_z F_0)(0) = 0$ and $L_{12}\left(\frac{\Gamma(\theta)}{c_*}\frac{2z^2}{(1+z)^3}\right)(0) = 1$.
Moreover, $L_{12}(\mathcal{N}_*)(0) = 0$ so long as $L_{12}(g)(0) = 0$[12]. Now we can proceed with the proof:

$$\begin{aligned} L_{12}(RHS)(0) &= L_{12}\left(\frac{3}{1+z}\sin(2\theta)\partial_\theta g\right)(0)L_{12}\left(\frac{\Gamma(\theta)}{c*}\frac{2z^2}{(1+z)^3}\right)(0) - \mu L_{12}(F_0)(0) \\ &\quad - (\mu+\lambda+\mu\lambda)L_{12}(z\partial_z F_0)(0) + L_{12}(\mathcal{N}_0)(0) + L_{12}(\mathcal{N})(0) + L_{12}(\mathcal{N}_*)(0) \\ &= L_{12}\left(\frac{3}{1+z}\sin(2\theta)\partial_\theta g\right)(0) - 2\alpha\mu + L_{12}(\mathcal{N}_0)(0) + L_{12}(\mathcal{N})(0) \\ &\overset{\heartsuit}{=} L_{12}\left(\frac{3}{1+z}\sin(2\theta)\partial_\theta g\right)(0) - 2\alpha\mu - L_{12}\left(\frac{3}{1+z}\sin(2\theta)\partial_\theta g\right)(0) + 2\alpha\mu \\ &= 0 \end{aligned} \tag{41}$$

where $\heartsuit$ is ensured by the declared choice of μ. □

[12] See footnote 11.

We can then reformulate Equation 39 as

$$\mathcal{L}_{\Gamma}^{T}(g) = -\bar{\mu}\frac{\Gamma(\theta)}{c*}\frac{2z^2}{(1+z)^3} + \mathcal{N}_0 + \mathcal{N} + \mathcal{N}_* =: \text{RHS}. \tag{42}$$

It can be proved that Equation 42 has a solution in $\mathcal{H}^4$ which grows at most like α^2. This is obtained by deriving a bound for the right-hand-side

$$|(\text{RHS}, g)|_{\mathcal{H}^4} \leq C\left(\alpha|g|_{\mathcal{H}^4}^2 + \frac{1}{\alpha^{3/2}}|g|_{\mathcal{H}^4}^3 + \frac{1}{\alpha^{5/2}}|g|_{\mathcal{H}^4}^4\right). \tag{43}$$

Then, coupling this result with the coercivity of the $\mathcal{L}_{\Gamma}^{T}$ operator and assuming that $|g|_{\mathcal{H}^4} \leq \alpha^{7/4}$, we can derive the desired inequality

$$c|g|_{\mathcal{H}^4}^2 \leq (\mathcal{L}_{\Gamma}^{T}g, g)_{\mathcal{H}^4} \leq C\left(\alpha|g|_{\mathcal{H}^4}^2 + \frac{1}{\alpha^{3/2}}|g|_{\mathcal{H}^4}^3 + \frac{1}{\alpha^{5/2}}|g|_{\mathcal{H}^4}^4\right) \quad \Rightarrow \quad |g|_{\mathcal{H}^4} \leq C\alpha^2.$$

The above result concerns stability; for the existence, the strategy is similar to the example in section 3: we introduce a fake time variable τ, and solve the auxiliary problem

$$\begin{cases} \partial_\tau g + \mathcal{L}_{\Gamma}^{T}(g) = \text{ RHS} \\ g(z,\theta,0) = 0. \end{cases} \tag{44}$$

We can obtain *a priori* estimates on g and $\partial_\tau g$, using the coercivity of $\mathcal{L}_{\Gamma}^{T}$ and the bound for the RHS in Equation 43, which read as

$$|g|_{\mathcal{H}^4} \leq C\alpha^2, \qquad |\partial_\tau g|_{\mathcal{H}^3} \leq C\alpha^2 \exp(-c\tau). \tag{45}$$

