Compressing images with Discrete Cosine Basis

In [1]:
```
%matplotlib inline
import numpy as np
import scipy.fftpack
import scipy.misc
import matplotlib.pyplot as plt
plt.gray()
```

```
<Figure size 432x288 with 0 Axes>
```

In [2]:
```
# Two auxiliary functions that we will use. You do not need to read them (but make sure

def dct(n):
    return scipy.fftpack.dct(np.eye(n), norm='ortho')

def plot_vector(v, color='k'):
    plt.plot(v,linestyle='', marker='o',color=color)
```

5.3.1 The canonical basis

The vectors of the canonical basis are the columns of the identity matrix in dimension n. We plot their coordinates below for $n = 8$.

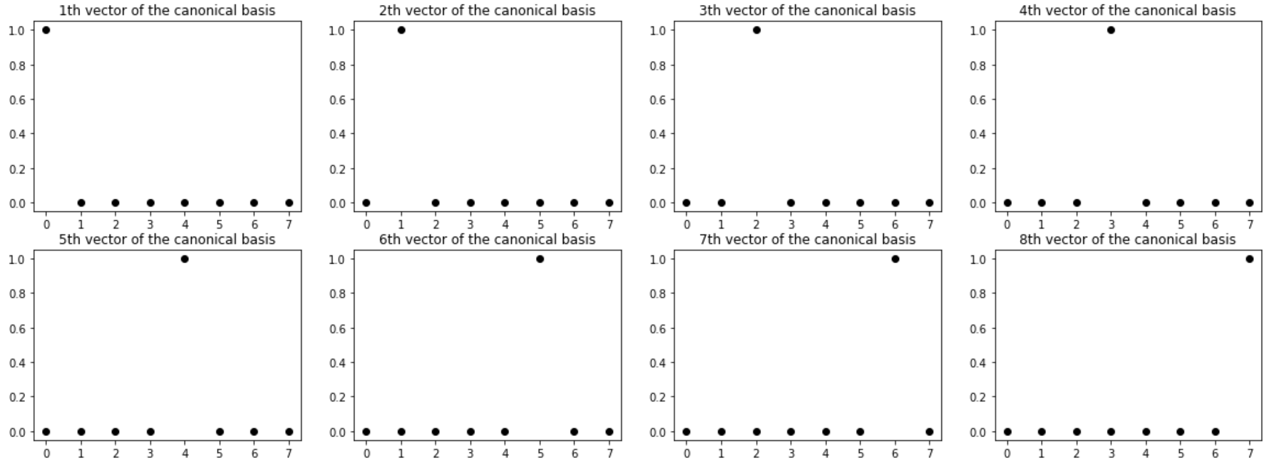
In [3]:
```
identity = np.identity(8)
print(identity)

plt.figure(figsize=(20,7))
for i in range(8):
    plt.subplot(2,4,i+1)
    plt.title(f"{i+1}th vector of the canonical basis")
    plot_vector(identity[:,i])

print('\n Nothing new so far...')
```

```
[[1. 0. 0. 0. 0. 0. 0. 0.]
 [0. 1. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1.]]

 Nothing new so far...
```



5.3.2 Discrete Cosine basis

The discrete Fourier basis is another basis of $\mathbb{R}^n$. The function `dct(n)` outputs a square matrix of dimension n whose columns are the vectors of the discrete cosine basis.

