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On the Laplacian coefficients of signed graphs

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ABSTRACT

Let $\Gamma = (G, \sigma)$ be a signed graph, where G is its underlying graph and σ its sign function (defined on edges of G). A signed graph Γ', the subgraph of Γ, is its signed TU-subgraph if the signed graph induced by the vertices of Γ' consists of trees and/or unbalanced unicyclic signed graphs. Let $L(\Gamma) = D(G) - A(\Gamma)$ be the Laplacian of Γ. In this paper we express the coefficient of the Laplacian characteristic polynomial of Γ based on the signed TU-subgraphs of Γ, and establish the relation between the Laplacian characteristic polynomial of a signed graph with adjacency characteristic polynomials of its signed line graph and signed subdivision graph. As an application, we identify the signed unicyclic graphs having extremal coefficients of the Laplacian characteristic polynomial.

1. Introduction

Let $G = (V(G), E(G))$ be a graph of order $n = |V(G)|$ and size $m = |E(G)|$, and let $\sigma : E(G) \to \{+, -\}$ be a mapping defined on the edge set of G. Then $\Gamma = (G, \sigma)$ is a

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signed graph (or sigraph for short). The graph G is its underlying graph, while σ its sign function (or signature). Furthermore, it is common to interpret the signs as the integers $\{+1, -1\}$. Hence, sometimes signed graphs are treated as weighted graphs, whose (edge) weights belong to $\{1, -1\}$. An edge e is *positive* (*negative*) if $\sigma(e) = +$ (resp. $\sigma(e) = -$). If all edges in Γ are positive (negative), then Γ is denoted by $(G, +)$ (resp. $(G, -)$).

Most of the concepts defined for graphs are directly extended to signed graphs. For example, the degree of a vertex v in G (denoted by $deg(v)$) is also its degree in Γ. So $\Delta(G)$, the maximum (vertex) degree in G, also stands for $\Delta(\Gamma)$, interchangeably. Furthermore, if some subgraph of the underlying graph is observed, then the sign function for the subgraph is the restriction of the previous one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$ (note, all edges incident to v are deleted). If $U \subset V(G)$ then $\Gamma[U]$ or $G(U)$ denotes the (signed) induced subgraph arising from U, while $\Gamma - U = \Gamma[V(G) \setminus U]$. Sometimes we also write $\Gamma - \Gamma[U]$ instead of $\Gamma - U$. A cycle of Γ is said to be *balanced* (or *positive*) if it contains an even number of negative edges, otherwise it is *unbalanced* (or, *negative*). A signed graph is said to be *balanced* if all its cycles are balanced; otherwise, it is *unbalanced*. For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) *switching equivalent* to Γ. In fact, switching equivalent signed graphs can be considered as *switching isomorphic* graphs and their signatures are said to be equivalent. Observe also that switching equivalent graphs have the same set of positive cycles.

Given a (simple) graph G, then its *line graph* $\mathcal{L}(G)$ has as its vertex set the edge set of G, with two vertices in $\mathcal{L}(G)$ being adjacent whenever the corresponding edges of G have a common vertex; its *subdivision graph* $\mathcal{S}(G)$ is obtained from G by inserting into each edge of G a vertex of degree 2. If G is a signed graph, then the resulting graphs are signed graphs (to be defined later).

Simple graphs are widely studied in the literature by means of the eigenvalues of several matrices associated to graphs. Among them, the most common are the adjacency and Laplacian matrix. Given a graph G, $A(G) = (a_{ij})$ is its adjacency matrix if $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise; $L(G) = D(G) - A(G)$ is its Laplacian matrix, where $D(G)$ is the diagonal matrix of vertex degrees. Since recently, the so-called signless Laplacian matrix, defined as $Q(G) = A(G) + D(G)$, has attracted much attention in the literature, see for example [4,6–8]. For signed graphs we consider the analogous matrices. Let $e = ij$ be an edge of G joining vertices i and j. The adjacency matrix $A(\Gamma) = (a^\sigma_{ij})$ with $a^\sigma_{ij} = \sigma(ij)a_{ij}$ is called the signed adjacency matrix; $L(\Gamma) = D(G) - A(\Gamma)$ is the corresponding Laplacian matrix. Note, $L(G, +) = L(G)$, while $L(G, -) = Q(G)$.

In this paper we will consider both, the characteristic polynomial of the adjacency matrix and of the Laplacian matrix of a signed graph Γ. To avoid a confusion we denote by

$$\phi(\Gamma, x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,$$

and

$$\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma),$$

the adjacency characteristic polynomial and the adjacency eigenvalues, respectively. For the Laplacian matrix, we denote by

$$\psi(\Gamma, x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n,$$

and

$$\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) \geq 0,$$

the Laplacian characteristic polynomial and the Laplacian eigenvalues, respectively. We will see later that the Laplacian matrices of signed graphs are positive-semidefinite.

Finally, it is important to observe that switching equivalent signed graphs have similar adjacency and Laplacian matrices. In fact, any switching arising from U can be realized by a diagonal matrix $S_U = \mathrm{diag}(s_i)$ having $s_i = 1$ for each $i \in U$, and $s_i = -1$ otherwise. Hence, $A(\Gamma) = S_U A(\Gamma^U) S_U$ and $L(\Gamma) = S_U L(\Gamma^U) S_U$; in this case we say that the matrices are *switching similar*. Similar effect features with eigenvectors. When we consider a signed graph Γ, from a spectral viewpoint, we are considering its switching isomorphism class $[\Gamma]$.

Signed graphs appear in the context of social (signed) networks describing the relation between people – positive (or negative) edges arise for friends (resp. enemies). For a (possibly) complete bibliography on signed graphs, we refer the reader to [24]. For a notation not given here, we refer the reader to [2,3,25].

The remainder of the paper is organized as follows: in Section 2, we derive a formula for computing the coefficient of the Laplacian characteristic polynomial of a signed graph; in Section 3, we give yet another matrix-tree theorem; in Section 4, we identify the unicyclic graphs whose coefficient are extremal in moduli.

2. Relations between the spectra of signed graphs

In this section we first give formulas which relate the characteristic polynomial of the Laplacian of signed graphs, with the characteristic polynomials of the adjacency matrices of their

- ◇ line graphs, and
- ◇ subdivision graphs.