For the above results to hold, it is necessary that

$$\partial_\tau L_{12}(g)(0,\tau) = -L_{12}(\mathcal{L}_{\Gamma}^{T}(g))(0,\tau) = 0,$$

so the above mentioned condition $L_{12}(\mathcal{L}_{\Gamma}^{T}(g))(0,\tau) = 0$ becomes relevant at this stage.
The former bound in Equation 45 implies the existence of a subsequential limit for g as $\tau \to \infty$, while the latter implies that g has a unique limit as $\tau \to \infty$.
This concludes Elgindi's argument in [7] for the construction of a blowing-up solution to the axi-symmetric 3D Euler Equation, which is seen as a perturbation of the fundamental model in Equation 35.

5 Conclusions

In this project, we have deepened into the question of regularity of solutions to the Euler Equation starting from Elgindi's works. In particular, we have developed the details of two examples of singular solutions that inspired the recent result of finite-time blow-up of $C^{1,\alpha}$ solutions to the incompressible Euler equation in $\mathbb{R}^3$ when α is small, and we have analyzed the intuition behind Elgindi's strategy.

What can be learned from the first example in section 2 is the clear physical intuition for the singularity formation, and the identification of behaviors that are helpful in obtaining a blow-up, such as the property that the vorticity is odd in the coordinate representing the vertical axis (which means that there is conflict of eddies rotating in opposite directions), the presence of a vortex stretching term (due to swirl velocity in this case) and of spatial boundaries. Moreover, we could experience the possibility to simplify *Biot-Savart* law to obtain a simpler model.

Then, we left a scenario in which the blow-up could be obtained by exact calculations, and set in an abstract, very general framework, in which we saw the *modulation strategy* at work; this method had already been applied in the study of other PDEs, as remarked in [8], but extended to Euler by Elgindi's et al. works.

We analyzed the building blocks of the strategy exploited in the main reference [7] in a simpler problem; as the operators involved are not always explicitly written, we made some additional hypothesis that can be recovered in the Euler case, where each operator has a declared form.
In section 3 we made an attempt to understand all the ingredients and the steps needed for the Euler case, since the intuition is the same in the simpler setting and in Euler, but with more complex operators and longer computations in the latter case. Apart from technicalities, the main difficulties that differentiate the Euler case from the example in section 3 are the presence of non-local effects, the choice of the scaling parameters, and of suitable spaces in which the reduced model could be set.
More in general, with the analysis of the Euler case, we could get a glimpse of how to find a bridge between the properties corresponding to physical effects that help singularities and the abstract compactness method, such as the relation between the choice of the weights and the possibility to consider transport, which has a regularizing effect, as a small perturbation. Thanks to such intuitions, the modulation strategy could be exploited to add a significant result to the study of regularity of solutions to the Euler equation.

APPENDICES

Appendix A

Notations

Notation	Name	Expression
$\nabla\cdot$	divergence	$\partial_x + \partial_y + \partial_z$
$\nabla\times$	curl	$\begin{bmatrix} (\partial_y(\cdot)^z - \partial_z(\cdot)^y) \\ (\partial_z(\cdot)^x - \partial_x(\cdot)^z) \\ (\partial_x(\cdot)^y - \partial_y(\cdot)^x) \end{bmatrix}$
$\lvert\cdot\rvert_0$	norm in C^0	$\sup\lvert\cdot\rvert$
$\frac{D}{Dt}$	material derivative	$\partial_t + u^x \frac{\partial}{\partial_x} + u^y \frac{\partial}{\partial y} + u^z \frac{\partial}{\partial z}$
$\frac{\tilde{D}}{Dt}$	material derivative (axi-symmetric flows)	$\partial_t + u^r \frac{\partial}{\partial_r} + u^{x_3} \frac{\partial}{\partial x_3}$
$K_2(x)$	homogeneous kernel of degree -1	$\frac{1}{2\pi}\left(\frac{-x_2}{\lvert x\rvert^2}, \frac{x_1}{\lvert x\rvert^2}\right)^T$
$K_3(x)h$	homogeneous kernel of degree -2	$\frac{1}{4\pi}\frac{x\times h}{\lvert x\rvert^3}$

