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STEREOGRAPHIC PROJECTION

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Б. А. Розенфельд
и Н. Д. Сергеева

СТЕРЕОГРАФИЧЕСКАЯ ПРОЕКЦИЯ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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INTRODUCTION

Projection of figures onto a plane is often used in mathematics. To produce an image of a figure one has to choose a point in space, called the centre of projection, then to draw straight lines connecting this point with all points of the projected figure and find the points of intersection of these lines with the given plane; the image obtained is called the *projection* of the figure onto a given plane.

If the projected figure is a circle then its projection is the line of intersection of the plane with the surface consisting of straight lines passing through the centre of projection and the points of the circle. Such a surface is called a *circular cone*, a *right cone* if the perpendicular dropped from the projection centre onto the plane of the circle passes through its centre, and an *oblique cone* in all other cases. In a general case the lines of intersection of such a surface with a plane are not circular and are called the *conic sections* and, if the cutting plane does not pass through the cone vertex, belong to one of the three species of curves: *ellipses*, if these curves are closed, *parabolas*, if they consist of a single branch extending to infinity, and *hyperbolas*, if these curves consist of two branches extending to infinity (on the assumption that the straight lines connecting the cone vertex to a given circle are infinitely long); circles can be considered as particular cases of ellipses.

But there is a remarkable projection in which circles are always projected as circles or as straight lines. We shall obtain this projection if we consider only those circles which lie on a certain sphere (these circles are the lines of intersection of this sphere with planes), select one of the points of this sphere as the projection centre and assume the plane tangent to the sphere in the diametrically opposite point, or any plane parallel to the first and not passing through the projection centre, to be the projection plane. When the plane of the circle passes through the projection centre it is projected as a straight line; in other cases a circle on the sphere is projected as a circle on the projection plane. This projection possesses one more unexpected property: the angles between the lines on the sphere are mapped in this projection as equal to them angles between the lines on the plane. The third important property of this projection is that when the sphere is rotated around its diameter passing through the projection centre, the projections onto the plane of all the figures on the sphere are rotated around the point of intersection of this plane with the diameter of the sphere by the same angle.

This projection which is normally called the *stereographic projection* is frequently used in various branches of mathematics as well as in astronomy, crystallography and geography.

The present booklet is devoted to proofs of the aforesaid properties of the stereographic projection and to the presentation of some of its applications. The booklet consists of eight sections. Sec. 1 gives a definition of the stereographic projection and proofs of its basic properties. In Sec. 2 we establish the connection between the stereographic projection and a remarkable transformation of a plane onto itself in which the circles are also transformed into circles or straight lines and the angles between the lines are transformed into the angles equal to them – this transformation is called the *inversion with respect to a circle*; in the same section we establish the relation of the stereographic projection to the similar transformation of space – the *inversion with respect to a sphere.* In Sec. 3 the basic properties of the stereographic projection are proved in a different way, namely by means of coordinates. Sec. 4 establishes the relation between the stereographic projection and the complex numbers: when the projection plane is considered to be a plane of a complex variable, mapping of complex numbers by the points on the sphere is realized by means of a stereographic projection. This mapping is frequently utilized in the theory of functions of complex variables since the so-called point at infinity of the plane of the complex variable, which cannot be mapped on the plane itself, is given on the sphere by the very projection centre. The same section discusses the so-called spherical metric on a plane when the distance between two points of the plane is assumed equal to a spherical distance between the corresponding points on the sphere; this distance is expressed in the simplest form by means of complex numbers. In Sec. 5 we show how the rotations of the sphere are mapped by the plane transformations in the stereographic projection; these transformations are also expressed most simply by means of complex numbers. Sec. 6 gives an account of the history of stereographic projection which was developed already in antiquity and was very popular in the Middle Ages. Sec. 7 describes how the stereographic projection applies to astronomy – medieval *astrolabes* were based on this projection – and to geography where this projection is used to draw nautical maps. Sec. 8 presents the definition of the *Lobachevskian plane*, demonstrates how a peculiar stereographic projection can yield a projection of the Lobachevskian plane onto an ordinary plane so that the circles and some other curves on the Lobachevskian plane are mapped as circles or straight lines while the angles between the lines of the Lobachevskian plane are mapped as the angles equal to them.

The booklet is aimed to be used in the senior grades of the high schools and by the first- and second-year students.

1. Definition and Basic Properties of the Stereographic Projection

The *stereographic projection* is a projection of a sphere from one of its points S onto the plane σ tangent to the sphere in the diametrically opposite point S' (Fig. 1). The properties of this projection do not change significantly if the plane σ is replaced by any other plane parallel to σ and not passing through the projection centre; the diametral plane of the sphere is often taken as the projection plane (if we call the projection centre and the diametrically opposite point of the sphere its poles, this plane may be referred to as the equatorial plane of the sphere).

Let us prove the following three properties of the stereographic projection.

A. *The circles lying on a sphere are projected onto the plane σ as circles or, if the circles on the sphere pass through the projection centre, as straight lines.*

Before we proceed to the proof of this property we shall note that the transformation from any point M of the sphere to its projection M' on the plane is carried out on a certain plane passing through the diameter SS' of the sphere.

So we shall consider first the stereographic projection of a circle onto a straight line in one of such planes (Fig. 2) and prove for the case the following

Lemma. *Let the points M and N of the circle projected onto a straight line by the stereographic projection be projected into the points M' and N' of this line. Then $\angle SMN = \angle SN'M'$, and $\angle SNM = \angle SM'N'$.*

Indeed, the right triangles SMS' and $SS'M'$ with a common acute angle MSS' are similar so that

$$\frac{SM}{SS'} = \frac{SS'}{SM'}, \text{ i.e. } SM \cdot SM' = (SS')^2.$$

Much as in considering the right triangles SNS' and $SS'N'$ with a common angle NSS', we obtain $SN \cdot SN' = (SS')^2$. Comparing the above equalities we derive

$$SM \cdot SM' = SN \cdot SN', \tag{1}$$

whence

$$\frac{SM}{SN'} = \frac{SN}{SM'}. \tag{2}$$

As follows from the proportion (2), the triangles SMN and $SN'M'$ with a common acute angle MSN are similar, so that $\angle SMN$

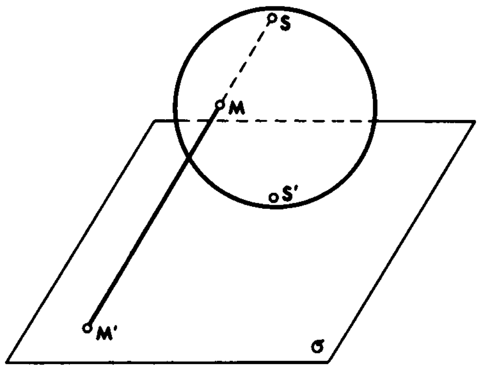


FIG 1

and $\angle SNM$ of the triangle SMN are equal to $\angle SN'M'$ and $\angle SM'N'$ of the triangle $SN'M'$ respectively.

We now proceed to prove the property (A) of the stereographic projection. If the circle on the sphere passes through the point S, it lies in the plane passing through this point, and its projection from the point S onto the plane σ is the line of intersection of these two planes, i.e. a straight line. If the circle on the sphere does not pass through the point S, we can assume that the plane passing through the straight line SS' and the centre of this circle is the plane of Fig. 2, and the diameter of this circle lying in this plane is the segment MN. Then the lines projecting the points of this circle are the rectilinear generators of the circular cone having the vertex in the point S.

While a right circular cone has only a single set of circular sections (sections by a set of planes parallel to its base), the oblique circular cone has two such sets. One of these sets also consists of sections by the planes parallel to the base of the cone. To obtain the second set of circular sections of the oblique circular cone we shall recall that if a perpendicular CD is dropped from an arbitrary point C of the circle onto its diameter AB (Fig. 3), then the following equality is valid:

$$AD \cdot DB = CD^2, \tag{3}$$

and, conversely if the equality (3) holds for each point C of the curve and for the line AB, then this curve is the circle, or a part of the circle.

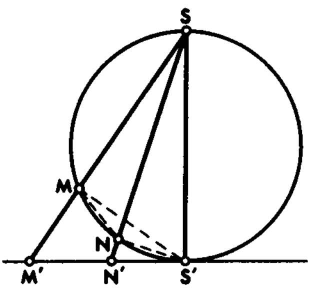


FIG 2

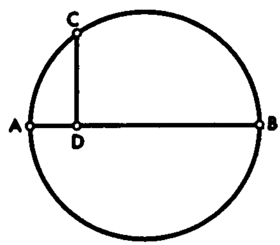


FIG. 3

Consider now an oblique circular cone with the vertex A and the base whose diameter is the segment BC; we shall assume that the line BC passes through the foot of the perpendicular dropped from the cone vertex onto its base (Fig. 4). We now cut the cone by a plane normal to the plane ABC and intersecting it along a line HK such that the points H and K lie on the surface of the cone, and $\angle AHK = \angle ACB$ and also $\angle AKH = \angle ABC$. This plane will cut the cone surface along the curve HJK. Let us show that this curve HJK is a circle. To achieve this we shall consider an arbitrary point J of this curve and an arbitrary point L on the cone base circumference and drop the perpendiculars JG and LM from these points onto the plane ABC. The straight lines JG and LM being the perpendiculars to the same plane are parallel to one another. Let us now draw through the point G the straight line DGE parallel to the line BC, and a plane passing through the straight lines DE and JG. Since the line DE is parallel to the line BC and the line JG is parallel to the line LM, the plane we have constructed is parallel to the plane of the cone base and hence cuts it by a circular section. But by virtue of the property (3) the following equality holds for this circular section

$$DG \cdot GE = JG^2. \tag{4}$$

On the other hand, since $\angle AHK = \angle ACB$ (which, due to the lines DE and BC being parallel, is equal to $\angle AED$) and $\angle AKH = \angle ABC$ (which due to the same reason is equal to $\angle ADE$), we obtain $\angle HDG = \angle EKG$ and $\angle DHG = \angle KEG$. Therefore the triangles EGK and HGD, having equal angles at the vertex G, are similar.

