Programming Languages

Lambda Calculus and Scheme

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λ-Calculus

- invented by Alonzo Church in 1932 as a model of computation
- predated any electronic digital computers
- even came before the Turing Machine (A.M. Turing)
- defines the same class of functions as Turing Machines
- later turned out to be Turing complete
- basis for functional languages (e.g., Lisp, Scheme, ML, Haskell)
- typed and untyped variants
- has *syntax* and *reduction rules*

Syntax

We will discuss the *pure*, *untyped* variant of the λ-calculus.

The syntax is simple:

$$
\begin{array}{rcll}
M & ::= & x & \text{variable} \\
  & | & \lambda x \,.\, M & \text{abstraction (function definition)} \\
  & | & M\, M & \text{application (function invocation)}
\end{array}
$$

Shorthands:

- We can use parentheses to indicate grouping
- We can omit parentheses when intent is clear
- $\lambda x\, y\, z \,.\, M$ is a shorthand for $\lambda x \,.\, (\lambda y \,.\, (\lambda z \,.\, M))$ (right-associative)
- $M_1\, M_2\, M_3$ is a shorthand for $(M_1\, M_2)\, M_3$ (left-associative)
- Application has precedence over abstraction:
  - $\lambda x \,.\, y\ \lambda x \,.\, z$ means: $\lambda x \,.\, (y\ (\lambda x \,.\, z))$
  - Not $(\lambda x \,.\, y)\ (\lambda x \,.\, z)$

Free and bound variables

- In a term $\lambda x \,.\, M$, the scope of x is M.
- We say that x is *bound* in M.
- Variables that are not bound are *free*.

Example:

$$(\lambda x \,.\, (\lambda y \,.\, (x\,(z\,y))))\,y$$

- The z is free.
- The last y is free.
- The x and remaining y are bound.

We can perform α-conversion at will:

$$\lambda y \,.\, (\ldots y \ldots) \quad \longrightarrow_\alpha \quad \lambda w \,.\, (\ldots w \ldots)$$

β-reduction

The main reduction rule in the λ-calculus is function application:

$$(\lambda x \,.\, M)\, N \quad \longrightarrow_\beta \quad [x \mapsto N]M$$

The notation $[x \mapsto N]M$ means:

M, with all bound occurrences of x replaced by N.

Restriction: N should not have any free variables which are bound in M.

Example:

$$(\lambda x \,.\, (\lambda y \,.\, (x\, y)))\, (\lambda y \,.\, y) \quad \longrightarrow_\beta \quad \lambda y \,.\, ((\lambda y.y)\ y)$$

An expression that cannot be β-reduced any further is a *normal form*.

β-reduction

Not everything has a normal form:

$$(\lambda z \,.\, zz)(\lambda z \,.\, zz)$$

reduces to itself. Application rule can be applied infinitely.

Evaluation strategies

We have the β-rule, but if we have a complex expression, where should we apply it first?

$$(\lambda x\,.\,\lambda y\,.\,y\,x\,x)\,((\lambda x\,.\,x)(\lambda y\,.\,z))$$

Two popular strategies:

- **normal-order**: Reduce the outermost "redex" first.

  $$[x \mapsto (\lambda x\,.\,x)(\lambda y\,.\,z)](\lambda y\,.\,y\,x\,x) \;\longrightarrow_\beta\; \lambda y\,.\,y\,((\lambda x\,.\,x)(\lambda y\,.\,z))\,((\lambda x\,.\,x)(\lambda y\,.\,z)$$

- **applicative-order**: Arguments to a function evaluated first, from left to right.

  $$(\lambda x\,.\,\lambda y\,.\,y\,x\,x)\,([x \mapsto (\lambda y\,.\,z)]x) \;\longrightarrow_\beta\; (\lambda x\,.\,\lambda y\,.\,y\,x\,x)\,((\lambda y\,.\,z))$$

Evaluation strategies

Some observations:

- Some lambda expressions do not terminate when reduced.
- If a lambda reduction terminates, it terminates to the same reduced expression regardless of reduction order.
- If a terminating lambda reduction exists, normal order evaluation will terminate.

