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Introduction to Supersymmery

Giuliano Panico

IFAE and BIST, Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Spain

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Preface

These notes have been prepared for the "Quantum Field Theory III" course I held in the winter semester 2010-2011 at ETH Zürich. The course contained a comprehensive introduction to supersymmetry in quantum field theories. The lectures covered most of the relevant theoretical aspects of the subject, as well as some more phenomenologically-oriented topics, including a presentation of the supersymmetric version of the Standard Model.

The course was offered in the Physics Master and in the Doctoral and Post-Doctoral Studies programme at ETH Zuürich. It included 14 lessons with 2 hours main lesson plus 1 hour exercise session.

I made a revision of the notes in 2017 when I adapted them for the course "Physics Beyond the Standard Model" I held at Universitat Autonoma de Barcelona together with Prof. A. Pomarol.

I thank Prof. C. Anastasiou for encouraging me to teach the course at ETH.

Barcelona, April 2017
Giuliano Panico

1

Introduction

The aim of the course is to provide a comprehensive introduction to supersymmetry (SUSY). We will discuss the theoretical basis of supersymmetry and some phenomenological applications in the context of beyond the Standard Model (BSM) physics.

1.1 Naturalness in the SM

The SM is a gauge theory based on the electroweak (EW) gauge group $SU(2)_L \times U(1)_Y$.[1] In nature we do not observe long-range interactions that respect the $SU(2)_L \times U(1)_Y$ invariance, so we know that this symmetry must be broken so that only the $U(1)_{em}$ electromagnetic subgroup is preserved, which corresponds to the massless photon.

In the SM this breaking occurs spontaneously and is due to a scalar field, the Higgs boson H, which is a complex doublet transforming in the $\mathbf{2}_{-1/2}$ representation of the EW group. The Higgs field acquires a vacuum expectation value (VEV) not invariant under the $SU(2)_L \times U(1)_Y$ group.

The Lagrangian that describes the Higgs dynamics contains a potential term

$$V(H) = -m^2|H|^2 + \frac{\lambda}{4}|H|^4 \,. \tag{1.1.1}$$

If $m^2 > 0$ (and $\lambda > 0$) the minimum of the potential corresponds to a configuration in which the Higgs acquires a VEV

$$\langle H \rangle = \sqrt{-\frac{2m^2}{\lambda}} \,. \tag{1.1.2}$$

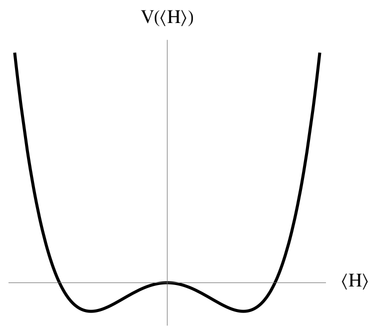


Figure 1.1: Shape of the Higgs potential.

The value of the Higgs VEV is a very important quantity in the SM. It determines the values of the masses of the SM gauge bosons as well as of the leptons and quarks. One gets experimentally

$$\langle H \rangle = v \simeq 246 \text{ GeV} \,. \tag{1.1.3}$$

This result, together with the measured valued for the Higgs mass

$$m_H = 2\, m = 125 \text{ GeV} \,, \tag{1.1.4}$$

[1] We neglect here the $SU(3)_c$ QCD group, which plays no role in the naturalness problem.

completely fixes the parameters in the potential $V(H)$:

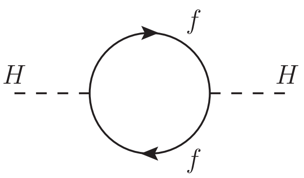
$$\lambda \simeq 0.13\,. \tag{1.1.5}$$

The presence of a light Higgs is an essential feature of the SM, which allows to extend the model up to (almost) arbitrarily high energy scales. In fact, in the absence of the Higgs, the amplitude of some scattering processes (most notably $WW \to WW$ and $WW \to HH$) grows with the energy, violating perturbative unitarity at energies of the order $E \sim 4\pi v \sim 3$ TeV. The presence of the Higgs boson provides additional contributions to the scattering amplitudes that cancel the growth with the energy, thus ensuring perturbative unitarity at high energy.

This picture, however, has a theoretical drawback. We know that the SM is not a complete description of nature. The lack of a full quantum description of gravity tells us that the SM must necessarily be modified at energies scales at most of the order of the Planck mass $E \lesssim 4\pi M_P \simeq 10^{19}$ GeV.[2] Apart from the lack of a full description of gravity, the SM has also additional issues. For instance it does not explain the origin and size of the neutrino masses, moreover it lacks a viable dark matter candidate. Although these additional issues might very well find an explanation at energy scales much larger than the ones experimentally accessible in the near future, they show that new physics is necessarily present beyond the SM.

When we try to extend the SM we have to cope with a theoretical obstruction to find a predictive theory. This problem is known as the *naturalness* (or *hierarchy*) problem, and stems from the fact that the Higgs mass, which is the only dimensionful parameter in the SM, is not stable under radiative corrections. Any new dynamics coupled to the Higgs would induce large loop corrections to the Higgs mass (of the order of the new-physics energy scale), whose size can be much larger than the measured Higgs mass. In order to allow for a light Higgs, a large amount of tuning would be needed in generic extensions of the SM.

The instability of the Higgs mass is deeply related to the fact that the Higgs mass is not protected by any symmetry, or, in other words, no additional symmetry is found in the limit $m_H \to 0$.[3] As an example let us consider a fermion with mass m_f which is coupled to the Higgs through a Yukawa interaction $-\lambda_f \overline{f} H f$. If we compute the one-loop contribution to the Higgs mass Δm_H^2 in a cut-off regularization we get



$$\Rightarrow \quad \Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\left[\Lambda_{UV}^2 - 3m_f^2 \log\left(\frac{\Lambda_{UV}^2 + m_f^2}{m_f^2}\right) + \text{finite}\right]. \tag{1.1.6}$$

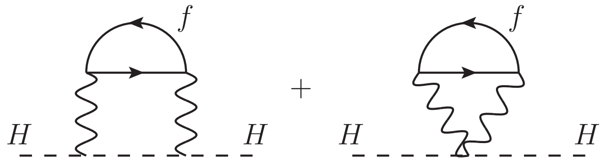
The cut-off Λ_{UV} can be interpreted as the energy at which some new physics is present that can correct the behavior of the loop integral and remove the divergence.

Notice that similar corrections can also come from particles that couple to the Higgs boson only indirectly, as for instance through the gauge interactions. For example a fermion charged under the SM gauge fields but not directly coupled to the Higgs give rise to a 2-loop contribution

[2] This can be seen from the fact that the effective gravity coupling strength grows like $g_G \simeq E/M_P$, thus reaching the perturbativity bound $g_G \lesssim 4\pi$ for $E \sim 4\pi M_P$.

[3] A possible symmetry that could protect the Higgs mass is conformal invariance. This symmetry is however believed to be broken by gravity, so it can not be really used to solve the naturalness problem.

to Δm_H:



$$\Rightarrow \quad \Delta m_H^2 \sim \left(\frac{g^2}{16\pi^2}\right)^2 \Lambda_{UV}^2 + \cdots \tag{1.1.7}$$

The physics Higgs mass is given by the sum of the bare mass $(m_H^2)_B$ (i.e. the mass term that appears in the Lagrangian) and the loop corrections

$$(m_H^2)_{phys} = (m_H^2)_B + \Delta m_H^2 \,. \tag{1.1.8}$$

If we assume that the cut-off of the SM is high, a large tuning is required in order to cancel the corrections from Δm_H^2 and the correct Higgs mass $m_H = 125$ GeV. If $\Delta m_H^2 \gg m_H^2 = (125 \text{ GeV})^2$ we need to fine-tune the bare mass against the loop corrections an an *unnatural* way.

To get a lower estimate of the tuning we can just consider the loop corrections due to the heaviest fermion in the SM, top quark. In this case we get

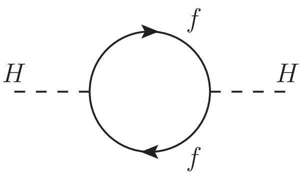
$$\Delta m_H^2 \simeq \frac{3y_t^2}{8\pi^2}\Lambda_{UV}^2 \,, \tag{1.1.9}$$

where $y_t = \sqrt{2}m_{top}/v$ is the top Yukawa coupling. This implies a minimal amount of fine-tuning

$$\Delta \geq \frac{\Delta m_H^2}{m_H^2} = \frac{3y_t^2}{8\pi^2}\frac{\Lambda_{UV}^2}{m_H^2} \simeq \left(\frac{\Lambda_{UV}}{450 \text{ GeV}}\right)^2 \,. \tag{1.1.10}$$

As a consequence, a completely *natural*, i.e. not fine-tuned, extension of the SM requires new physics around or below the TeV scale.

One could wonder about what happens with a different regularization. For instance with dimensional regularization the quadratic divergence is removed and we are left with only a logarithmic sensitivity to the renormalization scale. For example the one-loop contributions coming from a fermion loop read



$$\Rightarrow \quad \Delta m_H^2 \sim \frac{|\lambda_f|^2}{16\pi^2} m_f^2 \left[\frac{1}{\varepsilon} + \left(a + b\log\frac{m_f^2}{\mu^2}\right) + \cdots\right] \,. \tag{1.1.11}$$

This means that the Higgs mass term is quadratically sensitive to the mass of the fermion. In particular Δm_H^2 is of the order of the mass of the *heaviest* particle which couples (directly or indirectly) to the Higgs. In other words, the UV physics can not be decoupled from the Higgs sector.

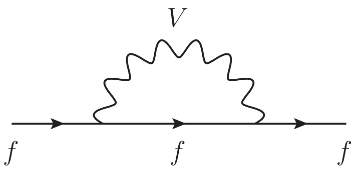
1.2 A lesson from fermions

How can we solve this problem? One possibility would be to realize the Higgs as a composite object (for instance a Goldstone boson as the pions in QCD). In this case the natural mass scale of the Higgs mass (and of the Higgs potential) is linked to the strong-coupling scale of the composite dynamics. A large hierarchy between the strong-coupling scale and a UV dynamics can be easily generated through dimensional transmutation. This mechanism, for instance, is at work in QCD, whose strong-coupling scale is determined by the position of the Landau pole of the QCD gauge coupling, which determines the typical mass scale of the mesonic and baryonic resonances $\Lambda_{\rm QCD} \sim 300$ MeV.

However the composite Higgs option is not the only possibility. We will see in the following that also in weakly-coupled theories with an elementary Higgs protection mechanisms for the EW scale can be implemented.

Let us look at the other fields in the SM, in particular to the fermions. Although the SM fermion masses span several orders of magnitude, they do not imply an additional tuning problem. The reason for that is the fact that, although small, the fermion masses are stable under radiative corrections, since these corrections are proportional to the masses themselves: small masses receive small loop corrections. This kind of feature is called *technical naturalness*.[4]

Let us give a closer look to the quantum corrections to the fermion masses

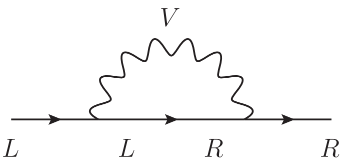


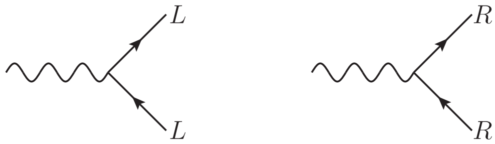
$$\Rightarrow \quad \Delta m_f \propto m_f \tag{1.2.1}$$

This result can be understood technically by looking at the chirality structure of the diagrams. A mass term has a L-R structure

$$m_f(\overline{f}_L f_R + \overline{f}_R f_L)\,. \tag{1.2.2}$$

Now we look at the diagram





$$\sim \frac{\not{p} + m_f}{p^2 - m_f^2}$$

To generate a LR term we need to use the $\frac{m_f}{p^2-m_f^2}$ part of the propagator, so the final result is necessarily proportional to m_f.

There is an equivalent (and more elegant) way to see the same thing. In the limit of $m_f = 0$ the theory has an enhanced symmetry: chiral invariance, i.e. independent U(1) rotations of f_L and f_R. The two chiralities do not talk to each other if $m_f = 0$, so no mass can be generated, even at loop level.

A third way to see this is by considering the number of degrees of freedom. A massless chiral fermion, eg. f_L, is a Weyl spinor and has 2 degrees of freedom. A massive Dirac fermion has 4 degrees of freedom. So if we start with only f_L we do not have enough degrees of freedom to create a massive spinor, so no mass can be radiatively generated, no matter the loop order.

[4]Notice that, even if the hierarchy in the fermion Yukawa's is not associated to a tuning, it is still a "mysterious" feature of the SM, which seems to need some explanation beyond the SM. However, due to the technical naturalness, this explanation could come from a UV dynamics present at energy scales much higher than the EW scale. On the contrary, the stabilization of the Higgs mass, being associated with a tuning, requires an explanation around the EW scale. In the absence of such explanation, the Higgs potential is necessarily tuned and a different *class* of theories, for instance *landscape* scenarios with many possible vacua must be considered.

A question raises naturally: can we use this mechanism to protect the Higgs mass as well? Unfortunately this can not be done directly. Real scalars always have one degree of freedom, irrespective of the fact that they are massless or massive. However there is a way to circumvent this limitation: we could try to build a symmetry that relates fermions and scalars, in such a way that they need to have the same mass. In this case, since the fermion masses are technically natural, the fermion–scalar symmetry would transfer the protection to the scalar mass as well.

1.3 Looking for a fermion-boson symmetry

Imagining a symmetry between bosons and fermions is not easy. For instance an internal symmetry can not be used for this purpose. Let us understand why.

An obstruction in building such a symmetry is the fact that the spin is a quantum number determined by the Poincaré invariance. Any symmetry that commutes with Poincaré, such as the internal symmetries, can not change the spin, so it can not connect bosons and fermions. We need something that does not commute with Poincaré.

There is also a stronger obstruction to build a symmetry between bosons and fermions: the *Coleman–Mandula theorem.*

Theorem (*Coleman–Mandula*, 1967). The only possible *Lie algebra* of symmetry generators consists of

- Poincaré invariance,
- "internal" global symmetries, that commute with the Poincaré group and act on physical states by multiplying them with spin- and momentum-independent Hermitian matrices.

This is true under the assumptions:

i) for any M there are only a finite number of particle types with mass less than M,

ii) the S matrix is non-trivial: any two-particle state undergoes some reaction at almost all energies (except perhaps an isolated set),

iii) the ampitudes for elastic two-body scattering are analytic functions of the scattering angle at almost all energies and angles.

A proof of the Coleman–Mandula theorem can be found in ref. [3] (chapter 24, appendix B).[5]

Notice that symmetries that connect particles with different spin can be present in non-relativistic theories. For example an SU(6) symmetry of this kind was proposed for the non-relativistic quark model (see ref. [3], chapter 24).

In order to evade a theorem, we can always try to weaken its assumptions. Most if the assumptions of the Coleman–Mandula theorem seem quite necessary for any sensible physical theory, eg. the non-triviality of the S matrix, or the analytic properties. Also the assumption on the spectrum seems reasonable, at least for what we know experimentally.

However, the assumption of restricting our analysis only to Lie algebras could be seen as not fully justified. And indeed if we weaken it we can find a more general group of symmetries. In particular if we allow for "anticommuting" generators, as well as the usual "commuting" ones, we can get an extended symmetry known as *supersymmetry* (or SUSY). Supersymmetry is defined

[5] In theories with *only* massless particles the most general Lie algebra of symmetries is given by the *conformal group* plus internal symmetries.

by the introduction of anticommuting symmetry generators that transform in the $(1/2, 0)$ and $(0, 1/2)$ (i.e. spinor) representations of the Lorentz group.[6]

Since the new generators are spinors (as opposed to scalars) they do not commute with the Lorentz generators. So SUSY is *not* an internal symmetry, but rather it is an extension of the Poincaré spacetime symmetries.

Later on, in 1975, *Haag, Lopuszański and Sohnius* proved that supersymmetry is the only additional symmetry of the S-matrix allowed by the weakened assumptions on the Coleman–Mandula theorem. So we can really consider SUSY as the only possible extension of the known spacetime symmetries.

1.4 Supersymmetry and the cancellation of divergences

Let us see how SUSY can help us with the hierarchy problem. SUSY connects bosons and fermions, so let us add some scalars that play the role of "partners" of the fermions we considered in the Higgs loop. We consider two scalars ϕ_L and ϕ_R described by

$$\mathcal{L}_S = -\frac{\lambda}{2} H^2(|\phi_L|^2 + |\phi_R|^2) - H(\mu_L|\phi_L|^2 + \mu_R|\phi_R|^2) - m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2 \,. \tag{1.4.1}$$

We get some new contribution to the Higgs self-energy, shown in the following diagrams

The contribution of these diagrams is

$$\Delta m_H^2\big|_a = \frac{\lambda}{16\pi^2}\left[2\Lambda_{\rm UV}^2 - m_L^2 \log\left(\frac{\Lambda_{\rm UV}^2 + m_L^2}{m_L^2}\right) - m_R^2 \log\left(\frac{\Lambda_{\rm UV}^2 + m_R^2}{m_R^2}\right) + \cdots\right] , \tag{1.4.2}$$

$$\Delta m_H^2\big|_b = -\frac{1}{16\pi^2}\left[\mu_L^2 \log\left(\frac{\Lambda_{\rm UV}^2 + m_L^2}{m_L^2}\right) + \mu_R^2 \log\left(\frac{\Lambda_{\rm UV}^2 + m_R^2}{m_R^2}\right) + \cdots\right] . \tag{1.4.3}$$

If we choose

$$\lambda = |\lambda_f|^2 \tag{1.4.4}$$

we cancel the quadratic divergence in Δm_H^2 from the fermion loop. Moreover if we choose

$$\begin{cases} m_L = m_R = m_f \\ \mu_L^2 = \mu_R^2 = 2\lambda m_f^2 \end{cases} \tag{1.4.5}$$

we also cancel the logarithmic pieces in the Higgs mass corrections.

Of course if we impose these conditions "by hand", we are just using a fine-tuning to eliminate the corrections to the Higgs mass. So the hierarchy problem is not solved. On the other hand, if we have a symmetry that relates the parameters of the fermionic and bosonic Lagrangian, then we really solve the hierarchy problem. We will see that SUSY implies exactly the relations in eqs. (1.4.4) and (1.4.5), so it is a good candidate to solve the hierarchy problem.

[6] We could think of anticommuting generators as "fermionic" generators, in contrast with the "commuting" generators which can be interpreted as bosonic.

1.5 Other phenomenological considerations

Another intriguing result regarding SUSY is the fact that in minimal SUSY extensions of the SM (the MSSM) one can naturally get the unification of the gauge couplings.

In the SM the SU(3), SU(2)$_L$ and U(1)$_Y$ gauge couplings run with the energy and at $E \sim 10^{14}$ GeV their strengths become comparable, although they do not fully coincide. This could be suggestive of a possible "unification" of the forces in a larger group.

In the MSSM the running is modified by the new states (with typical masses $m \sim$ TeV) and the couplings become of the same strength at the same energy $E \sim 2 \times 10^{16}$ GeV. The running of the gauge couplings in the SM and in the MSSM is shown in figure 1.2.

We will discuss better this aspect of the MSSM in section 13.

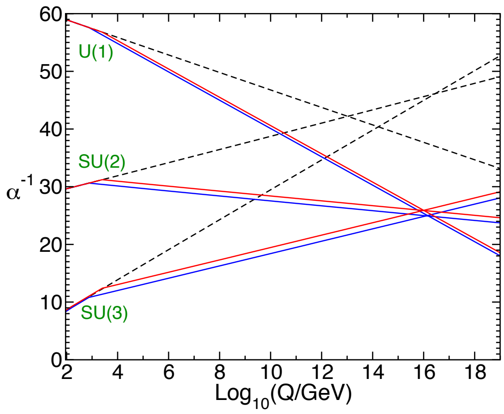


Figure 1.2: SM running couplings (from ref. [6]).

1.6 Theoretical relevance of supersymmetry

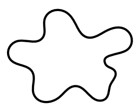
The importance of SUSY is not limited to its possible phenomenological relevance. It is also a valuable theoretical tool with a wide spectrum of applications.

1.6.1 String theory

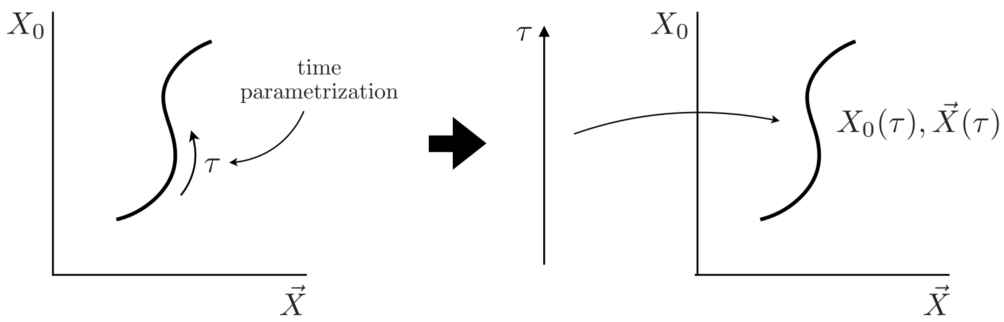
SUSY is an essential ingredient for building consistent string theories involving fermionic degrees of freedom.

String theory describes the dynamics of one-dimensional objects

open string 

closed string 

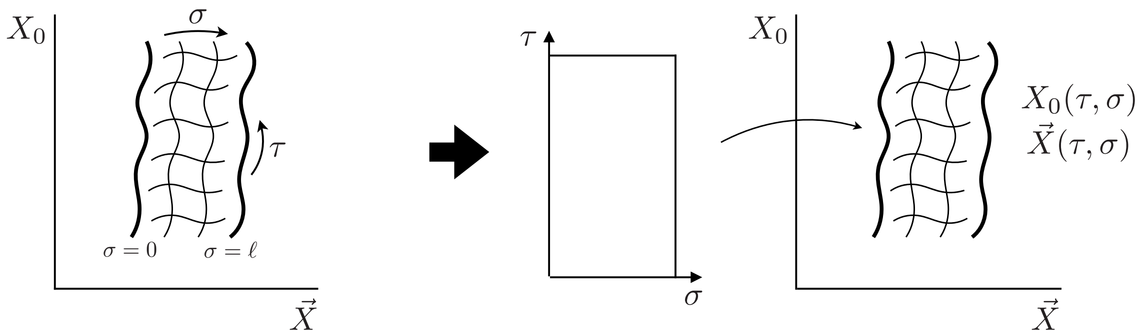
The action for a string can be constructed in analogy to the one for point particles. For a point particle we can use the following procedure



the action is

$$S_{pp} = \int d\tau \, \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} \,. \tag{1.6.1}$$

For a string we just add a coordinate along the string



The bosonic string action is ($\mu = 0, 1, \ldots, d-1$)

$$I[X] = \frac{T}{2} \int d\sigma \int d\tau \, \eta_{\mu\nu} \left[\frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} - \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\nu}{\partial \sigma} \right] \tag{1.6.2}$$

$$= T \int d\sigma^+ \int d\sigma^- \, \eta_{\mu\nu} \frac{X^\mu}{\partial \sigma^+} \frac{X^\nu}{\partial \sigma^-} \,, \tag{1.6.3}$$

where T is the *string tension* and

$$\sigma^\pm = \tau \pm \sigma \,. \tag{1.6.4}$$

We can also introduce fermions $\psi_1^\mu(\sigma, \tau)$ and $\psi_2^\mu(\sigma, \tau)$ in the action

$$I[X, \psi] = \int d\sigma^+ \int d\sigma^- \left[T \frac{\partial X^\mu}{\partial \sigma^+} \frac{\partial X^\nu}{\partial \sigma^-} \eta_{\mu\nu} + i\psi_2^\mu \frac{\partial}{\partial \sigma^+} \psi_{2\,\mu} + i\psi_1^\mu \frac{\partial}{\partial \sigma^-} \psi_{1\,\mu} \right] \,. \tag{1.6.5}$$

This action has an enhanced symmetry under the exchange of bosons and fermions (provided we choose the appropriate boundary conditions):

$$\begin{cases} \delta\psi_2^\mu(\sigma^+, \sigma^-) = iT\alpha_2(\sigma^-) \dfrac{\partial}{\partial \sigma^-} X^\mu(\sigma^+, \sigma^-) \\ \delta\psi_1^\mu(\sigma^+, \sigma^-) = iT\alpha_1(\sigma^+) \dfrac{\partial}{\partial \sigma^+} X^\mu(\sigma^+, \sigma^-) \\ \delta X^\mu(\sigma^+, \sigma^-) = \alpha_2(\sigma^-)\psi_2^\mu(\sigma^+, \sigma^-) + \alpha_1(\sigma^+)\psi_1^\mu(\sigma^+, \sigma^-) \end{cases} , \tag{1.6.6}$$

where $\alpha_{1,2}$ are infinitesimal *fermionic* functions (analogous to Grassmann variables). This is a *superconformal symmetry* for the 2d theory.

It can be shown that SUSY in strings can also be promoted to a 10-dimensional spacetime supersymmetry.

1.6.2 Amplitudes computations

Although is could seem that the presence of SUSY invariance makes a theory more complex, it is actually true the opposite: it makes a theory simpler! The reason is the fact that a theory with a larger symmetry group is much more constrained than a theory with less symmetry. So we can use the additional constraints to find relations among physical quantities and simplify our computations.

For instance we already saw that SUSY can be used to remove the dependence on the cut-off in Δm_H^2. This means that SUSY removes the divergences (at least in some cases) in loop computations. We will see that this is a quite generic feature of SUSY, namely the fact that rediative corrections get more constrained and more under control.

The fact that the loop computations get simpler has been extensively used to explore high-loop structures of QFT amplitudes. SUSY in some cases allows to get full all-loop results and even exact non-perturbative results.

Furthermore SUSY computations can also be used as building blocks or approximations for computations in non-SUSY theories. For instance several QCD amplitudes can be recast in an equivalent SUSY form, which simplifies the computation.

2

The Poincaré group

In this section we will describe the Poincaré invariance and its physical consequences. The Poincaré group contains the Lorentz group and the translations. Single-particle states are identified with irreducible representations of the Lorentz group.

2.1 The Lorentz algebra

The Lorentz group is defined as the group of transformations that leave the Minkowski metric invariant. In this course we will adopt the following convention for the spacetime metric

$$\eta_{\mu\nu} = (+1\,,\ -1\,,\ -1\,,\ -1)\,. \tag{2.1.1}$$

The Lorentz group, also denoted as $\mathrm{SO}(3,1)$, is a non-compact group with 6 generators

- 3 rotations: $J_i \quad i = 1, 2, 3\,,$
- 3 boosts: $K_i \quad i = 1, 2, 3\,.$

Their commutation relations are given by

$$\begin{cases} [J_i, J_j] = i\varepsilon_{ijk} J_k \\ [K_i, K_j] = -i\varepsilon_{ijk} J_k \\ [J_i, K_j] = i\varepsilon_{ijk} K_k \end{cases} . \tag{2.1.2}$$

In order to study the Lorentz group representations it is useful to introduce the following linear combinations of the rotations and boost generators

$$S_i \equiv \frac{1}{2}(J_i + iK_i)\,, \qquad T_i \equiv \frac{1}{2}(J_i - iK_i)\,. \tag{2.1.3}$$

These combinations satisfy the following commutation rules

$$[S_i, S_j] = i\varepsilon_{ijk} S_k\,, \qquad [T_i, T_j] = i\varepsilon_{ijk} T_k\,, \qquad [S_i, T_j] = 0\,. \tag{2.1.4}$$

The first two commutation rules correspond to the ones of the generators of the $\mathrm{SU}(2)$ algebra. However S_i and T_i are not Hermitian, instead $S_i^\dagger = T_i$, so that they define a *complexified version* of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ group. Nevertheless we can still classify the Lorentz group representations in terms of the representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

The Lorentz group is also equivalent to $\mathrm{Sl}(2, \mathbb{C})$, which is the universal cover of $\mathrm{SO}(3,1)$, and can be identified with the 2×2 complex matrices of unit determinant. To prove the equivalence we can just rewrite the spacetime coordinates X^μ in a matrix notation

$$x^\mu \to x^\mu \sigma_\mu\,, \tag{2.1.5}$$

where $\sigma_0 = 1_{2\times 2}$ and σ_i are the Pauli matrices. The determinant of $x^\mu\sigma_\mu$ coincides with the norm of the x^μ vector

$$\det(x^\mu\sigma_\mu) = x^\mu x_\mu\,. \tag{2.1.6}$$

The Lorentz transformations leave $x^\mu x_\mu$ invariant, so they are given by the transformations that leave $\det(x^\mu\sigma_\mu)$ unchanged, or explicitly

$$x^\mu\sigma_\mu \to A x^\mu\sigma_\mu A^\dagger\,, \tag{2.1.7}$$

where A is a complex 2×2 matrix with unit determinant (up to an irrelevant phase), that is an element of $\mathrm{Sl}(2,\mathbb{C})$.

2.2 The Poincaré algebra

If we add translations to the Lorentz transformations we get the Poincaré group. The spacetime translations generators are denoted by P_μ and they satisfy the additional commutation relations

$$\begin{aligned} &[P_\mu, P_\nu] = 0\,, \\ &[J_i, P_j] = i\varepsilon_{ijk}P_k\,, \qquad &&[J_i, P_0] = 0\,, \\ &[K_i, P_j] = i\delta_{ij}P_0\,, \qquad &&[K_i, P_0] = -iP_i\,. \end{aligned} \tag{2.2.1}$$

The Poincaré algebra can also be rewritten in a convariant form. We define the combinations $M_{\mu\nu}$ (which satisfy $M_{\mu\nu} = -M_{\nu\mu}$)

$$M_{0i} = K_i\,, \qquad M_{ij} = \varepsilon_{ijk}J_k\,. \tag{2.2.2}$$

The Poincaré algebra reads

$$\begin{cases} [P_\mu, P_\nu] = 0 \\ [M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} \\ [M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu \end{cases}. \tag{2.2.3}$$

The Poincaré group has two Casimirs

$$P^2 \equiv P_\mu P^\mu \tag{2.2.4}$$

and

$$W^2 \equiv W^\mu W_\mu\,, \tag{2.2.5}$$

where W^μ is the Pauli-Lubanski vector

$$W^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}\,. \tag{2.2.6}$$

The irreducible representations have definite values of P^2 and W^2. In particular, $P^2 = m^2$ gives the mass of the particles, while W^2 gives the spin for the massive particles and W^μ is related to the helicity in the massless case.

2.3 2-components spinors

The representations of the Lorentz group can be labeled by the corresponding representations of SU(2) × SU(2), namely they are identified by a pair

$$(a, b)$$

where a and b denote the representation under the first and second SU(2) subgroup and can take integer or half-integer values. The sum $a+b$ coincides with the spin (or helicity for massless states) of the representation. The bosonic representations thus coincide with the ones in which $a+b$ is integer, while the fermionic ones correspond to $a+b$ half-integer. In particular the spin-1/2 fermions are given by the representations

$$\begin{aligned} &\left(\frac{1}{2}, 0\right) \quad \Rightarrow \quad \text{“left-handed” Weyl fermion}\,, \\ &\left(0, \frac{1}{2}\right) \quad \Rightarrow \quad \text{“right-handed” Weyl fermion}\,. \end{aligned} \tag{2.3.1}$$

A *spinor* is an object with two complex components[1]

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{2.3.2}$$

The $(1/2, 0)$ representation of the Lorentz generators is simply given by the Pauli matrices, $J_i = \sigma_i/2$ and $K_i = -i\sigma_i/2$. Hence a generic element of the Lorentz group in this representation can be written as

$$\mathcal{M} = \exp\left(-i\frac{\vec{\theta}\vec{\sigma}}{2} - \frac{\vec{\zeta}\vec{\sigma}}{2}\right) , \tag{2.3.3}$$

and acts on the ψ spinor as

$$\psi_\alpha \to \psi'_\alpha = \mathcal{M}_\alpha{}^\beta \psi_\beta \,. \tag{2.3.4}$$

Notice that we denoted by α and β the spinor indices. They take the values $\alpha, \beta = 1, 2$.

The complex conjugate of the $(1/2, 0)$ representation gives an independent representation, the $(0, 1/2)$, not equivalent to the previous one. We denote a two-component object in this representation by a bar and a dotted index[2], namely $\bar{\psi}_{\dot{\alpha}}$. Notice that, by convention, the values of the dotted indices are also represented with a dot, $\dot{\alpha} = \dot{1}, \dot{2}$. The Lorentz generators in the $(0, 1/2)$ representation are given by $J_i = -\sigma_i^*/2$ and $K_i = -i\sigma_i^*/2$, thus a generic transformation takes the form

$$\mathcal{M}^* = \exp\left(-i\frac{\vec{\theta}\vec{\sigma}^*}{2} - \frac{\vec{\zeta}\vec{\sigma}^*}{2}\right) . \tag{2.3.5}$$

And the corresponding transformation rule for the $\bar{\psi}_{\dot{\alpha}}$ spinor is

$$\bar{\psi}_{\dot{\alpha}} \to \bar{\psi}'_{\dot{\alpha}} = \mathcal{M}^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \,. \tag{2.3.6}$$

Notice that, by convention, the $\mathcal{M}^*$ matrix carries dotted indices, since they correspond to the $(0, 1/2)$ representation. It is easy to see that we can identify

$$\bar{\psi}_{\dot{\alpha}} \equiv (\psi_\alpha)^\dagger \,. \tag{2.3.7}$$

[1] For a review of the two-component spinor formalism see ref. [12].

[2] In the literature the dotted spinors are alternatively denoted by a † and not a bar, namely $\bar{\psi}_{\dot{\alpha}} \equiv \psi^\dagger_{\dot{\alpha}}$.

One can easily see that the spin-1/2 fermions can equivalently be defined as the basic representations of $\mathrm{Sl}(2,\mathbb{C})$. The $\mathcal{M}$ and $\mathcal{M}^*$ matrices, in fact, are the most general complex 2×2 matrices with unit determinant. Therefore they provide the most general element of the $\mathrm{Sl}(2,\mathbb{C})$ group and correspond to the complexified SU(2) subgroups of the Lorentz group.

An important thing to remember is the fact that spinors are *anticommuting* objects, thus if we exchange two fermions in an expression we must change the sign. For example

$$\psi_1\chi_2 = -\chi_2\psi_1\,, \qquad \psi_1\bar{\chi}_{\dot{2}} = -\bar{\chi}_{\dot{2}}\psi_1\,. \tag{2.3.8}$$

2.4 Notations and conventions

In these lecture notes we will follow the notation of ref. [7]. We introduce the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{2.4.1}$$

These objects are used to rise and lower spinor indices

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta\,, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta\,, \qquad \bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}\,, \qquad \bar{\psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}\,. \tag{2.4.2}$$

This definition is consistent since

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \varepsilon^{\gamma\beta}\varepsilon_{\beta\alpha} = \delta_\alpha^\gamma\,, \qquad \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\gamma}} = \varepsilon^{\dot{\gamma}\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\alpha}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}\,. \tag{2.4.3}$$

As a convention, repeated indices contracted like

$${}^{\alpha}{}_{\alpha} \qquad \textit{undotted}\text{: from upper left to lower right}$$

$${}_{\dot{\alpha}}{}^{\dot{\alpha}} \qquad \textit{dotted}\text{: from lower left to upper right}$$

can be suppressed (eg. $\psi^\alpha\chi_\alpha \equiv \psi\chi$, $\bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \equiv \bar{\psi}\bar{\chi}$). Applying these rules we can see that

$$\psi\chi = \psi^\alpha\chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\varepsilon^{\alpha\beta}\chi_\alpha\psi_\beta = \varepsilon^{\beta\alpha}\chi_\alpha\psi_\beta = \chi^\beta\psi_\beta = \chi\psi\,, \tag{2.4.4}$$

and analogously

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}\,. \tag{2.4.5}$$

Moreover we get

$$(\psi\chi)^\dagger = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}\,. \tag{2.4.6}$$

The role of the γ^μ matrices used in the 4-component spinor notation is played in the 2-component notation by the σ^μ matrices

$$\begin{aligned} (\sigma^\mu)_{\alpha\dot{\alpha}} &= (1, -\sigma_i)_{\alpha\dot{\alpha}}\,, \\ (\overline{\sigma}^\mu)^{\dot{\alpha}\alpha} &= \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} = (1, \sigma_i)^{\dot{\alpha}\alpha}\,. \end{aligned} \tag{2.4.7}$$

By using these objects we can construct spinor bilinears

$$\psi\sigma^\mu\bar{\chi} \equiv \psi^\alpha\sigma^\mu{}_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}}\,, \qquad \bar{\psi}\bar{\sigma}^\mu\chi \equiv \bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\chi_\beta\,. \tag{2.4.8}$$

One can also check the following identities

$$\bar{\psi}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\psi} = (\bar{\chi}\bar{\sigma}^\mu\psi)^* = -(\psi\sigma^\mu\bar{\chi})^*\,, \tag{2.4.9}$$

and

$$\psi\sigma^\mu\bar{\sigma}^\nu\chi = \chi\sigma^\nu\bar{\sigma}^\mu\psi = (\bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\psi})^* = (\bar{\psi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi})^* \,. \tag{2.4.10}$$

A Fierz rearrangement identity can also be proven

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\chi\xi)\,, \tag{2.4.11}$$

moreover the following identities are valid

$$\begin{aligned}
&\sigma^\mu_{\alpha\dot{\alpha}}\overline{\sigma}^{\dot{\beta}\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}}\,,\\
&\sigma^\mu_{\alpha\dot{\alpha}}\sigma_{\mu,\beta\dot{\beta}} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\,,\\
&\overline{\sigma}^{\mu,\dot{\alpha}\alpha}\overline{\sigma}^{\dot{\beta}\beta}_\mu = 2\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\,,\\
&[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2\eta^{\mu\nu}\delta^\beta_\alpha\,,\\
&[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot{\beta}}{}_{\dot{\alpha}} = 2\eta^{\mu\nu}\delta^{\dot{\beta}}_{\dot{\alpha}}\,,\\
&\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = \eta^{\mu\nu}\overline{\sigma}^\rho + \eta^{\nu\rho}\overline{\sigma}^\mu - \eta^{\mu\rho}\overline{\sigma}^\nu + i\epsilon^{\mu\nu\rho\lambda}\overline{\sigma}_\lambda\,,\\
&\sigma^\mu\overline{\sigma}^\nu\sigma^\rho = \eta^{\mu\nu}\sigma^\rho + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu - i\epsilon^{\mu\nu\rho\lambda}\sigma_\lambda\,,
\end{aligned} \tag{2.4.12}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\varepsilon^{0123} = +1$.

2.5 Dirac spinors

A Dirac spinor transforms as the reducible representation $(1/2, 0) \oplus (0, 1/2)$. It can be built from a dotted and an undotted spinor as

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}\,. \tag{2.5.1}$$

The Dirac matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,, \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix}\,. \tag{2.5.2}$$

The Dirac spinor is formed by a left-handed and a right-handed Weyl spinor

$$\begin{aligned}
P_L\Psi_D &= \frac{1+\gamma_5}{2}\Psi_D = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}\,,\\
P_R\Psi_D &= \frac{1-\gamma_5}{2}\Psi_D = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}\,.
\end{aligned} \tag{2.5.3}$$

From the 2-components spinors we can also construct a Majorana spinor

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}\,, \tag{2.5.4}$$

by setting $\chi = \psi$ in eq. (2.5.1).

3

The supersymmetry algebra

The Coleman–Mandula theorem severely restricts the type of symmetries that can be present in a relativistic theory. The most general Lie algebra of symmetries that is allowed in a physical theory is given by the Poincaré group and a group of internal symmetries whose generators B_a satisfy

$$\begin{cases} [B_a, B_b] = i f_{ab}{}^c B_c \\ [P_\mu, B_a] = 0 \\ [M_{\mu\nu}, B_a] = 0 \end{cases} . \tag{3.0.1}$$

However, by weakening one of the assumptions of the Coleman–Mandula theorem, we can find some more general symmetries compatible with a relativistic theory. This can be done by relaxing the assumption that the symmetry is described by a Lie group and by allowing some of the symmetry generators to be of “fermionic” nature, thus satisfying anticommutation relations instead of the usual commutation rules.

In 1975 Haag, Łopuszański and Sohnius proved that, with this weakened assumptions, supersymmetry is the only possible extension of the usual Poincaré plus internal symmetries algebra. Moreover, the structure of the new symmetry is almost unique.

Before discussing the SUSY algebra, it is useful to introduce the definition of a *graded Lie algebra*, which we can use to describe the structure of the SUSY generators.

Let us start with the definition of a Lie algebra:

> A Lie algebra is given by a vector space L (over a field like $\mathbb{R}$ or $\mathbb{C}$) with a composition rule
>
> $$[\,.\,,\,.\,]\;:\;L \times L \to L\,, \tag{3.0.2}$$
>
> which satisfies the properties ($v_1, v_2, v_3 \in L$)
>
> $$\begin{array}{rll} (i) & [v_1\,, v_2] \in L\,, & \\ (ii) & [v_1\,, (v_2 + v_3)] = [v_1\,, v_2] + [v_1\,, v_3] & \text{(linearity)}\,, \\ (iii) & [v_1\,, v_2] = -[v_1\,, v_2] & \text{(antisymmetry)}\,, \\ (iv) & [v_1\,, [v_2\,, v_3]] + [v_3\,, [v_1\,, v_2]] + [v_2\,, [v_3\,, v_1]] = 0 & \text{(Jacobi identity)}\,. \end{array} \tag{3.0.3}$$

Note. When we consider a matrix realization of a Lie algebra, we identify the $[\,.\,,\,.\,]$ operation with the commutator. For example the space of complex 2×2 traceless and anti-Hermitian matrices form the Lie algebra of $\mathrm{SU}(2, \mathbb{C})$, if we define

$$[a\,, b] \equiv ab - ba\,, \qquad a\,, b \in su(2, \mathbb{C})\,. \tag{3.0.4}$$

A basis for this algebra is given by $\tau_i = i\sigma_i/2$, where σ_i are the Pauli matrices.

Now we can define a $\mathbb{Z}_2$-graded Lie algebra.

A $\mathbb{Z}_2$-graded Lie algebra is a vector space L that is the direct sum of two subspaces

$$L = L_0 \oplus L_1 \,, \tag{3.0.5}$$

with a composition law $[\,.\,,\,.\,\}$ satisfying

$$[L_0\,, L_0\} \subset L_0\,, \qquad [L_0\,, L_1\} \subset L_1\,, \qquad [L_1\,, L_1\} \subset L_0\,. \tag{3.0.6}$$

The composition law $[\,.\,,\,.\,\}$ must satisfy also the properties

$$\begin{aligned}
&(i) \quad [L_i\,, L_j\} = (-1)^{ij}[L_j\,, L_i\} \qquad \text{(supersymmerization)}\\
&(ii) \quad (-1)^{ik}[L_i\,,[L_j\,, L_k\}\} + (-1)^{ji}[L_j\,,[L_i\,, L_k\}\} + (-1)^{kj}[L_k\,,[L_i\,, L_j\}\} = 0\\
&\qquad\qquad \text{(generalized Jacobi identities)}\,.
\end{aligned} \tag{3.0.7}$$

Notice that the subspace L_0, with the composition rule $[\,.\,,\,.\,\}$ forms an ordinary Lie algebra.

We can assign a degree to the elements of the algebra

$$\begin{aligned}
\eta(x_a) &= 0 \qquad \text{if } x_a \in L_0 \qquad \textit{even} \text{ or } \textit{bosonic}\,,\\
\eta(x_a) &= 1 \qquad \text{if } x_a \in L_1 \qquad \textit{odd} \text{ or } \textit{fermionic}\,.
\end{aligned} \tag{3.0.8}$$

We can now define $[\,.\,,\,.\,\}$ as

$$[x_i\,, x_j\} = x_i x_j - (-1)^{\eta_i \eta_j} x_j x_i\,, \tag{3.0.9}$$

where $\eta_{i,j} \equiv \eta(x_{i,j})$. This composition rule satisfies the properties for a graded Lie algebra. More explicitly we get

$$\begin{aligned}
&\textit{commutators} \left\{ \begin{aligned} &[x_i\,, x_j\} = x_i x_j - x_j x_i = [x_i\,, x_j] \qquad x_i\,, x_j \in L_0\,,\\ &[x_i\,, y_j\} = x_i y_j - y_j x_i = [x_i\,, y_j] \qquad x_i \in L_0\,, y_j \in L_1\,, \end{aligned}\right.\\
&\textit{anticommutators} \left\{ \; [y_i\,, y_j\} = y_i y_j + y_j y_i = \{y_i\,, y_j\} \qquad y_i\,, y_j \in L_1\,. \right.
\end{aligned} \tag{3.0.10}$$

3.1 The supersymmetry algebra

The supersymmetry algebra is a $\mathbb{Z}_2$-graded Lie algebra with

- a *bosonic sector* corresponding to - Poincaré P_μ, $M_{\mu\nu}$,
  - internal symmetries G,
- a *fermionic sector* with generators Q^I_α, $\overline{Q}^I_{\dot\alpha}$, $I = 1\,, \ldots\,, N$.

The commutation/anticommutation relations characterizing the supersymmetry algebra are the following. Their derivation will be discussed in sec. 3.2.

i) The anticommutation relations among the supersymmetry generators are

$$\{Q^I_\alpha\,, \overline{Q}^J_{\dot\alpha}\} = 2\delta^{IJ}(\sigma^\mu)_{\alpha\dot\alpha}P_\mu\,, \tag{3.1.1}$$

and

$$\{Q^I_\alpha\,, Q^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ}\,, \qquad \{\overline{Q}^I_{\dot\alpha}\,, \overline{Q}^J_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta}(Z^{IJ})^*\,. \tag{3.1.2}$$

The Z^{IJ} are operators that commute with all generators of the full algebra, for this reason they are called *central charges*. One can show that $Z^{IJ} = -Z^{JI}$, so the central charges vanish identically if the simple algebra with only one supersymmery generator ($N = 1$).

ii) The commutation relations with the Poincaré genarators are

$$\begin{cases} [Q^I_\alpha\,, P_\mu] = 0 \\ [\overline{Q}^{I\dot\alpha}\,, P_\mu] = 0 \end{cases}, \qquad \begin{cases} [M_{\mu\nu}\,, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \\ [M_{\mu\nu}\,, \overline{Q}^{I\dot\alpha}] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{I\dot\beta} \end{cases}, \tag{3.1.3}$$

where $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ are the Lorentz generators in the spinorial representation and are given by

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{1}{4}\left(\sigma^\mu_{\alpha\dot\gamma}\,\bar\sigma^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\,\bar\sigma^{\mu\dot\gamma\beta}\right), \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \equiv \frac{1}{4}\left(\bar\sigma^{\mu\dot\alpha\gamma}\,\sigma^\nu_{\gamma\dot\beta} - \bar\sigma^{\nu\dot\alpha\gamma}\,\sigma^\mu_{\gamma\dot\beta}\right). \tag{3.1.4}$$

iii) The commutators with the generators of the internal symmetries are

$$\begin{cases} [Q^I_\alpha\,, B_l] = S^I_{l\;J} Q^J_\alpha \\ [\overline{Q}^I_{\dot\alpha}\,, B_l] = -\overline{Q}^J_{\dot\alpha}(S^I_{l\;J})^* \end{cases}, \tag{3.1.5}$$

where S_a form a representation of the internal symmetry group, namely

$$[S_a, S_b] = i\, f_{ab}{}^c S_c\,, \tag{3.1.6}$$

where $f_{ab}{}^c$ are the structure constants of the internal symmetry group $[B_a, B_b] = i\, f_{ab}{}^c B_c$.

3.2 Derivation of the supersymmetry algebra

We now want to derive the commutation/anticommutation relations among the SUSY generators.

i) We start from the commutator

$$\{Q^I_\alpha\,, \overline{Q}^J_{\dot\alpha}\}\,. \tag{3.2.1}$$

If we evaluate the anticommutator on a state of the Hilbert space of physical states we get, for the $I = J$ case,

$$\langle u|\{Q^I_\alpha\,, \overline{Q}^I_{\dot\alpha}\}|u\rangle = |\overline{Q}^I_{\dot\alpha}|u\rangle|^2 + |Q^I_\alpha|u\rangle|^2 > 0 \qquad \text{if } Q^I_\alpha \neq 0\,. \tag{3.2.2}$$

To derive this relation we chose the generators so that $\bar Q = Q^\dagger$. This means that $\{Q^I_\alpha\,, \overline{Q}^I_{\dot\alpha}\}$ can not vanish if $Q^I_\alpha \neq 0$.

Now we assume that Q^I transforms in the (j, j') representation of the Lorentz group. In this case $\overline{Q}^I$ is in the (j', j) representation and $\{Q^I\,, \overline{Q}^I\}$ will contain the representation $(j+j', j+j')$. The latter is a bosonic representation, so it is part of the bosonic subalgebra which satisfies the Coleman–Mandula theorem. The only bosonic generator of this kind is P_μ, which transforms in the $(1/2, 1/2)$ representation. As a consequence all the Q's must transform in the $(1/2, 0)$ or $(0, 1/2)$ representation.

