Session 4: Norms, Inner Products, Orthogonality

1 Norms

Euclidian Norm

$$||x||_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

Definition of Norms:(let V be a vector space)

1. Homogeneity: $||\alpha v|| = |\alpha| \times ||v||$ for all $\alpha \in \mathbb{R}^n$ and $v \in V$
2. Positive Definitiness: if $||v|| = 0$ for some v then $v = 0$
3. Triangular Inequality: $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in V$

2 Inner Products

Definition of Inner Products (let V be a vector space)

1. Symmetry: $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$
2. Linearity: $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$ for all $v, u, w \in V$ and $\alpha \in \mathbb{R}$
3. Positive Definiteness: $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$

Proposition: If $\langle \cdot, \cdot\rangle$ is an inner product on V then

$$||v|| = \sqrt{\langle v, v\rangle}$$

is a norm on V. We say that the norm $|| \cdot ||$ is induced by the inner product $\langle \cdot, \cdot\rangle$

Cauchy-Schwartz Inequality Let $|| \cdot ||$ be the norm induced by the inner product $\langle \cdot, \cdot\rangle$ on the vector space V. Then for all $x, y \in V$:

$$|\langle x, y\rangle| \leq ||x|| \times ||y||$$

Moreover, there is equality if and only if x and y are linearly dependent (i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$)

3 Orthogonality

Definition of Orthogonality: Let V be a vector space and $\langle \cdot, \cdot\rangle$ be an inner product on V

- We say that vectors x, y are orthogonal if $\langle x, y\rangle = 0$. We write $x \perp y$
- We say that vector x is orthogonal to the set of vectors A if x is orthogonal to all of the vectors in A. We write $x \perp A$

For a family of vectors $\{v_1, \ldots, v_n\}$:

- The family is orthogonal if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$
- The family is orthonormal if all the vectors are orthogonal and all of the v_i have unit norm $||v_1|| = \cdots = ||v_k|| = 1$

Proposotion: A vector space of finite dimension admits an orthonormal basis
Proposition: Assume that $dim(V) = n$ and let $v_1, \ldots, v_n$ be an orthonormal basis of V. Then the coordinates of a vector $x \in V$ in the basis $v_1, \ldots, v_n$ are

$$x = \langle x, v_1 \rangle v_1 + \cdots + \langle x, v_n \rangle v_n$$

Pythagorean Theorem: Let $|| \cdot ||$ be the norm induced by $\langle \cdot, \cdot \rangle$ for all $x, y \in V$ we have:

$$x \perp y \longleftrightarrow ||x + y||^2 = ||x||^2 + ||y||^2$$

Orthogonal Projection Let S be the subspace of R^n The orthogonal projection of a vector x onto S is defined as the vector $P_S(x)$ in S that minimizes the distance to x:

$$P_S(x) = argmin||x - y|| \quad for\ y \in S$$

The distance from x to the subspace S is defined by:

$$d(x, S) = min||x - y|| = ||x - P_S(x)||$$

Proposition

Let S be a subspace of $\mathbb{R}^n$ and let $(v_1, \ldots, v_k)$ be an **orthonormal basis** of S. Then for all $x \in \mathbb{R}^n$,

$$P_S(x) = \langle v_1, x \rangle v_1 + \cdots + \langle v_k, x \rangle v_k.$$

Consequences

Let $V = \overset{\mathbb{R}^{n \times k}}{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix}}$ gather the orthonormal basis-vectors of the subspace S

Proposition

The orthogonal projection is given by $P_S(x) = VV^\top x$.

- P_S is a linear transform.
- $VV^\top$ is its matrix.

4 Proofs:

PROBLEM 2.2:

Recall that to prove that to sets are equal $(A = B)$ can prove that $A \subseteq B$ and $B \subseteq A$.

- Let $x \in \mathrm{Im}(A)$, by definition there exists $v \in \mathbb{R}^n$ such that

$$x = Av = \begin{pmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= v_1 c_1 + v_2 c_2 + \cdots + v_n c_n$$

So that $x \in \mathrm{Span}(c_1, \ldots c_n)$.

