Problem set

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Abstract

Problem set solutions.

Contents

1 Distributions & probabilities

1.1 Given the PDF:

$$\mathrm{f}\,(t) = Ce^{-\frac{t}{\tau}} \quad (1)$$

in the range $t \in \; [t_0, \infty]$, the value of C for which the PDF is normalized is:

$$\int_{t_0}^{\infty} Ce^{-\frac{t}{\tau}}\, dt = 1 \longrightarrow C = \frac{e^{\frac{t_0}{\tau}}}{\tau} \quad (2)$$

I note a close resemblance of our PDF to the exponential distribution, and they coincide if $t_0 = 0$. The mean or better the expectation value is then:

$$\mathbb{E}[t] = \int_{t_0}^{\infty} t\,\mathrm{f}\,(t)dt \longrightarrow t_0 + \tau \quad (3)$$

It's nice to note that for $t_0 = 0$ it matches the well known result for the exponential distribution. The width could be quantified with the variance as:

$$\begin{aligned} Var(t) &= \mathbb{E}[x^2] - (\mathbb{E}[x])^2 = \\ &= \int_{t_0}^{\infty} t^2\,\mathrm{f}\,(t)dt - (\int_{t_0}^{\infty} t\,\mathrm{f}\,(t)dt)^2 = \\ &= 2\tau^2 - 2t_o\tau + t_0^2 - (t_0+\tau)^2 = \\ &= \tau^2 - 4\tau t_0 \end{aligned} \quad (4)$$

and once more if $t_0 = 0$ the variance of the starting PDF coincides withe the one of the exponential distribution. A graphical representation of all said above is found in Figure 1.

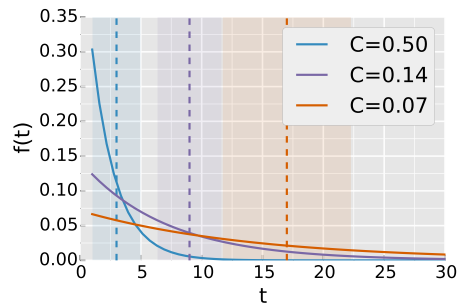

Figure 1: Plot of the given PDF, for different values of τ and t_0. Dashed lines are the calculated expectation values, and the shaded area represent $\pm$ one square root of the variance around the mean.

1.2 Assuming that little Peter is a fair pal, and is using fair coins then the probability of success (heads) follows the binomial distribution. If we call n the number of tries, $p = \frac{1}{2}$ the probability for each event to success, and r the number of success out of n tries, the probability is the following:

$$\begin{aligned} P(n,r) &= \frac{n!}{r!(n-r)!}p^r(1-p)^{n-r} \\ &= \binom{n}{r}p^r(1-p)^{n-r} \end{aligned} \quad (5)$$

The *chance* of getting 14 or more heads with 20 coin flips is low, as it is possible to see in Figure 2.

The probability of getting 14 heads is $\approx\;$ 0.036, and by integrating equation 5 from r=14 to infinity it is possible to quantify the aforementioned low chance.

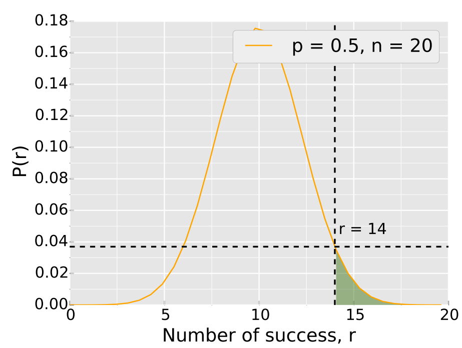

Figure 2: Probability of coin flips. The parameters are reported in the legend; the shadowed area quantifies the chance of getting 14 or more heads.

To find the chance that little Peter gets at least 18 coin at once when flipping 20 coins 100 times, first the probability P_{18} is calculated with equation 5 using $p = \frac{1}{2}$, $n = 20$, and $r = 18$. It is easy to see that the distribution of the chances of getting at least 18 successes is still binomial but with p $= P_{18}$, $n = 100$, and $r = 1$. The probability of getting one time a success is: $P(1) \approx 0.0177$. The chance of getting more than one is quantified as said before and it is rapresenented as the shaded area in Figure 3.

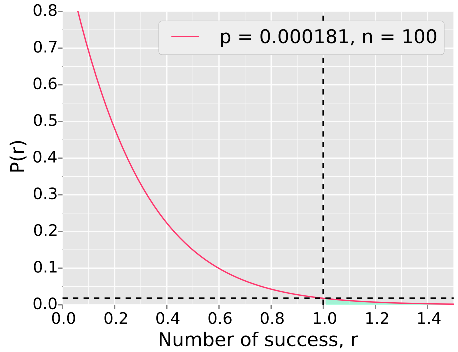

Figure 3: Probability of getting out of 100 tries 18 successes out of 20 coin filps. The parameters are reported in the legend, and the shadowed area quantifies the chance of getting one or more times 18 successes.

2 Error propagation

2.1 Three different ways are employed to calculate the average of the given measurements, namely the arithmetic mean, a error weighted mean and finally a fit with a zeroth order polynomial. The last two options require to accept and believe in the reported experimental errors. The results are reported in Figure 4.