By imposing Lorentz invariance we get

$$\{Q^I_\alpha\,, \overline{Q}^J_{\dot\alpha}\} = 2\delta^{IJ}(\sigma^\mu)_{\alpha\dot\alpha}P_\mu\,, \tag{3.2.3}$$

where we used the freedom in the choice of the generators to fix the normalization and to get a δ^{IJ} matrix (see ref. [4]).

ii) The commutators of Q^I and $\overline{Q}^I$ with the Lorentz generators $M_{\mu\nu}$ are completely determined by the fact that, as we saw before,

$$
\begin{aligned}
Q^I &\sim (1/2, 0) \quad \text{representation}\,,\\
\overline{Q}^I &\sim (0, 1/2) \quad \text{representation}\,.
\end{aligned}
$$

To write the commutators we need to rewrite the Lorentz generators by using the 2-components notation. The usual Lorentz generators are

$$
\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}\,, \tag{3.2.4}
$$

where

$$
\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}\,. \tag{3.2.5}
$$

We see that the dotted and undotted spinors transform separately:

ψ_α has generators $i\sigma^{\mu\nu}$

$$
(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\,\bar{\sigma}^{\nu\dot{\gamma}\beta} - \sigma^\nu_{\alpha\dot{\gamma}}\,\bar{\sigma}^{\mu\dot{\gamma}\beta}\right)\,, \tag{3.2.6}
$$

$\bar{\psi}^{\dot{\alpha}}$ has generators $i\bar{\sigma}^{\mu\nu}$

$$
(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\gamma}\,\sigma^\nu_{\gamma\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\gamma}\,\sigma^\mu_{\gamma\dot{\beta}}\right)\,. \tag{3.2.7}
$$

The commutators of Q and $\overline{Q}$ with $M_{\mu\nu}$ are given by

$$
\begin{cases}
[M_{\mu\nu}\,, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \\
[M_{\mu\nu}\,, \overline{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{Q}^{I\dot{\beta}}
\end{cases}\,. \tag{3.2.8}
$$

iii) To complete the transformation properties of the Q's under the Poincaré group, we need to find the commutation rules with P_μ. In the following we will prove that

$$
\begin{cases}
[Q^I_\alpha\,, P_\mu] = 0 \\
[\overline{Q}^{I\dot{\alpha}}\,, P_\mu] = 0
\end{cases}\,. \tag{3.2.9}
$$

Let us prove this statements. The commutator of Q with P_μ can contain the representations $(1/2, 1/2) \otimes (1/2, 0) = (1, 1/2) \oplus (0, 1/2)$. There are no $(1, 1/2)$ generators, so we get

$$
[Q^I_\alpha\,, P_\mu] = C^I{}_J(\sigma_\mu)_{\alpha\dot{\beta}}\overline{Q}^{J\dot{\beta}}\,, \tag{3.2.10}
$$

and its adjoint

$$
[\overline{Q}^{I\dot{\alpha}}\,, P_\mu] = (C^I{}_J)^*(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}Q^J_\beta\,. \tag{3.2.11}
$$

We have

$$
[[Q^I_\alpha\,, P_\mu]\,, P_\nu] = C^I{}_J(C^J{}_K)^*(\sigma_\mu)_{\alpha\dot{\beta}}(\bar{\sigma}_\nu)^{\dot{\beta}\gamma}Q^K_\gamma\,. \tag{3.2.12}
$$

By using the Jacobi identity

$$
[[Q^I_\alpha\,, P_\mu]\,, P_\nu] + [[P_\mu\,, P_\nu]\,, Q^I_\alpha] + [[P_\nu\,, Q^I_\alpha]\,, P_\mu] = 0\,, \tag{3.2.13}
$$

we get

$$C^I{}_J(C^J{}_K)^*(\sigma_\mu\bar\sigma_\nu)_\alpha{}^\gamma Q^K_\gamma - C^I{}_J(C^J{}_K)^*(\sigma_\nu\bar\sigma_\mu)_\alpha{}^\gamma Q^K_\gamma = C^I{}_J(C^J{}_K)^*(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu)_\alpha Q^K_\gamma$$
$$= 4(CC^*)^I{}_K(\sigma_{\mu\nu})_\alpha{}^\gamma Q^K_\gamma = 0\,. \qquad (3.2.14)$$

Give that $\sigma_{\mu\nu}$ and Q are non-zero, we get

$$CC^* = 0\,. \qquad (3.2.15)$$

Let us now consider the most general form of $\{Q^I_\alpha\,,Q^J_\beta\}$ (see discussion in point (iv))

$$\{Q^I_\alpha\,,Q^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ} + \text{term symmetric in } \alpha\beta\,, \qquad (3.2.16)$$

where $Z^{IJ} = -Z^{JI}$ is in the $(0,0)$ Lorentz representation (i.e a Lorentz scalar). Then we get

$$\begin{aligned} 0 &= \varepsilon^{\alpha\beta}[\{Q^I_\alpha\,,Q^J_\beta\}\,,P_\mu] = \varepsilon^{\alpha\beta}\left(\{Q^I_\alpha\,,[Q^J_\beta\,,P_\mu]\} + \{Q^J_\beta\,,[Q^I_\alpha\,,P_\mu]\}\right) \\ &= \varepsilon^{\alpha\beta}\left(C^J{}_K(\sigma_\mu)_{\beta\dot\gamma}\{Q^I_\alpha\,,\overline{Q}^{K\dot\gamma}\} + C^I{}_K(\sigma_\mu)_{\alpha\dot\gamma}\{Q^J_\beta\,,\overline{Q}^{K\dot\gamma}\}\right) \\ &= \varepsilon^{\alpha\beta}\left(C^J{}_K(\sigma_\mu)_{\beta\dot\gamma}\varepsilon^{\dot\gamma\dot\delta}2\delta^{JK}(\sigma^\rho)_{\alpha\dot\delta}P_\rho + C^I{}_K(\sigma_\mu)_{\alpha\dot\gamma}\varepsilon^{\dot\gamma\dot\delta}2\delta^{JK}(\sigma^\rho)_{\beta\dot\delta}P_\rho\right) \\ &= (C^J{}_I - C^I{}_J)(\sigma_\mu)_{\beta\dot\gamma}(\sigma^\rho)_{\alpha\dot\delta}\varepsilon^{\alpha\beta}\varepsilon^{\dot\gamma\dot\delta}P_\rho\,, \end{aligned} \qquad (3.2.17)$$

which follows from the fact that Z^{IJ} is a combination of the bosonic internal generators, which commute with P_μ. From the above relation we obtain

$$C^I{}_J = C^J{}_I\,, \qquad (3.2.18)$$

and therefore (from eq. (3.2.15) we get the result

$$CC^\dagger = 0 \qquad \Rightarrow \qquad C = 0\,. \qquad (3.2.19)$$

This completes the proof.

iv) We can now consider the $\{Q^I_\alpha\,,Q^J_\beta\}$ anticommutator. It must be a linear combination of the representations

$$(1/2,0)\otimes(1/2,0) = (0,0)\oplus(1,0)\,. \qquad (3.2.20)$$

The only Lorentz generator in the $(1,0)$ representation is $M_{\mu\nu}$, but it does not commute with P_μ, while

$$[\{Q^I_\alpha\,,Q^J_\beta\}\,,P_\mu] = 0\,, \qquad (3.2.21)$$

which follows from $[Q^I_\alpha\,,P_\mu] = 0$. Thus we are left with

$$\{Q^I_\alpha\,,Q^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ}\,, \qquad (3.2.22)$$

with Z^{IJ} a linear combination of the internal symmetry generators

$$Z^{IJ} = a^{rIJ}B_r\,, \qquad (3.2.23)$$

that satisfy $Z^{IJ} = -Z^{JI}$. As we will prove in the following

$$[Z^{IJ}\,,\text{any generator}] = 0\,, \qquad (3.2.24)$$

which means that the Z^{IJ} form an Abelian subalgebra. For this reason the Z^{IJ} are called *central charges*.

The central charges commute with the momentum generators P_μ because of the Coleman–Mandula theorem: Z^{IJ} are bosonic generators in the $(0,0)$ representation of the Lorentz group, thus they are a linear combination of internal generators and

$$[P_\mu\,, Z^{IJ}] = [P_\mu\,, (Z^{IJ})^*] = 0\,. \tag{3.2.25}$$

By using the Jacobi identity for Q^I_α, Q^J_β and $\overline{Q}^K_{\dot\gamma}$ we easily get

$$[\overline{Q}^K_{\dot\gamma}\,, Z^{IJ}] = 0\,, \tag{3.2.26}$$

and analogously

$$[Q^K_\gamma\,, (Z^{IJ})^*] = 0\,. \tag{3.2.27}$$

Let us now consider the Jacobi identity for Z^{IJ}, Q^K_α and $\overline{Q}^L_{\dot\beta}$:

$$-\cancel{[Z^{IJ}\,, \{Q^K_\alpha\,, \overline{Q}^L_{\dot\beta}\}]} + \{\overline{Q}^L_{\dot\beta}\,, [Z^{IJ}\,, Q^K_\alpha]\} - \cancel{\{Q^K_\alpha\,, [\overline{Q}^L_{\dot\beta}\,, Z^{IJ}]\}} = 0\,, \tag{3.2.28}$$

which implies

$$\{\overline{Q}^L_{\dot\beta}\,, [Z^{IJ}\,, Q^K_\alpha]\} = 0\,. \tag{3.2.29}$$

The most general form of the commutator between Z^{IJ} and Q^K_α is

$$[Z^{IJ}\,, Q^K_\alpha] = \sum_L M^{IJKL} Q^L_\alpha\,, \tag{3.2.30}$$

and substituting it in eq. (3.2.29) we get

$$0 = \{\overline{Q}^L_{\dot\beta}\,, \sum_N M^{IJKN} Q^N_\alpha\} = \sum_N 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{LN} = 2M^{IJKL}\sigma^\mu_{\alpha\dot\beta} P_\mu\,. \tag{3.2.31}$$

This result implies that $M^{IJKL} = 0$, hence

$$[Z^{IJ}\,, Q^K_\alpha] = 0\,. \tag{3.2.32}$$

Analogously one can prove that $[(Z^{IJ})^*, \overline{Q}^K_{\dot\alpha}] = 0$. By using the Jacobi identity among Z^{IJ}, Q^K_α and Q^L_β we obtain

$$[Z^{IJ}\,, Z^{KL}] = 0\,, \tag{3.2.33}$$

and analogously for the complex conjugate relation $[(Z^{IJ})^*\,, (Z^{KL})^*] = 0$. Moreover from the Jacobi identity involving Z^{IJ}, $\overline{Q}^K_{\dot\alpha}$ and $\overline{Q}^L_{\dot\beta}$

$$[Z^{IJ}\,, (Z^{KL})^*] = 0\,. \tag{3.2.34}$$

To prove that the central charges commute also with the internal generators we need first of all to derive the interplay between the supersymmetry and the internal symmetries.

v) Finally we must find the commutation relations of the supersymmetry generators with the generators of the internal symmetries. Given that the internal symmetry generators are in the $(0,0)$ Lorentz representation we can write

$$\begin{cases} [Q^I_\alpha\,, B_l] = S^I_{l\ J} Q^J_\alpha \\ [\overline{Q}^I_{\dot\alpha}\,, B_l] = -\overline{Q}^J_{\dot\alpha}(S^I_{l\ J})^* \end{cases} . \tag{3.2.35}$$

By considering the Jacobi identity with B_l, Q^I_α and $\overline{Q}^J_{\dot\beta}$ we get

$$\begin{aligned} 0 &= [B_l\,, \{Q^I_\alpha\,, \overline{Q}^J_{\dot\beta}\}] + \{Q^I_\alpha\,, [\overline{Q}^J_{\dot\beta}\,, B_l]\} - \{\overline{Q}^J_{\dot\beta}\,, [B_l\,, Q^I_\alpha]\} \\ &= 2\sigma^\mu_{\alpha\dot\beta}\delta^{IJ}\cancel{[B_l\,, P_\mu]} + \{Q^I\alpha\,, -\overline{Q}^K_{\dot\beta}(S^J_{l\ K})^*\} + \{\overline{Q}^J_{\dot\beta}\,, S^I_{l\ K}Q^K_\alpha\} \\ &= -(S^J_{l\ K})^* 2\sigma^\mu_{\alpha\dot\beta}\delta^{IK}P_\mu + S^I_{l\ K}2\sigma^\mu_{\alpha\dot\beta}\delta^{JK}P_\mu\,, \end{aligned} \tag{3.2.36}$$

from which we obtain

$$\left(S^I_{l\ J} - (S^j_{l\ I})^*\right) 2\sigma^\mu_{\alpha\dot\beta}P_\mu = 0\,. \tag{3.2.37}$$

This implies

$$S^I_{l\ J} = (S^I_{JI})^* = S^{\dagger\ J}_{jI} \qquad \Rightarrow \qquad S_l = S^\dagger_l\,. \tag{3.2.38}$$

From the above relations we get

$$\begin{cases} [Q^I_\alpha\,, B_l] = S^I_{l\ J} Q^J_\alpha \\ [\overline{Q}^I_{\dot\alpha}\,, B_l] = -\overline{Q}^J_{\dot\alpha}S_{lJ}{}^I \end{cases} . \tag{3.2.39}$$

From the Jacobi identity with B_a, B_b and Q^I_α one can prove that

$$[S_a\,, S_b] = i f_{ab}{}^c\, S_c\,, \tag{3.2.40}$$

where $f_{ab}{}^c$ are the structure constants of the internal symmetry group, namely

$$[B_a\,, B_b] = i f_{ab}{}^c\, B_c\,. \tag{3.2.41}$$

This means that the S_a form a representation of the internal symmetry algebra.

Now we can also prove that the central charges Z^{IJ} commute with the internal symmetry generators. We start from the Jacobi identity

$$\begin{aligned} 0 &= \{Q^I_\alpha\,, [Q^J_\beta\,, B_l\} - \{Q^J_\beta\,, [B_L\,, Q^I_\alpha]\} + [B_l\,, \{Q^I_\alpha\,, Q^J_\beta\}] \\ &= S^I_{l\ K}\{Q^I_\alpha\,, Q^K_\beta\} + S^I_{l\ K}\{Q^J_\beta\,, Q^K_\alpha\} + [B_l\,, \varepsilon_{\alpha\beta}Z^{IJ}] \\ &= \varepsilon_{\alpha\beta}\left([B_l\,, Z^{IJ}] + S^J_{l\ K}Z^{IJ} - S^I_{l\ K}Z^{JK}\right)\,, \end{aligned} \tag{3.2.42}$$

from which we get

$$[B_l\,, Z^{IJ}] = S^J_{l\ K}Z^{KI} - S^I_{l\ K}Z^{KJ}\,. \tag{3.2.43}$$

Together with the previous result $[Z^{IJ}\,, Z^{KL}] = 0$, we get that the Z^{IJ} form an invariant Abelian subalgebra of the bosonic symmetry algebra.

The Coleman–Mandula theorem states that the complete bosonic internal symmetries are given by the direct product of a semi-simple Lie algebra and several U(1) algebras. The only invariant Abelian subalgebras of such a Lie algebra are spanned by U(1) generators, so the Z^{IJ} must be U(1) generators and must commute with the B_l:

$$[B_l\,, Z^{IJ}] = 0\,. \tag{3.2.44}$$

Note. Although the central charges commute with all the generators, they are not just numbers. They are operators that can take different values on different physical states. Notice that on a supersymmetric vacuum the central charges vanish

$$\varepsilon_{\alpha\beta} Z^{IJ}|0\rangle = \{Q^I_\alpha\,, Q^J_\beta\}|0\rangle = 0 \tag{3.2.45}$$

because $Q^K_\gamma|0\rangle = 0$ for a supersymmetric vacuum.

3.3 The R-symmetry

In the absence of central charges the supersymmetry algebra is invariant under a group $\mathrm{U}(N)$ of internal symmetries, where N is the number of supersymmetry generators Q^I_α:

$$Q^I_\alpha \to \sum_J V^I{}_J Q^J_\alpha\,, \tag{3.3.1}$$

with $V^I{}_J$ an $N \times N$ unitary matrix.

A supersymmetry algebra with $N > 1$ is called an *N-extended supersymmetry*, while for $N = 1$ we have a *simple supersymmetry.* If we have just one fermionic generator Q_α ($N = 1$), we can not have central charges as a consequence of the relation

$$Z^{IJ} = -Z^{JI}\,. \tag{3.3.2}$$

In this case we have a simplified form of the supersymmetry algebra

$$\begin{cases} \{Q_\alpha\,, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \\ \{Q_\alpha\,, Q_\beta\} = \{\overline{Q}_{\dot\alpha}\,, \overline{Q}_{\dot\beta}\} = 0 \end{cases}. \tag{3.3.3}$$

In this case the R-symmetry is just a U(1) symmetry

$$Q_\alpha \to \exp(i\phi) Q_\alpha\,, \tag{3.3.4}$$

with ϕ a real phase.

3.4 Properties of the supersymmetry algebra

In this subsection we will discuss a few basic consequence of the supersymmetry algebra.

3.4.1 Positivity of the energy

In a supersymmetric theory the energy P_0 is always positive. To see this we consider any state $|\Phi\rangle$, then, by the positivity of the Hilbert space, we have

$$\begin{aligned} 0 &\le |Q^I_\alpha|\Phi\rangle|^2 + |(Q^I_\alpha)^\dagger|\Phi\rangle|^2 \\ &= \langle\Phi|(Q^I_\alpha)^\dagger Q^I_\alpha + Q^I_\alpha (Q^I_\alpha)^\dagger|\Phi\rangle = \langle\Phi|\{Q^I_\alpha \overline{Q}^I_{\dot\alpha}\}|\Phi\rangle \\ &= 2\sigma^\mu_{\alpha\dot\alpha}\langle\Phi|P_\mu|\Phi\rangle\,, \end{aligned} \tag{3.4.1}$$

where we used the fact that

$$(Q^I_\alpha)^\dagger = \overline{Q}^I_{\dot\alpha}\,. \tag{3.4.2}$$

Taking the trace over the α and $\dot\alpha$ indices, and using $\operatorname{tr}\sigma^\mu = 2\delta^{\mu 0}$, we get

$$0 \leq 4\langle\Phi|P_0|\Phi\rangle\,, \tag{3.4.3}$$

which proves the positivity of the energy on any physical state.

Notice that the relation

$$\{Q^I_\alpha\,,\overline{Q}^J_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu \tag{3.4.4}$$

has another important consequence. If the vacuum state of the theory $|0\rangle$ is invariant under supersymmetry, that is if supersymmetry is not spontaneously broken, then

$$Q^I_\alpha|0\rangle = \overline{Q}^I_{\dot\alpha}|0\rangle = 0 \tag{3.4.5}$$

thus

$$\langle 0|P_0|0\rangle = 0\,, \tag{3.4.6}$$

hence a supersymmetric vacuum has zero energy.

3.4.2 Casimirs of the supersymmetry algebra

It is easy to prove that P^2 is still a Casimir of the supersymmetry algebra. On the other hand W^2 does not commute with the supersymmetry generators, so it is not a Casimir of the supersymmetry algebra:

$$[W^2\,,Q] \neq 0\,. \tag{3.4.7}$$

This implies that all the states in a representation will have the same mass, but not the same spin.

4

Representations of the supersymmetry algebra

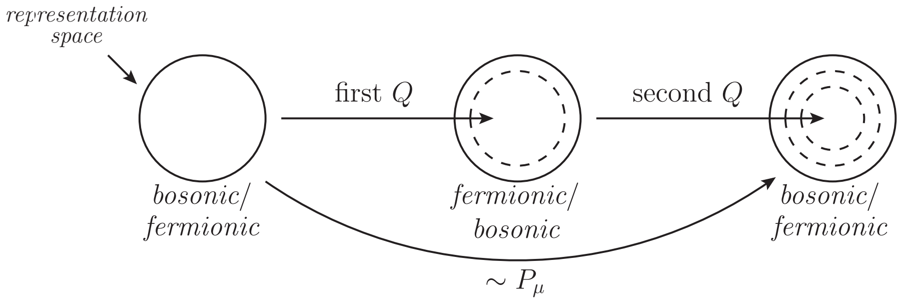
In this section we will discuss the representations of the supersymmetry algebra.

4.1 The "fermions" = "bosons" rule

Before discussing the supersymmetry representations it is useful to derive an important property of supersymmetry multiplets:

> A supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom.
>
> Proof. Let the fermion number be N_f equal to 1 on a fermionic state and 0 on a bosonic state, so that $(-1)^{N_f}$ is $+1$ for a boson and -1 for a fermion. We need to show that
>
> $$\mathrm{Tr}(-1)^{N_f} = 0 \tag{4.1.1}$$
>
> on a finite-dimensional representation of the SUSY algebra. From the fact the $(-1)^{N_f}$ anticommutes with Q, and using the cyclicity of the trace, one has
>
> $$\begin{aligned} 0 &= \mathrm{Tr}\left(-Q_\alpha(-1)^{N_f}\overline{Q}_{\dot\beta} + (-1)^{N_f}\overline{Q}_{\dot\beta}Q_\alpha\right) = \mathrm{Tr}\left((-1)^{N_f}\{Q_\alpha\,,\overline{Q}_{\dot\beta}\}\right) \\ &= 2\sigma^\mu_{\alpha\dot\beta}\,\mathrm{Tr}\left((-1)^{N_f}P_\mu\right)\ . \end{aligned} \tag{4.1.2}$$
>
> in any representation in which P_μ is non-zero, we get the wanted result.

We can also understand the "fermion" = "boson" rule pictorially. The anticommutators $\{Q\,,\overline{Q}\}$ are a combination of two mappings on the representation space



But the anticommutator $\{Q_\alpha\,, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu$ means that if P_μ maps the representation space onto itself, this implies that the bosonic and fermionics subspaces must have the same dimension and Q must map one onto the other.

For a large class of representations P_μ gives a map onto the representation space itself. So the "fermions" = "bosons" rule is verified. In particular this is true on quantum fields on which the momentum P_μ is the generator of translations and it is represented by the derivatives $P_\mu \sim i\partial_\mu$.[1]

In the following we will discuss the irreducible representations of the SUSY algebra on single particle states. This means that we will consider representations on asymptotic on-shell physical states. SUSY representations on quantum fields will be discussed in section 5.

4.2 Massless supermultiplets

We start by considering the representations for massless supermultiplets. Since $P^2 = m^2$ is a Casimir operator for the SUSY algebra, all the particles in a supermultiplet have the same mass. For massless states we can choose a reference frame with

$$P_\mu = (E, 0, 0, E)\,, \tag{4.2.1}$$

which, of course, satisfy $P_\mu P^\mu = 0$. In this frame we get

$$\{Q^I_\alpha\,, \overline{Q}^j_{\dot\beta}\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\beta} \delta^{IJ}\,. \tag{4.2.2}$$

In particular we have that

$$\{Q^I_1\,, \overline{Q}^J_{\dot 1}\} = 0 \qquad\quad \forall\; I, J\,. \tag{4.2.3}$$

This has an important consequence: on a positive definite Hilbert space we must set

$$Q^I_1 = \overline{Q}^I_{\dot 1} = 0 \qquad\quad \forall\; I\,, \tag{4.2.4}$$

as can be seen from

$$0 = \langle\Phi|\{Q^I_1\,, \overline{Q}^J_{\dot 1}\}|\Phi\rangle = |Q^I_1|\Phi\rangle|^2 + |\overline{Q}^I_{\dot 1}|\Phi\rangle|^2 \quad\Rightarrow\quad Q^I_1 = \overline{Q}^I_{\dot 1} = 0 \qquad \forall\; I\,. \tag{4.2.5}$$

Thus we are left with only N fermionic generators:

$$Q^I_2 \qquad \text{and} \qquad \overline{Q}^I_{\dot 2}\,. \tag{4.2.6}$$

We can rescale them as

$$a_I \equiv \frac{1}{\sqrt{4E}} Q^I_2\,, \qquad\quad a^\dagger_I \equiv \frac{1}{\sqrt{4E}} \overline{Q}^I_{\dot 2}\,, \tag{4.2.7}$$

in this case a_I and $a^\dagger_I$ are anticommuting annihilation and creation operators

$$\{a_I\,, a^\dagger_J\} = \delta_{IJ}\,, \qquad\quad \{a_I\,, a_J\} = \{a^\dagger_I\,, a^\dagger_J\} = 0\,. \tag{4.2.8}$$

We can construct a supermultiplet by acting with the Q^I_2 and $\overline{Q}^I_{\dot 2}$ on one of its states. Given that Q^I and $\overline{Q}^I$ commute with P_μ, all the states in a multiplet have the same P_μ.

[1]The only states for which the "fermions"="bosons" rule can be violated are the vacua $|0\rangle$ for which $P_\mu|0\rangle = 0$.

The building blocks to construct the supermultiplet are the massless representations of the Poincaré group, which are characterized by $P^2 = 0$ and by some helicity λ. The commutation relations of the helicity operator, which in the frame we chose is $J_3 = M_{12}$, with the Q^I_2 and $\overline{Q}^I_{\dot{2}}$

$$
\begin{aligned}
[M_{12}\,, Q^I_2] &= -\frac{1}{2} Q^I_2\,, \\
[M_{12}\,, \overline{Q}^J_{\dot{2}}] &= \frac{1}{2} \overline{Q}^J_{\dot{2}}\,,
\end{aligned}
\tag{4.2.9}
$$

tell us that Q^I_2 lowers the helicity by 1/2 and $\overline{Q}^I_{\dot{2}}$ reises it by 1/2:

$$
\begin{aligned}
Q^I_2|\lambda\rangle &= |\lambda - 1/2\rangle\,, \\
\overline{Q}^I_{\dot{2}}|\lambda\rangle &= |\lambda + 1/2\rangle\,.
\end{aligned}
\tag{4.2.10}
$$

To construct the supermultiplet we start from the state with lowest helicity $|\lambda_0\rangle$, which is annihilated by all the a_I (this state is called the *Clifford vacuum*)

$$
a_I|\lambda_0\rangle = 0\,. \tag{4.2.11}
$$

We assume that $|\lambda_0\rangle$ is a singlet of the $\mathrm{SU}(N)$ symmetry which acts on the I and J indices. The other states is in the supermultiplet can be be obtained by acting with $a^\dagger_I$ on $|\lambda_0\rangle$:

$$
\begin{aligned}
&|\lambda_0\rangle \\
a^\dagger_I|\lambda_0\rangle &= |\lambda_0 + 1/2\rangle_I \\
a^\dagger_I a^\dagger_J|\lambda_0\rangle &= |\lambda_0 + 1\rangle_{IJ} \\
&\vdots \\
a^\dagger_1 a^\dagger_2 \cdots a^\dagger_N|\lambda_0\rangle &= |\lambda_0 + N/2\rangle\,.
\end{aligned}
\tag{4.2.12}
$$

Due to antisymmetry in $I, J, \ldots$ there are $\binom{n}{k}$ states with helicity $\lambda = \lambda_0 + k/2$, $k = 0, 1, \ldots, N$. In total a supermultiplet contains 2^N states:

$$
\left.\begin{aligned}
&2^{N-1} \text{ bosons} \\
&2^{N-1} \text{ fermions}
\end{aligned}\right\} \; 2^N \text{ states}\,.
$$

In general in such supermultiplet, except if $\lambda_0 = -N/4$, the helicities will not be distributed symmetrically around zero. Such supermultiplets can not be invariant under CPT, since CPT flips the sign of the helicity. To satisfy CPT we then need to double these multiplets by adding their CPT conjugate with opposite helicities and opposite quantum numbers.

4.2.1 Simple supersymmetry $N = 1$

For simple supersymmetry each massless supermultiplet contains two states

$$
|\lambda_0\rangle\,, \qquad |\lambda_0 + 1/2\rangle\,. \tag{4.2.13}
$$

They can never be CPT self-conjugate, so we need to double them . We have the following possibilities

- *chiral multiplet*: $\lambda_0 = 0$, so we have the helicity states $(0, 1/2) \oplus (-1/2, 0)$, i.e. a Weyl fermion and a complex scalar;

- *vector multiplet*: $\lambda_0 = 1/2$, so we have the helicity states $(1/2, 1) \oplus (-1, -1/2)$, i.e. a massless vector and a Weyl fermion;

- *gravitino multiplet*: $\lambda_0 = 1$, so we have the helicity states $(1, 3/2) \oplus (-3/2, -1)$, i.e. a gravitino and a massless vector;

- *graviton multiplet*: $\lambda_0 = 3/2$, so we have the helicity states $(3/2, 2) \oplus (-2, -3/2)$, i.e. a graviton and a gravitino.

Massless particles with spin greater than 2 can not be consistently included in an interacting theory.[2] Thus the only allowed massless supermultiplets are the ones listed before. In a renormalizable theory without gravity we can have only chiral and vector supermultiplets. If we also consider gravity (getting a so-called supergravity theory) we also need the supermultiplets with higher helicity.

4.2.2 Extended supersymmetry $N > 1$

Because the Q_1^I and $\overline{Q}_1^I$ all vanish when applied to the states of a massless supermultiplet (including the states obtained by acting with Q_2^I and $\overline{Q}_2^I$ on any state of the multiplet), the central charges Z^{IJ} must also vanish on any state of the multiplet. This is the reason for which we did not include them in the previous discussion.

The algebra of the N raising operators $a_I^\dagger$ is invariant under an $\mathrm{SU}(N)$ R-symmetry. This implies that the states of given helicity in a supermultiplet form a representation of $\mathrm{SU}(N)$, namely the rank-n antisymmetric tensor representation (given that the $a_I^\dagger$ anticommute).

Now we briefly discuss the most relevant supermultiplets in the $N = 2$, $N = 4$ and $N = 8$ cases.

- $N = 2$

  There are two multiplets for global supersymmetry:

  - *vector multiplet*, which contains:
    - a gauge boson (massless vector) with helicity $+1$;
    - two fermions of helicity $+1/2$, which form a doublet under the $\mathrm{SU}(2)$ R-symmetry;
    - one boson of helicity 0.

    To get a CPT-invariant multiplet we must also add the conjugate multiplet with reversed helicities.

  - *hypermultiplet*, which contains
    - one fermion of each helicity $\pm 1/2$;
    - an $\mathrm{SU}(2)$ doublet of bosons with helicity 0.

    To have a CPT-invariant multiplet in a quantum field theory, we must add the conjugate multiplet. This is required because otherwise the scalars would be just two real scalar fields, which can not form an $\mathrm{SU}(2)$ doublet.

- $N = 4$

  If we are interested in global supersymmetry, we can use only one $N = 4$ supermultiplet. It is CPT self-conjugate and contains

[2] The reason for this is the fact that there are no conserved currents that can be coupled to massless particles with spin greater than 2.

- 1 massless vector with helicities ± 1;
- 4 fermions of each helicity $\pm 1/2$;
- 6 bosons of helicity 0.

- $N = 8$

  In this case there is only one multiplet with helicities $|\lambda| \leq 2$. It is CPT self-conjugate and contains

  - 1 graviton with helicities ± 2;
  - 8 gravitinos with helicities $\pm 3/2$;
  - 28 massless gauge bosons with helicities ± 1;
  - 56 fermions with helicities $\pm 1/2$;
  - 70 bosons with helicity 0.

  This means that $N = 8$ supersymmetry is necessarily necessarily a theory of gravity (so-called supergravity) and we can not build a theory with only global supersymmetry.

Note. The supermultiplets in the $N = 3$ and $N = 7$ extended supersymmetries, when CPT invariance is taken into account, have exactly the same particle content as the $N =$ and $N = 8$ supermultiplets respectively.

4.2.3 Chiral fermions

In simple SUSY we can have chiral fermions by using the supermultiplets containing just helicity $+1/2$ and 0. These can be in a complex representation of the gauge group, distinct from the representation of the CPT-conjugate supermultiplet.

On the other hand, in all the supermultiplets of extended SUSY (except for hypermultiplets of $N = 2$) fermions of spin 1/2 are always in multiplets which contain gauge bosons. This means that they are in the adjoint representation of the gauge group, which is a real representation. Thus they can not be chiral, which would require them to be in a complex representation.

In the SM the fermions are in chiral representations of the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge group, so extended SUSY is in conflict with the chiral nature of quarks and leptons.

The only possible exception could be the hypermultiplet in $N = 2$. However, also this supermultiplet can not give chiral fermions. Each multiplet contains fermions of helicity $+1/2$ and $-1/2$, therefore they must transform in the same way under gauge transformations that leave the supersymmetry generators invariant. They may belong to a complex representation, but then the CPT conjugate of this hypermultiplet would be in the complex-conjugate representation, and in that case the sum of the two representations would give again a real representation without chiral fermions.

4.3 Massive supermultiplets

Let us now discuss the massive supermultiplets. The building blocks to construct massive SUSY representations are the usual massive Lorentz representations, which are characterized by a certain mass $P^2 = m^2$ and a given spin J.

For massive particles we can choose the rest frame

$$P_\mu = (m, 0, 0, 0) \tag{4.3.1}$$

as a reference frame to build the supermultiplets.

First of all we will discuss the simple supersymmetry case and then we will consider the extended supersymmetry scenarios.

4.3.1 Simple supersymmetry $N = 1$

In this case the SUSY algebra becomes

$$\{Q_\alpha \,, \overline{Q}_{\dot\beta}\} = 2m\sigma^0_{\alpha\dot\beta} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{4.3.2}$$

This shows that none of the generators vanishes on the representation, so we have two pairs of raising and lowering operators. The Q_α and $\overline{Q}_{\dot\beta}$ operators act on massive Lorentz representations with a given spin j giving new states with spin $j \pm 1/2$

$$\begin{aligned} Q_\alpha|j\rangle \quad &\Rightarrow \quad |j+1/2\rangle \quad \text{and} \quad |j-1/2\rangle \,, \\ \overline{Q}_{\dot\alpha}|j\rangle \quad &\Rightarrow \quad |j+1/2\rangle \quad \text{and} \quad |j-1/2\rangle \,. \end{aligned} \tag{4.3.3}$$

Obviously if $j = 0$ we only get $Q_\alpha|0\rangle \Rightarrow |1/2\rangle$ and analogously for $\overline{Q}_{\dot\alpha}$.

We can define normalized raising and lowering operators as

$$\begin{cases} a_\alpha \equiv \dfrac{1}{\sqrt{2m}} Q_\alpha \\ a^\dagger_\alpha \equiv \dfrac{1}{\sqrt{2m}} \overline{Q}_{\dot\alpha} \end{cases} , \tag{4.3.4}$$

which satisfy the anticommutation relations

$$\begin{cases} \{a_\alpha \,, a^\dagger_\beta\} = \delta_{\alpha\beta} \\ \{a_\alpha \,, a_\beta\} = \{a^\dagger_\alpha \,, a^\dagger_\beta\} = 0 \end{cases} . \tag{4.3.5}$$

To build the representations we start from the Clifford vacuum $|\Omega\rangle$ which is defined by

$$a_\alpha|\Omega\rangle = 0 \qquad\qquad \alpha = 1, 2 \,. \tag{4.3.6}$$

Using the algebra of the a_α generators one can show that such a state always exists in a representation.

In this case $|\Omega\rangle$ is a massive Lorentz representation, this means that it is a state with a given spin j and has degeneracy $2j + 1$ since j_3 takes the values $-j \,, \dots \,, j$. Starting from Clifford vacua with different j we find different SUSY representations.

For a given $|\Omega\rangle$ the full massive SUSY representation is

$$\begin{gathered} |\Omega\rangle \\ a^\dagger_1|\Omega\rangle \\ a^\dagger_2|\Omega\rangle \\ \frac{1}{\sqrt{2}} a^\dagger_1 a^\dagger_2|\Omega\rangle = -\frac{1}{\sqrt{2}} a^\dagger_2 a^\dagger_1|\Omega\rangle \,. \end{gathered} \tag{4.3.7}$$

There are e total of $4(2j + 1)$ states in the representation. The spin of the states is

$$\begin{aligned} &|\Omega\rangle \,, \ \frac{1}{\sqrt{2}} a^\dagger_1 a^\dagger_2|\Omega\rangle \quad \Rightarrow \quad \text{spin } j \,, \\ &a^\dagger_1|\Omega\rangle \,, \ a^\dagger_2|\Omega\rangle \qquad\quad \Rightarrow \quad \text{spin } j + 1/2 \text{ and } j - 1/2 \quad (j - 1/2 \text{ is there only for } j \geq 1/2) \,. \end{aligned} \tag{4.3.8}$$

To prove that $\frac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega\rangle$ has spin j we can use the equivalent representation

$$\frac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega\rangle = \frac{1}{2\sqrt{2}}\varepsilon_{\alpha\beta}(a^\gamma)^\dagger a_\gamma^\dagger|\Omega\rangle\,. \tag{4.3.9}$$

One can show that the $(a^\gamma)^\dagger a_\gamma^\dagger$ operator is rotationally invariant, so it has spin zero. By applying it to $|\Omega\rangle$ we again get a state of spin j.

- If the Clifford vacuum has spin zero ($j = 0$) the supermultiplet is given by
  - two states of spin 0 (bosons) ,
  - one state of spin 1/2 (fermion) .
- If the Clifford vacuum has spin $j > 0$ the supermultiplet is given by
  - two states of spin j ,
  - one state of spin $j - 1/2$,
  - one state of spin $j + 1/2$.

4.3.2 Extended supersymmetry $N > 1$

For extended SUSY we have two possible situations depending on whether the central charges vanish or not.

Vanishing central charges

This situation is quite similar to the simple SUSY case. We have $2N$ pairs of rising and lowering operators

$$\begin{cases} a_\alpha^I = \frac{1}{\sqrt{2m}}Q_\alpha^I \\ (a_\alpha^I)^\dagger = \frac{1}{\sqrt{2m}}\overline{Q}_{\dot\alpha}^I \end{cases}, \tag{4.3.10}$$

which satisfy the anticommutation relations

$$\begin{cases} \{a_\alpha^I\,, (a_\beta^J)^\dagger\} = \delta_{\alpha\beta}\delta^{IJ} \\ \{a_\alpha^I\,, a_\beta^J\} = \{(a_\alpha^I)^\dagger\,, (a_\beta^J)^\dagger\} = 0 \end{cases}. \tag{4.3.11}$$

This means that we have N copies of the operators of simple SUSY and we can generate the representations in a similar way. Starting from a Clifford vacuum with spin j we get a representation with $2^{2N}(2j+1)$ states.

The algebra of generators in this case exhibits an $\mathrm{SU}(2)\times\mathrm{USp}(2N)$ symmetry. This can be seen be defining the new set of operators

$$\begin{cases} q_\alpha^l = a_\alpha^l \\ q_\alpha^{N+l} = \displaystyle\sum_{\beta+1}^{2}\varepsilon_{\alpha\beta}(a_\beta^l)^\dagger \end{cases} \qquad l = 1,\dots,N\,. \tag{4.3.12}$$

Under Hermitian conjugation

$$(q_\alpha^r)^\dagger = \varepsilon^{\alpha\beta}\Lambda^{rt}q_\beta^t\,, \tag{4.3.13}$$

where $r, t = 1,\dots,2N$ and

$$\Lambda = \begin{pmatrix} 0_{N\times N} & 1_{N\times N} \\ 1_{N\times N} & 0_{N\times N} \end{pmatrix}\,. \tag{4.3.14}$$

The anticommutation relations of the q's are

$$\{q^r_\alpha\,, q^t_\beta\} = -\varepsilon_{\alpha\beta}\Lambda^{rt}\,. \tag{4.3.15}$$

This shows the invariance under SU(2) (which acts on the α and β indices) and USp($2N$) (which acts on the r and t indices). This invariance group is useful because states of a given spin form irreducible representations of USp($2N$).[3]

Non-vanishing central charges

Now we consider the case in which the central charges are non-zero. The SUSY algebra with $P_\mu = (m, 0, 0, 0)$ is

$$\begin{cases} \{q^I_\alpha\,, (Q^J_\beta)^\dagger\} = 2m\delta_{\alpha\beta}\delta^{IJ} \\ \{Q^I_\alpha\,, Q^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ} \\ \{(Q^I_\alpha)^\dagger\,, (Q^J_\beta)^\dagger\} = \varepsilon_{\alpha\beta}(Z^{IJ})^* \end{cases}\,. \tag{4.3.16}$$

To satisfy the algebra we can use an appropriate U(N) rotation of the Q's which brings the antisymmetric Z^{IJ} matrix into a standard form

$$Z^{IJ} = \begin{pmatrix} \begin{matrix} 0 & q_1 \\ -q_1 & 0 \end{matrix} & 0_{2\times 2} & \cdots \\ 0_{2\times 2} & \begin{matrix} 0 & q_2 \\ -q_2 & 0 \end{matrix} & \\ & \vdots & \end{pmatrix} \qquad \text{with } q_N \geq 0\,, \tag{4.3.17}$$

with a last row and column of zeroes in the case of odd N. If N is odd the N-th generator satisfies

$$\begin{cases} \{Q^N_\alpha\,, (Q^N_\beta)^\dagger\} = 2m\delta_{\alpha\beta} \\ \{Q^N_\alpha\,, Q^N_\beta\} = \{(Q^N_\alpha)^\dagger\,, (Q^N_\beta)^\dagger\} = 0 \end{cases}\,. \tag{4.3.18}$$

So we have a pair of raising and lowering operators

$$\begin{cases} a^N_\alpha = \dfrac{1}{\sqrt{2m}}Q^N_\alpha \\ (a^N_\alpha)^\dagger = \dfrac{1}{\sqrt{2m}}(Q^N_\alpha)^\dagger \end{cases}\,. \tag{4.3.19}$$

These can be treated as in the case of vanishing central charges, so in the following we will focus on the case with even N. In this case we can define $2N$ pairs of raising and lowering operators

$$\begin{cases} a^L_\alpha = \dfrac{1}{\sqrt{2}}\left(Q^{2L-1}_\alpha + \varepsilon_{\alpha\beta}\overline{Q}^{2L\dot\beta}\right) \\ b^L_\alpha = \dfrac{1}{\sqrt{2}}\left(Q^{2L-1}_\alpha - \varepsilon_{\alpha\beta}\overline{Q}^{2L\dot\beta}\right) \end{cases} \qquad L = 1, \ldots, N/2\,, \tag{4.3.20}$$

and their Hermitian conjugates $(a^L_\alpha)^\dagger$, $(b^L_\alpha)^\dagger$.

Notice that the combination of Lorentz indices we chose in eq. (4.3.20) is weird, but the important point is that Q_α and $\overline{Q}^{\dot\alpha}$ transform in the same way under spatial rotations. This means that $(a^L_\alpha)^\dagger$ and $(b^L_\alpha)^\dagger$ create states of definite spin. This is consistent with the fact that

[3]The SU(2) × USp($2N$) symmetry is actually a subgroup of a larger symmetry group of the algebra: SU(2) × USp($2N$) ⊂ SO($4N$) (see for example ref. [1]).

we are already in a specific reference frame (the rest frame $P_\mu = (m,0,0,0)$), which obviously breaks the invariance under boosts but preserves the invariance under spatial rotations. We will discuss better this point at the end of section 4.3.3.

The anticommutation relations for the a^L_α and b^L_α are

$$\begin{aligned}
\{a^r_\alpha\,,(a^s_\beta)^\dagger\} &= (2m - q_r)\delta_{rs}\delta_{\alpha\beta}\,,\\
\{b^r_\alpha\,,(b^s_\beta)^\dagger\} &= (2m + q_r)\delta_{rs}\delta_{\alpha\beta}\,,\\
\{a^r_\alpha\,,(b^s_\beta)^\dagger\} &= \{a^r_\alpha\,,a^s_\beta\} = \cdots = 0\,.
\end{aligned} \tag{4.3.21}$$

Positivity of the Hilbert space implies that

$$2m \geq |q_n| \qquad \forall n\,. \tag{4.3.22}$$

Notice that this relation provides another proof of the fact that for massless representations all the central charges must be trivially represented, that is they must vanish on the representation states.

I some (or all) of the q_n saturate the bound, i.e. $|q_n| = 2m$, then the corresponding operators must be set to zero, as we did in the massless case with the Q^I_1.

When $2m > |q_n|$ for all n, the multiplicities of the massive irreducible representations are the same as for the case of no central charges. A multiplet will contain $2^{2N}(2j+1)$ states.

When same of the bounds are saturated we loose some of the creation operators. If r central charges saturate the bound we are left with a Clifford algebra of $2(N-r)$ raising and lowering operators. The corresponding representations are similar to the ones for the case without central charges and with N reduced by r.

When no central charge bound is saturated we get multiplets called *long multiplets*. Instead, if some of the bounds are saturated we have a *short multiplet*.

As an example let us compare the long and short representations for the $N=2$ case. For $q < 2m$ we have the long multiplet with $|\Omega\rangle$ of spin 0

spin	spin reps.	number of states
0	5	5
1/2	4	8
1	1	3

for $q = 2m$ we have the short multiplet

spin	spin reps.	number of states
0	2	2
1/2	1	2
1	0	0

As one can check, the short multiplet has the same number of states as the $N=2$ massless hypermultiplet.

For a Clifford vacuum with $j = 1/2$ we get the long multiplet

spin	spin reps.	number of states
0	4	4
1/2	6	12
1	4	12
3/2	1	4

for the short multiplet

spin	spin reps.	number of states
0	1	1
1/2	2	4
1	1	3
3/2	0	0

Again the short multiplet has the same degrees of freedom as the massless $N = 2$ vector multiplet.

The states that saturate the bounds and lead to short multiplets are also called BPS-states. This is because of their analogy with BPS monopoles in gauge theories (BPS stands for Bogomolnyi, Prasad and Sommerfeld).

The states that saturate some BPS bounds are also called *supersymmetric states*. This is due to the fact that they are invariant under a part of the SUSY algebra, namely for the operators that saturate the bounds $(a^I_\alpha)^\dagger|BPS\rangle = 0$. For example a state which saturates all the bounds preserves half of the supersymmetry.

A short multiplet is stable under radiative corrections. This means that its mass can not be changed by radiative corrections. The reason is simple: if the BPS bound is not satisfied any more, the multiplet should have more states than the short multiplet, but this is not possible because (small) perturbations can not change a discrete quantity like the number of states in a multiplet.

This is a strong property that follows from the fact that we related a physical quantity (the mass) to the symmetry algebra of the theory. Moreover this is an example of the protection mechanisms that are provided by supersymmetry.

4.3.3 The USp(2N) group

We have seen that in the case of massive representations without central charges we can define the following set of operators

$$\begin{cases} q^l_\alpha = a^l_\alpha \\ q^{N+l}_\alpha = \displaystyle\sum_{\beta=1}^{2} \varepsilon_{\alpha\beta}(a^l_\beta)^\dagger \end{cases} \qquad l = 1, \ldots, N\,, \tag{4.3.23}$$

which satisfy the relations

$$(q^r_\alpha)^\dagger = \varepsilon^{\alpha\beta}\Lambda^{rt}q^t_\beta\,, \tag{4.3.24}$$

and

$$\{q^r_\alpha\,, q^t_\beta\} = -\varepsilon_{\alpha\beta}\Lambda^{rt}\,, \tag{4.3.25}$$

where

$$\Lambda = \begin{pmatrix} 0_{N\times N} & 1_{N\times N} \\ -1_{N\times N} & 0_{N\times N} \end{pmatrix}\,. \tag{4.3.26}$$

One can easily check that taken $U_{\alpha\beta} \in$ SU(2), the algebra is invariant under

$$q^r_\alpha \to U_{\alpha\beta}q^r_\beta \equiv q'^r_\alpha\,. \tag{4.3.27}$$

This follows from

$$\begin{aligned} \{q'^r_\alpha\,, q'^t_\beta\} &= \{U_{\alpha\gamma}q^r_\gamma\,, U_{\beta\delta}q^t_\delta\} = -U_{\alpha\gamma}U_{\beta\delta}\varepsilon_{\gamma\delta}\Lambda^{rt} \\ &= -U_{\alpha\gamma}\varepsilon_{\gamma\delta}(U^T)_{\delta\beta}\Lambda^{rt} = -(U\varepsilon U^T)_{\alpha\beta}\Lambda^{rt} = -\varepsilon_{\alpha\beta}\Lambda^{rt}\,, \end{aligned} \tag{4.3.28}$$

where we used that $\varepsilon = i\sigma^2$ and $\sigma^2U^T = U^\dagger\sigma^2$.

Now we investigate what kind of invariance we have for the r and t indices. We can consider a transformation

$$q^r_\alpha \to q'^r_\alpha \equiv S^{rt} q^t_\alpha \,. \tag{4.3.29}$$

The algebra changes as

$$\{q'^r_\alpha \,, q'^t_\beta\} = \{S^{rm} q^m_\alpha \,, S^{tn} q^n_\beta\} = -\varepsilon_{\alpha\beta} S^{rm} S^{tn} \Lambda^{mn} = -\varepsilon_{\alpha\beta} (S \Lambda S^T)^{rt} \,, \tag{4.3.30}$$

to be invariant we need

$$S \Lambda S^T = \Lambda \,, \tag{4.3.31}$$

which is the definition of the symplectic group Sp($2N$). But we must also respect the relation in eq. (4.3.24), which imposes a "reality" condition which leads to the group USp($2N$). To see this we apply the Sp($2N$) transformation to eq. (4.3.24)

$$(S^{rn} q^n_\alpha)^\dagger = \varepsilon^{\alpha\beta} \Lambda^{rt} S^{tm} q^m_\beta \,. \tag{4.3.32}$$

From this we get

$$(S^{\dagger T})^{rn} (q^n_\alpha)^\dagger = \varepsilon^{\alpha\beta} \Lambda^{rt} S^{tn} q^m_\beta \tag{4.3.33}$$

$$\Rightarrow \quad (q^n_\alpha)^\dagger = ((S^{-1})^{\dagger T})^{nr} \varepsilon^{\alpha\beta} \Lambda^{rt} S^{tm} q^m_\beta = ((S^{-1})^{\dagger T} \Lambda S)^{nm} \varepsilon^{\alpha\beta} q^m_\beta \,. \tag{4.3.34}$$

To reproduce eq. (4.3.24) we need

$$\Lambda = (S^{-1})^{\dagger T} \Lambda S \,. \tag{4.3.35}$$

Using the relation $\Lambda^{-1} = -\Lambda$, we can manipulate the constraint $S \Lambda S^T = \Lambda$:

$$- S \Lambda^{-1} S^T = -\Lambda^{-1} \ \Rightarrow \ (S^{-1})^T \Lambda S^{-1} = \Lambda \ \Rightarrow \ \Lambda = S^T \Lambda S \,. \tag{4.3.36}$$

Comparing this equation with eq. (4.3.35) we get

$$S^T = (S^{-1})^{\dagger T} \ \Rightarrow \ S = (S^{-1})^\dagger \,, \tag{4.3.37}$$

which tells us that S must be unitary, or, in other words, the symmetry group is USp($2N$).

Note. An important point in this discussion is the fact that we can mix the Q^I_α generators with the $\overline{Q}^{I\dot\alpha}$ generators to build a single set of operators, like the ones we used to show the USp($2N$) symmetry. this is possible only because we have to respect only the subgroup of the Lorentz group which is unbroken in the representation. When we choose $P_\mu = (m, 0, 0, 0)$ we break the Lorentz invariance to the SU(2) subgroup of spatial rotations (the unbroken subgroup is usually called the *little group*), and we must preserve only this subgroup when we build multiplets. Q_α and $\overline{Q}^{\dot\alpha}$ transform in the same way under spatial rotations, so we can mix them in the algebra (this is also done for the case of massive representations with central charges).

In general the symmetry group of the algebra on a specific representation can be different from the symmetry group of the algebra itself. This is a consequence of the fact that some symmetries can be broken and some can take specific values (for example they can be trivially realized), which allows for a different symmetry group.

5

Supersymmetry representations on quantum fields

So far we have only seen representations of SUSY on physical on-shell states. However, in order to build supersymmetric field theories, we also need to find the supersymmetry representations on quantum fields. As we will see in the following, representations on fields are in general off-shell representations, hence they contain some unphysical degrees of freedom, which are nevertheless necessary for treating a quantum field theory at the radiative level. The extra degrees of freedom are, of course, removed when we require the fields to be on-shell, that is when the equations of motion are imposed.

By far, the simplest field representations of the SUSY algebra are the ones for simple supersymmetry ($N = 1$). In the following we will explicitly construct the simplest field representation of the $N = 1$ SUSY, the so-called *chiral superfield*.

As a first step to understand supersymmetry in field theory, we will construct the chiral superfield by brute force, then, in section 6, we will learn a more convenient technique, the *superspace* formalism that allows to find easily $N = 1$ SUSY representations and to build sypersymmetric field theories.

5.1 Direct construction of the chiral superfield

We want to build a field that corresponds to the simplest supermultiplet of arbitrary mass in $N = 1$. To build the multiplet we start from a scalar field that creates the spin zero states of the multiplet from the vacuum. To reproduce the condition that the Clifford vacuum is annihilated by a subset of the SUSY generators we impose our scalar field $\phi(x)$ to commute with the $\overline{Q}_{\dot\alpha}$ operators

$$[\overline{Q}_{\dot\alpha}\, , \phi(x)] = 0\,. \tag{5.1.1}$$

When the field $\phi(x)$ is applied to a SUSY-invariant vacuum $|0\rangle$ (i.e. $\overline{Q}_{\dot\alpha}|0\rangle = 0$), it creates states that are annihilated by the $\overline{Q}_{\dot\alpha}$, namely $\overline{Q}_{\dot\alpha}\phi(x)|0\rangle = [\overline{Q}_{\dot\alpha}, \phi(x)]|0\rangle = 0$.[1]

In order to build a non-trivial representation we must assume that $\phi(x)$ is a complex field. If it is real then by taking the conjugate of eq. (5.1.1) we get

$$[Q_\alpha\, , \phi(x)] = 0 \tag{5.1.2}$$

but then[2]

$$0 = [\{Q_\alpha\, , \overline{Q}_{\dot\alpha}\}, \phi(x)] = 2\sigma^\mu_{\alpha\dot\alpha}[P_\mu\, , \phi(x)] = -2\sigma^\mu_{\alpha\dot\alpha} i\partial_\mu \phi(x)\,, \tag{5.1.3}$$

[1] The choice of the $\overline{Q}_{\dot\alpha}$ is arbitrary, we could have chosen the some relation in eq. (5.1.1) with Q_α instead of $\overline{Q}_{\dot\alpha}$.

[2] We use the representation $P_\mu \equiv -i\partial_\mu$ on fields.

which imply $\phi(x) = \text{const.}$. We thus conclude that

$$[Q_\alpha\,, \phi(x)] \neq 0\,. \tag{5.1.4}$$

Note that eq. (5.1.3) has also another consequence: it forbids trivial representations of the SUSY algebra on fields. If we assume that a field ϕ is invariant under SUSY transformations, it would commute with Q_α and $\overline{Q}_{\dot\alpha}$ and thus it would be just constant.

To build a representation we must find a set of fields that transform one into each other under the action of a SUSY transformation. A simple strategy to do this is to use the commutators of the fields with the SUSY generators in order to obtain the various components of the multiplet. We continue to compute commutators until the representation is complete, that is until the set of fields we find remains in itself under a generic SUSY transformation.

We start from the commutator of $\phi(x)$ with Q_α. It is a fermionic field which we parametrize as

$$[\phi(x)\,, Q_\alpha] \equiv \sqrt{2} i \psi_\alpha\,. \tag{5.1.5}$$

We can then apply the SUSY transformations to ψ_α:

$$\begin{cases} \{\psi_\alpha\,, Q_\beta\} \equiv i\sqrt{2} F_{\alpha\beta} \\ \{\psi_\alpha\,, \overline{Q}_{\dot\beta}\} \equiv X_{\alpha\dot\beta} \end{cases}. \tag{5.1.6}$$

We can now use the SUSY algebra to find some constraints on $F_{\alpha\beta}$ and $X_{\alpha\dot\beta}$. From the Jacobi identity with ϕ, Q_α and $\overline{Q}_{\dot\beta}$ we get

$$\begin{aligned} 0 &= \{[\phi(x)\,, Q_\alpha]\,, \overline{Q}_{\dot\beta}\} + [\{Q_\alpha\,, \overline{Q}_{\dot\beta}\}\,, \phi(x)] - \{\cancel{[\overline{Q}_{\dot\beta}, \phi(x)]}\,, Q_\alpha\} \\ &= \{\sqrt{2} i \psi_\alpha\,, \overline{Q}_{\dot\beta}\} + [2\sigma^\mu_{\alpha\dot\beta} P_\mu\,, \phi(x)] \\ &= \sqrt{2} i X_{\alpha\dot\beta} - 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \phi(x)\,, \end{aligned} \tag{5.1.7}$$

which implies

$$X_{\alpha\dot\beta} = \sqrt{2}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\phi(x)\,. \tag{5.1.8}$$

This means that $X_{\alpha\dot\beta}$ is not a new degree of freedom, because it can be expressed in terms of the $\phi(x)$ field.