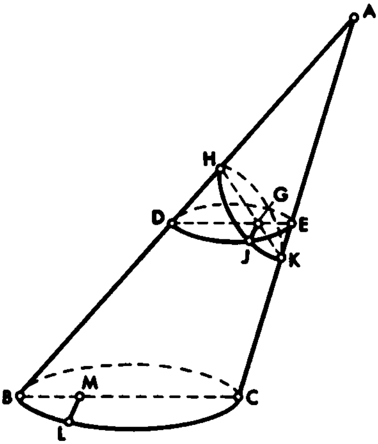


FIG 4

By virtue of similarity of these triangles the following proportion holds:

$$\frac{HG}{GD} = \frac{EG}{GK},$$

whence

$$DG \cdot GE = HG \cdot GK,$$

i.e. by Eq. (4)

$$HG \cdot GK = JG^2. \tag{5}$$

Since Eq. (5) also has the form of Eq. (3) and this equality is valid for any point of the curve HJK and of the straight line HK, the curve HJK is a circle. Since the section of the cone by any plane parallel to the plane HJK possesses the same property, we have obtained the second set of circular sections of the oblique circular cone.

Since the triangles $SM'N'$ and SNM in Fig. 2 are in the same positions as the triangles ABC and AHK in Fig. 4, then it follows from the equality of the angles of the triangles $SM'N'$ and SNM that the section of the oblique circular cone (whose rectilinear generators are the straight lines projecting a circle of a diameter MN on the sphere) by the plane tangent to the sphere in the point S' is the circle with the diameter $M'N'$. Thus the property (A) is proved.

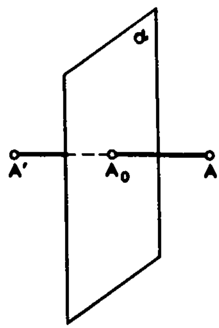


FIG 5

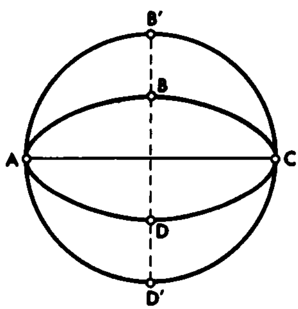


FIG 6

That the oblique circular cone has two sets of circular sections can also be proved in a different way, by using the symmetry plane of this cone. A figure is said to be symmetric with respect to the plane α (Fig. 5) if for any point A of this figure there exists another point A' of this same figure which is a specular reflection of the point A with respect to the plane α, i.e. the point A' lies on the perpendicular, dropped from the point A onto the plane α, at the same distance from the plane α as the point A but on the other side of this plane. In the case of a right circular cone any plane passing through its axis is the symmetry plane. It can be proved that one of the symmetry planes of the oblique circular cone shown in Fig. 4 is the plane ABC drawn through a straight line, connecting the cone vertex with the base centre, and through the perpendicular dropped from the cone vertex onto the base plane. This plane cuts the cone along two generators. The bisectrix of the angle between the obtained generators is called the axis of the oblique cone (which we imagine to go to infinity). The second symmetry plane of the oblique cone is the plane passing through the cone axis normally to the first plane. All circular sections of the cone are transformed into themselves by the reflection in the first plane, while the reflection in the second plane transforms the circular

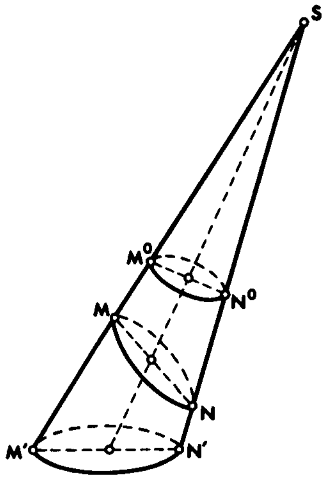


FIG 7

sections of the first set into those of the second set, and vice versa. The existence of two mutually perpendicular symmetry planes in the oblique circular cone is closely connected with the fact that a section of this cone by a plane normal to its axis gives the curve having two mutually perpendicular symmetry axes – the so-called ellipse which can be obtained by compressing a circle towards one of its diameters. Fig. 6 shows the ellipse $ABCD$ obtained by compressing the circle $AB'CD'$ towards its diameter AC. Fig. 7 shows two sections of the oblique circular cone in which MN is the base diameter: a circular section with the diameter $M'N'$ and an elliptic section in which the segment M^0N^0 is one of the symmetry axes.

It should be noted that any circle or straight line on the plane σ is the projection of a circle on the sphere: any straight line is the projection of the circle which is cut on the sphere by the plane passing through this line and the projection centre, and any circle on the plane σ is the base circle of an oblique circular cone with the vertex in the projection centre. Reasoning along the same line as in (A) we can prove that the line along which the surface of this cone intersects the sphere is a circle of the circular section belonging to the set of the circular

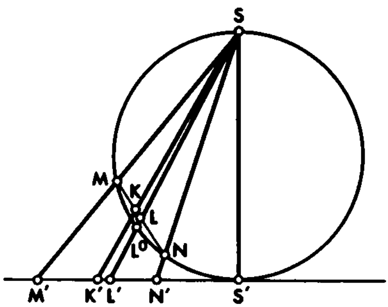


FIG 8

sections of the cone not parallel to the base. Due to this any circle on the plane σ is the projection of that very circle on the sphere along which the sphere is cut by the cone defined by the circle on the plane σ.

We also note that the centre of the circle of the diameter MN is not projected into the centre of the circle with the diameter $M'N'$. Indeed, let L be the middle point of the diameter MN and L' its projection onto the plane (Fig. 8). Since the straight line SL is not normal to the chord MN, it cuts the arc MN into unequal parts ML^0 and L^0N, with $ML^0 > L^0N$. Therefore $\angle MSL > \angle LSN$. Let us draw a straight line SK at an angle $\angle MSK$ to the line SM, equal to $\angle LSN$; this line intersects the line $M'N'$ in K'. The angles at S in the triangles $SM'K'$ and SNL are equal by construction, and the angles M' and N are equal as proved above. Therefore these triangles are similar, hence the following proportion $\dfrac{SN}{SM'} = \dfrac{NL}{M'K'}$ holds. On the other hand,

the similarity of the triangles SMN and $SN'M'$ yields the proportion $\frac{SN}{SM'} = \frac{MN}{M'N'}$ whence we obtain $\frac{NL}{M'K'} = \frac{MN}{M'N'}$ and $\frac{NL}{MN} = \frac{M'K'}{M'N'}$. But $NL = {}^1/_2\ MN$, therefore $M'K' = {}^1/_2\ M'N'$, i.e. K' is the middle point of the diameter $M'N'$. Hence the centre K' of the circle with the diameter $M'N'$ is not the projection of the centre L of the circle with the diameter MN but of the point K of this diameter for which $\angle MSK = \angle LSN$.

Let us prove the second property of the stereographic projection.

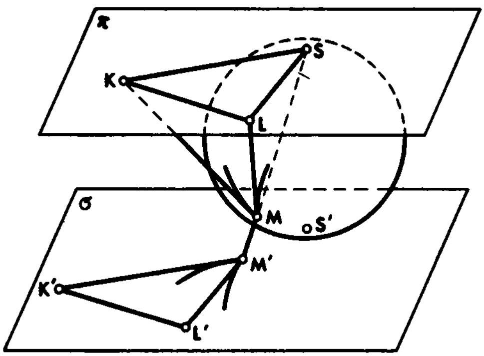


FIG 9

B. ***The stereographic projection maps the angles between the curves lying on a sphere as equal to them angles between the curves projected onto the plane σ.***

The angle between two curves is defined as the angle between the tangents to these curves at the point of their intersection. Let us draw two curves from the point M of the sphere. Let the tangents to these curves in the point M intersect the plane π, tangent to the sphere in the point S, and in the points K and L (Fig. 9). Let us connect the points K and L with the point S. Then $KM = KS$ as two tangents to the sphere drawn from the same point, and $LM = LS$ due to the same reason. Therefore all arms in the triangles KLM and KLS with the common arm KL are mutually equal, whence follows the equality of the angles in these triangles; in particular, $\angle KML = \angle KSL$. Our curves are projected onto the plane σ as two curves emerging from the point M', the angle between these curves being equal to that between the tangents. These tangents $M'K'$ and $M'L'$ are the projections of the tangents MK and ML and are, therefore, the intersections of the planes SKM and SLM with the projection plane σ. But the planes SKM and SLM intersect the plane π, parallel to the projection plane σ, along the straight lines SK and SL, therefore the straight lines $M'K'$ and $M'L'$ are parallel, respectively, to the lines SK and SL, and $\angle K'M'L' = \angle KSL$; and since $\angle KSL = \angle KML$, we obtain $\angle K'M'L' = \angle KML$. Thus the property (B) is complete.

The third property of the stereographic projection is as follows.