This shows that $\mathrm{Im}(A) \subseteq \mathrm{Span}(c_1, \ldots, c_n)$

- Let $x \in \mathrm{Span}(c_1, \ldots, c_n)$, so x is a linear combination of $c_1, \ldots c_n$: there exists $\alpha_1, \ldots \alpha_n$ in $\mathbb{R}$ such that

$$x = \alpha_1 c_1 + \cdots + \alpha_n c_n$$

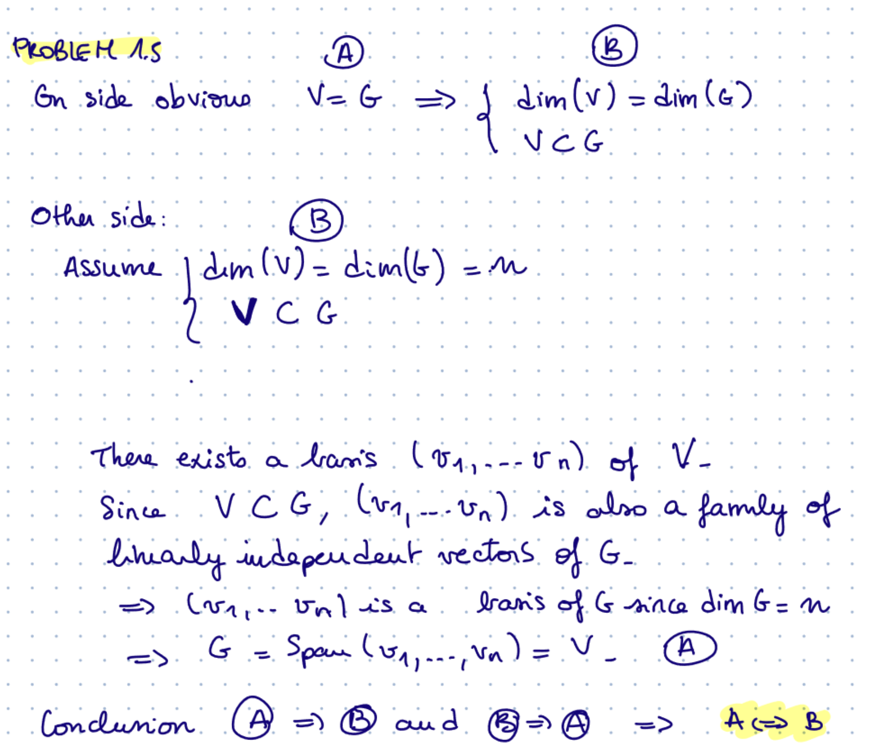
$$= \begin{pmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{pmatrix} \underbrace{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\text{call the vector } v \in \mathbb{R}^m}$$

$$= Av \implies x \in \mathrm{Im}(A)$$

This proves $\mathrm{Span}(c_1, \ldots, c_n) \subseteq \mathrm{Im}(A)$

- Overall conclusion: Since $\mathrm{Span}(c_1, \ldots c_n) \subseteq \mathrm{Im}(A)$ and $\mathrm{Im}(A) \subseteq \mathrm{Span}(c_1, \ldots c_n)$ we can conclude that $\mathrm{Im}(A) = \mathrm{Span}(c_1, \ldots c_n)$

Figure 1: Im(A) = span of columns proof


PROBLEM 1.5

On side obvious (A) $V = G \Rightarrow$ (B) $\begin{cases} \dim(V) = \dim(G) \\ V \subset G \end{cases}$

Other side: (B)

Assume $\begin{cases} \dim(V) = \dim(G) = n \\ V \subset G \end{cases}$

There exists a basis $(v_1, \dots v_n)$ of V.
Since $V \subset G$, $(v_1, \dots v_n)$ is also a family of linearly independent vectors of G.
$\Rightarrow (v_1, \dots v_n)$ is a basis of G since $\dim G = n$
$\Rightarrow G = \mathrm{Span}(v_1, \dots, v_n) = V$. (A)

Conclusion (A) $\Rightarrow$ (B) and (B) $\Rightarrow$ (A) $\Rightarrow$ A $\Leftrightarrow$ B