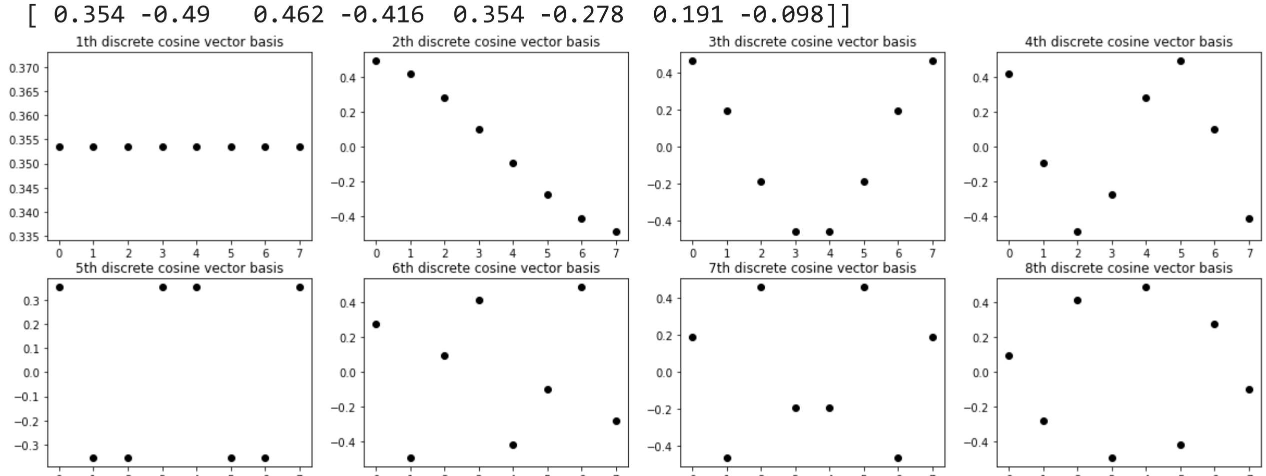
In [4]:

```
# Discrete Cosine Transform matrix in dimension n = 8
D8 = dct(8)
print(np.round(D8,3))

plt.figure(figsize=(20,7))

for i in range(8):
    plt.subplot(2,4,i+1)
    plt.title(f"{i+1}th discrete cosine vector basis")
    plot_vector(D8[:,i])
```

```
[[ 0.354  0.49   0.462  0.416  0.354  0.278  0.191  0.098]
 [ 0.354  0.416  0.191 -0.098 -0.354 -0.49  -0.462 -0.278]
 [ 0.354  0.278 -0.191 -0.49  -0.354  0.098  0.462  0.416]
 [ 0.354  0.098 -0.462 -0.278  0.354  0.416 -0.191 -0.49 ]
 [ 0.354 -0.098 -0.462  0.278  0.354 -0.416 -0.191  0.49 ]
 [ 0.354 -0.278 -0.191  0.49  -0.354 -0.098  0.462 -0.416]
 [ 0.354 -0.416  0.191  0.098 -0.354  0.49  -0.462  0.278]
 [ 0.354 -0.49   0.462 -0.416  0.354 -0.278  0.191 -0.098]]
```

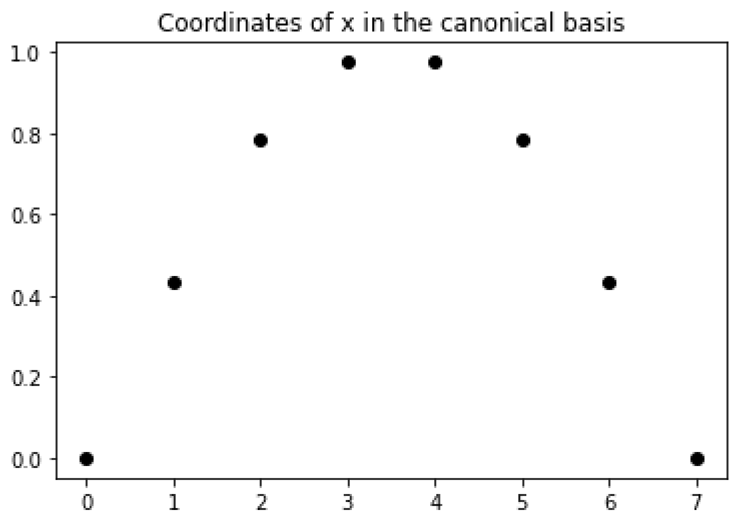


5.3 (a) Check numerically (in one line of code) that the columns of `D8` are an orthonormal basis of $\mathbb{R}^8$ (ie verify that the Haar wavelet basis is an orthonormal basis).

