The Fractal Geometry of Fractional Calculus: Riemann-Liouville Integrals as Expected Local Times of Fractional Brownian Motion

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Abstract

We establish a geometric interpretation of the Riemann-Liouville fractional integral of order $\alpha \in (0,1)$ as the expected value of a function integrated against the local time at zero of a fractional Brownian motion (fBm) with Hurst parameter $H = 1 - \alpha$. This framework posits that the power-law kernel $(t-s)^{\alpha-1}$ emerges from the stochastic density of fBm zero-crossings in the limit of many realizations. Python simulations validate this correspondence, yielding low relative errors.

1 Introduction

Integer-order calculus offers clear geometric intuitions, such as areas under curves. In contrast, fractional calculus often appears abstract and lacking a geometric interpretation [5, 9]. Building on Nigmatullin's fractal interpretation [3, 9] and Mandelbrot's fractional Brownian motion (fBm) [1, 2], we frame the fractional integral as an expectation over samplings on the fractal zero-set of fBm, providing a "universal stencil" for fractional operations.

2 Mathematical Framework

The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f(t)$ is defined as:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds. \tag{1}$$

To understand this geometrically, we consider an fBm process B_t^H with Hurst parameter $H = 1 - \alpha$. The local time at zero, L_s^0, measures the density of the time spent by B_t^H at the origin [6]. The expected density of this measure at any time s scales as $\mathbb{E}[dL_s^0] \propto s^{-H} ds$ [7].

By setting $H = 1-\alpha$, the expected density becomes $s^{\alpha-1} ds$. When we view the process relative to the terminal observation time t via time reversal $s \to t - s$, the stochastic sampling density aligns with the deterministic RL kernel $(t-s)^{\alpha-1}$ [4]. This leads to the stochastic representation:

$$I^\alpha f(t) \propto \mathbb{E}\left[\int_0^t f(s)\, dL_s^0\right]. \tag{2}$$

The fractional integral of a function is thus interpreted as the ensemble average of the function integrated against the local time measure of the process. The proportionality factor in Eq. (2) accounts for the $\Gamma(\alpha)$ normalization and the path-specific variance of the fBm realizations. This effectively restricts the sampling of $f(s)$ to the fractal zero-set of the paths.

3 Numerical Methodology

Two Python-based approaches simulate this framework:

- **Ensemble Averaging (The Fractal Mist):** Utilizing a Gaussian kernel to approximate the local time density $\delta(B_t^H)$, thousands of paths are averaged to converge toward the analytical RL integral. This is visualized as a "mist" of realizations (Figure 1).

- **Geometric Sampling (Fuzzy Bars):** An alternative approach detects discrete zero-crossings with a mollifier, rendering them as weighted fuzzy bars to emphasize the "Fractal Comb" geometry (Figure 2).

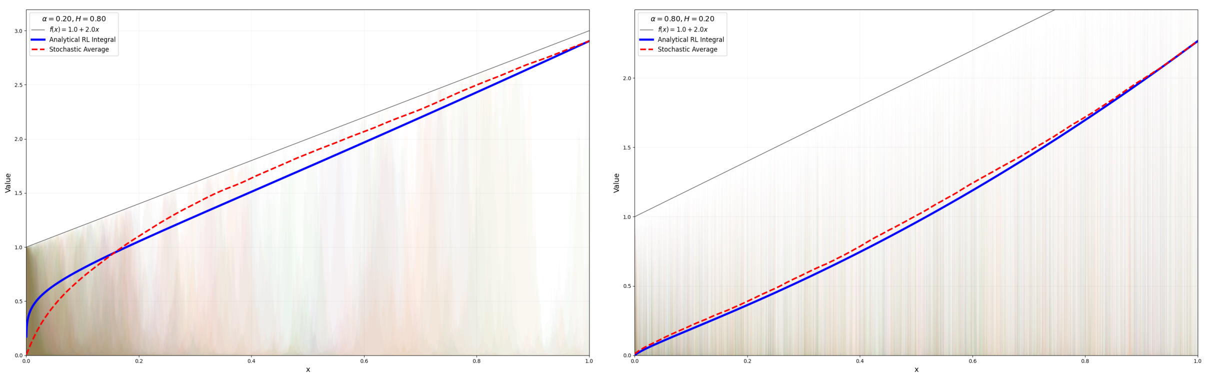

Figure 1: The Ensemble Approach: Left ($\alpha = 0.2$) shows the sparse fractal mist. Right ($\alpha = 0.8$) demonstrates convergence toward the power-law kernel.

