Mathematical Statistics Solutions: HW 1

October 19, 2022

Exercise 1

Part (a)

For all $t > 0$, consider the following random variable X:

$$\mathbb{P}[X = 0] = \mathbb{P}[X = t] = \frac{1}{2}$$

then

$$\mathbb{E}X = \frac{t}{2}, \quad \mathbb{P}[X \geq t] = \frac{1}{2} = \frac{\mathbb{E}X}{t}.$$

Part (b)

For all $t > 0$, consider the following random variable X:

$$\mathbb{P}[X = -t] = \mathbb{P}[X = t] = \frac{1}{2}$$

then

$$\mathbb{E}X = 0, \ \mathsf{Var}[X] = t^2, \ \mathbb{P}[|X - \mathbb{E}X| \geq t] = 1 = \frac{\mathsf{Var}[X]}{t^2}.$$

We conclude that both Markov's and Chebyshev's inequality can not be improved.

Exercise 2

Part (a)

For all random variables X and $c \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}(X-c)^2 &= \mathbb{E}(X - \mathbb{E}X + \mathbb{E}X - c)^2 \\
&= \mathbb{E}(X - \mathbb{E}X)^2 + \mathbb{E}(\mathbb{E}X - c)^2 + 2\mathbb{E}(X - \mathbb{E}X)(\mathbb{E}X - c) \\
&= \mathsf{Var}[X] + (\mathbb{E}X - c)^2
\end{aligned}$$

Since $(\mathbb{E}X - c)^2 \geq 0$, we know that

$$\mathsf{Var}[X] \leq \mathbb{E}(X-c)^2. \tag{1}$$

Part (b)

Let $c = \frac{a+b}{2}$ in (1), we get

$$\mathsf{Var}[X] \leq \mathbb{E}\left(X - \frac{a+b}{2}\right)^2 \leq \mathbb{E}\left(\frac{(b-a)^2}{4}\right) = \frac{(b-a)^2}{4}$$

where the second inequality is due to the fact that, for $X \in [a,b]$ we always have

$$\left|X - \frac{a+b}{2}\right| \leq \frac{b-a}{2}.$$

Part (c)

We only need to show that $q_\lambda(x) \geq 0$ for all x, and $\int_{\mathbb{R}} q_\lambda(x)dx = 1$. First, $q_\lambda(x) \geq 0$ holds because $p(x) \geq 0$ as a density, and $e^{\lambda x} > 0$ for all x. We also have

$$\int_{\mathbb{R}} q_\lambda(x)dx = \int_{\mathbb{R}} \frac{e^{\lambda x}}{\mathbb{E}e^{\lambda X}} p(x)dx = \frac{1}{\mathbb{E}e^{\lambda X}} \int_{\mathbb{R}} e^{\lambda x} p(x)dx = \frac{1}{\mathbb{E}e^{\lambda X}} \cdot \mathbb{E}e^{\lambda X} = 1.$$

We conclude that q_λ is a probability density.

Part (d)

For $K(\lambda) = \log \mathbb{E}e^{\lambda X}$, we have

$$\begin{aligned} K'(\lambda) &= \frac{1}{\mathbb{E}e^{\lambda X}} \left(\frac{d}{d\lambda} \mathbb{E}e^{\lambda X} \right) \\ &= \frac{1}{\mathbb{E}e^{\lambda X}} \mathbb{E}[Xe^{\lambda X}] \\ &= \frac{1}{\mathbb{E}e^{\lambda X}} \int_{\mathbb{R}} xe^{\lambda x} p(x)dx \\ &= \int_{\mathbb{R}} xq_\lambda(x)dx \end{aligned}$$

and

$$\begin{aligned} K''(\lambda) &= \int_{\mathbb{R}} x \left(\frac{d}{d\lambda} q_\lambda(x) \right) dx \\ &= \int_{\mathbb{R}} x \left(\frac{xe^{\lambda x}}{\mathbb{E}e^{\lambda X}} - \frac{e^{\lambda x} \cdot \mathbb{E}[Xe^{\lambda X}]}{(\mathbb{E}e^{\lambda X})^2} \right) p(x)dx \\ &= \int_{\mathbb{R}} x^2 \frac{e^{\lambda x}}{\mathbb{E}e^{\lambda X}} p(x)dx - \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}e^{\lambda X}} \cdot \int_{\mathbb{R}} x \frac{e^{\lambda x}}{\mathbb{E}e^{\lambda X}} p(x)dx \\ &= \int_{\mathbb{R}} x^2 q_\lambda(x)dx - \left(\int_{\mathbb{R}} xq_\lambda(x)dx \right)^2. \end{aligned}$$

Part (e)

$K(0) = \log \mathbb{E}e^0 = \log 1 = 0$; since X is centered, we also have

$$K'(0) = \int_{\mathbb{R}} xq_0(x)dx = \int_{\mathbb{R}} xp(x)dx = \mathbb{E}X = 0.$$

Let Y be a random variable with density q_λ; from the second part of (d) we know that $K''(\lambda) = \mathsf{Var}(Y)$. Note that, for X supported on $[a, b]$, Y is also supported on $[a, b]$. Thus Proposition 1.13 in the lecture notes gives

$$K''(\lambda) = \mathsf{Var}(Y) \leq \frac{(b-a)^2}{4}.$$

Part (f)