C. *Rotation of the sphere around the diameter passing through its pole results in the rotation on the plane σ around the tangency point by the same angle.*

This property stems directly from the fact that the transformation of any point M into its projection M' on the plane σ takes place in a plane passing through the diameter SS'; when the sphere is rotated by the angle φ, the line of intersection of this plane with the projection plane is rotated by the same angle.

2. Stereographic Projection and Inversion

Let us find which points of the stereographic projection on the plane σ map the diametrically opposite points of the sphere. Let the diametrically opposite points of the sphere M and N be projected by the stereographic projection into the points M' and N' of the plane σ (Fig. 10). Let us show that if R denotes the radius of the sphere then

$$S'M' \cdot S'N' = 4R^2. \tag{6}$$

Indeed, the angle MSN based on the diameter MN of the circle is right and therefore the angle S in the triangle $M'SN'$ is also right. Since the segment SS' is the height of the right-angled triangle $M'SN'$, the following equality holds:

$$S'M' \cdot S'N' = (SS')^2.$$

Substituting SS' in this equality by $2R$ we obtain Eq. (6).

If now M' is an arbitrary point of the plane σ, distinct from the point S', we can establish a correspondence between the point M' and a certain point N' of the plane in the following manner: we connect the point M' with S, find the intersection point M of the line $M'S$ with the sphere, find the point N of the sphere diametrically opposite to the point M and project the point N onto the point N' of the plane σ. We have obtained a transformation of the plane which sets up a correspondence between any point M' of the plane, distinct from the point S', and a certain point N' of the same plane.

Let us show that this transformation is closely connected with a well-known transformation of the plane, the so-called *inversion* with respect to a circle. Let a circle with the centre M_0 and the radius R be specified on the plane (Fig. 11). The transformation of a plane in which each point M of the plane, distinct from M_0, is transformed into a point M' on the straight line M_0M lying on the same side from M_0 as M, such that

$$M_0M \cdot M_0M' = R^2, \tag{7}$$

is called the inversion with reference to this circle.

The inversion transforms the points lying inside the circle into the points lying outside the circle, and vice versa, while the points of the circle itself are transformed into themselves.

The transformation considered above which maps the points M' of the plane σ, distinct from the point S', into the points N' differs from the inversion in a circle of radius $2R$ with the centre S' in that the point N' lies on the straight line $M'S'$ at a distance determined by Eq. (6) not on the same side from the point S' as the point M' but on the opposite side. This transforma-

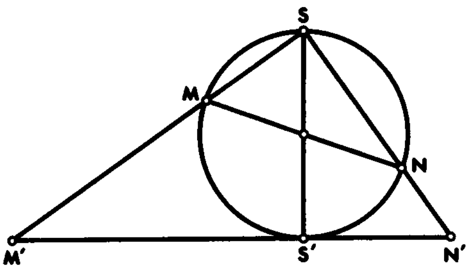


FIG 10

tion can be presented as two consecutively performed operations (in any sequence) of the aforesaid inversion and reflection with respect to the point S'. Hence the inversion in any circle of the plane with the centre M_0 and radius R can be presented as the result of four consecutively performed transformations: the transformation reciprocal of the stereographic projection with reference to the sphere of the radius $R/2$ tangent to the plane in the point M_0 whereby the point M of the plane σ is transformed into a point of the sphere; transformation from this point of the sphere into a diametrically opposite point of the same sphere; stereographic projection of the point obtained on the sphere onto the plane σ; and the reflection of the thus obtained point on the plane σ with respect to the point M_0.

Owing to the property (A) of the stereographic projection circles on the sphere are projected onto the plane σ as circles and straight lines, and conversely, any circle or straight line on the plane σ is the projection of a circle on the sphere; transition to the diametrically opposite points of the sphere (i. e. reflection of the sphere with respect to its centre) transforms the circles on the sphere into the circles; reflection of the plane with respect to a point transforms the circles and straight lines into circles and straight lines, so that the following property stems

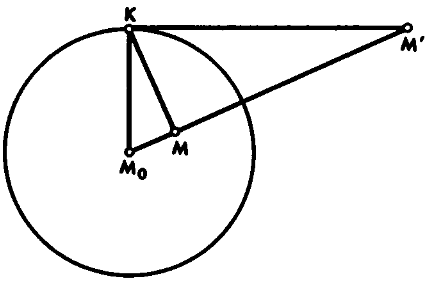


FIG 11

from our representation of the inversion as the result of the consecutive application of the four mentioned transformations:

A'. *The inversion transforms circles and straight lines into circles and straight lines.*

It is readily seen that circles and straight lines are transformed by inversion into straight lines if and only if they pass through the point M_0.

The same line of argument shows that by virtue of property (B) of the stereographic projection the angles between the curves on the sphere are mapped as equal to them angles between the corresponding curves on the plane σ. The reflection of the sphere with respect to its centre and the reflection in the plane from a point also transforms the angles between the curves on the sphere or on the plane into equal to them angles on the sphere or on the plane. As follows from four consecutive transformations the inversion is also characterized by the property

B'. *The inversion transforms the angles between the curves into equal to them angles between the transformed curves.*

The result of the consecutively performed inversion with respect to a circle of radius R with the centre M_0 and reflection of the plane from a point is referred to, due to reasons which will be discussed below (Sec. 4), as the inversion in an imaginary circle of an imaginary radius iR with the centre M_0.

We shall mention that quite similarly to the inversion with respect to a circle on a plane we can define the *spatial inversion with respect to a sphere* of radius R with the centre M_0 i.e. a transformation of space in which each point of the space,

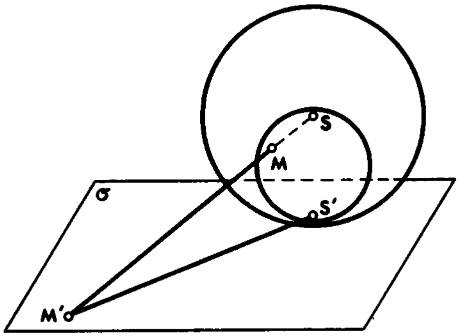


FIG 12

distinct from M_0, is transformed into point M' on the straight line M_0M and on the same side of M_0 as M, such that the relationship (7) is satisfied.

It can be proved that the spatial inversion in a sphere possesses the same properties (A') and (B') as the inversion with respect to a circle on a plane, and in addition, one more property quite similar to the property (A'):

A″. *The inversion transforms the spheres and planes into spheres and planes.*

It can be readily verified that the circles and straight lines are transformed by the inversion in a sphere into straight lines, and the spheres and planes are thereby transformed into planes if and only if they pass through the point M_0.

There exists a remarkable interconnection between the stereographic projection and the inversion with respect to a sphere. Namely, *if we perform the inversion with respect to the sphere with S as its centre and SS' as its radius, then the sphere with the diameter SS' will be transformed into a plane σ tangent to both spheres in the point S'; the transformation of the sphere on* *the plane σ thus obtained coincides with the stereographic projection of the sphere onto the plane σ* (Fig. 12). Indeed we have established in Sec. 1 that for any point M of the sphere projected onto the plane σ and the point M' corresponding to it on the plane σ we have

$$SM \cdot SM' = (SS')^2,$$

which shows that the point M' of the plane is obtained from the point M by the inversion with respect to a sphere of radius SS' with the centre S.

3. Proof of the Properties of the Stereographic Projection by Means of Coordinates

For those readers who mastered the method of coordinates we present the proof of the properties of the stereographic projection by means of coordinates (this section may be safely skipped without detriment to understanding of the sections that follow). We shall use the rectangular coordinates in space X, Y, Z. As will be recalled in these coordinates the distance d between the point M_1 with the coordinates X_1, Y_1, Z_1 and the point M_2 with the coordinates X_2, Y_2, Z_2, which we shall denote as $M_1(X_1, Y_1, Z_1)$ and $M_2(X_2, Y_2, Z_2)$, is equal to

$$d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}, \tag{8}$$

and the angle φ between the directed segments (vectors) $\overline{OM}_1$ and $\overline{OM}_2$ drawn from the origin of coordinates O is found via the formula

$$\cos \varphi = \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}\sqrt{X_2^2 + Y_2^2 + Z_2^2}}. \tag{9}$$

Let us consider the stereographic projection of a sphere of radius 1 through the origin O from the projection centre S, located on the axis OZ, onto the plane σ tangent to the sphere in the diametrically opposite point. In this case the equation for the sphere takes the form

$$X^2 + Y^2 + Z^2 = 1, \tag{10}$$

the point S has the coordinates 0, 0, 1, and the projection plane σ is the plane $Z = -1$. Let the point $M(X, Y, Z)$ of the sphere be projected stereographically into the point $M'(x, y, -1)$ of the plane. Let us find the relation between the coordinates x, y of the point M' and the coordinates X, Y, Z of the point M.

Since the points S, M and M' lie on the same straight line, the vectors $\overline{SM}$ and $\overline{SM'}$ are directed along the same line so that, as a consequence, the differences between the coordinates X, Y, $Z-1$ of the points S, M and x, y, -2 of the points S, M' are proportional

$$\frac{X}{x} = \frac{Y}{y} = \frac{1-Z}{2} = k.$$

Therefore

$$X = kx, \; Y = ky, \; Z = 1 - 2k.$$

Since the coordinates X, Y, Z satisfy the equation (10) of the sphere we obtain

$$k^2 (x^2 + y^2) + (1 - 2k)^2 = 1,$$

or

$$k^2 (x^2 + y^2 + 4) - 4k = 0. \tag{11}$$

The values of k, satisfying the condition (11), correspond to the points of intersection of the line SM with the sphere: the value $k = 0$ corresponds to the point S itself, and the value

$$k = \frac{4}{x^2 + y^2 + 4}$$

to the point M. Hence the coordinates of the point M, corresponding to the point M', are equal to

$$X = \frac{4x}{x^2 + y^2 + 4}, \; Y = \frac{4y}{x^2 + y^2 + 4}, \; Z = \frac{x^2 + y^2 - 4}{x^2 + y^2 + 4}. \tag{12}$$

Let us now prove the properties of the stereographic projection by means of coordinates.