1 $\mathcal{N}$ operators

$$\mathcal{N}_0 := \frac{3}{2\alpha}L_{12}(F_0)D_\theta F_0 - (\cos(2\theta) - \sin^2(\theta))L_{12}(F_0)D_z F_0 + \frac{1}{\alpha}L_{12}(g)g$$
$$+ \frac{3}{2\alpha}L_{12}(g)D_\theta F - (\cos(2\theta) - \sin^2(\theta))L_{12}(g)D_z F - (\cos(2\theta) - \sin^2(\theta))L_{12}(F_0)D_z g$$

$$\mathcal{N} := 2\mathcal{R}\left(\Phi - \frac{1}{4\alpha}\sin(2\theta)L_{12}(F)\right)F - 2U\left(\Phi - \frac{1}{4\alpha}\sin(2\theta)L_{12}(F)\right)\partial_\theta F$$
$$- 2\alpha V\left(\Phi - \frac{1}{4\alpha}\sin(2\theta)L_{12}(F)\right)D_z F - \sin(2\theta)(F,K)_{L^2_\theta}\partial_\theta F - 2\sin^2(\theta)(F,K)_{L^2_\theta}F$$

$$\mathcal{N}_* := -\mu g - (\mu + \lambda + \mu\lambda)D_z g$$

where

$$D_z = z\partial_z \qquad D_\theta = \sin(2\theta)\partial_\theta$$
$$K(\theta) = 3\sin(\theta)\cos^2(\theta)$$
$$F = F_0 + g$$
$$U(\Phi) := -3\Phi - \alpha D_z\Phi$$
$$V(\Phi) := \partial_\theta\Phi - \tan(\theta)\Phi$$
$$\mathcal{R}(\Phi) := \frac{1}{\cos(\theta)}(2\sin(\theta)\Phi + \alpha\sin(\theta)D_z\Phi + \cos(\theta)\partial_\theta\Phi)$$
$$\Phi \text{ is related to } \Psi \text{ by } \Psi = \frac{1}{1-(1+\mu)t}\Phi(z,\theta)$$

Appendix B

Minor Derivations

1 Derivation of the vortex-dynamics equation

The vortex-dynamics equation follows immediately from the following lemma:

Lemma 2. *Given a velocity field $u(x,t)$ in $\mathbb{R}^N$, let $h(x,t)$ be a vector field satisfying*

$$\frac{Dh}{Dt} = h \cdot \nabla u,$$

and let $f(x,t)$ be a scalar function solving

$$\frac{Df}{Dt} = 0.$$

Then,

$$\frac{D}{Dt}(\nabla f \cdot h) = 0.$$

Proof. By applying the chain rule, we get:

$$\frac{D}{Dt}(\nabla f \cdot h) = \frac{D}{Dt}\nabla f \cdot h + \nabla f \cdot \frac{D}{Dt}h. \tag{B.1}$$

We know, by assumption, that $\frac{Df}{Dt} = 0$, hence:

$$\begin{aligned}
0 = \nabla\left(\frac{D}{Dt}f\right) = \nabla(\partial_t f + (u\cdot\nabla)f) &= \partial_t \nabla f + \nabla u \cdot \nabla f + u \cdot \nabla\nabla f \\
&= \partial_t \nabla f + \nabla u \cdot \nabla\nabla f + \nabla(u \cdot \nabla f) - u \cdot \nabla\nabla f \\
&= \frac{D}{Dt}\nabla f + \nabla(u\cdot\nabla f) - u\cdot\nabla\nabla f \\
&= \frac{D}{Dt}\nabla f + u\cdot\nabla\nabla f + \nabla f\cdot\nabla u + u\times(\nabla\times\nabla f) + \nabla f\times(\nabla\times u) - u\cdot\nabla\nabla f \\
&= \frac{D}{Dt}\nabla f + \nabla f\cdot\nabla u + \nabla f\times(\nabla\times u) \\
&\Rightarrow \frac{D}{Dt}\nabla f = -\nabla f\cdot\nabla u + (\nabla\times u)\times\nabla f.
\end{aligned}$$

Then, by plugging this result into Equation B.1 and expanding the differential operators, we get that

$$\frac{D}{Dt}(\nabla f \cdot h) = -\nabla f\cdot\nabla u + (\nabla\times u)\times\nabla f + \nabla f\cdot(h\cdot\nabla u) = 0.$$

□

Since ω and the coordinate θ solve respectively

$$\frac{D\omega}{Dt} = \omega \cdot \nabla u, \qquad \frac{D\theta}{Dt} = \frac{v^\theta}{r} = 0,$$

by applying 2 with $h = \omega$ and $f = \theta$, we get

$$\frac{D}{Dt}(\nabla\theta \cdot \omega) = \frac{D}{Dt}\frac{\omega^\theta}{r} = \frac{\tilde{D}}{Dt}\xi = 0.$$

2 Extension of the result in section 2 for non-zero initial vorticity

In the case of intial datum not null, the solution of the Equation 10 reads as

$$\psi(\Phi(x,t),t) = \psi_0(x) + \int_0^t f(\Phi(x,s),s)ds,$$

hence, we simply add the term $\omega_0(x_1, x_2)$ and get

$$\omega(x_1, x_2, t) = \partial_1\rho_0\left(\mu(t)x_1, \frac{x_2}{\mu(t)}\right)\int_0^t \mu(s)ds.$$

We just have to check that inequalities (13,15) still hold. We start again by $\dot{\mu}(t) = \mu(t)\lambda(t)$, adding the ω_0 term in ω, and get

$$\frac{\dot{\mu}(t)}{\mu(t)} = \left(\int_0^t \mu(s)ds\right) 4\int_0^\infty \int_0^\infty \frac{y_1y_2}{|y|^4}\left[\partial_1\rho_0\left(\mu(t)y_1, \frac{y_2}{\mu(t)}\right) + \omega_0\left(\mu(t)y_1, \frac{y_2}{\mu(t)}\right)\right]dy_1dy_2.$$

To prove Equation 13, we can bound

$$\left|\frac{\dot{\mu}(t)}{\mu(t)}\right| \leq \left(\int_0^t \mu(s)ds\right) 4\int_0^\infty \int_0^\infty \frac{y_1y_2}{|y|^4}\left|\omega_0\left(\mu(t)y_1, \frac{y_2}{\mu(t)}\right)\right|dy_1dy_2$$

proceeding as done for $|\partial_1\rho|$, as the hypothesis on parity and compactness of support still hold.
To prove Equation 15, it is sufficient to take a non-negative ω_0, not vanishing on $x_2 = 0$, and noting that inequality (15) still holds as we are adding a non-negative term.

Appendix C

Insights

1 Hou-Luo's test case

In this section, we reflect more in detail upon the physical phenomenon leading to a singularity in the numerical example studied by Hou and Luo in [17].
We note in Figure 1a that the top and bottom half of the cylinder rotate in opposite directions (expressed in Equation 7 by the fact that ω is odd in the variable x_1); although this motion represents the main current, also more complicated currents cycling in the vertical direction (upwards and downwards) are detected. It is exactly thanks to these movements that the hyperbolic pattern at point $z = 0, r = 1$ originates (see also the recent post on Quanta Magazine[1] presenting the numerical experiments on this test case, from which Figure C.1 is taken).