η-reduction

We use η-reduction to eliminate useless variables:

$$(\lambda x \,.\, M \; x) \quad \longrightarrow_\eta \quad M$$

Since:

$$(\lambda x \,.\, M \; x) \; N \quad \Longleftrightarrow \quad M \; N$$

This type of reduction is mostly for notational convenience and does not add any expressive power to the calculus.

Computational power

Fact: The untyped λ-calculus is Turing complete. (Turing, 1937)

But how can this be?

- There are no built-in types other than “functions” (e.g., no booleans, integers, etc.)
- There are no loops
- There are no control structures (e.g., if-then-else, switch)
- There are no recursive definitions
- There are no imperative features (e.g., side effects)

Numbers and numerals

- *number*: an abstract idea
- *numeral*: the representation of a number

Example: 15, fifteen, XV, `0F`

These are different numerals that all represent the same *number*.

Alien numerals:

frobnitz $-$ frobnitz $=$ wedgleb
wedgleb $+$ taksar $=$?

Booleans in the λ-calculus

How can a value of "true" or "false" be represented in the λ-calculus?

Any way we like, as long as we define all the boolean operations correctly.

One reasonable definition:

- `true` takes two values and returns the first
- `false` takes two values and returns the second

$$
\begin{aligned}
\texttt{TRUE} &\equiv \lambda a\,.\,\lambda b\,.\,a \\
\texttt{FALSE} &\equiv \lambda a\,.\,\lambda b\,.\,b \\[1ex]
\texttt{IF} &\equiv \lambda c\,.\,\lambda t\,.\,\lambda e\,.\,(c\ t\ e) \\[1ex]
\texttt{AND} &\equiv \lambda m\,.\,\lambda n\,.\,\lambda a\,.\,\lambda b\,.\,m\,(n\ a\ b)\ b \\
\texttt{OR} &\equiv \lambda m\,.\,\lambda n\,.\,\lambda a\,.\,\lambda b\,.\,m\ a\,(n\ a\ b) \\
\texttt{NOT} &\equiv \lambda m\,.\,\lambda a\,.\,\lambda b\,.\,m\ b\ a
\end{aligned}
$$

Booleans in the λ-calculus

Let's try passing `TRUE` to `IF`.
We'll use `1,0` as shorthand to represent λ functions.

Evaluate the expression to `1` if `TRUE`, or `0` otherwise:

$$\begin{aligned} \texttt{TRUE} &\equiv \lambda a\,.\,\lambda b\,.\,a \\ \texttt{IF} &\equiv \lambda c\,.\,\lambda t\,.\,\lambda e\,.\,(c\;t\;e) \end{aligned}$$

$$\begin{aligned} &(\lambda c\,.\,\lambda t\,.\,\lambda e\,.\,(c\ t\ e))(\lambda a\,.\,\lambda b\,.\,a)\ 1\ 0 \\ &\longrightarrow_\beta (\lambda t\,.\,\lambda e\,.\,((\lambda a\,.\,\lambda b\,.\,a)\ t\ e))\ 1\ 0 \\ &\longrightarrow_\beta (\lambda e\,.\,((\lambda a\,.\,\lambda b\,.\,a)\ 1\ e))\ 0 \\ &\longrightarrow_\beta ((\lambda a\,.\,\lambda b\,.\,a)\ 1\ 0) \\ &\longrightarrow_\beta ((\lambda b\,.\,1)\ 0) \\ &\longrightarrow_\beta 1 \end{aligned}$$

Arithmetic in the λ-calculus

We can represent the number n in the λ-calculus by a function which maps f to f composed with itself n times: $f \circ f \circ \ldots \circ f$.