A. Since circles on a sphere are cut on it by planes, the coordinates of circles on the sphere fall under the same conditions as the coordinates of the points of the plane, i. e. by the equations of planes. Let us consider the plane determined by the equation

$$AX + BY + CZ + D = 0 \tag{13}$$

and find the locus of the points on the plane corresponding to the points of intersection of the plane (13) with the sphere (10). To do this we substitute into Eq. (13) the values of X, Y, Z from Eqs. (12). We obtain that the coordinates x, y of the points of this locus satisfy the condition

$$A\frac{4x}{x^2+y^2+4}+B\frac{4y}{x^2+y^2+4}+C\frac{x^2+y^2-4}{x^2+y^2+4}+D=0,$$

which can be rewritten in the form

$$4Ax+4By+C(x^2+y^2-4)+D(x^2+y^2+4)=0,$$

or

$$(C+D)(x^2+y^2)+4Ax+4By+4(D-C)=0. \tag{14}$$

Substituting the coordinates of the point S into Eq. (13) we obtain $C+D=0$ which is a necessary and sufficient condition for the plane (13) to pass through the point S. Therefore if the plane (13) does not pass through the point S, then $C+D\neq 0$ and Eq. (14) is the equation of a circle. If the plane (13) passes through the point S, then $C+D=0$ and Eq. (14) is the equation of a straight line.

B. The proof of this property by means of coordinates requires certain knowledge of the differential calculus. The angle between two curves on the sphere is equal to the angle between the tangents to these curves in the intersection point and, consequently, to the angle between the vectors directed along these tangents. But if we call the coordinates of the point M the coordinates of the vector $\overline{OM}$, then the vector whose coordinates are the differentials dX, dY, dZ of the coordinates of the point M can be taken as the vector directed along the tangent to the curve in the point $M(X, Y, Z)$. If such a vector, directed along the tangent to one of the two curves on the sphere, is denoted by $\{dX, dY, dZ\}$, then the vector directed along the other curve will be denoted by $\{\delta X, \delta Y, \delta Z\}$. By virtue of Eq. (9) the angle Φ between these vectors and, consequently, between the curves is found by the formula

$$\cos\Phi=\frac{dX\,\delta X+dY\delta Y+dZ\,\delta Z}{\sqrt{dX^2+dY^2+dZ^2}\,\sqrt{\delta X^2+\delta Y^2+\delta Z^2}}. \tag{15}$$

The angle φ between the two curves obtained by projecting onto the plane the two curves on the sphere is equal to the angle between the corresponding vectors $\{dx, dy\}$ and $\{\delta x, \delta y\}$, i.e.

$$\cos\varphi=\frac{dx\,\delta x+dy\delta y}{\sqrt{dx^2+dy^2}\,\sqrt{\delta x^2+\delta y^2}}. \tag{16}$$

We shall find the differentials dX, dY, dZ by differentiating the formulas (12). These differentials are equal to

$$dX = \frac{(x^2 + y^2 + 4)\,4\,dx - 2\,(xdx + ydy)\,4x}{(x^2 + y^2 + 4)^2} =$$

$$= \frac{4\,(y^2 - x^2 + 4)\,dx - 8\,xy\,dy}{(x^2 + y^2 + 4)^2},$$

$$dY = \frac{(x^2 + y^2 + 4)\,4\,dy - 2\,(xdx + ydy)\,4y}{(x^2 + y^2 + 4)^2} =$$

$$= \frac{4\,(x^2 - y^2 + 4)\,dy - 8\,xy\,dx}{(x^2 + y^2 + 4)^2},$$

$$dZ = \frac{(x^2 + y^2 + 4)\ \ 2\,(xdx + ydy) - (x^2 + y^2 - 4)\ \ 2\,(xdx + ydy)}{(x^2 + y^2 + 4)^2} =$$

$$= \frac{16(xdx + ydy)}{(x^2 + y^2 + 4)^2}.$$

Substituting these differentials into the expressions for the numerator and for the multipliers in the denominator of Eq. (15), we obtain

$$dX\,\delta X + dY\,\delta Y + dZ\,\delta Z = \frac{16\,(dx\,\delta x + dy\,\delta y)}{(x^2 + y^2 + 4)^2},$$

$$\sqrt{dX^2 + dY^2 + dZ^2} = \frac{4\sqrt{dx^2 + dy^2}}{x^2 + y^2 + 4},$$

$$\sqrt{\delta X^2 + \delta Y^2 + \delta Z^2} = \frac{4\sqrt{\delta x^2 + \delta y^2}}{x^2 + y^2 + 4}.$$

Therefore

$$\cos\Phi = \frac{dX\delta Y + dY\delta Y + dZ\delta Z}{\sqrt{dX^2 + dY^2 + dZ^2}\ \ \sqrt{\delta X^2 + \delta Y^2 + \delta Z^2}} =$$

$$= \frac{dx\delta x + dy\delta y}{\sqrt{dx^2 + dy^2}\ \ \sqrt{\delta x^2 + \delta y^2}} = \cos\varphi, \tag{17}$$

and, as a result, the angle Φ between the curves on the sphere is equal to the angle φ between the corresponding curves on the plane σ.

C. Rotation of the sphere around the OZ axis can be written as

$$\left.\begin{aligned} X' &= X\cos\Phi - Y\sin\Phi, \\ Y' &= X\sin\Phi + Y\cos\Phi, \end{aligned}\right\} \\ Z' = Z, \tag{18}$$

and the rotation of the plane around the origin of coordinates has the form

$$\left.\begin{aligned} x' &= x\cos\varphi - y\sin\varphi, \\ y' &= x\sin\varphi + y\cos\varphi. \end{aligned}\right\} \tag{19}$$

The coordinates of the point on the sphere, corresponding to the point in the plane with the coordinates x', y', -1 owing to a readily verified relation

$$x'^2 + y'^2 = x^2 + y^2,$$

have the form

$$X' = \frac{4(x\cos\varphi - y\sin\varphi)}{x^2 + y^2 + 4} = X\cos\varphi - Y\sin\varphi,$$

$$Y' = \frac{4(x\sin\varphi + y\cos\varphi)}{x^2 + y^2 + 4} = X\sin\varphi + Y\cos\varphi,$$

$$Z' = \frac{x^2 + y^2 - 4}{x^2 + y^2 + 4} = Z,$$

i.e. they coincide with the coordinates (18) at $\Phi = \varphi$, whence the validity of our statement follows.

4. Spherical Metric on a Plane. Application of Complex Numbers

Apart from an ordinary distance between the points on a plane we can also define other distances by quite different rules. The rules defining the distances between the points are called the *metrics* of the plane (from the Greek word "μετρεο" for "I measure").

Among other things in projecting a sphere onto a plane, we can transfer the metric of the sphere to the plane if we assume the distance between the points M' and N' of the plane to be equal to that between the corresponding points on the sphere measured along the great circle of the sphere – the so-called spherical distance; in case of the sphere of radius r the spherical distance between the points M and N is equal to the angle MON between the radii OM and ON of the sphere r times as large, and to the angle MON when $r = 1$.

The stereographic projection establishes a mutual one-to-one correspondence between the points of the plane σ and the points of the sphere, with the point S excluded.

To obtain a mutual one-to-one correspondence of the plane σ with the whole sphere, we have to supplement the plane σ by a point which we shall assume to correspond to the point S of the sphere. When a point of the sphere approaches the point S, the corresponding point of the plane moves off to infinity, then the complementary point is referred to as the *point at infinity*; we shall denote this point by the symbol ∞.

