Linear Algebra HW 5

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1 Problem 5.1

Give an othonormal basis of $\mathbb{R}^3$ using the Gram-Schmidt algorithm starting from the linearly independent family (v_1, v_2, v_3) where $v_1 = (1, 1, 1)$, $v_2 = (2, 1, 1)$ and $v_3 = (2, 0, 1)$.

Lets begin making an orthonomal set, x out of our linearly independent family, v. To start our algorithm, we will need to normalize v_1 such that its norm is equal to 1, to make our x_1. We can accomplish this with the following:

$$x_1 = \frac{v_1}{||v_1||} = (1,1,1) \times (\sqrt{\frac{1}{3}}) = (\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}) \text{ Now } ||x_1|| = ||\frac{v_1}{||v_1||}|| = \frac{||v_1||}{||v_1||} = 1$$

Although we did some operations v_1 and x_1, x_1 is still some linear combinations of v_1 still share the same span, and have the same dimension, and are the same subspace, as all we changed was the magnitude of the norm. That is to say

$$dim(span(v_1)) = 1 = dim(span(x_1)) \text{ and } span(x_1) \subset span(v_1) \text{ therefore } x_1 = v_1$$

Now that x_1 has an Euclidean norm of 1, we can begin to make the rest of our orthonormal basis, starting with x_2. To compute x_2 we will perform the following operaiton:

$$x_2 = x_2 - \langle x_1, v_2 \rangle \times x_1 = (2,1,1) - (4\sqrt{\frac{1}{3}}) \times (\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}) = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \tag{1}$$

Our x_2 is now orthogonal to x_1, but we need to normalize x_2 to ensure our basis is orthonormal

$$x_2 = \frac{x_2}{||x_2||} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \times \frac{1}{\sqrt{\frac{2}{3}}} = (\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}) \text{ and } ||x_2|| = ||\frac{x_2}{||x_2||}|| = \frac{||x_2||}{||x_2||} = 1 \tag{2}$$

x_2 is now normalized to have a Euclidian norm of 1. We can check that x_1 and x_2 are orthogonal to one another:

$$\langle x_1, x_2 \rangle = \langle x_2 - \langle x_1, v_2 \rangle \times x_1, x_1 \rangle = \langle x_2, x_1 \rangle - \langle x_1 \times \langle x_1, v_2 \rangle, x_1 \rangle = \langle x_2, x_1 \rangle - \langle x_2, x_1 \rangle \times \langle x_1, x_1 \rangle = 0$$

We have confirmed that $x_1 \perp x_2$ by checking that $\langle x_2, x_1 \rangle = 0$, and we made sure to normalize both x_1, x_2 so as of right now we have an orthonormal family of vectors. Now for the final piece, we must derive x_3 from v_3 by doing the following operation:

$$x_3 = v_3 - x_2\langle x_2, v_3\rangle - x_1\langle x_1, v_3\rangle \tag{3}$$

$$= (2,0,1) - ((\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}) \times \frac{\sqrt{6}}{2})) - ((\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \times \frac{1}{3} = (0, -\frac{1}{2}, \frac{1}{2}) \tag{4}$$

We now have our third orthogonal vector x_3, but we need to normalize it:

$$x_3 = \frac{x_3}{||x_3||} = (0, -\frac{1}{2}, \frac{1}{2}) \times \frac{1}{\frac{1}{\sqrt{2}}} = (0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \to ||x_3|| = \frac{||x_3||}{||x_3||} = 1$$

Now we have the following orthonormal basis:

$$\{x_1, x_2, x_3\} = \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Lets ensure that this is an orthonormal basis with the following:
x_3 is orthogonal to x_1, x_2

$$\langle x_3, x_2 + x_1\rangle = \langle x_3, x_2\rangle + \langle x_3, x_1\rangle = \langle v_3, x_2\rangle - \langle x_2, v_3\rangle \times \langle x_2, x_2\rangle + \langle v_3, x_1\rangle - \langle v_3, x_1\rangle \times \langle x_1, x_1\rangle = 0$$

We know that we have normalized all of our basis vectors to Euclidean norm equal to 1, so we have a valid orthonormal basis.