Now we consider the Jacobi identity with ϕ, Q_α and Q_β:

$$\begin{aligned} 0 &= [\cancel{\{Q_\alpha, Q_\beta\}}\,, \phi(x)] - \{[Q_\beta\,, \phi(x)\}\,, Q_\alpha] + \{[\phi(x)\,, Q_\alpha]\,, Q_\beta\} \\ &= \sqrt{2} i \{\psi_\beta\,, Q_\alpha\} + \sqrt{2} i \{\psi_\alpha\,, Q_\beta\} \\ &= -2F_{\beta\alpha} - 2F_{\alpha\beta}\,. \end{aligned} \tag{5.1.9}$$

This implies that $F_{\alpha\beta}$ is antisymmetric in α and β, so that

$$F_{\alpha\beta} = -F_{\beta\alpha} \qquad \Rightarrow \qquad F_{\alpha\beta} = \varepsilon_{\alpha\beta} F\,, \tag{5.1.10}$$

where $F(x)$ is a complex scalar field.

We can now compute the action of Q_α and $\overline{Q}_{\dot\alpha}$ on F. We define

$$\begin{cases} [F\,, Q_\alpha] \equiv \lambda_\alpha \\ [F\,, \overline{Q}_{\dot\alpha}] \equiv \overline{\chi}_{\dot\alpha} \end{cases}. \tag{5.1.11}$$

By using the Jacobi identity with ψ_α, Q_β and $\overline{Q}_{\dot\beta}$ we get

$$\begin{aligned} 0 &= [\{\psi_\alpha\,, Q_\beta\}\,, \overline{Q}_{\dot\beta}] + [\{Q_\beta\,, \overline{Q}_{\dot\beta}\}\,, \psi_\alpha] + [\{\overline{Q}_{\dot\beta}\,, \psi_\alpha\}\,, Q_\beta] \\ &= \sqrt{2}\varepsilon_{\alpha\beta}[F\,, \overline{Q}_{\dot\beta}] - 2i\sigma^\mu_{\beta\dot\beta}\partial_\mu\psi_\alpha + [X_{\alpha\dot\beta}\,, Q_\beta] \\ &= \sqrt{2}i\varepsilon_{\alpha\beta}\overline{\chi}_{\dot\beta} - 2i\sigma^\mu_{\beta\dot\beta}\partial_\mu\psi_\alpha + \sigma^\mu_{\alpha\dot\beta}2i\partial_\mu\psi_\beta\,. \end{aligned} \tag{5.1.12}$$

By contracting with $\varepsilon^{\alpha\beta}$ we get

$$2\sqrt{2}i\overline{\chi}_{\dot\beta} - 2i\partial_\mu\psi^\beta\sigma^\mu_{\beta\dot\beta}2i\partial_\mu\psi_\alpha\sigma^\mu_{\alpha\dot\beta} = 0\,, \tag{5.1.13}$$

which implies that $\overline{\chi}_{\dot\beta}$ can be expressed in terms of $\partial_\mu\psi_\alpha$:

$$\overline{\chi}_{\dot\beta} = \sqrt{2}\partial_\mu\psi_\beta\sigma^\mu_{\beta\dot\beta}\,. \tag{5.1.14}$$

We can show that λ_α vanishes by using the Jacobi identity

$$\begin{aligned} 0 &= [\{\psi_\alpha\,, Q_\beta\}\,, Q_\gamma] + [\cancel{\{Q_\beta\,, Q_\gamma\}}\,, \psi_\alpha] + [\{Q_\gamma\,, \psi_\alpha\}\,, Q_\beta] \\ &= i\varepsilon_{\alpha\beta}[F\,, Q_\gamma] + i\varepsilon_{\alpha\gamma}[F\,, Q_\beta] \\ &= i\varepsilon_{\alpha\beta}\lambda_\gamma + i\varepsilon_{\alpha\gamma}\lambda_\beta\,. \end{aligned} \tag{5.1.15}$$

By choosing $\alpha = \gamma \neq \beta$ we get

$$\lambda_\alpha = 0\,. \tag{5.1.16}$$

This shows that the set of fields (ϕ, ψ_α, F) from a representation of the $N = 1$ supersymmetry algebra. This multiplet is called *chiral multiplet.*

By taking the adjoint of the above multiplet we get the *antichiral multiplet* with components $(\phi^\dagger, \overline{\psi}_{\dot\alpha}, F^\dagger)$. This multiplet can also be obtained by starting from the condition $[Q_\alpha\,, \phi^\dagger] = 0$.

Let's count the degrees of freedom of the chiral multiplet. There are a total of 4 bosonic degrees of freedom $(\mathrm{Re}\,\phi, \mathrm{Im}\,\phi, \mathrm{Re}\,F, \mathrm{Im}\,F)$ and 4 fermionic degrees of freedom $(\mathrm{Re}\,\psi_1, \mathrm{Im}\,\psi_1, \mathrm{Re}\,\psi_2, \mathrm{Im}\,\psi_2)$. Therefore the multiplet has $4+4$ degrees of freedom. This is the smallest possible number in 4 space-time dimensions, since any multiplet must contain a spinor, and spinors have at least two complex (Weyl) or four real components (Majorana). The multiplet is thus irreducible. Notice that this counting gives the off-shell degrees of freedom. To find the on-shell degrees of freedom we need to take into account the constraints given by the equations of motion.

Usually one introduces anticommuting parameters ζ^α and $\overline{\zeta}^{\dot\alpha}$ that are defined to anticommute with everything fermionic (including themselves) and to commute with everything bosonic (including ordinary c-numbers). Mathematically speaking, they are given by Grassmann variables.

With the help of the spinor parameters we can define infinitesimal variations of a field X by

$$\delta X \equiv [\zeta Q + \overline{Q\zeta}, X]\,. \tag{5.1.17}$$

Notice that ζQ and $\overline{Q\zeta}$ are bosonic objects, so they commute and satisfy commutation relations.

For the chiral multiplet we get

$$\begin{cases} \delta\phi = -\sqrt{2}i\zeta\psi \\ \delta\psi = i\sqrt{2}F\zeta - \sqrt{2}\partial_\mu\phi\sigma^\mu\overline{\zeta} \\ \delta F = -\sqrt{2}\partial_\mu\psi\sigma^\mu\overline{\zeta} \end{cases}\,. \tag{5.1.18}$$

We can explicitly check that the above transformations satisfy the SUSY algebra. We compute the commutator of two transformations

$$\begin{aligned}(\delta_1\delta_2-\delta_2\delta_1)X &= [\zeta_1 Q+\overline{\zeta}_1\overline{Q}\,,[\zeta_2 Q+\overline{\zeta}_2\overline{Q},X]]-[\zeta_2 Q+\overline{\zeta}_2\overline{Q}\,,[\zeta_1 Q+\overline{\zeta}_1\overline{Q},X]]\\ &= [\zeta_1 Q+\overline{\zeta}_1\overline{Q}\,,[\zeta_2 Q+\overline{\zeta}_2\overline{Q},X]]+[\zeta_2 Q+\overline{\zeta}_2\overline{Q}\,,[X,\zeta_1 Q+\overline{\zeta}_1\overline{Q}]]\\ &= -[X,[\zeta_1 Q+\overline{\zeta}_1\overline{Q}\,,\zeta_2 Q+\overline{\zeta}_2\overline{Q}]]\\ &= [X,2i(\zeta_1\sigma^\mu\overline{\zeta}_2-\zeta_2\sigma^\mu\overline{\zeta}_1)\partial_\mu]\\ &= -2i(\zeta_1\sigma^\mu\overline{\zeta}_2-\zeta_2\sigma^\mu\overline{\zeta}_1)\partial_\mu X\,,\end{aligned} \tag{5.1.19}$$

and we can check that the above transformation rules satisfy the algebra.

5.2 The Wess–Zumino Lagrangian

With the chiral multiplet we can build a simple supersymmetric theory: the Wess–Zumino Lagrangian

$$\mathcal{L}_{\mathrm{WZ}}=\partial_\mu\phi^\dagger\partial^\mu\phi-i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi+F^\dagger F\,. \tag{5.2.1}$$

This is the simplest SUSY Lagrangian and contains only kinetic terms for a scalar and a fermion field. One can show that $\mathcal{L}_{\mathrm{WZ}}$ gives an action that is invariant under SUSY transformations

$$\mathcal{L}_{\mathrm{WZ}}\to\mathcal{L}_{\mathrm{WZ}}+\partial_\mu\xi^\mu \qquad\Rightarrow\qquad S_{\mathrm{WZ}}\equiv\int d^4x\;\mathcal{L}_{\mathrm{WZ}}\qquad \text{is invariant}\,. \tag{5.2.2}$$

The action contains also the auxiliary field F, which is a non-dynamical field, i.e. it has no kinetic term in the Lagrangian.

We can now determine the set of on-shell degrees of freedom. The equations of motion are

$$\begin{cases}\partial_\mu\partial^\mu\phi=0\\ i\overline{\sigma}^\mu\partial_\mu\psi=0\\ F=0\end{cases}. \tag{5.2.3}$$

The equations of motion eliminate the F field and two real degrees of freedom from ψ Thus the on-shell degrees of freedom are

$$\begin{cases}2\ \text{bosonic}\\ 2\ \text{fermionic}\end{cases}. \tag{5.2.4}$$

We can also use the $F=0$ equation to simplify the Lagrangian

$$\widetilde{\mathcal{L}}_{\mathrm{WZ}}=\partial_\mu\phi^\dagger\partial^\mu\phi-i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi\,, \tag{5.2.5}$$

at the same time the transformation rules for the fields become

$$\begin{cases}\delta\phi=-\sqrt{2}i\zeta\psi\\ \delta\psi=-\sqrt{2}\partial_\mu\phi\sigma^\mu\overline{\zeta}\end{cases}. \tag{5.2.6}$$

In this case the transformation rules satisfy the SUSY algebra only if we also impose the equations of motion

$$\begin{aligned}[][\delta_1\,,\delta_2]\psi &= -2i(\zeta_1\sigma^\mu\overline{\zeta}_2-\zeta_2\sigma^\mu\overline{\zeta}_1)\partial_\mu\psi-2i[(\partial_\mu\psi\sigma^\mu\overline{\zeta}_2)\zeta_1-(\partial_\mu\psi\sigma^\mu\overline{\zeta}_1)\zeta_2]\\ &= -2i(\zeta_1\sigma^\mu\overline{\zeta}_2-\zeta_2\sigma^\mu\overline{\zeta}_1)\partial_\mu\psi+2i[(\overline{\zeta}_2\overline{\sigma}^\mu\partial_\mu\psi)\zeta_1-(\overline{\zeta}_1\overline{\sigma}^\mu\partial_\mu\psi)\zeta_2]\,.\end{aligned} \tag{5.2.7}$$

To find the above equation we need to use the Fierz identity $\chi_\alpha(\xi\eta)=-\xi_\alpha(\eta\chi)-\eta_\alpha(\xi\chi)$ and the relation $\overline{\xi}\overline{\sigma}^\mu\chi=-\chi\sigma^\mu\overline{\xi}$. The equations of motion eliminate the terms in square brackets, thus leaving only the terms needed to satisfy the SUSY algebra.

6

Superspace and superfields

The direct-construction procedure to build supermultiplets is always possible, however it becomes usually very cumbersome, especially when we must build SUSY Lagrangians. What we did to build a Lagrangian was to just guess its form and then explicitly check that it was invariant under SUSY transformations.

Fortunately for $N = 1$ SUSY a very powerful method has been found to construct SUSY multiplets and SUSY Lagrangians. This method consists in encoding all the components of a SUSY multiplet in a single object, the *superfield*, which is a field in an extended *superspace* that includes the usual space-time coordinates plus some extra fermionic coordinates. Superfields can be easily combined to form new superfields and this property can be used to extract SUSY-invariant quantities from which we can build a SUSY Lagrangian.

In the following we will first introduce the concept of superspace and then we will see how to construct superfields defined on it.

6.1 The $N = 1$ rigid superspace

Analogously to the four-momentum operators P_μ, which are defined as the generators of translations in the ordinary space-time coordinates x^μ, the supersymmetry generators Q_α and $\overline{Q}^{\dot\alpha}$ may be regarded as the generators of 'translations' of some fermionic superspace coordinates θ_α and $\overline{\theta}_{\dot\alpha}$, which anticommute with each other and with fermionic fields and commute with the x^μ and all bosonic fields. The SUSY generators have non-vanishing anticommutators, so we can not take them as simply proportional to the supercoordinate translation operators.

To find the correct representation of the SUSY generators we start from the observation that the SUSY algebra may be viewed as a Lie algebra with anticommuting parameters. This motivates us to define a corresponding group element

$$G(x, \theta, \overline{\theta}) = e^{i(-x^\mu P_\mu + \theta Q + \overline{\theta Q})} . \tag{6.1.1}$$

To multiply group elements we can use the Hausdorff's formula in the form

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} , \tag{6.1.2}$$

which comes from the fact that higher commutators vanish for the P_μ, Q_α and $\overline{Q}_{\dot\alpha}$ generators. In this way we get

$$G(0, \varepsilon, \overline{\varepsilon})G(x^\mu, \theta, \overline{\theta}) = G(x^\mu + i\theta\sigma^\mu\overline{\varepsilon} - i\varepsilon\sigma^\mu\overline{\theta}, \theta + \varepsilon, \overline{\theta} + \overline{\varepsilon}) . \tag{6.1.3}$$

This means that a group element $G(0, \varepsilon, \overline{\varepsilon})$ induces the following motion in the superspace

$$(x^\mu, \theta, \overline{\theta}) \xrightarrow{G(0,\varepsilon,\overline{\varepsilon})} (x^\mu + i\theta\sigma^\mu\overline{\varepsilon} - i\varepsilon\sigma^\mu\overline{\theta}, \theta + \varepsilon, \overline{\theta} + \overline{\varepsilon}) . \tag{6.1.4}$$

This transformation may be generated by the differential operators Q and $\overline{Q}$ (we use the same name as the SUSY generators):

$$i\varepsilon Q + i\overline{\varepsilon}\overline{Q} = i\varepsilon^\alpha \left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\right) + i\left(i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu\right)\overline{\varepsilon}^{\dot\alpha}\,. \tag{6.1.5}$$

So that we can identify the superspace representation of the SUSY generators

$$\begin{cases} Q_\alpha = -i\dfrac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu \\ \overline{Q}_{\dot\alpha} = i\dfrac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \end{cases}. \tag{6.1.6}$$

One can easily check (with the help of the definition of the derivatives on the θ and the $\overline{\theta}$ coordinates which will be given below) that Q_α and $\overline{Q}_{\dot\alpha}$ satisfy the SUSY algebra

$$\{Q_\alpha\,, \overline{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\,. \tag{6.1.7}$$

It is useful to introduce some *covariant derivatives* that anticommute with the SUSY generators:

$$\begin{cases} D_\alpha = \dfrac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu \\ \overline{D}_{\dot\alpha} = \dfrac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \end{cases}, \tag{6.1.8}$$

where $\overline{D}_{\dot\alpha} = (D_\alpha)^\dagger$. One can check the following anticommutation rules

$$\begin{aligned} &\{D_\alpha\,, \overline{D}_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\,, \qquad \{D_\alpha\,, D_\beta\} = \{\overline{D}_{\dot\alpha}\,, \overline{D}_{\dot\beta}\} = 0\,, \\ &\{D_\alpha\,, Q_\beta\} = \{\overline{D}_{\dot\alpha}, Q_\beta\} = \{D_\alpha\,, \overline{Q}_{\dot\beta}\} = \{\overline{D}_{\dot\alpha}\,, Q_\beta\} = 0\,. \end{aligned} \tag{6.1.9}$$

A superfield is defined as a field on the superspace, namely

$$F(x,\theta,\overline{\theta})\,. \tag{6.1.10}$$

We can find the action of the SUSY generators on the superfield F for infinitesimal transformations

$$(1 + i\varepsilon Q + i\overline{\varepsilon}\overline{Q})F(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\beta}) = F(x^\mu - i\varepsilon\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\varepsilon}, \theta^\alpha + \varepsilon^\alpha, \overline{\theta}^{\dot\beta} + \overline{\varepsilon}^{\dot\beta})\,, \tag{6.1.11}$$

and the SUSY variation of a superfield is defined by

$$\delta_{\varepsilon,\overline{\varepsilon}}F = (i\varepsilon Q + i\overline{\varepsilon}\overline{Q})F\,. \tag{6.1.12}$$

The linearity of the representation of the SUSY generators Q_α and $\overline{Q}_{\dot\alpha}$ has the consequence that the sum and the product of two superfields are still superfields

$$\delta(F_1 + F_2) = (i\varepsilon Q + i\overline{\varepsilon}\overline{Q})F_1 + (i\varepsilon Q + i\overline{\varepsilon}\overline{Q})F_2 = \delta F_1 + \delta F_2\,, \tag{6.1.13}$$

and

$$\delta(F_1F_2) = (i\varepsilon Q + i\overline{\varepsilon}\overline{Q})(F_1F_2) = [(i\varepsilon Q + i\overline{\varepsilon}\overline{Q})F_1]F_2 + F_1[(i\varepsilon Q + i\overline{\varepsilon}\overline{Q})F_2] = \delta F_1 F_2 + F_1\delta F_2\,. \tag{6.1.14}$$

Note. Usually we call superfields only the functions on the superspace which carry a representation of the SUSY algebra.

6.1.1 Notations and conventions

Now we fix the notations and conventions that are needed to properly define the $N = 1$ rigid superspace. The superspace coordinates anticommute

$$\{\theta_\alpha\,,\theta_\beta\} = \{\overline{\theta}_{\dot\alpha}\,,\overline{\theta}_{\dot\beta}\} = \{Q_\alpha\,,\overline{\theta}_{\dot\beta}\} = 0\,. \tag{6.1.15}$$

Spinor indices are contracted in the usual way

$$\theta\theta \equiv \theta^\alpha\theta_\alpha\,, \qquad \overline{\theta\theta} \equiv \overline{\theta}_{\dot\alpha}\overline{\theta}^{\dot\alpha}\,. \tag{6.1.16}$$

One can prove the following identities

$$\begin{aligned}
&\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta\,, && \overline{\theta}^{\dot\alpha}\overline{\theta}^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\overline{\theta\theta}\,,\\
&\theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta\,, && \overline{\theta}_{\dot\alpha}\overline{\theta}_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\overline{\theta\theta}\,,\\
&(\theta\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\psi\chi)\,, && (\overline{\theta\psi})(\overline{\theta\chi}) = -\frac{1}{2}(\overline{\theta\theta})(\overline{\psi\chi})\,,\\
&(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta}) = \frac{1}{2}(\theta\theta)(\overline{\theta\theta})\eta^{\mu\nu}\,.
\end{aligned} \tag{6.1.17}$$

Derivatives in θ and $\overline{\theta}$ are defined as

$$\begin{aligned}
&\partial_\alpha\theta^\beta = \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha^\beta\,, && \partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha} = -\varepsilon^{\alpha\beta}\partial_\beta\,,\\
&\overline{\partial}_{\dot\alpha}\overline{\theta}^{\dot\beta} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\overline{\theta}^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}\,, && \overline{\partial}^{\dot\alpha} \equiv \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} = -\varepsilon^{\dot\alpha\dot\beta}\partial_{\dot\beta}\,.
\end{aligned} \tag{6.1.18}$$

Superspace derivatives act as left-derivatives, that is we must move to the leftmost position the variable we are differentiating, for example

$$\frac{\partial}{\partial\theta^\alpha}\xi^\beta\theta^\gamma = -\frac{\partial}{\partial\theta^\alpha}\theta^\gamma\xi^\beta = -\delta_\alpha^\gamma\xi^\beta\,. \tag{6.1.19}$$

We can prove the relations

$$\begin{aligned}
&\partial_\alpha(\theta\theta) = 2\theta_\alpha\,, && \overline{\partial}_{\dot\alpha}(\overline{\theta\theta}) = -2\overline{\theta}_{\dot\alpha}\,,\\
&\partial^2(\theta\theta) \equiv \partial^\alpha\partial_\alpha(\theta\theta) = 4\,, && \overline{\partial}^2(\overline{\theta\theta}) \equiv \overline{\partial}_{\dot\alpha}\overline{\partial}^{\dot\alpha}(\overline{\theta\theta}) = 4\,.
\end{aligned} \tag{6.1.20}$$

From the above definitions we get $(\partial_\alpha)^\dagger = \overline{\partial}_{\dot\alpha}$ with $\alpha = \dot\alpha$, notice the $+$ sign rather than the $-$ sign that appears in the relation $(\partial_\mu)^\dagger = -\partial_\mu$.

We can also introduce an integration on the superspace. It is defined as the Berezin integral for Grassmann variables η:

$$\begin{aligned}
&\int d\eta\,\eta = 1\,, \qquad \int d\eta\,1 = 0\\
&\int d\eta\,f(\eta) = \int d\eta\,(f_0 + \eta f_1) = f_1\,.
\end{aligned} \tag{6.1.21}$$

The Berezin integral is translationally invariant

$$\begin{aligned}
&\int d(\eta+\xi)\,f(\eta+\xi) = \int d\eta\,f(\eta)\,,\\
&\int d\eta\,\frac{\partial}{\partial\eta}f(\eta) = 0\,.
\end{aligned} \tag{6.1.22}$$

The Berezin integration is equivalent to the differentiation

$$\frac{d}{d\eta} f(\eta) - f_1 = \int d\eta f(\eta)\,, \tag{6.1.23}$$

and we can also define a Grassmann delta-function

$$\delta(\eta) \equiv \eta\,. \tag{6.1.24}$$

Note. Given that $\eta^2 = 0$ we have a finite series expansion in η: $f(\eta) = f_0 + \eta f_1$.

These properties are easily generalized to the superspace coordinates θ^α and $\overline{\theta}^{\dot\alpha}$. In this case we must pay attention to the order of integration: we integrate bringing the variable to the leftmost position and the first integration is the innermost one:

$$\begin{aligned} \int d\theta^1 d\theta^2\, \theta^2\theta^1 &\equiv \int d\theta^2 \left(\int d\theta^2\, \theta^2 \right) \theta^1 = \int d\theta^1\, \theta^1 = 1\,, \\ \int d\theta^2 d\theta^1\, \theta^2\theta^1 &= -\int d\theta^2 d\theta^1\, \theta^1\theta^2 = -\int d\theta^1 d\theta^2\, \theta^2\theta^1 = -1\,. \end{aligned} \tag{6.1.25}$$

Since

$$\theta\theta = 2\theta^2\theta^1\,, \overline{\theta\theta} = 2\overline{\theta}^{\dot 1}\overline{\theta}^{\dot 2}\,, \tag{6.1.26}$$

we define

$$d^2\theta \equiv \frac{1}{2} d\theta^1 d\theta^2 = -\frac{1}{4} d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta}\,, d^2\overline{\theta} \equiv \frac{1}{2} d\overline{\theta}^{\dot 2} d\overline{\theta}^{\dot 1} = -\frac{1}{4} d\overline{\theta}_{\dot\alpha} d\overline{\theta}_{\dot\beta} \varepsilon^{\dot\alpha\dot\beta} = [d^2\theta]^\dagger\,, \tag{6.1.27}$$

so that

$$\int d^2\theta (\theta\theta) = \int d^2\overline{\theta} (\overline{\theta\theta}) = 1\,. \tag{6.1.28}$$

It is easy to check that

$$\int d^2\theta = \frac{1}{4} \varepsilon^{\alpha\beta} \frac{\partial}{\partial\theta^\alpha} \frac{\partial}{\partial\theta^\beta}\,, \int d^2\overline{\theta} = -\frac{1}{4} \varepsilon^{\dot\alpha\dot\beta} \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} \frac{\partial}{\partial\overline{\theta}^{\dot\beta}}\,, \tag{6.1.29}$$

and

$$\int d^4\theta (\theta\theta)(\overline{\theta\theta}) \equiv \int d^2\theta d^2\overline{\theta}\, (\theta\theta)(\overline{\theta\theta}) = 1\,. \tag{6.1.30}$$

6.2 The general $N = 1$ scalar superfield

Now we can construct the most general $N = 1$ scalar superfield. We recall that the components of θ and $\overline{\theta}$ anticommute, so any product of their components vanishes if two of them are the same component. In total θ and $\overline{\theta}$ have only 4 components, so any function of θ and $\overline{\theta}$ has a power series that terminates with its quartic term.

The most generic component expansion of the superfield is

$$\begin{aligned} \Phi(x,\theta,\overline{\theta}) \;=\; & f(x) + \theta\phi(x) + \overline{\theta\lambda}(x) + \theta\theta m(x) + \overline{\theta\theta} n(x) \\ & + \theta\sigma^\mu\overline{\theta} v_\mu(x) + (\theta\theta)\overline{\theta\lambda}(x) + (\overline{\theta\theta})\theta\psi(x) + (\theta\theta)(\overline{\theta\theta}) d(x)\,. \end{aligned} \tag{6.2.1}$$

Notice that if $\Phi(x,\theta,\overline{\theta})$ is a scalar, as we chose before, then $f(x)$, $m(x)$, $n(x)$, $v_\mu(x)$ and $d(x)$ are bosonic fields, while $\phi(x)$, $\overline{\chi}(x)$, $\overline{\lambda}(x)$ and $\psi(x)$ are fermionic fields. To obtain the above expansion for Φ we used the spinor identities to remove some redundant terms, like $\overline{\theta}\overline{\sigma}^\mu\theta v_\mu$.

We can now compute the variation of the superfield components under a SUSY transformation:

$$
\begin{aligned}
\delta_\varepsilon \Phi(x,\theta,\overline{\theta}) &= (i\varepsilon Q + i\overline{\varepsilon}\overline{Q})\Phi(x,\theta,\overline{\theta}) \\
&= \left(\varepsilon^\alpha \frac{\partial}{\partial\theta^\alpha} - i\varepsilon^\alpha \sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu - \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}\overline{\varepsilon}^{\dot{\alpha}} + i\theta^\beta \sigma^\mu_{\beta\dot{\alpha}}\overline{\varepsilon}^{\dot{\alpha}}\partial_\mu\right)\Phi(x,\theta,\overline{\theta}) \\
&= \varepsilon\phi + \overline{\varepsilon\chi} + i\theta\sigma^\mu\overline{\varepsilon}\partial_\mu f + 2\varepsilon\theta m + \theta\sigma^\mu\overline{\varepsilon}v_\mu - i\varepsilon\sigma^\mu\overline{\theta}\partial_\mu f \\
&\quad + 2\overline{\varepsilon}\overline{\theta}n + \varepsilon\sigma^\mu\overline{\theta}v_\mu + i(\theta\sigma^\mu\overline{\varepsilon})\theta\partial_\mu\phi + (\theta\theta)(\overline{\varepsilon\lambda}) = -(\varepsilon\sigma^\mu\overline{\theta})\overline{\theta}\partial_\mu\overline{\chi} \\
&\quad + (\overline{\theta\theta})(\varepsilon\psi) - i(\varepsilon\sigma^\mu\overline{\theta})\theta\partial_\mu\phi + i(\theta\sigma^\mu\overline{\varepsilon})\overline{\theta}\partial_\mu\overline{\chi} + 2(\varepsilon\theta)(\overline{\theta\lambda}) + 2(\overline{\varepsilon\theta})(\theta\psi) \\
&\quad - i(\varepsilon\sigma^\mu\overline{\theta})(\theta\theta)\partial_\mu m + i(\theta\sigma^\mu\overline{\varepsilon})(\theta\sigma^\nu\overline{\theta})\partial_\mu v_\nu + 2(\theta\theta)(\overline{\varepsilon\theta})d \\
&\quad + i(\theta\sigma^\mu\overline{\varepsilon})(\overline{\theta\theta})\partial_\mu n - i(\varepsilon\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\partial_\mu v_\nu + 2(\varepsilon\theta)(\overline{\theta\theta})d \\
&\quad - i(\varepsilon\sigma^\mu\overline{\theta})(\theta\theta)\overline{\theta}\partial_\mu\overline{\lambda} + i(\theta\sigma^\mu\overline{\varepsilon})(\overline{\theta\theta})\theta\partial_\mu\psi\,.
\end{aligned}
\tag{6.2.2}
$$

Using the Fierz identities we find the component fields transformation rules

$$\delta_\varepsilon f = \varepsilon\phi + \overline{\varepsilon\chi}\,, \tag{6.2.3}$$

$$\delta_\varepsilon \phi_\alpha = 2\varepsilon_\alpha m + \sigma^\mu_{\alpha\dot{\beta}}\overline{\varepsilon}^{\dot{\beta}}(i\partial_\mu f + v_\mu)\,, \tag{6.2.4}$$

$$\delta_\varepsilon \overline{\chi}^{\dot{\alpha}} = 2\overline{\varepsilon}^{\dot{\alpha}} + \varepsilon^\beta\sigma^\mu_{\beta\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\alpha}}(i\partial_\mu f - v_\mu)\,, \tag{6.2.5}$$

$$\delta_\varepsilon m = \overline{\varepsilon}\overline{\lambda} - \frac{i}{2}\partial_\mu\phi\sigma^\mu\overline{\varepsilon}\,, \tag{6.2.6}$$

$$\delta_\varepsilon n = \varepsilon\psi + \frac{i}{2}\varepsilon\sigma^\mu\partial_\mu\overline{\chi}\,, \tag{6.2.7}$$

$$\delta_\varepsilon v_\mu = \varepsilon\sigma_\mu\overline{\lambda} + \frac{i}{2}\partial_\nu\phi\sigma_\mu\overline{\sigma}^\nu\varepsilon - \frac{i}{2}\partial_\nu\overline{\chi\sigma}_\mu\sigma^\nu\overline{\varepsilon}\,, \tag{6.2.8}$$

$$\delta_\varepsilon \overline{\lambda}^{\dot{\alpha}} = 2\overline{\varepsilon}^{\dot{\alpha}}d + i\overline{\sigma}^{\mu\,\dot{\alpha}\beta}\varepsilon_\beta\partial_\mu m - \frac{i}{2}\overline{\sigma}^{\nu\,\dot{\alpha}\beta}\sigma^\mu_{\beta\dot{\gamma}}\overline{\varepsilon}^{\dot{\gamma}}\partial_\mu v_\nu\,, \tag{6.2.9}$$

$$\delta_\varepsilon \psi_\alpha = 2\varepsilon_\alpha d + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\varepsilon}^{\dot{\beta}}\partial_\mu n - \frac{i}{2}\sigma^\nu_{\alpha\dot{\beta}}\overline{\sigma}^{\mu\,\dot{\beta}\gamma}\varepsilon_\gamma\partial_\mu v_\nu\,, \tag{6.2.10}$$

$$\delta_\varepsilon d = \frac{i}{2}\partial_\mu(-\psi\sigma^\mu\overline{\varepsilon} + \varepsilon\sigma^\mu\overline{\lambda})\,. \tag{6.2.11}$$

An important property of the above transformations is the fact that d transforms as a total derivative. This will be useful to construct supersymmetric actions.

The above transformation rules show that the general scalar superfield forms a basis for an (off-shell) linear representation of the $N = 1$ supersymmetry. However this representation is *reducible*. For example we could impose the constraints

$$
\begin{cases}
\chi = 0 \\
n = 0 \\
v_\mu = i\partial_\mu f \\
\overline{\lambda} = \dfrac{i}{2}\partial_\mu\phi\sigma^\mu \\
\psi = 0 \\
d = -\dfrac{1}{4}\Box f
\end{cases}
\,. \tag{6.2.12}
$$

One can easily check that the transformations of the fields under SUSY respect these constraints, so they define a SUSY representation (actually they define an irreducible representation).

Of course, it is not practical to guess all possible constraints which can give a SUSY representation from the general scalar superfield. In the following we will see how we can use some operators to set some SUSY-invariant constraints on the general superfield to get irreducible representations.

Notice that not any operation that we apply on a superfield gives another superfield. For a function on the superspace to be a superfield that carries a SUSY representation we must require that its form is not modified by SUSY transformations. For example summing or multiplying two superfields gives a superfield. In the same way a space-time derivative of a superfield is still a superfield. On the other hand, by taking a derivative with respect to the fermionic superspace coordinates (∂_α or $\overline{\partial}_{\dot\alpha}$) or by multiplying a field with θ or $\overline{\theta}$ we do not get a superfield that carries a SUSY presentation because we eliminate some component fields in a non-SUSY-invariant way.

7

Supersymmetric actions

In this section we discuss how supersymmetric theories can be constructed by using the superfield formalism. As a first step we will consider the analogous of the Wess–Zumino model, that is we will focus on theories that only contain chiral multiplets. The next step, which we will discuss in section 8, will be the introduction of gauge symmetries in supersymmetric theories.

To build SUSY theories with chiral multiplets using the superspace formalism we need first of all to identify the superfield that corresponds to the chiral multiplet. This superfield can be constructed by applying SUSY-invariant constraints on the general scalar $N = 1$ superfield discussed in section 6.2.

7.1 $N = 1$ chiral superfields

A chiral superfield can be obtained by imposing the constraints in eq. (6.2.12) on the superfield components. A more elegant way, equivalent to imposing "by hand" the constraints, is to impose on a superfield the *covariant* constraint

$$\overline{D}_{\dot{\alpha}}\Phi = 0\,. \tag{7.1.1}$$

This constraint is manifestly SUSY invariant given that

$$\{\overline{D}_{\dot{\alpha}}\,, Q_\beta\} = \{\overline{D}_{\dot{\alpha}}\,, \overline{Q}_{\dot{\beta}}\} = 0\,. \tag{7.1.2}$$

We must now find the most general solution to the covariant constraint. We can simplify our task by defining new bosonic coordinates in the superspace:

$$y^\mu \equiv x^\mu + i\theta\sigma^\mu\overline{\theta}\,, \qquad\qquad y^{\mu\dagger} \equiv x^\mu - i\theta\sigma^\mu\overline{\theta}\,. \tag{7.1.3}$$

Since

$$\overline{D}_{\dot{\alpha}} y^\mu = 0\,, \qquad\qquad \overline{D}_{\dot{\alpha}}\theta^\alpha = 0\,, \tag{7.1.4}$$

any function of y^μ and θ^α (but not $\overline{\theta}_{\dot{\alpha}}$) satisfies the covariant constraint

$$\overline{D}_{\dot{\alpha}}\Phi(y,\theta) = 0\,. \tag{7.1.5}$$

Furthermore, this is actually the most general solution, given that $\overline{D}_{\dot{\alpha}}$ satisfies the chain rule. Thus we can write the most general $N = 1$ chiral superfield as

$$\Phi(y,\theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y)\,, \tag{7.1.6}$$

where $z(y)$, $f(y)$ are complex scalar fields, while $\psi^\alpha(y)$ is a complex left-handed Weyl spinor.[1]

[1]The $\sqrt{2}$ in front of $\theta\psi$ is a convention.

We immediately see that the field content of this superfield is exactly the same as the one we found in section 5.1 with the direct construction method: we have 4 real bosonic and 4 real fermionic degrees of freedom off-shell. This is twice the number of degrees of freedom in the on-shell fundamental $N = 1$ massive representation.

As we will see, F is an auxiliary field, the on-shell degrees of freedom are given by a complex scalar and a Weyl fermion ($2 + 2 = 4$ degrees of freedom).

We can obtain the full θ, $\overline{\theta}$ component expansion for the chiral superfield (by using the Fierz identities):

$$\Phi(y,\theta) = z(x) + \sqrt{2}\theta\psi(x) - \theta\theta F(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu z(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\overline{\theta} - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\Box z(x)\,. \tag{7.1.7}$$

An infinitesimal $N = 1$ SUSY transformation on the chiral superfield yields

$$\begin{cases} \delta z = \sqrt{2}\varepsilon\psi \\ \delta\psi = -\sqrt{2}\varepsilon F + \sqrt{2}i\sigma^\mu\overline{\varepsilon}\partial_\mu z \\ \delta F = \sqrt{2}i\partial_\mu\psi\sigma^\mu\overline{\varepsilon} \end{cases} . \tag{7.1.8}$$

Notice that $\delta F(x)$ is a total derivative. These transformations are the same we found in section 5.1. To see this one needs to identify $\varepsilon = -i\zeta$.

We can also define *antichiral* superfields by imposing the condition

$$D_\alpha\overline{\Phi} = 0\,. \tag{7.1.9}$$

It is easy to see that if Φ is a chiral superfield, then $\Phi^\dagger$ is an antichiral superfield.

It is immediate to see that the sum of chiral superfields is still a chiral superfield. Another important property of chiral superfields is the fact that the product of chiral superfields is still chiral

$$\overline{D}_{\dot\alpha}\Phi^i = 0 \qquad \Rightarrow \qquad \overline{D}_{\dot\alpha}\left(\prod_i \Phi^i\right) = 0\,, \tag{7.1.10}$$

and analogously for antichiral superfields. Notice however that the sum or the product of chiral *and* antichiral superfields is *not* chiral (nor antichiral).

To derive the above relations it is useful to rewrite the SUSY generators and the derivatives in the y^μ, θ, $\overline{\theta}$ coordinates:

$$\begin{aligned} Q_\alpha &= -i\frac{\partial}{\partial\theta^\alpha}\,, & \overline{Q}_{\dot\alpha} &= i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + 2\theta^\beta\sigma^\mu_{\beta\dot\alpha}\frac{\partial}{\partial y^\mu}\,, \\ D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\frac{\partial}{\partial y^\mu}\,, & \overline{D}_{\dot\alpha} &= \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\,. \end{aligned} \tag{7.1.11}$$

7.2 $N = 1$ globally supersymmetric actions

To build a supersymmetric action we must find some quantities that are invariant under supersymmetry or whose variation is a spacetime derivative.

The only completely invariant quantities are spacetime constants (give that $\{Q_\alpha\,, \overline{Q}_{\dot\alpha}\} \propto P_\mu$), so the only possibility is to construct quantities whose variation is a total derivative. When we analyzed the general $N = 1$ superfield and the chiral superfield we already found two candidates:

- the *D-term* (i.e. the $(\theta\theta)(\overline{\theta\theta})$ term) of a general superfield,

- the *F-term* (i.e. the $(\theta\theta)$ term) of a chiral superfield (and analogously the $(\overline{\theta\theta})$ term for the antichiral case).

These two quantities are the building blocks to construct SUSY-invariant theories. Of course, we can take the D-term of a generic superfield, including possible products of superfields, and analogously for the F-term which can also come from a product of chiral superfields.

The most general supersymmetric Lagrangian for a set of chiral superfields Φ is

$$\begin{aligned}\mathcal{L} &= \left[\mathcal{K}(\Phi,\overline{\Phi})\right]_D + \left[\mathcal{W}(\Phi)\right]_F + \left[\mathcal{W}(\Phi)\right]_F^\dagger \\ &= \int d^2\theta d^2\overline{\theta}\, \mathcal{K}(\Phi,\overline{\Phi}) + \int d^2\theta\, \mathcal{W}(\Phi) + \int d^2\overline{\theta}\, \mathcal{W}^\dagger(\overline{\Phi})\,, \end{aligned} \tag{7.2.1}$$

where $\mathcal{K}(\Phi,\overline{\Phi})$ is a real scalar function of the chiral Φ and antichiral $\overline{\Phi}$ superfields ($\mathcal{K}$ is also known as *Kahler potential*), and $\mathcal{W}(\Phi)$ is an *holomorphic* function of the chiral fields only. When $\mathcal{W}$ is expressed as a function only of elementary chiral superfields (and not their superderivatives or spacetime derivatives) it is known as the *superpotential*.

Notice that we can always express an integral over $d^2\theta d^2\overline{\theta}$ as an integral on only $d^2\theta$:

$$\int d^2\theta d^2\overline{\theta} F = -\frac{1}{4}\int d^2\theta \overline{D}^2 F\,, \tag{7.2.2}$$

this is related to the fact that $\overline{D}^2 F$ is a chiral superfield (as can be easily checked explicitly). This means that a D-term can also be expressed as an F-term. The contrary, however, is in general not possible. Thus we usually call F-terms only the ones that can *not* be expressed as an integral over the whole superspace, while the others are called D-terms.

7.3 The Wess–Zumino model

To explore the world of SUSY theories and understand the role of the Kahler potential $\mathcal{K}$ and of the superpotential $\mathcal{W}$, we will consider the simple case with just one chiral superfield Φ.

The most general renormalizable SUSY Lagrangian with one chiral superfield Φ is

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\, \Phi\overline{\Phi} + \left[\int d^2\theta \left(\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\right) + \text{h.c.}\right]. \tag{7.3.1}$$

Let us consider the first term

$$\int d^2\theta d^2\overline{\theta}\, \Phi\overline{\Phi}\,. \tag{7.3.2}$$

One can compute the D-term of the product $\Phi\overline{\Phi}$:

$$\begin{aligned}\Phi\overline{\Phi}\big|_{\theta\theta\overline{\theta\theta}} &= -\frac{1}{4}(\Box z^\dagger)z - \frac{1}{4}z^\dagger\Box z + F^\dagger F + \frac{1}{2}\partial^\mu z^\dagger\partial_\mu z + \frac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\psi} - \frac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi} \\ &= \partial_\mu z^\dagger\partial^\mu z + \frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\overline{\psi} - \psi\sigma^\mu\partial_\mu\overline{\psi}\right) + F^\dagger F + \text{total derivative}\,. \end{aligned} \tag{7.3.3}$$

This shows that the Kahler potential contains the kinetic terms for the component fields (apart from a term containing the auxiliary field F). Any term coming from the Kahler potential contains derivatives, so it can only give kinetic terms (for the physical fields) but not interaction terms. To get the interactions we must consider the superpotential.

Instead of just studying the given superpotential, it is more interesting to treat the general case and then adapt it to the case at hand. Using the y^μ and θ variables it is easy to find the expansion

$$\mathcal{W}(\Phi) = \mathcal{W}(z(y)) + \sqrt{2}\frac{\partial \mathcal{W}}{\partial z}\theta\psi(y) - \theta\theta\left(\frac{\partial \mathcal{W}}{\partial z}F(y) + \frac{1}{2}\frac{\partial^2 \mathcal{W}}{\partial z \partial z}\psi(y)\psi(y)\right) \tag{7.3.4}$$

where $\frac{\partial \mathcal{W}}{\partial z}$ and $\frac{\partial^2 \mathcal{W}}{\partial z \partial z}$ are evaluated at $z(y)$.

The complete action reads

$$S = \int d^4x \left[|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} + F^\dagger F - \frac{\partial \mathcal{W}}{\partial z}F + \text{h.c.} - \frac{1}{2}\frac{\partial^2 \mathcal{W}}{\partial z \partial z}\psi\psi + \text{h.c.}\right]. \tag{7.3.5}$$

The F field is clearly an auxiliary field (it lacks a kinetic term), so we can integrate it out by solving its equations of motion

$$F^\dagger = \frac{\partial \mathcal{W}}{\partial z}. \tag{7.3.6}$$

By substituting into the action we get

$$S = \int d^4x \left[|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} - \left|\frac{\partial \mathcal{W}}{\partial z}\right|^2 - \frac{1}{2}\frac{\partial^2 \mathcal{W}}{\partial z \partial z}\psi\psi - \frac{1}{2}\left(\frac{\partial^2 \mathcal{W}}{\partial z \partial z}\right)^\dagger \overline{\psi\psi}\right]. \tag{7.3.7}$$

We can notice that the scalar potential V is determined by the superpotential $\mathcal{W}$:

$$V = \left|\frac{\partial \mathcal{W}}{\partial z}\right|^2. \tag{7.3.8}$$

We notice moreover that the potential is always *non-negative*, in agreement with the property that the energy in a SUSY theory must be positive or zero.

It is now straightforward to derive the explicit form of the action in the example we considered before with

$$\mathcal{W}(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3. \tag{7.3.9}$$

In this case

$$\frac{\partial \mathcal{W}}{\partial z} = mz + gz^2, \qquad \frac{\partial^2 \mathcal{W}}{\partial z^2} = m + 2gz, \tag{7.3.10}$$

so we get the following action, which is the interacting Wess–Zumino action

$$\begin{aligned}\mathcal{S}_{\text{WZ}} = \int d^4x \Big[&|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} - m^2|z|^2 - \frac{m}{2}(\psi\psi + \overline{\psi\psi}) \\ &- mg(z^\dagger z^2 + (z^\dagger)^2 z) - g^2|z|^4 + g(z\psi\psi + z^\dagger\overline{\psi\psi})\Big].\end{aligned} \tag{7.3.11}$$

This action describes a complex scalar and a fermion both of mass m. Note that the Yukawa interaction appears with a coupling g that is related by SUSY to the bosonic couplings mg and g^2.

We can easily check that the Kahler term is SUSY invariant

$$\delta_{\varepsilon,\overline{\varepsilon}} = i(\varepsilon Q + \overline{\varepsilon}\overline{Q}) = \frac{\partial}{\partial\theta^\alpha}(-\varepsilon^\alpha\mathcal{K}) + \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}(-\overline{\varepsilon}^{\dot\alpha}\mathcal{K}) + \partial_\mu[-i(\varepsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\overline{\varepsilon})\mathcal{K}]. \tag{7.3.12}$$

When we integrate over the whole superspace the first two terms give zero (they contain at most three θ's) and the last term is a total derivative.

Let now discuss a few relevant points of the model.

- The superpotential can be chosen not to contain superderivatives or spacetime derivatives, and this is its standard form. A piece with derivatives/superderivatives is either not chiral or can be rewritten as a D-term. For example if we act with D_α on a chiral field we get something which is non-chiral, on the other hand if we include a term like

$$\Phi \overline{D}\overline{D} S \tag{7.3.13}$$

where Φ is chiral, we can always rewrite it as

$$\overline{D}\overline{D}(\Phi S) \tag{7.3.14}$$

which can be expressed as a D-term.

- The Kahler potential is defined up to a chiral superfield. A chiral superfield has no D-term which can contribute to the action, so

$$\mathcal{K} \to \mathcal{K} + \Lambda(\Phi) + \Lambda^\dagger(\overline{\Phi}) \equiv \mathcal{K}' \tag{7.3.15}$$

gives the same action.

- From the component expansion of chiral superfields we can see that the Kahler term, written as a function of Φ and $\overline{\Phi}$ but not their derivatives, contains terms with at most two derivatives

$$[\mathcal{K}]_D = \cdots + g(\phi, \overline{\phi})\partial_\mu \overline{\phi} \partial^\mu \phi + \cdots \tag{7.3.16}$$

7.4 Renormalizability conditions

The Lagrangian density in a renormalizable theory can only contain operators with dimension (counting powers of energy) four or less. From the relation

$$\{Q_\alpha\,, \overline{Q}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \tag{7.4.1}$$

we get that the energy dimension of the superderivatives is

$$\left[\frac{\partial}{\partial\theta^\alpha}\right] = \left[\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\right] = \frac{1}{2}\,, \tag{7.4.2}$$

so that

$$\left[\theta^\alpha\right] = \left[\overline{\theta}^{\dot\alpha}\right] = -\frac{1}{2}\,. \tag{7.4.3}$$

The F- and D-terms in a superfield contain two and four θ powers respectively, so, if $d(\Phi)$ is the dimension of the superfield Φ, the dimension of its F-term and D-term is

$$[\Phi_D] = d(\Phi) + 2\,, \qquad [\Phi_F] = d(\Phi) + 1\,. \tag{7.4.4}$$

Thus in a renormalizable field theory $\mathcal{W}$ and $\mathcal{K}$ consist of operators with dimension at most three and two, respectively:

$$[\mathcal{K}] \leq 2\,, \qquad [\mathcal{W}] \leq 3\,. \tag{7.4.5}$$

The dimension of an elementary scalar superfield is that of an elementary scalar field $[\Phi] = 1$, so the superpotential $\mathcal{W}$ can contain at most three factors of Φ, without superderivatives or spacetime derivatives. As we discussed before, terms with superderivatives in $\mathcal{W}$ can be expressed as D-terms. Moreover terms with spacetime derivatives in renormalizable theories would necessarily be of the form $\partial_\mu \partial^\mu \Phi$, as a consequence of Lorentz invariance, so they do not contribute to the action.

A similar analysis shows that in renormalizable theories $\mathcal{K}$ is at most a quadratic function of Φ and $\overline{\Phi}$, without derivatives. But any term which contain just Φ or just $\overline{\Phi}$ would be chiral, so it would not contribute to the action with a D-term. Therefore $[\mathcal{K}(\Phi, \overline{\Phi})]_D$ receives contributions only from terms that involve both Φ and $\overline{\Phi}$, namely

$$\mathcal{K}(\Phi, \overline{\Phi}) = \overline{\Phi}\Phi \,. \tag{7.4.6}$$

This analysis shows that the interacting Wess–Zumino model we described before is the most general renormalizable SUSY Lagrangian of one chiral superfield.

7.5 Generalization to an arbitrary set of chiral superfields

The above construction and results can be easily generalized to the case of a set of chiral superfields Φ_i, $i = 1, \ldots, n$. The general form of the Lagrangian is

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\, \mathcal{K}(\Phi_i, \overline{\Phi}_j) + \int d^2\theta\, \mathcal{W}(\Phi_i) + \int d^2\overline{\theta}\, \mathcal{W}^\dagger(\overline{\Phi}_i) \,. \tag{7.5.1}$$

Where again $\mathcal{K}$ is a real function of $\{\Phi_i\}$ and $\{\overline{\Phi}_i\}$ and $\mathcal{W}$ is an holomorphic function of $\{\Phi_i\}$.

In the renormalizable case we have

$$\mathcal{K} = K^i{}_j \Phi_i \overline{\Phi}^j \,, \tag{7.5.2}$$

which, by a redefinition of the fields (notice that $K^i{}_j$ must be positive definite to have a well-defined action) can be brought to the standard form

$$\mathcal{K} = \sum_i \overline{\Phi}^i \Phi_i \,. \tag{7.5.3}$$

For the superpotential $\mathcal{W}$ we have

$$\mathcal{W}(\Phi_i) = \sum_i a_i \Phi_i + \sum_{i,j} m_{i,j} \Phi_i \Phi_j + \sum_{i,j,k} g_{ijk} \Phi_i \Phi_j \Phi_k \,, \tag{7.5.4}$$

where m_{ij} and g_{ijk} are totally symmetric in the indices (because the Φ_i fields commute among themselves).

The action is given by

$$S = \int d^4x \left[|\partial_\mu z_i|^2 - i\psi_i \sigma^\mu \partial_\mu \overline{\psi}_i + F^\dagger F - \frac{\partial \mathcal{W}}{\partial z_i} F_i + \text{h.c.} - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial z_i \partial z_j} \psi_i \psi_j + \text{h.c.} \right] . \tag{7.5.5}$$

We can again solve the equations of motion for the auxiliary fields:

$$S = \int d^4x \left[|\partial_\mu z_i|^2 - i\psi_i \sigma^\mu \partial_\mu \overline{\psi}_i - \left| \frac{\partial \mathcal{W}}{\partial z_i} \right|^2 - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial z_i \partial z_j} \psi_i \psi_j - \frac{1}{2} \left(\frac{\partial^2 \mathcal{W}}{\partial z_i \partial z_j} \right)^\dagger \overline{\psi}_i \overline{\psi}_j \right] . \tag{7.5.6}$$

and the potential is

$$V = \sum_i F^\dagger F = \sum_i \left| \frac{\partial \mathcal{W}}{\partial z_i} \right|^2 . \tag{7.5.7}$$

Note. In the case of a theory of one chiral superfield the linear term $a\Phi$ in the superpotential can always be eliminated by a field redefinition, provided that m and/or g are non-vanishing. If $m = g = 0$ the $a\Phi$ term gives just a contribution to the auxiliary field Lagrangian and is irrelevant.