The spherical distance ω between the points M and N of the sphere (10) of the unit radius, with the coordinates X, Y, Z and X', Y', Z', is equal to the angle between the radii OM and ON, i.e.

$$\cos \omega = XX' + YY' + ZZ'. \tag{20}$$

By substituting the expressions (12) for the coordinates X, Y, Z via the coordinates x, y of the point on the plane and similar expressions for the coordinates X', Y', Z' via x', y' the distance ω will be expressed via the coordinates x, y and x', y' by the formula

$$\cos\omega = \frac{16(xx' + yy') + (x^2 + y^2 - 4)(x'^2 + y'^2 - 4)}{(x^2 + y^2 + 4)(x'^2 + y'^2 + 4)} \tag{21}$$

or

$$\cos^2\frac{\omega}{2} = \frac{1 + \cos\omega}{2} =$$

$$= \frac{(x^2 + y^2 + 4)(x'^2 + y'^2 + 4) + 16(xx' + yy') + + (x^2 + y^2 - 4)(x'^2 + y'^2 - 4)}{2(x^2 + y^2 + 4)(x'^2 + y'^2 + 4)},$$

i. e.

$$\cos^2\frac{\omega}{2} = \frac{(x^2 + y^2)(x'^2 + y'^2) + 8(xx' + yy') + 16}{(x^2 + y^2 + 4)(x'^2 + y'^2 + 4)}. \tag{22}$$

The formulae (12) and (22) can be further simplified if the plane σ with the complimentary point at infinity ∞ is considered as the plane of the complex variable, i. e. if we put in correspondence each point $M(x, y)$ of the plane σ and the complex number

$$z = x + iy.$$

Substituting x and y by $\frac{z + \bar{z}}{2}$ and $\frac{z - \bar{z}}{2i}$ where $\bar{z} = x - iy$, we can rewrite the expressions (12) in the following form

$$\left.\begin{aligned} X &= \frac{2(z + \bar{z})}{z\bar{z} + 4}, \\ Y &= \frac{2(z - \bar{z})}{i(z\bar{z} + 4)}, \\ Z &= \frac{z\bar{z} - 4}{z\bar{z} + 4}, \end{aligned}\right\} \tag{23}$$

and the formula (22) in the form

$$\cos^2 \frac{\omega}{2} = \frac{(z\bar{z}' + 4)(z'\bar{z} + 4)}{(z\bar{z} + 4)(z'\bar{z}' + 4)}. \tag{24}$$

Let us find which points on the plane σ map the diametrically opposite points of the sphere in the stereographic projection. If the points M and M' of the sphere (10) are diametrically opposite, the spherical distance ω between them is π and $\cos \frac{\omega}{2} = \cos \frac{\pi}{2} = 0$. Therefore in this case the numerator of the expression (24), equal to $|z\bar{z}' + 4|^2$, is zero, i. e.

$$z\bar{z}' + 4 = 0,$$

and hence

$$z' = -\frac{4}{\bar{z}}. \tag{25}$$

We now proceed to express the inversion in a circle of radius R with the centre M_0 by means of complex numbers. If the points M_0, M and M' are specified by the complex numbers z_0, z and z', the condition (7) can be written as

$$|z - z_0| \cdot |z' - z_0| = R^2,$$

and since the vectors $\overline{M_0M}$ and $\overline{M_0M'}$ are both directed along the same straight line and are different only in the positive multiplier, the same relation is valid for the complex numbers $z - z_0$ and $z' - z_0$ mapping these vectors, i. e.

$$z' - z_0 = \frac{|z' - z_0|}{|z - z_0|}(z - z_0) = \frac{R^2}{|z - z_0|^2}(z - z_0) =$$

$$= \frac{R^2}{(z - z_0)(\bar{z} - \bar{z}_0)}(z - z_0) = \frac{R^2}{\bar{z} - \bar{z}_0},$$

i. e. the inversion transforming points M into points M', related to them by the condition (7) can be written by means of complex numbers in the form

$$z' - z_0 = \frac{R^2}{\bar{z} - \bar{z}_0}. \tag{26}$$

Therefore the transformation (25) consists of the inversion with respect to a circle of radius 2

$$z\bar{z} = 4 \tag{27}$$

with the centre O and the reflection $z' = -z$.

Since the transformation (25) can be considered as the transformation (26) for $z_0 = 0$ and $R^2 = -4$, this transformation is called

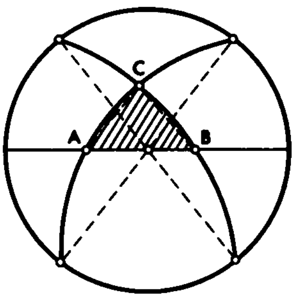


FIG 13

the inversion with respect to an imaginary circle

$$z\bar{z} = -4 \tag{28}$$

of the imaginary radius $2i$ with the centre O.

A remark is in order that the circle (27) is the projection of the great circle on the sphere, cut by the diametric plane parallel to the projection plane, i. e. of the equator of the sphere if the points S and S' are referred to as its poles. Since each two great circles of the sphere intersect in the diametrically opposite points, and the diametrically opposite points of the equator are projected into the diametrically opposite points of the circle (27) on the plane, hence the great circles of the sphere are projected onto the plane as circles or straight lines such that intersect the circle (27) in the diametrically opposite points.

As will be recalled the sum of the angles in a spherical triangle, i. e. in a triangle on a sphere, whose arms are the arcs of great circles, always exceeds π (it can be proved that the area of a spherical triangle is equal to the product of the excess of the sum of its angles over π by the squared radius of the sphere). Now we can present a perfect demonstration of this: let us construct a spherical triangle ABC in the stereographic projection (in Fig. 13 the arm AB of this triangle is shown by the segment of the diameter of the circle (27), while the arms AC and BC are shown by the arcs of the circles intersecting this circle in diametrically opposite points). Then owing to the property (B) of the stereographic projection the angles of the triangle ABC are mapped on the plane in a one-to-one correspondence. Let us connect the vertices of the triangle on the plane by straight lines. The sum of the angles of the right-angled triangle on the plane is equal to π, and, as is clearly seen in Fig. 13, the sum of the angles of the spherical triangle ABC is greater than that of the angles in the right-angled triangle drawn in the Figure, i. e. greater that π.

A circle of radius R with the centre M_0 can be characterized by the equation

$$|z - z_0| = R$$

or

$$(z - z_0)(\bar{z} - \bar{z}_0) = R^2. \tag{29}$$

We shall now give the proof of the properties (A′) and (B′) of the inversion with reference to a circle, by means of complex numbers. To do this we consider the inversion

$$z' = \frac{r^2}{\bar{z}} \tag{30}$$

with respect to the circle

$$z\bar{z} = r^2 \tag{31}$$

with the centre O. To prove the property (A′) we have to write Eq. (29) in the form

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0 \tag{32}$$

(by multiplying both parts of Eq. (29) by A and assuming $B = - Az_0$, $C = A(z_0\bar{z}_0 - R^2)$) and to replace z in this equation by its expression via z' from Eq. (30). Thus we obtain

$$\frac{Ar^4}{z'\bar{z}'} + \frac{Br^2}{z'} + \frac{\bar{B}r^2}{\bar{z}'} + C = 0,$$

i. e.

$$Cz'\bar{z}' + BR^2\bar{z}' + \bar{B}R^2 z' + AR^4 = 0. \tag{33}$$

The circle or straight line passes through the point O when $C = 0$.

The property (B′) can be proved in a way quite similar to that used in Sec. 3 to prove the property (B) by means of coordinates. The property (B′) also stems from the fact that the inversion (26) is the result of the consecutive application of the transformation $z' = \bar{z}$ and of the transformation

$$z'' - z_0 = \frac{R^2}{z' - z_0}. \tag{34}$$

But the transformation $z' = \bar{z}$ is the reflection with respect to the real axis and thus transforms any angle into an angle equal to it. As for the transformation (34) which is identical to the function

$$w = \frac{R^2}{z - z_0} + z_0, \tag{35}$$

anyone acquainted with the differential calculus of functions of complex numbers knows that the function (35) has the derivative

$$\frac{dw}{dz} = -\frac{R^2}{(z - z_0)^2};$$

and, if we denote this derivative by k, then the differentials dz and dw are related by the formula

$$dw = k\,dz. \tag{36}$$

Therefore if two curves, differentials along which are equal to dz and δz, radiate from the point $M(z)$, then these curves are transformed by the inversion into two curves, radiating from the point $M'(w)$, with respective differentials $dw = k\,dz$, $\delta w = k\,\delta z$. But the transformation

$$w = az \tag{37}$$

on the complex variable plane at $|a| = 1$, i.e. at $a = \cos\varphi + i\sin\varphi$, is the rotation by the angle φ (in this case the transformation (37) in coordinates takes the form (19)), at $a = \bar{a} = r$ this is a homothetic transformation with the coefficient r, and in the general case it consists of a rotation and of a homothety; thus, if the transformation (37) is performed over dz and δz at $a = k$, we obtain the transformation not affecting the angles and, as a result, the angle between the differentials dw and δw is equal to the angle between the differentials dz and δz.

It should be noted that the inversion with reference to a circle (32) can be written in the form

$$z' = \frac{B\bar{z} + C}{A\bar{z} + B}. \tag{38}$$

5. Mapping of Sphere Rotations on a Plane

By virtue of the property (B) of the stereographic projection the rotation of the sphere about the diameter SS' is mapped on the plane σ by the rotation (19) which, as we have seen, can be written by means of complex numbers in the form (37) at $a = \cos\varphi + i\sin\varphi$.

Let us find the transformation which maps on the plane an arbitrary rotation of the sphere. Since circles on the sphere are transformed by an arbitrary rotation of the sphere into circles, and since due to the property (A) of the stereographic projection they are mapped on the plane as circles or straight lines, the rotations of the sphere are mapped on the extended plane of a complex variable by mutually single-valued transformations of this plane which map the circles into circles or straight lines.

The transformations (37) and more general linear transformations

$$w = az + b, \tag{39}$$

consisting of the transformations (31) and of translations

$$w = z + b,$$

as well as the reflection $w = \bar{z}$ and some more general transformations

$$w = a\bar{z} + b, \tag{40}$$

consisting of the linear transformations (39) and reflection $w = \bar{w}$, belong to this type of transformations on a plane. To the same group also belong the inversions with respect to circles, the transformations (35), and the most general linear fractional transformations

$$w = \frac{az + b}{cz + d} \tag{41}$$

and

$$w = \frac{a\bar{z} + b}{c\bar{z} + d}, \tag{42}$$

consisting of the linear transformations (39) and (40) and of the inversions or transformations (35); the fact that the transformation (41) consists of those mentioned above is demonstrated by giving it the following form

$$w = \frac{\frac{a}{c}cz + \frac{a}{c}d - \frac{a}{c}d + b}{cz + d} = \frac{a}{c} + \frac{bc - ad}{c(cz + d)},$$

and replacing there z by $\bar{z}$, we obtain the same form as in the transformation (42).