Since we have 3 lineraly independent vectors, the span of our new basis is equal to the span of the linearly independent basis we started with. That is to say:

$$dim(span(x_1, x_2, x_3)) = 3 = dim(span(v_1, v_2, v_3))$$

Lastly, we know that $\{x_1, x_2, x_3\}$ are linearly combinations of $\{v_1, v_2, v_3\}$ which means:

$$span(x_1, x_2, x_3) \subset span(v_1, v_2, v_3)$$

By lecture 1, since $dim(span(x_1, x_2, x_3)) = dim(span(v_1, v_2, v_3))$ and $span(x_1, x_2, x_3) \subset span(v_1, v_2, v_3)$ then the family of vectors x is equal to the family of vectors $= v$

2 Problem 5.2

Consider $U = span((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}))$ and $V = span((1,0,0,0),(0,1,0,0))$, two subspaces of $\mathbb{R}^4$.

a) Compute the canonical matrix $M_U \in \mathbb{R}^{4\times4}$ of orthogonal projection $P_U(\cdot)$ onto subspace U. What is the rank of M_U?

We can compute the canonical matrix $M_U \in \mathbb{R}^{4\times4}$ of the orthogonal projection $P_U(\cdot)$ onto the subspace U with the matrix projection formula of UU^T

$$M_U = UU^T = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \times (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \text{Row echelon: } \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ By doing } R_{2,3,4} - R_1$$

After we calculated the row echelon form, we can see that there is only 1 linearly independent column $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and three free variables. Therefore, $Dim(Ker(M_U)) = 3$ and $Rank(M_U) = Dim(Span(M_U)) = 1$

b) Compute the canonical matrix $M_V \in \mathbb{R}^{4\times4}$ of orthogonal projection $P_V(\cdot)$ onto subspace V. What is the rank of M_V?

We can calculate the canonimal matrix $M_V \in \mathbb{R}^{4\times4}$ of orthogonal projection $P_V(\cdot)$ onto subspace V with the same process we used in part 1.

$$M_V = VV^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can see clearly that we have 2 linearly independent columns in M_V $(1,0,0,0)$ and $(0,1,0,0)$. That means $Dim(Ker(M_V)) = 2$ and $Rank(M_V) = Dim(Span(M_V)) = 2$

c) Let $x = (1,2,3,4)$ in $\mathbb{R}^4$, compute $y = P_U \circ P_V(x)$ and $z = P_V \circ P_U(x)$. Do we have $y = z$?

Lets compute $y = P_U \circ P_V(x)$ first.

$$y = P_U \circ P_V(x) = M_U \circ M_V(x) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} .75 \\ .75 \\ .75 \\ .75 \end{pmatrix}$$

Now lets compute $z = P_V \circ P_U(x)$

$$z = P_V \circ P_U(x) = M_V \circ M_U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5 \\ 0 \\ 0 \end{pmatrix}$$

We can clearly see that $y = (.75, .75, .75, .75) \neq z = (2.5, 2.5, 0, 0)$

d) Compute the matrix products $M_U M_V$ and $M_V M_U$. Do M_U and M_V "commute", meaning do we have $M_U M_V = M_V M_U$. Can you give an intuition of why it is the case looking the definitions of U and V?

Lets firstly compute $M_U M_V$:

$$M_U M_V = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} .25 & .25 & 0 & 0 \\ .25 & .25 & 0 & 0 \\ .25 & .25 & 0 & 0 \\ .25 & .25 & 0 & 0 \end{pmatrix}$$

Lets now compute $M_V M_U$

$$M_V M_U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can see that the two matrix compositions do not commute, that is to say that $M_V M_U \neq M_U M_V$ This is because the two sub-spaces we are projecting into have no span that overlaps, and furthermore, have different dimensions. Therefore, the order in which we project matters. If we first project using P_V then project into P_U, we are essentially projecting a vector first into a two dimensional subspace, then a one dimensional line. If we project a vector into P_U then project it into P_V we are firstly projecting a vector onto a one dimensional subspace (a line), and then onto a two dimensional subspace (a hyper-plane). In each sequence of projections, we lose different amounts of information at different times due to the dimension of M_U and M_V and also we end up in different sub-spaces. Therefore, it makes intuitive sense that the matrices do not commute.

e) Considering now $U' = span\left((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)\right)$. Compute $M_{U'}$. Do we have $M_{U'} M_V = M_V M_{U'}$? Can you give an intuition why?