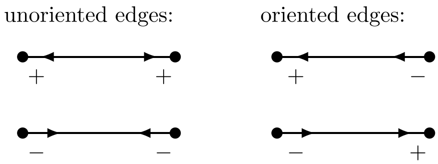

Fig. 1. Bidirected edges.

For this aim, recall first that the orientation of signed graphs is somewhat different from the standard orientation of edges in digraphs. Namely, instead of one arrow we can use two arrows assigned to edges. This yields to *bi-directed graphs*. More precisely, an *oriented signed graph* is an ordered pair $\Gamma_\eta = (\Gamma, \eta)$, where

$$\eta : V(G) \times E(G) \to \{-1, +1, 0\} \tag{1}$$

satisfies the following three conditions:

(i) $\eta(u, vw) = 0$ whenever $u \neq v, w$;
(ii) $\eta(v, vw) = +1$ (or -1) if an arrow at v is going into (resp. out of) v (cf. Fig. 1);
(iii) $\eta(v, vw)\eta(w, vw) = -\sigma(vw)$.

So we have that positive edges are *oriented* edges, while negative *unoriented* (see also Fig. 1). Thus each bi-directed graph is a signed graph. The converse is also true, but then one arrow (at any end) can be taken arbitrarily, while not the other arrow (in view of (iii) from above). At this place, we observe that in the literature there is also the notion of *mixed graphs*, namely graphs whose edges are either oriented or unoriented. On the other hand, it is clear that the theory of mixed graphs is the same as that of signed graphs, see, for example, [26].

The *incidence matrix* of Γ_η is the matrix $B_\eta = (b_{ij})$, whose rows correspond to vertices and columns to edges of G, with $b_{ij} = \eta(v_i, e_j)$ (here $v_i \in V(G)$, $E_j \in E(G)$). Usually, only Γ is given, and then we use the arbitrary orientation (as explained above). So each row of the incidence matrix corresponding to vertex v_i contains $deg(v_i)$ non-zero entries, each equal to $+1$ or -1. On the other hand, each column of the incidence matrix corresponding to edge e_j contains two non-zero entries, each equal to $+1$ or -1. Therefore, even in the case that multiple edges exist, we easily obtain that

$$B_\eta B_\eta^T = D(G) - A(\Gamma_\eta) = L(\Gamma_\eta), \tag{2}$$

where $D(G)$ is the diagonal matrix of vertex degrees of G. In particular, if $\Gamma = (G, +)$ (or $\Gamma = (G, -)$) then we obtain the standard Laplacian (resp. signless Laplacian) matrix of G. Needless to add, multiple edges, but not loops, if exist in the underlying graph are treated as usual edges. Observe also (see (2)) that $L(\Gamma_\eta)$ is positive-semidefinite, as already noted in the previous section.

It is also easy to obtain that

$$B_\eta^T B_\eta = 2I + A(\mathcal{L}(\Gamma_\eta)), \tag{3}$$

where $\mathcal{L}(\Gamma_\eta)$ is line graph of an oriented signed graph. It is noteworthy to say here that $\mathcal{L}(\Gamma_\eta)$ has $\mathcal{L}(G)$ as its underlying graph, while the sign of the edge ef $(e, f \in E(G))$ in the resulting signed graph is equal to $\sigma_l(ef) = \eta(w, e)\eta(w, f)$ (w is a common vertex of edges e and f in G). So this is rather a matrix than combinatorial definition of line graphs of signed graphs (tailored for the spectral graph theory). It is important to observe that in the literature (see, for example, [10,22]) there is an alternative version of signed line graphs, which differs from the above one up to signs.

We now consider the subdivision graphs. As with line graphs, we will now extend to signed graphs the well-known matrix representation of the adjacency matrix of subdivision graphs, which now reads in the block form

$$A(\mathcal{S}(\Gamma_\eta)) = \begin{pmatrix} O_n & B_\eta \\ B_\eta^\top & O_m \end{pmatrix}, \tag{4}$$

where O_t is the $t \times t$ zero matrix. It is easy to see that the underlying graph of $\mathcal{S}(\Gamma_\eta)$ is $\mathcal{S}(G)$, while the signature σ_s is defined by $\sigma_s(v_i e_j) = \eta_{ij}$ (note that $V(S(G)) = V(G) \cup E(G)$).

Remark 2.1. It is important to observe that any (random) orientation η to the edges of Γ gives rise to the same matrices $A(\Gamma_n) = A(\Gamma)$ and $L(\Gamma_\eta) = L(\Gamma)$, while the matrices $A(\mathcal{L}(\Gamma_\eta))$ and $A(\mathcal{S}(\Gamma_\eta))$ do depend on η. Let S be a ± 1-diagonal matrix such that $B'_\eta = B_\eta S$. It can be easily seen that $A(\mathcal{L}(\Gamma_{\eta'})) = SA(\mathcal{L}(\Gamma_{\eta'}))S$, so the two matrices are switching similar. For matrices $A(\mathcal{S}(\Gamma_{\eta'}))$ and $A(\mathcal{S}(\Gamma_\eta))$, the similar relation holds with the matrix $I_n \dot{+} S$ in the role of S (the symbol $\dot{+}$ denotes the direct sum of two matrices). From now on, the index η will be not specified anymore.

An example of subdivision and line graphs of a signed graph is depicted in Fig. 2. Note that positive edges are denoted by bold lines, while negative edges are dotted lines.

From (3) and (4) we obtain:

Theorem 2.2. *Let Γ be a signed graph of order n and size m, and let $\phi(\Gamma, x)$ and $\psi(\Gamma, x)$ be its adjacency and Laplacian characteristic polynomials, respectively. Then it holds*

1^o $\phi(\mathcal{L}(\Gamma), x) = (x + 2)^{m-n}\psi(\Gamma, x + 2)$,
2^o $\phi(\mathcal{S}(\Gamma), x) = x^{m-n}\psi(\Gamma, x^2)$.

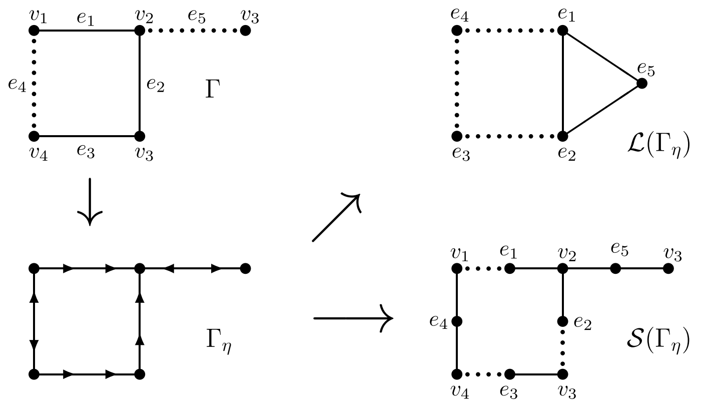

Fig. 2. A signed graph and the corresponding signed subdivision and line graphs.