It can be shown that, conversely, any mutually single-valued transformation of the complex variable plane, supplemented by the point ∞, in which circles are transformed into circles or straight lines, has the form of Eq. (41) or (42). Actually, let the transformation T map the point ∞ into the point S. Now let us consider the inversion J with respect to a circle with the centre S. Then the transformation U, consisting of two such transformations will map the point ∞ into itself and, hence, the straight lines into straight lines. We shall assume it known that any mutually single-valued transformation of a plane, mapping straight lines into straight lines, – such transformations are called *affine* – can be written in the form

$$\left.\begin{aligned} x' &= Ax + By + E, \\ y' &= Cx + Dy + F. \end{aligned}\right\} \tag{43}$$

Since the transformation U realizes, in addition, the transformation of circles into circles, it is a similarity, i. e. consists of motion and homothety, and therefore, can be written on the complex variable plane in the form of Eq. (39) or (40). Therefore the transformation T, comprising the transformation U and the inversion J, consists of the transformation (39) or (40) and the inversion (38) and hence, it has either the form (41) or (42).

Therefore, the rotation of the sphere is effected by the stereographic projection of the sphere onto the plane via the transformation either of the type (41) or (42). Since the rotation of the sphere transforms the diametrically opposite points of this sphere into similarly located points, and since the stereographic projection transforms these points into points related by Eq. (25), the rotations of the sphere are mapped on the plane by the transformations (41) or (42) which are commutative with the transformation (25), i. e. the results obtained by performing these operations in a different sequence are identical. Since the results of performing such operations in a different sequence are

$$\frac{-\dfrac{4a}{\bar{z}}+b}{-\dfrac{4c}{\bar{z}}+d}=\frac{-4a+b\bar{z}}{-4c+d\bar{z}}, \quad -\frac{4}{\dfrac{\overline{\bar{a}\bar{z}+\bar{b}}}{\bar{c}\bar{z}+\bar{d}}}=\frac{-4\bar{c}\bar{z}-4\bar{d}}{\bar{a}\bar{z}+\bar{b}},$$

then, comparing the free terms and coefficients at $\bar{z}$ in the numerators and denominators of these fractions we arrive at the relationships

$$d=\bar{a}, \; c=-\frac{1}{4}\bar{b}. \tag{44}$$

We shall obtain the same relations (44) by performing the transformations (42) and (25) in a different sequence and comparing the absolute terms and the coefficients at z in the fractions obtained. Therefore the rotations of the sphere are mapped on the plane by the transformations

$$z'=\frac{az+b}{-\frac{1}{4}\bar{b}z+\bar{a}} \tag{45}$$

and

$$z'=\frac{a\bar{z}+b}{-\frac{1}{4}\bar{b}\bar{z}+\bar{a}} \tag{46}$$

Rotation around the diameter SS' is realized by the transformation (45), in which from $z=0$ follows $z'=0$. In this case $b=0$ and the transformation (45) takes the form

$$z'=\frac{a}{\bar{a}}z=(\cos\varphi+i\sin\varphi)\,z.$$

6. History of the Stereographic Projection

The earliest surviving evidence of the stereographic projection is *Planisphaerium* by Claudius Ptolemy, a well-known savant from Alexandria (II century A. D.). *Planisphaerium* described the instrument for measuring coordinates of stars on the celestial sphere, the so-called astrolabe, in which the stereographic projection was utilized (to get an insight into the stereographic projection used in this instrument refer to Sec. 7). The properties (A), (B) and (C) are used, though without proof, in the text of *Planisphaerium* that survived till nowadays. The earliest existing presentation of the theory of the stereographic projection with complete proofs of the property (A) was written by Ahmad al-Fergani, a IX-century scientist from Central Asia, who was born in Fergana and was working in Baghdad. This theory is presented in Chapter I of al-Fergani's *Book on Constructing the Astrolabe* which is to be published soon in Tashkent (translated into Russian by N. D. Sergeeva) along with the Russian translation of al-Fergani's famous book *Elements of Astronomy*, in collected articles of al-Fergani's astronomical treatises. The latest scientists of the Orient indicated that al-Fergani's book on astrolabe gives one of the best presentations of the theory of this instrument; this theory seems to have already been known to Ptolemy but was not mentioned in *Planisphaerium* available to the scientists of the Middle Ages.

Al-Fergani's book contains the proof of the lemma which was given at the beginning of Sec. 1 of the present booklet, as well as the proof of the property (A) also given in the same section; then al-Fergani showed, just as we did it in Sec. 1, that the point of the plane into which the centre of the circle on the sphere is projected, does not coincide with the centre of the circle on the plane. Al-Fergani's proof of the property (A), as cited above in this book, is quite close to the proof of the fifth proposition in the first volume of the famous treatise on conics, written by Apollonios of Perga, Greak geometer of the Alexandrian school, (about III century B. C.) which described the second set of circular sections of the oblique circular cone. Therefore it appears probable that the property (A) of the stereographic projection was already known to Apollonios. It should be noted that in his treatise *On Plane Loci* Apollonios comes out with a similar property (A′) of the inversion (ancient Greeks meant by the plane loci the curves one could draw by using a straightedge and a compass, i. e. straight lines and circles).

In the treatise *On Plane Loci* Apollonios states that if "two straight lines" (i. e., two rectilinear segments) are drawn from one point of a straight line and "contain a given rectangle" (i. e. the product of these segments is constant) and "if the endpoint of one of these straight lines traces a plane locus then the endpoint of another straight line also traces a plane locus of the same or other type". Apollonios also says that the same is true if the straight lines are drawn from different points parallel or at a certain angle to one another, i. e. when one "plane locus" is obtained from the other locus by a transformation consisting of inversion and translation (here Apollonios also discusses homothety as well as the transformation consisting of homothety and translation). The foregoing proposition of Apollonios' *Conic Sections* made it possible for the Greeks to rigorously prove the property (A) of the stereographic projection; this has apparently been achieved, if not in the lifetime of Apollonios himself but undoubtedly in the course of four centuries between Apollonios and Ptolemy. Al-Fergani must be highly credited for his reconstruction of the proof of the property (A) since he possessed only the statement and no proof of this property.

In the Middle Ages the stereographic projection was referred to as the "astrolabe projection". The term "stereographic projection" was introduced in 1831 by a German mathematician L. Magnus (1790-1861) to whom the discovery of this remarkable projection is sometimes ascribed. This term originates with the Greek words "στερεον", "a solid body", (to which the word "stereometry" is traced), and "γραφη", "draw, write", which in its former sense is incorporated into our word "photography" ("drawing by light") and in the latter sense, into our words "geography" ("description of the Earth") and "biography" ("description of life").

7. Application of the Stereographic Projection to Astronomy and Geography

First of all we shall get acquainted with the design of the medieval astrolabe based on the use of the stereographic projection. Nowadays each schoolboy is acquainted with the school astrolabe which is a disk mounted horizontally on a tripod (Fig. 14 *a*). Degrees of arc are marked along the rim of the disk. The "alidade" (or sight rule), i. e. a straightedge with two diopters by means of which one can sight a direction at a certain point, is rotatable about the centre of the disk. Sighting directions at various points by means of such astrolabe one can measure angles between the directions on the Earth surface. In the Middle Ages the astrolabe was meant for quite a different use which is indicated by the term itself originating with the Greek words "αστερ" – "a star" and "λαβη" – "to grasp"; the astrolabe served to fix the coordinates of celestial bodies. The fundamental part of the existing school astrolabe, viz., the circle of degrees and the alidade with the diopters comprised only the front side of the astrolabe. The astrolabe was to be suspended by a ring, the alidade (this term is a distorted arabic word al-idada, i. e. "a device") was aimed at a celestial body and its pointer indicated on the disk limb the altitude of the body (Fig. 14 *b*).

The second coordinate of the celestial body is found with the help of the back side of the astrolabe. Here a stationary disk (tympanum) is fixed and a fretted "spider"-disk is mounted rotatable around the centre of the stationary disk. The circles of the celestial sphere which are not altered by its apparent diurnal motion are shown on the tympanum by the stereographic projection: the celestial equator, transformed by this motion unto itself, the Tropic of Cancer and the Tropic of Capricorn – the two parallels of the celestial equator tangent to the ecliptic, i. e. a great circle onto which is projected the apparent yearly motion of the Sun (located along the ecliptic are twelve Zodiacal constellations; the Sun crosses the celestial equator during vernal and autumnal equinoxes and is the farthest from it in the days of summer and winter solstices when it enters into the Cancer and Capricorn constellations whence originate the names of tropics; the term "Tropic" originates with the Greek word "τροπε" – "a turn"), the horizon and its parallels – almucantars (from the arabic word al-mucantars, i. e. "constructed with a vault"), the zenith point ("zenith" is a distorted arabic word "samth" – "direction", "upward direction" is meant; the word "samth" was transformed into "zenith" by a mistake of a medieval copyist who has read "m" in the word "zemth" as "ni") and the verticals, i. e. great circles passing through zenith perpendicularly to the horizon. By the property (A) all the enumerated circles on the sphere

