Homework 0

Due September 12 at 11 pm

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1. (Sets) We will use set theory to define probability spaces. Are these statements true or false? Provide a proof if they are true (you can use Venn diagrams to gain intuition, but also write down a formal proof), or a counterexample if they are false.

   A partition of a set Ω is a collection of sets $S_1, \ldots, S_n$ such that $\Omega = \cup_i S_i$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

   (a) If $S_1, \ldots, S_n$ is a partition of Ω, then for any subset $A \subseteq \Omega$, $S_1 \cap A, \ldots, S_n \cap A$ is a partition of A.

   This statement is true, as $A = \cup_i S_i \cap A$ (I'm not sure if I wrote this correctly mathematically, but what I'm trying to say is that if we add (by taking the union) all the intersections between A and $S_1, \ldots, S_n$ they will add up to A).

   Also, $(A \cap S_i) \cap (A \cap S_j) = \emptyset$ for $i \neq j$ (Again, not sure if I wrote this correctly, but what I am trying to express is that since $S_1, \ldots, S_n$ is a partition of Ω by definition it is disjoint, so every $A \cap S_i$ will be disjoint from every other $A \cap S_j$ where $i \neq j$).

   (b) For any sets A and B, $A^c \cup B^c = (A \cup B)^c$.

   This statement is false, which will be proven by counter example. Consider the sample space where $A, B \in \Omega$ and $A \cap B \neq \emptyset$ (I am trying to describe the classic ven-diagram where A and B exist in a sample space and share some area). In this case the statement $A^c \cup B^c$ is the area outside of the ven diagrams. The statement $(A \cup B)^c$ however, contains all the area of the sample space except for $A \cap B$. Therefore, the two areas are not equal. See picture at the end of pdf for illustration.

   (c) For any sets A, B, and C, $(A \cup B) \cap C = A \cup (B \cap C)$.

   This is false by counter example. Consider the sample space Ω where $A, B, C \in \Omega$ and $A \cap B, A \cap C, B \cap C$ and $A \cap B \cap C \neq \emptyset$. (Here I am trying to illustrate a sample space with the triple ven diagram, where A,B,C have overlap). The statement $(A \cup B) \cap C$ contains the areas $A \cap B, A \cap C, A \cap B \cap C$. The statement $A \cup (B \cap C)$ however, contains the areas $A, A \cap B, A \cap C, A \cap B \cap C$ Therefore what our statement of says is:

$$A \cap B + A \cap C + A \cap B \cap C = A + B \cap C \tag{1}$$

   Which we can clearly see is not true. See picture at the end of pdf for illustration.

2. (Series) We will need series to compute probabilities and expectations related to discrete quantities.

   (a) Assuming $r \neq 1$, derive a simple expression for

   $$S_n := \sum_{i=m}^{n} r^i \tag{2}$$

   as a function of r, m and n, and prove that it holds. Assume m and n are positive integers with $m \leq n$.

   To solve this geometric series we're going to do the following 5 steps:
   1) Unwrap the series into a form we can perform algebraic manipulations on
   2) Multiply both sides by r
   3) Subtract our S_n statement by our rS_n statement
   4) Factor out $(1 - r)$ from both sides
   5) Isolate S_n and we have our answer

   $$\begin{aligned} S_n &:= r^m + r^{m+1} + \cdots + r^n \\ r \times S_n &:= r \times r^m + r \times r^{m+1} + \cdots + r \times r^n \\ S_n - r \times S_n &:= r - r \times r^n \\ S_n(1 - r) &= r(1 - r^n) \\ Sn &= \frac{r(1 - r^n)}{1 - r} \end{aligned} \tag{3}$$

   (b) Under what condition on r does the infinite series

   $$\sum_{i=m}^{\infty} r^i = \lim_{n \to \infty} S_n \tag{4}$$

   converge (where again m is a positive integer)?

   The series converges at k where $-1 < k < 1$ holds true. If $k = 1$ the series is undefined (division by 0), and if $k = -1$ Otherwise the series expands to $\pm\infty$ and never converges.

   (c) Use induction to prove the identity

   $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \tag{5}$$

   where n is a nonnegative integer greater than 1.