Some numerals:

$$
\begin{aligned}
\ulcorner 0 \urcorner &\equiv \lambda f x \,.\, x \\
\ulcorner 1 \urcorner &\equiv \lambda f x \,.\, f x \\
\ulcorner 2 \urcorner &\equiv \lambda f x \,.\, f (f x) \\
\ulcorner 3 \urcorner &\equiv \lambda f x \,.\, f (f (f x))
\end{aligned}
$$

Some operations:

$$
\begin{aligned}
\texttt{ISZERO} &\equiv \lambda n \,.\, n\, (\lambda x \,.\, \texttt{FALSE})\, \texttt{TRUE} \\
\texttt{SUCC} &\equiv \lambda n\, f\, x \,.\, f\, (n\, f\, x) \\
\texttt{PLUS} &\equiv \lambda m\, n\, f\, x \,.\, m\, f\, (n\, f\, x) \\
\texttt{MULT} &\equiv \lambda m\, n\, f \,.\, m\, (n\, f) \\
\texttt{EXP} &\equiv \lambda m\, n \,.\, n\, m \\
\texttt{PRED} &\equiv \lambda n \,.\, n\, (\lambda g\, k \,.\, (g\, \ulcorner 1 \urcorner)\, (\lambda u \,.\, \texttt{PLUS}\, (g\, k)\, \ulcorner 1 \urcorner)\, k)\, (\lambda v \,.\, \ulcorner 0 \urcorner)\, \ulcorner 0 \urcorner
\end{aligned}
$$

Arithmetic in the λ-calculus

Let's try passing $\ulcorner 0 \urcorner$ to SUCC:

$$\begin{array}{lcl} \ulcorner 0 \urcorner & \equiv & \lambda f x \,.\, x \\ \mathtt{SUCC} & \equiv & \lambda n\, f\, x \,.\, f\,(n\, f\, x) \end{array}$$

$$\begin{array}{l} (\lambda n f x \,.\, f(n f x))(\lambda f x \,.\, x) \\ \longrightarrow_\beta \lambda f x \,.\, f((\lambda f x \,.\, x)\ f x) \\ \longrightarrow_\beta \lambda f x \,.\, f((\lambda x \,.\, x)\ x) \\ \longrightarrow_\beta \lambda f x \,.\, f x \\ \equiv \ulcorner 1 \urcorner \end{array}$$

Recursion

How can we express recursion in the λ-calculus?

Example: the factorial function

$$fact(n) \equiv \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n * fact(n-1)$$

In the λ-calculus, we can start to express this as:

$$fact \equiv \lambda n\,.\,(\texttt{ISZERO}\,n)\,\ulcorner 1 \urcorner\,(\texttt{MULT}\,n\,(fact\,(\texttt{PRED}\,n)))$$

But we need a way to give the factorial function a name.

Idea: Pass in $fact$ as an extra parameter somehow:

$$\lambda fact\,.\,\lambda n\,.\,(\texttt{ISZERO}\,n)\,\ulcorner 1 \urcorner\,(\texttt{MULT}\,n\,(fact\,(\texttt{PRED}\,n)))$$

We want the *fix-point* of this function:

$$\texttt{FIX}(f) \equiv f(\texttt{FIX}(f))$$

Fix point combinator, rationale

Definition of a fix-point operator:

$$\mathtt{FIX}(f) \equiv f(\mathtt{FIX}(f))$$

One step of `fact` is: $\lambda f\,.\,\lambda x\,.\,(\mathtt{ISZERO}\,x)\,\ulcorner 1 \urcorner\,(\mathtt{MULT}\,x\,(f\,(\mathtt{PRED}\,x)))$

Call this F. If we apply `FIX` to this, we get

$$\begin{aligned}
\mathtt{FIX}(F)(n) &= F\,(\mathtt{FIX}(F))\,(n)\\
\mathtt{FIX}(F)(n) &= \lambda x\,.\,(\mathtt{ISZERO}\,x)\,\ulcorner 1 \urcorner\,(\mathtt{MULT}\,x\,(\mathtt{FIX}(F)\,(\mathtt{PRED}\,x)))(n)\\
\mathtt{FIX}(F)(n) &= (\mathtt{ISZERO}\,n)\,\ulcorner 1 \urcorner\,(\mathtt{MULT}\,n\,(\mathtt{FIX}(F)\,(\mathtt{PRED}\,n)))
\end{aligned}$$

If we rename "$\mathtt{FIX}(F)$" as "`fact`", we have exactly what we want:

$$\mathtt{fact}(n) = (\mathtt{ISZERO}\,n)\,\ulcorner 1 \urcorner\,(\mathtt{MULT}\,n\,(\mathtt{fact}\,(\mathtt{PRED}\,n)))$$

Conclusion: $\mathtt{fact} = \mathtt{FIX}(F)$. (But we still need to define `FIX`.)

