Mathematical Statistics Solutions: HW 3

October 19, 2022

Exercise 1

Part (a)

If $\mathbb{E}|X_n - X| \to 0$, then by Markov's inequality, for any $\varepsilon > 0$,

$$\mathbb{P}\left[|X_n - X| \geq \varepsilon\right] \leq \frac{\mathbb{E}|X_n - X|}{\varepsilon} \to 0.$$

Therefore, $X_n \xrightarrow{p} X$. If instead we have $\mathbb{E}|X_n - X|^r \to 0$ for some $r > 0$, using Proposition 1.4, for any $\varepsilon > 0$, we still have

$$\mathbb{P}\left[|X_n - X| \geq \varepsilon\right] \leq \frac{\mathbb{E}|X_n - X|^r}{\varepsilon^r} \to 0,$$

so we still have $X_n \xrightarrow{p} X$.

Part (b)

For any $\varepsilon \in (0, 1)$, we have

$$\mathbb{P}\left[|X_n| \geq \varepsilon\right] = \mathbb{P}\left[X_n = 1\right] = \lambda_n \to 0,$$

so $X_n \xrightarrow{p} 0$.

Part (c)

For any $\varepsilon \in (0, 1)$, we have

$$\Pr\left[|Y_n| \geq \varepsilon\right] = \mathbb{P}\left[|X_n| \geq \lambda_n \varepsilon\right] = \mathbb{P}\left[X_n = 1\right] = \lambda_n \xrightarrow{n \to \infty} 0,$$

therefore $Y_n \xrightarrow{p} 0$. However, notice that $\mathbb{E}|Y_n - 0| = \lambda_n^{-1}\mathbb{E}|X_n| = \lambda_n^{-1}\lambda_n = 1 \neq 0$. We conclude that the inverse argument of Part (a) does not hold: $Y_n \xrightarrow{p} 0$ does not imply $\mathbb{E}|Y_n - 0| \to 0$ as $n \to \infty$.

Part (d)

For any $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}
\mathbb{P}\left[|X_n - X| \geq \varepsilon\right] &= \mathbb{P}\left[X_n = 1, X = 0 \text{ or } X_n = 0, X = 1\right] \\
&= \mathbb{P}\left[X_n = 1, X = 0\right] + \mathbb{P}\left[X_n = 0, X = 1\right] \\
&= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2}.
\end{aligned}$$

Therefore, X_n does not converge to X in probability. Meanwhile, for any $t \in [0,1]$,

$$\mathbb{P}\left[X_n \leq t\right] = \frac{1}{2} - \frac{1}{n} \to \frac{1}{2} = \mathbb{P}\left[X \leq t\right];$$

for any $t > 1$,

$$\mathbb{P}\left[X_n \leq t\right] = 1 = \mathbb{P}\left[X \leq t\right];$$

for any $t < 0$,

$$\mathbb{P}\left[X_n \leq t\right] = 0 = \mathbb{P}\left[X \leq t\right].$$

We conclude that $X_n \xrightarrow{d} X$.

Exercise 2

Let $T_n \sim \mathrm{Unif}([-n^{-1}, n^{-1}])$, with

$$F_{T_n}(t) = \frac{t + n^{-1}}{2n^{-1}} = \frac{1}{2} + t\frac{n}{2},$$

for $t \in [-n^{-1}, n^{-1}]$. Note that for $t = 0$, $F_{T_n}(0) \to \frac{1}{2}$ as $n \to \infty$.

Let limiting distribution $T = \delta_0$, a Dirac mass at zero. Clearly, $F_{T_n}(0) \not\to F_T(0) = 1$.

However, for any f bounded and continuous,

$$\begin{aligned}
|\mathbb{E}[f(T_n)] - \mathbb{E}f(T)| &= |\mathbb{E}[f(T_n)] - f(0)| \\
&\leq \frac{n}{2}\int_{-n^{-1}}^{n^{-1}} |f(x) - f(0)| \mathrm{d}x
\end{aligned}$$

By continuity of f and the fact that $|x| \leq n^{-1}$, for all $\varepsilon > 0$, there exists an n_0 such that for all $n > n_0$, $|f(x) - f(0)| \leq \varepsilon$. Thus, as $n \to \infty$,

$$\begin{aligned}
|\mathbb{E}[f(T_n)] - \mathbb{E}f(T)| &\leq \frac{n}{2}\int_{-n^{-1}}^{n^{-1}} |f(x) - f(0)| \mathrm{d}x \\
&\leq \varepsilon\frac{n}{2}\int_{-n^{-1}}^{n^{-1}} \mathrm{d}x \\
&= \varepsilon.
\end{aligned}$$

Exercise 3

Part (a)

For x, y such that $|x - y| \leq \epsilon$, we have $|g(x) - g(y)| \leq L|x - y| \leq \varepsilon$, since g is L-Lipschitz. This implies that

$$\mathbb{P}(|g(x) - g(y)| \geq \epsilon) \leq \mathbb{P}(|x - y| \geq \varepsilon L^{-1}).$$

Replacing x and y with T_n and T respectively, and noting that $T_n \xrightarrow{p} T$, it must be that $g(T_n) \xrightarrow{p} g(T)$ also.