7.6 R-charge

There is a symmetry which is often important in supersymmetric theories: the R-symmetry. We already saw that the $N = 1$ SUSY algebra is invariant under a U(1) R-symmetry under which

$$\begin{cases} [Q_\alpha\,, R] = Q_\alpha \\ [\overline{Q}_{\dot\alpha}\,, R] = -\overline{Q}_{\dot\alpha} \end{cases}. \tag{7.6.1}$$

From the representation of Q_α and $\overline{Q}_{\dot\alpha}$ on the superspace we see that θ has R-charge -1, while $\overline{\theta}$ has R-charge $+1$. This implies that, under a finite R-symmetry transformation, chiral fields transform as

$$\begin{aligned} R\Phi(\theta, y) &= e^{ir\alpha}\Phi(e^{-i\alpha}\theta, y)\,, \\ R\overline{\Phi}(\overline{\theta}, y^\dagger) &= e^{-ir\alpha}\overline{\Phi}(e^{i\alpha}\overline{\theta}, y^\dagger)\,, \end{aligned} \tag{7.6.2}$$

where r is the R-charge of the superfield. For the component fields we get

$$\begin{aligned} &z \to e^{i\alpha r} z &\qquad& R(z) = r\,, \\ &\psi \to e^{i(r-1)\alpha}\psi && R(\psi) = r - 1\,, \\ &F \to e^{i(r-2)\alpha} F && R(F) = r - 2\,. \end{aligned} \tag{7.6.3}$$

For the superpotential $\mathcal{W}(\Phi)$ to respect R-symmetry its F-term must have vanishing R-charge, that is

$$R\left(\mathcal{W}(\Phi)\right) = +2\,. \tag{7.6.4}$$

On the other hand, the D-term of a superpotential has the same R-charge as the superfield itself. Thus the Kahler potential respects R-symmetry if it has vanishing R-charge

$$R\left(\mathcal{K}(\Phi, \overline{\Phi})\right) = 0\,. \tag{7.6.5}$$

For a single chiral field this is achieved if all the terms in the Kahler potential contain an equal number of Φ and $\overline{\Phi}$ terms.

Notice that in general it is *not* necessary to impose R-symmetry in a supersymmetric theory, or, even if it is present, it could be spontaneously broken.

7.7 General Kahler potentials[2]

In some cases it is useful to consider non-renormalizable Lagrangians of the form

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\,\mathcal{K}(\Phi_i, \overline{\Phi}_j) + \int d^2\theta\,\mathcal{W}(\Phi_i) + \int d^2\overline{\theta}\,\mathcal{W}^\dagger(\overline{\Phi}_i)\,, \tag{7.7.1}$$

where $\mathcal{W}(\Phi_i)$ and $\mathcal{K}(\Phi_i, \overline{\Phi}_j)$ are arbitrary functions of Φ_i and $\overline{\Phi}_j$ but not of their derivatives. We already described the Lagrangian that comes from a generic superpotential, so now we will concentrate on the Kahler potential.

In general, the D-term of an arbitrary Kahler potential can be written in the form

$$\begin{aligned} \int d^2\theta d^2\overline{\theta}\,\mathcal{K}(\Phi_i, \overline{\Phi}_j) \,=\, &K_i{}^j\left(\partial_\mu z^i \partial^\mu z_j^\dagger - \frac{i}{2}\psi^i\sigma^\mu\partial_\mu\overline{\psi}_j + \frac{i}{2}\partial_\mu\psi^i\sigma^\mu\overline{\psi}_j + F^i F_j^\dagger\right) \\ &+ \frac{i}{4}K_{ij}{}^k\left(\psi^i\sigma^\mu\overline{\psi}_k\partial_\mu z^j + \psi^j\sigma^\mu\overline{\psi}_k\partial_\mu z^i - 2i\psi^i\psi^j F_k^\dagger\right) + \text{h.c.} \\ &+ \frac{1}{4}K_{ij}{}^{kl}\psi^i\psi^j\overline{\psi}_k\overline{\psi}_l + \text{total derivative}\,, \end{aligned} \tag{7.7.2}$$

[2] For more details see refs. [5, 7].

where we defined

$$K_i{}^j \equiv \frac{\partial^2}{\partial z^i \partial z_j^\dagger} \mathcal{K}(\Phi_i, \overline{\Phi}_j)\,. \tag{7.7.3}$$

$K_i{}^j$ can be interpreted as the metric of a complex Riemannian manifold, called Kahler manifold. $K_i{}^j$ is called the Kahler metric. Notice that, since $\mathcal{K}$ is real, $K_i{}^j$ is Hermitian, moreover, in order to have a consistent theory we must also require it to be positive definite and non-singular. The interpretation of $K_i{}^j$ as a metric of a complex manifold implies that models with complicated Lagrangians can be characterized by the algebraic geometry of the Kahler manifold. Models of this kind are a supersymmetric version of ordinary σ-models, and are useful to describe the Goldstones coming from a spontaneously broken global symmetry.

8
Supersymmetric gauge theories

In this section we will show how we can construct theories that satisfy both supersymmetry and gauge invariance.

As a first step we will try to introduce a local symmetry in a theory that contains chiral multiplets. This will give us the possibility to understand how to put together a gauge symmetry and supersymmetry and will show is the basic ingredients we need to build a gauge theory, namely the introduction of a new kind of superfield that corresponds to the gauge multiplet representation of $N = 1$ supersymmetry.

For simplicity in this section we will consider only $N = 1$ theories, moreover we will assume that the gauge generators *commute* with the SUSY generators Q. Notice that for simple supersymmetry this is a mild assumption, give that there is just one SUSY generator Q_α, which can only furnish a trivial representation of any semi-simple gauge group. The only symmetries that can act non-trivially on the Q's are Abelian symmetries (recall the R-symmetry).

8.1 Gauge-invariant actions for chiral superfields

If the gauge symmetry commutes with the Q generators, as we assumed, then each component field in a supermultiplet must transform in the same way under a gauge transformation. For a chiral superfield

$$\begin{cases} z_n(x) \to \sum_m \left[\exp\left(i\sum_A t^A\lambda^A(x)\right)\right]_{nm} z_m(x) \\ \psi_n(x) \to \sum_m \left[\exp\left(i\sum_A t^A\lambda^A(x)\right)\right]_{nm} \psi_m(x) \\ F_n(x) \to \sum_m \left[\exp\left(i\sum_A t^A\lambda^A(x)\right)\right]_{nm} F_m(x) \end{cases}, \tag{8.1.1}$$

where t^A are Hermitian matrices representing the generators of the gauge group and $\lambda^A(x)$ are *real* functions of x^μ that parametrize a finite gauge transformation.

To use the superfield formalism we need to generalize the above transformation to find a corresponding superfield transformation. The main difficulty comes from the fact that a chiral superfield contains some terms that are given by derivatives of the component fields (the $\theta\sigma^\mu\overline{\theta}$, the $\theta\theta\overline{\theta}$ and the $\theta\theta\overline{\theta\theta}$ components), so we can not just apply the gauge transformation rule for the z, ψ and F components to the whole superfield (recall that $\lambda^A(x)$ depends on x^μ).

However we can use a shortcut to derive the correct form of the transformation. We notice that a chiral superfield rewritten in terms of the $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$ and θ coordinates does not contain derivatives:

$$\Phi_n(y,\theta) = z_n(y) + \sqrt{2}\theta\psi_n(y) - \theta\theta F_n(y)\,. \tag{8.1.2}$$

This implies that its transformation properties must be

$$\Phi_n(x,\theta,\overline{\theta}) = \Phi(y,\theta) \to \sum_n \left[\exp\left(i\sum_A t^A\lambda^A(y)\right)\right]_{nm} \Phi_m(y,\theta)\,, \tag{8.1.3}$$

where the gauge parameters now depend on y^μ and not just x^μ.

If a term in the action depends only on chiral superfields, and not their derivatives or complex conjugates, like the superpotential term $\int d^2\theta \mathcal{W}(\Phi)$, then it will be invariant under the *local* transformation in eq. (8.1.3), if it is invariant under *global* transformations with $\lambda^A(x)$ independent of x^μ.

However a term that contains $\overline{\Phi}$ as well as Φ does not have the same property and will force us to introduce gauge superfields in order to make it gauge invariant. A term of this kind is the Kahler potential. Let us see closer where the problem originates. Since the matrices t^A are Hermitian we get that eq. (8.1.3) implies

$$\begin{aligned}\overline{\Phi}_n(x,\theta,\overline{\theta}) = \overline{\Phi}(y^\dagger,\overline{\theta}) \to &\sum_m \overline{\Phi}(y^\dagger,\overline{\theta}) \left[\exp\left(-i\sum_A t^A\left(\lambda^A(y)\right)^*\right)\right]_{mn} \\ &= \sum_m \overline{\Phi}(y^\dagger,\overline{\theta}) \left[\exp\left(-i\sum_A t^A\lambda^A(y^\dagger)\right)\right]_{mn}\,. \end{aligned}\tag{8.1.4}$$

In general $\left(\lambda^A(y)\right)^* = \lambda^A(y^\dagger)$ and $\lambda^A(y)$ are different, so $\overline{\Phi}$ does not transform with the inverse of the transformation matrix for Φ. As a consequence $\overline{\Phi}\Phi$ is *not* gauge invariant, hence the Kahler potential is not gauge invariant if we do not modify it. This is not unexpected: the Kahler potential gives rise to the kinetic terms for the chiral superfield components and we know that in gauge theories the derivatives in the kinetic terms must be modified in order to preserve gauge invariance.

To solve the problem we must introduce a gauge connection matrix $\Gamma_{mn}(x,\theta,\overline{\theta})$, with the transformation property

$$\Gamma_{mn}(x,\theta,\overline{\theta}) \to \exp\left(i\sum_A t^A\lambda^A(y^\dagger)\right)\Gamma_{mn}(x,\theta,\overline{\theta})\exp\left(-i\sum_A t^A\lambda^A(y)\right)\,. \tag{8.1.5}$$

By multiplying $\overline{\Phi}$ on the right with Γ we get a superfield that transforms as

$$\left[\overline{\Phi}(x,\theta,\overline{\theta})\Gamma_{mn}(x,\theta,\overline{\theta})\right]_n \to \sum_m \left[\overline{\Phi}(x,\theta,\overline{\theta})\Gamma_{mn}(x,\theta,\overline{\theta})\right]_m \left[\exp\left(-i\sum_A t^A\lambda^A(y)\right)\right]_{mn}\,, \tag{8.1.6}$$

so any *globally* gauge-invariant function constructed from Φ and $\overline{\Phi}\Gamma$ (and *not* their derivatives or complex conjugates) will also be *locally* gauge invariant. One obvious example is the gauge-invariant version $(\overline{\Phi}\Gamma\Phi)_D$ of the D-term in the renormalizable Lagrangian for a chiral superfield.

Any $\Gamma(x,\theta,\overline{\theta})$ that transforms as in eq. (8.1.5) allows us to construct gauge-invariant Lagrangians of the chiral superfields. The choice is not unique: if we multiply Γ on the right with a chiral superfield Y with the transformation rule

$$Y \to \exp\left(i\sum_A t^A\lambda^A(y)\right) Y \exp\left(-i\sum_A t^A\lambda^A(y)\right)\,, \tag{8.1.7}$$

the we get a new gauge connection that also satisfies eq. (8.1.5).

8.1.1 The $N = 1$ vector superfield

In the following we will use the freedom in the choice of Γ to find a simple form for the gauge connection and to identify the gauge multiplet related to the gauge symmetry.

A first simplification is to take Γ to be Hermitian

$$\Gamma^\dagger(x,\theta,\overline{\theta}) = \Gamma(x,\theta,\overline{\theta})\,. \tag{8.1.8}$$

This is always possible because if γ satisfies the transformation rule in eq. (8.1.5), then also $\Gamma^\dagger$ transforms according to the same rule. This means that if Γ is not Hermitian, we can replace it with its Hermitian part $(\Gamma+\Gamma^\dagger)/2$ (or, if it vanishes, with its anti-Hermitian part $(\Gamma-\Gamma^\dagger)/2i$).

Another simplification is to express Γ in terms of fields whose transformation properties are independent of the specific representation t^A of the gauge algebra. For this purpose we choose Γ to be of the form

$$\Gamma(x,\theta,\overline{\theta}) = \exp\left(2\sum_A t^A V^A(x,\theta,\overline{\theta}\right)\,, \tag{8.1.9}$$

where $V^A(x,\theta,\overline{\theta})$ are a set of *real superfields* (so that Γ is Hermitian), not depending on the representation of the gauge algebra furnished by the t^A. This can be understood by using the Baker–Hausdorff formula

$$e^A e^B = \exp\left(a+b+\frac{1}{2}[a,b]+\frac{1}{12}[a,[a,b]]+\frac{1}{12}[b,[b,a]]+\cdots\right)\,, \tag{8.1.10}$$

which implies

$$\exp\left(\sum_A a^A t^A\right)\exp\left(\sum_A b^A t^A\right) = \exp\left(\sum_A g^A(a,b)t^A\right)\,, \tag{8.1.11}$$

where

$$g^A(a,b) = a^A + b^A = \frac{1}{2}i\sum_{B,C} f^A{}_{BC}a^B b^C - \frac{1}{12}f^A{}_{BC}f^C{}_{DE}a^B a^D b^E - \frac{1}{12}f^A{}_{BC}f^C{}_{DE}b^B b^D a^E + \cdots \tag{8.1.12}$$

The expression for $g^A(a,b)$ depends only on the Lie algebra

$$\left[t^B,t^C\right] = i\sum_A f^A{}_{BC}t^A\,, \tag{8.1.13}$$

but not on the particular representation furnished by the t^A.

We can obtain a further simplification by exploiting an additional symmetry of supersymmetric gauge theories. If a function of Φ and $\overline{\Phi}\Gamma$ is invariant under global gauge transformations, then it will automatically be invariant not only under the local gauge transformations, but also under the larger group of *extended gauge transformations*

$$\Phi_n(x,\theta,\overline{\theta}) \to \sum_m\left[\exp\left(-i\sum_A t^A\Lambda^A(x,\theta,\overline{\theta}\right)\right]\Phi_m(x,\theta,\overline{\theta})\,, \tag{8.1.14}$$

and

$$\Gamma(x,\theta,\overline{\theta}) \to \left[\exp\left(-i\sum_A t^A\overline{\Lambda}^A(x,\theta,\overline{\theta})\right)\right]\Gamma(x,\theta,\overline{\theta})\left[\exp\left(i\sum_A t^A\Lambda^A(x,\theta,\overline{\theta})\right)\right]\,, \tag{8.1.15}$$

where $\Lambda^A(x,\theta,\overline{\theta})$ is an arbitrary chiral superfield. Under this transformation

$$V^A(x,\theta,\overline{\theta}) \to V^A(x,\theta,\overline{\theta}) + \frac{i}{2}\left[\Lambda^A(x,\theta,\overline{\theta}) - \overline{\Lambda}^A(x,\theta,\overline{\theta}\right] + \cdots\,, \tag{8.1.16}$$

where we omitted terms arising from the commutators in the Baker–Hausdorff formula that are of first or higher order in the gauge coupling constants.

Let us expand $V^a(x,\theta,\overline{\theta})$ in components. We recall that we chose the condition $V^\dagger = V$, so we get

$$\begin{aligned} V^a(x,\theta,\overline{\theta}) \;=\;& f^A(x) + \theta\chi^A(x) + \overline{\theta}\overline{\chi}^A(x) + \theta\theta m^A(x) + \overline{\theta\theta} m^{A\dagger}(x) \\ &+ \theta\sigma^\mu\overline{\theta} v^A_\mu + i(\theta\theta)\overline{\theta}\left(\overline{\lambda}^A + \frac{1}{2}\overline{\sigma}^\mu\partial_\mu\chi^A\right) - i(\overline{\theta\theta})\theta\left(\lambda^A - \frac{1}{2}\sigma^\mu\partial_\mu\overline{\chi}^A\right) \\ &+ \frac{1}{2}(\theta\theta)(\overline{\theta\theta})\left(D^A - \frac{1}{2}\Box f^A\right). \end{aligned} \tag{8.1.17}$$

Notice that in the above expansion we changed the notation with respect to the previously used expansion of the general scalar superfield. In particular we redefined the cubic and quartic terms in the θ expansion by subtracting suitable functions of χ and f. This will be useful to simplify the notation later on.

To find the effect of an extended SUSY transformation on the vector superfield components we also need the expansion of the chiral superfield $\Lambda^A(x,\theta,\overline{\theta})$. We find

$$\begin{aligned} \frac{1}{2}i\left(\Lambda^A - \overline{\Lambda}^A\right) \;=\;& \frac{1}{2}(z^A - z^{A\dagger}) + i\frac{1}{\sqrt{2}}\theta\psi^A - i\frac{1}{\sqrt{2}}\overline{\theta\psi}^A - \frac{i}{2}\theta\theta F^A + \frac{i}{2}\overline{\theta\theta}F^{A\dagger} \\ &- \frac{1}{2}\theta\sigma^\mu\overline{\theta}\partial_\mu(z^A + z^{a\dagger}) - \frac{1}{2\sqrt{2}}\theta\theta\partial_\mu\psi^A\sigma^\mu\overline{\theta} - \frac{1}{2\sqrt{2}}\overline{\theta\theta}\partial_\mu\overline{\psi}^A\overline{\sigma}^\mu\theta \\ &- \frac{i}{8}\theta\theta\overline{\theta\theta}\Box(z^A - z^{A\dagger})\,. \end{aligned} \tag{8.1.18}$$

Substituting the explicit expressions of V^a and Λ^A into eq. (8.1.16) we find the following transformation rules

$$\begin{aligned} f^A &\to f^A - \mathrm{Im}\, z^A + \cdots \\ \chi^A &\to \chi^A + \frac{i}{\sqrt{2}}\psi^A + \cdots \\ m^A &\to m^A - \frac{i}{2}F^A + \cdots \\ v^A_\mu &\to v^A_\mu - \partial_\mu \mathrm{Re}\, z^A + \cdots \\ \lambda^A &\to \lambda^A + \cdots \\ D^A &\to D^A + \cdots \end{aligned} \tag{8.1.19}$$

where we omitted terms that arise from the structure constants of the gauge algebra and are therefore proportional to one or more factors of gauge coupling constants.

We can use such an extended gauge transformation to put the gauge superfields into a convenient form, known as *Wess–Zumino gauge* in which

$$f^A = \chi^A = m^A = 0\,, \tag{8.1.20}$$

so that

$$V^A_{\mathrm{WZ}}(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}v^A_\mu + i(\theta\theta)\overline{\theta\lambda}^A - i(\overline{\theta\theta})\theta\lambda^A + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})D^A\,. \tag{8.1.21}$$

To do this at zero order in the coupling constants we just need to choose

$$\begin{cases} \operatorname{Im} z^A = f^A \\ \psi^A = \sqrt{2}\, i\, \chi^A \\ F^A = -2\, i\, m^A \end{cases} . \tag{8.1.22}$$

For Abelian gauge theories this is enough because the structure constants of the group vanish, so one has no extra terms in the transformation rules for the V^A component fields. For non-Abelian theories one needs to cancel the f, χ and m components taking into account also the terms in the transformation rules that depend on the structure constants. Thus the actual values of $\operatorname{Im} z^A$ ψ^A and F^A needed to reach the Wess–Zumino gauge are power series in the gauge coupling constants. There is however no need to find the complete form of the transformation needed to reach the Wess–Zumino gauge: the important point is that it is possible to find such a transformation.

The Wess–Zumino gauge condition is not invariant under supersymmetry transformations[1] Once we adopt the Wess–Zumino gauge, the action is no longer invariant under either extended gauge transformations or under supersymmetry. However it is invariant under supersymmetry transformations, which take us out of the Wess–Zumino gauge, followed by a suitable extended gauge transformation that takes us back to the Wess–Zumino gauge.

8.1.2 Gauge transformations

Now we will investigate the transformations of the vector superfield under ordinary gauge transformations. First of all we notice that there are some residual gauge symmetries that are not fixed by the Wess–Zumino gauge: they are given by the transformations with Λ^A given by chiral superfields satisfying

$$\begin{cases} \psi^A = 0 \\ F^A = 0 \\ z^A = z^{A\dagger} \end{cases} , \tag{8.1.23}$$

or, explicitly

$$\begin{cases} \Lambda^A + \overline{\Lambda}^A = (z^A + z^{A\dagger}) - \dfrac{1}{4}(\theta\theta)(\overline{\theta\theta})\Box(z^A + z^{A\dagger}) = 2z^A - \dfrac{1}{2}(\theta\theta)(\overline{\theta\theta})\Box z^A \\ \Lambda^A - \overline{\Lambda}^A = i\theta\sigma^\mu\overline{\theta}\partial_\mu(z^A + z^{A\dagger}0 = 2i\theta\sigma^\mu\overline{\theta}\partial_\mu z^A \end{cases} . \tag{8.1.24}$$

To find the infinitesimal transformation of V^A, we can apply the Baker–Hausdorff formula. To first order in Λ^A we get

$$\delta V = \frac{i}{2} L_V(\Lambda + \overline{\Lambda}) + \frac{i}{2} L_V \coth L_V(\Lambda - \overline{\Lambda})\,, \tag{8.1.25}$$

where we defined

$$V \equiv \sum_A t^A V^A\,, \qquad \Lambda \equiv \sum_A t^A \Lambda^A\,, \tag{8.1.26}$$

and $L_X Y$ denotes the Lie derivative

$$\begin{aligned} L_X Y &= [X, Y] \\ (L_X)^2 Y &= [X, [X, Y]] \\ &\vdots \end{aligned} \tag{8.1.27}$$

[1] Unless $v_\mu^A = \lambda^A = 0$, and the condition $\lambda^A = 0$ is not supersymmetric unless also $D^A = 0$, in which case the whole superfield vanishes.

The term $L_{V/2}\coth L_{V/2}$ is meant to be expanded in series

$$x\coth x = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \cdots \tag{8.1.28}$$

The transformation rule becomes much simpler in the Wess–Zumino gauge. Given that all the terms in V_{WZ} contain at least two powers of θ, only terms with at most two powers of V_{WZ} survive in the expansion. Moreover, with the previously defined Λ, $\Lambda - \overline{\Lambda}$ already contains at least two powers of θ, so only the constant term survives in the expansion of $x\coth x$. Thus we get for an infinitesimal gauge transformation

$$\delta V_{\mathrm{WZ}} = \frac{i}{2}(\Lambda - \overline{\Lambda}) - \frac{i}{2}[(\Lambda + \overline{\Lambda}), V_{\mathrm{WZ}}]\,. \tag{8.1.29}$$

This transformation gives the usual non-Abelian gauge transformations for the components fields

$$\begin{cases} \delta v_\mu^A = -\partial_\mu z^A - f^A{}_{BC} v_\mu^B z^C \\ \delta\lambda^A = -f^A{}_{BC}\lambda^B z^C \\ \delta D^A = -f^A{}_{BC} D^B z^C \end{cases} . \tag{8.1.30}$$

We can see that v_μ^A is the non-Abelian gauge field. λ^A are a set of fermions that transform in the adjoint representation of the gauge group (they are called *gaugino* fields). D^A are real scalars again in the adjoint representation of the gauge group and they will turn to be a set of auxiliary fields.

To construct gauge-invariant actions with chiral superfields we need to know explicitly the expression for $\Gamma(x, theta, \overline{\theta})$. Fortunately, in the Wess–Zumino gauge, we can easily compute $\Gamma(x,\theta,\overline{\theta})$ by expanding in series of V_{WZ}, the particular form of V_{WZ} ensures that terms containing V_{WZ}^3 are vanishing (they would contain at least 6 factors of θ), so we are left with the first three terms in the expansion

$$\Gamma(x,\theta,\overline{\theta}) = \exp(2V_{\mathrm{WZ}}) = 1 + 2V_{\mathrm{WZ}} + 2V_{\mathrm{WZ}}^2\,. \tag{8.1.31}$$

We find

$$V_{\mathrm{WZ}}^2 = \frac{1}{2}(\theta\theta)(\overline{\theta\theta})\sum_{A,B} t^A t^B v^{A\mu} v_\mu^B\,. \tag{8.1.32}$$

We can use this result to compute the gauge-invariant version of the renormalizable Kahler potential for chiral superfields. Writing only the terms that contribute to the D-term we get (we use the notation $v_\mu \equiv \sum_A t^A v_\mu^A$)

$$\overline{\Phi} V_{\mathrm{WZ}}\Phi\big|_{\theta\theta\overline{\theta\theta}} = \frac{i}{2}z^\dagger v^\mu\partial_\mu z - \frac{i}{2}\partial_\mu z^\dagger v^\mu z + \frac{1}{2}z^\dagger D z - \frac{1}{2}\overline{\psi}\overline{\sigma}^\mu v_\mu\psi + \frac{i}{\sqrt{2}}(z^\dagger\lambda\psi - \overline{\psi\lambda}z)\,, \tag{8.1.33}$$

$$\overline{\Phi} V_{\mathrm{WZ}}^2\Phi\big|_{\theta\theta\overline{\theta\theta}} = \frac{1}{2}z^\dagger v^\mu v_\mu z\,. \tag{8.1.34}$$

Putting the various pieces together we get

$$\overline{\Phi} e^{2V_{\mathrm{WZ}}}\Phi\big|_{\theta\theta\overline{\theta\theta}} = (D_\mu z)^\dagger D^\mu z - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + F^\dagger F + \sqrt{2}iz^\dagger\lambda\psi - \sqrt{2}i\overline{\psi\lambda}z + z^\dagger D z + \text{total derivat.}\,, \tag{8.1.35}$$

where we defined the *gauge covariant derivatives*

$$\begin{cases} D_\mu z = \partial_\mu z - iv^\mu z \\ D_\mu\psi = \partial_\mu\psi - iv_\mu\psi \end{cases} . \tag{8.1.36}$$

8.2 Gauge-invariant action for Abelian gauge superfields

We can now use the above defined objects to construct a gauge-invariant supersymmetric action for the gauge superfields. We will first consider the Abelian case and then we will generalize it to the non-Abelian gauge symmetries.

In an Abelian gauge theory the gauge-invariant field constructed from v_μ is the fields-strength tensor

$$f_{\mu\nu}(x) = \partial_\mu v_\nu(x) - \partial_\nu v_\mu(x)\,. \tag{8.2.1}$$

This will be the starting point to find a supersymmetric generalization, that is a supermultiplet that contains the field-strength tensor as a component field. The transformation rule of $f_{\mu\nu}$ under a SUSY transformation is

$$\delta f_{\mu\nu} = \varepsilon(\sigma_\mu\partial_\nu - \sigma_\nu\partial_\mu)\overline{\lambda} - \overline{\varepsilon}(\overline{\sigma}_\mu\partial_\nu - \overline{\sigma}_\nu\partial_\mu)\lambda\,. \tag{8.2.2}$$

The transformation for λ is

$$\delta\lambda = i\varepsilon D - \sigma^{\mu\nu}\varepsilon f_{\mu\nu}\,, \tag{8.2.3}$$

and the one for D is

$$\delta D = -\partial_\mu(\lambda\sigma^\mu\overline{\varepsilon} + \varepsilon\sigma^\mu\overline{\lambda})\,. \tag{8.2.4}$$

None of this depends on whether or not the superfield $V(x,\theta,\overline{\theta})$ is taken to be in the Wess–Zumino gauge. We see that $f_{\mu\nu}(x)$, $\lambda(x)$ and $D(x)$ form a complete supersymmetry multiplet.

Note. The above transformation rules can be derived from the ones for the general scalar supermultiplet applied to $V(x,\theta,\overline{\theta})$. Notice that the λ, $\overline{\lambda}$ and D components of V are redefined with respect to the form we used to write the general scalar supermultiplet.

At this point we could just guess a supersymmetric Lagrangian for $f_{\mu\nu}$, λ and D, however it is more convenient to find a superfield that contains these components and use it to build a Lagrangian. The superfield that does the job is a *spinor superfield* W_α (with its conjugate $\overline{W}_{\dot\alpha} = (W_\alpha)^\dagger$), which is defined as

$$\begin{aligned} W_\alpha &= -\frac{1}{4}(\overline{D}\overline{D})D_\alpha V(x,\theta,\overline{\theta})\,,\\ \overline{W}_{\dot\alpha} &= -\frac{1}{4}(DD)\overline{D}_{\dot\alpha} V(x,\theta,\overline{\theta})\,. \end{aligned} \tag{8.2.5}$$

W_α is a chiral superfield as a consequence of $(\overline{D})^3 = 0$, in the same way $\overline{W}_{\dot\alpha}$ is an antichiral superfield. However W_α is not a general chiral spinor superfield, because W_α and $\overline{W}_{\dot\alpha}$ are related by an additional covariant constraint

$$\overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha} = D^\alpha W_\alpha\,. \tag{8.2.6}$$

This constraint can be easily proven

$$\begin{aligned} \overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha} &= \varepsilon^{\dot\alpha\dot\beta}\overline{D}_{\dot\alpha}\overline{W}_{\dot\beta} - \frac{1}{4}\varepsilon^{\dot\alpha\dot\beta}\overline{D}_{\dot\alpha}(DD)\overline{D}_{\dot\beta}V\\ &= -\frac{1}{4}D^\alpha\overline{D}^2 D_\alpha V = D^\alpha W_\alpha\,. \end{aligned} \tag{8.2.7}$$

An important property of W_α (and $\overline{W}_{\dot\alpha}$) is that it is invariant under gauge transformations. For Abelian gauge theories an extended gauge transformation takes the simple form

$$V(x,\theta,\overline{\theta}) \to V(x,\theta,\overline{\theta}) + \frac{i}{2}\left(\Lambda(x,\theta,\overline{\theta}) - \overline{\Lambda}(x,\theta,\overline{\theta})\right)\,, \tag{8.2.8}$$

with Λ a chiral superfield. We get

$$\begin{aligned} W_\alpha \to & -\frac{1}{4}(\overline{DD})D_\alpha\left(V+\frac{i}{2}\Lambda-\frac{i}{2}\overline{\Lambda}\right) \\ &= W_\alpha - \frac{i}{8}(\overline{DD})D_\alpha\Lambda \qquad\qquad (\text{since } D_\alpha\overline{\Lambda}=0) \\ &= W_\alpha + \frac{i}{8}\overline{D}^{\dot\beta}\{\overline{D}_{\dot\beta}, D_\alpha\}\Lambda \qquad (\text{since } \overline{D}_{\dot\beta}\Lambda=0) \\ &= W_\alpha\,. \end{aligned} \tag{8.2.9}$$

Since W_α and $\overline{W}_{\dot\alpha}$ are both gauge invariant, there is no loss of generality in computing their components in the Wess–Zumino gauge:

$$\begin{cases} W_\alpha = -\frac{1}{4}(\overline{DD})D_\alpha V_{\rm WZ}(x,\theta,\overline{\theta}) \\ \overline{W}_{\dot\alpha} = -\frac{1}{4}(DD)\overline{D}_{\dot\alpha} V_{\rm WZ}(x,\theta,\overline{\theta}) \end{cases}. \tag{8.2.10}$$

To compute the explicit expression for W_α it is more convenient to express the superfield in terms of y^μ, θ^α and $\overline{\theta}^{\dot\alpha}$:

$$V_{\rm WZ} = \theta\sigma^\mu\overline{\theta}v_\mu(y) + i\theta\theta\overline{\theta}\overline{\lambda}(y) - i\overline{\theta}\overline{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\left(D(y) - i\partial_\mu v^\mu(y)\right)\,. \tag{8.2.11}$$

Then, by using $\sigma^\nu \overline{s}igma^\mu = 2\sigma^{\nu\mu}$, it is straightforward to find

$$\begin{aligned} D_\alpha W_{\rm WZ} = & (\sigma^\mu\overline{\theta})_\alpha v_\mu(y) + 2i\theta_\alpha\overline{\theta}\overline{\lambda}(y) - i\overline{\theta}\overline{\theta}\lambda_\alpha(y) + \theta_\alpha\overline{\theta}\overline{\theta}D(y) \\ & + 2i(\sigma^{\mu\nu}\theta)_\alpha\overline{\theta}\overline{\theta}\partial_\mu v_\nu(y) + \theta\theta\overline{\theta}\overline{\theta}(\sigma^\mu\partial_\mu\overline{\lambda}(y))_\alpha\,, \end{aligned} \tag{8.2.12}$$

and then, by using $\overline{DD\theta\theta} = -4$, we get

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu v_\nu(y)\partial_\nu v_\mu(y)) + \theta\theta(\sigma^\mu\partial_\mu\overline{\lambda}(y))_\alpha\,. \tag{8.2.13}$$

We see that this chiral spinor superfield contains the component fields we found before by applying the SUSY transformations to $f_{\mu\nu}$. This supermultiplet is usually called *curl supermultiplet* or *field strength supermultiplet.*

Since W_α is a chiral superfield $\int d^2\theta W^\alpha W_\alpha$ is a SUSY invariant Lagrangian. Explicitly

$$W^\alpha W_\alpha|_{\theta\theta} = -2i\lambda\sigma^\mu\partial\mu\overline{\lambda} + D^2 - \frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta}f_{\mu\nu}f_{\rho\sigma}\,. \tag{8.2.14}$$

by using

$$(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\,, \tag{8.2.15}$$

we get

$$\int d^2\theta W^\alpha W_\alpha = -\frac{1}{2}f_{\mu\nu}f^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + D^2 + \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}\,. \tag{8.2.16}$$

Note that the first three terms are real, while the last one is purely imaginary.

The supersymmetric action for Abelian gauge fields is

$$\frac{1}{4}\left(\int d^2\theta W^\alpha W_\alpha + \int d^2\overline{\theta}\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\right) = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + \frac{1}{2}D^2\,. \tag{8.2.17}$$

Notice that the above action does not come from an F-term, it is indeed a D-term:

$$\begin{aligned}\int d^2\theta W^\alpha W_\alpha &= \int d^2\theta \left(-\frac{1}{4}\overline{DD}D^\alpha V\right) W_\alpha \\ &= -\frac{1}{4}\int d^2\theta \overline{DD}(D^\alpha V W_\alpha) \\ &= \int d^2\theta d^2\overline{\theta} D^\alpha V W_\alpha\,.\end{aligned} \tag{8.2.18}$$

Another interesting aspect of the expression of $W^\alpha W_\alpha|_{\theta\theta}$ in eq. (8.2.16) is the presence of the $\varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}$ term. This term is a total derivative and we will discuss its role in the more general context of non-Abelian gauge theories. Notice that the action in eq. (8.2.17) shows that D is an auxiliary field.

8.2.1 The Fayet-Iliopoulos term

In Abelian theories we can add also an extra term to the action. Under a gauge transformation

$$V \to V + \frac{i}{2}\Lambda - \frac{i}{2}\overline{\Lambda} \tag{8.2.19}$$

the D term of the V field is invariant, so we can add it to the action

$$\mathcal{L}_{\text{FI}} = \xi \int d^2\theta d^2\overline{\theta} V = \frac{1}{2}\xi D\,, \tag{8.2.20}$$

where ξ is an arbitrary constant. This is called the *Fayet-Iliopoulos term*, and it can be added to the action *only* for the U(1) subgroups of the gauge group (we have seen that for a non-Abelian group the D term is not gauge invariant since it transforms in the adjoint representation of the gauge group).

8.3 Gauge-invariant action for general gauge superfields

We now want to consider the case of a general gauge symmetry. To construct the gauge superfield Lagrangian we need, first of all, to find the non-Abelian generalization of the field strength superfield W_α.

We notice that the Abelian definition of W_α can be rewritten as

$$\begin{cases} W_\alpha = -\frac{1}{8}(\overline{DD})e^{-2V}D_\alpha e^{2V} \\ \overline{W}_{\dot\alpha} = \frac{1}{8}(DD)e^{2V}\overline{D}_{\dot\alpha}e^{-2V} \end{cases}. \tag{8.3.1}$$

These definitions can be adopted also in the general non-Abelian case. Let us compute how W_α and $\overline{W}_{\dot\alpha}$ transform under a gauge transformation. First we notice that

$$e^{-2V}D_\alpha e^{2V} \to e^{-i\Lambda}e^{-2V}\left(D_\alpha e^{2V}\right)e^{i\Lambda} + e^{-i\Lambda}D - \alpha e^{i\Lambda}\,. \tag{8.3.2}$$

Using the fact that $\overline{D}_{\dot\alpha}$ commutes with Λ, we get

$$W_\alpha \to e^{-i\Lambda}W_\alpha e^{i\Lambda} - \frac{1}{8}e^{-i\Lambda}(\overline{DD})D_\alpha e^{i\Lambda}\,, \tag{8.3.3}$$

the second term vanishes, so we obtain that W_α and $\overline{W}_{\dot\alpha}$ transform conveniently in the non-Abelian case:

$$\begin{cases} W_\alpha \to e^{i\Lambda} W_\alpha e^{i\Lambda} \\ \overline{W}_{\dot\alpha} \to e^{-i\overline{\Lambda}} \overline{W}_{\dot\alpha} e^{i\overline{\Lambda}} \end{cases} . \tag{8.3.4}$$

Again we can explicitly compute W_α in the Wess–Zumino gauge:

$$\begin{aligned} W_\alpha &= -\frac{1}{8}(\overline{DD})e^{-2V_{\mathrm{WZ}}} D_\alpha e^{2V_{\mathrm{WZ}}} \\ &= -\frac{1}{4}(\overline{DD})D_\alpha V_{\mathrm{WZ}} + \frac{1}{2}(\overline{DD})V_{\mathrm{WZ}} D_\alpha V_{\mathrm{WZ}} - \frac{1}{4}(\overline{DD})D_\alpha V_{\mathrm{WZ}}^2 . \end{aligned} \tag{8.3.5}$$

By using the expression for V_{WZ} in the coordinates y^μ, θ^α, $\overline{\theta}^{\dot\alpha}$ we get

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i\sigma_\alpha^{\mu\nu\,\beta}\theta_\beta F_{\mu\nu}(y) + (\theta\theta)\sigma^\mu_{\alpha\dot\beta} D_\mu \overline{\lambda}^{\dot\beta}(y) , \tag{8.3.6}$$

where

$$F_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu] , \, D_\mu \overline{\lambda}^{\dot\beta} \equiv \partial_\mu \overline{\lambda}^{\dot\beta} - i[v_\mu, \overline{\lambda}^{\dot\beta}] , \tag{8.3.7}$$

that is $F_{\mu\nu}$ is the Yang–Mills field strength, while D_μ is the Yang–Mills gauge covariant derivative.

A gauge-invariant supersymmetric action can be obtained from

$$\frac{1}{2}\int d^2\theta \, \mathrm{tr}\, W^\alpha W_\alpha = \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} - i\lambda\sigma^\mu D_\mu \overline{\lambda} + \frac{1}{2}D^2\right] . \tag{8.3.8}$$

Notice that the above term is not real and it has no coupling constant. To solve these problems we introduce a complex gauge coupling τ:

$$\tau = \frac{\theta_{\mathrm{YM}}}{2\pi} + \frac{4\pi i}{g^2} , \tag{8.3.9}$$

where g is the gauge coupling constant and θ_{YM} is the Tang–Mills theta parameter. The $N = 1$ Yang–Mills action we want is

$$\begin{aligned} \frac{1}{8\pi}\mathrm{Im}\left[\tau \int d^4x \int d^2\theta \, \mathrm{tr}\, W^\alpha W_\alpha\right] &= \frac{1}{g^2}\int d^4x \, \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \overline{\lambda} + \frac{1}{2}D^2\right] \\ &\quad - \frac{\theta_{\mathrm{YM}}}{32\pi^2}\int d^4x \, \mathrm{tr}\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} . \end{aligned} \tag{8.3.10}$$

Note. We could have written this action as the real part of $\int d^2\theta W^\alpha W_\alpha$ just by multiplying τ by a factor of i. We chose the above convention to follow the literature.

The $\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ term is a total derivative, so it does not contribute to the action unless some topologically non-trivial instantonic solutions exist. In Abelian theories such solutions are not there, so the integral of this term on the whole space always vanishes. For non-Abelian theories one can have instanton solutions and in this case the term can have a non-zero integral over spacetime. This integral is related to the topological properties of the instanton solution, namely to its winding number. Given that the θ_{YM} term is related to topological properties and is a total derivative, it has only non-perturbative effects and plays no role in perturbation theory.

8.4 Renormalizable gauge theories with chiral superfields

Now we have all the ingredients we need to construct the general renormalizable gauge theory with chiral superfields. It is given by

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{FI}} \\
&= \frac{1}{8\pi}\text{Im}\left[\tau \int d^2\theta \, \text{tr}\, W^\alpha W_\alpha\right] + 2\sum_{Abelian} \xi^a \int d^2\theta d^2\overline{\theta} V^a \\
&\quad + \int d^2\theta d^2\overline{\theta}\,\overline{\Phi} e^{2V}\Phi + \int d^2\theta \mathcal{W}(\Phi) + \int d^2\overline{\theta}\mathcal{W}^\dagger(\overline{\Phi}) \\
&= \frac{1}{g^2}\text{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda} + \frac{1}{4}D^2\right) - \frac{\theta_{\text{YM}}}{32\pi^2}\text{tr}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \sum_{Abelian}\xi^a D^a \\
&\quad + (D_\mu z)^\dagger D^\mu z - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + F^\dagger F + \sqrt{2}iz^\dagger\lambda\psi - \sqrt{2}i\overline{\psi}\overline{\lambda}z \\
&\quad + z^\dagger D z - \left(\frac{\partial\mathcal{W}}{\partial z^i}F^i + \text{h.c.}\right) - \left(\frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial z^i\partial z^j}\psi^i\psi^j + \text{h.c.}\right) + \text{total derivative}\,.
\end{aligned} \tag{8.4.1}$$

As we did in the case without gauge symmetries, we can eliminate the auxiliary fields D and F from the Lagrangian by using the equations of motion:

$$F_i^\dagger = \frac{\partial\mathcal{W}}{\partial z^i}\,, \tag{8.4.2}$$

and

$$D^a = -z^\dagger T^a z - \xi^a\,, \tag{8.4.3}$$

where it is understood that $\xi^a = 0$ is a does not take values in an Abelian factor of the gauge group. By substituting back into the Lagrangian one finds

$$\begin{aligned}
\mathcal{L} &= \frac{1}{g^2}\text{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda}\right) - \frac{\theta_{\text{YM}}}{32\pi^2}\text{tr}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\
&\quad + (D_\mu z)^\dagger D^\mu z - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + F^\dagger F + \sqrt{2}iz^\dagger\lambda\psi - \sqrt{2}i\overline{\psi}\overline{\lambda}z \\
&\quad - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial z^i\partial z^j}\psi^i\psi^j - \frac{1}{2}\left(\frac{\partial^2\mathcal{W}}{\partial z^i\partial z^j}\right)^\dagger\overline{\psi}^i\overline{\psi}^j - V(z^\dagger, z) + \text{total derivative}\,,
\end{aligned} \tag{8.4.4}$$

where the scalar potential $V(z^\dagger, z)$ is now given by

$$V(z^\dagger, z) = F^\dagger F + \frac{1}{2}D^2 = \sum_i \left|\frac{\partial\mathcal{W}}{\partial z^i}\right|^2 + \frac{1}{2}\sum_a \left|z^\dagger T^a z + \xi^a\right|^2\,. \tag{8.4.5}$$

In this case the potential is the sum of two contributions: one coming from the F auxiliary field and one from the D auxiliary field. Notice that the two contributions are both non-negative, so the positivity of energy is respected.

8.5 General gauge theories with chiral superfields

If we do not require renormalizability we can write the generalization of the σ-model we described for chiral superfields in section 7.7 adapted to a theory with gauge invariance. The most generic matter Lagrangian gauge invariant under a group G is

$$\mathcal{L}_{\text{matter}} = \int d^2\theta d^2\overline{\theta}\,\mathcal{K}(\overline{\Phi}e^{2V}, \Phi) + \int d^2\theta\,\mathcal{W}(\Phi) + \int d^2\overline{\theta}\,\mathcal{W}^\dagger(\overline{\Phi})\,, \tag{8.5.1}$$

where $\mathcal{K}(\overline{\Phi}_i e^{2V}, \Phi_i)$ is a real (globally) G-invariant function, and $\mathcal{W}(\Phi_i)$ is a G-invariant function of the Φ_i.

The Gauge Lagrangian is usually generalized as

$$\mathcal{L}_{\text{gauge}} = \frac{1}{16g^2} \int d^2\theta \, h_{ab}(\Phi^i) W^{a\alpha} W^b_\alpha + \text{h.c.} \,, \tag{8.5.2}$$

where $h_{ab} = h_{ba}$ transforms under G as the symmetric product of the adjoint representation with itself. To get back the standard renormalizable gauge Lagrangian we only need to take $\frac{1}{g^2} h_{ab} = \frac{\tau}{\pi i} \mathrm{tr}\, T^a T^b$.

The components expansion of the Lagrangian can be found in section 7.2 of ref. [7] or in section 27.4 of ref. [3].

9

Spontaneous supersymmetry breaking

Up to this point we have seen how to construct supersymmetric theories, however in nature we do not observe the degeneracy of masses between bosons and fermions predicted by supersymmetry. This means that, if supersymmetry is really a symmetry of nature, it must be broken. There are two common ways to break supersymmetry:

- *explicit breaking*, obtained by introducing (soft) SUSY-breaking terms in the action;

- *spontaneous breaking*, due to the existence of a non-SUSY-invariant vacuum.

Often explicit breaking by soft terms is a way to parametrize in a low-energy effective theory the effects of a spontaneous supersymmetry breaking at high energy. An example of this procedure is the parametrization of SUSY breaking in the minimal supersymmetric Standard Model (MSSM) (see section 13), which is done by introducing soft SUSY-breaking terms in the theory.

In this section we will discuss the mechanism of spontaneous SUSY breaking.

9.1 Supersymmetry breaking and vacuum energy

We already saw that in SUSY theories energy is non-negative. This means that, in particular, the vacuum has non-negative energy. Let's use the SUSY algebra to write

$$\langle 0|P_0|0\rangle = \frac{1}{4}|Q_\alpha|0\rangle|^2 + \frac{1}{4}|\overline{Q}_{\dot{\alpha}}|0\rangle|^2 \geq 0\,, \tag{9.1.1}$$

where $|0\rangle$ is the vacuum. If the vacuum is invariant under SUSY transformations, that is SUSY is *not* spontaneously broken, we have

$$Q_\alpha|0\rangle = \overline{Q}_{\dot{\alpha}}|0\rangle = 0 \qquad \Rightarrow \qquad \langle 0|P_0|0\rangle = 0\,. \tag{9.1.2}$$

On the other hand, if the vacuum is not invariant, that is SUSY *is* spontaneously broken, we get

$$Q_\alpha|0\rangle \neq 0 \qquad \Rightarrow \qquad \langle 0|P_0|0\rangle > 0\,. \tag{9.1.3}$$

The above statements can also be reversed, that is we get

$$Q_\alpha|0\rangle = \overline{Q}_{\dot{\alpha}}|0\rangle = 0 \qquad \Leftrightarrow \qquad \langle 0|P_0|0\rangle = 0\,, \tag{9.1.4}$$

and

$$Q_\alpha|0\rangle \neq 0 \qquad \Leftrightarrow \qquad \langle 0|P_0|0\rangle > 0\,. \tag{9.1.5}$$

This gives an important restatement of the condition for spontaneous supersymmetry breaking: supersymmetry is spontaneously broken if and only if the vacuum energy is positive.

9.2 Vacua in supersymmetric theories

Another criterion of spontaneous supersymmetry breaking can be given in terms of vacuum expectation values of the auxiliary fields. Let us make a step backwards and discuss first of all the properties of the vacuum and how it can be found.

A first important property comes from the Poincaré invariance, so we must always require that the vacuum configuration respects this symmetry. In other words, we need to impose that all the fields must have constant values on Minkowski space (that is their spacetime derivatives must vanish) and all the fields that are non-scalars must have zero expectation value. Only scalar fields z^i can have a non-vanishing VEV $\langle z^i \rangle$.

The vacuum configuration is the one that minimizes the value of the Euclidean action (or equivalently minimizes the energy), this is equivalent to the requirement that the scalar potential V is minimized by the vacuum configuration. Thus we have for a vacuum

$$\langle v_\mu^a \rangle = \langle \lambda^a \rangle = \langle \psi^i \rangle = \partial_\mu \langle z^i \rangle = 0 \,, \qquad V\left(\langle z^i \rangle, \langle z^{i\dagger} \rangle\right) = \text{minimum} \,. \tag{9.2.1}$$

The minimum may be the global minimum of V, in which case one has the true vacuum, or it may be only a local minimum, in which case one has a false (metastable) vacuum. In any case, for a true or false vacuum one has

$$\frac{\partial V}{\partial z^i}\left(\langle z^i \rangle, \langle z^{i\dagger} \rangle\right) = \frac{\partial V}{\partial z^{i\dagger}}\left(\langle z^i \rangle, \langle z^{i\dagger} \rangle\right) = 0 \,. \tag{9.2.2}$$

This shows that the vacuum is a solution of the equations of motion.

In a supersymmetric theory the scalar potential is given by

$$V(z, z^\dagger) = F_i^\dagger F_i + \frac{1}{2} D^a D^a \,, \tag{9.2.3}$$

where

$$F_i^\dagger = \frac{\partial \mathcal{W}(z)}{\partial z^i} \,, \tag{9.2.4}$$

and

$$D^a = -z^\dagger T^a z - \xi^a \,. \tag{9.2.5}$$

The potential is non-negative, so it will certainly be at a global minimum, namely $V = 0$, if

$$F^i(\langle z^\dagger \rangle) = D^a(\langle z \rangle, \langle z^\dagger \rangle) = 0 \,. \tag{9.2.6}$$

In this case supersymmetry is unbroken, given that the vacuum energy is zero.

Notice that, however, the system of equations (9.2.6) does not necessarily have a solution. We can have two cases

i) If eq. (9.2.6) has a solution, this solution is a global minimum with $V = 0$, hence supersymmetry is unbroken. Note that there can be many solutions of eq. (9.2.6), and all of them correspond to degenerate SUSY-invariant vacua. In addition there could be some false vacua corresponding to local minima of the potential with $V \neq 0$. These are metastable vacua with broken supersymmetry.

ii) If eq. (9.2.6) has no solution, the scalar potential can never vanish and its minimum is strictly positive: $V > 0$. This means that the vacuum is not SUSY-invariant and supersymmetry is necessarily broken.

As we have seen, spontaneous SUSY breaking is related directly to the values of the auxiliary fields F^i and D^a on the vacuum. We have that SUSY is broken if and only if some of the F^i and/or D^a has non-zero value on the vacuum.

There is also another way to see that SUSY breaking is related to F^i and D^a. By looking at the SUSY transformations of the chiral superfields components we get

$$\begin{cases} \delta\langle z^i\rangle = \sqrt{2}\varepsilon\langle\psi^i\rangle \\ \delta\langle\psi^i\rangle = \sqrt{2}i\partial_\mu\langle z^i\rangle\sigma^\mu\overline{\varepsilon} - \sqrt{2}\langle F^i\rangle\varepsilon \\ \delta\langle F^i\rangle = \sqrt{2}i\partial_\mu\langle\psi^i\rangle\sigma^\mu\overline{\varepsilon} \end{cases}, \tag{9.2.7}$$

which, taking into account the conditions given by the unbroken Poincaré group, become

$$\begin{aligned} &\delta\langle z^i\rangle = 0\,, \\ &\delta\langle\psi^i\rangle = -\sqrt{2}\langle F^i\rangle\varepsilon = 0\,, \\ &\delta\langle F^i\rangle = 0\,. \end{aligned} \tag{9.2.8}$$

The two conditions on $\delta\langle\psi^i\rangle$ come, on one side, from the SUSY transformations and, on the other side, from the fact that $\langle\psi^i\rangle$ must vanish (as well as its variation) on a Poincaré-invariant vacuum.

The above conditions, which encode the requirements to have a SUSY-invariant vacuum, can be consistent only if $\langle F^i\rangle = F^i(\langle z^{i\dagger}\rangle) = 0$. Differently, if $\langle F^i\rangle \neq 0$ for some i, the vacuum is not SUSY invariant and SUSY is spontaneously broken.

We can find a similar argument for D^a. Under SUSY transformations

$$\begin{cases} \delta f^a_{\mu\nu} = \varepsilon(\sigma_\mu\partial_\nu - \sigma_\nu\partial_\mu)\overline{\lambda}^a - \overline{\varepsilon}(\overline{\sigma}_\mu\partial_\nu - \overline{\sigma}_\nu\partial_\mu)\lambda^a \\ \delta\lambda^a = i\varepsilon D^a - \sigma^{\mu\nu}\varepsilon f^a_{\mu\nu} \\ \delta D^a = -\partial_\mu(\lambda^a\sigma^\mu\overline{\varepsilon} + \varepsilon\sigma^\mu\overline{\lambda}^a) \end{cases}. \tag{9.2.9}$$

Evaluating these expressions on the vacuum we get the conditions for SUSY (and Poincaré) invariance

$$\begin{aligned} &\delta\langle f^a_{\mu\nu}\rangle = 0\,, \\ &\delta\langle\lambda^a\rangle = i\varepsilon\langle D^a\rangle = 0\,, \\ &\delta\langle D^a\rangle = 0\,. \end{aligned} \tag{9.2.10}$$

Again we see that the auxiliary field D^a is related to the breaking of SUSY. if $\langle D^a\rangle \neq 0$ on the vacuum, then the vacuum is not SUSY-invariant and SUSY is spontaneously broken.