Figure 2: subspaces are the same example 2

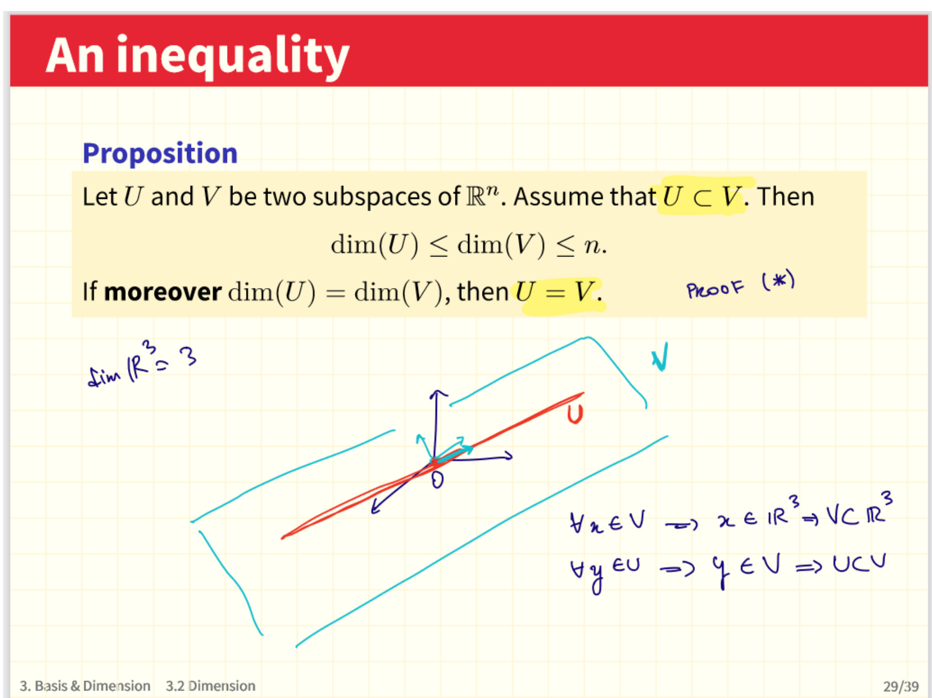


Figure 3: subspaces are the same

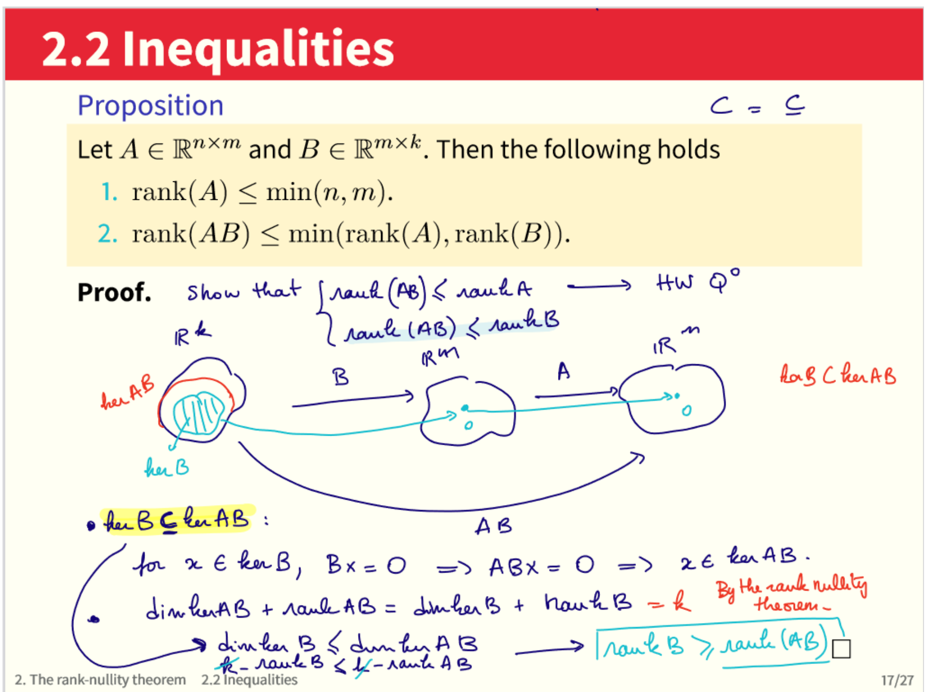


Figure 4: rank nullity

PROBLEM 3.4

(a) $\begin{cases} \text{rank}(A) = \dim \text{Im}(A) \\ \text{rank}(AB) = \dim \text{Im}(AB) \end{cases}$

yet $\text{Im}(AB) \subset \text{Im}(A) \Rightarrow \text{rank}(AB) \leq \text{rank}\, A$

(b) • trivial to show $\ker(L) \subset \ker(L^T L)$

• now for any $x \in \ker(L^T L)$ $\quad L^T L x = 0$

$$x^T L^T L x = 0$$
$$\Rightarrow \|Lx\| = 0$$
$$\Rightarrow Lx = 0$$
$$\Rightarrow x \in \ker(L)$$

so $\ker(L^T L) \subset \ker L$

Conclusion: $\ker(L^T L) = \ker L$.