Proof. First, 1^o immediately follows from (3).

Secondly, to prove 2^o, recall, that the formula for computing the determinant of a 2×2 block matrix reads

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D||A - BD^{-1}C|,$$

where A and D are square blocks and D is nonsingular. So we have,

$$\begin{aligned} \phi(\mathcal{S}(\Gamma), x) &= \begin{vmatrix} xI_n & -B \\ -B^\top & xI_m \end{vmatrix} = x^m|(xI_n) - B(xI_m)^{-1}B^\top| = x^m|\frac{1}{x}(x^2I_n - BB^\top)| \\ &= x^{m-n}|x^2I_n - L(\Gamma)| = x^{m-n}\psi(\Gamma, x^2). \end{aligned} \tag{5}$$

This completes the proof. □

Let G be a multi-graph with multiple edges appearing only as petals, where a petal is a pendant double edge (see, for example, [5, p. 6]). If p is a petal (so $p = \{e, e'\}$ and $e = e' = vv'$) let, say v', be its pendant vertex. Define an orientation of G as follows:

$$\eta(v, e) = \begin{cases} + & \text{if } v \text{ is not a pendant vertex of any petal, or otherwise,} \\ \pm & \text{if } e \text{ is the first (resp. second) edge in the petal.} \end{cases}$$

It turns now that the signed graph Γ that arises is this way is actually a root (signed) graph of the line graph $\mathcal{L}(\Gamma)$, defined by (3), or actually as well, of the generalized line graph. So we can say that the root graph of any generalized line graph is a signed multi-graph (for more details, see again [5]).

We conclude this section by providing a formula which expresses the adjacency characteristic polynomial of a signed graph in terms of the (signed) matching polynomial.

Prior to this we need to recall the coefficient theorem (Sachs formula) for the adjacency characteristic polynomial of signed graphs. Recall, elementary figures are the graphs K_2 and C_n (i.e. the complete graph of order the 2 and the cycle of order n); a basic figure is the disjoint union of elementary figures.

Theorem 2.3. *Let $\Gamma = (G, \sigma)$ and $\phi(\Gamma, x) = x^n + a_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + a_n$ be a signed graph and its adjacency characteristic polynomial, respectively. Then*

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} 2^{|c(B)|} \sigma(B), \tag{6}$$

where $\mathcal{B}_i$ is the set of basic figures on i vertices in G, $p(B)$ is the number of components of B, $c(B)$ the set of cycles in B, and $\sigma(B) = \prod_{C \in c(B)} \sigma(C)$.

A k-matching in a (simple) graph is a collection of k vertex-disjoint edges. The matching polynomial of a graph G, denoted by $\omega(G, x)$, is defined as

$$\omega(G, x) = \sum_{k=0}^{m(G)} (-1)^k m_k(G) x^{n-2k},$$

where $m(G)$ is the size of the largest matching in G, and $m_k(G)$[1] is the number of k-matchings in G. The following formula was first given in [11]; here we extend it to signed graphs. Note that it is a reformulation of Theorem 2.3, in fact a basic figure consists of disjoint cycles and matchings.

Theorem 2.4. *Let $\mathcal{C}$ be the set of 2-regular signed graphs of $\Gamma = (G, \sigma)$. Then*

$$\phi(\Gamma, x) = \omega(G, x) + \sum_{C \in \mathcal{C}} \sigma(C)(-2)^{p(C)} \omega(G - V(C), x),$$

where $p(C)$ and $\sigma(C)$ denote the number of cycles and the products of cycles signs of C, respectively.

3. Yet another matrix-tree theorem for signed graphs

One of the most important results on the Laplacian matrix of (simple) graphs is the *Matrix-Tree Theorem*, i.e. the formula which says that for any connected graph G, $\tau(G) = |L_i(G)|$, where $\tau(G)$ is its *complexity* (i.e. the number of spanning trees in G), while $L_i(G)$ is the principal minor of $L(G)$ obtained by deleting both the i-th row and column. This formula appeared first in [15]. On the other hand, $b_{n-1} = \sum_{i=1}^{n} |L_i(G)| = n\tau(G)$. More than a century later, the latter formula was extended by Kel'mans in [14]. It

[1] In sequel we assume that $m_0 = 1$ and $m_k = 0$ if $k < 0$.

includes all coefficients of the Laplacian characteristic polynomial, and is widely known as *Kel'mans formula.*

Theorem 3.1. *Let G be a simple graph. Then, the coefficients of the Laplacian characteristic polynomial are given by*

$$b_{n-k} = (-1)^{n-k} \sum_{F \in \mathcal{F}_k} \gamma(F), \tag{7}$$

where $k \geq 1$ (for $k = 0$, $b_n = 0$), $\mathcal{F}_k$ denotes the set of forests in G on k components, and $\gamma(F) = \prod_{i=1}^{k} |F_i|$ is the product of the orders of the components of the forest F.

Many authors have published variants of formula (7) covering any minor of $L(G)$. In 1982, the most general (and complicated) formula was given by Chaiken [1], which refers to weighted digraphs, signed graphs and voltage graphs. In the Chaiken formula minors are obtained by summing the weight of forest on some prescribed number of components, with much additional work on summands involved. Almost in the same time of Chaiken, Dedò published in [9] a slightly different variant of Kel'mans–Chaiken type formula for the signless Laplacian characteristic polynomial. In the Dedò's formula, the coefficients of the signless Laplacian characteristic polynomial of a graph G are obtained through the TU-subgraphs, that are subgraphs whose components are tree and/or odd unicyclic graphs.

Theorem 3.2. *(See [9].) Let G be a simple graph. Denote the signless Laplacian characteristic polynomial of G by $\varphi(G, x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$. Then*

$$c_i = (-1)^k \sum_{H \in \mathcal{H}_i} w(H) \quad (i = 1, 2, \ldots, n), \tag{8}$$

where $\mathcal{H}_i$ is the set of all TU-subgraphs of G on i edges, and $w(H) = 4^c \prod_{j=1}^{t} |T_j|$, with c being the number of odd unicyclic components and T_j's are trees, i.e. the remaining components of H (t in total).