   Lets take a base case with $n = 2$. We have: $(2(2 + 1))/2 = 3$ which if we check $1+2 = 3$ holds true. Now assume the expression holds for n where $n = k$. We would have the following:

   $$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}, \tag{6}$$

when $n = k + 1$

$$\begin{aligned}
\sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2} &= \sum_{i=1}^{k} i + (k+1) \\
\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) &= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)(k+1+1)}{2}
\end{aligned} \tag{7}$$

We've shown that the sequence holds for $n = 1$, $n = k$, and $n = k + 1$ therefore it holds for all positive integers greater than 1.

3. (Derivatives)

(a) Briefly explain why the derivative of a function can be interpreted as an *instantaneous rate of change*.

Answer to 3a): Derivatives calculate the slope of a tangent line between two points of a function, f(x) and f(x+h). Using the definition of the derivative, as h approaches 0, the distance between x and x+h on the x-axis becomes increasingly small, and the tangent line becomes an increasingly accurate estimate of slope. When h becomes minute, the tangent line essentially measures a rate of change a specific point, which can be interpreted as an instantaneous rate of change at that point.

(b) Use the definition to derive the derivative of the function x^2.

Answer to part 3b) Use the definition of a derivative given to us, where $f(x) = x^2$. Once we get to $2x + h$, we can drop h as $h \to 0$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} \Longrightarrow \frac{(x^2 + 2xh + h^2) - (x^2)}{h} \Longrightarrow \tag{8}$$

$$\frac{x^2}{h} - \frac{x^2}{h} + \frac{2xh}{h} + \frac{h^2}{h} \Longrightarrow 2x + h \Longrightarrow 2x \tag{9}$$

(c) We would like to approximate a differentiable function f at y using a linear function $L_y(x) := ax + b$. We set a and b so that f and L_y have the same value and the same derivative at y (i.e., $L_y(y) = f(y)$ and $L_y'(y) = f'(y)$). Give an expression for $L_y(x)$ in terms of y, $f(y)$, and $f'(y)$.

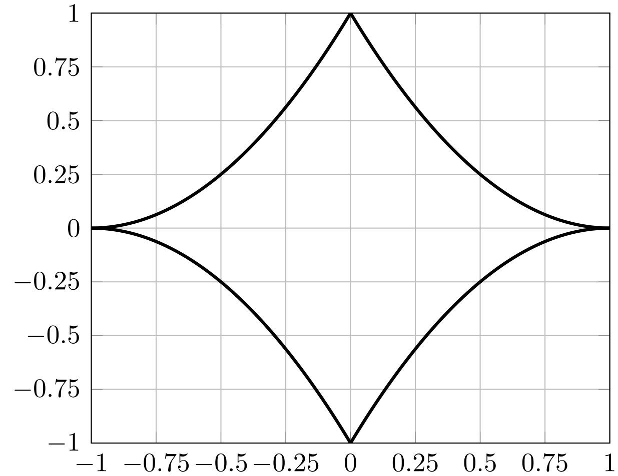
Start with $L_y(x) := ax + b$. We know a represents the slope at the given point, so we will replace that with the derivative at that point. $a = f'(y)$
Let us remember, we are using the point y to approximate a point x. We will need to account for this by scaling our ax term with the following: $x - y$ and therefore $L_y(x) := f'(y)(x - y) + b$.
Lastly, b is our starting point, in this case, $f(y)$
Thus we have $L_y(x) := f'(y)(x - y) + f(y)$

(d) Let $f(x) = 4x^2e^x$. Plot f and L_2 between 1 and 3.

Using the formula derived in the previous step, $f'(y)$ will be set to $f'(x) = 4x(x+2)e^x$ and $f'(2) = 32e^X$ and $b = f(2) = 16e^x$. All together we have $L_y(x) = 32e^2 \times (x - 2) + 16e^2$ which we will then plot along with f from the range $x = 1 - 3$. **The graph of the functions can be found on the last page of this pdf.**

4. (Integrals)

(a) Express the area of the following shape in terms of an integral and solve it.