9.3 The Goldstone theorem for supersymmetry

Goldstone's theorem states that, whenever a continuous *global* symmetry is spontaneously broken, there is a massless mode in the spectrum, i.e. a massless particle. The quantum numbers carried by the Goldstone particle are related to the broken symmetry. Similarly if supersymmetry is spontaneously broken there is a massless spin one-half particle, i.e. a *massless fermion*, usually called *Goldstino*.

Now we will explicitly prove that, in the presence of spontaneously broken SUSY, we get a massless fermion. A vacuum that breaks SUSY is such that

$$\begin{cases} \dfrac{\partial V}{\partial z^i}\left(\langle z^i\rangle, \langle z^{i\dagger}\rangle\right) = 0 \\ \langle F^i\rangle \neq 0 \quad \text{and/or} \quad \langle D^a\rangle \neq 0 \end{cases}. \tag{9.3.1}$$

From the explicit form of V we get

$$\frac{\partial V}{\partial z^i} = F^j \frac{\partial^2 \mathcal{W}}{\partial z^i \partial z^j} - D^a z_j^\dagger (T^a)^j{}_i \,, \tag{9.3.2}$$

and this vanishes on the vacuum. The statement that the superpotential is gauge invariant can be written as

$$0 = \delta^{(a)}_{gauge} \mathcal{W} = \frac{\partial \mathcal{W}}{\partial z^i} \delta^{(a)}_{gauge} z^i = F_i^\dagger (T^a)^j{}_i z^j \,. \tag{9.3.3}$$

We can now combine the above equations into the matrix equation

$$M = \begin{pmatrix} \dfrac{\partial^2 \mathcal{W}}{\partial z^i \partial z^j} & -\langle z_l^\dagger \rangle (T^a)^l{}_i \\ -\langle z_l^\dagger \rangle (T^b)^l{}_i & 0 \end{pmatrix} \,, \qquad M \begin{pmatrix} \langle F^j \rangle \\ \langle D^a \rangle \end{pmatrix} \,, \tag{9.3.4}$$

which states that the matrix M has a zero eigenvalue.

Both the matrix M is exactly the fermion mass matrix. We already discussed the form of the gauge Lagrangian with chiral superfields and we found that the fermion mass term (in the presence of a VEV for the scalar fields) is

$$\left(i\sqrt{2} \langle z_j^\dagger \rangle (T^a)^j{}_i \lambda^a \psi^i - \frac{1}{2} \langle \frac{\partial^2 \mathcal{W}}{\partial z^i \partial z^j} \rangle \psi^i \psi^j \right) + \text{h.c.} = -\frac{1}{2} (\psi^i, \sqrt{2} i \lambda^b) M \begin{pmatrix} \psi^j \\ \sqrt{2} i \lambda^a \end{pmatrix} + \text{h.c.} \tag{9.3.5}$$

This mass matrix has a zero eigenvalue and this means that there is a massless fermion: this is the Goldstone fermion or Goldstino.

The presence of a massless Goldstino as a consequence of spontaneously supersymmetry breaking can also understood non-perturbatively, by an argument related to pairing of bosonic and fermionic states. For this discussion see chapter 29 of ref. [3].

9.4 Mechanisms of supersymmetry breaking

Now we are ready to analyze the possible mechanisms that can generate a spontaneous breaking of rigid $N = 1$ supersymmetry. In the following we will start analyzing the case of a theory with only chiral superfields and then we will consider the issue of SUSY breaking in gauge theories.

9.4.1 Supersymmetry breaking in theories with only chiral superfields

As we have seen before, SUSY is spontaneously broken if the scalar potential of the theory is always strictly positive. In the case of a theory with only chiral superfields the scalar potential come sonly from the superpotential $\mathcal{W}(\Phi)$, that is it is generated by an F-term. In general we have

$$V = \sum_i F_i^\dagger F_i = \sum_i \left| \frac{\partial \mathcal{W}}{\partial z^i} \right|^2 \,, \tag{9.4.1}$$

and SUSY is spontaneously broken if and only if the system of equations

$$F_i^\dagger = \frac{\partial \mathcal{W}}{\partial z^i} = 0 \tag{9.4.2}$$

admits no solution. Notice that there are as many independent variables as there are equations to satisfy, so we generally expect a solution to exist. In order for supersymmetry to be broken in these theories, it is necessary to impose restrictions on the form of the superpotential.

In the simplest possible case, the theory of one chiral superfield, we have a complex equation

$$F^{\dagger} = \frac{\partial \mathcal{W}}{\partial z} = 0 \,. \qquad (9.4.3)$$

In general $\frac{\partial \mathcal{W}}{\partial z}$ will be a complex polynomial in z, so it always has at least one zero in the complex plane and supersymmetry can not be broken.[1]

To get a theory with SUSY breaking we thus need to consider more than one chiral superfield. The most interesting cases, clearly, are the ones that involve a renormalizable theory, so we will concentrate on this case in the following discussion. For a renormalizable theory we have

$$\mathcal{W}(\Phi_i) = a_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k \,, \qquad (9.4.4)$$

thus the set of equations that determine the breaking of SUSY are

$$F_i^{\dagger} = \frac{\partial \mathcal{W}}{\partial z^i} = a_i + m_{ij} z^j + g_{ijk} z^j z^k = 0 \,. \qquad (9.4.5)$$

We can notice that, if $a_i = 0$ for each i, the system of equations has a trivial solution $\langle z^i \rangle = 0$ for each i, so SUSY is unbroken. Thus a necessary condition to have SUSY breaking is that some of the a_i should be non-zero. This however is *not* a sufficient condition for SUSY breaking as can be easily checked.

Notice that we can redefine the fields in our theory by shifting the fields by a constant: $\Phi_i \to \Phi_i + b_i$. This transformation changes only the z^i scalar field component of Φ_i in a way that respects SUSY transformations (the variation of z^i is not charged by adding a constant, $\delta(z^i + b^i) = \delta z^i$, and z^i appears only in the variation of ψ with a spacetime derivative, $\delta\psi^i = \sqrt{2} i \sigma^{\mu} \bar{\varepsilon} \partial_{\mu} z^i - \sqrt{2} \varepsilon F^i$). Moreover, in a renormalizable theory also the kinetic term is unchanged by the shift. With this transformation we get a new superpotential with parameters

$$\begin{cases} a'_i = a_i + m_{ij} b_j + g_{ijk} b_j b_k \\ m'_{ij} = m_{ij} + 2\, g_{ijk} b_k \\ g'_{ijk} = g_{ijk} \end{cases} . \qquad (9.4.6)$$

Notice that he condition to have unbroken SUSY is equivalent to the possibility of setting all the $a'_i = 0$ by a shift in the fields. The reason is simple: by the shift we can move the minimum of the potential to the origin, $\langle z^i \rangle = 0 \ \forall i$, if the minimum does not break SUSY, then we get a new potential that vanishes for $\langle z^i \rangle = 0$, so all the a'_i must be zero.

It can be shown that, in renormalizable theories with only two chiral superfields, SUSY can not be spontaneously broken, or, in other words, the potential has at least one zero. The simplest models that exhibit spontaneous SUSY breaking heve three chiral superfields and have been proposed by O'Raifeartaigh. In the following we will describe a generalization of the class of models due to O'Raifeartaigh. We consider a superpotential that is a linear combination of a set Y_i of chiral superfields, with coefficients given by functions $h_i(X)$ of a second set of chiral superfields X_n:

$$\mathcal{W}(X, Y) = \sum_i Y_i h_i(X) \,. \qquad (9.4.7)$$

[1] An exception to the above statement is obviously given by the trivial case in which $\frac{\partial \mathcal{W}}{\partial z} = \text{const.} \neq 0$, or, in other words $\mathcal{W}(\Phi) = \lambda \Phi$. Moreover in this case it is easy to see that the theory is just a free field theory of a scalar and a fermion field without mass terms or interactions, so we can neglect this uninteresting case.

The conditions for SUSY to be unbroken are

$$\begin{cases} 0 = \dfrac{\partial \mathcal{W}(x,y)}{\partial y_i} = h_i(x) \\ 0 = \dfrac{\partial \mathcal{W}(x,y)}{\partial x_n} = \displaystyle\sum_i y_i \frac{\partial h_i(x)}{\partial x_n} \end{cases} . \tag{9.4.8}$$

The second set of equations can always be solved by taking $y_i = 0$, with no effect in the problem of solving the first set of equations. But, if the number of X_n superfields is smaller than the number of Y_i superfields, then the first set of equations imposes more conditions on the x_n than the number of free variables, so, without fine-tuning, a solution does not exist and supersymmetry is broken.

Notice that the initial assumption on the form of the superpotential can be naturally obtained (without fine-tuning) by assuming a suitable R-symmetry. We can obtain the wanted structure for the superpotential by requiring R-invariance, with R-charges $+2$ for the Y_i and 0 for the X_n (remember that the superpotential must contain terms with R-charge $+2$ to get an R-invariant Lagrangian).

The scalar potential of this class of models is

$$V(x,y) = \sum_i |h_i(x)|^2 + \sum_n \left| \sum_i y_i \frac{\partial h_i(x)}{\partial x_n} \right|^2 . \tag{9.4.9}$$

This potential is always minimized by choosing the x_n to minimize the first term, while the second term can always be minimized by choosing $y_i = 0$. These models, however, have a peculiar feature: there are always directions in the space of fields in which the minimum of the potential is flat. If $x_n = \langle x_n \rangle$ are a set of values that minimize the first term, the second term vanishes not only for $y_i = 0$ but also for any vector y_i in a direction orthogonal to all the vectors $(v^n)_i = (\partial h_i(x)/\partial x_n)_{x=\langle x \rangle}$. If there are N_X superfields X_n and N_Y superfields Y_i with $N_Y > N_X$, then there will be at least $N_Y - N_X$ flat directions. Notice that for any non-vanishing value of the y_i R-symmetry is broken.

The simplest example of this class of models is provided by the case with one X and two Y superfields. Renormalizability requires the coefficients $h_i(X)$ to be quadratic functions of X, and, by taking suitable linear combinations of the Y_i and shifting and rescaling X, we can choose these functions as

$$h_a(X) = X - a\,, \qquad h_2(X) = X^2\,, \tag{9.4.10}$$

with an arbitrary constant a. SUSY is clearly broken unless $a = 0$. The potential is

$$V(x,y) = |x|^4 + |x-a|^2 + |y_1 + 2xy_2|^2 . \tag{9.4.11}$$

The first two terms have a unique global minimum x_0. The flat direction is the one for which $y_1 + 2x_0 y_2 = 0$. For $a = 0$ we get unbroken SUSY and the minima have $x_0 = 0$, $y_1 = 0$ and y_2 arbitrary.

Note. The coordinates that parametrize the degenerate vacua are called *moduli.* The set of degenerate vacua is the *moduli space.*

9.4.2 Supersymmetry breaking in gauge theories

Let us now discuss what happens in supersymmetric gauge theories. In this case the scalar potential is

$$V(z,z^\dagger) = F_i^\dagger F_i + \frac{1}{2} D^a D^a\,, \tag{9.4.12}$$

where

$$F_i^\dagger = \frac{\partial \mathcal{W}(z)}{\partial z^i}\,, \qquad D^a = -z^\dagger T^a z - \xi^a\,, \tag{9.4.13}$$

where the ξ^a can be non-zero only for the Abelian subgroups of the gauge group. Naively one could think that supersymmetry breaking is very common given that there are more equations than variables. However, for a gauge group of dimension D the superpotential $\mathcal{W}(\Phi)$ is subject to the D constraints

$$\sum_m \frac{\partial \mathcal{W}(z)}{\partial z^m}(T^a z)_m = 0 \tag{9.4.14}$$

for all a. Hence if there are n independent chiral superfields, then the number of independent conditions is exactly n, equal to the number of variables. With the number of conditions equal to the number of free variables, it is likely to find solutions for generic superpotentials, thus supersymmetry is usually not broken.

It is easy to see that, if all the Fayet-Iliopoulos constants ξ^a vanish, if there exists a solution to the conditions $F_i^\dagger = 0$, then there is always (at least) another configuration that satisfies *all* the conditions for SUSY to be unbroken. This means that in this case the mechanism of SUSY breaking is similar to the one for theories with only chiral superfields, that is SUSY breaking can be induced only by non-vanishing F_i terms and it can not happen a breaking induced *only* by the D^a terms. Let us show this. We notice that the superpotential $\mathcal{W}(\Phi)$ does *not* involve $\overline{\Phi}$, so it is invariant not only under ordinary gauge transformations $\Phi \to \exp(i\sum_A T^A\lambda^A)\Phi$ with λ^A arbitrary real numbers, but also under transformations with with λ^A arbitrary complex numbers. Under all these transformations the $F_i^\dagger$ (and F_i) terms transform linearly, so if z_0^i satisfies $F_i^\dagger = 0$, then so does $z^\lambda \equiv \exp(i\sum_A T^A\lambda^A)z_0$. On the other hand, the scalar product $z^\dagger z$ is *not* invariant under the transformations with λ^A complex, but $z^{\dagger\lambda}z^\lambda$ remains real and positive for complex λ, so it is bounded from below and, therefore, has a minimum. For $\xi^a = 0$, the condition that $z^{\dagger\lambda}z^\lambda$ is at a minimum is just

$$z^{\dagger\lambda}T^a z^\lambda = 0\,, \tag{9.4.15}$$

which tells us that the D^a term must vanish. Thus in the absence of Fayet-Iliopoulos terms the question of SUSY breaking is entirely determined by the superpotential.

Now we can present some explicit models that show different patterns of symmetry breaking. We will see that the breaking of supersymmetry and the breaking of the gauge symmetry are independent phenomena and we can build models in which both symmetries or just one of them is spontaneously broken.

There is an interesting connection between the auxiliary fields D^a and the breaking of gauge symmetry. The D^a auxiliary components are *not* gauge invariant (unless they correspond to an Abelian subgroup), so if D^a gets non-vanishing VEV one has spontaneous gauge symmetry breaking. Notice that this is *not* a necessary condition for gauge symmetry breaking, one can have gauge breaking also if SUSY is unbroken (and thus the D^a have zero VEV), in this case it is the VEV of some scalar field z^i that breaks the gauge invariance.

9.4.3 Fayet–Iliopoulos supersymmetry breaking

The existence of Fayet-Iliopoulos terms for U(1) gauge subgroup gives another mechanism to break supersymmetry. The simplest case is a theory with a U(1) gauge group.[2]

[2] In order to avoid U(1) − U(1) − U(1) and U(1)-graviton-graviton anomalies in the theory it is necessary that the sum of the U(1) quantum numbers of all the chiral superfields and the sum of their cubes vanish.

The model we consider is the supersymmetric version of QED. The field content is given by two chiral superfields Φ_+ and Φ_- with U(1) quantum numbers $\pm e$ and an Abelian vector superfield. The Lagrangian is given by

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\left(\overline{\Phi}_+ e^{2eV}\Phi_+ + \overline{\Phi}_- e^{2eV}\Phi_- + \xi V\right) + \left[\int d^2\theta\left(\frac{1}{4}W^\alpha W_\alpha + m\Phi_+\Phi_-\right) + \text{h.c.}\right]. \tag{9.4.16}$$

Notice that we explicitly introduced the gauge coupling e by rescaling the gauge field ($V \to eV$). The scalar potential is

$$V = \frac{1}{8}(\xi + 2e(|z_+|^2 - |z_-|^2))^2 + m^2(|z_+|^2 + |z_-|^2), \tag{9.4.17}$$

and the equations of motion for the auxiliary fields are

$$\begin{cases} F_\pm^\dagger = m z_\mp \\ D = -\dfrac{1}{2}(\xi + 2e(|z_+|^2 - |z_-|^2)) \end{cases}. \tag{9.4.18}$$

Unless ξ vanishes, it is not possible to find a SUSY-invariant vacuum.

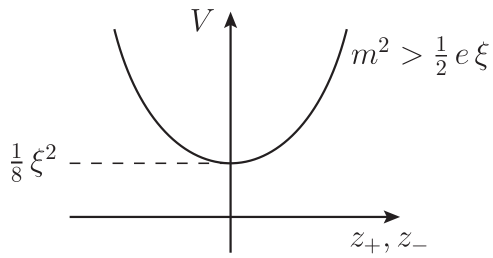
There are two regimes with different qualitative behaviors.

i) If $m^2 > \frac{1}{2}e\xi$, the minimum of the potential occurs for $\langle z++\rangle = \langle z_-\rangle = 0$ and the model describes two complex scalars with masses

$$m_\pm^2 = m^2 \mp \frac{1}{2}e\xi. \tag{9.4.19}$$

The fermion masses do not change (they are equal to m) and the gauge field and the gaugino remain massless. Notice that SUSY is broken and the gaugino plays the role of the Goldstino. Gauge symmetry is unbroken.

This situation is schematically described by the following potential



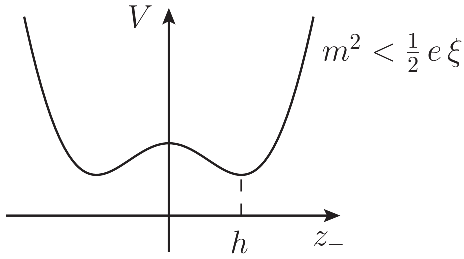
ii) If $m^2 < \frac{1}{2}e\xi$, the minimum of the potential is at

$$\langle z_+\rangle = 0, \qquad \langle z_-\rangle = h, \qquad \text{where} \qquad |h|^2 = \frac{\xi}{2e} - \frac{m^2}{e^2}. \tag{9.4.20}$$

There are infinite degenerate vacua corresponding to the phase of h. We can arbitrarily chose h to be real and positive, so that

$$\langle z_-\rangle = \sqrt{\frac{\xi}{2e} - \frac{m^2}{e^2}}. \tag{9.4.21}$$

The situation is now described by the potential



We want to find the masses of the scalars. By expanding the potential around the minimum we get

$$\begin{aligned}&|z_+|^2\left(m^2+\frac{1}{2}e\xi-e^2h^2\right)+|\mathrm{Im}\,z_-|^2\left(m^2-\frac{1}{2}e\xi+e^2h^2\right)+|\mathrm{Re}\,z_-|^2\left(m^2-\frac{1}{2}e\xi+3e^2h^2\right)\\&=\ |z_+|^2(2m^2)+|\mathrm{Re}\,z_-|^2(e\xi-2m^2)\,.\end{aligned} \tag{9.4.22}$$

So we get the following masses

$$\begin{cases} m_{z_+}=\sqrt{2}m \\ m_{z_-}=\sqrt{e\xi-2m^2}=\sqrt{2}eh \end{cases}. \tag{9.4.23}$$

Notice that $\mathrm{Im}\,z_-$ has no mass term: this field is absorbed by the gauge field which becomes massive as a consequence of gauge symmetry breaking. ($\mathrm{Im}\,z_-$ is the would-be Goldstone of the broken $\mathrm{U}(1)$ symmetry.)

For the gauge field we get

$$(D_\mu z_-)^\dagger D^\mu z_- \quad\Rightarrow\quad e^2h^2A_\mu A^\mu \qquad (D_\mu=\partial_\mu+ieA_\mu)\,, \tag{9.4.24}$$

thus

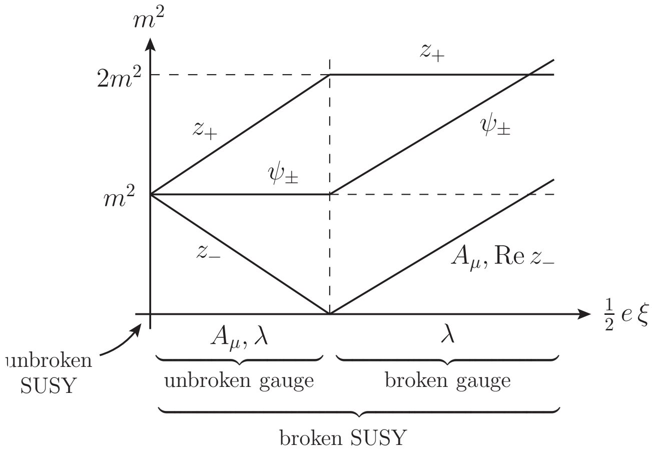
$$m_{A_\mu}=\sqrt{\frac{1}{2}e\xi-m^2}=\sqrt{2}eh\,. \tag{9.4.25}$$

One can also compute the fermion masses. The mass matrix for the λ and $\psi_\pm$ fermions can be diagonalized to get the eigenvectors $(\widetilde{\lambda},\widetilde{\psi}_\pm)$ with masses

$$\begin{cases} m_{\widetilde{\lambda}}=0 \\ m_{\widetilde{\psi}_\pm}=\sqrt{e\xi-m^2}=\sqrt{2e^2h^2+m^2} \end{cases}, \tag{9.4.26}$$

notice the presence of a Goldstino as a consequence of supersymmetry breaking.

The behavior of the masses as a function of ξ is given by



9.5 Mass formula

If supersymmetry is unbroken all particles within a supermultiplet have the same mass. This is no longer true if SUSY is spontaneously broken, but, as we will see, one can still find a relation among the masses. In the following we will analyze the mass spectrum at tree level in a generic SUSY theory with possible SUSY and/or gauge spontaneous symmetry breaking. To derive the mass matrices it is more convenient to canonically normalize the gauge fields and reintroduce the gauge couplings in the covariant derivatives. For this purpose it is sufficient to rescale the gauge superfields as $V \to gV$.

Let us start with the masses of the gauge fields. These are non-vanishing only if the gauge symmetry is broken. The vector fields, in this case, acquire an extra degree of freedom through the Higgs mechanism. The mass term comes from the term $(D_\mu z)^\dagger D^\mu z$, thus we set the mass term $g^2\langle z^\dagger T^a T^b z\rangle v^a_\mu v^{b\mu}$ and thus the mass matrix

$$\left(\mathcal{M}_1^2\right)^{ab} = 2g^2\langle z^\dagger T^a T^b z\rangle\,. \tag{9.5.1}$$

It is convenient to introduce the notations

$$D^a_i \equiv \frac{\partial D^a}{\partial z^i} = -g(z^\dagger T^a)_i\,, \qquad D^{i\,a} \equiv \frac{\partial D^a}{\partial z^{i\dagger}} = -g(T^a z)_i\,, \qquad D^{a\,i}_j \equiv -gT^{a\,i}_j\,, \tag{9.5.2}$$

and

$$F^{ij} \equiv \frac{\partial F^i}{\partial x^{j\dagger}} = \frac{\partial^2 \mathcal{W}^\dagger}{\partial z^{j\dagger}\partial z^{i\dagger}}\,, \qquad F_{ij} \equiv \frac{\partial F^\dagger_i}{\partial x^j} = \frac{\partial^2\mathcal{W}}{\partial z^j\partial z^i}\,. \tag{9.5.3}$$

With this notation

$$\left(\mathcal{M}_1^2\right)^{ab} = 2\langle D^a_i D^{b\,i}\rangle = 2\langle D^a_i\rangle\langle D^{b\,i}\rangle\,. \tag{9.5.4}$$

The mass term for the fermions is

$$-\frac{1}{2}(\psi^i\ \lambda^a)\mathcal{M}_{1/2}\begin{pmatrix}\psi^j\\ \lambda^b\end{pmatrix} + \text{h.c.}\,, \qquad \mathcal{M}_{1/2} = \begin{pmatrix}\langle F_{ij}\rangle & \sqrt{2}i\langle D^b_i\rangle\\ \sqrt{2}i\langle D^a_j\rangle & 0\end{pmatrix} \tag{9.5.5}$$

with the squared masses of the fermions given by the eigenvalues of the Hermitian matrix

$$\left(\mathcal{M}_{1/2}\mathcal{M}^\dagger_{1/2}\right) = \begin{pmatrix}\langle F_{il}\rangle\langle F^{jl}\rangle + 2\langle D^c_i\rangle\langle D^{cj}\rangle & -\sqrt{2}i\langle F_{il}\rangle\langle D^{bl}\rangle\\ \sqrt{2}i\langle D^a_l\rangle\langle F^{jl}\rangle & 2\langle D^a_l\rangle\langle D^{bl}\rangle\end{pmatrix}\,. \tag{9.5.6}$$

Finally for the scalars the mass terms are

$$-\frac{1}{2}(z^i\ z^\dagger_j)\mathcal{M}_0^2\begin{pmatrix}z^\dagger_k\\ z^l\end{pmatrix} \tag{9.5.7}$$

with

$$\mathcal{M}_0^2 = \begin{pmatrix}\langle\dfrac{\partial^2 V}{\partial z^i\partial z^\dagger_k}\rangle & \langle\dfrac{\partial^2 V}{\partial z^i\partial z^l}\rangle\\ \langle\dfrac{\partial^2 V}{\partial z^\dagger_j\partial z^\dagger_k}\rangle & \langle\dfrac{\partial^2 V}{\partial z^\dagger_j\partial z^l}\rangle\end{pmatrix}\,. \tag{9.5.8}$$

We find that

$$\mathcal{M}_0^2 = \begin{pmatrix}\langle F_{ip}\rangle\langle F^{kp}\rangle + \langle D^{ak}\rangle\langle D^a_i\rangle + \langle D^a\rangle D^{a\,k}_i & \langle F_p\rangle\langle F_{i\,lp}\rangle + \langle D^a_i\rangle\langle D^a_l\rangle\\ \langle F^\dagger_p\rangle\langle F_{j\,kp}\rangle + \langle D^{aj}\rangle\langle D^{ak}\rangle & \langle F_{lp}\rangle\langle F^{jp}\rangle + \langle D^{aj}\rangle\langle D^a_l\rangle + \langle D^a\rangle D^{a\,j}_l\end{pmatrix}\,. \tag{9.5.9}$$

We can now compute the traces of the squared mass matrices, which yield the sum of the squared masses of the fields.

$$\begin{aligned}
\mathrm{tr}\,\mathcal{M}_1^2 &= \langle D_i^a\rangle\langle D^{ai}\rangle\,, && (9.5.10)\\
\mathrm{tr}\,\mathcal{M}_{1/2} &= \langle F_{il}\rangle\langle F^{il}\rangle + 4\langle D_i^a\rangle\langle D^{ai}\rangle\,, && (9.5.11)\\
\mathrm{tr}\,\mathcal{M}_0^2 &= 2\langle F_{ip}\rangle\langle F^{ip}\rangle + 2\langle D_i^a\rangle\langle D^{ai}\rangle - 2g\langle D^a\rangle \mathrm{tr}\,T^a\,. && (9.5.12)
\end{aligned}$$

So we get

$$\mathrm{Str}\mathcal{M}^2 \equiv 3\,\mathrm{tr}\,\mathcal{M}_1^2 - 2\,\mathrm{tr}\,\mathcal{M}_{1/2}\mathcal{M}_{1/2}^\dagger + \mathrm{tr}\,\mathcal{M}_0^2 = -2g\langle D^a\rangle \mathrm{tr}\,T^a\,. \quad (9.5.13)$$

in this equation Str is the so-called *supertrace* and is defined as the difference of the trace over the bosonic states and the trace over the fermionic states taking into account the multiplicity of the various fields. Explicitly, for a massive vector we have three degrees of freedom[3], for a fermion we have two degrees of freedom, for a real scalar one degree of freedom.

We see that if $\langle D^a\rangle = 0$ or $\mathrm{tr}\,T^a = 0$ (that is no U(1) subgroups of the gauge group are present) the supertrace $\mathrm{Str}\,\mathcal{M}^2$ vanishes, stating that the sum of tree-level squared masses of all bosonic degrees of freedom equals the sum for all fermionic ones. Without SUSY breaking the above statement is trivially true, since any SUSY multiplet has the same number of bosonic and fermionic degrees of freedom all with the same mass. In the case of SUSY breaking this supertrace formula is still a strong constraint on the mass spectrum.

Notice that, if there are unbroken symmetries (as for example conservation of charge, color, baryon and lepton number), the mass matrices can not have elements linking particles with different values of the conserved quantum numbers. In this case the supertrace formula holds separately for each subset of states with the same conserved quantum numbers.

9.6 The Witten index

An important concept in the determination of supersymmetry breaking is the *Witten index.* It is a quantity that can help to determine when supersymmetry is *not* broken. We consider the Hilbert space $\mathcal{H}$ of the states of a supersymmetric theory, and we define the Witten index as

$$I(\beta) = \mathrm{Str}_{\mathcal{H}} e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{\beta H}\,, \quad (9.6.1)$$

where β is a positive real number and F is the fermionic number operator (i.e. -1^F is $+1$ on a bosonic state and -1 on a fermionic state). We will see that $I(\beta)$ is actually independent of β and is determined uniquely by the set of zero-energy states.

The critical observation is the fact that in a supersymmetric theory there are an equal number of bosonic and fermionic states with any given positive energy (see section 29.1 of ref. [3] for a proof of this statement). Hence the trace of $(-1)^F e^{-\beta H}$ on any given set of states with energy $H = E > 0$ vanishes. Therefore $I(\beta)$ receives contributions only from the $E = 0$ states. In formulae

$$I(\beta) = \sum_{E>\geq 0} e^{-\beta E} n(E)\,, \quad (9.6.2)$$

where

$$n(E) = \mathrm{Tr}_{\mathcal{H}_E}(-1)^F = n_+(E) - n_-(E) \quad (9.6.3)$$

is the difference between the number of bosonic and fermionic states with energy E. Given that

$$n_+(E) = n_-(E) \qquad \text{for} \quad E > 0\,, \quad (9.6.4)$$

[3]In the above equation we used 3 degrees of freedom also for possible massless vector states which only have two degrees of freedom on-shell, however massless states do not give a contribution to $\mathrm{tr}\,\mathcal{M}^2$.

we get

$$I(\beta) = \mathrm{Tr}_{\mathcal{H}_0}(-1)^F = n_+(0) - n_-(0)\,, \tag{9.6.5}$$

and $I(\beta)$ is independent of β.

The above result implies that a non-zero value for the Witten index signals the existence of some zero-energy state, which implies that supersymmetry is *not* broken. (Recall that SUSY is unbroken if and only if there exists a vacuum with zero energy.) In contrast, a zero value for the Witten index does not allow to conclude anything, since this implies only that there are an equal number of bosonic and fermionic zero-energy states, but this number could be zero (broken SUSY) or non-zero (unbroken SUSY).

Notice that the Witten index is a kind of "topological" invariant, in the sense that it does not vary in we change the parameters of the theory. This follows from the fact that we can not create or destroy states by continuously varying the parameters, we can only change their energy. But this would not change the Witten index (even if a state with $E > 0$ goes to $E = 0$ it always gives an equal number of fermionic and bosonic states with $E = 0$, so $n_+(0) - n_-(0)$ is unchanged).

Note. This is no longer true if we add a perturbation that changes the asymptotic behavior of the Lagrangian for large values of the fields: this can produce or destroy states.

As a simple example of the use of the Witten index, we can consider the Wess–Zumino model with one chiral superfield and a superpotential

$$\mathcal{W}(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{1}{6}g\Phi^3\,. \tag{9.6.6}$$

This model does not exhibit SUSY breaking at tree level, but the Witten index will allow is to conclude that SUSY breaking does not happen even at higher loop order in perturbation theory nor if we consider non-perturbative effects.

Perturbation theory is trustable if m is large and g is small. In this regime there are two zero-energy bosonic states corresponding to the configurations

$$i) \qquad \langle z\rangle = 0 \tag{9.6.7}$$

$$ii) \qquad \langle z\rangle = -\frac{2m^2}{g} \tag{9.6.8}$$

but there are no zero-energy fermionic states (the lowest-energy fermionic state is a zero-momentum one-fermion state with energy $|m|$).

Thus for large m and small g the Witten index is $I = 2$ and SUSY must be unbroken. Because the Witten index does not change under changes in the parameters it remains equal to 2 also when g is large and we have a strongly-coupled theory, or when $m = 0$ and the two potential wells merge (in this case there are massless bosons and fermions and it is not easy to compute the Witten index directly). Since the Witten index is not zero, supersymmetry remains strictly unbroken in the Wess–Zumino model, whatever the values of its parameters.[4]

What happens for gauge theories? In the supersymmetric version of QED with a Fayet-Iliopoulos term we saw that SUSY is broken, so $I = 0$. However one can show that, without Fayet-Iliopoulos terms, a generalized version of the Witten index is non-zero, so SUSY us unbroken.

For a non-Abelian gauge theory without chiral superfields, one can again show that a generalized Witten index exists that is non-zero, so SUSY in unbroken. If we ass *massive* chiral superfields to this theory the Witten index does not change, so SUSY is still unbroken. On the

[4]For other examples and a deeper discussion see section 29.1 of ref. [3].

other hand there is no difficulty in finding theories with additional *massless* chiral superfields in which supersymmetry is broken.[5]

9.7 The supersymmetric Higgs mechanism

We want now to construct a model in which only the gauge symmetry is broken. This is the analogous of the Higgs mechanism in supersymmetric models.

We start by discussing a model with only chiral superfields with a spontaneously broken global U(1) symmetry. We consider three chiral superfields Φ_0, Φ_+ and Φ_-, that have charges 0, +1 and −1 under the U(1) symmetry. The superpotential is given by

$$\mathcal{W}(\Phi) = \frac{1}{2} m\, \Phi_0^2 + \mu\, \Phi_+ \Phi_- + \lambda\, \Phi_0 + g\, \Phi_0 \Phi_+ \Phi_- \,, \tag{9.7.1}$$

and is manifestly U(1) invariant. The conditions to have a supersymmetric vacuum are

$$\begin{cases} F_0^\dagger = \lambda + m\langle z_0\rangle + g\langle z_+\rangle\langle z_-\rangle = 0 \\ F_+^\dagger = \langle z_-\rangle(\mu + g\langle z_0\rangle) = 0 \\ F_-^\dagger = \langle z_+\rangle(\mu + g\langle z_0\rangle) = 0 \end{cases} . \tag{9.7.2}$$

This set of equations has two solutions:

$$i) \qquad \langle z_+\rangle = \langle z_-\rangle = 0\,, \quad \langle z_0\rangle = -\frac{\lambda}{m}\,, \tag{9.7.3}$$

$$ii) \qquad \langle z_+\rangle\langle z_-\rangle = -\frac{1}{g}\left(\lambda - \frac{m\mu}{g}\right)\,, \quad \langle z_0\rangle = -\frac{\mu}{g}\,. \tag{9.7.4}$$

The first vacuum does not break the U(1) global symmetry, but the second does. Notice that, in the second solution, only the product $\langle z_+\rangle\langle z_-\rangle$ is determined, so we have a continuum set of vacua. For any solution $\langle z_+\rangle$, $\langle z_-\rangle$ that satisfies the condition in eq. (9.7.4), there exists an entire class of solutions $e^\eta\langle z_+\rangle$, $e^{-\eta}\langle z_-\rangle$, for arbitrary *complex* η. The ground state has a larger degeneracy than required by the initial symmetry group: this stems from the fact that the theory is invariant under the U(1) group, but also under its complex extension.

Now we want to introduce gauge invariance in the model, so we add an Abelian vector superfield V that gauges the U(1) symmetry. This changes the kinetic terms as

$$\overline{\Phi}_+ e^{2eV}\Phi_+ + \overline{\Phi}_- e^{2eV}\Phi_- \,. \tag{9.7.5}$$

We get the following equation for the D auxiliary field

$$D = -e(\langle z_+\rangle^\dagger\langle z_+\rangle - \langle z_-\rangle^\dagger\langle z_-\rangle + \xi/e) = 0\,, \tag{9.7.6}$$

where we added the contribution from a possible Fayet-Iliopoulos term. This extra condition can be always solved by choosing an appropriate value for the η parameter that parametrizes the vacua in the global U(1) case. Notice that the D-term is still invariant under U(1) global symmetries, $\langle z_+\rangle \to e^{i\eta}\langle z_+\rangle$, $\langle z_-\rangle \to e^{i\eta}\langle z_-\rangle$ with η real, as well as the F conditions, so the gauged model still has a set of degenerate supersymmetric vacua. In this model the Fayet-Iliopoulos term does not induce spontaneous supersymmetry breaking. We get a mass term for the gauge fields

$$2e^2(\langle z_+\rangle^\dagger\langle z_+\rangle + \langle z_-\rangle^\dagger\langle z_-\rangle)V^2\,. \tag{9.7.7}$$

Notice that in this case supersymmetry is not broken, so one gets a complete massive vector supermultiplet. This is the supersymmetric action of the Higgs mechanism.

[5] For more details in SUSY breaking in gauge theories and the Witten index see section 29.4 of ref. [3].

9.8 Spontaneous breaking in extended supersymmetry

Up to now we only considered spontaneous SUSY breaking in simple supersymmetry. In this section we will briefly discuss what happens in extended supersymmetry. In general the supersymmetry related to a particular Q^I_α and $\overline{Q}^I_{\dot\alpha}$ pair of generators is unbroken if the vacuum is invariant, that is

$$Q^I_\alpha|0\rangle = \overline{Q}^I_{\dot\alpha}|0\rangle = 0\,. \tag{9.8.1}$$

We have already seen that SUSY breaking in simple supersymmetry is related to the vacuum energy. This is also true in extended SUSY because (we do not sum over I)

$$\langle 0|P_0|0\rangle = \frac{1}{4}\sum_{\alpha=\dot\alpha}\langle 0|\{Q^I_\alpha, \overline{Q}^I_{\dot\alpha}\}|0\rangle = \frac{1}{4}\left(\sum_\alpha |Q^I_\alpha|0\rangle|^2 + \sum_{\dot\alpha}|\overline{Q}^I_{\dot\alpha}|0\rangle|^2\right)\,. \tag{9.8.2}$$

Notice that this relation is true for each I. So the supersymmetry generated by a given pair of Q^I_α, $\overline{Q}^I_{\dot\alpha}$ is unbroken if and only if the vacuum energy vanishes. But this obviously implies that we can either preserve all the supersymmetry (if the vacuum energy vanishes) or break SUSY completely (if the vacuum energy is positive). Thus the only spontaneous SUSY breaking pattern is

$$N \xrightarrow[]{\substack{\textit{spontaneous}\\ \textit{breaking}}} N = 0\,, \tag{9.8.3}$$

but we can not have a theory in which SUSY is only partially broken, for example

$$N \to N = 1 \to N = 0 \qquad \text{is forbidden}\,. \tag{9.8.4}$$

Note. The above statement is not true in supergravity theories, in which one can have partial SUSY breaking. Moreover one can have partial SUSY breaking in some modified theories in which part of the supersymmetry is non-linearly realized.

10

The supercurrent

Like any continuous global symmetry, supersymmetry leads to the existence of a conserved current. The conservation and commutation properties of the supercurrent are operator equations that remain valid even when supersymmetry is spontaneously broken.

The presence of an ordinary global symmetry of the Lagrangian density under an infinitesimal transformation

$$\chi^l \to \chi^l + \delta\chi^l \equiv \chi^l + \varepsilon\mathcal{F}^l \,, \tag{10.0.1}$$

where ε is an infinitesimal parameter (χ^l is a generic canonical or auxiliary field and $\mathcal{F}^l$ is a function of the canonical and auxiliary fields), leads to the existence of a Noether current

$$j^\mu = \sum_l \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi^l)}\mathcal{F}^l \,. \tag{10.0.2}$$

The current is conserved for fields satisfying the field equations

$$\partial_\mu\left(\frac{\partial_L\mathcal{L}}{\partial(\partial_\mu\chi^l)}\right) = \frac{\partial_L\mathcal{L}}{\partial\chi^l} \,, \tag{10.0.3}$$

notice that here we chose the convention to take left-derivatives (∂_L) when we are deriving with respect to a fermion field. The spatial integral of the conserved current density, j^0, gives the generators of the symmetry

$$Q = \int d^3x\, j^0(x) \,, \tag{10.0.4}$$

whose commutators computed by using the canonical commutation relations give

$$\left[\int d^3x\, j^0(x)\,, \chi^l\right] = \mathcal{F}^l(x) \,. \tag{10.0.5}$$

The supersymmetry current requires a somewhat more complicated treatment bacause supersymmetry is *only* a symmetry of the action and not of the Lagrangian. Indeed the variation of the Lagrangian under a SUSY transformation does not vanish but is a total spacetime derivative

$$\delta\mathcal{L} = \varepsilon^\alpha\partial_\mu K^\mu_\alpha + \partial_\mu\overline{K}^\mu_{\dot\alpha}\overline{\varepsilon}^{\dot\alpha} \,, \tag{10.0.6}$$

with K^μ_α (and $\overline{K}^\mu_{\dot\alpha}$) four-vectors whose components are spinors. If we compute the usual Noether current we get

$$\sum_l \delta\chi^l \frac{\partial_L\mathcal{L}}{\partial(\partial_\mu\chi^l)} \equiv -\varepsilon^\alpha N^\mu_\alpha - \overline{N}^\mu_{\dot\alpha}\overline{\varepsilon}^{\dot\alpha} \,, \tag{10.0.7}$$

whose divergence is given by

$$\begin{aligned}\varepsilon^\alpha \partial_\mu N^\mu_\alpha + \partial_\mu \overline{N}^\mu_{\dot\alpha} \overline{\varepsilon}^{\dot\alpha} &= -\sum_l (\partial_\mu \delta\chi^l) \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^l)} - \sum_l \delta\chi^l \partial_\mu \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^l)} \\ &= -\sum_l (\partial_\mu \delta\chi^l) \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^l)} - \sum_l \delta\chi^l \frac{\partial_L \mathcal{L}}{\partial \chi^l} \\ &= -\delta\mathcal{L}\,. \end{aligned} \tag{10.0.8}$$

To obtain a conserved current, we need to define the supersymmetry current as

$$S^\mu \equiv N^\mu + K^\mu\,, \tag{10.0.9}$$

which, according to the above results, satisfies

$$\partial_\mu S^\mu = 0\,. \tag{10.0.10}$$

The generators of the SUSY transformations can be expressed as

$$Q_\alpha = \int d^3x\, S^0_\alpha\,, \qquad \overline{Q}_{\dot\alpha} = \int d^3x\, \overline{S}^0_{\dot\alpha}\,. \tag{10.0.11}$$

The variation of a field X under SUSY transformations is given, as usual, by

$$\left[i\varepsilon^\alpha Q_\alpha + i\overline{Q}_{\dot\alpha}\overline{\varepsilon}^{\dot\alpha}, X\right] = \delta X\,. \tag{10.0.12}$$

As an example we can compute the supercurrent for the massless Wess–Zumino model. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu z)^\dagger \partial^\mu z - i\psi\sigma^\mu\partial_\mu\overline{\psi} + F^\dagger F\,. \tag{10.0.13}$$

The SUSY transformations for the components of the chiral superfield are

$$\begin{cases} \delta z = \sqrt{2}\varepsilon\psi \\ \delta\psi_\alpha = -\sqrt{2}\varepsilon_\alpha F + \sqrt{2} i \sigma^\mu_{\alpha\dot\beta}\overline{\varepsilon}^{\dot\beta}\partial_\mu z \\ \delta F = \sqrt{2} i \partial_\mu \psi\sigma^\mu\overline{\varepsilon} \end{cases}. \tag{10.0.14}$$

By a straightforward computation we get

$$\begin{aligned}\varepsilon^\alpha N^\mu_\alpha + \overline{N}^\mu_{\dot\alpha}\overline{\varepsilon}^{\dot\alpha} &= -\delta z^\dagger \partial^\mu z - \delta z \partial^\mu z^\dagger - i\delta\overline{\psi}\overline{\sigma}^\mu\psi \\ &= -\sqrt{2}\varepsilon\psi\partial^\mu z^\dagger - \sqrt{2}\overline{\varepsilon}\overline{\psi}\partial^\mu z - i\sqrt{2}\psi\sigma^\mu\overline{\varepsilon}F^\dagger - \sqrt{2}\varepsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu z^\dagger \end{aligned} \tag{10.0.15}$$

and

$$\begin{aligned}\delta\mathcal{L} &= (\partial_\mu\delta z)^\dagger\partial^\mu z + (\partial_\mu z)^\dagger\partial^\mu\delta z + i\partial_\mu\overline{\psi}\overline{\sigma}^\mu\delta\psi + i(\partial_\mu\delta\overline{\psi})\overline{\sigma}^\mu\psi + \delta F^\dagger F + F^\dagger\delta F \\ &= \sqrt{2}\partial_\mu\left[\varepsilon\psi\partial^\mu z^\dagger + iF^\dagger\psi\sigma^\mu\overline{\varepsilon} - 2\overline{\psi}\overline{\sigma}^{\mu\nu}\overline{\varepsilon}\partial_\nu z\right]\,, \end{aligned} \tag{10.0.16}$$

which implies

$$\varepsilon^\alpha K_\alpha + \overline{K}_{\dot\alpha}\overline{\varepsilon}^{\dot\alpha} = \sqrt{2}\varepsilon\psi\partial^\mu z^\dagger + \sqrt{2}i\psi\sigma^\mu\overline{\varepsilon}F^\dagger - 2\sqrt{2}\overline{\psi}\overline{\sigma}^{\mu\nu}\overline{\varepsilon}\partial_\nu z\,. \tag{10.0.17}$$

The supercurrent is thus given by

$$\begin{aligned}\varepsilon S^\mu + \overline{S}^\mu\overline{\varepsilon} &= \varepsilon N^\mu + \overline{N}^\mu\overline{\varepsilon} + \varepsilon K^\mu + \overline{K}^\mu\overline{\varepsilon} \\ &= -\sqrt{2}\varepsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu z^\dagger - \sqrt{2}\overline{\psi}\overline{\sigma}^\mu\sigma^\nu\overline{\varepsilon}\partial_\nu z\,, \end{aligned} \tag{10.0.18}$$

or, explicitly,

$$\begin{cases} S^\mu = -\sqrt{2}\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu z^\dagger \\ \overline{S}^\mu = -\sqrt{2}\overline{\psi}\overline{\sigma}^\mu\sigma^\nu\partial_\nu z \end{cases} . \tag{10.0.19}$$

There is another definition of symmetry currents in terms of the response of the action to a local symmetry transformation. In the absence of supergravity, the action is not invariant under local SUSY transformations. If we make such a transformation with a spacetime-dependent parameter $\varepsilon(x)$, the action will change by an amount that, in order to vanish when $\varepsilon(x)$ is constant, must (even when the field equations are *not* satisfied) be of the form

$$\delta I = -\int d^4x \left[(\partial_\mu\varepsilon(x))S^\mu(x) + \overline{S}^\mu(x)(\partial_\mu\overline{\varepsilon}(x))\right] , \tag{10.0.20}$$

where $S^\mu(x)$ is a four-vector of fermion operators. To completely define S^μ we need to generalize the global SUSY transformations to a local version. There is a choice of this generalization that guarantees that S^μ coincides with the previously computed supersymmetry current. This choice is to impose that derivatives of $\varepsilon(x)$ do not appear in the transformation of the canonical or auxiliary fields. For example, for chiral superfields

$$\begin{cases} \delta z = \sqrt{2}\varepsilon(x)\psi \\ \delta\psi_\alpha = -\sqrt{2}\varepsilon_\alpha(x)F + \sqrt{2}i\sigma^\mu_{\alpha\dot{\beta}}\overline{\varepsilon}^{\dot{\beta}}(x)\partial_\mu z \\ \delta F = \sqrt{2}i\partial_\mu\psi\sigma^\mu\overline{\varepsilon}(x) \end{cases} . \tag{10.0.21}$$

With this choice the change of the action under the local SUSY transformation has two pieces. The first piece comes from the terms containing derivatives of the canonical fields. When we take their variation we get a contribution proportional to $\partial_\mu\varepsilon(x)$. One can easily check that (compare eq. (10.0.7))

$$\delta_1 I = -\int d^4x \left[(\partial_\mu\varepsilon(x))N^\mu + \overline{N}^\mu(\partial_\mu\overline{\varepsilon}(x))\right] . \tag{10.0.22}$$

The second term arises from the fact that the Lagrangian density is not invariant even under the part of the SUSY transformation that does not involve derivatives of $\varepsilon(x)$. In this case we simply get

$$\delta_2 I = \int d^4x \left[\varepsilon(x)\partial_\mu K^\mu + (\partial_\mu\overline{K}^\mu)\overline{\varepsilon}(x)\right] - \int d^4x \left[(\partial_\mu\varepsilon(x))K^\mu + \overline{K}^\mu(\partial_\mu\overline{\varepsilon}(x))\right] . \tag{10.0.23}$$

Adding the tow contributions $\delta_1 I$ and $]delta_2 I$ we get a total change

$$\delta I = -\int d^4x \left[(\partial_\mu\varepsilon(x))(N^\mu + K^m u) + (\overline{N}^\mu + \overline{K}^\mu)(\partial_\mu\overline{\varepsilon}(x))\right] , \tag{10.0.24}$$

which shows that S^μ defined by eq. (10.0.20) coincides with the supersymmetry current.

10.1 Some remarks on the supercurrent

The above definitions of the supercurrent do not uniquely specify the form of the supercurrent, because we can always introduce a modified current

$$S^\mu_{new} \equiv S^\mu + \partial_\nu A^{\mu\nu} , \tag{10.1.1}$$

with $A^{\mu\nu} = -A^{\nu\mu}$ an arbitrary antisymmetric tensor of spinors. The term $\partial_\nu A^{\mu\nu}$ is conserved whether or not the field equations are satisfied ($\partial_\mu \partial_\nu A^{\mu\nu} = 0$ because of the antisymmetry property), and its time component is a space derivative, so

$$\int d^3x\, S^0_{new} = \int d^3x\, S^0 \tag{10.1.2}$$

gives the same supercharge.

There is a particular choice of $A^{\mu\nu}$ with the feature that $\overline{\sigma}_\mu S^\mu_{new}$ turns out to be a measure of the violation of scale invariance in the theory. This is obtained in the Wess–Zumino model by choosing

$$A^{\mu\nu} = -\frac{\sqrt{2}}{3}(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu)\psi z^\dagger \,. \tag{10.1.3}$$

Let us compute explicitly the case with only a kinetic term, which is obviously scale invariant. We get

$$\overline{\sigma}^\mu S^\mu = 2\sqrt{2}\overline{\sigma}^\nu \psi \partial_\nu z^\dagger \,, \tag{10.1.4}$$

and using the equation of motion for ψ (that is $\overline{\sigma}^\nu \partial_\nu \psi = 0$), one gets

$$\overline{\sigma}^\mu S^\mu = 2\sqrt{2}\overline{\sigma}^\nu \partial_\nu (\psi z^\dagger) \,. \tag{10.1.5}$$

We also have

$$\overline{\sigma}_\mu \partial_\nu A^{\mu\nu} = -\frac{\sqrt{2}}{3}\overline{\sigma}_\mu(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu)\partial_\mu(\psi z^\dagger) = -2\sqrt{2}\overline{\sigma}^\nu \partial_\nu(\psi z^\dagger) \,, \tag{10.1.6}$$

that is

$$\overline{\sigma}_\mu S^\mu_{new} = 0 \,, \tag{10.1.7}$$

in agreement with the scale invariance of the theory.

Another remarkable property of the supercurrent is the fact that it can also be interpreted as one of the components of a *supercurrent supermultiplet*. This supermultiplet, in addition to the supercurrent, also contains the energy–momentum tensor $T^{\mu\nu}$. This fact has relevant consequences in supergravity: in a gravity theory we know that the graviton is coupled to the conserved energy–momentum tensor, the presence of supersymmetry implies that the superpartner of the graviton, the gravitino, will be coupled to the conserved supercurrent S^μ which is in the same supermultiplet as $T^{\mu\nu}$. For more details see section 26.7 or ref. [3].

11

$N = 2$ GLOBALLY SUPERSYMMETRIC ACTIONS

In this section we will discuss the simplest examples of theories with extended global supersymmetry, namely the $N = 2$ globally supersymmetric actions. Theories with unbroken extended supersymmetry are not useful to build realistic extensions of the Standard Model, because they can not contain a chiral spectrum for the fermions. Nevertheless gauge theories with extended supersymmetry are interesting because they provide examples of the use of powerful mathematical methods to solve dynamical problems. Moreover extended SUSY theories present very peculiar features under renormalization, giving also the possibility to build completely finite quantum field theory models.

There are some special formalisms that have been proposed to construct Lagrangians with $N = 2$ supersymmetry, but we can also build them using the $N = 1$ formalism we already know. Any theory with $N = 2$ supersymmetry has obviously also $N = 1$ supersymmetry, so its Lagrangian must be a special case of the Lagrangians we already constructed for $N = 1$ SUSY. To build an $N = 2$ SUSY Lagrangian we need only to write down the most general Lagrangian with $N = 1$ SUSY whose $N = 1$ supermultiples contain physical fields for the particles in the $N = 2$ supermultiplets, and then impose a discrete R-symmetry on the Lagrangian. The Lagrangian will then be invariant under a second supersymmetry, whose supermultiplets are given by acting on the original $N = 1$ supermultiplets with the R-symmetry.