In [5]:
```
print(np.round(D8.T @ D8),2)
```

```
[[ 1. -0.  0. -0.  0. -0. -0.  0.]
 [-0.  1. -0.  0. -0. -0. -0.  0.]
 [ 0. -0.  1. -0.  0. -0.  0. -0.]
 [-0.  0. -0.  1. -0.  0. -0. -0.]
 [ 0. -0.  0. -0.  1. -0. -0. -0.]
 [-0. -0. -0.  0. -0.  1.  0. -0.]
 [-0. -0.  0. -0. -0.  0.  1.  0.]
 [ 0.  0. -0. -0. -0. -0.  0.  1.]] 2
```

In [6]:
```
# Let consider the following vector x
x = np.sin(np.linspace(0,np.pi,8))
plt.title('Coordinates of x in the canonical basis')
plot_vector(x)
```

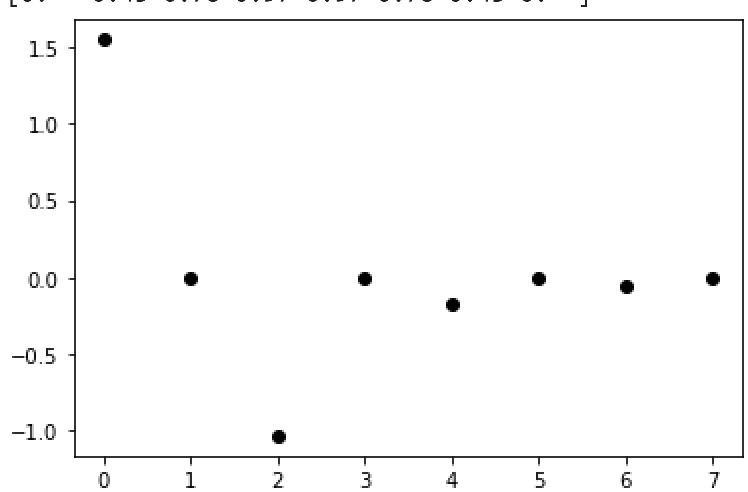


5.3 (b) Compute the vector $v \in \mathbb{R}^8$ of DCT coefficients of x. (1 line of code!), and plot them.

How can we obtain back x from v ? (1 line of code!).

In [7]:
```
v = D8.T @ x
plot_vector(v)
x = D8 @ v
print(np.round(x,2))
```

```
[0.   0.43 0.78 0.97 0.97 0.78 0.43 0.  ]
```



5.3.3 Image compression

In this section, we will use DCT modes to compress images. Let's use one of the template images of python.

In [8]:

```
image = scipy.misc.face(gray=True)
h,w = image.shape
print(f'Height: {h}, Width: {w}')

plt.imshow(image)
```

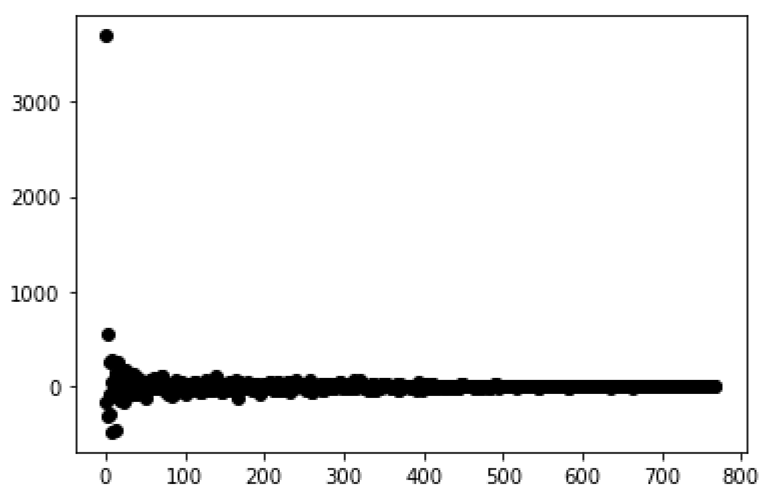
```
Height: 768, Width: 1024
```

Out[8]: `<matplotlib.image.AxesImage at 0x1d762d404c0>`

5.3 (c) We will see each column of pixels as a vector in $\mathbb{R}^{768}$, and compute their coordinates in the DCT basis of $\mathbb{R}^{768}$. Plot the entries of `x`, the first column of our image.