Fix point combinator, definition

There are many fix-point combinators. Here is the simplest, due to Haskell Curry:

$$\texttt{FIX} = \lambda f\,.\,(\lambda x\,.\,f\,(x\,x))\,(\lambda x\,.\,f\,(x\,x))$$

Let's prove that it actually works:

$$\begin{aligned}
\texttt{FIX}(g) = &\quad (\lambda f\,.\,(\lambda x\,.\,f\,(x\,x))\,(\lambda x\,.\,f\,(x\,x)))\,g \\
\longrightarrow_\beta &\quad ((\lambda x\,.\,g\,(x\,x))\,(\lambda x\,.\,g\,(x\,x))) \\
\longrightarrow_\beta &\quad g\,((\lambda x\,.\,g\,(x\,x))\,(\lambda x\,.\,g\,(x\,x)))
\end{aligned}$$

But this is exactly $g(\texttt{FIX}(g))$!

Scheme overview

- related to Lisp, first description in 1975
- designed to have clear and simple semantics (unlike Lisp)
- statically scoped (unlike Lisp)
- dynamically typed
  - types are associated with values, not variables
- functional: first-class functions
- garbage collection
- simple syntax; lots of parentheses
  - homogeneity of programs and data
- continuations
- hygienic macros

A sample Scheme session

```
(+ 1 2)
⇒ 3
(1 2 3)
⇒ procedure application: expected procedure; given: 1
a
⇒ reference to undefined identifier: a
(quote (+ 1 2))   ; a shorthand is '(+ 1 2)
⇒ (+ 1 2)
(car '(1 2 3))
⇒ 1
(cdr '(1 2 3))
⇒ (2 3)
(cons 1 '(2 3))
⇒ (1 2 3)
```

Uniform syntax: lists

- expressions are either atoms or lists
- atoms are either constants (e.g., numeric, boolean, string) or symbols
- lists nest, to form full trees
- syntax is simple because programmer supplies what would otherwise be the internal representation of a program:

```
(+ (* 10 12) (* 7 11))   ; means (10*12 + 7*11)
```

- a program is a list:

```
(define (factorial n)
        (if (= n 0)
            1
            (* n (factorial (- n 1)))))
```

List manipulation

Three primitives and one constant:

- `car`: get head of list
- `cdr`: get rest of list
- `cons`: prepend an element to a list
- `'()` null list

Add equality (`=` or `eq`) and recursion, and you've got yourself a universal model of computation

Rules of evaluation

- a *number* evaluates to itself
- an *atom* evaluates to its current binding
- a *list* is a computation:
  - must be a form (e.g., if, lambda), or
  - first element must evaluate to an operation
  - remaining elements are actual parameters
  - result is the application of the operation to the evaluated actuals

Quoting data

Q: If every list is a computation, how do we describe data?

A: Another primitive: `quote`

```
(quote (1 2 3 4))
⇒ (1 2 3 4)
(quote (Baby needs a new pair of shoes)
⇒ (Baby needs a new pair of shoes)
'(this also works)
⇒ (this also works)
```

List decomposition

```
(car '(this is a list of symbols))
⇒ this

(cdr '(this is a list of symbols))
⇒ (is a list of symbols)

(cdr '(this that))
⇒ (that) ; a list

(cdr '(singleton))
⇒ () ; the empty list

(car '())
```

⇒ *car: expects argument of type <pair>; given ()*

List building

```
(cons 'this '(that and the other))
⇒ (this that and the other)
(cons 'a '())
⇒ (a)
```

useful shortcut:

```
(list 'a 'b 'c 'd 'e)
⇒ (a b c d e)
```

equivalent to:

```
(cons 'a
      (cons 'b
            (cons 'c
                  (cons 'd
                        (cons 'e '())))))
```

List decomposition shortcuts

Operations like:

```
(car (cdr xs))
(cdr (cdr (cdr ys)))
```

are common. Scheme provides shortcuts:

`(cadr xs)`	is	`(car (cdr xs))`
`(cddr xs)`	is	`(cdr (cdr (cdr xs)))`

Up to 4 `a`'s and/or `d`'s can be used.