It is sufficient to choose the discrete R-transformation so that

$$Q^1_\alpha \to Q^2_\alpha \,, \qquad Q^2_\alpha \to Q^1_\alpha \,. \tag{11.0.1}$$

If the central charges were zero, the SUSY algebra would be invariant under an SU(2) R-symmetry group (our discrete transformation is just the element $\exp(i\pi\sigma_2/2)$), but requiring invariance under the discrete symmetry is enough to build an $N = 2$ theory, so we do not need to assume that the central charges are zero. However we will see that the Lagrangian we will build by this method will have an SU(2) R-symmetry, and not just the discrete symmetry we considered.

11.1 The vector multiplet Lagrangian

We saw that in $N = 2$ a massless gauge boson belongs to a multiplet also containing a pair of massless fermions that transform as a doublet under the SU(2) R-symmetry and a pair of real SU(2)-singlet scalars. In an $N = 1$ Language, this multiplet is clearly formed by a vector multiplet V and a chiral multiplet Φ. However, now all the fields are in the same $N = 2$ multiplet, so they must be in the *same* representation of the gauge group, namely the adjoint representation. The two supermultiplets have the components

$$V^a = (v^a_\mu \,, \lambda^a \,, D^a) \tag{11.1.1}$$

and

$$\Phi^a = (z^a\,,\psi^a\,,F^a)\,. \tag{11.1.2}$$

Under the discrete R-transformation

$$\psi^a \to \lambda^a \qquad \text{and} \qquad \lambda^a \to -\psi^a\,. \tag{11.1.3}$$

The $N = 1$ Lagrangian for the chiral multiplet is (we explicitly write the coupling constant g)

$$\begin{aligned}\mathcal{L}^{N=1}_{matter} &= \int d^2\theta d^2\overline{\theta}\mathrm{Tr}\overline{\Phi}e^{2gV}\Phi \qquad\qquad (11.1.4)\\ &= \mathrm{Tr}\left[(D_\mu z)^\dagger D^\mu z - i\psi\sigma^\mu D_\mu\overline{\psi} + F^\dagger F + i\sqrt{2}gz^\dagger\{\lambda,\psi\} - i\sqrt{2}g\{\overline{\psi},\overline{\lambda}\}z + gD[z,z^\dagger]\right]\,,\end{aligned}$$

where we defined

$$z = z^a T^a\,, \qquad \psi = \psi^a T^a\,, \qquad F = F^a T^a\,, \qquad a = 1,\ldots,\dim G \tag{11.1.5}$$

in addition to

$$\lambda = \lambda^a T^a\,, \qquad D = D^a T^a\,, \qquad v_\mu = v^a_\mu T^a\,, \tag{11.1.6}$$

and

$$\begin{cases} D_\mu z = \partial_\mu z - igv^a_\mu T^a z \\ D_\mu\psi = \partial_\mu\psi - igv^a_\mu T^a\psi \end{cases}. \tag{11.1.7}$$

The commutators and anticommutators arise since the generators in the adjoint representation are given by

$$(T^a_{adj})_{bc} = -if_{abc}\,, \tag{11.1.8}$$

and we normalize the generators as

$$\mathrm{Tr}\,T^aT^b = \delta^{ab}\,, \tag{11.1.9}$$

so that

$$\begin{aligned} x^\dagger\lambda\psi \,\to\, & z^\dagger_b\lambda^a(T^a_{adj})_{bc}\psi^c = -iz^\dagger_b\lambda^a f_{abc}\psi^c = iz^\dagger_b f_{abc}\lambda^a\psi_c \\ =\, & z^\dagger_b\lambda^a\psi^c\mathrm{Tr}\,T^b[T^a,T^c] = \mathrm{Tr}z^\dagger\{\lambda,\psi\}\,, \end{aligned} \tag{11.1.10}$$

and

$$z^\dagger Dz \to z^\dagger_b D^a(T^a_{adj})_{bc}z^c = -if_{abc}Z^\dagger_b D^a z^c = -\mathrm{Tr}\,D[z^\dagger,z] = \mathrm{Tr}\,D[z,z^\dagger]\,. \tag{11.1.11}$$

We must now add the action for the gauge superfield $\mathcal{L}^{N=1}_{gauge}$. We get

$$\begin{aligned}\mathcal{L}^{N=2}_{\mathrm{YM}} &= \frac{1}{32\pi}\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{Tr}\,W^aW_a\right) + \int d^2\theta d^2\overline{\theta}\,\mathrm{Tr}\,\overline{\Phi}e^{2gV}\Phi \\ &= \mathrm{Tr}\bigg(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda} - i\psi\sigma^\mu D_\mu\overline{\psi} + (D_\mu z)^\dagger D^\mu z \\ &\quad + \frac{\theta}{32\pi^2}g^2\mathrm{Tr}\,F_{\mu\nu}\widetilde{F}^{\mu\nu} + \frac{1}{2}D^2 + F^\dagger F \\ &\quad + i\sqrt{2}gz^\dagger\{\lambda,\psi\} - i\sqrt{2}g\{\overline{\psi},\overline{\lambda}\}z + gD[z,z^\dagger]\bigg)\,. \end{aligned} \tag{11.1.12}$$

As one can simply check, the relative coefficients of $\mathcal{L}^{N=1}_{gauge}$ and $\mathcal{L}^{N=1}_{matter}$ in the above Lagrangian have been chosen in such way to respect the discrete R-transformation: the λ and ψ kinetic terms have the same coefficient and the Yukawa couplings $z^\dagger\{\lambda,\psi\}$ and $\{\overline{\psi},\overline{\lambda}\}z$ also exhibit the symmetry. Notice that the Lagrangian has an even larger invariance: it is invariant under the SU(2) R-symmetry under which the fermions ψ and λ form a doublet (while the other fields are singlets).

The Lagrangian $\mathcal{L}_{\rm YM}^{N=2}$ has $N = 1$ supersymmetry with multiplets

$$(z^a, \psi^a, F^a) \quad \text{and} \quad (v_\mu^a, \lambda^a, D^a)\,, \tag{11.1.13}$$

and also a *second* independent $N = 1$ supersymmetry with multiplets

$$(z^a, \lambda^a, F^a) \quad \text{and} \quad (v_\mu^a, -\psi^a, D^a)\,, \tag{11.1.14}$$

it therefore satisfies the conditions imposed by $N = 2$ supersymmetry.

Notice that we can not add a superpotential term for Φ, because it would give ψ^a interactions or mass terms that are absent for λ^a.

The auxiliary fields equations are

$$F^a = 0\,, \qquad D^a = -g[z, z^\dagger]^a\,, \tag{11.1.15}$$

leading to the scalar potential

$$V(z, z^\dagger) = \frac{1}{2} g^2 \mathrm{Tr}\left([z, z^\dagger]\right)^2\,. \tag{11.1.16}$$

This potential has a minimum value of zero, which is reached not only for $z^a = 0$, but also for any set of z for which

$$\left[\sum_a t^a \,\mathrm{Re}\, z^a\,,\ \sum_b t^b \,\mathrm{Im}\, z^b\right] = 0\,. \tag{11.1.17}$$

That is, the minimum of the potential is reached for those scalar fields for which all generators $\sum_a t^a \,\mathrm{Re}\, z^a$ and $\sum_a t^a \,\mathrm{Im}\, z^a$ belong to a Cartan subalgebra of the full gauge algebra, all of whose generators commute with each other. All such value of z give zero potential, and hence unbroken $N = 2$ supersymmetry, but they are not physically equivalent, as shown for instance by the different masses they give to the gauge bosons associated to the broken gauge symmetries.

An important property of the massive states associated to the breaking of the gauge invariance is the fact that they are necessarily given by short $N = 2$ supermultiplets, that is their masses saturate the bound given by the $N = 2$ central charge. This can be easily checked by counting the degrees of freedom associated to these states.

First of all we recall that, even if the gauge symmetry is broken, SUSY is unbroken, so all the states must fill complete $N = 2$ supermultiplets. Given that the breaking of gauge invariance is spontaneous, the number of degrees of freedom associated to the fields is exactly the same as for the gauge-invariant vacuum, or, in other words, the number of degrees of freedom is the same as for massless $N = 2$ vector supermultiplets. The states that get a mass from gauge breaking will be contained in massive $N = 2$ vector supermultiplets, but the only massive multiplets that have the same number of physical states as the massless multiplets are the short supermultiplets. (Recall that long massive $N = 2$ supermultiplets have 4 times as many states as the corresponding massless multiplets.)

One can also explicitly verify that the massive states form short multiplets by computing the central charges for these multiplets. There is a way to use the supercurrents to extract the central charges (see section 28.9 of ref. [3] for a detailed discussion), the final result for the SU(2) case is

$$Z = 2\sqrt{2} v [q + ig]\,, \tag{11.1.18}$$

where $v = \langle z_3 \rangle$ is theVEV that breaks the SU(2) symmetry to U(1) and q and g are the electric and magnetic charges with respect to the unbroken U(1):

$$q = \int dS_i\, E^i\,, \qquad g = \int dS_i\, B^i \tag{11.1.19}$$

(the integrals are on an S_3 surface at spatial infinity).

By computing the masses of the states, one gets a massless multiplet related to the unbroken U(1) and a pair of massive multiplets with U(1) charges $\pm e$ (corresponding to the broken gauge generators) whose masses are given by

$$m = \sqrt{2}|ev|\,, \tag{11.1.20}$$

and whose magnetic charge is zero. For these states we saturate the BPS bound

$$m = \frac{1}{2}|Z|\,, \tag{11.1.21}$$

so they live in short $N = 2$ supermultiplets.

Note. If one relaxes the renormalizability requirement, a much more general action for $N = 2$ supermultiplets can be obtained. It turns out that its general form is

$$\mathcal{L} = \frac{1}{16\pi i}\int d^2\theta\, \mathcal{F}_{ab}(\Phi) W^{a\alpha} W^b_\alpha + \frac{1}{32\pi i}\int d^2\theta d^2\overline{\theta}\, \left(\overline{\Phi} e^{2V}\right)^a \mathcal{F}_a(\Phi) + \text{h.c.}\,, \tag{11.1.22}$$

where $\mathcal{F}_{ab}$ and $\mathcal{F}_a$ are given by

$$\mathcal{F}_a(\Phi) = \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi_a}\,, \qquad \mathcal{F}_{ab}(\Phi) = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi_a \partial \Phi_b}\,. \tag{11.1.23}$$

$\mathcal{F}(\Phi)$ is an holomorphic functional called the $N = 2$ *prepotential* and it determines the complete Lagrangian. Choosing $\mathcal{F} = \frac{\tau}{2}\mathrm{Tr}\,\Phi^2$ we get the renormalizable $N = 2$ Lagrangian.

11.2 The hypermultiplet Lagrangian

In $N = 2$ theories we can also introduce hypermultiplets in the Lagrangian. In terms of $N = 1$ superfields, an hypermultiplet is built by taking two chiral superfields

$$H_1 = (H^+, \psi^+_\alpha, F^+)\,, \qquad H_2 = (H^-, \psi^-_\alpha, F^-)\,. \tag{11.2.1}$$

The scalar components H^+, H^- form an SU(2) doublet, while the spinors $\psi^\pm_\alpha$ are singlets. We can build an $N = 2$ Lagrangian by imposing the discrete R-transformation

$$\begin{cases} H^+ \to -(H^-)^* \\ H^- \to (H^+)^* \end{cases}, \tag{11.2.2}$$

together with the transformation for the spinor components we used in the vector supermultiplet case ($\psi^+_a \to \psi^-_a$, $\psi^-_a \to -\psi^+_a$).

The Lagrangian is given by

$$\mathcal{L}^{N=2}_{hyper} = \int d^2\theta d^2\overline{\theta}\, \left(\overline{H}_1 e^{2V} H_1 + \overline{H}_2 e^{2V} H_2\right) + \int d^2\theta\, \overline{H}_1 \Phi H_2 + \text{h.c.} \tag{11.2.3}$$

One also finds that $N = 2$ supersymmetry requires H_1 and H_2 to be in complex-conjugate representations of the gauge group.[1]

[1] A mass term $\mu H^+ H^-$ can also be introduced in the $N = 2$ Lagrangian for the hypermultiplet.

11.3 $N = 4$ supersymmetric theories

We can construct an $N = 4$ SUSY theory by considering an $N = 2$ Lagrangian that contains a vector multiplet and an hypermultiplet both in the adjoint representation of the gauge group. Moreover no mass term is allowed for the hypermultiplet. To get an $N = 4$ SUSY we must also impose an SU(4) R-symmetry. Notice that $N = 4$ theories have essentially only one free parameter given by the gauge coupling. See section 27.9 of ref. [3] for a detailed discussion.

12

Non-renormalization theorems

As we briefly discussed in the introduction, an important feature of supersymmetry is the fact that it leads to cancellation of divergences in loop diagrams. For instance, this is the reason that suggested to use SUSY generalizations of the Standard Model of electroweak interactions to solve the problem of the instability of the Higgs mass against radiative corrections. In this section we will discuss more extensively the properties of SUSY theories from the point of view of renormalization. We will see that the behavior of those theories under renormalization is extremely simple and the "finiteness" of the theories improves foe extended supersymmetry leading to the result that $N = 4$ super-Yang–Mills theories are free of ultraviolet divergences.

12.1 Non-renormalization theorems in $N = 1$

In this section we discuss a method proposed by Seiberg, who showed how the non-renormalization theorems may be obtained by simple symmetry considerations.

We consider a general renormalizable SUSY theory with chiral superfields Φ_n and gauge superfields V_A. The Lagrangian is of the form

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\ \overline{\Phi} e^{2V} \Phi + \int d^2\theta\ \mathcal{W}(\Phi) + \frac{1}{4g^2} \int d^2\theta\ W_A^\alpha W_{A\alpha} + \text{h.c.}\,, \qquad (12.1.1)$$

where $\mathcal{W}(\Phi)$ is a cubic polynomial and we ignored possible θ_{ym}-terms, which have no effect in perturbation theory.

We will see that, under radiative corrections

- the Kahler potential is renormalized at all loop orders,
- the kinetic term of the gauge fields is renormalized *only at one loop*,
- the superpotential $\mathcal{W}(\Phi)$ is *not renormalized.*

To prove the theorem we reinterpret this theory as a special case of a theory with two additional external gauge-invariant chiral superfields

$$X = (x, \psi_x, F_x)\,, \qquad Y = (y, \psi_y, F_y)\,, \qquad (12.1.2)$$

with Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\ \overline{\Phi} e^{2V} \Phi + \int d^2\theta\ Y\mathcal{W}(\Phi) + \frac{1}{4} \int d^2\theta\ X W_A^\alpha W_{A\alpha} + \text{h.c.} \qquad (12.1.3)$$

This Lagrangian becomes equal to the original one when the scalar components of the extra fields are taken to be

$$x = \frac{1}{g^2}\,, \qquad\qquad y = 1\,, \tag{12.1.4}$$

and the spinors $\psi_{x,y}$ and the auxiliary fields $F_{x,y}$ are set equal to zero.

To analyze the renormalization properties of the theory we impose a cut-off λ on the momenta in the loops. We can find a local Wilsonian effective Lagrangian $_\lambda$ by integrating out all the momenta above λ

$$\exp\left(i\int d^4x\,\mathcal{L}_\lambda\right) = \int_{|p|>\lambda}\mathcal{D}\phi\,\exp\left(i\int d^4x\,\mathcal{L}\right). \tag{12.1.5}$$

The effective Lagrangian gives the same result as the original one for S-matrix elements of processes at momenta below λ. We want to study the form of this effective Lagrangian. If we consider a cut-off procedure that preserves supersymmetry and gauge invariance we get

$$\mathcal{L}_\lambda = \int d^2\theta d^2\overline{\theta}\;J(\Phi,\overline{\Phi},V,X,Y,D_\alpha,\ldots) + \int d^2\theta\;H(\Phi,X,Y,W^\alpha) + \text{h.c.} \tag{12.1.6}$$

with J and H both gauge invariant.

The dependence on X and Y is severely limited by two additional symmetries of the action (these symmetries are only broken by non-perturbative effects). The first symmetry os a U(1) R-symmetry under which Φ, V and X are unchanged, while Y has charge 2 (notice that, given that θ has R-charge -1 and $\overline{\theta}$ has R-charge $+1$, W^α has R-charge $+1$ because it is obtained by acting with $\overline{D}^2D_\alpha$ on V), summarizing

	Φ_n	V	X	Y	θ	$\overline{\theta}$	W^α
R-charge	0	0	0	$+2$	-1	$+1$	-1

Given that $H(\Phi,X,Y,W^\alpha)$ must be chiral, it must be an *holomorphic* function of its arguments, so it can be only of first order in Y or of second order in W^α, with coefficients depending on the R-neutral superfields Φ and/or X:

$$H(\Phi,X,Y,W^\alpha) = Y\mathcal{W}_\lambda(X,\Phi) + h_{\lambda,AB}(X,\Phi)W^\alpha_A W_{B\alpha}\,. \tag{12.1.7}$$

The second symmetry is the translation of X by an imaginary constant

$$X \to X + i\xi\,, \tag{12.1.8}$$

with ξ real. This changes the Lagrangian by an amount proportional to $\text{Im}(W^\alpha_A W_{A\alpha})$, which is a total derivative and, hence, has no effect in perturbation theory. This symmetry prevents X from appearing anywhere in the effective Lagrangian in eq. (12.1.6) except where it appears in the original Lagrangian in eq. (12.1.1). So we conclude that $\mathcal{W}_\lambda$ is independent of X while $h_{\lambda\,AB}$ has the form

$$h_{\lambda\,AB} = c_\lambda\delta_{AB}X + \ell_{\lambda\,AB}(\Phi)\,, \tag{12.1.9}$$

where c_λ is a real (cut-off dependent) constant.

Writing the terms explicitly

$$H(\Phi,X,Y,W^\alpha) = Y\mathcal{W}_\lambda(\Phi) + (c_\lambda\delta_{AB}X + \ell_{\lambda\,AB}(\Phi))W^\alpha_A W_{B\alpha}\,. \tag{12.1.10}$$

We can now determine the coefficients of the effective Lagrangian by setting the auxiliary superfields X and Y to suitable values and then using perturbation theory where it is trustable.

In particular we set the spinor and auxiliary components of X and Y to zero and we take the limits

$$x \to \infty\,, \qquad y \to 0\,, \tag{12.1.11}$$

so that the gauge coupling constant vanishes as $1/\sqrt{x}$ and all Yukawa and scalar couplings derived from the superpotential vanish as y.

In this limit the only diagram that contributes to the term $Y\mathcal{W}_\lambda(\Phi)$ has a single vertex arising from the term $\int d^2\theta\, Y\mathcal{W}(\Phi) + \text{h.c.}$ in eq. (12.1.1), so we get

$$\mathcal{W}_\lambda(\Phi) = \mathcal{W}(\Phi)\,. \tag{12.1.12}$$

Moreover, with $Y = 0$, there is a conservation law that requires every term in $\mathcal{L}_\lambda$ to have an equal number of Φ and $\overline{\Phi}$, so, since $\overline{\Phi}$ can not appear in $\ell_{\lambda\,AB}$ neither can Φ. Gauge invariance then requires for a simple gauge group:

$$\ell_{\lambda\,AB} = \delta_{AB} L_\lambda\,. \tag{12.1.13}$$

Let us now count the degree of divergence of the diagrams. We have the following scaling in x:

$$\begin{cases} \text{gauge propagators} \sim 1/x \\ \text{pure gauge interactions} \sim x \\ \text{scalar propagators} \sim 1 \\ \text{scalar interactions} \sim 1 \end{cases} \tag{12.1.14}$$

With $y = 0$ the number of powers of x in a diagram with V_W pure gauge boson vertices, I_W internal gauge boson lines and any number of scalar–gauge boson vertices and scalar propagators is

$$N_x = V_W - I_W\,. \tag{12.1.15}$$

The number of loops is given by

$$L = I_W + I_\Phi - V_W - V_\Phi + 1\,, \tag{12.1.16}$$

where I_Φ is the number of internal Φ lines and V_Φ is the number of Φ-V interaction vertices. All the Φ-V vertices have two Φ lines attached, so with no external Φ lines $I_\Phi = V_\Phi$, so that

$$L = I_W - V_W + 1\,, \tag{12.1.17}$$

and we get

$$N_x = 1 - L\,. \tag{12.1.18}$$

Terms linear in X come only from the tree-level diagrams ($L = 0$), so the coefficient c_λ in the effective Lagrangian is correctly given by the tree approximation, and therefore it is equal to its value in the original Lagrangian

$$c_\lambda = 1\,. \tag{12.1.19}$$

On the other hand, the coefficient L_λ of the X-independent term is given by one-loop diagrams ($L = 1$) only.

Putting everything together

$$\mathcal{L}_\lambda = \int d^2\theta d^2\overline{\theta}\; J(\Phi, \overline{\Phi}, V, X, Y, D_\alpha, \ldots) + \int d^2\theta\; Y\mathcal{W}(\Phi) + \frac{1}{4}\int d^2\theta (X + L_\lambda) W_A^\alpha W_{A\alpha} + \text{h.c.} \tag{12.1.20}$$

where L_λ is the one-loop contribution. Setting $Y = 1$ and $X = 1/g^2$ we obtain

$$\mathcal{L}_\lambda = \int d^2\theta d^2\overline{\theta}\, J(\Phi, \overline{\Phi}, V, X, Y, D_\alpha, \ldots) + \int d^2\theta\, \mathcal{W}(\Phi) + \frac{1}{4g_\lambda^2}\int d^2\theta\, W_A^\alpha W_{A\alpha} + \text{h.c.} \quad (12.1.21)$$

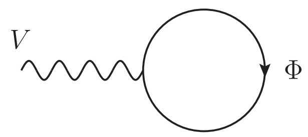
where the renormalized gauge coupling is $g_\lambda^{-2} = g^{-2} + L_\lambda$ and is corrected *only* at one loop. The Kahler potential is instead renormalized at all orders.

12.2 The Fayet–Iliopoulos term

We have seen that in the presence of U(1) gauge factors we can also introduce a Fayet–Iliopoulos term in the Lagrangian

$$\mathcal{L}_{\text{fi}} = \int d^2\theta d^2\overline{\theta}\, \xi V\,. \quad (12.2.1)$$

It can be shown that the radiative corrections to this term come only from tadpole diagrams in which a single external line is attached to a chiral loop.



The contribution from these diagrams is proportional to the sum of the gauge couplings of all chiral superfields, that is to the trace of the U(1) generator. But this trace must vanish in order to avoid gravitational anomalies, so we get that

- the Fayet–iliopoulos term is *not renormalized.*

12.3 Comments

The above results on the non-renormalization properties of SUSY theories can be extended to non-renormalizable theories as well. In such theories the $\int d^2\theta d^2\overline{\theta}\, \overline{\Phi} e^{2V}\Phi$ term is replaced by a general Kahler potential $\int d^2\theta d^2\overline{\theta}\, \mathcal{K}(\overline{\Phi}e^{2V}, \Phi)$, while the superpotential and the gauge kinetic term are replaced by an arbitrary globally gauge-invariant scalar function $f(\Phi, W_\alpha)$. It can be shown that

- $f_\lambda(\Phi, W_\alpha)$ is the same as $f(\Phi, W_\alpha)$ to all orders in perturbation theory, except for the one-loop renormalization of the term quadratic in W_α.

An important application of the non-renormalization theorem is related to supersymmetry breaking. It can be shown that, if there are no Fayet–Iliopoulos terms and if the superpotential $\mathcal{W}(\Phi)$ allows solutions of the equations $\frac{\partial \mathcal{W}(z)}{\partial z_n} = 0$, then supersymmetry is not broken at any finite order in perturbation theory. For a more detailed discussion see section 27.6 of ref. [3].

12.4 Non-renormalization theorems in extended supersymmetry

For theories with extended rigid supersymmetry even stronger non-renormalization properties are found.

- In $N = 2$ theories the full perturbative action does not contain ant correction at more than one loop.

  This result has also the consequence that, in an $N = 2$ theory is finite at one-loop level, then it is finite to all orders.

- $N = 4$ theories are *finite* at all orders in perturbation theory. These theories also have a conformal invariance and their β function ($\beta(g) = \mu d\, g/d\, \mu$) vanishes:

$$\beta(g) = 0\,. \tag{12.4.1}$$

13
The Minimal Supersymmetric Standard Model

In this section we will discuss am important application of supersymmetry to the physics of electroweak and strong interactions. The possibility of obtaining, through supersymmetry, a better UV behavior for quantum field theory and thus cancellation of divergences in loop diagrams is the main ingredient that makes supersymmetry an appealing framework to go beyond the Standard Model (SM) of electroweak interactions.

In the following we will present a minimal implementation of the SUSY framework in the context of electroweak theories, the so-called Minimal Supersymmetric Standard Model, also known as MSSM.

13.1 The SM and its problems

Before presenting the MSSM, it is useful to briefly recall the structure of the SM. This will also give us the possibility to better appreciate the theoretically unsatisfactory aspects of the SM that are the motivation for considering supersymmetry as a possible phenomenologically appealing option.

The SM is an extremely successful description of the properties and behavior of elementary particles up to the energy scale of the present collider experiments (eg. LHC). It contains three sectors whose particle content has been observed in high-energy collider experiments.

- *Gauge interactions mediators*: namely the three non-gravitational interactions, which are described by a *gauge theory* based on an internal symmetry

$$G_{\text{SM}} = \underbrace{\text{SU}(3)_c}_{\text{strong}} \times \underbrace{\text{SU}(2)_L \times \text{U}(1)_Y}_{\text{electroweak}} . \tag{13.1.1}$$

  $\text{SU}(3)_c$ is the QCD part of the SM gauge group and describes the *strong interactions*, while $\text{SU}(2)_L \times \text{U}(1)_Y$ is the *electroweak* subgroup. The label L in the $\text{SU}(2)_L$ subgroup specifies that only the left-handed fermions are charged under this subgroup, while the right-handed ones are uncharged.

  The corresponding gauge particles (which have spin 1) are the gluons, for the strong interactions, the W triplet for the $\text{SU}(2)_L$ and the B for the hypercharge $\text{U}(1)_Y$. As we will see the W and B bosons are not mass eigenstates, the latter are instead given by the $W^\pm$, the Z^0 and the photon γ. The quantum numbers of the SM gauge fields are summarized in the following table

	$\text{SU}(3)_c$	$\text{SU}(2)_L$	$\text{U}(1)_Y$
G_μ^a	8 (adj)	1	0
W_μ^a	1	3 (adj)	0
B_μ	1	1	0

- *Matter particles*: namely quarks and leptons. They come in three families with identical properties and differing only for their masses. All matter particles are fermions with spin 1/2. The fermions can be classified in terms of their quantum numbers under the $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry. For each family we have

		$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$
quarks	$Q_i = (u_L, d_L)_i$	3	2	1/6
	$\overline{u}_i = u^\dagger_{Ri}$	$\overline{3}$	1	−2/3
	$\overline{d}_i = d^\dagger_{Ri}$	$\overline{3}$	1	1/3
leptons	$L_i = (\nu, \ell_L)_i$	1	2	−1/2
	$\overline{\ell}_i = \ell^\dagger_{Ri}$	1	1	1

As we said, the SM contains three families:

$$\text{leptons:} \begin{cases} \text{- electron } e\,,\nu_e \\ \text{- muon } \mu\,,\nu_\mu \\ \text{- tau } \tau\,,\nu_\tau \end{cases} \qquad \text{quarks:} \begin{cases} \text{- up } u\,, \text{ down } d \\ \text{- charm } c\,, \text{ strange } s \\ \text{- top } t\,, \text{ bottom } b \end{cases}$$

Since all the fermions have at least a non-trivial charge, we can not include Majorana mass terms.[1] Moreover, it is easy to see that, given that left-handed and right-handed fermions are in different representations, we can not write down Dirac mass terms as well.

To be able to give masses to the matter fields we need to break the electroweak symmetry.[2]

There is also another reason for which the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry must be broken: an exact symmetry implies that the corresponding gauge bosons are massless, in which case they would mediate long-range forces. But in nature we do not observe $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ forces at long range, we only observe the $\mathrm{U}(1)_{em}$ electromagnetic force mediated by the photon. Hence we must have a (spontaneous) breaking

$$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{em}\,. \tag{13.1.2}$$

The electromagnetic group is a subgroup of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ and the electromagnetic charge is given by $T_{3L}+Y$, where T_{3L} is the charge under the Abelian subgroup of $\mathrm{SU}(2)_L$ generated by σ_3, while Y is the hypercharge.

By a spontaneous breaking the electroweak gauge bosons acquire a mass, giving the mass eigenstates $W^\mp$ and Z^0. Being massive these gauge bosons mediate only short-range forces with a typical range of order $1/m_{\mathrm{W}} \sim 1/m_{\mathrm{Z}}$. The photon on the other hand is associated to the unbroken $\mathrm{U}(1)_{em}$ subgroup and remains massless.

- *Higgs sector*. The breaking of the electromagnetic symmetry in the SM is induced by a scalar field, the Higgs boson H. it has the following quantum numbers

[1] We neglect here neutrino masses, as well as the possibility that a right-handed neutrino, neutral with respect to the whole SM gauge group, is present in the field content.

[2] Notice that most of the mass of the ordinary matter we see in Nature does not come from the elementary fermions masses. The mass of the atoms is dominated by the mass of the nucleus (formed by protons and neutrons), whereas the mass contribution coming from the electrons is three orders of magnitude smaller. Most of the nucleon masses is not due to the constituent quarks (up and down), but instead comes from QCD effects,and would be there even in the limit of unbroken electroweak symmetry.

	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
H	1	2	1/2

that is, it is a complex $SU(2)_L$ doublet. The most general renormalizable Lagrangian for the Higgs field includes a potential

$$V = m_{\mathrm{H}}^2 |H|^2 + \lambda |H|^4 \,. \tag{13.1.3}$$

Electroweak symmetry breaking is induced by a non-vanishing vacuum expectation value (VEV) for H. This is realized if $m_{\mathrm{H}}^2 < 0$ and $\lambda > 0$, in which case

$$\langle H \rangle = \sqrt{-\frac{m_{\mathrm{H}}^2}{2\lambda}} \,. \tag{13.1.4}$$

Through the gauge interactions, the Higgs VEV generates a mass term for the gauge fields

$$(D_\mu H)^\dagger (D^\mu H) \to \frac{g^2}{4} \langle H \rangle^2 W_\mu^+ W^{-\mu} + \frac{g^2}{8 \cos^2 \theta_{\mathrm{W}}} \langle H \rangle^2 Z_\mu Z^\mu \,. \tag{13.1.5}$$

From the experimental values of the W and Z masses we get

$$\langle H \rangle \simeq 246 \text{ GeV} \,. \tag{13.1.6}$$

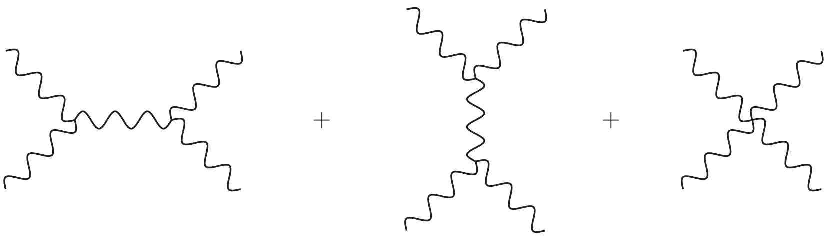
The Higgs induces also mass terms for the fermions through the Yukawa interactions

$$y_{ij}^u Q_i H \overline{u}_j + y_{ij}^d Q_i H^c \overline{d}_j + \text{h.c.} \,, \tag{13.1.7}$$

where we denoted by H^c the charge conjugate of the Higgs doublet, namely $H^c \equiv i\sigma_2 H^*$.

13.1.1 The WW scattering

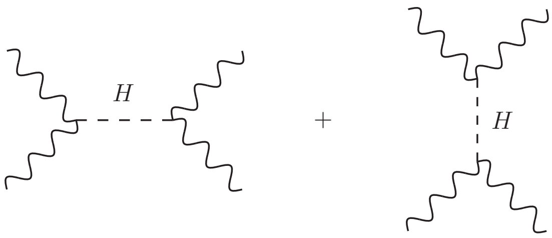
An important consequence of the presence of a Higgs doublet in the SM is the unitarization of the WW scattering amplitude. If we consider a model with spontaneous breaking of the electroweak symmetry, but without the physical Higgs mode, we get that the scattering of the longitudinal components of the W's has an amplitude that grows with the energy. The sum of the diagrams (plus the ones obtained from crossings)



leads to the amplitude

$$\mathcal{A}(W_L W_L \to W_L W_L) \simeq \frac{g^2}{4 m_{\mathrm{W}}^2} (s + t) \propto E^2 \,, \tag{13.1.8}$$

which would induce a violation of perturbative unitarity at energies $E \sim 4\pi v \simeq 3$ TeV. This behavior is modified by the presence of the physical Higgs mode, through diagrams of the type



Adding the contribution of these diagrams one finds that, for energies above the Higgs mass,

$$\mathcal{A}(W_L W_L \to W_L W_L) \sim \text{const.} \tag{13.1.9}$$

Hence, the presence of a Higgs with a mass below the TeV regularized the WW scattering process, allowing the SM to remain perturbative at (almost) arbitrarily high energies. On the other hand, if the Higgs would have been absent (or too heavy), the theory would have lost perturbativity at energies $E \gtrsim 3$ TeV.

Note. The physical Higgs mode is not necessary to write an effective theory with electroweak symmetry breaking. What is strictly needed are the Goldstone modes that correspond to the breaking of the global $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry to the $\mathrm{U}(1)_{em}$ subgroup. These modes are necessary because they play the role of the longitudinal components of the $W^{\pm}$ and of the Z^0 bosons, which are present when the gauge bosons acquire a mass. (Recall that a massive vector has 3 degrees of freedom, whereas a massless vector has only 2 degrees of freedom.)

13.1.2 The hierarchy problem

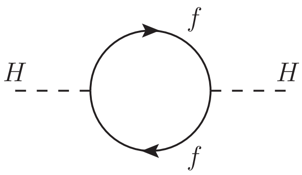
The Higgs mass term is the only mass term allowed in the SM Lagrangian (before electroweak symmetry breaking) and is also the only superrenormalizabe term. A s we already discussed in the introduction, the Higgs mass is sensitive to the UV physics through loop effects. This sensitivity comes from the fact that there is *no symmetry* that protects the Higgs mass, i.e. in the limit of vanishing Higgs mass the SM Lagrangian does not acquire any additional invariance.

The situation for the gauge boson masses and for the fermion masses is radically different, because in these cases there are some symmetries that protect them from large radiative corrections.

- Gauge invariance does not allow to write a mass term for the gauge fields. The masses arise only when the symmetry is spontaneously broken, so that they are determined by the scale of symmetry breaking (and by the gauge couplings).

- The chiral structure of the SM does not allow mas terms for the fermions. Moreover, if the Yukawa couplings are absent, the fermionic sector has a larger invariance under chiral transformations, i.e. transformations that rotate independently the left-handed and the right-handed fermions. Chiral transformations protect the Yukawa couplings from large radiative corrections because loop corrections to the couplings are necessarily proportional to the tree-level couplings, so that they are small if the tree-level couplings are small.

Let us now consider the Higgs field. We can compute the radiative corrections to the Higgs mass induced by loops of fermions (eg. the top quark) by using a cut-off regularization. We find for a fermion of mass m_f with a Yukawa interaction $\mathcal{L}_{Yuk} = -\frac{y}{\sqrt{2}}\overline{f}_L H f_R + \text{h.c.}$,



$$\Rightarrow \quad \Delta m_H^2 = -\frac{|y|^2}{8\pi^2}\left[\Lambda_{UV}^2 - 3m_f^2 \log\left(\frac{\Lambda_{UV}^2 + m_f^2}{m_f^2}\right) + \text{finite}\right]. \tag{13.1.10}$$

This shows that there are quadratically and logarithmically divergent corrections to the Higgs mass. The physical Higgs mass is given by

$$(m_H^2)_{phys} = (m_H^2)_B + (\Delta M_H^2)_{f_L, f_R}. \tag{13.1.11}$$

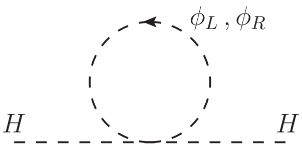
This implies that, if we assume that the SM has a high cut-off, in order to get a light Higgs we need to fine-tune the bare Higgs mass $(m_H^2)_B$ against the radiative corrections (and we need to tune the bare mass at each loop level). For example, if we assume that the SM is valid up to energies of the order of the Planck mass ($M_{Pl} \sim 10^{19}$ GeV), in order to get $(m_H)_{phys} = 125$ GeV, we need a tuning on $(m_H^2)_B$ of the order of 10^{-30}. This kind of tuning seems very unnatural, although it is *not* an inconsistency of the theory.

To obtain a theory without this unwanted tuning, we expect that some new physics should appear at an energy scale $E \lesssim 1$ TeV, which could screen the Higgs mass from radiative corrections and stabilize the electroweak symmetry breaking scale.

As we already anticipated in the introduction, supersymmetry can provide a symmetry that protect the Higgs potential from large corrections by relating the Higgs boson to a set of supersymmetric fermionic partners. To understand the mechanism, we consider the contribution to the Higgs mass coming from loops of scalar particles. We take a pair of scalars ϕ_L and ϕ_R with interactions

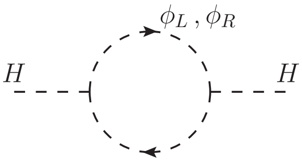
$$\mathcal{L}_{scalar} = -\frac{\lambda}{2}H^2(|\phi_L|^2 + |\phi_R|^2) - H(\mu_L|\phi_L|^2 + \mu_R|\phi_R|^2) - m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2. \tag{13.1.12}$$

We get



$$\Rightarrow \quad \begin{aligned} \Delta m_H^2\big|_1 = -\frac{\lambda}{16\pi^2}\Bigg[2\,\Lambda_{UV}^2 &- m_L^2 \log\left(\frac{\Lambda_{UV}^2 + m_L^2}{m_L^2}\right) \\ &- m_R^2 \log\left(\frac{\Lambda_{UV}^2 + m_R^2}{m_R^2}\right) + \text{finite}\Bigg]. \end{aligned} \tag{13.1.13}$$

and



$$\Rightarrow \quad \begin{aligned} \Delta m_H^2\big|_2 = -\frac{1}{16\pi^2}\Bigg[\mu_L^2 \log\left(\frac{\Lambda_{UV}^2 + m_L^2}{m_L^2}\right) & \\ + \mu_R^2 \log\left(\frac{\Lambda_{UV}^2 + m_R^2}{m_R^2}\right) + \text{finite}\Bigg]. \end{aligned} \tag{13.1.14}$$

We notice that

- if $\lambda = |y|^2$ the *quadratic divergences* are canceled,
- if, in addition, $m_f = m_L = m_R$ and $\mu_L^2 = \mu_R^2 = 2\lambda m_f^2$ also the *logarithmic divergences* are canceled.

		bosons	fermions	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
quarks	Q_i	$(\widetilde{u}_L, \widetilde{d}_L)_i$	$(u_L, d_L)_i$	3	2	1/6
	$\overline{u}_i$	$\widetilde{u}^*_{Ri}$	$\overline{u}_i = u^\dagger_{Ri}$	$\overline{3}$	1	−2/3
	$\overline{d}_i$	$\widetilde{d}^*_{Ri}$	$\overline{d}_i = d^\dagger_{Ri}$	$\overline{3}$	1	1/3
leptons	L_i	$(\widetilde{\nu}, \widetilde{e}_L)_i$	$(\nu, e_L)_i$	1	2	−1/2
	$\overline{e}_i$	$\widetilde{e}^*_{Ri}$	$\overline{e}_i = e^\dagger_{Ri}$	1	1	1
gauge bosons	G	G^a_μ	$\widetilde{G}^a$	8 (adj)	1	0
	W	$W^3_\mu, W^\pm_\mu$	$\widetilde{W}^3, \widetilde{W}^\pm$	1	3 (adj)	0
	B	B_μ	$\widetilde{B}$	1	1	0
Higgs	H_u	(H^+_u, H^0_u)	$(\widetilde{H}^+_u, \widetilde{H}^0_u)$	1	2	1/2
	H_d	(H^0_d, H^-_d)	$(\widetilde{H}^0_d, \widetilde{H}^-_d)$	1	2	−1/2

Table 13.1: Field content of the MSSM. The last three columns list the representation of the fields under $SU(3)_c$ and $SU(2)_L$ and the hypercharge $U(1)_Y$.

We have already seen that supersymmetry ensures the above relations, so if SUSY is unbroken all the divergences are canceled.

Of course, supersymmetry must be broken for phenomenological reasons. However, if we want to solve the fine-tuning problem it is sufficient to cancel the quadratic sensitivity to the cut-off Λ_{UV}, since the logarithmic divergences only introduce a very mild dependence on Λ_{UV}. To ensure this cancellation we need to break SUSY in a *controlled* way, so that the relation $\lambda = |y|^2$ is preserved. As we will see, this is obtained by assuming that SUSY is broken at low energy only by soft-breaking terms (i.e. by superrenormalizable terms, as for instance supersymmetry breaking mass terms that introduce a mass split between the SM articles and their superpartners).

13.2 The particle content of the MSSM

The field content of the MSSM consists of the SM fields and the corresponding superpartners. Only the Higgs sector requires to be enlarged, as we will discuss in the following. The MSSM field content (and our notation) is summarized in table 13.1. The bosonic superpartners get their names from the corresponding SM particles plus a prefix 's-' (eg. electron → selectron, top → stop). The fermionic superpartners are instead obtained from the SM particle names with the addition of the suffix '-ino' (eg. gluon → gluino, Higgs → Higgsino.). In the formulae we denote the superpartners by adding a $\widetilde{\ }$ to the SM particle symbol. Notice that the bar ($^-$) is part of the names of the conjugates of the right-handed fields. The particle generations are the same as in the SM:

$$\begin{aligned} u_i &= (u, c, t) \qquad & d_i &= (d, s, b) \\ \nu_i &= (\nu_e, \nu_\mu, \nu_\tau) \qquad & e_i &= (e, \mu, \tau) \end{aligned} \tag{13.2.1}$$

The Higgs sector needs to be modified with respect to the SM. We need at least twice as many Higgs doublets as in the SM. Let us see why this is the case.

If we introduce a supersymmetric Higgs doublet, its fermionic superpartner (the Higgsino) generates $U(1)^3_Y$ and $U(1)_Y\, SU(2)^2_L$ anomalies. To cancel them we need to introduce an extra doublet with opposite hypercharge. We can also include several Higgs doublets, but anomaly cancellation forces us to put them in pairs with opposite hypercharges.

With two doublets (H_u and H_d), the superpotential contains the terms

$$\mathcal{W}_{Higgs} = \overline{u} Y_u Q H_u - \overline{d} Y_d Q H_d - \overline{e} Y_e L H_d + \mu H_u H_d \,. \tag{13.2.2}$$

The dimensionless Yukawa coupling parameters $Y_{u,d,e}$ are 3×3 matrices in family space. The μ term can be written out as

$$\mu (H_u)_\alpha (H_d)_\beta \varepsilon^{\alpha\beta} \,, \tag{13.2.3}$$

where $\varepsilon^{\alpha\beta}$ is used to tie together the $\mathrm{SU}(2)_L$ indices in a gauge-invariant way. Analogously the term $\overline{u} Y_u Q H_u$ can be written as

$$\overline{u}^{ia} (Y_u)_i{}^j Q_{j\alpha a} (H_u)_\beta \varepsilon^{\alpha\beta} \,, \tag{13.2.4}$$

where $i, j = 1, 2, 3$ are family indices and $a = 1, 2, 3$ is a color index.

Notice that in the SM we can write a mass term for the up-type and down-type fermions with only one Higgs doublet, since we can use H and $H^c = i\sigma_2 H^*$ to write a Yukawa coupling. In a supersymmetric theory this is *not* possible, because $\overline{H}_u$ and $\overline{H}_d$ can not appear in the superpotential (since it must be holomorphic). Hence we are forced to introduce at least two doublets to give mass to all the quarks.

The heaviest fermions are the ones contained in the third generation, so we can approximate the Yukawa matrices as

$$Y_u \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix} , \qquad Y_d \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b \end{pmatrix} , \qquad Y_e \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau \end{pmatrix} . \tag{13.2.5}$$

In this limit only the third family contributes to the MSSM superpotential:

$$\mathcal{W}_{\text{MSSM}} \approx y_t (\bar{t} t H_u^0 - \bar{t} b H_u^+) - y_b (\bar{b} t H_d^- - \bar{b} b H_d^0) - y_\tau (\bar{\tau} \nu_\tau H_d^- - \bar{\tau} \tau H_d^0) + \mu (H_u^+ H_d^- - H_u^0 H_d^0) \,. \tag{13.2.6}$$

When H_u^0 and H_d^0 get a VEV, the Yukawa couplings generate the masses for the fermions. The minimal Higgs sector is summarized in table 13.1.

In a supersymmetric theory the Yukawa-like terms in the superpotential generate not only the usual Yukawa interactions as in the SM (Higgs-quark-quark, Higgs-lepton-lepton), but also interactions involving the superpartners (eg. squark-Higgsino-quark and slepton-Higgsino-lepton). by integrating out the auxiliary F components, one also gets quartic couplings (eg. (squark)^4, (slepton)^4, (squark)^2(Higgs)^2).

The dimensionful couplings in the supersymmetric part of the MSSM Lagrangian are all dependent on μ. Integrating out the auxiliary fields we get the Higgsinos and Higgs mass terms

$$\mathcal{L}_{Higgsino} = -\mu (\widetilde{H}_u^+ \widetilde{H}_d^- - \widetilde{H}_u^0 \widetilde{H}_d^0) + \text{h.c.} \tag{13.2.7}$$

and

$$\mathcal{L}_{Higgs} = -|\mu|^2 (|H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2) \,. \tag{13.2.8}$$

The Higgs potential has a quadratic term with a positive squared mass, so there is a stable minimum at the origin with $\langle H_u \rangle = \langle H_d \rangle = 0$. To have electroweak symmetry breaking we need to add *soft SUSY breaking terms* in the Lagrangian. A discussion of this aspect will be presented in section 13.6.

Let us now comment on the size of the μ term. To obtain a reasonable electroweak breaking scale we need $\mu \sim \mathcal{O}(m_{soft}) \sim \mathcal{O}(\text{TeV})$, otherwise unnatural cancellations are needed. In principle, there is no reason for which μ should be of TeV order, it could be as well of the order of the Planck mass (or of a grand-unification scale), so that a hierarchy of scales is reintroduced in the theory. This is known as the μ *problem*. Notice that this problem is very different from the hierarchy problem in the SM. In the latter the Higgs mass is *sensitive* to the UV physics, so it is driven to large values by radiative corrections. In supersymmetry, instead, the divergences are cancelled, so μ is only slightly (logarithmically) sensitive to the UV dynamics and receives small radiative corrections. This means that it is *technically natural*, or, in other words, once we take it to be of a certain order, its value is only mildly changed by radiative corrections, so we do *not* need a fine tuning of the bare parameter against loop corrections. The μ problem is thus only the lack of an explanation for the size of μ.

Note. A somewhat similar fact happens for the SM Yukawa couplings. The Yukawa's of the light families (eg. the electron) are much smaller than the others (eg the top quark). This difference is not explained by the SM (or the MSSM). Nevertheless, as we discussed in the introduction, small Yukawa's are stable against radiative corrections due to the chiral symmetry that is present on the limit of vanishing Yukawa's. Thus there is no need for a tuning to cancel loop corrections.

Integrating out the auxiliary fields we also get cubic scalar interactions proportional to μ

$$\begin{aligned}\mathcal{L}_{\mu,cubic} = \mu^*(&\widetilde{u}_R^* Y_u \widetilde{u}_L H_d^{0*} + \widetilde{d}_R^* Y_d \widetilde{d}_L H_u^{0*} + \widetilde{e}_R^* Y_e \widetilde{e}_L H_u^{0*} \\ &+ \widetilde{u}_R^* Y_u \widetilde{d}_L H_d^{-*} + \widetilde{d}_R^* Y_d \widetilde{u}_L H_u^{+*} + \widetilde{e}_R^* Y_e \widetilde{\nu}_L H_u^{+*}) + \text{h.c.}\end{aligned} \tag{13.2.9}$$

Note. Various solutions have been proposed to solve the μ problem. They work by assuming that the μ term is absent at tree level and arises from the VEV of some field. This VEV is generated by a potential that depends on the soft SUSY-breaking terms, so the value of μ is related to the value of the soft breaking parameters and is naturally of the same order.

13.3 R-parity and its consequences

The superpotential we wrote contains the minimal set of terms needed to produce a phenomenologically viable model. However there are other terms that one can write that are gauge invariant, holomorphic and renormalizable and could be included in the superpotential. These terms are

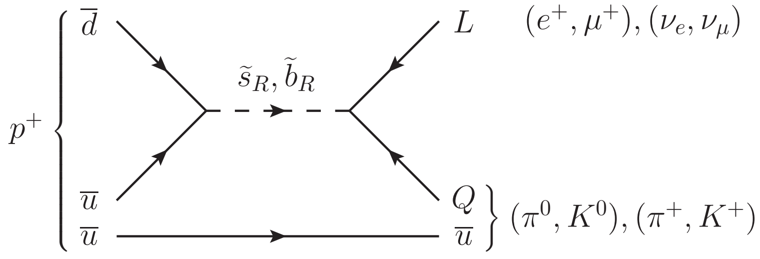
$$\mathcal{W}_{\Delta L=1} = \frac{1}{2}\lambda^{ijk} L_i L_j \overline{e}_k + \lambda'^{ijk} L_i Q_j \overline{d}_k + \mu'^j L_i H_u \tag{13.3.1}$$

$$\mathcal{W}_{\Delta B=1} = \frac{1}{2}\lambda''^{ijk} \overline{u}_i \overline{d}_j \overline{d}_k \tag{13.3.2}$$

where i, j, k are family indices. Here we conider the usual assignment of baryon and lepton numbers, that is equal numbers for all the components of a supermultiplet with

- $B = +1/3$ for Q_i; $B = -1/3$ for $\overline{u}_i$, $\overline{d}_i$; $B = 0$ for all others;
- $L = +1$ for L_i; $L = -1$ for $\overline{e}_i$; $L = 0$ for all the others.

The terms in $\mathcal{W}_{\Delta L=1}$ violate lepton number by 1, while those in $\mathcal{W}_{\Delta B=1}$ violate baryon number by 1. This violation can give rise to proton decay. The decay to a lepton and a meson is described at tree level by the Feynman diagram



We can estimate the proton decay width by noticing that it must be proportional to $\lambda''\lambda'/m_{\tilde{q}}^2$, where $m_{\tilde{q}}^2$ is the mass of the exchanged squark. By dimensional analysis we get

$$\Gamma_p \sim m_p^5 \frac{|\lambda''\lambda'|^2}{m_{\tilde{q}}^4} \,. \tag{13.3.3}$$

The proton lifetime can be estimated as

$$\tau_p = \frac{1}{\Gamma_p} \sim \frac{1}{|\lambda''\lambda'|^2} \left(\frac{m_{\tilde{q}}}{1 \text{ TeV}}\right)^4 10^{-12} \text{ s} \,. \tag{13.3.4}$$

The experimental bound on the proton lifetime is $\tau_p > 10^{32}$ years $\simeq 3 \times 10^{39}$ s. So we would need $|\lambda''\lambda'| < 10^{-25}$ (if $m_{\tilde{q}} \sim 1$ TeV), which is an unnaturally small coupling.

To solve this problem one could assume B and L conservation in the MSSM. However this is a step back with respect to the SM, where conservation of these quantum numbers is *not* assumed, but rather is an *accidental* symmetry coming from the fact that there are no possible renormalizable terms that violate B or L. Moreover it is known that B and L are violated by non-perturbative effects, so that it seems unnatural to impose B and L as symmetries by hand in the MSSM. A more elegant solution is to introduce a new discrete symmetry in the MSSM. This symmetry is called *matter parity*.

Matter parity is a multiplicatively conserved quantum number defined as

$$P_M = (-1)^{3(B-L)} \,, \tag{13.3.5}$$

for each particle in the theory. It is easy to check that the quark and lepton supermultiplets have $P_M = -1$, while the Higgs supermultiplets have $P_M = +1$. Gauge fields and gauginos have $B = L = 0$, so they have $P_M = +1$. An operator is allowed in the Lagrangian only if it is even under matter parity. It is easy to see that all the terms in $\mathcal{W}_{\Delta L=1}$ and $\mathcal{W}_{\Delta B=1}$ are forbidden, while all the terms we previously included in the superpotential are allowed.