Firstly, lets compute M_U'

$$M_{U'} = U' U^{T\prime} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \times (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now lets compute $M_U' M_V$

$$M_U' M_V = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now lets compute $M_V M_U'$:

$$M_V M_U' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we do have $M_U' M_V = M_V M_U'$. In this case, the matrix multiplication is commutative because the Span of M_U' is a subset of the Span of M_V. Therefore, when we project a vector into the subspace spanned by M_U' with the transformation $P_{U'}(x)$, we are also projecting the vector into the span of M_V as well. If we try to project the same vector we projected into M_U' into M_V using $P_V(x)$, because we are already in the subspace, the projection of the vector will just be the vector itself, and we will essentially be scaling the vector by 1.

3 Problem 5.3

Consider L a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and denote by $\tilde{L} \in \mathbb{R}^{n\times n}$ its canonical matrix. Let $(u_1, \cdot, u_n)$ be any orthonormal basis of $\mathbb{R}^n$ and

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \in \mathbb{R}^{n\times n}$$

.

Show that $\tilde{L}' = U^\top \tilde{L} U$ computes the transformation of vectors in $\mathbb{R}^n$ using coordinates in the basis $(u_1, \cdots, u_n)$.

Our expression $\tilde{L}' = U^\top \tilde{L} U$ performs the following transformation: firstly, it takes in a vector with coordinates in the basis U, (let's call these coordinates x'), which are then transformed into coordinates into the standard basis (lets call these coordinates x). Then, the transformation L is applied to the coordinates with the standard basis, producing a vector, (lets call it y). Lastly, the vector y is converted into coordinates the basis U, by transforming the vector y by the U^T matrix to produce y'. To illustrate this transformation, consider the following.

Let U be any orthonormal basis of $\mathbb{R}^n$ and let $x' = U^T x$, and $y' = U^T y$ with x, y being two vectors written in the standard canonical basis of $\mathbb{R}^n$, and x', y' are the vectors x, y with their coordinates expressed in the basis U and $\{x, y, x', y'\} \in \mathbb{R}^n$

$$\tilde{L}'x' = U^T \tilde{L} U U^T x \text{ We begin by transforming a vector with coordinates in U basis} \tag{5}$$

$$\tilde{U}^T \tilde{L} U U^T x = U^T \tilde{L} x \text{ The coordinates get converted into the standard basis} \tag{6}$$

$$\tilde{U}^T \tilde{L} x = U^T y \text{ The transformation L is applied to } x \text{ to produce } y \tag{7}$$

$$\tilde{U}^T y = y' \text{ The vector } y \text{ has its coordinates converted to the basis } U \tag{8}$$

Therefore the transformation $\tilde{L}' = U^\top \tilde{L} U$ computes the transformation of vectors in $\mathbb{R}^n$ using coordinates in the basis $(u_1, \cdots, u_n)$

Alternatively, we can show this the other way around, a little bit more concisely. Lets use the same values of $x, y, x', y', U, \tilde{L}'$

$$\tilde{L}'x' = y' \tag{9}$$
$$\tilde{L}'x' = U^T y \tag{9}$$
$$\tilde{L}'x' = U^T \tilde{L} x \tag{9}$$
$$\tilde{L}'x' = U^T \tilde{L} U x' \tag{9}$$
$$\tilde{L}'x' = \tilde{L}'x'(9)$$

We see this works both ways, and that the transfromation $\tilde{L}' = U^\top \tilde{L} U$ computes the transformation of vectors in $\mathbb{R}^n$ using coordinates in the basis $(u_1, \cdots, u_n)$.

4 Problem 5.4

In this problem, we will see how to compress, by using a particular orthonormal basis called a "discrete cosine basis".

All the questions are in the jupyter notebook `DCT.ipynb` and have to be answered directly in the notebook. (Submit only a pdf export of your notebook: Print → Save as pdf)

You have to use `Python` and its library `numpy`. A useful command: `A @ B` : performs the matrix product of the matrix `A` with the matrix `B`.