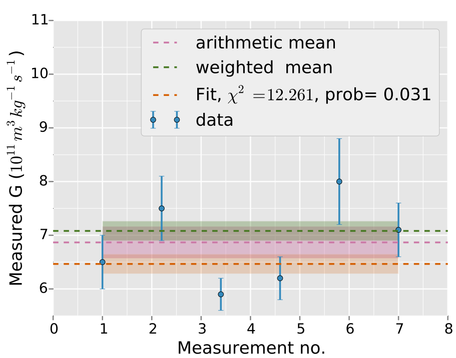

Figure 4: Graphical presentation of the given measurements and three differently calculated averages. The shadowed areas around each average value are plus and minus σ_μ, the associated error.

The χ^2 probability density function is reported in Figure 5 along with the probability of getting the same χ^2 found in the aforementioned zeroth

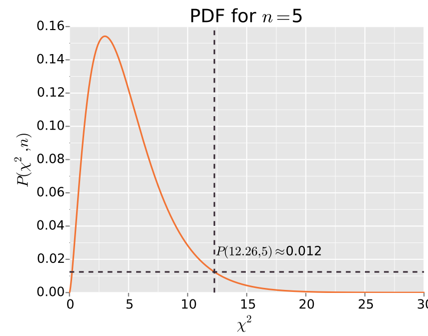

Figure 5: χ^2 distribution for five degrees of freedom.

order polynomial fit.

The result obtained are summarized in Table 1.

Table 1: Results. The averages and their errors are reported in units of: $10^{-11} m^3 kg^{-1} s^{-2}$.

method	**average**	**uncertainty**	χ^2**(prob)**
fit	6.5	.2	12.261 (0.031)
$< G >$	6.9	.3	-
$< G >_W$	7.1	.2	-

As it possible to see the fit is the method that is better representing the data with an average value. Moreover it is also compatible with the *true* value of G ($G = 6.6738 \pm .0008 \times 10^{-11}$ m^3 kg^{-1} s^{-2}). The probability of the fit is not particularity high but is expected due to the low number of measurements. Lastly it is possible to note that the probability to get $\chi^2 \gtrsim 12.261$ is $\lesssim 0.012$ according the the χ^2 distribution (Figure 5).

2.2 The uncertainty on a time measurement with a gridiron pendulum is:

$$\sigma_t = 2\pi \sqrt{(\frac{1}{2\sqrt{gL}})^2 \sigma_L^2 + \sigma_g^2 (\frac{\sqrt{L}}{2} g^{-\frac{3}{2}})^2} \qquad (6)$$

where L is the length of the pendulum, g is the gravitational constant with an error σ_g. If one considers that the error on the measurement on the length is **only** related to the thermal expansion of the materials making the pendulum then

σ_L can be calculated as:

$$\sigma_L = \sum_i \alpha_i \Delta T L_i \tag{7}$$

where α_i is the linear coefficient of expansion for the *ith* material, L_i its length and ΔT the temperature fluctuation of the system.

If the pendulum is made of a single component:

$$\sigma_t = \pi \sqrt{L \left(\frac{DT^2 \left(a_1\right)^2}{g} + \frac{\sigma_g^2}{g^3} \right)} \tag{8}$$

whereas if the pendulum is made of two components with a length ratio of λ:

$$\sigma_t = \pi \sqrt{L \left(\frac{DT^2 \left(a_1 + a_2\lambda\right)^2}{g} + \frac{\sigma_g^2}{g^3} \right)} \tag{9}$$

It's easy to see that the error coming from the thermal expansion can be killed by choosing the correct material combination such that $(a_1 + a_2\lambda) \approx 0$. Strangely enough that's the case for the iron/brass combo.

2.3 To calculate the index of refraction (IOR) of a solution (n_{sol}) with respect to air one has to rearrange Snell's law and to calculate the error ($\sigma_{n_{sol}}$) one has to consider the IOR of air a value ($n_{air} = 1$) without error.

$$n_{sol} = \frac{sin(\theta_{air})}{sin(\theta_{sol})} \tag{10}$$

$$\sigma_{n_{sol}} = (\left[\frac{cos(\theta_{air})}{sin(\theta_{sol})} \sigma_{air} \right]^2 + + \left[\frac{2cos(\theta_{sol})}{1 + cos(\theta_{sol}) - 1} sin(\theta_{air}) \sigma_{sol} \right]^2)^{\frac{1}{2}} \tag{11}$$

To determine the percentage and its error of sugar in the solution, using two known and errorless standard $n_1 = 1.3330$, and $n_2 = 1.4774$ for a 0% and a 75% solution respectively one has to use the following equations:

$$\% sugar = (n_{sol} - 1.3330) \frac{75}{(1.4774 - 1.3330)} \tag{12}$$

$$\sigma_{\%} = \frac{75}{(1.4774 - 1.3330)} \sigma_{n_{sol}} \tag{13}$$

The results are reported in Table 2.

Table 2: Snell's law results

	n_{sol}	**% sugar**
value	1.385	27
error	0.008	4

3 Monte Carlo

4 Statistical tests

5 Fitting data

6 notes

A git-hub repo with all the source code is available at: `http://giulioungaretti.github.io/stats2013`