In the remainder of the this section we extend (8) to signed graphs. The proof follows the method given by Cvetković et al. in [6]. For this aim, we will need the following preparatory lemmas. For the sake of completeness, we will briefly outline their proofs.

Lemma 3.3. *Let $\Gamma = (G, \sigma)$ be a signed graph and $\phi^{(k)}(x)$ be the k-th derivative of $\phi(x)$. Then*

$$\phi^{(k)}(\Gamma, x) = k! \sum_{|U|=k} \phi(\Gamma - U, x),$$

where the summation is taken over all k-vertex subsets U of G.

Proof. Note first that for any $U \subset V(G)$, $A(\Gamma - U)$ is equal to the principal submatrix $A(\Gamma)$ obtained by deleting the rows and columns indexed by the vertices of U. Hence, for $k = 1$ and by row-by-row differentiation, we have the following chain of equalities:

$$\phi'(\Gamma, x) = \sum_{i=1}^{n} |(xI_n - A(\Gamma))_i| = \sum_{i=1}^{n} |xI_{n-1} - A_i(\Gamma)| = \sum_{i=1}^{n} \phi(\Gamma - v_i).$$

The general case $k \geq 2$ can be proved by induction. $\square$

The following lemma is well-known and it has been proved in more general settings [21,23].

Lemma 3.4. *If B is the incident matrix of a connected signed graph $\Gamma = (G, \sigma)$ then*

$$\operatorname{rank}(B) = \begin{cases} n-1 & \textit{if } \Gamma \textit{ is balanced,} \\ n & \textit{if } \Gamma \textit{ is unbalanced.} \end{cases}$$

Proof. Let $B = (b_{ij})$, with rows $B_1, B_2, \dots, B_n$, and assume that the rows are linearly dependent, say

$$c_1B_1 + c_2B_2 + \cdots + c_nB_n = (0, 0, \dots, 0) \quad \text{and} \quad (c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0). \tag{9}$$

If two vertices v_s and v_t of G are joined by an edge e_j, then $b_{kj} = \pm 1$ if $k \in \{s, t\}$, while $b_{kj} = 0$ for all $k \neq s, t$. Next, each column of B contains two $+1$'s if the corresponding edge is negative; otherwise, $+1$ and -1 appear if the corresponding edge is positive. Consequently, from the latter consideration and from (9) we obtain $c_s = c_t$ if $\sigma(e_{st}) = 1$ and $c_s = -c_t$ if $\sigma(e_{st}) = -1$. Note that, due to connectivity, $|c_i| = |c_j| = c$ for every $i, j = 1, 2, \dots, n$, and the vector $(c_1, c_2, \dots, c_n)$ is unique up to a scalar factor. Further, we have that the vertices can be divided in two classes: those corresponding to $+c$ or $-c$. It is not difficult to check that the latter partitioning (one class might be empty) make vertices within the same class having only positive edges, while negative edges are those going from one class to the other one. The latter implies that Γ is balanced. $\square$

Lemma 3.5. *Let $\Gamma = (G, \sigma)$ be a signed graph of order n and size m, and let $\mathcal{L}(\Gamma)$ be the line graph of Γ. Let $m_\Gamma(a)$ be the multiplicity of the scalar a in the spectrum of $A(\Gamma)$. Then*

$$m_{\mathcal{L}(\Gamma)}(-2) = \begin{cases} m-n+1 & \textit{if } \Gamma \textit{ is balanced,} \\ m-n & \textit{if } \Gamma \textit{ is unbalanced.} \end{cases}$$

Proof. According to (3), $2I_m + A(\mathcal{L}(\Gamma))$ is positive-semidefinite. Hence, $\lambda_m(\mathcal{L}(\Gamma)) \geq -2$. On the other hand, the multiplicity of -2 is equal to the nullity of the matrix $B^\top B$.

The nullity of $B^\top B$ is given by $m - \operatorname{rank}(B)$, and the rest of the proof follows by Lemma 3.4. $\square$

Corollary 3.6. *Let $\Gamma = (G, \sigma)$ be a signed graph and $\mathcal{L}(\Gamma)$ be its signed line graph. We have that -2 is not an eigenvalue of $A(\mathcal{L}(\Gamma))$ if and only if G is a tree or Γ is an unbalanced unicyclic graph.*

The following lemma is well-known and can be found in several places (see, for example, [21]).

Lemma 3.7. *Let B be the incidence matrix of a signed graph $\Gamma = (G, \sigma)$, and let B_i be the submatrix of B obtained by deleting its i-th row. If Γ is a tree then $|B_i| = \pm 1$, and if Γ is an unbalanced unicyclic graph then $|B| = \pm 2$.*

Lemma 3.8. *If Γ is a connected signed graph on m edges then*

$$(-1)^m \phi(\mathcal{L}(\Gamma), -2) = \begin{cases} 4 & \text{if } \Gamma \text{ is an unbalanced unicyclic graph,} \\ m+1 & \text{if } \Gamma \text{ is a tree,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In view of Corollary 3.6, if Γ is not a tree nor an unbalanced unicyclic graph, then $\phi(\mathcal{L}(\Gamma), -2) = 0$. In the remaining two cases, let B be the incidence matrix of Γ, so that $(-1)^m \phi(\mathcal{L}(\Gamma), -2) = |B^\top B|$. If Γ is an unbalanced unicyclic graph, then, by Lemma 3.7, $\det(B) = \pm 2$, and consequently $(-1)^m \phi(\mathcal{L}(\Gamma), -2) = 4$. Finally, if Γ a tree of order n, then, by Lemma 3.7, $|B_i| = \pm 1$, where B_i is the matrix obtained from B by deleting the i-th row. Hence, by Binet–Cauchy formula, $|B^\top B| = \sum_{i=1}^n |B_i^\top B_i| = n$.

This completes the proof. $\square$

We now define TU-subgraphs in the context of signed graphs. A *signed TU-subgraph* of a signed graph Γ is a signed subgraph whose components are trees or unbalanced unicyclic graphs, namely the unique cycle contains an odd number of negative edges. If H is a signed TU-subgraph, then $H = T_1 \cup T_2 \cup \cdots \cup T_r \cup U_1 \cup U_2 \cup \cdots \cup U_s$, where the T_i's are trees and the U_i's are unbalanced unicyclic graphs. The weight of the signed TU-subgraph H is defined as $w(H) = 4^s \prod_{i=1}^r |T_i|$.

And we are now in position to give Dedo's formula for signed graphs.

Theorem 3.9. *Let Γ be a signed graph. Denote the Laplacian characteristic polynomial of Γ by $\psi(\Gamma, x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n$. Then*

$$b_i = (-1)^i \sum_{H \in \mathcal{H}_i} w(H) \quad (i = 1, 2, \ldots, n),$$

where $\mathcal{H}_i$ denotes the set of signed TU-subgraphs of Γ containing i edges.

Proof. As in [4], from (2) and by the MacLaurin development we obtain

$$
\begin{aligned}
\psi(\Gamma, x) &= x^{n-m}\phi(\mathcal{L}(\Gamma), x-2) \\
&= x^{n-m}\sum_{k=0}^{m}\phi^{(k)}(\mathcal{L}(\Gamma), -2)\frac{x^k}{k!} \\
&= x^{n-m}\sum_{k=m-n}^{m} x^k \frac{1}{k!}\phi^{(k)}(\mathcal{L}(\Gamma), -2),
\end{aligned}
$$

due to the fact that -2 is an eigenvalue of $A(\mathcal{L}(\Gamma))$ with multiplicity at least $m-n$. In view of Lemma 3.3 we arrive to

$$
\psi(\Gamma, x) = x^{m-n}\sum_{k=m-n}^{m} x^k \sum_{|U|=k}\phi(\mathcal{L}(\Gamma) - U, -2). \tag{10}
$$

Clearly, $\mathcal{L}(\Gamma) - U$ is still a signed line graph. From Lemma 3.5, signed line graphs have -2 as an eigenvalue unless all components are line graphs of trees or unbalanced unicyclic graphs. Hence, non-zero contributions in (10) come from signed line graphs whose roots are signed TU-subgraphs. Therefore, by Lemma 3.8, we have

$$
\sum_{|S|=k}\phi(\mathcal{L}(\Gamma) - S, -2) = (-1)^{m-k}\sum_{H\in\mathcal{H}_{m-k}} w(H).
$$

Next, from (10) we have

$$
\psi(\Gamma, x) = x^{m-n}\sum_{k=m-n}^{m} x^k(-1)^{m-k}\sum_{H\in\mathcal{H}_{m-k}} w(H),
$$

and by putting $i = m-k$ we obtain

$$
\psi(\Gamma, x) = \sum_{i=0}^{n} x^{n-i}(-1)^i \sum_{H\in\mathcal{H}_i} w(H).
$$

This completes the proof. □

Corollary 3.10. *Let (G, σ) and (G, σ') be two signed graphs, on the same underlying graph G. Let $\psi((G,\sigma), x) = \sum_{i=1}^{n} b_i x^{n-i}$ and $\psi((G,\sigma'), x) = \sum_{i=1}^{n} b'_i x^{n-i}$. If the girth of G is g then $b_i = b'_i$ for $i = 1, 2, \ldots, g-1$.*

If $\Gamma = (G, +)$ or $\Gamma = (G, -)$ we can immediately deduce the coefficients for the Laplacian and signless Laplacian characteristic polynomials of G.

4. Laplacian coefficients of signed unicyclic graphs

As an application of our results from the previous sections, we consider now a classic problem in the spectral graph theory, related to the coefficients of characteristic polynomials. So far the identification of graphs whose coefficients (in moduli) are extremal, either minimal or maximal, are known for trees and unicyclic graphs, for both Laplacian and signless Laplacian spectra. Here we will extend these results to Laplacian of signed graphs.

During the last ten years, the problem of identifying the graphs (within the graphs of fixed order and size) whose coefficients of the Laplacian or signless Laplacian characteristic polynomials are extremal, has attracted attention of several researchers. It is introduced by Gutman and Pavlović in [12], where the authors conjectured that in the class of n-trees the extremal graphs are the path P_n and the star S_n. This conjecture was proved by Gutman and Zhou in [27], and independently, by Mohar in [18], who introduced two graph perturbations under which the moduli of coefficients monotonically change. Mohar's transformations were extended to unicyclic graphs by Ilić and Stevanović in [20]. Several authors have used these ideas to achieve similar results for the Laplacian and signless Laplacian of graphs with prescribed order and size. Here, we consider the same problem but in the context of Laplacian of signed graphs. The proofs given for simple graphs are now adapted to signed graphs. Our aim is not only to reproduce such proofs technically, but rather to show that in the context of Laplacian of signed graphs, everything turns to be more natural.

The following graphs will be considered in the present section: the n-path P_n, the n-star S_n, the n-cycle C_n, S_n^+ as the star with an additional edge, the lollipop graph $L_{g,n}$ consisting of the cycle C_g with a pendant path of length $n-g$ attached at some vertex of C_g, and $C_{g,n}$ consisting of the cycle C_g with $n-g$ pendant edges attached at some vertex of C_g. Note that $S_n^+ = C_{3,n}$. By $g(G) = g$ we denote the girth of the graph G, and $\mathcal{U}_{n,g}$ is the class of signed unicyclic graphs of girth g.

Recall, given $\Gamma = (G, \sigma)$, let

$$\psi(\Gamma, x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n.$$

As shown in Section 3, the coefficients b_i's have alternating sign according to the parity of n. Further on, let

$$\beta_i = |b_i| \quad (i = 1, \ldots, n).$$

The result of Zhou and Gutman reads:

Theorem 4.1. *(See [27].) Let T be a tree of order n and let $\beta_i(T)$ be the modulus of the i-th coefficient of its Laplacian characteristic polynomial. Then for any $i = 1, 2, \ldots, n$ it holds*

$$\beta_i(S_n) \leq \beta_i(T) \leq \beta_i(P_n).$$

The above theorem also holds for signed trees, since not only their Laplacian and signless Laplacian spectra coincide, but also their Laplacian spectra for signed graphs. So, the first non-trivial step is to consider signed unicyclic graphs. Unicyclic graphs have already been considered in the context of Laplacian and signless Laplacian. As we will soon see, the corresponding results for signed graphs can be obtained without too much efforts.

Theorem 4.2. *(See [20].) Let U be a unicyclic graph of order n and let $\beta_i(U)$ be the modulus of the i-th coefficient of its Laplacian characteristic polynomial. Then for any $i = 1, \ldots, n$ it holds*

$$\beta_i(S_n^+) \leq \beta_i(U) \leq \beta_i(C_n).$$

For the signless Laplacian, an analogous result has been given in [16], but it is more complicated than the Laplacian counterpart. Let κ_n be the unique real root of the polynomial

$$f_n(x) = 3x^3 + (7 - 10n)x^2 + 2(6n^2 - 11n + 8)x - (4n^3 - 6n^2 - 10n + 24).$$

In [13] it is proved that κ_n is an irrational number. By taking into account this fact, we can restate Theorem 1.1 from [16] as follow:

Theorem 4.3. *(See [16].) If U is a unicyclic graph of order n then*

$$\begin{aligned} \beta_i(U, -) &\leq \beta_i(C_n, -), \quad \textit{for } i < \kappa_n, \\ \beta_i(U, -) &\leq \beta_i(L_{3,n}, -), \ \textit{for } n \geq i > \kappa_n. \end{aligned}$$

In contrast to maximizers, the analogue result for minimizers (see also [16]) is more complicated and reads:

Theorem 4.4. *(See [16].) Let U be a unicyclic graph of order n, and let $(U_n, -)$ be the signed graph such that $\beta_i(U_n, -) \leq \beta_i(U, -)$. Then the following holds*

$$U_n = \begin{cases} C_{4,n} & i = 2, 3, \ldots, n-4, \\ C_{4,n} & i = n-3 \textit{ and } 5 \leq n \leq 24, \\ S_n^+ & i = n-3 \textit{ and } n \geq 25, \\ C_{4,n} & i = n-2 \textit{ and } 5 \leq n \leq 8, \\ S_n^+ & i = n-2 \textit{ and } n \geq 9, \\ S_n^+ & i = n-1. \end{cases}$$

The above theorem sounds rather complicated, but in view of Lemma 4.5 (see below), it becomes more understandable. Indeed, for a unicyclic graph we have just two non-switching-equivalent signatures, $\sigma = +$ and $\bar{\sigma}$, the latter meaning that the unique cycle is unbalanced. On the other hand, from Theorem 3.9, unbalanced cycles strictly contribute to the magnitude of the coefficients. So, we deduce the following lemma:

Lemma 4.5. *Let $\Gamma = (G, \sigma)$ be a signed graph whose underlying graph G is unicyclic. Then for any $i = 1, 2, \ldots, n$ it holds*

$$\beta_i(G, +) \leq \beta_i(G, \bar{\sigma}).$$

The equality holds if and only if $i < g$, with g being the girth of G.

Now it is evident that the signed unicyclic graph with largest β_i's are unbalanced with signature $\bar{\sigma}$, while those minimizing β_i's have signature $\sigma = +$. In Theorem 4.4, from the point of view of signed graphs, the authors are comparing two signed graphs $(C_{4,n}, -)$ and $(S_n^+, -)$, but $(C_{4,n}, -)$ is switching equivalent to $(C_{4,n}, +)$ so the coefficients are not "pumped up" by unbalanced cycles. The latter does not apply to $(S_n^+, -)$ which, according to Lemma 4.5, has larger coefficients than $(S_n^+, +)$.

In fact, by combining Theorem 4.2 and Lemma 4.5, we obtain:

Theorem 4.6. *Let Γ be a signed graph whose underlying graph is unicyclic of order n, and let $\beta_i(\Gamma)$ be the modulus of its i-th coefficient of the Laplacian characteristic polynomial. Then for any $i = 1, \ldots, n$ it holds*

$$\beta_i(S_n^+, +) \leq \beta_i(U, \sigma),$$

with equality if and only if $i = 1$, or $i = n - 1$, $g(U) = 3$ and $\sigma = +$, or $i = n$ and $\sigma = +$, or (U, σ) is switching equivalent to $(S_n^+, +)$.

Proof. If $i = 1$ we know that the coefficients just depend on the order of the underlying graph, so they are the same. If $i = n - 1$, we have that $\beta_{n-1} = ng$ for $\sigma = +$, and $\beta_{n-1} = ng + 4\sum_{e \in F(U)} |T_e|$, where $F(U)$ is the set of $n - g$ edges that do not lie on the cycle of U and T_e is the tree component obtained by deleting an edge $e \in F(U)$. If $i = n$ then β_n equals either 0 if $\sigma = +$, or 4 if $\sigma = \bar{\sigma}$. For the remaining cases, in view of Lemma 4.5 we can restrict ourself to graphs with positive signatures, but then Theorem 4.2 applies, and the proof follows. □

For the maximizers, we can restrict ourselves to signed unicyclic graphs with unbalanced signature $\bar{\sigma}$. In fact, the proofs given in [16] (see also [17]), can be used in the context of the Laplacian of signed graphs. However, it is necessary to adapt some arguments and formulas to this more general case.

First we extend to signed graphs the π-transformation introduced by Mohar in [18]:

Theorem 4.7. *Let $\Gamma = (G, \sigma)$ a connected, non-trivial and rooted signed graph with root u_0. Denote by $\Gamma(p, q)$ such a signed graph having two hanging paths at u_0 of length p and q, respectively. If $p, q \geq 1$ then for any $i = 2, \dots, n-2$ it holds*

$$\beta_i(\Gamma(p, q)) < \beta_i(\Gamma(p + q, 0)).$$

Proof. Recall that the signs on the edges of hanging trees do not affect the spectrum. Let $P = \{u_1, u_2, \dots, u_p\}$ and $Q = \{v_1, v_2, \dots, v_q\}$ $(p, q \geq 1)$ be the two hanging paths at u_0. Clearly $\Gamma' = \Gamma - u_0v_1 + u_pv_1$. Consequently any signed TU-subgraph (sTU-graph, for short) of Γ which does not contain the edge u_0v_1 is a sTU-graph of Γ' as well. For any sTU-graph H of Γ containing u_0v_1, let $H' = H - u_0v_1 + u_pv_1$, so that H' is the corresponding sTU-graph of Γ'. Finally, let $\overline{\mathcal{H}}_i \subseteq \mathcal{H}_i$ (see Theorem 3.2) be the set of sTU-graphs of Γ on i edges containing the edge u_0v_1.