Note. Even if matter parity is an exact symmetry, baryon and lepton number conservation could be violated in the MSSM. However the MSSM does not have renormalizable interactions that violate B or L, if matter parity conservation is assumed.

It is often useful to recast matter parity in terms of *R-parity*, defined as

$$P_R = (-1)^{3(B-L)+2s} \,, \tag{13.3.6}$$

where s is the spin of the particle. Matter parity and R-parity are equivalent, since the product of $(-1)^{2s}$ for the particles involved in an interaction vertex in a theory that conserves angular momentum is always equal to $+1$ (equivalently one can notice that all the terms in the Lagrangian have an even number of fermions). Particles in the same multiplet do not have the same R-parity, so this symmetry does not commute with supersymmetry, it is an R-symmetry. R-parity transforms the particles as

$$\begin{aligned}
(\text{SM particle}) &\to (\text{SM particle}) \,, \\
(\text{superpartner}) &\to -(\text{superpartner}) \,.
\end{aligned}$$

R-parity conservation has a series of extremely important phenomenological consequences:

- The lightest superpartner with $P_R = -1$, called the *lightest supersymmetric particle* or LSP, must be absolutely stable. If the LSP is electrically neutral, it interacts only weakly with ordinary matter and could be a dark matter candidate.

- Each superpartner other than the LSP must eventually decay into a state that contains an odd number of LSP's.

- In collider experiments, superpartners can only be produced in even numbers (usually in pairs).

Note. R-parity or matter parity could originate from a gauged U(1) symmetry that is spontaneously broken at high energy to a discrete subgroup.

13.4 Where is supersymmetry broken?

Obviously supersymmetry is not present in the low-energy particle spectrum that has been experimentally explored. None of the superpartners of the SM fields that are included in the MSSM has been seen, and this implies that supersymmetry, if present in nature, must be necessarily broken at low energy.

The simplest possibility to make the MSSM compatible with the experiment would be to have spontaneous SUSY breaking arising at tree level. However this possibility can be definitely ruled out. There are several considerations that allow us to exclude all possible direct SUSY-breaking mechanisms in the MSSM.

A first, simple argument is based on the tree-level supertrace rule:

$$\mathrm{Str}\,\mathcal{M}^2 = -2g\langle D^a\rangle\,\mathrm{Tr}\,T^a\,. \tag{13.4.1}$$

In the MSSM a VEV for D^a (in particular for the $\mathrm{U}(1)_Y$ component) is not phenomenologically viable (for example if we include a FI term we would get a VEV for the squarks or sleptons, which do not have a superpotential mass term, and this would lead to color or electric charge breaking), so we can take

$$\mathrm{Str}\,\mathcal{M}^2 = 0\,. \tag{13.4.2}$$

This is valid for each set of fields with given color representation and electric charge. In the color triplet sector with electric charge $-1/3$, the only known fermions are the d, s and b quarks, for which

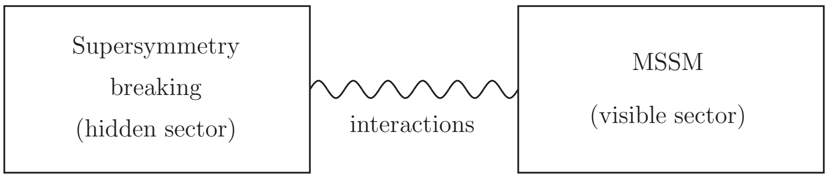
$$m_d^2 + m_s^2 + m_b^2 \simeq (5\ \mathrm{GeV})^2\,. \tag{13.4.3}$$

According to the sum rule, if there are no other fermions with this color and charge, then the sum of all squared masses for bosons with the same color and charge must equal about $2\times(5\ \mathrm{GeV})^2$. This implies that each of the squarks with this color and charge must have a mass not greater than ~ 7 GeV. The existence of such states is definitely ruled out experimentally. This argument, however, could be invalidated if there were a fourth generation of quarks.

A much stronger argument can be obtained by more carefully studying the possible SUSY-breaking mechanisms in the MSSM. The unbroken conservation of electric charge and color tells us that the only non-zero D-terms are the ones for the $\mathrm{U}(1)_Y$ generator (and at most also the t_3 generator of $\mathrm{SU}(2)_L$, which is allowed when $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ is broken to $\mathrm{U}(1)_{em}$). Moreover all the F-terms must vanish for the quark fields (given that they are color triplets) and the VEV's for the squarks must vanish as well.

One can prove that, depending on the sign of the D-terms, there must be either a squark of charge $2/3$ lighter than the u quark, or a squark of charge $-1/3$ lighter than the d quark. Both these possibilities are experimentally ruled out. So we are forced to reject the possibility of having SUSY spontaneously broken at tree level in the MSSM.

The above results are not the only arguments that show that SUSY breaking can not easily happen in the MSSM. The obvious consequence is that we need to assume that SUSY is broken by some unknown mechanism in a *hidden* sector, which has only small direct coupling with the *visible* sector given by the MSSM.



The breaking of SUSY in the hidden sector is communicated to the visible sector through some shared interactions, that induce only *soft* SUSY-breaking terms in the visible sector. The assumption that the generated terms are soft (i.e. superrenormalizable) comes form the fact that we want to preserve the protection mechanisms of the MSSM, which avoid the presence of quadratic divergences in the masses coming from loop corrections. We will discuss in more details this aspect in sections 13.5 and 13.6.

This mechanism of SUSY breaking through an hidden sector has some nice advantages. One of these is the fact that, if the mediating interactions are flavor-blind, then the soft terms appearing in the MSSM will not introduce large flavor-changing effects which are not seen experimentally.

Moreover we could also think to a mechanism that explains the difference of scale of the SUSY breaking scale and for example the Planck scale of a possible unification scale M_X. This can be obtained if SUSY is broken by non-perturbative effects. In particular, if there is a gauge field with an asymptotically free gauge coupling $g(\mu)$ at the renormalization scale μ and if $g^2(\mu)/8\pi^2 \ll 1$ for $\mu \approx M_X$, then, by running, this gauge interaction will become strong at an energy of order

$$\Lambda_S \simeq M_X \exp(-8\pi^2 b/g^2(M_X)) , \tag{13.4.4}$$

where b is a number of order unity. If SUSY is broken by this coupling that becomes strong, then a big difference between Λ_S and M_X can be naturally explained.

Notice that a similar mechanism, known as *dimensional transmutation*, is realized in QCD, whose strong-coupling scale $\Lambda_{\text{QCD}} \sim 300$ MeV is dynamically generated through the running of the $\mathrm{SU}(3)_c$ gauge coupling. The Λ_{QCD} scale is hierarchically smaller than any UV scale at which we imagine to start the QCD running. Moreover Λ_{QCD} determines the typical mass scale of the QCD composite resonances, such as the mesons and the baryons.

For the mechanism that can mediate SUSY breaking there are two leading candidates. One is provided by the $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge interactions themselves, the other is gravitation. These two mechanisms give rather different phenomenological predictions for the scale of SUSY breaking and for the masses of some superpartners. In the following we will summarize some general results largely independent if the exact implementation of the SUSY breaking mechanism.

In *gauge-mediated supersymmetry breaking* the mediating interactions are the usual electroweak and QCD gauge interactions. The MSSM soft terms come from loop diagrams involving some *messenger* fields. The messengers are new chiral supermulitplets that couple to a SUSY-breaking VEV $\langle F \rangle$, and also have $SU(3)_c \times SU(2)_L \times U(1)_Y$ interactions, which provide the necessary connection to the MSSM. Performing a rough estimate we get for the induced soft terms

$$m_{soft} \sim \frac{\alpha_a}{4\pi} \frac{\langle F \rangle}{M_{mess}}, \tag{13.4.5}$$

where $\alpha_a/4\pi = g_a^2/16\pi^2$ is a loop factor for Feynman diagrams involving gauge interactions and M_{mess} is a characteristic scale of the masses of the messenger fields. If $\sqrt{\langle F \rangle}$ and M_{mess} are roughly comparable, then, in order to have $M_{soft} \sim 1$ TeV, we need a SUSY-breaking scale

$$\sqrt{\langle F \rangle} \sim 100 \text{ TeV}. \qquad \text{(gauge mediated)} \tag{13.4.6}$$

In *Planck-mediated* (or *gravity-mediated*) *supersymmetry breaking* the mediating interactions are the gravitational interactions. The estimate of the SUSY-breaking scale varies according to the SUSY-breaking mechanism.

- If SUSY is broken in the hidden sector by a VEV $\langle F \rangle$, then the soft terms in the visible sector can be estimated as

$$m_{soft} \sim \frac{\langle F \rangle}{M_{Pl}} \tag{13.4.7}$$

by dimensional analysis. This is because we know that m_{soft} must vanish in the limit $\langle F \rangle \to 0$ when SUSY is unbroken, and also in the limit $M_{Pl} \to \infty$, in which gravity becomes irrelevant. This leads to the estimate

$$\sqrt{\langle F \rangle} \sim 10^{11} \text{GeV}. \qquad \text{(gravity mediated I)} \tag{13.4.8}$$

- Another possibility is that the SUSY-breaking order parameter is a *gaugino condensate* $\langle 0|\lambda^a \lambda^b|0\rangle = \delta^{ab}\Lambda^3 \neq 0$. if the composite field $\lambda^a\lambda^b$ is part of an auxiliary field F for some (possibly composite) chiral superfields, then by dimensional analysis we expect

$$m_{soft} \sim \frac{\Lambda^3}{M_{Pl}^2}, \tag{13.4.9}$$

with, effectively, $\langle F \rangle \sim \Lambda^3/M_{Pl}$. In this case, the scale of SUSY breaking should be roughly

$$\Lambda \sim 10^{13} \text{ GeV}. \qquad \text{(gravity mediated II)} \tag{13.4.10}$$

The large difference in the estimates of the SUSY-breaking scale M_S for gauge and gravitational mediation makes an important difference in particle phenomenology and cosmology. Supersymmetry dictates that the graviton must have a partner of spin 3/2, the *gravitino.* When SUSY is broken the gravitino acquires a mass of order M_S^2/M_{Pl} and it 'eats' the Goldstino in order to acquire the additional spin 1/2 components needed to form a massive spin 3/2 field.

For gauge-mediated SUSY breaking the gravitino mass is very small

$$m_g \sim \frac{M_S^2}{M_{Pl}} \approx 1 \text{ eV}. \tag{13.4.11}$$

Thus the gravitino would be by far the lightest of the new particles required by supersymmetry. Conservation of R-parity would also imply that the gravitino is stable.

On the other hand, for gravity-mediated SUSY breaking the gravitino mass is just of the same order of magnitude of the mass splitting between the known SM particles and their superpartners, so it would have roughly the same mass as the squarks, sleptons and gauginos. In this case the gravitino may or may not be the lightest supersymmetric partner.

If the gravitino is the lightest SUSY particle it could play the role of dark matter. From this observation one can find limits on the SUSY-breaking scale. In the case of gauge mediation, we have a very light gravitino ($m_g \sim 1$ eV) and the bounds require a dark matter candidate to be lighter than ~ 1 keV. Thus this limit is well satisfied, but at the same time the gravitino is too light to give an appreciable contribution to the mass density of the universe. In the case of gravity mediation, the experimental constraints require the gravitino to be heavy enough, roughly $m_g > 10$ TeV. This implies that the SUSY-breaking scale should be $M_S > 10^{11}$ GeV in the case of breaking through a VEV $\langle F \rangle$, while it should be $M_S > 10^{13}$ GeV in the case of breaking through a gaugino condensate.

13.5 Soft supersymmetry-breaking interactions

Let us now discuss what kind of SUSY-breaking terms can be introduced in a SUSY theory if we want to preserve the cancellation of quadratic divergences in the loop corrections to the masses of the scalars.

The soft SUSY-breaking terms in a general theory are

$$\begin{aligned} \mathcal{L}_{soft} &= -\left(\frac{1}{2}M_a\lambda^a\lambda^a + \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b^{ij}\phi_i\phi_j + t^i\phi_i\right) + \text{h.c.} - (m^2)^i{}_j\phi^{j*}\phi_i\,, \\ \mathcal{L}_{maybe\text{-}soft} &= -\frac{1}{2}c_i^{jk}\phi^{i*}\phi_j\phi_k + \text{h.c.} \end{aligned} \tag{13.5.1}$$

They consist of

- gaugino masses M_a for each gauge group,
- scalar squared-mass terms $(m^2)^i{}_j$ and b^{ij},
- (scalar)^3 couplings a^{ijk} and c_i^{jk},
- “tadpole” couplings t^i (they can occur only if ϕ_i is a gauge singlet, so it is absent in the MSSM).

Note. We could also include soft mass terms for the chiral supermultiplets fermions, like $\mathcal{L} = -\frac{1}{2}m^{ij}\psi_i\psi_j + \text{h.c.}$. However, including such terms would be redundant: they can always be absorbed into a redefinition of the superpotential and terms $(m^2)^i{}_j$ and c_i^{jk}.

It can be shown that a softly-broken SUSY theory with $\mathcal{L}_{soft}$ is free of quadratic divergences in quantum corrections to scalar masses to all orders in perturbation theory. The situation is more subtle if one tries to include the terms in $\mathcal{L}_{maybe\text{-}soft}$. If any of the chiral supermultiplets in the theory are singlets under all gauge symmetries, then non-zero c_i^{jk} terms can lead to quadratic divergences, despite the fact that they are superrenormalizable and, therefore, formally soft. This constraint does not apply to the MSSM, which does not include gauge-singlet chiral superfields. Nevertheless, the possibility of c_i^{jk} terms is nearly always neglected, because it is difficult to construct models of spontaneous SUSY breaking in which the c_i^{jk} coefficients are not negligibly small.

Note. In the special case of a theory that has chiral supermultiplets that are singlets or in the adjoint representation of a simple factor of the gauge group, there are also possible soft SUSY-breaking mass terms between the corresponding fermions ψ_a and the gauginos

$$\mathcal{L} = -M_D^a \lambda^a \psi_a + \text{h.c.} \tag{13.5.2}$$

This is not relevant for the MSSM, which does not have chiral multiplets in the adjoint representation.

A few comments.

- The gaugino masses M_a are always allowed by gauge symmetry.
- The $(m^2)^i{}_j$ terms are allowed for i, j such that ϕ_i, ϕ^{j*} transform in complex conjugate representations of each other under all gauge symmetries. in particular this is true if $i = j$, so every scalar is eligible to get a mass in this way if SUSY is broken.
- The a^{ijk}, b^{ij} and t^i terms have the same form of some terms in the superpotential, so they are allowed by gauge invariance if and only if a corresponding superpotential term is allowed.

13.6 Soft supersymmetry-breaking in the MSSM

We have discussed the form of the most general set of soft supersymmetry-breaking terms. Now we want to understand what happens in the MSSM. Applying the general recipe to the MSSM we get the following set of soft SUSY-breaking terms

$$\begin{aligned}
\mathcal{L}_{soft}^{\text{MSSM}} = & -\frac{1}{2}(M_3\, \widetilde{g}\, \widetilde{g} + M_2\, \widetilde{W}\, \widetilde{W} + M_1\, \widetilde{B}\, \widetilde{B} + \text{h.c.}) \\
& - (\widetilde{\overline{u}}\, a_u\, \widetilde{Q}\, H_u - \widetilde{\overline{d}}\, a_d\, \widetilde{Q}\, H_d - \widetilde{\overline{e}}\, a_e\, \widetilde{L}\, H_d + \text{h.c.}) \\
& + \widetilde{Q}^\dagger m_Q^2 \widetilde{Q} - \widetilde{L}^\dagger m_L^2 \widetilde{L} - \widetilde{\overline{u}}\, m_{\overline{u}}^2\, \widetilde{\overline{u}}^\dagger - \widetilde{\overline{d}}\, m_{\overline{d}}^2\, \widetilde{\overline{d}}^\dagger - \widetilde{\overline{e}}\, m_{\overline{e}}^2\, \widetilde{\overline{e}}^\dagger \\
& - m_{H_u}^2 H_u^* H_u - m_{H_d}^2 H_d^* H_d - (b H_u H_d + \text{h.c.})\,.
\end{aligned} \tag{13.6.1}$$

In the above expression

- M_3, M_2 and M_1 are the gluino, wino and bino mass terms,
- the second line contains the (scalar)^3 couplings; each of a_u, a_d, a_e is a complex 3×3 matrix in family space, with dimension of mass, and they are in one-to-one correspondence with the Yukawa couplings of the superpotential,
- the third line consists of squark and slepton mass terms of the $(m^2)^i{}_j$ type; each of m_Q^2, $m_{\overline{u}}^2$, $m_{\overline{d}}^2$, m_L^2, $m_{\overline{e}}^2$ is a 3×3 matrix in family space that can have complex entries, but they must be Hermitian, so that the Lagrangian is real,
- in the last line we have supersymmetry-breaking contributions to the Higgs potential; $m_{H_u}^2$ and $m_{H_d}^2$ are squared-mass terms of the $(m^2)^i{}_j$ type, while b is the only squared mass term of the type b^{ij} that can occur in the MSSM.

On dimensional grounds, we expect

$$M_1\,,\ M_2\,,\ M_3\,,\ a_u\,,\ a_d\,,\ a_e \sim m_{soft}\,, \tag{13.6.2}$$

$$m_Q^2\,,\ m_L^2\,,\ m_{\overline{u}}^2\,,\ m_{\overline{d}}^2\,,\ m_{\overline{e}}^2\,,\ m_{H_u}^2\,,\ m_{H_d}^2\,,\ b \sim m_{soft}^2\,, \tag{13.6.3}$$

with a characteristic mass scale m_{soft} that is not much larger than 1 TeV.

Unlike the supersymmetry-preserving part of the MSSM Lagrangian, the above $\mathcal{L}_{soft}^{\mathrm{MSSM}}$ introduces many new parameters that were not present in the ordinary SM Lagrangian. A careful count reveals that there are 105 parameters (masses, phases and mixings) in the MSSM Lagrangian that can not be rotated away by redefining the phases and flavor basis of the quark and lepton supermultiplets, and that have no counterpart in the ordinary SM. Thus, in principle, supersymmetry breaking appears to introduce a tremendous arbitrariness in the Lagrangian.

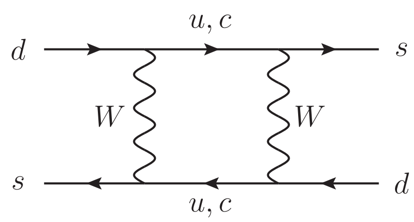
13.7 Constraints on the supersymmetry-breaking terms

In analyzing the phenomenological implications of the MSSM we must deal not only with the search for new particles, but also with two classes of severe empirical constraints on processes involving known particles: the experimental upper bounds on various flavor changing processes and on CP non-conservation.

13.7.1 Flavor-changing processes

In the SM there is an automatic suppression of flavor-changing processes like $K^0\overline{K}^0$ oscillations and $K^0 \to \mu + \mu_-$. This is due to a peculiar feature of the SM, the fact that it is only the mass splittings among the quarks that prevents them from being defined so that each flavor is separately conserved. Hence the amplitude of these flavor-changing processes must be proportional to several factors of small masses (this is known as the GIM mechanism, from Glashow, Iliopoulos and Maiani, who pointed out this feature).

In the SM $K^0 - \overline{K}^0$ oscillations, which are produced by effective operators of the form $(\overline{s}_L\gamma^\mu d_L)(\overline{s}_L\gamma_\mu d_L)$, can be induced by diagrams like

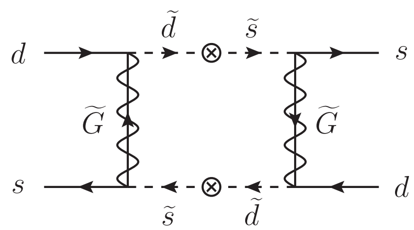


This gives the amplitude for $d_L\overline{s}_L \to s_L\overline{d}_L$, which is proportional to

$$\frac{g^4 \sin^2\theta_c \cos^2\theta_c}{m_{\mathrm{W}}^4}(m_c - m_u)^2\,, \tag{13.7.1}$$

where θ_c is the Cabibbo angle that appears in the CKM matrix $V_{ud} = \cos\theta_c$ ($\theta_c \simeq 0.2$). Computing the $K^0 - \overline{K}^0$ oscillations from this result one finds a prediction in good agreement with the experiments, it is then reasonable to require that the new physics contributions to the $d_L\overline{s}_L \to s_L\overline{d}_L$ transition should be smaller than the SM ones.

The squarks contribute to this process through diagrams of the type



If the superpartners of the d_L and s_L quarks are given by $\sum_i V_{di}\mathcal{D}_i$ and $\sum_i V_{si}\mathcal{D}_i$ of the squarks $\mathcal{D}_i$ of definite mass, we get an amplitude proportional to

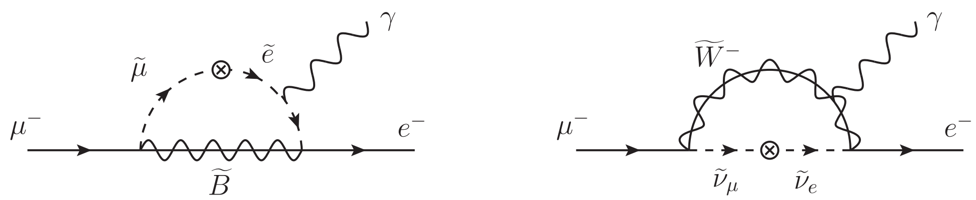
$$\frac{g_s^4}{\widetilde{M}^6}\left(\sum_i V_{di}V_{si}^*\Delta\widetilde{M}_i^2\right)^2 , \tag{13.7.2}$$

where $\widetilde{M}$ is the largest of M_{squark} squark and m_{gluino}, ΔM_i are the squark mass differences, while g_s is the strong coupling constant. Imposing that this amplitude is smaller then the SM one, we get the constraint

$$\left|\sum_i V_{di}V_{si}^*\frac{\Delta M_i}{\widetilde{M}^2}\right| < 1.5\times 10^{-3}\left(\frac{\widetilde{M}}{100\text{ GeV}}\right) . \tag{13.7.3}$$

The squark masses are unlikely to be much less than the gluino mass, so we can conclude that the squark masses are nearly degenerate, or the non-diagonal terms in the V_{ji} matrix must be small, or the squarks are heavier than about 10 TeV. This is one of the strongest constraints on the MSSM parameters.

Another constraint comes from the process $\mu\to e\gamma$. In the SM lepton flavor is automatically conserved, so that processes like $\mu\to e\gamma$ are forbidden.[3] In the MSSM this process is generated by the diagrams



The branching ratio for this process can be estimated as

$$\text{Br}(\mu\to e\,\gamma)\sim 3\times 10^{-4}\left(\frac{500\text{ GeV}}{\widetilde{M}}\right)^4\left(\frac{\Delta m^2}{\widetilde{M}^2}\right)^2 , \tag{13.7.4}$$

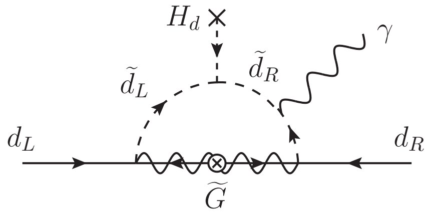
where $\widetilde{M}$ is the mass scale of the superpartners, while Δm^2 is the mass splitting. Experimentally the branching ratio has an upper bound $\text{Br}(\mu\to e\,\gamma) < 5.7\times 10^{-13}$, thus setting strong constraints on the superpartner masses and/or on the mass splittings.

13.7.2 CP-violation

The second important class of constraints provided by experimental data has to do with CP-violating effects, such as the electric dipole moments of the neutron and electron. These effects are rather small in the SM. This is because all CP-violating phases in the mass matrix of quarks and leptons and their interactions with gauge bosons could be absorbed into the definition of the quark and lepton fields if there were only two generations of quarks and leptons. Moreover, although there is a third generation, its mixing with the first two generations is quite small. The electric dipole moment of the neutron in the SM is thus expected to be less than about $10^{-30}e\,\text{cm}$, well below the experimental upper bound, $2.9\times 10^{-26}e\,\text{cm}$.

On the contrary, the parameters of the MSSM include dozens of CP-violating phases. For example the following one-loop diagram

[3]The presence of neutrino masses induced small lepton flavor violation. However these effects are tiny, since they are suppressed by the small neutrino masses.



gives rise to an electric dipole moment for the d quark, and hence for the neutron. In the above diagram the $\times$ denotes an insertion of the H_d VEV, which arises after electroweak symmetry breaking. Since the electric dipole moment is a CP-violating quantity, the amplitude needs a non-trivial complex phase which can be supplied by the soft SUSY-breaking parameters. If we call the overall phase δ then the electric dipole moment is approximately given by

$$\mathcal{M}_{\text{EDM}} \sim \frac{g^2}{16\pi^2} \frac{e\langle H_d\rangle (a_d)_{11}\delta}{\widetilde{M}^2} . \tag{13.7.5}$$

The experimental bound on the electric dipole moment of the neutron translates into the bound

$$(a_d)_{11}\delta \left(\frac{500 \text{ GeV}}{\widetilde{M}}\right)^2 \lesssim 1.5 \times 10^{-17} , \tag{13.7.6}$$

with $(a_d)_{11}$ in GeV units.

13.7.3 Soft supersymmetry-breaking universality

All the potentially dangerous flavor-changing and CP-violating effects in the MSSM can be avoided if one assumes that SUSY breaking is suitably *universal*. If the squarks and slepton squared mass matrices are flavor-blind

$$m_Q^2 = m_{\tilde{Q}}^2 \mathbb{1} , \qquad m_{\bar{u}}^2 = m_{\tilde{\bar{u}}}^2 \mathbb{1} , \qquad m_{\bar{d}}^2 = m_{\tilde{\bar{d}}}^2 \mathbb{1} , \qquad m_L^2 = m_{\tilde{L}}^2 \mathbb{1} , \qquad m_{\bar{e}}^2 = m_{\tilde{\bar{e}}}^2 \mathbb{1} , \tag{13.7.7}$$

then all squarks and sleptons mixing angles are rendered trivial, because squarks and sleptons with the same quantum numbers are degenerate in mass and can be rotated into each other. SUSY contributions to flavor-changing processes are very small in such a limit, up to mixing induced by a_u, a_d and a_e. With the further assumption that the (scalar)^3 couplings are proportional to the corresponding Yukawa coupling matrices

$$a_u = A_{u0} y_u , \qquad a_d = A_{d0} y_d , \qquad a_e = A_{e0} y_e , \tag{13.7.8}$$

only the squarks and sleptons of the third family can have large (scalar)^3 couplings. Finally, we can avoid large CP-violating effects by assuming that the soft parameters do not introduce new complex phases. This is automatic for $m_{H_u}^2$ and $m_{H_d}^2$, and for m_Q^2, $m_{\bar{u}}^2$, etc. if eq. (13.7.7) is assumed; if they were not real numbers the Lagrangian would not be real. One can also fix μ in the superpotential and b to be real, by appropriate phase rotations of fermions and scalar components of the H_u and H_d supermultiplets. If one then assumes

$$\arg(M_1) , \arg(M_2) , \arg(M_3) , \arg(A_{u0}) , \arg(A_{d0}) , \arg(A_{e0}) = 0 \text{ or } \pi , \tag{13.7.9}$$

then the only CP-violating phase in the theory is the usual CKM phase present in the ordinary Yukawa couplings.

The conditions in eqs. (13.7.7), (13.7.8) and (13.7.9) implement the hypothesis of *soft supersymmetry-breaking universality*.

Note. Often the universality conditions on the soft breaking terms are imposed at a high-energy scale (usually the coupling unification scale $\sim 10^{16}$ GeV). The soft terms at low energy are obtained by a renormalization group evolution.

There are other possible types of explanations for the suppression of flavor violation in the MSSM, that could replace the universality hypothesis. One possibility is the so called *alignment hypothesis*, that is the idea that the squark squared-mass matrices do not respect the flavor blindness in eq. (13.7.7), but are arranged in flavor space to be aligned with the relevant Yukawa matrices in such a way to avoid large flavor-changing effects. The alignment models typically require rather special flavor symmetries.

13.8 Gauge coupling unification

Now we consider an application of supersymmetry in one context in which the mechanism for the breakdown of supersymmetry is relatively unimportant, and in which supersymmetry has scored a great empirical success.

If the $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge group of the SM is embedded in a simple group G that has the known quarks and leptons (plus perhaps some fermions neutral under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$) as a representation, then at energies at or above the scale M_X at which G is spontaneously broken, the $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ coupling constants will be related by

$$g_s^2 = g^2 = \frac{5}{3} g'^2 \qquad \text{at energies } \geq M_X \,. \tag{13.8.1}$$

This, for example, happens if we embed the SM group into SU(5) or SO(10). At energies far below M_X, these couplings are seriously affected by renormalization corrections. If measured at a scale $\mu < M_X$, the couplings have values $g_s^2(\mu)$, $g^2(\mu)$, $g'^2(\mu)$ governed by the one-loop renormalization group equations

$$\mu \frac{d}{d\mu} g_i(\mu) = \beta_i(g_i(\mu)) \equiv \frac{1}{16\pi^2} b_i g_i^3 \,. \tag{13.8.2}$$

This implies that $\alpha_i^{-1} \equiv (g_i^2/4\pi)^{-1}$ run linearly with the energy at one loop:

$$\mu \frac{d}{d\mu} \alpha_i^{-1} = -\frac{b_i}{2\pi} \,. \tag{13.8.3}$$

Since M_X will turn out to be many orders of magnitude larger than the energies accessible with today accelerators, it seems reasonable to suppose that supersymmetry is unbroken over most of the range below M_X, in which case all the superpartners need to be included in the computation of the $\beta(g_i)$ functions.

In the SM the β function coefficients are

$$\begin{cases} b(g') = \dfrac{41}{6} \\ b(g) = -\dfrac{19}{6} \\ b(g_s) = -7 \end{cases} . \tag{13.8.4}$$

With these values the coupling constants become almost equal at an energy scale $\mu \sim 10^{13} - 10^{15}$ GeV, but they do not exactly unify (see figure 13.1).

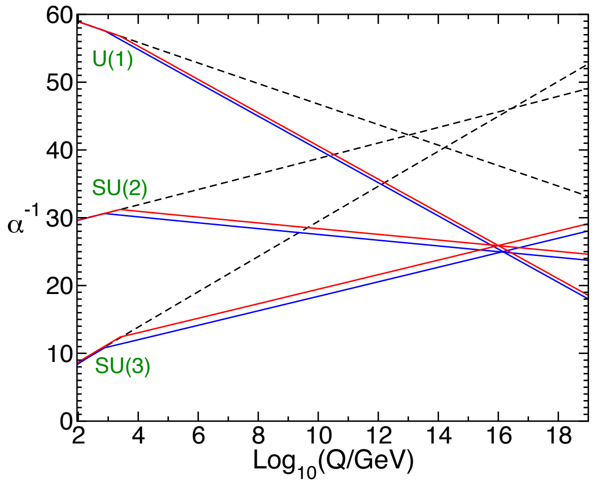


Figure 13.1: Renormalization group evolution of the $\mathrm{SU}(3)_c$, $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge couplings in the SM (dashed lines) and in the MSSM (solid lines). The figure il taken from ref. [6].

On the other hand, in the MSSM we get

$$\begin{cases} b(g') = 10 + \dfrac{n_s}{2} \\ b(g) = \dfrac{n_s}{2} \\ b(g_s) = -3 \end{cases}, \tag{13.8.5}$$

where we considered a slight generalization of the MSSM in which n_s Higgs doublets are present.

We can express the electroweak couplings in terms of the weak mixing angle θ and the positron charge e:

$$g(m_{\mathrm{Z}}) = e(m_{\mathrm{Z}})/\sin\theta\,, \qquad g'(m_{\mathrm{Z}}) = e(m_{\mathrm{Z}})/\cos\theta\,. \tag{13.8.6}$$

If we assume unification at a scale M_X, we can solve the renormalization group expressions for $g'(m_{\mathrm{Z}})$, $g(m_{\mathrm{Z}})$ and $g_s(m_{\mathrm{Z}})$, in order to find $\log(M_X/m_{\mathrm{Z}})$ and $\sin^2\theta$ as functions of $e(m_{\mathrm{Z}})$ and $g_s(m_{\mathrm{Z}})$

$$\sin^2\theta = \frac{18 + 3n_s + (e^2(m_{\mathrm{Z}})/g_s^2(m_{\mathrm{Z}}))(60 - 2n_s)}{108 + 6n_s}\,, \tag{13.8.7}$$

$$\log\left(\frac{M_X}{m_{\mathrm{Z}}}\right) = \left(\frac{8\pi^2}{e^2(m_{\mathrm{Z}})}\right)\left(\frac{1 - 8e^2(m_{\mathrm{Z}})/(3g^2(m_{\mathrm{Z}}))}{18 + n_s}\right)\,. \tag{13.8.8}$$

Using the values

$$\begin{aligned} \frac{e^2(m_{\mathrm{Z}})}{4\pi} &= \frac{1}{128}\,, \\ \frac{g_s^2(m_{\mathrm{Z}})}{4\pi} &= 0.118\,, \\ m_{\mathrm{Z}} &= 91.19 \text{ GeV}\,, \end{aligned} \tag{13.8.9}$$

we get

n_s	$\sin^2\theta$	M_X(GeV)
0	0.203	8.7×10^{17}
2	0.231	2.2×10^{16}
4	0.253	1.1×10^{15}

Remarkably, the value $n_s = 2$ for the simplest plausible theory yields a value $\sin^2\theta = 0.231$, which is in perfect agreement with the experimentally observed value $\sin^2\theta = 0.23$. The value of M_X is 20 times greater than the one computed in this way in non-SUSY great unification theories (GUT's), leading to a decrease by a factor 20^{-4} in the rate for proton decay processes like $p \to \pi^0\, e^+$, thus removing a conflict with the experimental non-observation of such processes.

13.9 Electroweak symmetry breaking and the Higgs boson

In the MSSM, the description of electroweak symmetry breaking is slightly complicated by the fact that there are two complex Higgs doublets $H_u = (H_u^+, H_u^0)$ and $H_d = (H_d^0, H_d^-)$ rather than just one as in the SM. The classical scalar potential for the Higgs scalar fields in the MSSM is given by

$$\begin{aligned} V &= (|\mu|^2 + m_{H_u}^2)(|H_u^0|^2 + |H_u^+|^2) + (|\mu|^2 + m_{H_d}^2)(|H_d^0|^2 + |H_d^-|^2) + \,[b(H_u^+ H_d^- - H_u^0 H_d^0) + \text{h.c.}] \\ &\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 + \frac{1}{2} g^2 |H_u^+ H_d^{0*} + H_u^0 H_d^{-*}|^2 \,. \end{aligned} \tag{13.9.1}$$

The terms proportional to $|\mu|^2$ come from F-terms, while the terms proportional to g^2 and g'^2 are the D-terms contributions. Finally, the terms proportional to $m_{H_u}^2$, $m_{H_d}^2$ and b are just a rewriting of the terms of soft SUSY breaking.

Note. The full scalar potential also includes many terms involving the squarks and slepton fields. We can ignore them since they do not get VEV's (which would break the gauge symmetries) because they have large positive squared masses.

We must now require that the minimum of the potential breaks the electroweak group to electromagnetism: $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{em}$. We can use the freedom to make $\mathrm{SU}(2)_L$ gauge transformations to set $\langle H_u^+\rangle = 0$ at the minimum of the potential. If we look for a stable minimum along the charged directions we find

$$\left.\frac{\partial V}{\partial H_u^+}\right|_{\langle H_u^+\rangle = 0} = bH_d^- + \frac{g^2}{2} H_d^{0*} H_d^- H_u^{0*} \,, \tag{13.9.2}$$

which does not vanish for non-zero H_d^- for generic values of the parameters. This implies that $\langle H_d^-\rangle = 0$ and $\mathrm{U}(1)_{em}$ is unbroken. We are left with

$$V = (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 + (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 - (bH_u^0 H_d^0 + \text{h.c.}) + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2 \,. \tag{13.9.3}$$

By a phase redefinition of H_u aor H_d we can set b to be real. Then, at the minimum of the potential, $\langle H_u^0\rangle$ and $\langle H_d^0\rangle$ have opposite phases, and, since they have opposite $\mathrm{U}(1)_Y$ hypercherges, we can set these phases to zero by a $\mathrm{U}(1)_Y$ gauge transformation. Thus CP symmetry is not spontaneously broken in the MSSM.

In order to have a sensible theory we need to impose that V is bounded from below. Note that this would be automatically true in a SUSY-invariant theory, but it is not so when we introduce soft SUSY-breaking terms. The scalar quartic interaction in V stabilizes the potential for all directions except when $H_u^0 = H_d^0$. So we need the quadratic part to be positive along this direction. This requirement gives

$$2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2 \,. \tag{13.9.4}$$

Moreover we need to require that one linear combination of H_u^0 and H_d^0 has a negative squared mass at the origin:

$$b^2 > (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2) \,. \tag{13.9.5}$$

Notice that, if $m_{H_u}^2 = m_{H_d}^2$, the above constraints can not both be satisfied. in models with minimal supergravity or gauge mediated SUSY breaking one usually finds $m_{H_u}^2 = m_{H_d}^2$ at the SUSY-breaking scale. But then the renormalization group evolution for $m_{H_u}^2$ naturally pushes it to negative or small values $m_{H_u}^2 < m_{H_d}^2$ at the electroweak scale. So in these models electroweak symmetry breaking is induced by radiative corrections.

We can now study the electroweak-breaking minimum. Let us write

$$\langle H_u^0 \rangle = \frac{v_u}{\sqrt{2}} \qquad \text{and} \qquad \langle H_d^0 \rangle = \frac{v_d}{\sqrt{2}} \,. \tag{13.9.6}$$

Such VEV's break electroweak symmetry, hence giving W and Z bosons masses

$$m_{\mathrm{W}}^2 = \frac{1}{4} g^2 v^2 \,, \qquad m_{\mathrm{Z}}^2 = \frac{1}{4}(g^2 + g'^2) v^2 \,, \tag{13.9.7}$$

where

$$v^2 = v_u^2 + v_d^2 \simeq (246 \text{ GeV})^2 \,. \tag{13.9.8}$$

We can rewrite the ratio of the VEV's in terms of an angle β

$$s_\beta \equiv \sin\beta \equiv \frac{v_u}{v} \,, \qquad c_\beta \equiv \cos\beta \equiv \frac{v_d}{v} \,, \tag{13.9.9}$$

with $0 < \beta < \pi/2$. The VEV's can be related to the parameters in the potential by imposing the minimum conditions $\partial V/\partial H_u^0 = \partial V/\partial H_d^0 = 0$, which give

$$|\mu|^2 + m_{H_u}^2 = b \cot\beta + \frac{m_{\mathrm{Z}}^2}{2} \cos 2\beta \tag{13.9.10}$$

$$|\mu|^2 + m_{H_d}^2 = b \tan\beta - \frac{m_{\mathrm{Z}}^2}{2} \cos 2\beta \,. \tag{13.9.11}$$

The Higgs scalar consists of eight real scalar degrees of freedom. When the electroweak symmetry is broken, three of them are the would-be Goldstone bosons that are 'eaten' by the Z^0 and $W^\pm$ bosons. This leaves five degrees of freedom, A^0, $H^\pm$, h^0 and H^0. The h^0 and H^0 are CP even and the A^0 is CP odd. It is convenient to shift the fields by their VEV's

$$H_u^0 = \frac{v_u}{\sqrt{2}} + \mathcal{H}_u^0 \,, \tag{13.9.12}$$

$$H_d^0 = \frac{v_d}{\sqrt{2}} + \mathcal{H}_d^0 \,, \tag{13.9.13}$$

then we can read off the mass terms for the various physical Higgs components.

For the imaginary parts of the neutral fields we get

$$V \supset \left(\mathrm{Im}\, \mathcal{H}_u^0, \mathrm{Im}\, \mathcal{H}_d^0\right) \begin{pmatrix} b \cot\beta & b \\ b & b \tan\beta \end{pmatrix} \begin{pmatrix} \mathrm{Im}\, \mathcal{H}_u^0 \\ \mathrm{Im}\, \mathcal{H}_d^0 \end{pmatrix} \,. \tag{13.9.14}$$

Diagonalizing, we find the two mass eigenstates

$$\begin{pmatrix} \pi^0 \\ A^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} s_\beta & -c_\beta \\ c_\beta & s_\beta \end{pmatrix} \begin{pmatrix} \mathrm{Im}\ \mathcal{H}_u^0 \\ \mathrm{Im}\ \mathcal{H}_d^0 \end{pmatrix} , \tag{13.9.15}$$

where we defined $c_\beta = \cos\beta$ and $s_\beta = \sin\beta$. π^0 is massless and is the would-be Goldstone eaten by the Z^0, while A^0 has a mass

$$m_A^2 = \frac{b}{s_\beta c_\beta} = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2 . \tag{13.9.16}$$

Considering the charged components

$$V \supset \left(H_u^{+*} , H_d^-\right) \begin{pmatrix} b\cot\beta + m_{\mathrm{W}}^2 c_\beta^2 & b + m_{\mathrm{W}}^2 c_\beta s_\beta \\ b + m_{\mathrm{W}}^2 c_\beta s_\beta & b\tan\beta + m_{\mathrm{W}}^2 s_\beta^2 \end{pmatrix} \begin{pmatrix} H_u^+ \\ H_d^{-*} \end{pmatrix} , \tag{13.9.17}$$

and the mass eigenstates are

$$\begin{pmatrix} \pi^+ \\ H^+ \end{pmatrix} = \begin{pmatrix} s_\beta & -c_\beta \\ c_\beta & s_\beta \end{pmatrix} \begin{pmatrix} H_u^+ \\ H_d^{-*} \end{pmatrix} , \tag{13.9.18}$$

where $\pi^- = \pi^{+*}$ are the would-be Goldstones eaten by the $W^\pm$ and $H^- = H^{+*}$. So the mass of the charged Higgs is

$$m_{H^\pm}^2 = m_A^2 + m_{\mathrm{W}}^2 . \tag{13.9.19}$$

Finally, for the real parts of the neutral fields we have

$$V \supset \left(\mathrm{Re}\, H_u^0 , \mathrm{Re}\, H_d^0\right) \begin{pmatrix} b\cot\beta + m_{\mathrm{W}}^2 s_\beta^2 & -b - m_{\mathrm{W}}^2 c_\beta s_\beta \\ -b - m_{\mathrm{W}}^2 c_\beta s_\beta & b\tan\beta + m_{\mathrm{W}}^2 c_\beta^2 \end{pmatrix} \begin{pmatrix} \mathrm{Re}\, H_u^0 \\ \mathrm{Re}\, H_d^0 \end{pmatrix} , \tag{13.9.20}$$

which has mass eigenstates

$$\begin{pmatrix} h^0 \\ H^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{Re}\, H_u^0 \\ \mathrm{Re}\, H_d^0 \end{pmatrix} , \tag{13.9.21}$$

with masses

$$m_{h,H}^2 = \frac{1}{2}\left(m_A^2 + m_{\mathrm{Z}}^2 \mp \sqrt{(m_A^2 + m_{\mathrm{Z}}^2)^2 + 4 m_{\mathrm{Z}}^2 m_A^2 \sin^2 2\beta} \right) , \tag{13.9.22}$$

and the mixing angle α is determined by

$$\frac{\sin 2\alpha}{\sin 2\beta} = -\frac{m_A^2 + m_{\mathrm{Z}}^2}{m_H^2 - m_h^2} , \qquad \frac{\cos 2\alpha}{\cos 2\beta} = -\frac{m_A^2 - m_{\mathrm{Z}}^2}{m_H^2 - m_h^2} . \tag{13.9.23}$$

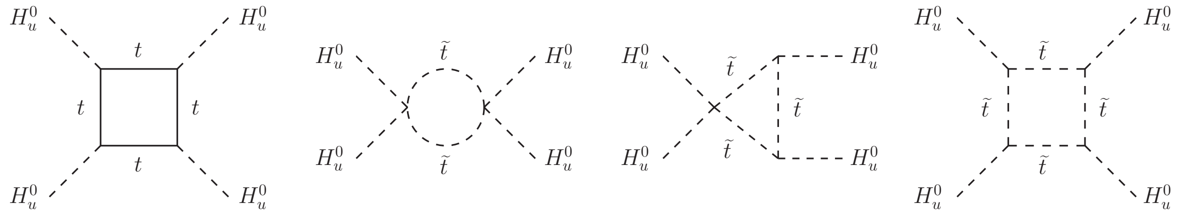
Note that m_A, $m_{H^\pm}$ and m_H grow as $b \to \infty$, but m_h is maximized at $m_A \to \infty$, so at tree level there is an upper bound on the Higgs mass

$$m_h < |\cos 2\beta\,| m_{\mathrm{Z}} = \frac{g^2 + g'^2}{4} |v_d^2 - v_u^2| , \tag{13.9.24}$$

which is in conflict with the measured Higgs mass $m_h \simeq 125$ GeV.

For $m_A \gg m_{\mathrm{Z}}$, then A^0, H^0 and $H^\pm$ are much heavier than h^0, forming a nearly degenerate isospin doublet. In this decoupling limit, the angle α is fixed to be approximately $\alpha = \beta - \pi/2$, and h^0 has SM couplings to quarks, leptons and gauge bosons.

The upper bound on the Higgs mass is relaxed if we also include radiative corrections. One can compute the contribution to the Higgs mass by evaluating the one-loop corrections to the quartic coupling of the potential. The largest contribution comes from top-stop loops. If the stops are heavy with respect to the top the loop diagrams do not exactly cancel. Some of the most important diagrams are



This leads to a shift in the physical Higgs mass

$$\Delta m_h^2 = \frac{3}{4\pi^2} v^2 y_t^4 \sin^2\beta \log \frac{m_{\tilde{t}_1} m_{\tilde{t}_2}}{m_t^2} \approx \frac{(90\ \mathrm{GeV})^2}{\sin^2\beta}\,, \tag{13.9.25}$$

valid for not too small $\sin\beta$ (otherwise the top Yukawa coupling becomes non-perturbative). In this way we get the weaker upper bound

$$m_h \lesssim 130\ \mathrm{GeV}\,, \tag{13.9.26}$$

which is compatible with the experimental measurement of the Higgs mass.

After electroweak symmetry breaking the superpartners with the same $\mathrm{SU}(3)_c$ and electroweak quantum numbers can mix to give new mass eigenstates. In particular all the squarks (sleptons) with the same electric charge can mix among themselves. For the gauginos we get that

- the gluinos not mix with other states,
- the charged winos ($\widetilde{W}^{\pm}$) mix with the charged Higgsinos ($\widetilde{H}_u^{+}$ and $\widetilde{H}_d^{-}$) giving rise to the *charginos* $\widetilde{C}_1^{\pm}$, $\widetilde{C}_2^{\pm}$,
- the neutral wino ($\widetilde{W}^0$) and the bino ($\widetilde{B}$) mix with the neutral Higgsinos ($\widetilde{H}_u^0$ and $\widetilde{H}_d^0$) giving rise to the *neutralinos* $\widetilde{N}_1$, $\widetilde{N}_2$, $\widetilde{N}_3$, $\widetilde{N}_4$.

Note. For the possible collider signatures of the MSSM see for example ref. [6].

14

Introduction to supergravity

If supersymmetry is present in nature, then for sure we will need also a supersymmetric description of gravity, at least at high energy scales. As we already explained, supersymmetry and gravity are deeply linked by the fact that the Poincaré algebra and the supersymmetry charges are tied together in a unique superalgebra. This leads to the fact that, when we try to build a supersymmetric version of gravity, at the same time we need to construct a local version of supersymmetry, which is known as *supergravity*.

Supergravity plays also a fundamental role in string theory, which needs supergravity to allow a consistent embedding of the fermions. It can be shown that the low-energy effective description of superstring models is given by supergravity theories.

Let us try to understand better the relation between supersymmetry and gravity. First of all we recall that gravity can be seen as a theory that is invariant under *local* Poincaré transformations, that is, it is a gauge theory for the Poincaré group. We can start by considering what happens if we look for a theory with local supersymetry invariance. The anticommutation relation

$$\{Q_\alpha\,,\overline{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu \tag{14.0.1}$$

implies that, on a quantum field X,

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)]X = 2i(\varepsilon_2\sigma^\mu\overline{\varepsilon}_1 - \varepsilon_1\sigma^\mu\overline{\varepsilon}_2)\partial_\mu X\,. \tag{14.0.2}$$

If we try to naively extend this transformation rule to a *local* supersymmetry transformation by considering spacetime-dependent parameters $\varepsilon_1(x)$, $\varepsilon_2(x)$ we expect that

$$[\delta(\varepsilon_1(x)),\delta(\varepsilon_2(x))]X \sim 4i\varepsilon_2(x)\sigma^\mu\overline{\varepsilon}_1(x)\partial_\mu X\,. \tag{14.0.3}$$

The right hand side of the above equation can be seen as a translation with a parameter that depends on the spacetime coordinates. This coincides with the notion of general coordinate transformation and leads one to expect that gravity must be present. Thus *local* supersymmetry should lead to gravity.

The reverse is also expected: global supersymmetry in the presence of gravity implies local supersymmetry. This can be seen, for example, from the commutation rule

$$[M_{\mu\nu},Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta\,. \tag{14.0.4}$$

This implies that, if we start with a supersymmetry transformation with a constant parameter ε and perform a local Lorentz transformation, then this parameter will in general become spacetime dependent as a result of the Lorentz transformation. Hence local supersymmetry and gravity (plus global supersymmetry) imply each other.

14.1 The vierbein formalism in general relativity

The formalism of gravity in terms of a metric tensor $g_{\mu\nu}$ is adequate for theories with fields restricted to scalars, vectors and tensors, but not for supergravity, where spinors are an indispensable ingredient. Unlike vectors and tensors, spinors have a Lorentz transformation rule that has no natural generalization to arbitrary coordinate systems. Instead, to deal with spinors, we have to introduce systems of coordinates $\xi^a_X(x)$ with $a = 0, 1, 2, 3$ that are *locally inertial* at a given point X in an arbitrary coordinate system (we can see this inertial system as the tangent space to the spacetime manifold at the point X). The principle of equivalence tells us that gravitation has no effect in these locally-inertial coordinates, so the action may be expressed in terms of matter fields like spinors, vectors, etc., that are defined in these locally inertial frames. Moreover we need a *vierbein* $e^a_\mu(X)$, which arises from the transformation between the locally-inertial and the general coordinates

$$e^a_\mu(X) \equiv \left.\frac{\partial \xi^a_X(x)}{\partial x^\mu}\right|_{x=X} . \tag{14.1.1}$$

The action needs to be invariant under

- *general coordinate* transformations $\quad x^\mu \to x'^\mu$
- *local Lorentz* transformations $\quad \xi^a \to \xi'^a = \Lambda^a{}_b(x)\xi^b$ with $\Lambda^a{}_c(x)\Lambda^b{}_d(x)\eta_{ab} = \eta_{cd}$.

Under general coordinate transformations the vierbein transforms as

$$e^a_\mu(x) \to e'^a_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu(x) , \tag{14.1.2}$$

while for a local Lorentz transformation $\xi^a(x) \to \Lambda^a{}_b(x)\xi^b(x)$, it transforms as

$$e^a_\mu(x) \to \Lambda^a{}_b(x) e^b_\mu(x) . \tag{14.1.3}$$

Theories with pure gravitation can be expressed in terms of a field, that is invariant under local Lorentz transformations and transforms as a tensor under general coordinate transformations, namely the metric

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} . \tag{14.1.4}$$

Note. The spacetime indices $\mu, \nu, \ldots$ are raised and lowered with the metric $g_{\mu\nu}$, while the Lorentz indices $a, b, \ldots$ are raised and lowered with the Lorentz metric η_{ab}.

The set of vierbeins $\{e^a_\mu(x)\}$, $a = 0, 1, 2, 3$, can be equivalently seen as a basis of the tangent vectors to the point x. μ labels the components of a vector tangent to the spacetime manifold, and a is the "name" of the vector. The condition in eq. (14.1.4) implies that the vectors are orthonormal

$$e^a_\mu(x) e^b_\nu(x) g^{\mu\nu}(x) = \eta^{ab} . \tag{14.1.5}$$

Local Lorentz transformations, $e^a_\mu(x) \to \Lambda^a{}_b(x) e^b_\mu(x)$, simply correspond to a change of basis in the tangent space at the point x.