Compressing images with Discrete Cosine Basis

In [1]:
```
%matplotlib inline
import numpy as np
import scipy.fftpack
import scipy.misc
import matplotlib.pyplot as plt
plt.gray()
```

```
<Figure size 432x288 with 0 Axes>
```

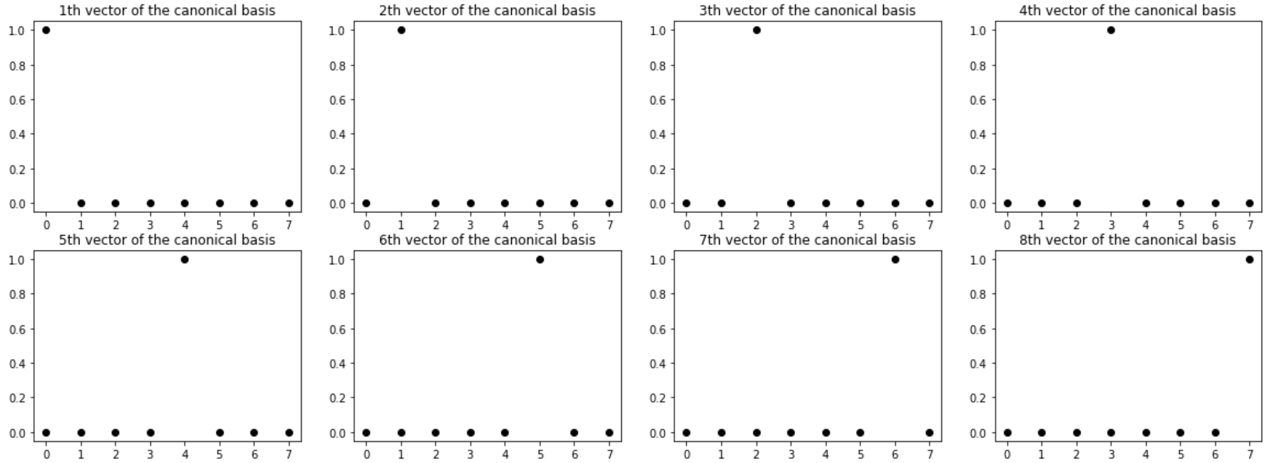
In [2]:
```
# Two auxiliary functions that we will use. You do not need to read them (but make sure

def dct(n):
    return scipy.fftpack.dct(np.eye(n), norm='ortho')

def plot_vector(v, color='k'):
    plt.plot(v,linestyle='', marker='o',color=color)
```

5.3.1 The canonical basis

The vectors of the canonical basis are the columns of the identity matrix in dimension n. We plot their coordinates below for $n = 8$.

In [3]:
```
identity = np.identity(8)
print(identity)

plt.figure(figsize=(20,7))
for i in range(8):
    plt.subplot(2,4,i+1)
    plt.title(f"{i+1}th vector of the canonical basis")
    plot_vector(identity[:,i])

print('\n Nothing new so far...')
```

```
[[1. 0. 0. 0. 0. 0. 0. 0.]
 [0. 1. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1.]]

 Nothing new so far...
```