Hence,

$$\beta_i(\Gamma') - \beta_i(\Gamma) = \sum_{H \in \mathcal{H}_i} [w(H') - w(H)] = \sum_{H \in \overline{\mathcal{H}}_i} [w(H') - w(H)].$$

We are done if we prove that the above sum is positive. We first show that it is non-negative. Observe that H consists of one or more components, but if u_0 and u_p are in the same component, then $w(H) = w(H')$. So we shall assume that u_0 and u_p are in two different components, say R and S, respectively. Now, let a be the number of vertices of Q that belong to R, b the number of vertices of P that belong to R together with u_0, c the number of vertices of $R \setminus \{u_0\}$ that belong to G, and d the number of vertices of S. In view of assumptions, we have $a, b, d \geq 1$ and $c \geq 0$.

If R is an unbalanced unicyclic component of H, then $w(H) = 4dN$, where $N = w(H - (R \cup S))$, or $N = 1$ whenever the latter graph is empty. Similarly, for H' we get $w(H') = 4(a + d)N$. So $w(H') - w(H) = 4aN \geq 1$.

If R is a tree component of H, then $w(H) = (a+b+c)dN$ and $w(H') = (b+c)(a+d)N$ which leads to $w(H') - w(H) = a(b + c - d)N$. In order to show that the latter equality is non-negative, we partition $\overline{\mathcal{H}}_i$ in subsets $\mathcal{T}$, such that in each class all sTU-graphs share the same components with the exception of R and S, for which the vertices of P being in the components R and S is a fixed number $b + d = K$, with $2 \leq K \leq p + 1$. In a few words, we can say that sTU-graphs in the same class have the same gap (along P) of $p - K$ vertices. Note also that $1 \leq b,\ d \leq K - 1$. Under the above notation we have that

$$\sum_{H \in \mathcal{T}} [w(H') - w(H)] = \sum_{b=1}^{K-1} a(b + c - d)N = aN \sum_{b=1}^{K-1} (2b - K + c) = acN(K - 1).$$

Since, all above quantities are non-negative we have proved that $\beta_i(\Gamma') - \beta_i(\Gamma) \geq 0$. In fact, the above quantity is 0 if and only if $c = 0$, namely, there is no sTU-graph with u_0 and u_p in different components and $R - Q \neq \{u_0\}$. This last possibility always happens for sTU-graphs on 1 edge and n edges, and with $n-1$ edges if $\Gamma = (G, \sigma)$ is balanced (sTU-graphs on $n-1$ edges are just spanning trees of G). In all other cases, we always have a sTU-graph, satisfying the above conditions, with $d \geq 0$. In fact, take any spanning tree T of G (recall that G is connected and non-trivial) and consider $T - u_0u_1$, the latter is sTU-graph on $n-2$ edges with $d \geq 1$. For $2 \leq i \leq n-2$ we can appropriately remove edges in order to get a suitable sTU-graph with $d \geq 1$. Finally, for $i = n-1$ if Γ is unbalanced we have at least a sTU-graph consisting of an unbalanced unicyclic graph (passing through u_0) and the path P.

This completes the proof. □

The above proof was based on the proof of Theorem 4.1 from [16]. In the same paper a more general result was proved, related to a variant of the π-transformation (called *generalized π-transformation*), which can be proved similarly for the Laplacian of signed graphs.

In view of Lemma 4.5 and Theorem 4.7, we have that for maximizers the unique cycle is negative, and all hanging trees are paths. Let $C_g(P_{r_1}, P_{r_2}, \ldots, P_{r_g})$ be the graph consisting of a cycle C_g, labelled in a natural way, having at each vertex v_i a pendant path of length $r_i \geq 0$. The next step is to show that the maximizer cannot have two non-trivial pendant paths. The proof again follows [16] and it relies on the fact the coefficients of the Laplacian of a signed graph can be obtained by counting the matchings in the subdivision graph.

Recall that $G - U = G - [U]$. The theorem below is a direct consequence of (5) and Theorem 2.4:

Theorem 4.8. *Let $\Gamma = (G, \sigma)$ be a signed graph whose underlying graph is unicyclic of order n and girth g. Let m_k ($k \leq \frac{n}{2}$) be the number of k-matchings in G. Then for any $i = 1, \ldots, n$ it holds*

$$\beta_i(\Gamma) = m_i(\mathcal{S}(G)) - \sigma(\Gamma)\, 2m_{i-g}(\mathcal{S}(G) - C_{2g}). \tag{11}$$

The next lemma is taken from [16]. Its proof is computationally too involved.

Lemma 4.9. *(See [16].) Let G be a unicyclic graph of order n and girth g. If $G \neq L_{g,n}$ then for any $k = 2, \ldots n-1$ it holds*

$$m_k(\mathcal{S}(G)) < m_k(\mathcal{S}(L_{g,n})).$$

Lemma 4.10. *Let u and v be two adjacent vertices of G. Then*

$$m_k(G) = m_k(G - uv) + m_{k-1}(G - u - v).$$

Lemma 4.11. *(See [16].) For any non-negative integer $k \leq \frac{n+m}{2}$, we have*

$$m_k(P_n) = \binom{n-k}{k}, \qquad m_k(P_n \cup P_m) = \sum_{l=0}^{r} (-1)^l \binom{n+m-k-l}{k-l},$$

where $r = \min\{k, n, m\}$.

Lemma 4.12. *Let $\Gamma = (G, \sigma)$ be a signed graph whose underlying graph is unicyclic of order n and girth g. If $\Gamma \neq (L_{g,n}, \bar{\sigma})$ and $G \neq C_n$, then for any $i = 2, \ldots, n-1$ it holds*

$$\beta_i(\Gamma) < \beta_i(L_{g,n}, \bar{\sigma}).$$

Proof. We may assume that $\Gamma = (C_g(P_{r_1}, P_{r_2}, \ldots, P_{r_g}), \bar{\sigma})$. In view of Lemma 4.8 (note all cycles are unbalanced) we have

$$\beta_i(G, \bar{\sigma}) = m_i(\mathcal{S}(G)) + 2m_{i-g}(\mathcal{S}(G) - C_{2g}),$$
$$\beta_i(L_{g,n}, \bar{\sigma}) = m_i(\mathcal{S}(L_{g,n})) + 2m_{i-g}(\mathcal{S}(L_{g,n}) - C_{2g}).$$

From Lemma 4.9 we have $m_i(\mathcal{S}(G)) < m_i(\mathcal{S}(L_{g,n}))$.