Now we have to ensure that, using the above formalism, we can build theories with the wanted local invariance. Requiring local Poincaré invariance is analogous to construct a *gauge* symmetry with the Lorentz group. This suggests that, to achieve local Lorentz invariance, we need to introduce a gauge field $\omega^a_{\mu b}(x)$ of the Lorentz group SO(3, 1). Here μ is a vector index tangent to the spacetime manifold, while a and b are SO(3, 1) indices. The gauge field $\omega^a_{\mu b}$, which is usually called the *spin connection*, transforms under local Lorentz transformations in the standard way

$$\omega_\mu \to \Lambda\omega_\mu\Lambda^{-1} - (\partial_\mu\Lambda)\Lambda^{-1}\,. \tag{14.1.6}$$

Let us now discuss the minimal choice for ω_μ that gives general relativity. Preliminarily we consider the covariant derivative of a vector field V^μ which is usually defined as

$$D_\lambda V^\mu = \partial_\lambda V^\mu + \Gamma^\mu_{\lambda\nu}V^\nu\,, \tag{14.1.7}$$

with $\Gamma^\mu_{\lambda\nu}$ being the Christoffel symbols. On the other hand, once a vierbein is introduced one could work with $V^a(x) \equiv e^a_\mu(x)V^\mu(x)$. The V^a contains the same amount of information as V^μ since $V^\mu(x) = e^\mu_a V^a(x)$. In terms of V^a the natural covariant derivative is

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b}V^b\,. \tag{14.1.8}$$

If we want to get the standard content of general relativity we need to ensure that

$$D_\mu V^a = e^a_\nu D_\mu V^\nu\,, \tag{14.1.9}$$

which is guaranteed if

$$D_\mu e^a_\nu \equiv \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu}e^a_\lambda + \omega^a_{\mu b}e^b_\nu = 0\,. \tag{14.1.10}$$

This equation completely determines $\Gamma^\lambda_{\mu\nu}$ and $\omega^a_{\mu b}$; one finds

$$\omega^a_{\mu b} = \frac{1}{2}e^{\nu\,b}(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu) - \frac{1}{2}e^{\nu\,b}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) - \frac{1}{2}e^{\rho\,a}e^{\sigma\,b}(\partial_\rho e_{\sigma\,c} - \partial_\sigma e_{\rho\,c})e^c_\mu\,, \tag{14.1.11}$$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\,. \tag{14.1.12}$$

Having defined the spin connection, we can form the gauge-covariant field strength

$$R_{\mu\nu}{}^a{}_b = \partial_\mu\omega^a_{\nu b} - \partial_\nu\omega^a_{\mu b} + [\omega_\mu, \omega_\nu]^a{}_b\,. \tag{14.1.13}$$

It has the same content as the Riemann tensor $R_{\mu\nu}{}^\sigma{}_\tau$ conventionally defined in terms of the Christoffel symbols and their derivative; in fact it follows that

$$R_{\mu\nu}{}^a{}_b = e^a_\sigma e^\tau_b R_{\mu\nu}{}^\sigma{}_\tau\,. \tag{14.1.14}$$

With the aid of the spin connection we can couple spinors to general relativity. As any gauge field, the spin connection can be coupled to a field $\psi(x)$ in any representation of the gauge group. Letting $\sigma^a{}_b$ be the generators of the Lorentz group in the spinor representation, the covariant derivative of ψ is

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\omega^{ab}_\mu\sigma_{ab}\psi\,. \tag{14.1.15}$$

Under a local Lorentz transformation $\Lambda(x)$ we require

$$\psi(x) \to \mathcal{D}(\Lambda(x))\psi(x)\,, \tag{14.1.16}$$

then the covariant derivative also transforms homogeneously

$$D_\mu\psi(x) \to \mathcal{D}(\Lambda(x))D_\mu\psi(x)\,. \tag{14.1.17}$$

To define the γ matrices, we first introduce the standard flat space gamma matrices γ_a that obey

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\,. \tag{14.1.18}$$

Curved-space gamma matrices are then defined as

$$\gamma_\mu(x) = e_\mu^a(x)\gamma_a\,, \tag{14.1.19}$$

and they obey

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2g_{\mu\nu}(x)\,. \tag{14.1.20}$$

Notice that in general it is useful to consider other "non-minimal" spin connections, which normally depend on fields other than the metric. The non minimality of a spin connection is conveniently measured by the *torsion* $T^a_{\mu\nu}$ defined as

$$T^a_{\mu\nu} = D_\mu e^a_\nu - D_\nu e^a_\mu\,. \tag{14.1.21}$$

The torsion depends only on the spin connection and not on the Christoffel symbols, given that $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$. Examples of theories with torsion are obtained by just adding fermions to general relativity.

14.2 Local supersymmetry in the massless Wess–Zumino model

To understand better why local supersymmetry implies gravity, it is useful to consider a simple example, the massless Wess–Zumino model.

Note. In this section we will adopt the 4-component spinor notation instead of the 2-component notation used so far. We take the discussion from ref. [10].

The action we consider is

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\overline{\lambda}\not{\partial}\lambda \qquad (\text{with } \overline{\lambda} = \lambda^\dagger\gamma^4)\,. \tag{14.2.1}$$

This is invariant under the global SUSY transformations

$$\begin{aligned} \delta A &= \frac{1}{2}\overline{\varepsilon}\lambda \\ \delta B &= -\frac{i}{2}\overline{\varepsilon}\gamma_5\lambda \qquad\qquad (\gamma-5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4)\,. \\ \delta\lambda &= \frac{1}{2}\not{\partial}(A - iB\gamma_5)\varepsilon \end{aligned} \tag{14.2.2}$$

As we know, the Lagrangian is SUSY invariant apart from a total derivative:

$$\delta\mathcal{L} = \partial_\mu K^\mu = \partial_\mu\left(-\frac{1}{4}\overline{\varepsilon}\gamma^\mu[\not{\partial}(A - iB\gamma_5)]\lambda\right)\,. \tag{14.2.3}$$

We now turn to local supersymmetry and ask what happens if we make ε local, hence $\varepsilon(x)$. For example we define

$$\delta\lambda = \frac{1}{2}\not{\partial}(A - iB\gamma_5)\varepsilon(x)\,. \tag{14.2.4}$$

Notice that we do not want to introduce terms with $\partial_\mu\varepsilon(x)$ in the variation of the fields except for the gravitino, which corresponds to a "gauge field" for the supersymmetry transformations.

We already discussed what happens for the variation of the action: it is the alternative procedure to get the supersymmetry current we discussed in section 10:

$$\delta I = \int d^4x\,(\partial_\mu\overline{\varepsilon}(x))j^\mu_N\,, \tag{14.2.5}$$

where j^μ_N is the supercurrent.

To cancel this variation of the action we need to introduce an extra ingredient: a gauge field that corresponds to supersymmetry. In this case the gauge parameter $\varepsilon(x)$ is a spinor (and not a scalar as in ordinary gauge theories), so we need a spinor field with an index μ as gauge field: ψ_μ. One can easily realize that such a field has spin 3/2, that is, it could be the partner of the graviton, the gravitino. We will see afterwards that this expectation is correct.

To cancel the δI variation we need a term of the form

$$I^N = \int d^4x\,(-\kappa\overline{\psi}_\mu j^\mu_N) \tag{14.2.6}$$

and we must require that

$$\delta\psi_\mu = \frac{1}{\kappa}\partial_\mu\varepsilon(x) + \text{more}\,. \tag{14.2.7}$$

The coupling κ is a coupling with dimension of a mass.

Now, when we compute the variation of the action, we must vary also I^N. In this way we get some terms quadratic in A and B:

$$\delta(I + I^N) \supset \int d^4x\,\frac{\kappa}{2}(\overline{\psi}_\mu\gamma_\nu\varepsilon)(T^{\mu\nu}(A) + T^{\mu\nu}(B))\,, \tag{14.2.8}$$

where $T_{\mu\nu}(A) = \partial_\mu A\partial_\nu A - \frac{1}{2}\delta_{\mu\nu}(\partial_\lambda A)^2$ is the energy-momentum tensor of the field A. The above variation can only be cancelled by adding a second coupling of a new field $g_{\mu\nu}$ to the Noether current of translations, $\frac{1}{2}T^{\mu\nu}$, and requiring that under local supersymmetry,

$$\delta g_{\mu\nu} = -\frac{\kappa}{2}\overline{\psi}_\mu\gamma_\nu\varepsilon - \frac{\kappa}{2}\overline{\psi}_\nu\gamma_\mu\varepsilon\,. \tag{14.2.9}$$

This tells us that we *need gravity* to get a theory that is locally supersymmetry invariant.

If we rewrite the theory in terms of the vierbein e^a_μ we get the following SUSY transformation

$$\delta e^a_\mu = \frac{1}{2}\kappa\bar{\varepsilon}\gamma^a\psi_\mu\,. \tag{14.2.10}$$

moreover, for the gravitino we expect

$$\delta\psi^a_\mu = \frac{1}{\kappa}D_\mu\varepsilon\,, \qquad \text{where} \qquad D_\mu\varepsilon = \partial_\mu\varepsilon + \frac{1}{2}\omega^{mn}_\mu\sigma_{mn}\varepsilon\,. \tag{14.2.11}$$

To be sure that what we found is meaningful we can consider the graviton multiplet in global supersymmetry. It is given by the two physical fields e^a_μ and ψ_μ (with helicities ± 2 and $\pm 3/2$) The global SUSY transformation for this multiplet are

$$\delta g_{\mu\nu} = \frac{\kappa}{2}(\bar{\varepsilon}\gamma_\mu\psi_\nu + \bar{\varepsilon}\gamma_\nu\psi_\mu)\,, \qquad \delta\psi_\mu = \frac{1}{2\kappa}(\omega^{mn}_\mu)_L\sigma_{mn}\varepsilon\,, \tag{14.2.12}$$

where $(\omega^{mn}_\mu)_L$ is a "linearized" form of the spin connection and ε is a constant spinorial parameter. This transformations exactly match with the form of the local SUSY transformations specialized to the case of constant ε, so we are confident that it is reasonable to identify the $(\psi_\mu\,, e^a_\mu)$ system we found in local supersymmetry with the graviton $N = 1$ supermultiplet.

14.3 The action for simple supergravity

As we have seen we have three objects in the pure supergravity action: the vierbein e^a_μ, the gravitino ψ_μ and the spin connection. Since e^a_μ and ψ_μ already describe the gravity supermultiplet, the field ω_μ^{mn} should not be physical. Indeed ω_μ^{mn} is a non-physical field, which is useful to write a simple action, but then is eliminated by solving its non-propagating equations of motion.

For pure gravity, without the gravitino, we already gave the form of the spin connection:

$$\omega_\mu^{mn} = \omega_\mu^{mn}(e)\,. \tag{14.3.1}$$

As we will see, when the gravitino is introduced, we get some extra ψ_μ-dependent contributions. For the graviton we have the usual Hilbert action (with $e = \det e_\mu^m$)

$$\mathcal{L}^{(2)} = -\frac{1}{2\kappa} e\, R(e,\omega) = -\frac{1}{2\kappa} e\, e^{m\,\nu} e^{\nu,\mu} R_{\mu\nu mn}(\omega)\,. \tag{14.3.2}$$

If we solve for ω we get the $\omega_\mu^{mn}(e)$ given in the previous discussion on general relativity.

We now turn to the fermionic part of the action. this is given by the Rarita–Schwinger action in curved space

$$\mathcal{L}^{(3/2)} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma\,, \tag{14.3.3}$$

where we have

$$D_\rho \psi_\sigma = \left(\partial_\rho + \frac{1}{2}\omega_\rho^{mn}\sigma_{mn}\right)\psi_\sigma\,. \tag{14.3.4}$$

Notice that we did not include a term with $\Gamma^\mu_{\nu\rho}$ in the expression for $D_\rho\psi_\sigma$, because in any case it would have been cancelled by the $\varepsilon^{\mu\nu\rho\sigma}$ factor.

One can check that the total action

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3/2)} \tag{14.3.5}$$

is invariant under the local supersymmetry transformations

$$\begin{cases} \delta e_\mu^m = \dfrac{\kappa}{2}\bar{\varepsilon}\gamma^m\psi_\mu \\ \delta\psi_\mu = \dfrac{1}{\kappa}D_\mu\varepsilon \end{cases}. \tag{14.3.6}$$

As we already anticipated, the spin connection obtained by solving the equations of motion contains a term that depends on the gravitino:

$$\omega_{\mu mn}(e,\psi) = \omega_{\mu mn}(e) + \frac{\kappa^2}{4}(\overline{\psi}_\mu\gamma_m\psi_n - \overline{\psi}_\mu\gamma_n\psi_m + \overline{\psi}_m\gamma_\mu\psi_n)\,. \tag{14.3.7}$$

14.4 Auxiliary fields for the supergravity action

Auxiliary fields are needed if we want the supersymmetry algebra to close also off-shell. Moreover in supergravity they are needed in order that the transformation rules of the graviton supermultiplet do not depend on the matter fields. If they did, without further modifications, two matter actions, each of which has been coupled to gravity in an invariant way, could not be put together. The reason is that the field transformation rules of the system I would not work for system II and viceversa.

However, if one adds auxiliary fields, the field transformation rules are always the same, independent of the matter fields and valid for any matter coupling system.

Let us start by counting how many auxiliary fields we need. In supergravity there are three local invariances:

- general coordinate transformations G,

- local Lorentz rotations L,

  local supersymmetry transformations Q.

Thus the counting of field components in the pure supergravity action is

$16\ e_\mu^m - 4$ general coordinate $- 6$ local Lorentz $= 6$ bosonic fields,

$16\ \psi_\mu^a - 4$ local supersymmetry $= 12$ fermionic fields.

Hence there is a mismatch of 6 bosonic components. The algebra is thus not closed and we need to add $6+n$ bosonic auxiliary fields and n fermionic auxiliary fields. There exist several sets of auxiliary fields, the most prominent one is the minimal set with $n=0$, consisting of an *auxiliary vector* A_m, a *scalar* S and a *pseudoscalar* P.

The action is

$$\mathcal{L} = \mathcal{L}^{(2)}(e,\omega) + \mathcal{L}^{(3/2)}(e,\psi,\omega) - \frac{e}{3}(S^2 + P^2 - A_m^2)\,, \tag{14.4.1}$$

so S, P and A_m are non-propagating fields, as expected. This action is invariant under the local supersymmetry transformations

$$\begin{aligned}
\delta e_\mu^m &= \frac{\kappa}{2}\bar{\varepsilon}\gamma^m\psi_\mu \\
\delta\psi_\mu &= \frac{1}{\kappa}\left(D_\mu + i\frac{\kappa}{2}A_\mu\gamma_5\right)\varepsilon + \frac{1}{6}\gamma_\mu(S - i\gamma_5 P - i\not{A}\gamma_5)\varepsilon \\
\delta S &= \frac{1}{4}\bar{\varepsilon}\gamma\cdot R^{cov} \\
\delta P &= -\frac{i}{4}\bar{\varepsilon}\gamma_5\gamma\cdot R^{cov} \\
\delta A_m &= \frac{3i}{4}\bar{\varepsilon}\gamma_5\left(R_m^{cov} - \frac{1}{3}\gamma_m\gamma\cdot R^{cov}\right)
\end{aligned} \tag{14.4.2}$$

where R_μ^{cov} is the gravitino field equation ($R^\mu = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_n u D_\rho\psi_\sigma$) but with the supercurrent derivatives:

$$R^{\mu,cov} = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\left(D_\rho\psi_\sigma - \frac{i}{2}A_\sigma\gamma_5\psi_\rho - \frac{1}{6}\gamma_\sigma(S - i\gamma_5 P - i\not{A}\gamma_5)\psi_\rho\right)\,. \tag{14.4.3}$$

As expected, with auxiliary fields the algebra closes. In this case the $\{Q,Q\}$ anticommutator can be written in a form valid for all fields

$$[\delta_Q(\varepsilon_1),\delta_Q(\varepsilon_2)] = \delta_G(\xi^a) + \delta_Q(-\xi^a\psi_a) + \delta_L\left[\xi^\mu\widehat{\omega}_\mu^{mn} + \frac{1}{3}\bar{\varepsilon}_2\sigma^{mn}(S - i\gamma_5 P)\varepsilon_1\right]\,, \tag{14.4.4}$$

where

$$\begin{aligned}
\widehat{\omega}_\mu^{mn} &= \omega_\mu^{mn} - \frac{i}{3}\varepsilon_{\mu abc}A^c\,, \\
\xi^\mu &= \frac{1}{2}\bar{\varepsilon}_2\gamma^\mu\varepsilon_1\,.
\end{aligned} \tag{14.4.5}$$

In the above expressions δ_Q denotes a supersymmetry transformation, δ_G a general coordinate transformation and δ_L a local Lorentz transformation with the parameters given in brackets. Notice that the parameters of the transformation depend on the fields (and auxiliary fields), this is a peculiar property of supergravity, which is not present in gravity or Yang–Mills theories.

A

Collection of exercises

In this appendix a collection of exercises used during the course are presented. The exercises are organized in exercise sheets that correspond roughly to single lessons. The exercises denoted by *** are slightly more complex and provide starting points for the exploration of additional topics not covered in the course.

A.1 Exercise Sheet 1

Exercise 1:
Show that for two 2-components spinors ψ and χ, we have (notice that $\overline{\psi}_{\dot{\alpha}} = (\psi_\alpha)^\dagger$)

$$\begin{aligned}
&a)\ \psi\chi = \chi\psi \\
&b)\ \overline{\psi}\overline{\chi} = \overline{\chi}\overline{\psi} \\
&c)\ (\psi\chi)^\dagger = \overline{\psi}\overline{\chi} \\
&d)\ \overline{\psi}\overline{\sigma}^\mu\chi = -\chi\sigma^\mu\overline{\psi} = (\overline{\chi}\overline{\sigma}^\mu\psi)^* = -(\psi\sigma^\mu\overline{\chi})^* \\
&e)\ \psi\sigma^\mu\overline{\sigma}^\nu\chi = \chi\sigma^\nu\overline{\sigma}^\mu\psi = (\overline{\chi}\overline{\sigma}^\nu\sigma^\mu\overline{\psi})^* = (\overline{\psi}\overline{\sigma}^\mu\sigma^\nu\overline{\chi})^*
\end{aligned}$$

Exercise 2:
Prove the Fierz rearrangement identity

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\chi\xi)\,.$$

Exercise 3:
Prove the following reduction identities

$$\begin{aligned}
&a)\ \sigma^\mu_{\alpha\dot{\alpha}}\overline{\sigma}^{\dot{\beta}\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \\
&b)\ \sigma^\mu_{\alpha\dot{\alpha}}\sigma_{\mu,\beta\dot{\beta}} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} \\
&c)\ \overline{\sigma}^{\mu,\dot{\alpha}\alpha}\overline{\sigma}^{\dot{\beta}\beta}_\mu = 2\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} \\
&d)\ [\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2\eta^{\mu\nu}\delta^\beta_\alpha \\
&e)\ [\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot{\beta}}{}_{\dot{\alpha}} = 2\eta^{\mu\nu}\delta^{\dot{\beta}}_{\dot{\alpha}} \\
&f)\ \overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = \eta^{\mu\nu}\overline{\sigma}^\rho + \eta^{\nu\rho}\overline{\sigma}^\mu - \eta^{\mu\rho}\overline{\sigma}^\nu + i\epsilon^{\mu\nu\rho\lambda}\overline{\sigma}_\lambda \\
&g)\ \sigma^\mu\overline{\sigma}^\nu\sigma^\rho = \eta^{\mu\nu}\sigma^\rho + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu - i\epsilon^{\mu\nu\rho\lambda}\sigma_\lambda
\end{aligned}$$

Exercise 4:
In this exercise we will learn how 2-components spinors are connected to the Dirac spinors. A Dirac spinor transforms in the reducible $(1/2, 0) \oplus (0, 1/2)$ Lorentz representation. It can be built from the dotted and undotted spinors as

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

The Dirac gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}.$$

The Dirac spinor is formed by a left- and a right-handed Weyl spinor

$$\begin{aligned}
P_L\Psi_D &= \frac{1+\gamma_5}{2}\Psi_D = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \\
P_R\Psi_D &= \frac{1-\gamma_5}{2}\Psi_D = \begin{pmatrix} 0 \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}.
\end{aligned}$$

From the 2-components spinors we can also form a Majorana spinor

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix}$$

by setting $\chi = \psi$.

a) Show that the Lagrangian for a Dirac fermion

$$\mathcal{L}_D = i\overline{\Psi}_D \partial\!\!\!/ \Psi_D - M\overline{\Psi}_D\Psi_D$$

written in 2-components notation up to a total derivative is

$$\mathcal{L}_D = i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + i\overline{\chi}\overline{\sigma}^\mu\partial_\mu\chi - M(\psi\chi + \overline{\psi}\overline{\chi})\,,$$

note that $\overline{\Psi}_D = (\chi^\alpha \quad \overline{\psi}_{\dot\alpha})$.

b) Prove the following identities

$$\begin{aligned}
\overline{\Psi}_i P_L \Psi_j &= \chi_i\psi_j \\
\overline{\Psi}_i P_R \Psi_j &= \overline{\psi}_i\overline{\chi}_j \\
\overline{\Psi}_i \gamma^\mu P_L \Psi_j &= \overline{\psi}_i\overline{\sigma}^\mu\psi_j \\
\overline{\Psi}_i \gamma^\mu P_R \Psi_j &= \chi_i\sigma^\mu\overline{\chi}_j
\end{aligned}$$

c) Show that the Lagrangian for Majorana fermions

$$\mathcal{L}_M = \frac{i}{2}\overline{\Psi}_M \partial\!\!\!/ \Psi_M - \frac{1}{2}M\overline{\Psi}_M\Psi_M$$

written in 2-components notation up to a total derivative is

$$\mathcal{L}_M = i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}M(\psi\psi + \overline{\psi\psi})\,.$$

A.2 Exercise Sheet 2

Exercise 1:
In the lecture we derived the commutation relations $\{Q^I_\alpha, \overline{Q}^J_{\dot\alpha}\} = 2\delta^{IJ}(\sigma^m u)_{\alpha\dot\alpha}P_\mu$ and the definition of the central charges $\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ}$. By using these relations plus Jacobi identity show the following commutation relations

a) $[\overline{Q}^K_{\dot\gamma}, Z^{IJ}] = 0$,

b) $[\overline{Q}^K_{\gamma}, (Z^{IJ})^*] = 0$.

Exercise 2:
In the lecture we showed the commutation relations $[Z^{IJ}, Q^K_\alpha] = 0$ and $[(Z^{IJ})^*, \overline{Q}^K_{\dot\alpha}] = 0$. By using these relations plus the Jacobi identity show the following commutation relations

a) $[Z^{IJ}, Z^{KL}] = 0$,

b) $[(Z^{IJ})^*, (Z^{KL})^*] = 0$,

c) $[Z^{IJ}, (Z^{KL})^*] = 0$.

Exercise 3:
The internal symmetry group fulfills the relation $[B_a, B_b] = if_{ab}{}^cB_c$, where $f_{ab}{}^c$ are the structure constants. Moreover the internal symmetries do not commute with the fermionic generators, but obey $[Q^I_\alpha, B_l] = S_l{}^I{}_J Q^J_\alpha$ and $[\overline{Q}^I_{\dot\alpha}, B_l] = -\overline{Q}^J_{\dot\alpha} S_{lJ}{}^I$. Using the Jacobi identity of B_a, B_b and Q^I_α prove that

$$[S_a, S_b] = if_{ab}{}^cS_c\,.$$

Exercise 4:
Show that P^2 is a Casimir of the supersymmetry algebra, i.e. show that it commutes with all operators of the supersymmetry algebra.

Exercise 5:
Show that W^2 does not commute with the fermionic generators, i.e. $[W^2, Q^I_\alpha] \neq 0$, and thus W^2 is not a Casimir of the supersymmetry algebra.

A.3 Exercise Sheet 3

Exercise 1:
Write the supersymmetry algebra in 4-components spinor notation.

HInt: As done in exercise 3 on sheet 1 with the 2-components spinors, the fermionic generators Q^I_α and $\overline{Q}^{I\dot{\alpha}}$ have to be combined to a 4-components generators, i.e.

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \Rightarrow \quad \mathcal{Q}^I \equiv \begin{pmatrix} Q^I_\alpha \\ \overline{Q}^{I\dot{\alpha}} \end{pmatrix}.$$

Similarly $\overline{\mathcal{Q}}^J \equiv (Q^{J\beta}, \overline{Q}^J_{\dot{\beta}})$. Use the 2-components algebra

- $\{Q^I_\alpha, \overline{Q}^J_{\dot{\beta}}\} = 2\delta^{IJ}(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\,,$
- $\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ}\,,$
- $\{\overline{Q}^I_{\dot{\alpha}}, \overline{Q}^J_{\dot{\beta}}\} = \varepsilon_{\dot{\alpha}\dot{\beta}}(Z^*)^{IJ}\,,$

to calculate $\{\mathcal{Q}^I, \overline{\mathcal{Q}}^J\}$.

Exercise 2:
Show that the massless supermultiplets with $N = 3$ and $N = 7$ have exactly the same particle content as the $N = 4$ and $N = 8$ supermultiplets respectively when CPT invariance is taken into account. For the $N = 3$ case you only need to consider global supersymmetry, i.e. only helicities $|\lambda| \leq 1$.

Exercise 3:
The raising and lowering operators of the massive multiplets in $N = 1$ supersymmetry obey the following commutation relations

$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}\,, \qquad \{a_\alpha, a_\beta\} = \{a^\dagger_\alpha, a^\dagger_\beta\} = 0\,.$$

The Clifford vacuum is defined by

$$a_\alpha|\Omega\rangle\,, \quad \alpha = 1, 2\,.$$

Show that such a state always exists in a representation.

Exercise 4*: Supersymmetric generalization of the Pauli–Lubanski vector**
In exercise 5 on sheet 2 it was shown that the square of the Pauli–Lubanski vector W does not commute with the fermionic generators and therefore is not a Casimir operator of the SUSY algebra. There is a generalization of the Pauli–Lubanski vector

$$\begin{aligned} C^2 &= C_{\mu\nu}\,, \\ C_{\mu\nu} &= B_\mu P_\nu - B_\nu P_\mu\,, \\ B_\mu &= W_\mu - \frac{1}{4}\overline{Q}_{\dot{\alpha}}\overline{\sigma}^{\dot{\alpha}\beta}_\mu Q_\beta\,, \end{aligned}$$

where $W_\mu = \varepsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ is the ordinary Pauli–Lubanski vector. Show that C^2 is a Casimir operator in supersymmetry theories. Moreover show that a particular state of a massive particle at rest is an eigenstate of the new Casimir operator with an eigenvalue that can be interpreted as a generalized spin, the *superspin*.

A.4 Exercise Sheet 4

Exercise 1:
The *little group* is the subgroup of the Lorentz group, that leaves the four-momentum invariant. Find the little group for

a) a massive particle,

b) a massless particle,

c) the vacuum.

Exercise 2:
The Wess–Zumino Lagrangian for massless fields is given by

$$\mathcal{L}_{\mathrm{WZ}} = \partial_\mu \phi^\dagger \partial^\mu \phi - i\overline{\psi}\overline{\sigma}^\mu \partial_\mu \psi + F^\dagger F \,.$$

Show that $\mathcal{L}_{\mathrm{WZ}}$ is invariant under the supersymmetry transformations we discussed in section 5 up to a total derivative. Show that the following mass term is also invariant under supersymmetry transformations

$$\Delta\mathcal{L} = \left(m\phi F + \frac{1}{2} m\psi\psi \right) + \text{h.c.}$$

What happens if the equations of motion for the auxiliary field F are used?

Exercise 3*: Supersymmetric quantum mechanics**
Consider a quantum mechanical system with commuting (x, y: bosonic) and anticommuting (ξ, η: fermionic) coordinates

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - V(x) + \eta\dot{\xi} + W(x)\eta\xi \,.$$

To quantize canonically, we postulate the (anti-)commutation relations

$$[\hat{p}, \hat{x}] = i\hbar \qquad \{\hat{\eta}, \hat{\xi}\} = \hbar \qquad (\hat{\eta}^2 = \hat{\xi}^2 = 0)\,.$$

The corresponding Hamiltonian is

$$H = \frac{1}{2}\hat{p}^2 + V(\hat{x}) - \frac{1}{2} W(\hat{x})[\hat{\eta}, \hat{\xi}]\,.$$

Consider the following representation for the fermionic operators

$$\hat{\xi} = \sigma_+ \quad \text{and} \quad \hat{\eta} = \hbar\sigma_- \,.$$

The wavefunction is a 2-component spinor

$$\psi(x,t) = \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix} ,$$

where ϕ_1 is the bosonic and ϕ_2 the fermionic component.

a) Rewrite the Hamiltonian in terms of Pauli matrices.

b) What is the relevance of the operator $N_F = \hat{\eta}\hat{\xi}$?

c) Consider the following potentials

$$V(\hat{x}) = \frac{1}{2}v^2(\hat{x}) \qquad \text{and} \qquad W(\hat{x}) = \frac{\partial v(\hat{x})}{\partial_x} = v'(\hat{x})$$

and the operators

$$Q_i = \frac{1}{2}\sigma_i[\hat{p} + i\sigma_3 v(\hat{x})]\,, \qquad \text{where} \quad i = 1, 2\,.$$

Calculate $[H, Q_i]$ and $\{Q_i, Q_j\}$ and then show that the Lagrangian is indeed supersymmetric.

d) Consider the Q_i matrices. How do they act on $\psi(x,t)$?

e) Consider the vacuum state $|0\rangle$, with $H|0\rangle = 0$. What can we deduce for $Q_i|0\rangle$?

f) Show that $|f\rangle = -\sqrt{2/E}Q_1|b\rangle = i\sqrt{2/E}Q_2|b\rangle$ is a fermionic state. Which is the corresponding bosonic state?

g) What happens if the vacuum gets a positive energy?

h) What is the spectrum of supersymmetric quantum mechanics?

A.5 Exercise Sheet 5

Exercise 1:
The most general component expansion of a superfield is

$$\begin{aligned}\Phi(x,\theta,\overline{\theta}) \,=\, & f(x)+\theta\phi(x)+\overline{\theta}\overline{\chi}(x)+\theta\theta m(x)+\overline{\theta\theta}n(x)+\theta\sigma^{\mu}\overline{\theta}v_{\mu}(x) \\ & +(\theta\theta)\overline{\theta\lambda}(x)+(\overline{\theta\theta})\theta\psi(x)+(\theta\theta)(\overline{\theta\theta})d(x)\,.\end{aligned}$$

Calculate $\Phi_1(x,\theta,\overline{\theta})\Phi_2(x,\theta,\overline{\theta})$. Use the spinor identities to bring it to the same form as the component expanded superfield.

Exercise 2:
Prove the following superspace identities:

a) $\theta^{\alpha}\theta^{\beta}=-\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta\,,$

b) $\theta_{\alpha}\theta_{\beta}=-\frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta\,,$

c) $(\theta\psi)(\theta\chi)=-\frac{1}{2}(\theta\theta)(\psi\chi)\,,$

d) $(\theta\sigma^{\mu}\overline{\theta})(\theta\sigma^{\nu}\overline{\theta})=\frac{1}{2}(\theta\theta)(\overline{\theta\theta})\eta^{\mu\nu}\,.$

Exercise 3:
The superspace covariant derivatives are defined as

$$\begin{aligned}D_{\alpha} &= \frac{\partial}{\partial\theta^{\alpha}}+i\sigma^{\mu}_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_{\mu}\,,\\ \overline{D}_{\dot{\alpha}} &= \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}+i\theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\partial_{\mu}\,.\end{aligned}$$

Show that $\overline{D}^{2}\mathcal{F}(x,\theta,\overline{\theta})$ is a chiral superfield, where $\mathcal{F}(x,\theta,\overline{\theta})$ is a general superspace function and $\overline{D}^{2}\equiv\overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}}$. Equivalently show that $D^{2}\mathcal{F}(x,\theta,\overline{\theta})$ is antichiral. *Hint:* What are $\overline{D}_{\dot{\alpha}}\overline{D}^{2}$ and $D_{\alpha}D^{2}$ respectively?

Exercise 4:
A chiral superfield is the most general function of the bosonic coordinate $y^{\mu}=x^{\mu}+i\theta\sigma^{\mu}\overline{\theta}$ and θ^{α}, which can be parametrized as

$$\Phi(y,\theta)=z(y)+\sqrt{2}\theta\psi(y)-\theta\theta F(y)\,.$$

It is obvious that the product of two (anti) chiral superfields is again a (anti)chiral superfield. To construct the kinetic term of the Lagrangian we need the D-term (the $\theta\theta\overline{\theta\theta}$ term) of the product of a chiral and an antichiral superfield. Calculate

$$\overline{\Phi}\Phi\big|_{\theta\theta\overline{\theta\theta}}\,.$$

Hint: To do this you need to Taylor expand the bosonic coordinate, i.e. you need to show that

$$\Phi(y,\theta)=z(x)+\sqrt{2}\theta\psi(x)-\theta\theta F(x)+i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}z(x)-\frac{i}{\sqrt{2}}(\theta\theta)\partial_{\mu}\psi(x)\sigma^{\mu}\overline{\theta}-\frac{1}{4}(\theta\theta)(\overline{\theta\theta})\Box z(x)\,.$$

A.6 Exercise Sheet 6

Exercise 1:
Show that in a renormalizable theory with a set of chiral superfields with a superpotential of the form $W(\Phi) = a_i\Phi_i + m_{ij}\Phi_i\Phi_j + g_{ijk}\Phi_i\Phi_j\Phi_k$ one can always eliminate the $a_i\Phi_i$ term by a redefinition of the field (assuming that m_{ij} and/or g_{ijk} are non-vanishing). *Hint:* Look at a theory a chiral superfield with $W(\Phi) = a\Phi + m\Phi^2 + g\Phi^3$. How does the superpotential change under a transformation $\Phi \to \Phi + b$?

Exercise 2:
In a previous exercise you showed that the component expansion of a chiral superfield is

$$\Phi(y,\theta) = z(x) + \sqrt{2}\theta\psi(x) - \theta\theta F(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu z(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\overline{\theta} - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\Box z(x)\,.$$

Calculate $\Phi|_{\theta=\overline{\theta}=0}$, $D_\alpha\, \Phi|_{\theta=\overline{\theta}=0}$ and $D^2\, \Phi|_{\theta=\overline{\theta}=0}$.

A.7 Exercise Sheet 7

Exercise 1:
We saw that an Abelian gauge superfield in the Wess–Zumino gauge has the form

$$V(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}v_\mu + i(\theta\theta)\overline{\theta\lambda} - i(\overline{\theta\theta})\theta\lambda + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})D\,.$$

This gauge choice is not invariant under supersymmetry. This means that, after a supersymmetry variation, a gauge superfield in the Wess–Zumino gauge does not respect any more this gauge choice. In order to get it back to the Wess–Zumino gauge it is necessary to perform a compensating gauge transformation. Compute the supersymmetry transformation of an Abelian gauge superfield in the Wess–Zumino gauge and then determine the compensating gauge transformation to go back to the Wess–Zumino gauge.

Exercise 2*: Superconformal invariance**
The goal of this exercise is to investigate the behavior of simple supersymmetry under transformations of the conformal algebra.

The conformal algebra of Minkowski spacetime contains the Poincaré algebra as a subalgebra, and, in addition, has five other generators: the *dilation* D and the *special conformal transformations* K_μ. These generators satisfy the following commutation relations

$$\begin{aligned}
[P_\mu, D] &= P_\mu\,,\\
[M_{\mu\nu}, D] &= 0\,,\\
[K_\mu, D] &= -K_\mu\,,\\
[P_\mu, K_\nu] &= 2\eta_{\mu\nu}D - 2M_{\mu\nu}\,,\\
[M_{\mu\nu}, K_\rho] &= \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu\,.
\end{aligned}$$

a) How do the generators of the conformal algebra (P_μ, $M_{\mu\nu}$, D, K_μ) have to act on a scalar field S, a pseudoscalar field P and a spinor ψ, such that the conformal algebra is fulfilled?

b) Prove that the action of the massless Wess–Zumino Lagrangian

$$\mathcal{L}_{\rm WZ} = -\frac{1}{2}(\partial S)^2 - \frac{1}{2}(\partial P)^2 - \frac{1}{2}\overline{\psi}\partial_\mu\gamma^\mu\psi - \lambda\overline{\psi}(S - P\gamma_5)\psi - \frac{1}{2}\lambda^2(S^2+P^2)^2$$

is conformal invariant.

c) We know that the Wess–Zumino Lagrangian is invariant under both supersymmetry and conformal transformations. The commutator of an infinitesimal supersymmetry and an infinitesimal special conformal transformation is, by definition, a *conformal supersymmetry.* These are generated by a spinorial generator S_a, defined by

$$[K_\mu, Q_a] = (\gamma_\mu)_a{}^b S_b\,.$$

The infinitesimal conformal symmetry is $\delta_\zeta\phi = \overline{\zeta}S\cdot\phi$, where ζ is an anticommuting Majorana spinor. How does δ_ζ act on S, P and ψ?

d) The last step is to show that the superconformal algebra on the fields S, P and ψ closes on-shell. For this we need to introduce the R-symmetry generator R. How does this generator act on S, P and ψ? What additional commutation relations do you get for the superconformal algebra, consisting of the generators P_μ, $M_{\mu\nu}$, D, K_μ, Q_a, S_a and R?

e) How can one interpret the dilation D and the special conformal transformations K_μ physically? How can you view them pictorially?

f) Are there Lagrangians that are invariant under superconformal transformations? *Hint:* What happens when you add gauge fields? Can a Lagrangian with massive fields or dimensionful couplings be invariant under dilation?

A.8 Exercise Sheet 8

Exercise 1:
Compute the non-Abelian generalization of the field strength superfield

$$W_\alpha = -\frac{1}{8}(\overline{DD})e^{-2V}D_\alpha e^{2V}$$

in the Wess–Zumino gauge, i.e.

$$V = V_{\mathrm{WZ}} = \theta\sigma^\mu\overline{\theta}v_\mu + i(\theta\theta)\overline{\theta\lambda} - i(\overline{\theta\theta})\theta\lambda + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})D\,.$$

Here it is understood that all fields are contracted with the Hermitian group generators, i.e. $V \equiv V^a t^A$, implying $v_\mu \equiv v_\mu^A t^A$, $\lambda \equiv \lambda^A t^A$ and $D \equiv D^A t^A$.

Exercise 2:
Consider the Lagrangian of supersymmetric QED

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}(\overline{\Phi}_+ e^{2eV}\Phi_+ + \overline{\Phi}_- e^{-2eV}\Phi_-) + \left[\int d^2\theta\left(\frac{1}{4}W^\alpha W_\alpha + m\Phi_+\Phi_-\right) + \mathrm{h.c.}\right]\,.$$

This Lagrangian is invariant under U(1) gauge transformations

$$\begin{aligned} \Phi_\pm &\to e^{\pm ie\Lambda}\Phi_\pm \\ V &\to V - \frac{i}{2}(\Lambda - \overline{\Lambda}) \\ W^\alpha &\to W^\alpha \end{aligned}$$

where $\pm e$ is the charge of the chiral superfields $\Phi_\pm$. Notice that we need to oppositely charged chiral superfields Φ_+ and Φ_- to write a gauge-invariant mass term. The corresponding massless theory is also invariant with only one chiral superfield Φ.

Expand this Lagrangian in components and verify that it describes a massless gauge boson (the photon) and a charged massive fermion (the electron), as well as a massless neutral fermion (the photino) and a massive charged scalar (the selectron).

A.9 Exercise Sheet 9

Exercise 1:
In this exercise we will explore the supersymmetric Higgs mechanism.

Consider a model with three chiral superfields Φ_0, Φ_+ and Φ_- with quantum numbers 0, +1 and −1 under a global U(1) symmetry. The superpotential is given by

$$W(\Phi) = \frac{1}{2}m\Phi_0^2 + \mu\Phi_+\Phi_- + \lambda\Phi_0 + g\Phi_0\Phi_+\Phi_- \,.$$

a) Show that the superpotential is invariant under the global U(1) symmetry.

b) Supersymmetry remains unbroken, if the vacuum is supersymmetric, in terms of the scalar potential this means $V = 0$. Show that there are two sets of vacua for the scalar components of the superfields, which preserve supersymmetry. Show that one set of vacua leaves U(1)-symmetry unbroken, while the remaining ones break it.

c) Calculate the masses of the chiral states and verify that bosons and their fermionic superpartners have the same mass.

d) Now we introduce gauge invariance in the theory: The complete Lagrangian consists of the kinetic term, the superpotential term, the Fayet–Iliopoulos term and the field strength for the gauge bosons. Write down the Lagrangian in components and find the equations of motion for the auxiliary fields D and F. *Hint:* Use the results of Exercise sheet 8, Exercise 2. The only differences are the Fayet–Iliopoulos contribution and the neutral field Φ_0, which does not couple to the gauge fields.

e) Show that supersymmetry remains unbroken and calculate the mass term of the gauge fields. Note that the masses of the chiral states stay the same before and after imposing gauge invariance.

A.10 Exercise Sheet 10

Exercise 1:

a) Compute the supercurrent for the massless Wess–Zumino model, with Lagrangian

$$\mathcal{L} = (\partial_\mu z)^\dagger(\partial^\mu z) - i\psi\sigma^\mu\partial_\mu\overline{\psi} + F^\dagger F\,.$$

To do this first compute the usual Noether current, which is given by

$$\sum_l \delta\chi^l \frac{\partial_L \mathcal{L}}{\partial(\partial_\mu \chi^l)} \equiv -(\varepsilon^\alpha N^\mu_\alpha + \overline{N}^\mu_{\dot{\alpha}}\overline{\varepsilon}^{\dot{\alpha}}\,,)$$

where $\delta\chi^l$ are the supersymmetry variations of the various fields. For a chiral multiplet the variations are given by

$$\begin{aligned}
\delta z &= \sqrt{2}\varepsilon\psi\,,\\
\delta\psi_\alpha &= -\sqrt{2}\varepsilon_\alpha F + \sqrt{2}i\sigma^\mu_{\alpha\dot{\beta}}\overline{\varepsilon}^{\dot{\beta}}\partial_\mu z\,,\\
\delta F &= \sqrt{2}i\partial_\mu\psi\sigma^\mu\overline{\varepsilon}\,.
\end{aligned}$$

We know that the action is invariant under supersymmetry, but the variation of the Lagrangian is a total derivative

$$\delta\mathcal{L} = \varepsilon^\alpha\partial_\mu K^\mu_\alpha + \partial_\mu\overline{K}^\mu_{\dot{\alpha}}\overline{\varepsilon}^{\dot{\alpha}}\,.$$

It is easy to show that the divergence of the usual Noether current is also related to the variation of the Lagrangian

$$\varepsilon^\alpha\partial_\mu N^\mu_\alpha + \partial_\mu\overline{N}^\mu_{\dot{\alpha}}\overline{\varepsilon}^{\dot{\alpha}} = -\delta\mathcal{L}\,.$$

Putting everything together we obtain a conserved quantity, which we call the supercurrent

$$S^\mu \equiv N^\mu + K^\mu\,, \qquad \text{with} \quad \partial_\mu S^\mu = 0\,.$$

b) What happens when you add to the massless Wess–Zumino model the superpotential term

$$\mathcal{L}_{\mathrm{W}} = \left(\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\right)_{F-term} + \text{h.c.}$$

A.11 Exercise Sheet 11

Exercise 1:
In $N = 2$ supersymmetric theories we have not only vector multiplets but also hypermultiplets. A hypermultiplet is built by taking two chiral superfields[1]

$$
\begin{aligned}
H_1 &= (H^+, \eta^+_\alpha, F^+)\,,\\
H_2 &= (H^-, \eta^-_\alpha, F^-)\,.
\end{aligned}
$$

The scalar components H^+ and H^- form an SU(2) doublet, while the other components are SU(2) singlets.

a) Write down the most general renormalizable supersymmetric Lagrangian with a discrete R-symmetry

$$
\begin{aligned}
H^+ &\to -(H^-)^*\,,\\
H^- &\to (H^+)^*\,.
\end{aligned}
$$

and without gauge interactions.

b) In $N = 2$ supersymmetry we can introduce a gauge theory by combining two $N = 1$ supermultiplets, a vector multiplet $V^A = (v^A_\mu, \lambda^A, D^A)$ and a chiral multiplet $\Phi^A = (Z^a, \psi^A, F^A)$, and imposing a discrete R-symmetry

$$\psi^A \to \lambda^A \qquad \lambda^A \to -\psi^A\,.$$

The Yang–Mills Lagrangian for this theory reads

$$\mathcal{L}^{N=2}_{\mathrm{YM}} = \frac{1}{32\pi}\mathrm{Im}\left(\tau\int d^2\theta\ \mathrm{Tr}W^\alpha W_\alpha + \mathrm{h.c.}\right) + \int d^2\theta d^2\overline{\theta}\ \mathrm{Tr}\,\overline{\Phi}e^{2gV}\Phi\,.$$

How can we introduce gauge invariance for the hypermultiplet Lagrangian in exercise 1a)?

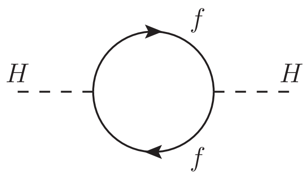
c) Why does $N = 2$ supersymmetry require H_1 and H_2 to be in a complex conjugate representations of the gauge group? *Hint:* Consider the easy case of a U(1) gauge theory and then generalize to arbitrary gauge groups.

[1] Here, the plus superscripts, e.g. H^+ denote the components of H_1. Complex conjugates are denoted by $\overline{H}^+$.

A.12 Exercise Sheet 12

Exercise 1:
The goal of this exercise is to check explicitly how supersymmetry improves the UV behavior of a quantum theory. We will look at the radiative corrections to the Higgs mass coming from fermion and scalar loops.

a) The Yukawa Lagrangian $\mathcal{L} = -\frac{y}{\sqrt{2}}H\overline{f}_L f_R + \text{h.c.}$ induces corrections to the Higgs mass through the following Feynman diagram

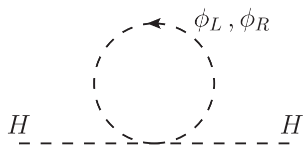


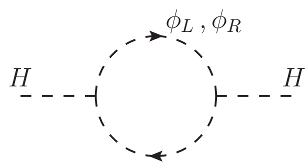
Compute the above Feynman diagram, by using a cut-off regularization. We are only interested in the divergent parts.

b) The Higgs can also interact with scalar particles through the following Lagrangian

$$calL = -\frac{\lambda}{2}H^2(|\phi_L|^2 + |\phi_R|^2) - H(\mu_L|\phi_L^2 + \mu_R|\phi_R|^2) - m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2\,.$$

Compute the divergent part of the following two Feyman diagrams contributing to the Higgs mass corrections using the cut-off regularization.





c) What relations among the couplings $(\lambda, y, \mu_L, \mu_R)$ and the masses (m_f, m_L, m_R) do we need to impose to cancel the quadratic divergences? What about the logarithmic divergences?

d) Are these relations fulfilled in supersymmetric theories? What happens to these relations if supersymmetry is broken explicitly by soft-breaking terms, i.e. mass terms and terms with couplings with positive mass dimension?

Exercise 2:
When extending the SM supersymmetrically, the following terms arise in the superpotential[2]

$$\mathcal{W}_{\Delta L=1} = \frac{1}{2}\lambda^{ijk}L_iL_j\overline{e}_k + \lambda'^{ijk}L_iQ_j\overline{d}_k + \mu'^{i}L_iH_u\,,$$

$$\mathcal{W}_{\Delta B=1} = \frac{1}{2}\lambda''^{ijk}\overline{u}_i\overline{d}_j\overline{d}_k\,.$$

a) Assume that the usual baryon number ($+1/3$ for Q_i, $-1/3$ for $\overline{u}_i$ and $\overline{d}_i$ and 0 for the other fields) is conserved. Show that the $\mathcal{W}_{\Delta B=1}$ term is forbidden.

[2]The superpotential has to be a gauge-invariant, holomorphic and renormalizable function of chiral superfields.

b) Find an assignment of lepton numbers that coincides with the usual one for the SM particles and is conserved by the $\mathcal{W}_{\Delta L=1}$ superpotential. *Hint:* The lepton number assignment can be different for the various components in a superfield. That is it can come from an R-symmetry that does not commute with the supersymmetry transformations.

A.13 Exercise Sheet 13

Exercise 1: Supersymmetry and Strong-Electroweak Unification
Assume that the SM gauge group $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is embedded into a simple group G, which is spontaneously broken at the GUT scale M_X. The gauge couplings g_s of $\mathrm{SU}(3)_c$, g of $\mathrm{SU}(2)_L$ and g' of $\mathrm{U}(1)_Y$ will be related by[3]

$$g_s^2 = g^2 = \frac{5}{3}g'^2 \qquad \text{at} \quad E \simeq M_X \,.$$

At energies far below M_X, these couplings change, according to the renormalization group equations (RGE's), which at one loop read

$$\mu\frac{d}{d\mu}g'(\mu) = \beta_1(g'(\mu))\,, \qquad \mu\frac{d}{d\mu}g(\mu) = \beta_2(g(\mu))\,, \qquad \mu\frac{d}{d\mu}g_s(\mu) = \beta_3(g_s(\mu))\,.$$

The one-loop beta functions are given by

$$\begin{aligned} \beta_1 &= \frac{g'^3}{4\pi^2}\left(\frac{5n_g}{6} + \frac{n_s}{8}\right), \\ \beta_2 &= \frac{g^3}{4\pi^2}\left(-\frac{3}{2} + \frac{n_g}{2} + \frac{n_s}{8}\right), \\ \beta_3 &= \frac{g_s^3}{4\pi^2}\left(-\frac{9}{4} + \frac{n_g}{2}\right), \end{aligned}$$

where n_g is the number of generations of quarks and leptons and n_s is the number of Higgs doublets.

a) Solve the one-loop RGE's.

b) Find an expression for the unification scale M_X and the weak mixing angle θ_w by imposing the conditions
$$g(m_Z) = \frac{-e(m_Z)}{\sin\theta_w} \qquad \text{and} \qquad g'(m_Z) = \frac{-e(m_Z)}{\cos\theta_w}\,.$$
These expressions will still depend on n_g and n_s. Imposing the MSSM conditions $n_g = 3$ and $n_s = 2$, compute the numerical values for M_X and $\sin\theta_w$. How these values agree with the experimental constraints coming from the proton decay and the electroweak precision measurements?

Exercise 2: Superpotential couplings in the MSSM
The most general MSSM superpotential respecting R-parity is

$$\mathcal{W}_{\text{MSSM}} = \overline{u}Y_uQH_u - \overline{d}Y_dQH_d - \overline{e}Y_eLH_d + \mu H_uH_d\,,$$

where Y_u, Y_d and Y_e are 3×3 Yukawa matrices in family space. Approximating them by setting all but the $(3,3)$ entries to zero, we obtain

$$\mathcal{W}_{\text{MSSM}} \sim y_t(\overline{t}tH_u^0 - \overline{t}bH_u^+) - y_b(\overline{b}tH_d^- - \overline{b}bH_d^0) - y_\tau(\overline{\tau}\nu_\tau H_d^- - \overline{\tau}\tau H_d^0) + \mu(H_u^+H_d^- - H_u^0H_d^0)\,.$$

What interactions does this superpotential generate? What are the coupling constants for these interactions?

[3] For an Abelian gauge group the coupling is not quantized. The value 5/3 in front of g'^2 is a convention useful to match the predictions of unified groups such as SU(5) or SO(10).

References

Books

[1] J. Wess and J. Bagger, *"Supersymmetry and supergravity"*, Princeton, USA, University Press (1992) 259 p.

[2] P. C. West, *"Introduction to supersymmetry and supergravity"*, Singapore, World Scientific (1990) 425 p.

[3] S. Weinberg, *"The quantum theory of fields. Vol. 3: Supersymmetry"*, Cambridge University Press (2000) 419 p.

Reviews

[4] M. F. Sohnius, *"Introducing Supersymmetry"*, Phys. Rept. **128** (1985) 39.

[5] J. D. Lykken, *"Introduction to supersymmetry"*, hep-th/9612114.

[6] S. P. Martin, *"A Supersymmetry primer"*, Adv. Ser. Direct. High Energy Phys. **21** (2010) 1 [Adv. Ser. Direct. High Energy Phys. **18** (1998) 1] [hep-ph/9709356].

[7] A. Bilal, *"Introduction to supersymmetry"*, hep-th/0101055.

[8] M. Drees, *"An Introduction to supersymmetry"*, hep-ph/9611409.

[9] J. M. Figueroa-O'Farrill, *"Busstepp lectures on supersymmetry"*, hep-th/0109172.

[10] P. Van Nieuwenhuizen, *"Supergravity"*, Phys. Rept. **68** (1981) 189.

[11] H. P. Nilles, *"Supersymmetry, Supergravity and Particle Physics"*, Phys. Rept. **110** (1984) 1.

[12] H. K. Dreiner, H. E. Haber and S. P. Martin, *"Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry"*, Phys. Rept. **494** (2010) 1 [arXiv:0812.1594 [hep-ph]].