We can use the three points $(0,1), (1,0), (\frac{1}{2}, \frac{1}{4})$ to determine the equation of the quadratic function in the top right corner. The equation is $y = x^2 - 2x + 1$, which if we integrate, we get $\frac{x^3}{3} - x^2 + x + C$ and when evaluated from 0 to 1, we get $\frac{1}{3}$ as the area under the curve. However, since this is only the top right quadrant, we can leverage the x,y symmetry of the graph and multiply by 4 to find the total area. Thus the total area of the shape is $\frac{4}{3}$

(b) Use change of variables to derive a closed-form expression for the function

$$f(t) := \int_0^t \frac{x}{1+x^2}\,\mathrm{d}x. \tag{10}$$

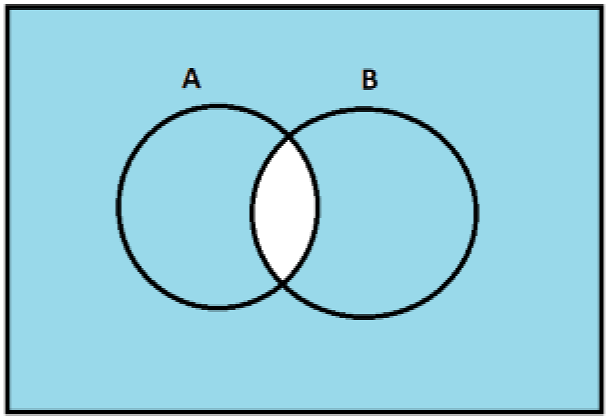
So to solve this we're going to use u-substitution. Let $u = (1 + x^2)$ and take the derivative of each side:

$$\begin{aligned} u &= 1 + x^2 \\ du &= 2xdx \end{aligned} \tag{11}$$

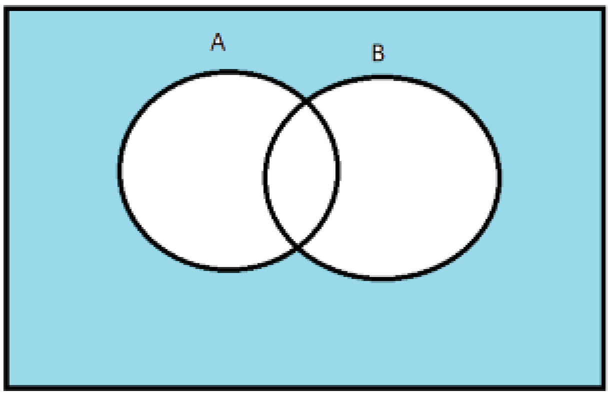
To solve the equation, start by substituting u in for $1 + x^2$, then substitute $\frac{\mathrm{d}(u)}{2}$ for $x\,\mathrm{d}x$, then integrate and sub $(1 + x^2)$ for u

$$\begin{aligned} f(t) := &\int_0^t \frac{x}{u}\,\mathrm{d}x \\ &\int_0^t \frac{1}{2}\frac{1}{u}\,\mathrm{d}u \\ &\frac{\ln(u)}{2} + c \\ &\frac{\ln(1+x^2)}{2} + c \end{aligned} \tag{12}$$

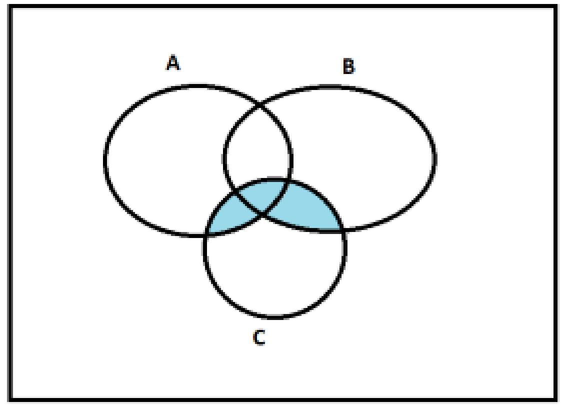
1 b) A^c Union B^C Contains everything this color