In [9]:

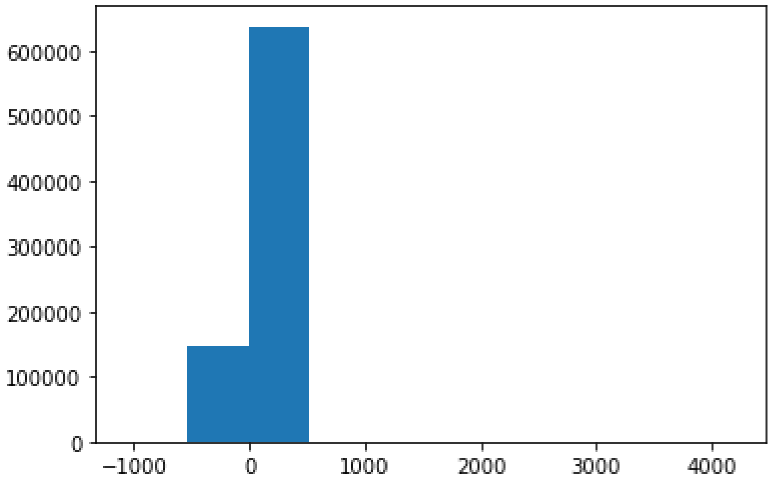
```
D768 = dct(768)
transformed = D768.T @ image
x = transformed[:,0]
plot_vector(x)
```



5.3 (d) Compute the 768 x 1024 matrix `dct_coeffs` whose columns are the dct coefficients of the columns of `image`. Plot an histogram of there intensities using `plt.hist`.

In [13]:
```
dct_coeffs = transformed
plt.hist(dct_coeffs.flatten())
```

Out[13]:
```
(array([6.36000e+02, 1.47189e+05, 6.37024e+05, 5.19000e+02, 4.00000e+01,
        0.00000e+00, 2.67000e+02, 2.41000e+02, 2.69000e+02, 2.47000e+02]),
 array([-1064.43123878,  -537.21884715,   -10.00645553,   517.20593609,
         1044.41832772,  1571.63071934,  2098.84311097,  2626.05550259,
         3153.26789421,  3680.48028584,  4207.69267746]),
 <BarContainer object of 10 artists>)
```



Since a large fraction of the dct coefficients seems to be negligible, we see that the vector `x` can be well approximated by a linear combination of a small number of discrete cosines vectors.

Hence, we can 'compress' the image by only storing a few dct coefficients of largest magnitude.

Let's say that we want to reduce the size by 98%: Store only the top 2% largest (in absolute value) coefficients of `wavelet_coeffs` .

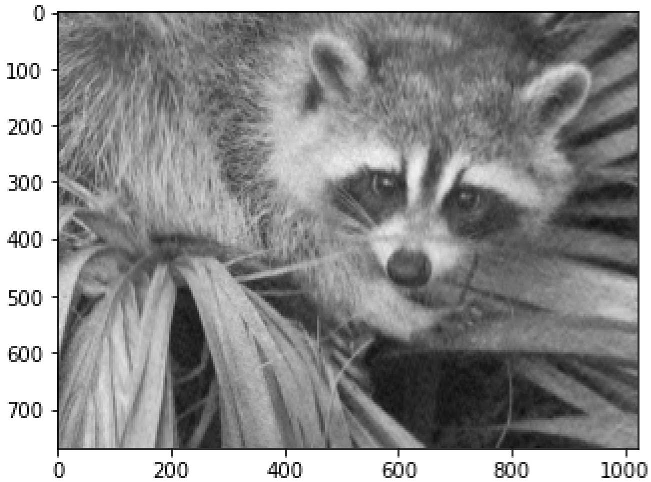
5.3 (e) Compute a matrix `thres_coeffs` who is the matrix `dct_coeffs` where about 97% smallest entries have been put to 0.

In [14]:
```
dct_coeffs[abs(dct_coeffs) < np.quantile(dct_coeffs, .97)] = 0
thres_coeffs = dct_coeffs
```

5.3 (f) Compute and plot the `compressed_image` corresponding to `thres_coeffs` .

In [20]:
```
compressed_image = D768 @ thres_coeffs
plt.imshow(compressed_image)
```

Out[20]: <matplotlib.image.AxesImage at 0x1d7625c9430>



In [ ]:

In [ ]: