"Opinions and Conflicts in Social Networks"

Final Exam for the Bertinoro International Spring School 2022

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Math / analytical questions

Question 1

1.1 We start by decomposing the formula into the difference of two products:

$$x^T L x = x^T (I - \frac{1}{d} A) x = x^T I x - \frac{1}{d} x^T A x$$

The I matrix in the first term has no effect, thus it is equivalent to:

$$x^T x = \sum_{v \in V} x_v^2 = \frac{1}{d} \sum_{(u,v) \in E} x_u^2 + \frac{1}{d} \sum_{(u,v) \in E} x_v^2$$

where the $\frac{1}{d}$ term is justified by the assumption that the graph G is d-regular, hence each node x_v will be counted d times having exactly d edges coming in and d edges going out. The second term represents instead the product of each two vertices connected by an edge, therefore:

$$x^T A x = 2 \sum_{(u,v) \in E} x_u x_v$$

where the 2 factor accounts for both the direction of the edges in the undirected graph G.

Combining together all the previous equations, we get that:

$$x^T L x = \frac{1}{d} \sum_{(u,v) \in E} x_u^2 + \frac{1}{d} \sum_{(u,v) \in E} x_v^2 - \frac{2}{d} \sum_{(u,v) \in E} x_u x_v$$

which can be eventually rewritten as:

$$x^T L x = \frac{1}{d} \sum_{(u,v) \in E} (x_u^2 - 2x_u x_v + x_v^2) = \frac{1}{d} \sum_{(u,v) \in E} (x_u - x_v)^2$$

in order to prove our claim.

1.2 A matrix $M \in R^{NxN}$ is defined as positive semidefinite if:

$$x^T M x \geq 0$$

for each vector $x \in R^N$. Having proved in **1.1** that:

$$x^T L x = \frac{1}{d} \sum_{(u,v) \in E} (x_u - x_v)^2$$

we can easily conclude that L is positive semidefinite since its value can be computed as the sum of squared terms which are, by definition, non-negative.

1.3 Considering the equation:

$$x^T L x = \frac{1}{d} \sum_{(u,v) \in E} (x_u - x_v)^2$$

a trivial way to minimize it consists in taking a constant vector. Besides, if G has more than one connected component, it is sufficient to take a vector $x \in R^N$ where nodes belonging to the same component share the same value, so that $\nexists (u,v) \in E$ s. t. $(x_u - x_v)^2 > 0$. Accordingly, the non-trivial vector x_* is defined as the one minimizing $x^T L x$ such that its value is *strictly* positive.

Given that the eigenvalues of a symmetric matrix A satisfy the following principles w.r.t. Rayleigh quotient:

$$\lambda_i = \min_{\substack{x \neq 0 \\ x \perp x_j, \forall j < i}} \frac{x^T A x}{x^T x}$$

it is clear that x_* is strictly linked to the first non-trivial eigenvalue - i.e., λ_{k+1}, where k is the number of connected components in G - of the normalized laplacian, showing how this vector maintains the highest informative content of the laplacian matrix and it represents ipso facto the most suitable embedding of the graph vertices onto a one-dimensional space.

Question 2

2.1 A function f is said to be submodular if it holds:

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$$

where $A \subseteq B$ and $x \in U \setminus B$. By definition, the $cut(\cdot)$ function is such that:

$$cut(S \cup \{x\}) = \mid \{(u, v) \in E \text{ such that } u \in S \cup \{x\} \text{ and } v \in V \setminus (S \cup \{x\})\} \mid$$

which can be rewritten as:

$$cut(S) + cut(\{x\}) - 2 \mid \{(u, x) \in E, u \in S\} \mid$$

or rather, as the sum between the function evaluated in the sets S and $\{x\}$ separately, minus the number of edges linking x to nodes in S[1].

We can now evaluate the LHS of the submodularity condition as:

$$cut(A) + cut(\{x\}) - 2 \mid \{(u, x) \in E, u \in A\} \mid - cut(A)$$

and its RHS as:

$$cut(B) + cut(\{x\}) - 2 \mid \{(u, x) \in E, u \in B\} \mid - cut(B)$$

which, after some simple algebra, leads to the following inequality:

$$\mid \{(u, x) \in E, u \in B\} \mid \geq \mid \{(u, x) \in E, u \in A\} \mid$$

This last condition can be easily be proven to hold. Indeed, being A a subset of B, the number of nodes $v \in B$ which are connected to x must be at least the same as the number of nodes $v \in A$ connected to x.

2.2 The $cut(\cdot)$ function is not monotone. To prove that, let us consider the function $cut(V)$ as a counterexample, which evaluates to 0 for each possible graph $G = (V, E)$ since there cannot be any edge $(v, w) \in E$ where $v \in V$ and $w \in V \setminus V = \emptyset$. On the contrary, unless the set E itself is empty, there will always be a strict subset S containing at least one node $v \in S$ that is connected to a node $w \in V \setminus S$. It follows that $cut(S) > cut(V)$ even though $S \subset V$, which is against the definition of monotonicity.

[1] Undirected graphs are being considered here for the sake of brevity, but the same reasoning can be extended to directed graphs using two different sets: one for edges (u, x) and one for edges (x, u).

2.3 In order to find an approximation for the maximum cut in a graph, we may use an algorithm that iteratively refines a solution starting from a random one. The algorithms steps will be:

1. build a random set S by taking each node $v \in V$ with probability $\frac{1}{2}$;

2. for nodes $x \notin S$, compute their gain using the equations from **2.1.** as:

   $gain(x) = cut(S \cup \{x\}) - cut(S) = cut(\{x\}) - 2 \mid \{(u,x) \in E, x \in S\} \mid$

   while for nodes $y \in S$ the gain is defined as:

   $gain(y) = cut(S \backslash \{y\}) - cut(S) = 2 \mid \{(u,y) \in E, u \in S \backslash \{y\}\} \mid - cut(\{y\})$

3. select the node which introduces the maximal - positive - gain and move it either inside or outside S, depending on its original position;

4. repeat steps (2) and (3) until a stopping criteria is met, e.g., convergence - no positive gains - or fixed number of iterations reached.

Since this algorithm improves the random assignment, its approximation factor will be $\geq \frac{1}{2}$.

Question 3

3.1 For this proof, we will rely on a different - but equivalent - definition of submodularity, namely for each $A, B \subseteq U$:

$$f(A) + f(B) \geq f(A \cup B) - f(A \cap B)$$

Let us consider the independent-cascade model. When a node $v \in V$ is activated, its influence may be spread to its adjacent - inactive - nodes w such that $\exists (v,w) \in E$ with probability $p(v,w)$. Since the action timestep is not relevant for the purposes of the activation, we could precompute the activation of each *edge* $e \in E$ as the **result of a random process** and obtain a **fixed set** $E' \subseteq E$ of all the activated nodes. Eventually, we can quantify $\sigma(\cdot)$ as:

$$\sigma(S) = \mid S \mid + \mid \{v \in V \setminus S \text{ s.t. } \exists path(s,v,G'), s \in S, G' = (V,E')\} \mid$$

namely the number of seeds - i.e., the initially active nodes -, plus the number of all the remaining nodes that are connected to the seeds through a path involving active edges only.

The sumbodularity of $\sigma(\cdot)$ can be easily proved by stating that, given a pair of sets $A, B \subseteq U$, any node $v \notin A \cup B$ that gets activated using both the sets as seeds, gets activated as well using either of the two sets as seeds. Indeed, let us consider an active node $v \in V \setminus (A \cup B)$, then there must exist an active path $\{n, n_1, ..., n_k, v\}$ with $n \in A \cup B$ and $n_i \notin A \cup B, \forall i \in \{1, ..., k\}$. This last condition is guaranteed because, if there would have been a node $n_i \in A \cup B$ then we could have replaced the whole path with the sub-path $\{n_i, n_{i+1}, ..., n_k, v\}$ without loss of generality. Now, since none of the internal nodes were part of the two initial sets, either $n \in A$ and therefore this path would have existed even if using as seed the A set only, otherwise it would have existed if using as seed the B set only. This proves that:

$$\sigma(A) + \sigma(B) \geq \sigma(A \cup B) \geq \sigma(A \cup B) - \sigma(A \cap B)$$

hence $\sigma(\cdot)$ is submodular. At this point, the submodularity of $f(\cdot)$ comes as a direct consequence since the influence function $f(\cdot)$ is defined on the expected value of $\sigma(\cdot)$.

3.2 The influence function $f(\cdot)$ is monotone. Indeed, let us start by proving that the same property holds for $\sigma(\cdot)$ and eventually extend this result to $f(\cdot)$. As we showed in **3.1**, for a fixed subset $E' \subseteq E$ we get that:

$$\sigma(A) = \mid A \mid + \mid \{v \in V \setminus A \text{ s.t. } \exists path(a, v, G'), a \in A, G' = (V, E')\} \mid$$

from which it follows that, for any $A' = A \cup \{x\}$:

$$\sigma(A') = 1 + \mid A \mid + \mid \{v \in V \setminus A' \text{ s.t. } \exists path(a, v, G'), a \in A', G' = (V, E')\} \mid$$

Now, given that the introduction of a new origin can only increment the number of reachable nodes, the number of active nodes obtained using A' as seed must be at least the same as the number of active nodes obtained using A as seed, minus one element to account for the additional node x which may be present in the latter set but certainly not in the former. It follows that $\sigma(A \cup \{x\}) \geq \sigma(A)$, thus proving the monotonicity of $\sigma(\cdot)$. Accordingly, $f(\cdot)$ is monotone as well since it is defined on its expected value.