(c) $\text{rank}(L^T L) \leq \text{rank}(L^T)$

$\Rightarrow m - \dim\ker(L^T L) \leq \text{rank}(L^T)$ rank nullity theorem

$\Rightarrow m - \dim\ker(L) \leq \text{rank}(L^T)$

$\Rightarrow \text{rank}(L) \leq \text{rank}(L^T)$

⊕ apply the same inequality to L^T:

$$\text{rank}(L^T) \leq \text{rank}(\underbrace{(L^T)^T}_{=L})$$

conclusion $\quad \text{rank}(L) \leq \text{rank}(L^T) \leq \text{rank}(L)$

$\hookrightarrow \text{rank}(L) = \text{rank}(L^T)$.

Figure 5: rank L = rank LT proof

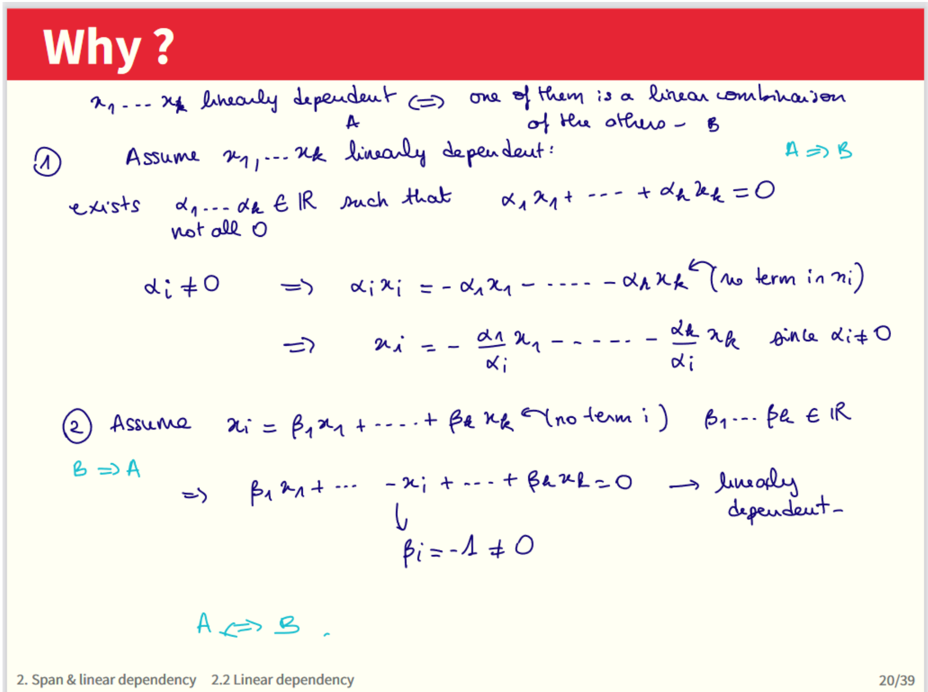


Figure 6: how to prove linear dependence.png

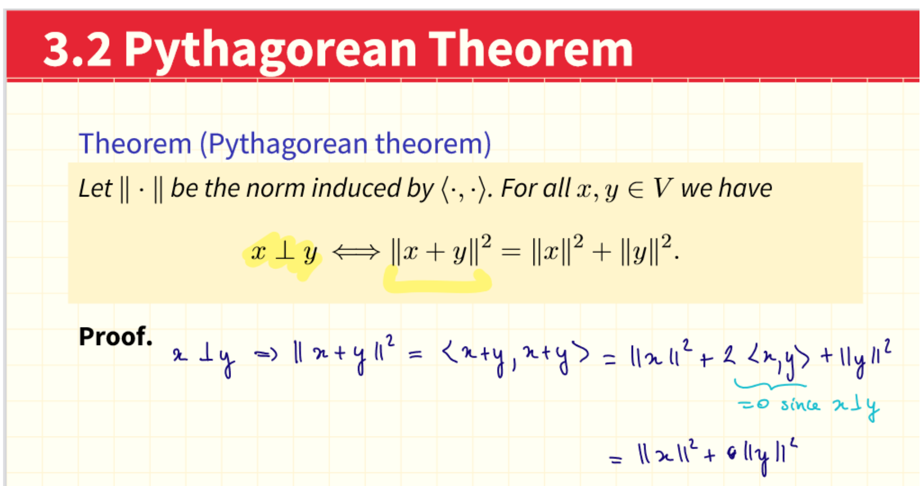


Figure 7: pythagorean theorem

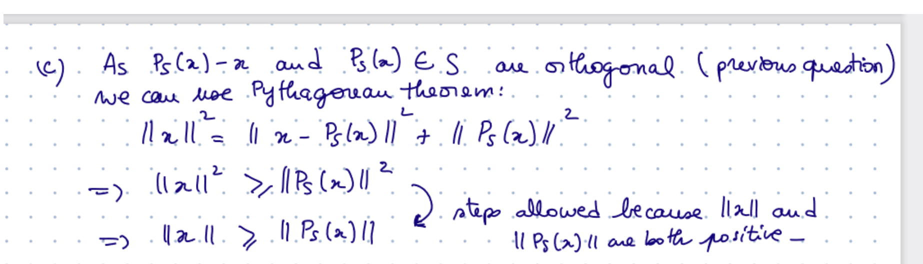


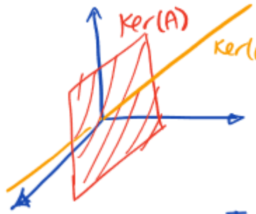
Figure 8: norm of proj x leq norm of x

6. Let $A \in \mathbb{R}^{n\times m}$. Assume the Euclidean inner product. Prove that $Ker(A)^{\perp} = Im(A^T)$.

row space

$A = [\equiv]$ $\quad A^T = [\,|||\,]$

rows of A



$Ker(A)^{\perp} = Im(A^T)$

Not intuitive!

(i) show $Im(A^T) \subseteq Ker(A)^{\perp} \iff \forall x \in Im(A^T) \Rightarrow x \in Ker(A)^{\perp}$

for $\forall x \in Im(A^T)$ $\exists y$ s.t. $x = A^T y$

$\forall z \in Ker(A)$, $\langle x, z\rangle = x^T z = (A^T y)^T z = y^T \underbrace{Az}_{0} = 0$

That is, x is orthogonal to all vectors in $Ker(A)$

$\Rightarrow x \in Ker(A)^{\perp}$

(ii) show $Ker(A)^{\perp} \subseteq Im(A^T)$ $\leftarrow$ Not easy to show

