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On the definiteness of graph Laplacians with negative weights: Geometrical and passivity-based approaches

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***Abstract*— The positive semidefiniteness of Laplacian matrices corresponding to graphs with negative edge weights is studied. Two alternative proofs to a result by Zelazo and Bürger (Theorem 3.2), which provides upper bounds on the magnitudes of the negative weights in terms of effective resistances within which to ensure definiteness of the Laplacians, are provided. Both proofs are direct and intuitive. The first employs purely geometrical arguments while the second relies on passivity arguments and the laws of physics for electrical circuits. The latter is then used to establish consensus in multi-agent systems with generalized high-order dynamics. A numerical example is given at the end of the paper to highlight the result.**

I. INTRODUCTION

Consensus in multi-agent systems has been a very popular topic in the control community in last couple of decades, attracting wide research attention following the seminal works [1], [2]. In the discrete-time setting, stochastic matrices are an essential element in establishing consensus. In continuous-time, it is Laplacian matrices that play a similar role. Various continuous-time linear consensus algorithms for single and double integrator multi-agent systems rely on definiteness of the Laplacians corresponding to the underlying graphs [3], [4]. For instance, in the case of single integrator consensus, the dynamics of the agents can be described by

$$\dot{x}(t) = -Lx(t), \tag{1}$$

where L denotes the Laplacian. Hence, the dynamics rely on the spectral properties of L.

It is well-known that the Laplacian of an undirected graph with positive weights is positive semidefinite. Indeed, it belongs to the class of negated Metzler matrices, for which many useful properties are known [5]. If, in addition, the graph is connected, then the Laplacian has only one eigenvalue at 0 with the corresponding eigenvector being the n-vector 1_n with all elements equal to 1. In this case, the solution of (1) converges to $\alpha 1_n$ as t tends to infinity [4], [6], where the value of $\alpha \in \mathbb{R}$ depends on the initial conditions. In contrast, when a graph contains negative edge weights, properties of the corresponding Laplacian are not entirely well understood. One motivation to consider negative weights is that they can be used to model antagonistic interactions in a network [7], which may represent perturbations in consensus problems, inhibition and repression of expressions in gene regulatory networks, etc. Thus, negative edge weights represent "local" inhibitions between respective nodes, and positive weights represent "local" activations.

Recently in [8], the authors gave a necessary and sufficient condition on the semidefinitenesss of Laplacian matrices under certain non-cyclic assumptions on the negative weights (Theorem 3.2). In particular, they showed that the Laplacian matrix is positive semidefinite if and only if the absolute values of the negative edge weights are less than or equal to the reciprocals of the corresponding effective resistances. However, the proof in [8] is rather involved and difficult to follow. One contribution of the present paper is that we provide alternative proofs to this main result in [8] which we believe are more direct and intuitive. Specifically, the first proof uses geometrical arguments and the second resistive circuit theory.

One consequence of the new proofs of the Zelazo-Bürger result is that they provide insights on the passivity of a system that is defined by a graph Laplacian. In particular, we utilize this in the context of a unifying framework for robust synchronization based on integral quadratic constraints [9] that has been put forth in [10], [11]. In fact, combining the new insights on passivity with the results in [10], [11] we establish a consensus result in multi-agent systems with individual high-order passive dynamics, *while allowing* the presence of negative edge weights in the network topology (Theorem 4.4).

The paper is organized as follows. Section II establishes notation and preliminaries. Section III details two alternative proofs of the Zelazo-Bürger result, a geometrical proof and a passivity-based proof. The problem of consensus between agents with passive dynamics, albeit allowing negative interaction through graph edges, is considered in Section IV and a corresponding robust consensus result is established. The paper concludes with a numerical example (Section V).

II. NOTATION

A. Matrices

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively. $j\mathbb{R}$ denotes the imaginary axis, $\mathbb{C}_+$ (resp. $\bar{\mathbb{C}}_+$) the open (resp. closed) right half complex plane, and $\|\cdot\|$ the Euclidean norm. Given an $A \in \mathbb{C}^{m\times n}$ (resp. $\mathbb{R}^{m\times n}$), $A^* \in \mathbb{C}^{n\times m}$ (resp. $A^T \in \mathbb{R}^{n\times m}$) denotes its complex conjugate transpose (resp. transpose). a_{ij} denotes the (i,j) entry of A

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and $A^\dagger$ the Moore-Penrose pseudoinverse. Let $\oplus$ denote the direct sum of matrices; thus $\bigoplus_{i=1}^n A_i := A_1 \oplus A_2 \oplus \ldots \oplus A_n$ may be thought of as a block diagonal matrix with diagonal blocks the A_i's. I_n denotes the identity matrix of size $n \times n$. Denote by $\{e_i\}_{i=1}^n$ the set of canonical basis vectors of $\mathbb{R}^n$, i.e. e_i has 1 as its i^{th} entry and zeros everywhere else. Also define $e_{ij} := e_i - e_j$.