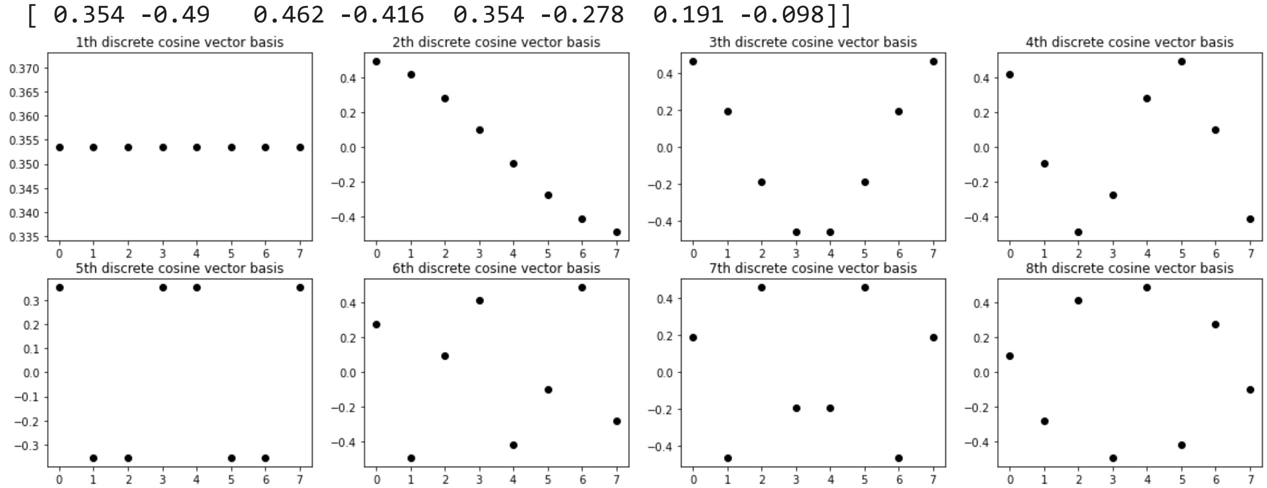
5.3.2 Discrete Cosine basis

The discrete Fourier basis is another basis of $\mathbb{R}^n$. The function `dct(n)` outputs a square matrix of dimension n whose columns are the vectors of the discrete cosine basis.

In [4]:

```
# Discrete Cosine Transform matrix in dimension n = 8
D8 = dct(8)
print(np.round(D8,3))

plt.figure(figsize=(20,7))

for i in range(8):
    plt.subplot(2,4,i+1)
    plt.title(f"{i+1}th discrete cosine vector basis")
    plot_vector(D8[:,i])
```

```
[[ 0.354  0.49   0.462  0.416  0.354  0.278  0.191  0.098]
 [ 0.354  0.416  0.191 -0.098 -0.354 -0.49  -0.462 -0.278]
 [ 0.354  0.278 -0.191 -0.49  -0.354  0.098  0.462  0.416]
 [ 0.354  0.098 -0.462 -0.278  0.354  0.416 -0.191 -0.49 ]
 [ 0.354 -0.098 -0.462  0.278  0.354 -0.416 -0.191  0.49 ]
 [ 0.354 -0.278 -0.191  0.49  -0.354 -0.098  0.462 -0.416]
 [ 0.354 -0.416  0.191  0.098 -0.354  0.49  -0.462  0.278]
 [ 0.354 -0.49   0.462 -0.416  0.354 -0.278  0.191 -0.098]]
```

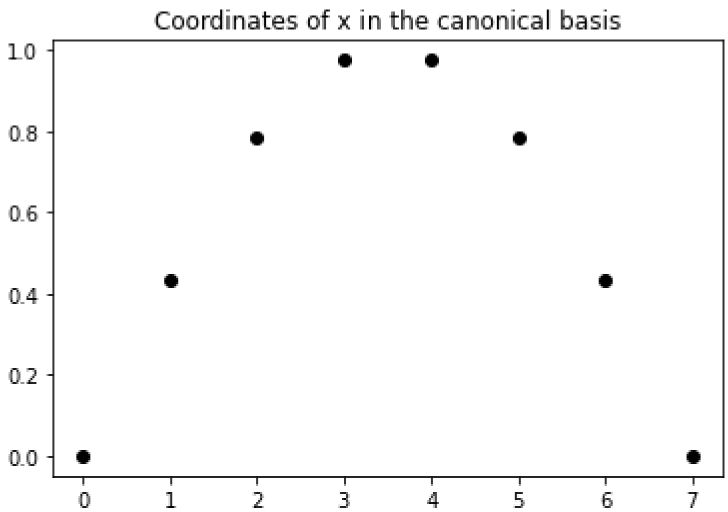


5.3 (a) Check numerically (in one line of code) that the columns of `D8` are an orthonormal basis of $\mathbb{R}^8$ (ie verify that the Haar wavelet basis is an orthonormal basis).

In [5]:
```
print(np.round(D8.T @ D8),2)
```

```
[[ 1. -0.  0. -0.  0. -0. -0.  0.]
 [-0.  1. -0.  0. -0. -0. -0.  0.]
 [ 0. -0.  1. -0.  0. -0.  0. -0.]
 [-0.  0. -0.  1. -0.  0. -0. -0.]
 [ 0. -0.  0. -0.  1. -0. -0. -0.]
 [-0. -0. -0.  0. -0.  1.  0. -0.]
 [-0. -0.  0. -0. -0.  0.  1.  0.]
 [ 0.  0. -0. -0. -0. -0.  0.  1.]] 2
```

In [6]:
```
# Let consider the following vector x
x = np.sin(np.linspace(0,np.pi,8))
plt.title('Coordinates of x in the canonical basis')
plot_vector(x)
```