What lists are made of

```
(cons 'a '(b))    ⇒   (a b)       a list
(car '(a b))      ⇒   a
(cdr '(a b))      ⇒   (b)

(cons 'a 'b)      ⇒   (a . b)     a dotted pair
(car '(a . b))    ⇒   a
(cdr '(a . b))    ⇒   b
```

A list is a special form of dotted pair, and can be written using a shorthand:

`'(a b c)` is shorthand for `'(a . (b . (c . ())))`

We can mix the notations:

`'(a b . c)` is shorthand for `'(a . (b . c))`

A list not ending in `'()` is an *improper list*.

Booleans

Scheme has true and false values:

- `#t` – true
- `#f` – false

However, when evaluating a condition (e.g., in an `if`), any value not equal to `#f` is considered to be true.

Simple control structures

- Conditional

```
(if condition expr1 expr2)
```

- Generalized form

```
(cond
  (pred1 expr1)
  (pred2 expr2)
  ...
  (else exprn))
```

Evaluate the `pred`'s in order, until one evaluates to true. Then evaluate the corresponding `expr`. That is the value of the `cond` expression.

`if` and `cond` are not regular functions

Global definitions

`define` is also special:

```
(define (sqr n) (* n n))
```

The body is not evaluated; a binding is produced: `sqr` is bound to the body of the computation:

```
(lambda (n) (* n n))
```

We can `define` non-functions too:

```
(define x 15)
(sqr x)
⇒ 225
```

`define` can only occur at the top level, and creates global variables.

Recursion on lists

```
(define (member elem lis)
   (cond
      ((null? lis) #f)
      ((= elem (car lis)) lis)
      (else (member elem (cdr lis)))))
```

Note: every non-false value is true in a boolean context.

Convention: return rest of the list, starting from `elem`, rather than `#t`.

Standard predicates

If variables do not have associated types, we need a way to find out what a variable is holding:

- `symbol?`
- `number?`
- `pair?`
- `list?`
- `null?`
- `zero?`

Different dialects may have different naming conventions, e.g., `symbolp`, `numberp`, etc.

Functional arguments

```
(define (map fun lis)
   (cond
      ((null? lis) '())
      (else (cons (fun (car lis))
                  (map fun (cdr lis))))))

(map sqr (map sqr '(1 2 3 4)))
⇒ (1 16 81 256)
```

Locals

Basic `let` skeleton:

```
(let
    ((v1 init1) (v2 init2) ... (vn initn))
    body)
```

To declare locals, use one of the `let` variants:

- `let` : Evaluate all the *inits* in the current environment; the *vs* are bound to fresh locations holding the results.
- `let*` : Bindings are performed sequentially from left to right, and each binding is done in an environment in which the previous bindings are visible.
- `letrec` : The *vs* are bound to fresh locations holding undefined values, the *inits* are evaluated in the resulting environment (in some unspecified order), each *v* is assigned to the result of the corresponding *init*. This is what we need for mutually recursive functions.

Tail recursion

> "A Scheme implementation is properly tail-recursive if it supports an unbounded number of active tail calls."

```
(define (factorial n)
  (if (zero? n) 1
      (* n (factorial (- n 1))))) ; not tail recursive
                                  ; stack grows to size n
(define (fact-iter prod count var)
  (if (> count var) prod
      (fact-iter (* count prod)  ; tail recursive
                 (+ count 1)     ; implemented as loop
                 var)))
(define (factorial n) (fact-iter 1 1 n)) ; OK
```