Optimization of Projectile Range

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1 Introduction

Given a projectile thrown with initial velocity $\vec{v_0}$ from a height y_0, what is the optimal angle θ^* between such velocity and the horizontal plane in order to maximize its range?

We will show that the optimal angle is, more generally, not always $45°$; such angle is indeed dependent from the initial state of the projectile (i.e. its initial velocity and height).

2 Initial State

Let's consider a projectile with initial velocity $\vec{v_0}$. We'll study its motion in the plane x, y from its initial position $(0, y_0)$ at time t_0, with $y_0 > 0$, to the ground.

As such, $\vec{v_0} = (v_{0x}, v_{0y})$ with $v_{0x} > 0$ and $v_{0y} > 0$. The angle θ between $\vec{v_0}$ and the horizontal plane is $\theta = \arctan\left(\frac{v_{0y}}{v_{0x}}\right)$.

The projectile will be subject to an uniform gravitational acceleration $\vec{g}$ with $g \approx 9.81 \frac{m}{s^2}$ pointing downwards. We'll neglect other forces such as air resistance.

3 Equations of Motion

The equations of motion for the projectile are

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2 \end{cases} , \quad t \geq t_0 = 0 \tag{1}$$

From the system above we can derive the function of the projectile's trajectory $y(x)$. Remembering that $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$, we get

$$y(x) = y_0 + \tan\theta \cdot x - \frac{g}{2v_0^2\cos^2\theta}x^2 \tag{2}$$

Let r be the range of the projectile. The point of contact with the ground is given by $(r, 0)$, thus

$$\begin{aligned} y(r) = 0 \implies r &= \frac{-\tan\theta + \sqrt{\tan^2\theta + \frac{2gy_0}{v_0^2\cos^2\theta}}}{-\frac{g}{v_0^2\cos^2\theta}} \\ &= \frac{v_0^2}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gy_0}{v_0^2}}\right)\cos\theta \end{aligned} \tag{3}$$

For the sake of simplicity, let $A = \frac{v_0^2}{g}$ and $B = \frac{2gy_0}{v_0^2}$. So that, writing r as a function of θ,

$$r(\theta) = A\left(\sin\theta + \sqrt{\sin^2\theta + B}\right)\cos\theta \tag{4}$$

4 Degenerate Case $y_0 = 0$

Let's first consider the degenerate case $y_0 = 0$. The term B goes to zero, thus $r(\theta)$ simplifies to

$$r(\theta) = A\sin 2\theta \tag{5}$$

The range is maximized when the sine is maximum, i.e. $\max r(\theta) = r(\theta^*) = A$ when $\theta^* = \frac{\pi}{4}$.

5 General Case $y_0 > 0$

Considering the general case $y_0 > 0$, we evaluate the derivative of $r(\theta)$

$$\begin{aligned}\frac{dr}{d\theta} &= A\left[-\sin\theta\left(\sin\theta + \sqrt{\sin^2\theta + B}\right) + \cos\theta\left(\cos\theta + \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + B}}\right)\right] \\ &= A\left(-\sin^2\theta - \sin\theta\sqrt{\sin^2\theta + B} + \cos^2\theta + \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + B}}\right)\end{aligned} \tag{6}$$

Since $\max r(\theta) = r(\theta^*)$, the derivative must be zero when $\theta = \theta^*$.

$$\begin{aligned}\frac{dr}{d\theta}(\theta^*) = 0 \implies & A\left(-\sin^2\theta^* - \sin\theta^*\sqrt{\sin^2\theta^* + B} + \cos^2\theta^* + \frac{\sin\theta^*\cos^2\theta^*}{\sqrt{\sin^2\theta^* + B}}\right) = 0 \\ \implies & -\sin^2\theta^* \cdot \sqrt{\sin^2\theta^* + B} - \sin\theta^* \cdot \left(\sin^2\theta^* + B\right) \\ & + \cos^2\theta^* \cdot \sqrt{\sin^2\theta^* + B} + \sin\theta^*\cos^2\theta^* = 0 \\ \implies & \left(\cos^2\theta^* - \sin^2\theta^*\right)\sqrt{\sin^2\theta^* + B} = \sin^3\theta^* + B\sin\theta^* - \sin\theta^*\cos^2\theta^* \\ \implies & \left(\cos^2\theta^* - \sin^2\theta^*\right)^2\left(\sin^2\theta^* + B\right) = \left(\sin^3\theta^* + B\sin\theta^* - \sin\theta^*\cos^2\theta^*\right)^2 \\ \implies & \sin^2\theta^*\cos^4\theta^* + B\cos^4\theta^* + \sin^6\theta^* + B\sin^4\theta^* - 2\sin^4\theta^*\cos^2\theta^* \\ & - 2B\sin^2\theta^*\cos^2\theta^* = \sin^6\theta^* + B^2\sin^2\theta^* + \sin^2\theta^*\cos^4\theta^* + 2B\sin^4\theta^* \\ & - 2B\sin^2\theta^*\cos^2\theta^* - 2\sin^4\theta^*\cos^2\theta^* \\ \implies & B - 2B\sin^2\theta^*\cos^2\theta^* = B^2\sin^2\theta^* + 2B\sin^4\theta^*\end{aligned} \tag{7}$$

Dividing both sides by B and remembering that $\cos^2\theta^* = 1 - \sin^2\theta^*$

$$\begin{aligned} & 1 - 2\sin^2\theta^*\left(1 - \sin^2\theta^*\right) = B\sin^2\theta^* + 2\sin^4\theta^* \\ \implies & 1 - 2\sin^2\theta^* = B\sin^2\theta^* \\ \implies & \theta^* = \arcsin\sqrt{\frac{1}{B+2}}\end{aligned} \tag{8}$$

Note that by setting $B = 0$ we get $\theta^* = \frac{\pi}{4}$, as expected from the case $y_0 = 0$.

6 Conclusion

The optimal take-off angle θ^* to maximize the projectile range is given by

$$\theta^* = \arcsin\sqrt{\frac{1}{\frac{2gy_0}{v_0^2} + 2}} \tag{9}$$

For example, a baseball thrown with initial velocity $v_0 = 20\frac{m}{s}$ from a $y_0 = 20m$ 5th floor window will have an optimal take-off angle of $\theta^* \approx 35.3°$.