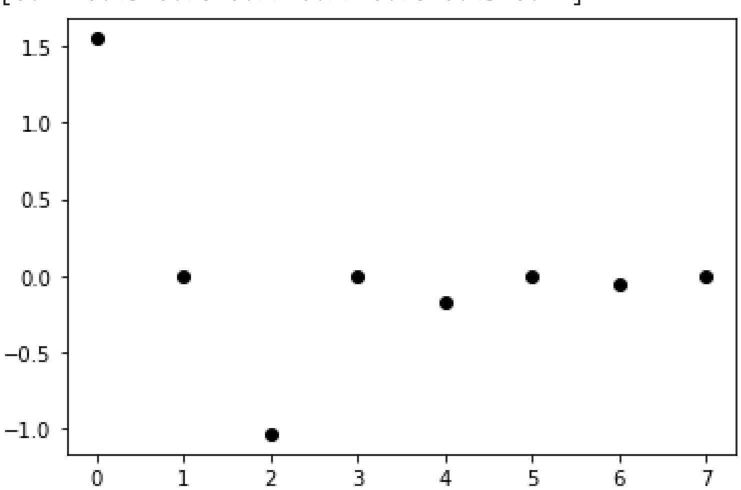


5.3 (b) Compute the vector $v \in \mathbb{R}^8$ of DCT coefficients of x. (1 line of code!), and plot them.

How can we obtain back x from v ? (1 line of code!).

In [7]:
```
v = D8.T @ x
plot_vector(v)
x = D8 @ v
print(np.round(x,2))
```

```
[0.   0.43 0.78 0.97 0.97 0.78 0.43 0.  ]
```



5.3.3 Image compression

In this section, we will use DCT modes to compress images. Let's use one of the template images of python.

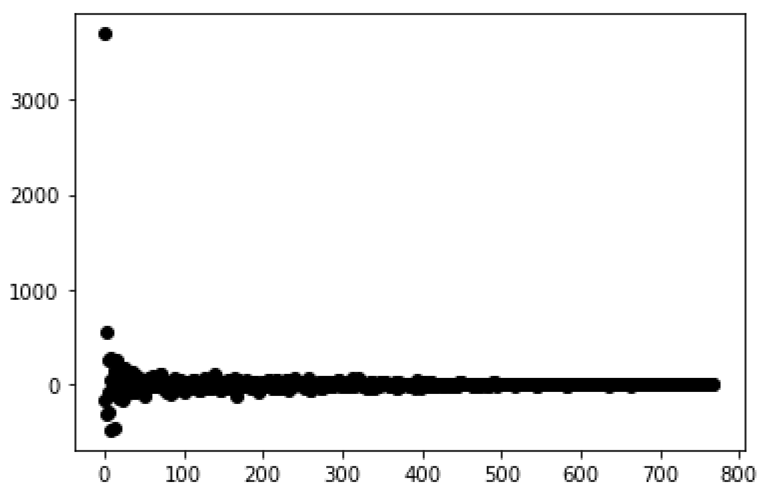
In [8]:
```
image = scipy.misc.face(gray=True)
h,w = image.shape
print(f'Height: {h}, Width: {w}')

plt.imshow(image)
```

```
Height: 768, Width: 1024
```

Out[8]:
```
<matplotlib.image.AxesImage at 0x1d762d404c0>
```

5.3 (c) We will see each column of pixels as a vector in $\mathbb{R}^{768}$, and compute their coordinates in the DCT basis of $\mathbb{R}^{768}$. Plot the entries of `x` , the first column of our image.