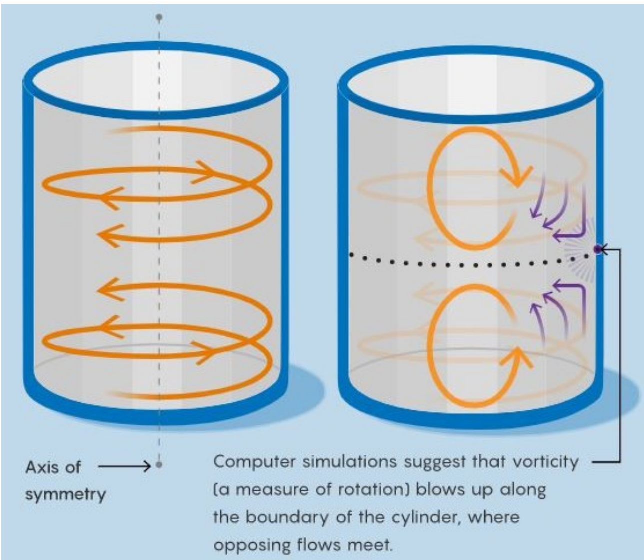


Figure C.1
Creation of the singularity at $z = 0, r = 1$

2 Fixed Point Method

The procedure is similar to the one adopted in subsection 3.1. In this case, we search for a self-similar solution with only one free parameter δ, since it will be enough for the condition we need to impose. Let this solution be of the form

$$f(x,t) = \frac{1}{1-t} F_\varepsilon\left(\frac{x}{(1-t)^{1+\delta}}\right);$$

[1]https://www.quantamagazine.org/deep-learning-poised-to-blow-up-famed-fluid-equations-20220412

inserting it into Equation 17, we obtain the equation

$$F_\varepsilon + (1+\delta)z\partial_z F_\varepsilon = F_\varepsilon^2 + \varepsilon\mathcal{N}(F_\varepsilon).$$

Proceeding as above, we can view F_ε as a perturbation of F_0, and substitute it with $F_0 + g$. Hence, we arrive to the equation

$$\begin{aligned}\mathcal{L}(g) &:= g + z\partial_z g - \frac{2}{1+z}g \\ &= -\delta z\partial_z F_0 + g^2 + \varepsilon\mathcal{N}(F_0+g) =: \text{RHS}.\end{aligned}$$

If $\mathcal{L}$ would be an invertible operator, then we could put $\delta = 0$ and solve

$$g = \mathcal{L}^{-1}(g^2 + \varepsilon\mathcal{N}(F_0+g)).$$

However, $\mathcal{L}$ is not invertible, hence we need the free parameter δ to exploit the following result and solve for g.

Lemma 3. *Let $f \in H^k$, $k \geq 2$. Then, $\mathcal{L}(g) = f$ is solvable in $C^1 \iff f'(0) + 2f(0) = 0$.*
Moreover, in this case we can write

$$g(z) = -f(0) + \frac{z}{(1+z)^2}\int_0^z \frac{(1+t)^2}{t^2}\left(f(t) + f(0)\frac{t-1}{t+1}\right)dt$$

and $|g|_{H^k} \leq C_k |f|_{H^k}$.

Proof. ($\Rightarrow$) Evaluating $\mathcal{L}(g) = f$ and its derivative at $z = 0$, we get

$$g(0) - 2g(0) = -g(0) = f(0),$$

$$g'(0) + g'(0) - 2g'(0) + 2g(0) = 2g(0) = f'(0).$$

Hence, C^1 solutions need to satisfy $f'(0) + 2f(0) = 0$.
($\Leftarrow$) Re-writing the equation as

$$\frac{z-1}{z+1}g + z\partial_z g = f, \tag{C.1}$$

we can start solving as for common ODEs. Let us introduce

$$G := g - g(0) \quad \text{and} \quad F := f + f(0)\frac{z-1}{z+1},$$

where $F(0) = 0$ and $F'(0) = f'(0) + 2f(0) = 0$ by hypothesis. Note that Equation C.1 can be written as