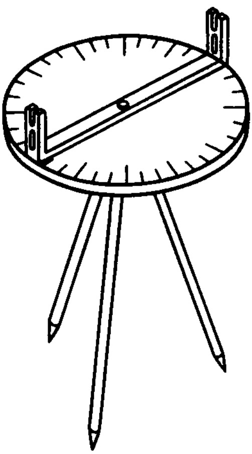


FIG 14a

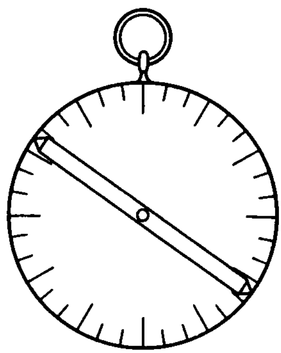


FIG 14b

are mapped on the tympanum by arcs of circles or by segments of straight lines. The Southern Pole of the celestial sphere is usually taken as the point S. Therefore the equator and the tropics are mapped on the tympanum by concentric circles. The tympanum is usually cut off on a circle mapping the tropic (Fig. 15). Since in a locality with a geographical latitude φ the celestial equator makes the angle $\frac{\pi}{2} - \varphi$ with the horizon (this equator is perpendicular to the horizon at the earth equator and coincides with it at the poles), then by virtue of the property (B) the horizon is projected as a circle intersecting the equator projection in two diametrically opposite points at an angle $\frac{\pi}{2} - \varphi$. It can be shown that the almucantars are mapped as circles such that form together with the circle mapping the horizon a bundle of circles

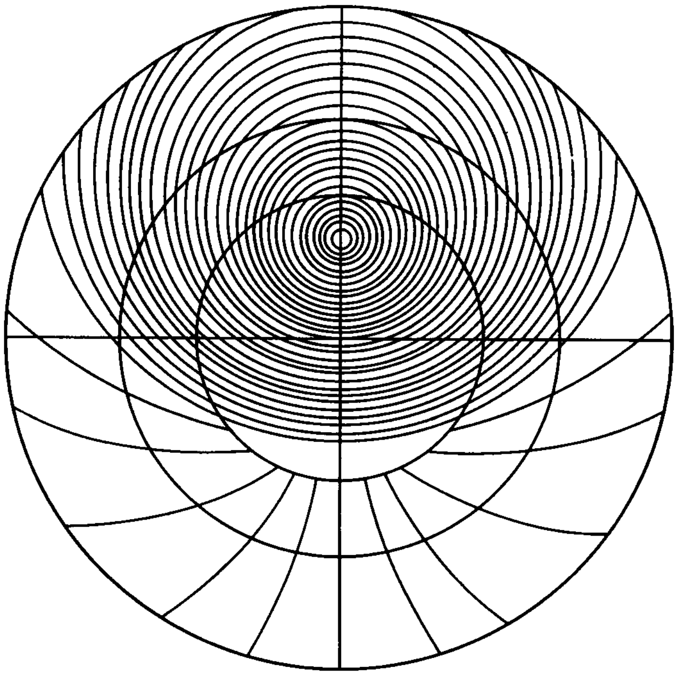


FIG 15

presenting the loci of points for which the ratio of the distances from the point Z projecting the zenith to the point mapping the diametrically opposite point of the celestial sphere ("nadir") is constant. The verticals are projected as circles passing through the point Z perpendicularly to the circle mapping the horizon. The so-called hour lines, serving to determine time in "season hours" (equal to $^1/_{12}$ of the light or dark daily interval), are traced on the tympanum under the horizon. Projections of almucantars and verticals form a "cobweb", along which the "spider" moves. Ecliptic and the brightest stars rotating during the apparent diurnal motion of the celestial sphere are marked on the "spider". Evidently, the ecliptic is mapped as a circle tangent to the tropic projection. The mapping of the ecliptic circle is divided into twelve segments for twelve zodiacal constellations in each of which the Sun remains for a month; these segments are subdivided further to allow for the determination of the Sun's position any day of the year. The stars are shown by pointers fixed at the rim or at the ecliptic projection (Fig. 16).

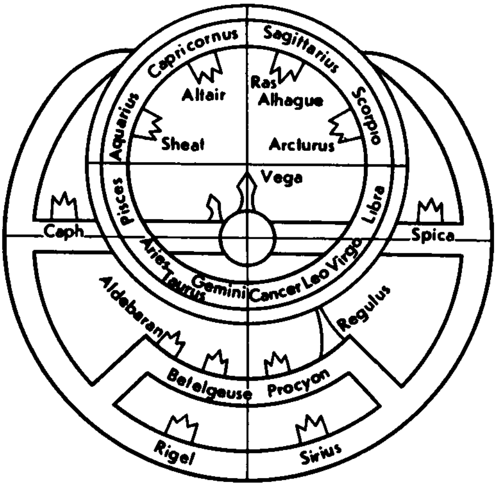


FIG 16

The astrolabe enables one to determine the azimuths of only the celestial bodies which are shown on its "spider", i. e. the Sun and the stars marked thereon.

Having measured by means of the alidade the altitude of the Sun or a star, the astrolabe is turned the tympanum-upward and the spider is rotated to an angle such that the celestial body might be seen on the almucantar at the same altitude. This operation is based on the property (C) of the stereographic projection according to which the diurnal rotation of the celestial sphere is mapped by the rotation of the "spider". Having turned the "spider" we obtain the exact position of the celestial sphere on a plane at a given moment. At this moment the azimuth of a celestial body is equal to the angle between the vertical whose mapping passes through the body's projection and a certain vertical serving as the reference line. The angle of "spider"'s rotation yields the exact time in astronomical hours elapsed from the beginning of the day or night; this starting moment corresponds to the position of the "spider" at which the projection of the body is on the projection of the horizon; the hour lines, mentioned above, are used to find the time in "season hours" which were widely used in the Middle Ages both to determine the time of prayers and in civic life.

The stereographic projection is used to project the surface of the terestrial globe onto a plane, i. e. for geographical mapping. By virtue of the property (B) of this projection, the angles between the lines are reproduced on such maps in a one-to-one correspondence. Such maps are especially important for seamen since in such a case the angle of rudder turning is exactly equal to that measured on a map. The well-known investigations of Leonard Euler (1707-1783), the great Swiss mathematician working in Petersburg and in Berlin, were devoted to the application of the stereographic projection to drawing of maps. In his works *On the Presentation of a Spherical Surface on a Plane*, *On the Geographical Projection of a Spherical Surface* and *On the Delil Geographical Projection as Applied to the General Map of the Russian Empire*, Euler sets up a problem of the most general transformation of a sphere onto a plane, conserving the values of the angles between the lines.

To achieve this, Euler performs the stereographic projection of the sphere onto a plane and then, considering the plane to be the plane of a complex variable, performs on this plane the transformation by means of a function $w = f(z)$ having the derivative $\frac{dw}{dz}$, or by means of a conjugated function $w = \overline{f(z)}$: for such functions the differentials dz and dw are related by Eq. (36), whence we obtain that the property (B′) holds for both transformations mentioned above.

8. Application of the Stereographic Projection to the Lobachevskian Geometry

One of the simplest methods of defining the Lobachevskian plane is as follows: let us change the rule of determining the distances ("the metric") in our space in such a way that the distance M_1M_2 between the points $M_1(X_1, Y_1, Z_1)$ and $M_2(X_2, Y_2, Z_2)$ be expressed not by the formula (8) but by the expression

$$d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 - (Z_2 - Z_1)^2}, \tag{47}$$

and the angle φ between the vectors $\overline{OM_1}$ and $\overline{OM_2}$ not by the formula (9) but by the expression

$$\cos\varphi = \frac{X_1X_2 + Y_1Y_2 - Z_1Z_2}{\sqrt{X_1^2 + Y_1^2 - Z_1^2} \cdot \sqrt{X_2^2 + Y_2^2 - Z_2^2}}. \tag{48}$$

This space is called the *pseudo-Euclidean space.* In contrast to an ordinary Euclidean space the pseudo-Euclidean space contains the segments of real, zero and imaginary length, three types of straight lines – with real, zero ("isotropic lines") and purely imaginary length of segments, three types of planes – with Euclidean, pseudo-Euclidean and intermediate between these two the "isotropic" geometry, and three types of spheres – of real, purely imaginary and zero radius. The equations of these three types of spheres with the centre at the origin have, respectively, the following forms

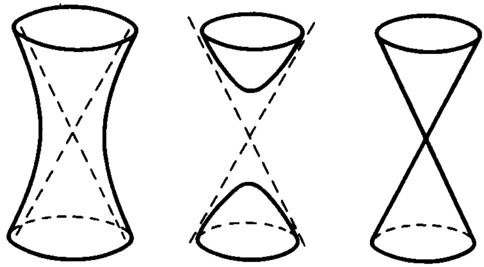
$$X^2 + Y^2 - Z^2 = R^2, \tag{49}$$
$$X^2 + Y^2 - Z^2 = -R^2, \tag{50}$$

and

$$X^2 + Y^2 - Z^2 = 0. \tag{51}$$

Therefore the real-radius spheres in the pseudo-Euclidean space are one-sheet hyperboloids (Fig. 17 *a*), the spheres of purely imaginary radius are two-sheet hyperboloids (Fig. 17 *b*), while the zero-radius spheres are cone-shaped (Fig. 17 *c*). The cone (51) is called the asymptotic cone of the spheres (49) and (50).

The *Lobachevskian plane* can be defined as the imaginary radius sphere in the pseudo-Euclidean space with identified diametrically opposite points (or as one of the sheets of this sphere).