B. Function spaces and transfer functions

We consider the usual Lebesgue spaces

$$\mathbf{L}_2^n := \left\{ f : [0,\infty) \to \mathbb{R}^n \mid \|f\|_2^2 := \int_0^\infty \|f(t)\|^2 \, dt < \infty \right\},$$
$$\mathbf{L}_\infty := \{ \phi : j\mathbb{R} \to \mathbb{C} \mid \|\phi\|_\infty := \operatorname{ess\,sup}_{\omega\in\mathbb{R}} |\phi(j\omega)| < \infty \},$$

and the Hardy space

$$\mathbf{H}_\infty := \left\{ \phi \in \mathbf{L}_\infty \;\middle|\; \begin{array}{l} \phi \text{ has analytic continuation into } \mathbb{C}_+ \\ \text{with } \sup_{s\in\mathbb{C}_+} |\phi(s)| = \|\phi\|_\infty < \infty \end{array} \right\}.$$

We denote by $\mathbf{C} \subset \mathbf{L}_\infty$ the class of continuous functions on $j\bar{\mathbb{R}}$, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is the compactified real line. We denote by $\mathbf{R}$ the set of proper real-rational transfer functions and by $\mathbf{RH}_\infty := \mathbf{R} \cap \mathbf{H}_\infty$ [12]. For technical reasons we use a slightly stronger notion of passivity than usual, in that we call passive a function $H(s) \in \mathbf{R}^{n\times n}$ such that

(i) $H(s)$ has no poles in $\Re[s] \geq 0$ with a possible exception the origin,
(ii) $H(j\omega) + H(j\omega)^* \geq 0$ for all $\omega > 0$,
(iii) if $s = 0$ is a pole of $H(s)$, then it is a simple pole and the residue matrix $\lim_{s\to 0} sH(s)$ is positive semidefinite.

The above definition of passivity is more restrictive than the usual one in [13] where, besides the origin, additional simple poles on $j\mathbb{R}$ with positive residue are also allowed.

C. Graph theory

A *weighted graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ consists of a set of nodes $\mathcal{V} = \{1, \ldots, n\}$, a set of edges $\mathcal{E} = \{\epsilon_1, \ldots, \epsilon_m\} \subset \mathcal{V} \times \mathcal{V}$ where $\epsilon_k = (i,j) \in \mathcal{E}$ if node i is connected to node j, and a weight function $\mathcal{W} : \mathcal{E} \to \mathbb{R}\backslash\{0\}$ that corresponds each edge to a scalar weight. By a slight abuse of notation, sometimes we use $k \in \mathcal{E}$ instead of $\epsilon_k \in \mathcal{E}$ and, for notational simplicity, we abbreviate the weight of edge (i,j) as $\mathcal{W}(i,j)$. We do not restrict these weights to assume only positive values. Let W be a diagonal matrix with all edge weights on the diagonal, i.e., $w_{kk} := \mathcal{W}(\epsilon_k)$, $\epsilon_k \in \mathcal{E}$. The cardinality $\operatorname{card}(\mathcal{E})$ of $\mathcal{E}$ is assumed finite.

A graph is *undirected* if $(i,j) \in \mathcal{E}$ then $(j,i) \in \mathcal{E}$. A *path* on $\mathcal{G}$ of length N is an ordered set of distinct vertices $\{n_0, n_1, \ldots, n_N\}$ such that $(n_i, n_{i+1}) \in \mathcal{E}$ for all $i \in \{0, 1, \ldots, N-1\}$. The path is called a *cycle* if $n_0 = n_N$. An undirected graph is said to be *connected* if any two nodes in $\mathcal{V}$ are connected by a path. The weighted *adjacency matrix* $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined by $a_{ij} = \mathcal{W}(i,j)$ if $(i,j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Note that A is symmetric for an undirected graph. The weighted *Laplacian* matrix $L = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ is defined as

$$\ell_{ii} := \sum_{j=1, j\neq i}^n a_{ij}, \qquad \ell_{ij} := -a_{ij},\ i \neq j.$$

Note that L has a zero eigenvalue corresponding to the vector of ones $1_n \in \mathbb{R}^n$. In the case where all the weights are positive, the multiplicity of the zero eigenvalue of L is equal to the number of connected components in the graph [14]. This is not true in general in the presence of negative weights. The Laplacian matrix can be factorised as $L = DWD^T$, where $D = [d_{ik}] \in \mathbb{R}^{n\times m}$ is the oriented *incidence matrix*. It is defined by associating an orientation to every edge of the graph: for each $\epsilon_k = (i,j) = (j,i)$, one of i,j is defined to be the head and the other tail of the edge:

$$d_{ik} := \begin{cases} +1 & \text{if } i \text{ is the head of } \epsilon_k \\ -1 & \text{if } i \text{ is the tail of } \epsilon_k \\ 0 & \text{otherwise.} \end{cases}$$