Also, note that $\mathcal{S}(G) - C_{2g} = P_{2r_1} \cup P_{2r_2} \cup \ldots \cup P_{2r_g}$, $\mathcal{S}(L_{g,n}) - C_{2g} = P_{2n-2g}$, and $\sum_{i=k}^{g} 2r_k = 2n - 2g$. Obviously, $m_k(P_n \cup P_m) < m_k(P_{n+m})$, consequently, $m_{i-g}(\mathcal{S}(G) - C_{2g}) < m_{i-g}(\mathcal{S}(L_{g,n}) - C_{2g})$.

This completes the proof. □

From the above lemma we have now restricted the candidates for maximizers to unbalanced lollipop graphs and signed cycles (which are a kind of degenerate lollipops). Also, all basic lemmas, analogous to those from [16] used for proving Theorem 4.4, are now at disposal for proving its signed counterpart.

Theorem 4.13. *Let $\Gamma = (G, \sigma)$ be a signed graph whose underlying graph is unicyclic of order n. Then it holds*

$$\beta_i(\Gamma) \leq \beta_i(C_n, \sigma), \quad \text{for } i < \kappa_n,$$
$$\beta_i(\Gamma) \leq \beta_i(L_{3,n}, \bar{\sigma}), \quad \text{for } \kappa_n < i \leq n.$$

However, we omit here the proof since it is essentially the same as its counterpart from [16]. Instead, we will show how the coefficients of unbalanced lollipops are changing if the length of the cycle in $L_{g,n}$ is shortened by one. In fact, such result can be considered as an ordering of the signed graphs with maximal Laplacian coefficients.

Theorem 4.14. *Let $\Gamma = (L_{n,g}, \bar{\sigma})$ and $\Gamma' = (L_{n,g-1}, \bar{\sigma})$ two signed lollipops on $n > g \geq 4$ vertices. Then, provided that $n \geq \frac{3}{2}(g-1)$, for any $i = 1 \dots n$ it holds*

$$\beta_i(\Gamma) \leq \beta_i(\Gamma'),$$

with equality if and only if $i < g$ or $i = n$.

Proof. We compute the values of $\beta_i(L_{g,n}, \bar{\sigma})$ by looking to the matchings in the subdivision graph. In view of Theorem 4.4 and Lemmas 4.10 and 4.11, we get

$$\begin{aligned}\beta_i(L_{n,g}, \bar{\sigma}) &= m_i(L_{2g,2n}) + 2m_{i-g}(P_{2n-2g}) \\ &= m_i(P_{2n}) + m_{i-1}(P_{2n-2g} \cup P_{2g-2}) + 2m_{i-g}(P_{2n-2g}) \\ &= \binom{2n-i}{i} + \sum_{l=0}^{r(g)} (-1)^l \binom{2n-1-i-l}{i-1-l} + 2\binom{2n-g-i}{i-g}, \end{aligned} \tag{12}$$

where $r(g) = \min\{i-1, 2n-2g, 2g-2\}$. If we consider the Laplacian coefficients of $(L_{g-1,n}, \bar{\sigma})$, we have that the first summand is the same, the second has the same sequence but with a, possibly, different coefficient in the sum (note, $|r(g-1) - r(g)| \leq 2$) and the third summand appears from $i = g - 1$. It is not difficult to check that

$$\begin{aligned}\binom{2n-g-i+1}{i-g+1} - \binom{2n-g-i}{i-g} &= \binom{2n-g-i+1}{2n-2i} - \binom{2n-g-i}{2n-2i} \\ &= \frac{(2n-g-i)(2n-g-i-1)\cdots(i-g+2)}{(2n-2i-1)!} \\ &= N(n,g,i) > 0.\end{aligned}$$

The second summand needs more computations, and we need to calculate both $r(g-1)$ and $r(g)$. With a little effort, the reader can check that we have the following cases:

a) $n \leq 2g - 3$:
- $\diamond$ $r(g) = r(g-1) = i - 1$, when $i \leq 2n - 2g + 1$,
- $\diamond$ $r(g) = 2n - 2g < 2n - 2g + 1 = r(g-1)$, when $i = 2n - 2g + 2$,
- $\diamond$ $r(g) = 2n - 2g < 2n - 2g + 2 = r(g-1)$, when $i \geq 2n - 2g + 3$.

b) $n = 2g - 2$:
- $\diamond$ $r(g) = r(g-1) = 2g - 4$;

c) $n \geq 2g + 1$:
- $\diamond$ $r(g) = r(g-1) = i - 1$, when $i \leq 2g - 3$,
- $\diamond$ $r(g) = 2g - 3 > 2g - 4 = r(g-1)$, when $i = 2g - 2$,
- $\diamond$ $r(g) = 2g - 2 > 2g - 4 = r(g-1)$, when $i \geq 2g - 1$.

From the above cases we obtain

$$\sum_{l=0}^{r(g-1)}(-1)^l\binom{2n-1-i-l}{i-1-l}-\sum_{l=0}^{r(g)}(-1)^l\binom{2n-1-i-l}{i-1-l}$$
$$=\begin{cases} M(n,g,i)=-\dfrac{(2g-i-3)(2g-i-4)\cdots(2g+i-2n-1)}{(2n-2i-1)!}, \\ \quad n\le 2g-3, i\ge 2n-2g+2; \\ M'(n,g,i)=\dfrac{(2n-2g-i+1)(2n-2g-i)\cdots(i-2g+3)}{(2n-2i-1)!}, \\ \quad n\ge 2g-1, i\ge 2g-2; \\ 0, \quad \text{otherwise.} \end{cases}$$

So in most cases we have that $\beta_i(L_{g-1,n},\bar{\sigma})\ge\beta_i(L_{g,n},\bar{\sigma})$. However, when $n\le 2g-3$ and $i\ge 2n-2g+2$ we have that $\beta_i(L_{g-1,n},\bar{\sigma})-\beta_i(L_{g,n},\bar{\sigma})=M(n,g,i)+2N(n,g,i)$ and the second member is negative for $2n<3g-3$ and $2n-2g+2\le i<n-1$.

This completes the proof. □

5. Conclusion

In this paper we have put forward more arguments to the be-liveness that the theory of Laplacian of signed graphs is a right generalization of both theories which address Laplacian and signless Laplacian spectrum of simple graphs. Needless to add, we also believe that many researches will support us in this respect.

Finally, we were aware that most of our results, as also pointed by the referee, could be generalized a bit further, so to include, say the voltage/gain graphs. For more details on such possibilities, the readers are referred to [19], and the references therein.

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