$$\frac{z-1}{z+1}(g - g(0)) + z\partial_z(g - g(0)) = f + f(0)\frac{z-1}{z+1}.$$

Thanks to the fact that $-g(0) = f(0)$. Therefore we can solve

$$\frac{z-1}{z+1}G + z\partial_z G = F,$$

$$\frac{z-1}{z(z+1)}G + \partial_z G = \frac{F}{z}. \tag{C.2}$$

Since

$$\int \frac{z-1}{z(z+1)}dz = \int \frac{2z-z-1}{z(z+1)} = 2\log(z+1) - \log(z) = \log\left(\frac{(1+z)^2}{z}\right),$$

we can re-write Equation C.2 as

$$\partial_z\left(\frac{(1+z)^2}{z}G\right) = \frac{(1+z)^2}{z^2}F,$$

and since $G(0) = 0$ by definition of G, the solution of the ODE is

$$G(z) = \frac{z}{(1+z)^2} \int_0^z \frac{(1+t)^2}{t^2} F(t)dt.$$

Going back to the original unknown g, we have

$$g(z) = g(0) + G(z) = -f(0) + \frac{z}{(1+z)^2} \int_0^z \frac{(1+t)^2}{t^2} F(t)dt.$$

This is well defined, because, when t is small, we can prove that

$$f(t) + f(0)\frac{t-1}{t+1} = O(t^2).$$

Indeed, with a Taylor expansion of f around $t = 0$, we have: $f(t) = f(0) + tf'(0) + O(t^2)$. Hence,

$$\begin{aligned}
\frac{1}{1+t}[tf(t) + f(t) + tf(0) - f(0)] &\overset{\heartsuit}{=} tf(0) + t^2 f'(0) + O(t^3) + f(0) + tf'(0) + O(t^2) + tf(0) - f(0) \\
&= t[2f(0) + f'(0)] + O(t^2) \\
&= O(t^2).
\end{aligned}$$

where $\heartsuit$ holds since $\frac{1}{1+t} \sim 1$ when $t \to 0$.
On the other hand, for $z \to \infty$, a cancellation between the terms $-f(0)$ and $f(0)\frac{t-1}{t+1}$ occurs:

$$\begin{aligned}
-f(0) + \frac{z}{(1+z)^2} \int_1^z \frac{(1+t)^2}{t^2} f(0)\frac{t-1}{t+1}dt &= -f(0) + \frac{z}{(1+z)^2} \int_1^z \frac{(t+1)(t-1)}{t^2} f(0)dt \\
&= -f(0) + \frac{z}{(1+z)^2} f(0) \int_1^z 1 - \frac{1}{t^2}dt \qquad \text{(C.3)} \\
&= -f(0) + \frac{z^2}{(1+z)^2} f(0) + \frac{1}{(1+z)^2} f(0) - \frac{2z}{(1+z)^2} f(0).
\end{aligned}$$

Note that

$$\frac{z^2}{(1+z)^2} f(0) \sim f(0)$$

for $z \to \infty$, and that

$$\frac{1}{(1+z)^2} f(0) - \frac{2z}{(1+z)^2} f(0) = f(0) O\left(\frac{1}{z}\right).$$

The H^k−norm of g can be treated with Hardy's inequality, using the generalization made by Levinson in [16]. □

The parameter δ is needed exactly to impose the condition $2f(0) + f'(0) = 0$ on the RHS. In particular,

$$2\text{RHS}(0) = 2g(0)^2 + 2\varepsilon\mathcal{N}(F_0 + g)(0),$$

$$\text{RHS}'(0) = -\delta F_0'(0) + 2g(0)g'(0) + \varepsilon\mathcal{N}(F_0 + g)'(0).$$

Since $F_0'(0) = -1$, we get

$$\delta = -2g^2(0) - 2\varepsilon\mathcal{N}(F_0 + g)(0) - 2g(0)g'(0) - \varepsilon\mathcal{N}(F_0 + g)'(0).$$

Hence, $\mathcal{L}(g) = \text{RHS}$ can be solved:

$$g = \mathcal{K}(g) := \mathcal{L}^{-1}(-\delta z\partial_z F_0 + g^2 + \varepsilon\mathcal{N}(F_0 + g)),$$

and we conclude by applying the following theorem.