D. Graphs as electrical networks

Insightful observations can be made by associating graphs with electrical networks [15], [16], [17]. Given a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, associate with each edge ϵ_k with a resistor of resistance value $r_k := 1/w_k$, where w_k denotes the weight on the edge ϵ_k. The resistance matrix $R \in \mathbb{R}^{m\times m}$ is defined to be the inverse of the weight matrix W, i.e. $R = W^{-1}$. Note that by definition, w_k represents the electrical conductance of each edge.

Assume that the entries of $\boldsymbol{c} \in \mathbb{R}^n$ denote the amount of current entering at each vertex from external independent sources and that the sum of all such currents is zero, i.e., $\boldsymbol{c}^T 1_n = 0$ which means that there is no charge "building" on the network. Denote by $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{i} \in \mathbb{R}^m$ the voltage at the various nodes and current at edges, respectively. Kirchoff's current law asserts that the difference between the outgoing and the incoming currents through the edges of every vertex equals external input current injection, i.e.,

$$D\boldsymbol{i} = \boldsymbol{c}. \tag{2}$$

On the other hand, Ohm's law states that the current through each edge is given by the potential difference across the associated vertices divided by the resistance of that edge, i.e.

$$WD^T\boldsymbol{v} = \boldsymbol{i}. \tag{3}$$

Combining the last two equalities yields

$$DWD^T\boldsymbol{v} = L\boldsymbol{v} = \boldsymbol{c}. \tag{4}$$

The solutions of this equation are of the form $\boldsymbol{v} = L^\dagger\boldsymbol{c} + \alpha 1_n$, $\alpha \in \mathbb{R}$, since the graph is assumed connected and where $L^\dagger$ denotes the pseudo-inverse. Since $e_{ij}^T 1_n = 0$, the voltage difference across the edge $\epsilon = (i,j)$ is given by

$$e_{ij}^T\boldsymbol{v} = e_{ij}^T L^\dagger \boldsymbol{c}. \tag{5}$$

Accordingly, we have the following definition.

Definition 2.1 ([15]): Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with

Laplacian L and two nodes $i, j \in \mathcal{V}$, the *effective resistance* across the nodes is

$$\mathcal{R}_{ij}(\mathcal{G}) := e_{ij}^T L^\dagger e_{ij},$$

and the *effective conductance* across the nodes is

$$\mathcal{W}_{ij}(\mathcal{G}) := \mathcal{R}_{ij}(\mathcal{G})^{-1}.$$

It is seen from (5) that $\mathcal{R}_{ij}(\mathcal{G})$ is also the voltage across (i,j) when a unit current is injected at i and taken out from j, i.e. $\boldsymbol{c} = e_{ij}$. In this case, the total power dissipated through the network is also $\mathcal{R}_{ij}(\mathcal{G})$ (in consistent units).

III. Positive semidefiniteness of graph Laplacians

This section provides two alternative proofs to the main result in [8] on the positive semidefiniteness of Laplacians for weighted graphs (Theorem 3.2).

Given a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, we separate the positive and negative edges into $\mathcal{E}_+$ and $\mathcal{E}_-$, respectively, thus, $\mathcal{E}_+ \cup \mathcal{E}_- = \mathcal{E}$. Similarly, $\mathcal{W}$ restricted to $\mathcal{E}_+$ (resp. $\mathcal{E}_-$) is denoted by $\mathcal{W}_+$ (resp. $\mathcal{W}_-$). Then, $\mathcal{G}_+ = (\mathcal{V}, \mathcal{E}_+, \mathcal{W}_+)$ and $\mathcal{G}_- = (\mathcal{V}, \mathcal{E}_-, \mathcal{W}_-)$ represent the positively and negatively weighted parts of the graph, respectively. For the main result, Theorem 3.2, we need the following assumption.

Assumption 3.1: Given a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, let $(i,j) \in \mathcal{E}_-$ and $(i',j') \in \mathcal{E}_-$ be any two distinct pairs of nodes with negative weights. Then there exists no cycle in $\mathcal{G}$ containing i, j, i', and j'.