Using results from Part (e), we get

$$K'(\lambda) = K'(0) + \int_0^\lambda K''(u)du \leq \lambda \frac{(b-a)^2}{4}$$

and

$$K(\lambda) = K(0) + \int_0^\lambda K'(u)du \leq \int_0^\lambda udu \cdot \frac{(b-a)^2}{4} = \frac{\lambda^2}{2} \frac{(b-a)^2}{4}.$$

Exercise 3

For all $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}e^{\lambda(X-\mathbb{E}X)} &= \int_{\mathbb{R}} e^{\lambda(x-\mu)} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} e^{\lambda y - \frac{y^2}{2\sigma^2}} dy \\
&= e^{\frac{\lambda^2\sigma^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\lambda\sigma^2)^2}{2\sigma^2}} dy \\
&= e^{\frac{\lambda^2\sigma^2}{2}}.
\end{aligned}$$

where the second equation applied a change of variable $y = x - \mu$, and the final equation used the fact that the integrated function is the density of $\mathcal{N}(\lambda\sigma^2, \sigma^2)$ so the integral equals to 1.

Exercise 4

Part (a)

Suppose X is σ^2-subgaussian. Writing the definition and expanding in a Taylor series, we have

$$\begin{aligned}
\exp\left(\frac{\sigma^2\lambda^2}{2}\right) &\geq \mathbb{E}\exp\Big(\lambda(X - \mathbb{E}X)\Big) && \text{(definition of subgaussian)} \\
&= \mathbb{E}\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(X - \mathbb{E}X)^k \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\mathbb{E}(X - \mathbb{E}X)^k \\
&= 1 + \frac{\mathsf{Var}[X]}{2}\lambda^2 + \sum_{k=3}^{\infty} \frac{\lambda^k}{k!}\mathbb{E}(X - \mathbb{E}X)^k
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{\mathsf{Var}[X]}{2} &\leq \lim_{\lambda\to 0} \frac{\exp(\sigma^2\lambda^2/2) - 1}{\lambda^2} \\
&= \lim_{\lambda\to 0} \frac{\sigma^2\lambda\exp(\sigma^2\lambda^2/2)}{2\lambda} && \text{(l'Hôpital's rule)} \\
&= \lim_{\lambda\to 0} \frac{\sigma^2\exp(\sigma^2\lambda^2/2) + \sigma^4\lambda^2\exp(\sigma^2\lambda^2/2)}{2} && \text{(l'Hôpital's rule)} \\
&= \frac{\sigma^2}{2}, && \text{(evaluating at } \lambda = 0\text{)}
\end{aligned}$$

and multiplying by 2 on both sides completes the proof.

Part (b)

Recall that $X \sim \mathsf{Exp}(1)$ has the density $e^{-x}\mathbb{1}\{x \geq 0\}$. Therefore, we calculate the first two moments by integrating by parts:

$$\begin{aligned}
\mathbb{E}X &= \int_0^\infty x e^{-x} dx \\
&= \int_0^\infty e^{-x} dx && \text{(integration by parts)} \\
&= -e^{-x}\Big|_0^\infty \\
&= 1, \\
\mathbb{E}X^2 &= \int_0^\infty x^2 e^{-x} dx \\
&= \int_0^\infty 2x e^{-x} dx && \text{(integration by parts)} \\
&= 2. && \text{(from previous calculation)}
\end{aligned}$$

Therefore, $\mathsf{Var}[X] = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 1$.

On the other hand, the moment generating function is

$$\begin{aligned}
\mathbb{E}\exp(\lambda(X - \mathbb{E}X)) &= \int_0^\infty e^{-x} e^{\lambda(x-1)} dx \\
&= e^{-\lambda} \int_0^\infty e^{(\lambda-1)x} dx,
\end{aligned}$$

which in particular is infinite whenever $\lambda \geq 1$ (then the integrand is always at least 1). Therefore, the subgaussian bound cannot hold for any finite σ^2.

Exercise 5

Part (a)

By the rules for scaling variance and adding variances of independent random variables, we have

$$\begin{aligned}
\mathsf{Var}[\bar{X}] &= \mathsf{Var}\left[\frac{1}{n}\sum_{i=1}^n X_i\right] \\
&= \frac{1}{n^2}\mathsf{Var}\left[\sum_{i=1}^n X_i\right] \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathsf{Var}[X_i] \\
&= \frac{1}{n}\mathsf{Var}[X_1] \\
&= \frac{p(1-p)}{n}.
\end{aligned}$$

Part (b)

Note that $\mathbb{E}\bar{X} = p$. By Proposition 1.10, noting that here σ^2 is the variance of X_i which is $p(1-p)$, we find taking $s = 1/(2\sigma) = 1/(2\sqrt{p(1-p)}$ we have

$$\mathbb{P}\left[|\bar{X} - p| \geq \frac{1}{2\sqrt{n}}\right] \leq \frac{1}{s^2} = 4p(1-p).$$

By Proposition 1.17, taking $a = 0$, $b = 1$, and $s = 1$, we have

$$\mathbb{P}\left[|\bar{X} - p| \geq \frac{1}{2\sqrt{n}}\right] \leq 2e^{-1/2}.$$

Part (c)

The bound from Proposition 1.17 is a constant that is approximately $2e^{-1/2} \approx 1.21 > 1$, which is a vacuous upper bound on a probability, since all probabilities are at most 1. The bound from Proposition 1.10 is at most $4p$, and thus when $p = \lambda/n$ for any $\lambda > 0$ and sufficiently large n, the bound is smaller than 1 (indeed, it tends to zero as $O(1/n)$)), and thus gives a non-trivial bound.