So use result from problem (5). The above is equivalent to $Im(A^T)^{\perp} \subseteq Ker(A) \iff \forall x \in Im(A^T)^{\perp} \Rightarrow x \in Ker(A)$

Show this!

Let $y \in Im(A^T)$ & $x \in Im(A^T)^{\perp}$ $\Rightarrow \langle x, y\rangle = x^T y = 0$

and $\exists v$ s.t. $y = A^T v$ $\Rightarrow \langle A^T v, x\rangle = (A^T v)^T x = v^T A x = 0$

not zero

$\Rightarrow Ax = 0$

$\Rightarrow x \in Ker(A)$

Figure 9: complement to ker(a) = im(a^t)

PROBLEM(4)

(a) as usual

(b) We can use the rank nullity theorem for the linear transformation corresponding to the orthogonal projector on S.

$$\begin{cases} Im(P_S) = S \\ Ker(P_S) = S^{\perp} \end{cases}$$

(c) For any $u \in \mathbb{R}^n$ $P_S(u) \in S$ and $u - P_S(u) \in S^{\perp}$

$$u = P_S(u) + (u - P_S(u))$$
$$= x + y. \qquad \blacksquare$$

Figure 10: dim s complement stuff (not that great tbh)

Let y be a vector of S, and $(v_1, \dots, v_n)$ an orthonormal basis of S.

We have,

$$y = \langle v_1, y\rangle v_1 + \dots + \langle v_n, y\rangle v_n$$

$$P_S(x) = \langle v_1, x\rangle v_1 + \dots + \langle v_n, x\rangle v_n$$

So that $\langle y, x\rangle = \langle \langle v_1, y\rangle v_1 + \dots + , x\rangle$

$$= \langle v_1, y\rangle\langle v_1, x\rangle + \dots \langle v_n, y\rangle\langle v_n, x\rangle$$

and $\langle y, P_S(x)\rangle = \underbrace{\langle \langle v_1, y\rangle v_1 + \dots , \langle v_1, x\rangle v_1 + \dots \rangle}$

would have all crossed terms: $\langle v_i, y\rangle \underbrace{\langle v_i, v_j\rangle}\langle v_j, x\rangle$