Modulo a sign, $De_\ell = e_{ij}$ where i,j are the indices of nodes corresponding to edge ϵ_ℓ. Hence, given a positively weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ and any edges $\epsilon_k, \epsilon_\ell$ for which no cycle in $\mathcal{G}$ contains their associated nodes, we have

$$e_k^T D^T (DWD^T)^\dagger De_\ell = e_k^T D^T L^\dagger De_\ell = e_{i'j'}^T L^\dagger e_{ij} = 0,$$

since i, j, i', j' are not in a cycle. In other words, view De_ℓ as an external current injection $\boldsymbol{c}$ to the graph circuit. Since there is no cycle going through ϵ_k and ϵ_ℓ, the resulting $\boldsymbol{v} = L^\dagger De_\ell$ must have equal potentials at the two nodes associated with the edge ϵ_k. The fact that $e_k^T D^T L^\dagger De_\ell = 0$ follows. With this argument, we can see that Assumption 3.1 implies

$$e_k^T D^T (D|W|D^T)^\dagger De_\ell = 0, \ k \neq \ell$$

for all $k, \ell \in \mathcal{E}_-$. Here $|W|$ is the absolute value of W. We are now ready to state the theorem.

Theorem 3.2: Given a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, suppose Assumption 3.1 holds. Then the Laplacian $L = DWD^T$ is positive semidefinite if, and only if, $|\mathcal{W}(i,j)| \leq \mathcal{W}_{ij}(\mathcal{G}_+)$ for all $(i,j) \in \mathcal{E}_-$.

Two different proofs to the theorem are provided below.

A. Geometrical proof

Proof: Consider the positive semidefiniteness of the Laplacian L, i.e.,

$$x^T L x = x^T DWD^T x \geq 0, \ \forall x. \tag{6}$$

Let $\hat{D} = D\sqrt{|W|}$ and $\hat{W} = \text{sign}(W)$ where $|W|$, $\sqrt{|W|}$, and the signum function $\text{sign}(W)$ are applied entrywise (and $\text{sign}(0) := 0$). Then $L = \hat{D}\hat{W}\hat{D}^T$ and therefore (6) is equivalent to

$$x^T L x = x^T \hat{D}\hat{W}\hat{D}^T x \geq 0, \ \forall x. \tag{7}$$

The above can be rewritten as

$$y^T \hat{W} y \geq 0$$

for any vector y in the range of $\hat{D}^T$. That is,

$$\sum_{i \in \mathcal{E}_+} y_i^2 \geq \sum_{i \in \mathcal{E}_-} y_i^2, \ \forall y \in \text{ran}\left(\hat{D}^T\right). \tag{8}$$

Let $\mathcal{A}$ be the span of $\{e_i \mid i \in \mathcal{E}_-\}$ and $\mathcal{B}$ be the range of $\hat{D}^T$, then (8) is equivalent to

$$\|y - P_\mathcal{A} y\|_2^2 \geq \|P_\mathcal{A} y\|_2^2, \ \forall y \in \mathcal{B}.$$

This is saying that for any element in $\mathcal{B}$, its distance to the space $\mathcal{A}$ is greater than the length of its projection onto $\mathcal{A}$. From geometric point of view, this is equivalent to saying there is no nonzero vectors $y \in \mathcal{B}$ and $z \in \mathcal{A}$ such that the angle between them is strict less than $\pi/4$. This angle condition holds if and only if

$$\|x\|^2 \geq 2\|P_\mathcal{B} x\|^2, \ \forall x \in \mathcal{A}. \tag{9}$$

Since

$$P_\mathcal{B} x = \hat{D}^T (\hat{D}\hat{D}^T)^\dagger \hat{D} x = \sqrt{|W|} D^T (D|W|D^T)^\dagger D \sqrt{|W|} x,$$

by Assumption 3.1 and the discussion following it we have

$$e_i^T P_\mathcal{B} e_j = 0, \forall i, j \in \mathcal{E}_-, \, i \neq j.$$

Inequality (9) is therefore equivalent to

$$\|e_i\|^2 \geq 2\|P_\mathcal{B} e_i\|^2, \ \forall i \in \mathcal{E}_-. \tag{10}$$

We complete the proof by establishing the equivalence between (10) and the condition on effective resistance as stated in the theorem. Inequality (10) can be explicitly expressed as

$$e_i^T \sqrt{|W|} D^T (D|W|D^T)^\dagger D \sqrt{|W|} e_i \leq \frac{1}{2}, \ \forall i \in \mathcal{E}_-,$$

or equivalently,

$$(e_{ij}^T L_1^\dagger e_{ij})^{-1} \geq 2|\mathcal{W}(i,j)|, \ \forall (i,j) \in \mathcal{E}_-,$$

where L_1 is the Laplacian corresponding to $|\mathcal{G}|$. In view of Definition 2.1 this step completes the proof. ∎

B. Passivity-based proof

We begin with a lemma on graphs with a single negatively weighted edge and then establish a general result on graphs with multiple negative edges.

Lemma 3.3: Given a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, suppose there is only one edge (i,j) with negative weight $\mathcal{W}(i,j)$. Then the Laplacian L is positive semidefinite if, and only if, $|\mathcal{W}(i,j)| \leq \mathcal{W}_{ij}(\mathcal{G}_+)$.