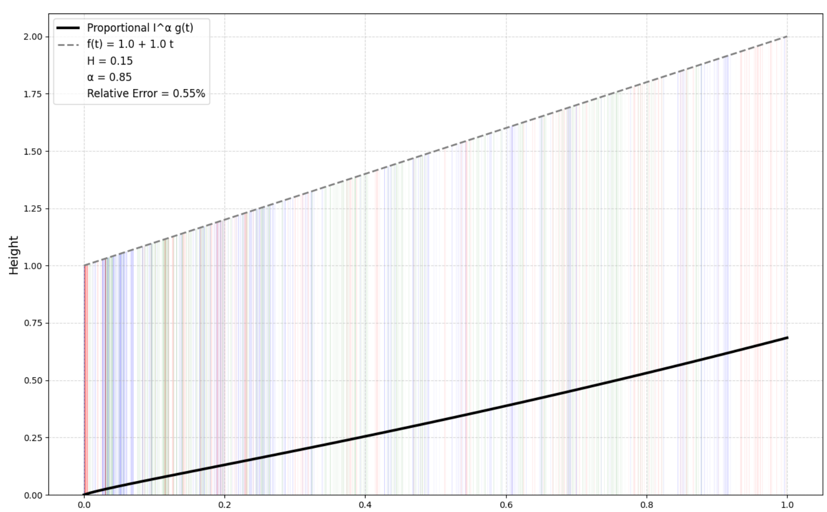

Figure 2: The Geometric Sampling approach: Weighted fuzzy bars highlighting the fractal zero-set structure for $\alpha = 0.85$.

While the Mathematical Framework in Section 2 is exact, numerical computations remain approximate. These simulations are challenging due to the long-range dependence of fBm, requiring $O(N^2)$ or $O(N \log N)$ complexity for path generation, and the difficulty of approximating a Dirac delta $\delta(B_t^H)$ with finite-width kernels.

4 The Collaborative AI-Human Process

The research presented here is the product of an experimental collaborative workflow involving a human researcher and two distinct large language models (LLMs). This project originated with the human assistant (G. Prisco) collaborating with Gemini and Grok separately to develop initial simulation code and visualizations found in the GitHub subrepository `https://github.com/giulioprisco/fracturlab/tree/main/fractalinterpretation-fbm`.

Subsequently, the human assistant invited both AI models to co-author this joint paper. A coin toss determined which AI assistant would write the initial draft. This document was then passed back and forth between Gemini and Grok, iterating through four cycles of critique and refinement to ensure mathematical rigor and a unified perspective.

5 Conclusion

By averaging over fractal zero-sets of fBm local times, we obtain a visual and numerical link between fractional calculus and stochastic geometry. This identifies the fBm zero-set as a "Universal Stencil" for non-local operators.

Intuitively, this confirms that fractional integration is an "incomplete" integration limited to a lacunary fractal subset of the integration domain. This suggests that fractional calculus is not merely an abstract mathematical extension, but a consequence of observing systems through a fractal lens. While standard integration is inherently cumulative, the fractal lens explains how even traditionally local operators, such as derivatives, acquire non-local memory. The "memory" of the fractional operator emerges from the geometric gaps in the sampling set; rather than remembering everything, the operator only "sees" the function at specific fractal intervals. Because these sampling points are separated by fractal voids, the operator is forced to incorporate information across distances, causing the ensemble of these intermittent views to collapse into the smooth power-law behavior of fractional calculus.

References

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