$= 0$ if $i \neq j$

$= 1$ if $= j$

because $v_1, \dots v_n$ orthonormal

so that only remains:

$$\langle y, P_S(x)\rangle = \langle v_1, y\rangle\langle v_1, x\rangle + \dots + \langle v_n, y\rangle\langle v_n, x\rangle$$
$$= \langle y, x\rangle$$

Figure 11: dot product of xy

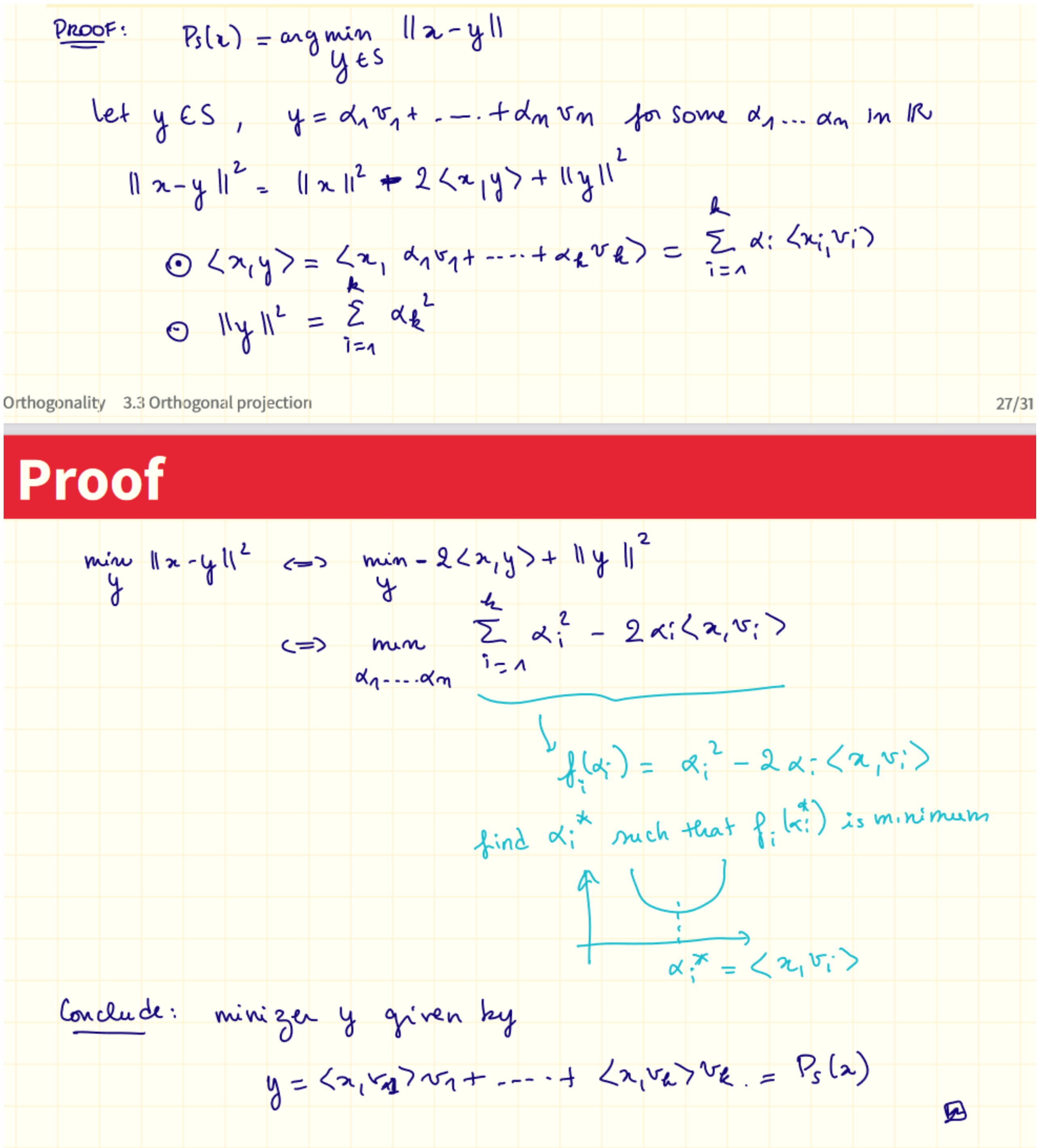


Figure 12: argmin projection proof