Proof: Recall that positive semidefiniteness of L is equivalent to $\boldsymbol{v}^T L \boldsymbol{v} \geq 0$ for all $\boldsymbol{v} \in \mathbb{R}^n$. By (4), this is thus equivalent to $\boldsymbol{v}^T \boldsymbol{c} \geq 0$ for all $\boldsymbol{v} \in \mathbb{R}^n$. Note that physically, this means the electrical network is passive [18].

(Necessity) Let $\boldsymbol{c} := e_{ij}$, then

$$\boldsymbol{v} = L^\dagger \boldsymbol{c} = L^\dagger e_{ij},$$

and it follows that

$$\boldsymbol{v}^T \boldsymbol{c} = e_{ij}^T L^\dagger e_{ij} = \boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}).$$

Hence a necessary condition for the positive semidefiniteness of L is $\boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}) \geq 0$. By elementary circuit theory, $\boldsymbol{\mathcal{R}}_{ij}(\mathcal{G})$ is simply formed by the parallel connection of the negative resistance $r_{ij} = 1/\mathcal{W}(i,j)$ and a single lumped resistance $\boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}_+)$ between nodes i and j. In other words,

$$\boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}) = \frac{1}{\mathcal{W}(i,j) + \boldsymbol{\mathcal{W}}_{ij}(\mathcal{G}_+)}. \tag{11}$$

Nonnegativity of $\boldsymbol{\mathcal{R}}_{ij}(\mathcal{G})$ is hence equivalent to

$$|\mathcal{W}(i,j)| \leq \boldsymbol{\mathcal{W}}_{ij}(\mathcal{G}_+).$$

(Sufficiency) Note that any power injected to the electrical network is equal to the sum of all the powers dissipated through the resistors, i.e.

$$\boldsymbol{v}^T \boldsymbol{c} = \sum_{l=1}^m \boldsymbol{i}_l^2 r_l;$$

see, for instance, the conservation of energy result in [19]. Without loss of generality, assume that $r_k < 0$. Then the positive semidefiniteness of L is equivalent via (3) to

$$\sum_{l=1}^m \boldsymbol{i}_l^2 r_l = \sum_{l \neq k} \boldsymbol{i}_l^2 r_l + \boldsymbol{i}_k^2 r_k \geq 0 \;\forall \boldsymbol{i} \in \mathcal{I}, \tag{12}$$

where

$$\mathcal{I} := \{\boldsymbol{i} \in WD^T \boldsymbol{v} : \boldsymbol{v} \in \mathbb{R}^n\}.$$

We show below this is guaranteed by $|\mathcal{W}(i,j)| \leq \boldsymbol{\mathcal{W}}_{ij}(\mathcal{G}_+)$.

Let $\boldsymbol{v}(\gamma) \in \mathbb{R}^n$ be such that $L\boldsymbol{v}(\gamma) = \gamma e_{ij}$ and $\boldsymbol{i}(\gamma) = WD^T \boldsymbol{v}(\gamma)$ to be the corresponding current, where γ is some positive constant. In other words, $\boldsymbol{v}(\gamma)$ and $\boldsymbol{i}(\gamma)$ correspond to the potentials and currents in the electrical circuit when γ amount of current is injected at i and the same taken out from j. Recall that the power dissipated through the circuit is simply given by

$$\sum_{l=1}^m \boldsymbol{i}_l(\gamma)^2 r_l = \gamma^2 \boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}) \geq 0, \tag{13}$$

where the inequality follows from $|\mathcal{W}(i,j)| \leq \boldsymbol{\mathcal{W}}_{ij}(\mathcal{G}_+)$.

Now given any $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{c} = L\boldsymbol{v}$ and $\boldsymbol{i} = WD^T\boldsymbol{v}$, select γ so that $\boldsymbol{i}(\gamma)_k = \boldsymbol{i}_k$. Let $\boldsymbol{d} := \boldsymbol{i} - \boldsymbol{i}(\gamma)$. Then

$$\begin{aligned}
\sum_{l=1}^m \boldsymbol{i}_l^2 r_l &= \sum_{l=1}^m (\boldsymbol{i}_l(\gamma) + \boldsymbol{d}_l)^2 r_l \\
&= \sum_{l=1}^m (\boldsymbol{i}_l(\gamma))^2 r_l + 2\sum_{l=1}^m \boldsymbol{i}_l(\gamma) r_l \boldsymbol{d}_l + \sum_{l=1}^m \boldsymbol{d}_l^2 r_l \qquad (14) \\
&= \gamma^2 \boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}) + 2\sum_{l=1}^m \boldsymbol{i}_l(\gamma) r_l \boldsymbol{d}_l + \sum_{l \neq k} \boldsymbol{d}_l^2 r_l,
\end{aligned}$$

where (13) and $\boldsymbol{i}(\gamma)_k = \boldsymbol{i}_k$ have been used in the last equality. Direct calculation gives

$$\sum_{l=1}^m \boldsymbol{i}_l(\gamma) r_l \boldsymbol{d}_l = \boldsymbol{i}(\gamma)^T R \boldsymbol{d} = 0.$$

Therefore, it follows from (14) that

$$\sum_{l=1}^m \boldsymbol{i}_l^2 r_l \geq \gamma^2 \boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}) \geq 0.$$

Since $v \in \mathbb{R}^n$ was arbitrary, (12) holds and the positive semidefiniteness of L follows. ∎

The passivity-based proof for Theorem 3.2 is now in order.

Proof: [of Theorem 3.2] Note that the injection of a unit current at node i and the withdrawal of a unit current at node j does not induce any current through any other negatively weighted edges because there exists no cycle containing two or more such edges by Assumption 3.1. It thus follows that for any $(i,j) \in \mathcal{E}_-$ and $\mathcal{G}_e := (\mathcal{V}, \mathcal{E} \setminus (i,j), \mathcal{W})$,

$$\boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}_e) = \boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}_+).$$

(Necessity) By the same arguments in the proof of Lemma 3.3, it follows that the positive semidefiniteness of L implies $\boldsymbol{\mathcal{R}}_{ij}(\mathcal{G}) \geq 0$ for all $(i,j) \in \mathcal{E}_-$. The latter is equivalent to $|\mathcal{W}(i,j)| \leq \boldsymbol{\mathcal{W}}_{ij}(\mathcal{G}_+)$ for all $(i,j) \in \mathcal{E}_-$.

(Sufficiency) By Assumption 3.1, any current through a negative edge does not flow to/from any other negative edges in the electrical network. Thus, given any $\boldsymbol{v} \in \mathbb{R}^n$ and the corresponding $\boldsymbol{i} = WD^T\boldsymbol{v}$, the total dissipated power can be decomposed into a sum of $N := \mathrm{card}(\mathcal{E}_-)$ terms, where in each term only one negative edge is involved and the current is zero through all other negative edges. That is, given $\{\epsilon_1, \ldots, \epsilon_N\} = \mathcal{E}_-$, there exist $\boldsymbol{i}(\epsilon_1), \ldots, \boldsymbol{i}(\epsilon_N)$ such that

$$\sum_{l=1}^m \boldsymbol{i}_l^2 r_l = \sum_{l \in \mathcal{E}_+ \cup\, \epsilon_1} \boldsymbol{i}_l(\epsilon_1)^2 r_l + \ldots + \sum_{l \in \mathcal{E}_+ \cup\, \epsilon_N} \boldsymbol{i}_l(\epsilon_N)^2 r_l \tag{15}$$

and

$$\sum_{l \in \mathcal{E}_+ \cup\, \epsilon_j} \boldsymbol{i}_l(\epsilon_j)^2 r_l = \sum_{l=1}^m \boldsymbol{i}_l(\epsilon_j)^2 r_l \quad \forall j = 1, 2, \ldots, N.$$

It holds by Lemma 3.3 that $|\mathcal{W}(\epsilon_i)| \leq \boldsymbol{\mathcal{W}}_{\epsilon_i}(\mathcal{G}_+)$ for all $\epsilon_i \in \mathcal{E}_i$ guarantees that each of the term on the right of (15) is nonnegative, which in turn implies that the total dissipated power is nonnegative. Passivity of the electrical network, and hence the positive semidefiniteness of L, thus follow. ∎

IV. Consensus

A. Algebraic multiplicity of the zero eigenvalue

In the case of a connected graph with positive weights only, since, the Laplacian is a positive semidefinite matrix and has only one zero eigenvalue with corresponding eigenvector 1_n, the system will reach consensus. This property of L is necessary for consensus. When negative weights are present, provided these are sufficiently small and Assumption 3.1 is satisfied, the Laplacian is still positive semidefinite (Section III). Here we further establish that, provided the inequalities in Theorem 3.2 are strict, the Laplacian has only one zero eigenvalue and it corresponds to the eigenvector 1_n.

Lemma 4.1: Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, if the negative weights in $\mathcal{E}_-$ satisfy

$$|\mathcal{W}(i,j)| < \boldsymbol{\mathcal{W}}_{ij}(\mathcal{G}_+) \tag{16}$$

and Assumption 3.1, then its Laplacian L is positive semidefinite with only one zero eigenvalue with corresponding eigenvector 1_n.