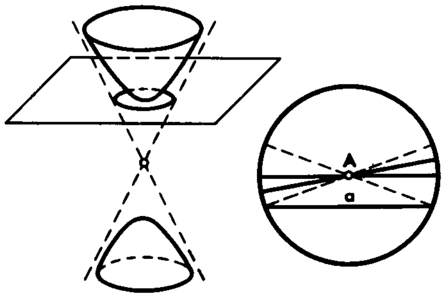


(a) (b) (c)

FIG 17

Diametric sections of the sphere, similar to great circles of ordinary spheres, play the role of straight lines on the Lobachevskian plane. It can be readily verified that the tangent planes to this sphere are Euclidean; hence, the geometry within small areas of such spheres, as within small areas of ordinary spheres, differs only slightly from the Euclidean geometry (on the contrary, the planes tangent to the real-radius spheres in the pseudo-Euclidean space are pseudo-Euclidean, while those tangent to the zero-radius sphere are "isotropic"). On the other hand, if we project the imaginary-radius sphere from its centre onto the tangent plane (Fig. 18 *a*) then the Lobachevskian plane as a whole will be mapped as the internal area of the circle (produced by intersecting the asymptotic cone of a sphere by the projection plane), and the diametric sections of the sphere, i. e. the straight lines of the Lobachevskian plane, will be projected as chords of this circle (this projection is called the *Beltrami-Klein interpretation* of the Lobachevskian plane). Fig. 18 *b* clearly shows that through the point A of this projection one can draw more than one chord not intersecting the specified chord a; this complies with the Lobachevskian axiom according to which we can draw through a point of the plane more than one straight line not intersecting the specified straight line of the given plane; on the other hand, we can verify that all axioms of the Euclidean geometry are satisfied in this plane, except for the parallel postulate, according to which not more than one straight line, not intersecting the specified line on the plane, can be drawn through a point of this plane.

Just as for the Euclidean space, we can define in the pseudo-Euclidean space the stereographic projection onto the plane for



(a) (b)

FIG 18

spheres of both real and imaginary radius. In particular, the stereographic projection from the point $S(0, 0, 1)$ onto the plane $Z = -1$ of a sphere with the radius i, described by the equation

$$X^2 + Y^2 - Z^2 = -1, \tag{52}$$

(Fig. 19) is realized by the formulas similar to Eq. (12):

$$X = \frac{4x}{4 - x^2 - y^2}, \quad Y = \frac{4y}{4 - x^2 - y^2}, \quad Z = \frac{x^2 + y^2 + 4}{x^2 + y^2 - 4}. \tag{53}$$

The properties (A), (B) and (C) of the stereographic projection of the imaginary-radius sphere onto the plane are proved by means of these formulas exactly as it was done in Sec. 3. The lower sheet of the imaginary-radius sphere is projected thereby into the inner area of the circle

$$x^2 + y^2 = 4, \tag{54}$$

cut on the plane by the cone obtained from the asymptotic cone of the imaginary-radius sphere by translating its vertex from the centre of the sphere into the point S; the upper sheet of this sphere is mapped by the area external to the circle. If the plane is considered as the plane of a complex variable then the circle (54) coincides with the circle (27); we can show, just as in Sec. 4, that the diametrically opposite points of the imaginary-radius sphere are projected onto the plane as the points inversed relative to this circle; hence, all diametric sections of the sphere, i. e. the straight lines of the Lobachevskian plane, are mapped as circles transformed

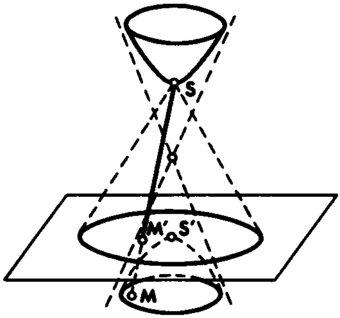


FIG. 19

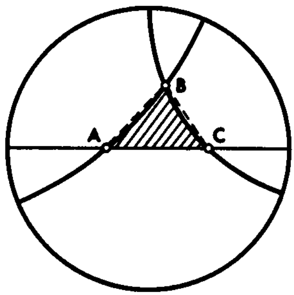


FIG 20

by this inversion into themselves, i. e. as circles orthogonal to the circle (54). Thus we have obtained the so-called *Poincaré interpretation* of the Lobachevskian plane.

As will be recalled the sum of the angles of a triangle in the Lobachevskian plane is always less than π (it can be proved that the area of a triangle in the Lobachevskian plane is equal to the deficiency of its angles sum as compared to π, times the squared module of the radius of the corresponding imaginary-radius sphere). Now, just as we did in Sec. 4, we can obtain this result in a perfectly clear manner: let us map in the Poincaré interpretation the triangle ABC (the arm AC of this triangle is mapped in Fig. 20 as a segment of the diameter of the circle (54), and the arms AB and BC, by the arcs of the circles orthogonal to the same circle). Then by virtue of the property (B) the angles of the triangle are projected onto the plane in a one-to-one correspondence. We now connect the vertices of the triangle in a plane by straight lines; the sum of the angles of the obtained rectilinear triangle in a plane is π, and Fig. 20 clearly shows that the sum of the angles of the triangle in the Lobachevskian plane is less than that in the constructed rectilinear triangle, i. e. less than π.

Since in the Lobachevskian plane we can draw through a point more than one straight line not intersecting the given straight line, then the set of these lines contains two boundary straight lines separating the straight lines, passing through the given point and intersecting the given line, from those passing through this point and not intersecting the given straight line. These two straight lines are called the straight lines *parallel* to the given line; all other straight lines, not intersecting the given line are

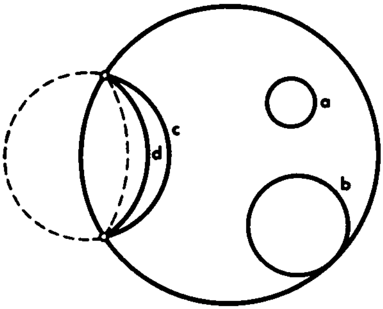


FIG 21

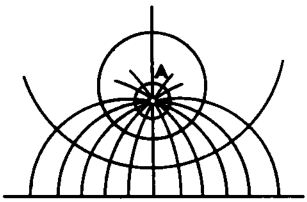


FIG 22

called *divergent* relative to the given line. It can be shown that in the stereographic projection *parallel straight lines are mapped as arcs of the circles tangent to each other in a point located on the circle* (54).

The Lobachevskian plane contains three classes of extraordinary curves: circles, equidistants and oricycles.

1. A *circle* is, as in an ordinary plane, the locus of points of the plane, lying at the same distance from a specified point. As in an ordinary plane, circles in the Lobachevskian plane can also be defined as curves such that in each of their points intersect at right angles a bundle of straight lines intersecting in a common point.

2. An *equidistant curve* is a curve separated by equal distances from all points of a given straight line, called the base of the equidistant curve. In the Lobachevskian plane such loci in contrast to an ordinary plane are not the pairs of straight lines but the curves consisting of two branches. The equidistant curve can also be defined as a curve which in each of its points intersects at right angles the straight lines perpendicular to one given straight line, viz. the base of the equidistant curve.

3. *Oricycle* (or horocycle) is defined as a curve which in each of its points intersects at right angles the straight lines parallel to each other (the word "oricycle" means "the limiting circle").

It can be shown that our stereographic projection maps the circles of the Lobachevskian plane as circles enclosed inside the circle (54) (Fig. 21 *a*). Oricycles are projected as circles tangent to the circle (54) (Fig. 21 *b*) in the point through which pass the arcs of circles projecting the parallel lines perpendicular to the

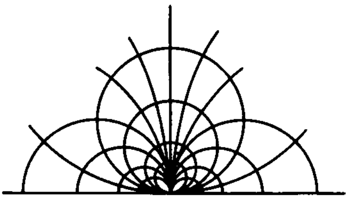


FIG 23

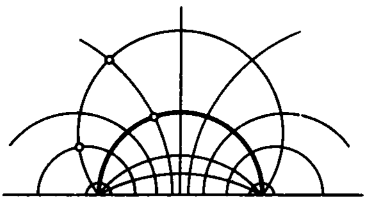


FIG 24

oricycle. Equidistant curves are projected as circles intersecting the circle (54) in two points (Fig. 21 *c*), namely the two points in which the circle, mapping the base of the equidistant curve (Fig. 21 *d*), intersects the circle (54). (In identifying the diametrically opposite points of the imaginary-radius sphere, each point of the plane is identified with the point into which it is transformed by the inversion with reference to the circle (54), and the part of the circle mapping the equidistant curve located outside the circle (54) can be substituted by an arc inside this circle, into which the arc under consideration is transformed by the foregoing inversion.) That all these curves are stereographically projected as circles is due to the fact that on the imaginary-radius sphere these curves are projected as plane sections: the circles – as sections of the pseudo-Euclidean space by Euclidean planes, equidistant curves – as sections of this space by pseudo-Euclidean planes, and oricycles – as sections by "isotropic" planes.

It should be noted that Poincaré suggested his own interpretation, in which the role of the circle (54) is played by the upper semiplane of the complex variable plane, and the role of the circumference (54) – by the real axis of this plane. In this interpretation the straight lines of the Lobachevskian plane are mapped as semi-circles with the centres on the real axis, and the circles (Fig. 22), oricycles (Fig. 23) and equidistant curves (Fig. 24) are mapped as circles not intersecting the real axis, tangent to this axis, and intersecting this axis in two points not at the right angle, respectively (if the circle intersects the real axis at a right angle, it maps a straight line). In this case the movements of the Lobachevskian plane are mapped by the linear-fractional transformations (41) in which all four numbers a, b, c, d are real.

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