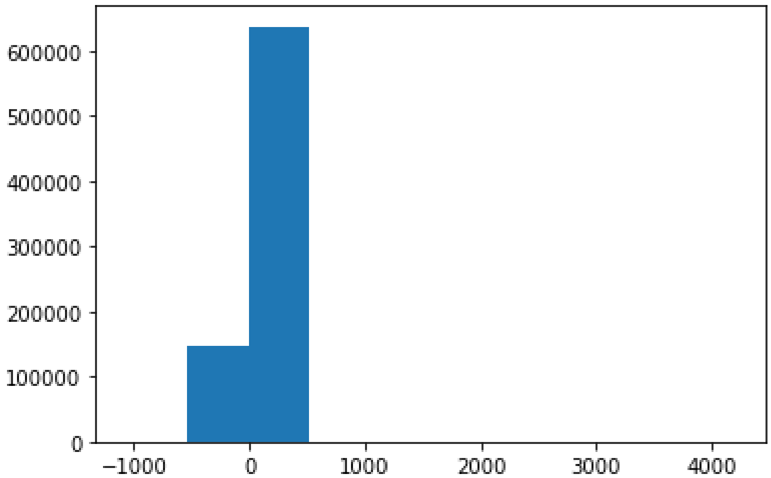
In [9]:
```
D768 = dct(768)
transformed = D768.T @ image
x = transformed[:,0]
plot_vector(x)
```



5.3 (d) Compute the 768 x 1024 matrix `dct_coeffs` whose columns are the dct coefficients of the columns of `image` . Plot an histogram of there intensities using `plt.hist` .

In [13]:
```
dct_coeffs = transformed
plt.hist(dct_coeffs.flatten())
```

Out[13]:
```
(array([6.36000e+02, 1.47189e+05, 6.37024e+05, 5.19000e+02, 4.00000e+01,
        0.00000e+00, 2.67000e+02, 2.41000e+02, 2.69000e+02, 2.47000e+02]),
 array([-1064.43123878,  -537.21884715,   -10.00645553,   517.20593609,
         1044.41832772,  1571.63071934,  2098.84311097,  2626.05550259,
         3153.26789421,  3680.48028584,  4207.69267746]),
 <BarContainer object of 10 artists>)
```



Since a large fraction of the dct coefficients seems to be negligible, we see that the vector `x` can be well approximated by a linear combination of a small number of discrete cosines vectors.

Hence, we can 'compress' the image by only storing a few dct coefficients of largest magnitude.

Let's say that we want to reduce the size by 98%: Store only the top 2% largest (in absolute value) coefficients of `wavelet_coeffs`.

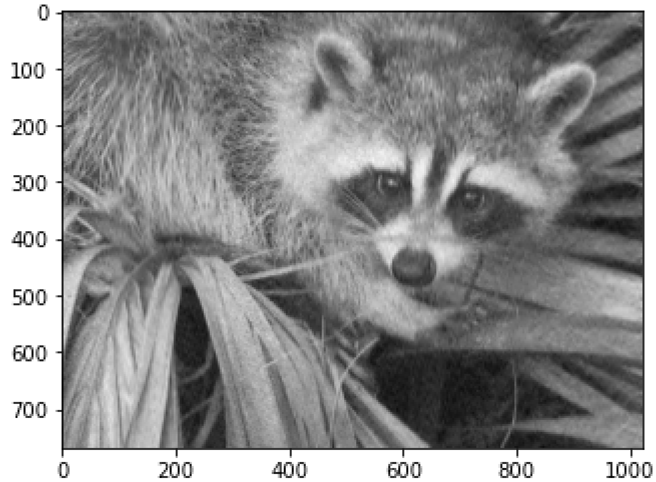
5.3 (e) Compute a matrix `thres_coeffs` who is the matrix `dct_coeffs` where about 97% smallest entries have been put to 0.

In [14]:
```
dct_coeffs[abs(dct_coeffs) < np.quantile(dct_coeffs, .97)] = 0
thres_coeffs = dct_coeffs
```

5.3 (f) Compute and plot the `compressed_image` corresponding to `thres_coeffs`.

In [20]:
```
compressed_image = D768 @ thres_coeffs
plt.imshow(compressed_image)
```

Out[20]: `<matplotlib.image.AxesImage at 0x1d7625c9430>`



In [ ]:

```

```

In [ ]:

```

```

5 Problem 5.5

Let S be a subspace of $\mathbb{R}^n$. We define the orthogonal complement of S by

$$S^{\perp}\big\{x \in^n \bigm| x \perp S\big\} = \big\{x \in^n \bigm| \forall y \in S,\, \langle x, y\rangle = 0\big\}.$$

a) Show that $S^{\perp}$ is a subspace of ^{n}.

b)Show that $\dim(S^{\perp}) = n - \dim(S)$.
c) Show that for any $u \in^n$, we can find $x \in S$ and $y \in S^{\perp}$ such that $u = x + y$.