Proof: The positive semidefinite part follows directly from Theorem 3.2. To see that L has only one zero eigenvalue with corresponding eigenvector 1_n, we need only to show that $x^T L x > 0$ for any nonzero vector $x \neq 1_n$. Assume not, that is, $x^T L x = 0$ for some $x \neq 1_n$, then $x^T(L_p + (L - L_p))x = 0$. Here L_p is the Laplacian of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\mathcal{W}|)$. It follows

$$x^T(L - L_p)x = -x^T L_p x < 0. \tag{17}$$

On the other hand, since the negative weights satisfy the strict inequality (16), the slightly perturbed graph corresponding to the Laplacian $L_\varepsilon = L_p + (1+\varepsilon)(L - L_p)$ for some small $\varepsilon > 0$ also satisfies (16), and L_ε is thus positive semidefinite. In view of (17), we conclude

$$x^T L_\varepsilon x = \varepsilon x^T (L - L_p)x < 0,$$

which contradicts the fact that L_ε is positive semidefinite. This completes the proof. ∎

B. Consensus for passive dynamics

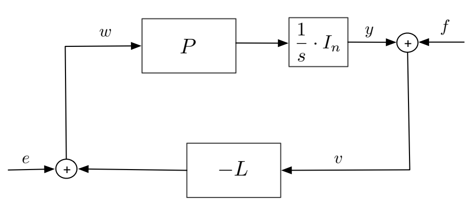

Fig. 1. Feedback interconnection for consensus.

Consider the feedback interconnection in Figure 1. There, $P := \bigoplus_{i=1}^n P_i \in \mathbf{R}^{n\times n}$ is a diagonal transfer matrix with the single-input-single-output (SISO) dynamical agents $P_i \in \mathbf{R}$ and L is the Laplacian corresponding to a weighted graph. Such a setup models the problem of consensus of a network of heterogeneous agents interconnected through a Laplacian. The network topology, i.e. the arrangement of the agents and their communicating neighbours, is captured by the underlying weighted connected graph. A negative weight may represent the case of an antagonistic attack intended to jeopardise the objective of reaching consensus, defined as follows.

Definition 4.2: The interconnection in Figure 1 is said to reach consensus if $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$ for all $i, j \in \{1, 2, \ldots, n\}$ and external disturbances $e, f \in \mathbf{L}_2^n$.

That is, consensus means that the agents/nodes asymptotically reach agreement in their output y_i, i.e., $\lim_{t\to\infty} y(t)$ lies in the subspace spanned by 1_n. Next we present conditions ensuring consensus that can be obtained from the vantage point of the theory of integral quadratic constraints [9].

Proposition 4.3: The feedback configuration in Figure 1, with $P := \bigoplus_{i=1}^n P_i : P_i \in \mathbf{RH}_\infty; P_i(0) \neq 0$ and a Laplacian $L \in \mathbb{R}^{n\times n}$ corresponding to a weighted connected graph, reaches consensus if the zero eigenvalue of L is simple and there exists a multiplier $\Pi \in \mathbf{C}^{2n\times 2n}$ such that

(i) $\begin{bmatrix} I_n \\ \tau \frac{1}{j\omega} P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ \tau \frac{1}{j\omega} P(j\omega) \end{bmatrix} \geq \gamma I_n \ \forall \omega \in \mathbb{R} \setminus \{0\}, \tau \in [0,1]$, and some $\gamma > 0$;

(ii) $\begin{bmatrix} -L \\ I_n \end{bmatrix}^T \Pi(j\omega) \begin{bmatrix} -L \\ I_n \end{bmatrix} \leq 0 \ \forall \omega \in \mathbb{R} \setminus \{0\}$.

Proof: The claim in the proposition is a special case of [11, Thm. 4.5] or [10, Thm. 4.3]. ∎

We now present a main result on the robustness of consensus for high-order passive nodal-dynamics. We employ the notation explained at the beginning of Section III.

Theorem 4.4: Consider $P := \bigoplus_{i=1}^n P_i$ with $P_i \in \mathbf{RH}_\infty$, $P(0) \neq 0$ and a Laplacian matrix $L \in \mathbb{R}^{n\times n}$ corresponding to a weighted connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ that satisfies Assumption 3.1. If $\frac{1}{s}P$ is passive and $|\mathcal{W}(i,j)| < \boldsymbol{\mathcal{W}}_{ij}(\mathcal{G}_+)$ for all $(i,j) \in \mathcal{E}_-$, then the feedback interconnection of Figure 1 reaches consensus.