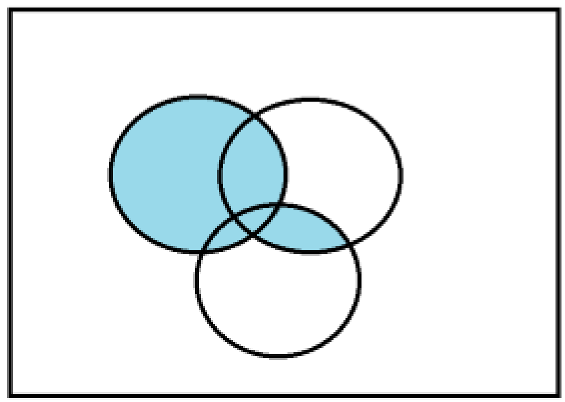
1 b) (A union B)^C Contains everything this color

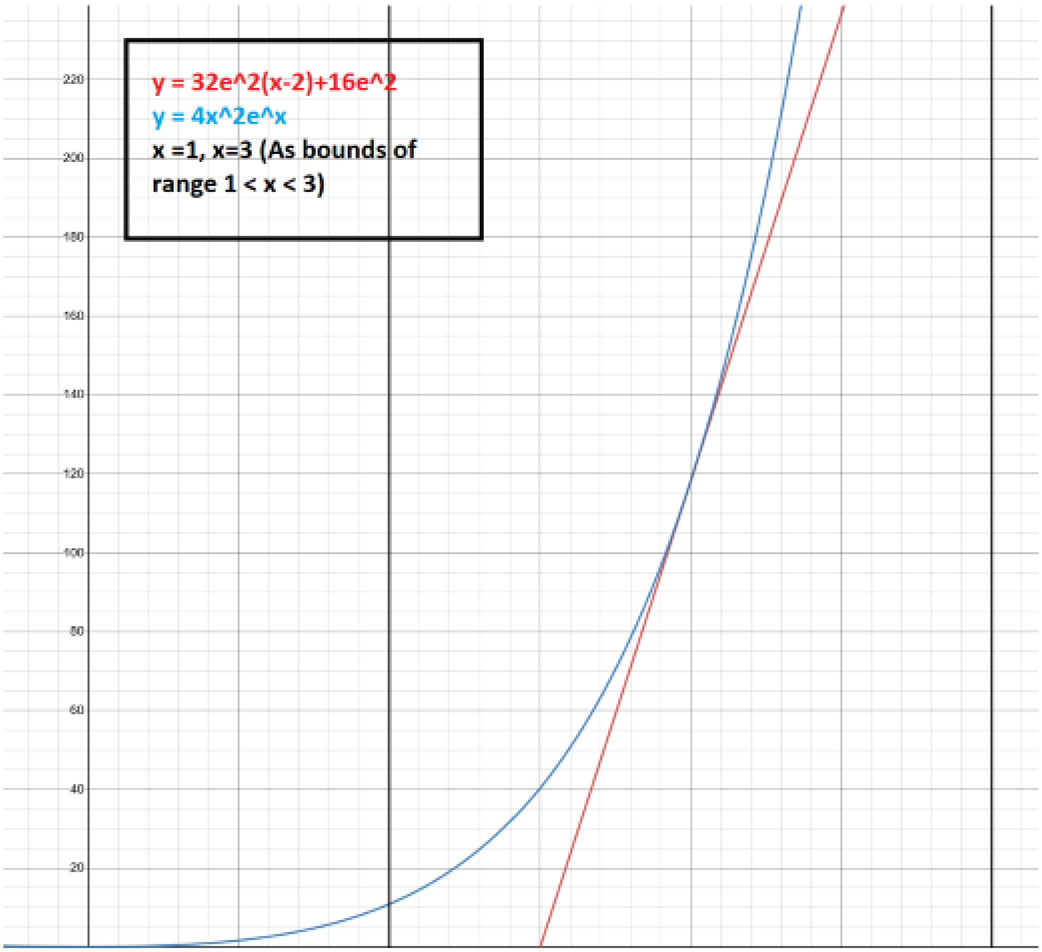


1 c) (A Union B) Intersect C Contains everything this color



1 c) A Union (B Intersect C) Contains everything this color




y = 32e^2(x-2)+16e^2
y = 4x^2e^x
x =1, x=3 (As bounds of
range 1 < x < 3)
220
200
180
160
140
120
100
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