Part (b)

The hint tells us that $\varepsilon \leq |g(T_n) - g(T)| \leq \omega(|T_n - T|)$, and thus

$$\mathbb{P}(|g(T_n) - g(T)| \geq \varepsilon) \leq \mathbb{P}(\omega(|T_n - T|) \geq \varepsilon) = \mathbb{P}(|\omega(|T_n - T|)| \geq \varepsilon),$$

where in the last equality we used the fact that ω is a non-negative valued function. From the definition of continuity, we have that for all $\varepsilon > 0$, there exists a $\delta_0 > 0$ such that $|\omega(x) - \omega(y)| \geq \varepsilon$ implies that $|x - y| \geq \delta_0$. Taking $y = 0$ and $x = T_n - T$, it follows that

$$\mathbb{P}(\omega(|T_n - T|) \geq \varepsilon) \leq \mathbb{P}(|T_n - T| \geq \delta_0)$$

. Since $T_n \xrightarrow{p} T$, we have $\mathbb{P}(|T_n - T| \geq \delta) \to 0$ for all $\delta > 0$. Picking $\delta = \delta_0$, this completes the claim since

$$\mathbb{P}(|g(T_n) - g(T)| \geq \varepsilon) \leq \mathbb{P}(|\omega(|T_n - T|)| \geq \varepsilon) \leq \mathbb{P}(|T_n - T| \geq \delta_0) \to 0\,.$$

Exercise 4

For some $\sigma^2 > 0$, we have that

$$\sqrt{n}(T_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)\,.$$

Fix $\varepsilon > 0$ and apply the hint and the Berry-Esseen CLT:

$$\begin{aligned}
\mathbb{P}(|T_n - \mu| \geq 0) &\leq \mathbb{P}(T_n - \mu \leq -\varepsilon) + 1 - \mathbb{P}(T_n - \mu \leq \varepsilon/2) \\
&= \mathbb{P}(\sqrt{n}(T_n - \mu) \leq -\sqrt{n}\varepsilon) + 1 - \mathbb{P}(\sqrt{n}(T_n - \mu) \leq \sqrt{n}\varepsilon/2) \\
&\leq \left(\Phi(-\varepsilon\sqrt{n}/\sigma) + Cn^{-1/2}\right) + 1 - \left(\Phi(\varepsilon\sqrt{n}/\sigma) + C'n^{-1/2}\right),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the CDF of the standard normal Gaussian. Taking limits, we see that the right-hand side converges to zero for all choice of ε. By squeeze theorem, this completes the proof.

Exercise 5

Part (a)

Claim: If Z is σ^2-subgaussian, then Z/σ is 1-subgaussian.

Proof. WLOG assume Z has mean zero. The claim follows from a straightforward computation: For $\lambda \in \mathbb{R}$

$$\mathbb{E}[e^{\lambda Z/\sigma}] = \mathbb{E}[e^{(\lambda/\sigma)Z} = e^{(\lambda/\sigma)^2\sigma^2/2} = e^{\lambda^2/2}\,,$$

where λ/σ is just another real number. □

Part (b)

$$\begin{aligned}
\int_0^\infty 3t^2\mathbb{P}(|Z| \geq t)dt &= \int_0^\infty \mathbb{E}[3t^2\mathbf{1}_{\{|Z|\geq t\}}]dt \\
&= \mathbb{E}\left[\int_0^\infty 3t^2\mathbf{1}_{\{|Z|\geq t\}}dt\right] \\
&= \mathbb{E}\left[\int_0^{|Z|} 3t^2 dt\right] \\
&= \mathbb{E}[|Z|^3]
\end{aligned}$$

Part (c)

We use sub-gaussian tail bounds in the first inequality, and symmetry of the integrand in the second equality:

$$\begin{aligned}
\mathbb{E}[|Z|^3] &= \int_0^\infty 3t^2 \mathbb{P}(|Z| \geq t) dt \\
&\leq \int_0^\infty 6t^2 e^{-\frac{t^2}{2}} dt \\
&= 3 \int_{-\infty}^\infty t^2 e^{-t^2/2} dt \\
&= \sqrt{2\pi} 3 \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty t^2 e^{-t^2/2} dt \right) \\
&= \sqrt{2\pi} 3 \mathbb{E}[|Z|^2] \\
&= \sqrt{2\pi} 3,
\end{aligned}$$

as desired.

Exercise 6

Part (a)

The delta-method states that we have $\sqrt{n}(g(T_n) - g(\mu)) \xrightarrow{d} \mathcal{N}(0, g'(\mu)\sigma^2(\theta))$. We have the additional information that $g'(\mu) = \frac{1}{\sigma^2(\theta)}$, which completes the claim.

Part (b)

Setting $T_n = \frac{1}{n}\sum_{i=1}^n Z_i^2$, we see that $\mathbb{E}[T_n] = \sigma^2$ and $\mathrm{Var}(T_n) = \frac{1}{n} 2\sigma^4$. By CLT, this completes the claim.

Part (c)

Define $g(x) = \log(x)$ with $g'(x) = \frac{1}{x}$. Then $g'(\sigma^2) = \frac{1}{\sigma^2}$. Combining this with **(b)** and **(a)** yields the proof.