Proof: First note that the graph Laplacian L has a simple zero eigenvalue by hypothesis and Lemma 4.1. It is shown below that the conditions in Proposition 4.3 are satisfied with respect to a constant matrix

$$\Pi = \Pi^T := \begin{bmatrix} \gamma I_n & I_n \\ I_n & 0 \end{bmatrix}.$$

To this end, observe that

$$\begin{aligned} &\begin{bmatrix} I_n \\ \tau \frac{1}{j\omega} P(j\omega) \end{bmatrix}^* \Pi \begin{bmatrix} I_n \\ \tau \frac{1}{j\omega} P(j\omega) \end{bmatrix} \\ &= \gamma I_n + \tau \left(\frac{1}{j\omega} P(j\omega) \right)^* + \tau \frac{1}{j\omega} P(j\omega) \\ &\geq \gamma I_n \quad \forall \omega \in \mathbb{R} \setminus \{0\}, \tau \in [0,1], \end{aligned}$$

where passivity of $\frac{1}{s}P$ has been used in the last inequality. On the other hand,

$$\begin{bmatrix} -L \\ I_n \end{bmatrix}^T \Pi \begin{bmatrix} -L \\ I_n \end{bmatrix} = \gamma L^2 - 2L \leq 0$$

for sufficiently small $\gamma > 0$, where the positive semidefiniteness of L as guaranteed by Theorem 3.2 has been exploited. Consensus thus follows from Proposition 4.3. ∎

Remark 4.5: Convergence of $\dot{x} = -Lx$ in (1) in the introduction is a special case of the above result since $\frac{1}{s} I_n$ is a passive system.

V. Numerical examples

In this section we study an academic example to illustrate how the magnitudes of negative weights affect the consensus result. Consider a four-agent system as in Figure 1 with

$$P_i = \frac{2s + a_i}{s + a_i}, \ 1 \leq i \leq 4,$$

where $(a_1, a_2, a_3, a_4) = (1, 2, 3, 4)$, and underlying graph structrue as in Figure 2. Obviously $\frac{1}{s}P$ is passive since

$$\frac{1}{s}P_i = \frac{1}{s} + \frac{1}{s + a_i}$$

is the sum of two passive transfer functions for all $1 \leq i \leq 4$.

Let the weights $(w_1, w_2, w_4) = (3, 2, 4)$ be positive and we study the consensus property of the system when w_3 takes a negative value. As shown in Theorem 4.4, the system will reach consensus if the absolute value of w_3 is strictly less than the effective conductance between node 1 and 4, namely,

$$|w_3| = |\mathcal{W}(1, 4)| < \boldsymbol{\mathcal{W}}_{14}(\mathcal{G}_+) = 1.2.$$

Here, we take two values of $w_3 = -1.15, -1.25$, one less than the critical value and one greater than the critical value. The simulation results are shown in Figure 3 and Figure 4, respectively. As we can see, the system reaches consensus when $|w_3| < 1.2$, while this is not the case when $|w_3| > 1.2$.

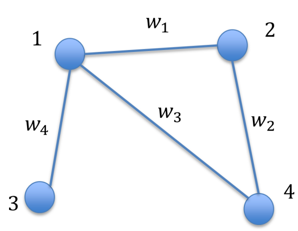

Fig. 2. Network topology.

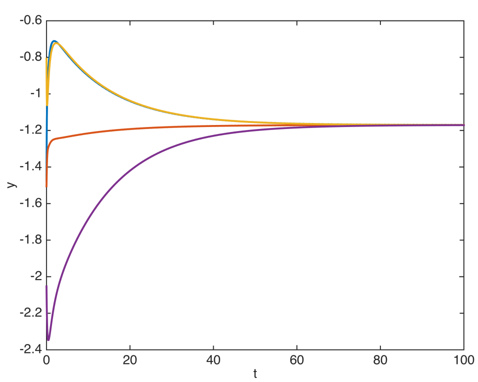

Fig. 3. Sample trajectories for $w_3 = -1.15$.

VI. Conclusions

The present paper provides two alternative proofs to the result by Zelazo and Bürger regarding the positive definiteness of the graph Laplacian when negative edge weight is allowed. Both of these two proof are intuitive and easy to follow. The second proof relies on passivity argument, which is then used to establish more general consensus conditions in multi-agent systems with generalized high-order dynamics.

It is envisioned that the methods developed in the paper will facilitate the study of similar problems defined on directed graphs.

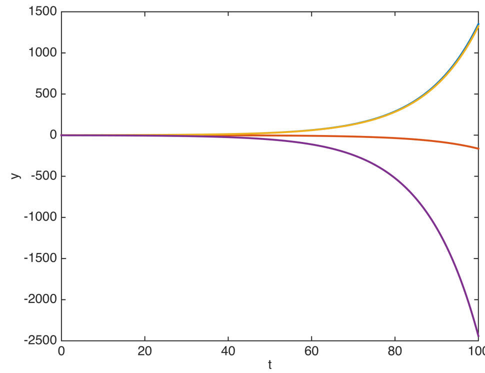

Fig. 4. Sample trajectories for $w_3 = -1.25$.

VII. Acknowledgement

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