Theorem 3. *Let $\mathcal{N}$ be a bounded operator in H^k for some $k \geq 2$. Then, $\exists\delta > 0$ s.t. $\mathcal{K} : B_\delta(0) \to B_\delta(0)$ is a contraction.*

3 Stability among general solutions

Another problem that can be studied is whether the self-similar solution of Equation 16 is stable among *more general* solutions, and not just solutions of the form

$$f(x,t) = \frac{1}{1-(1+\mu)t} F_\varepsilon\left(\frac{x}{(1-(1+\mu)t)^{1+\lambda}}\right)$$

as done previously.
This case is analyzed in a posterior work by Elgindi, Ghoul and Masmoudi [8], and we will briefly sketch how to adapt the previous arguments in this setting. The relevance of this study relies in the fact that it enables to localize the self-similar solutions obtained in [7] to locally self-similar solutions with compactly supported vorticity and without external force, since the blow-up is stable to certain kinds of perturbations that allow us to construct an $L^2 \cap C^{1,\alpha}$ classical solution that becomes singular in finite time.
Hence, the last *not too physical* characteristics of Elgindi's solution in [7] (the fact that it has infinite energy) can be dropped.

Let us suppose that the solution to Equation 16 can be written as

$$f(x,t) = \frac{1}{\bar{\lambda}(t)} F\left(\frac{\bar{\mu}(t)x}{\bar{\lambda}(t)}, t\right),$$

where $\bar{\mu}$ and $\bar{\lambda}$ are given. Note that here the scaling parameters μ, λ are determined *dynamically*, and they are time-dependent. Defining the new variables

$$z = \frac{\bar{\mu}(t)x}{\bar{\lambda}(t)}, \qquad \frac{ds}{dt} = \frac{1}{\bar{\lambda}},$$

Equation 18 can be rewritten as

$$\partial_s F - \frac{\lambda_s}{\lambda} F + \left(\frac{\mu_s}{\mu} - \frac{\lambda_s}{\lambda}\right) z\partial_z F = F^2,$$

with $\lambda_s = \frac{d\lambda}{ds}$ and $\mu_s = \frac{d\mu}{ds}$. The self-similar solution already presented has $F(z,t) = F_0(z)$, $\lambda_s = -\lambda$ and $\mu \equiv 1$; looking for solutions of the form $F = F_0 + g$, we get

$$\begin{aligned} \partial_s g + \mathcal{L}(g) &= \partial_s g + g + z\partial_z g - \frac{2g}{1+z} \\ &= \underbrace{\left(\frac{\lambda_s}{\lambda}+1\right)(F_0 + z\partial_z F_0) - \frac{\mu_s}{\mu} z\partial_z F_0}_{\text{linear terms, chosen to have } g \text{ in the space where } \mathcal{L} \text{ is positive}} + \underbrace{\left(\frac{\lambda_s}{\lambda}+1\right)(g + z\partial_z g) - \frac{\mu_s}{\mu} z\partial_z g + g^2}_{\text{non-linear, to be treated with energy arguments}} . \end{aligned} \tag{C.4}$$

Provided that for g_0 initial datum of Equation C.4 we choose λ, μ such that

$$\left|\frac{\lambda_s}{\lambda} + 1\right| + \left|\frac{\mu_s}{\mu}\right| + |g|_X \leq C\varepsilon \exp(-cs),$$

results similar to the case dealt with in the project can be obtained.

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