3.3 The easiest way to solve the *influence-maximization* problem is to start from an empty set $A = \emptyset$ and iteratively insert a new node $x \in V \setminus A$ that maximizes the greedy policy $gain(x) = f(A \cup \{x\} - f(A)$ until $\mid A \mid = k$.

It has been proven that the greedy policy is able to find the solution of a non-negative monotone submodular function with an approximation factor of $1-\frac{1}{e}$, provided that the function itself can be evaluated with certainty. Even though this is not the case for the influence function, since it is probabilistic rather then deterministic, we may still evaluate it numerically up to a certain error ε, leading to an overall approximation factor of $1-\frac{1}{e}-\varepsilon$.

Question 4

One way to measure polarization exploiting both the users' opinions and the network structure may rely on the average opinion of the neighborhood of each node. E.g., for each node $v \in V$ having a set of adjacent nodes $adj(v) = \{w \text{ s.t. } \exists(v,w) \in E\}$ with opinions $z_w \in [-1,1]$, we may express this value as:

$$\mathcal{Z}(v,G) = \sum_{w \in adj(v)} \frac{z_w}{\mid adj(v) \mid}$$

This formula tries to describe the source of information of a node; indeed, when the neighborhood opinion is near-zero it means that the node is either surrounded by moderate users or by users with balancing extreme opinions, and vice versa. Accordingly, we can define the polarization score of the entire graph $G = (V, E)$ as the average squared neighborhood opinion of each node, namely:

$$\mathcal{P}(G) = \sum_{v \in V} \frac{\mathcal{P}(v,G)^2}{\mid V \mid}$$

As an example, let us compute this metric for a fully connected graph with $2N$ nodes, half of which have opinion $z=-1$ while the other half has opinion $z=1$. Independently from its belief, each neighborhood will have an average opinion:

$$\mathcal{Z}(v,G) = -\frac{z_v}{2N-1}$$

since all the nodes' beliefs but one are balanced. As a consequence, the overall polarization will be:

$$\mathcal{P}(G) = (\frac{-z_v}{2N-1})^2 = \frac{1}{(2N-1)^2} \approx 0$$

Similarly, we can compute the polarization for another reference graph, where all the nodes share the same opinion z. In this case, each node will be such that $\mathcal{Z}(v, G) = z$, leading to a polarization $\mathcal{P}(G) = z^2$. Even though this may be debatable, I believe it to be correct since, despite minimizing the conflict, this kind of network may be steered towards either one or the other pole, showing in fact a certain polarization.

Question 5

Some may believe that social-media companies should not intervene as moderators within their platforms, and that the emergence of filter bubbles do not justify them to forcibly steer users towards contents coming from users having a different opinion from their own. Arguments opposing this top-down intervention generally follow two principles. The first, a practical one, states that companies should not be appointed to handle extremist drifts, but rather this should be a responsibility of the ruling class; moreover, it is believed to be unfair to force private companies to perform actions against their interests, since many studies proved that a to polarized community corresponds a higher engagement on the platform and, as a consequence, an increment in the company's profits. The second one, instead, leans on the idea that the presence of an external moderator would prevent freedom of though, speech, and association, which on the contrary should be able to rise on their own within an unregulated virtual space.

I do not share neither of those positions. Firstly, because I completely reject the idea that a private company must not be held responsible for anything that happens on its platforms and the communities it creates. Not to mention that, in my opinion, economic interests cannot be considered neither more important than nor as important as ethical questions, especially when they may lead to strong shifts in both people lives and society as a whole. Secondly, even though I may share the idea that any form of moderation would prevent any freedom, it is well known that this moderation is currently happening, since our expositions are already driven by algorithms aimed at expressly creating filter bubbles in order to maximize the profits of companies, which means that people are not free at all to be exposed to any information they wish to receive without any algorithmic intervention.

Programming project

Part 1

I found the paper of Chitra and Musco very interesting and well written. All the concepts and ideas that they presented came out clear, as well as the experiments they have done and the results they have obtained.

In my opinion, the strongest point of this work lies in the correct balance between a plain theoretical expositions - with new definitions and proofs on the mathematical guarantees of their findings - and a deep experimental analysis with multiple results demonstrating both on synthetic and real data the correctness of the *filter bubble* hypothesis. Finally, the last section, where a newly-introduced methodology to reduce the effects of the filter bubble issue is proposed, is not only interesting from a practical point of view, but it poses as well new technical and ethical questions about how and why hi-tech corporations are currently adopting certain methodologies rather than other on their social-media platforms.

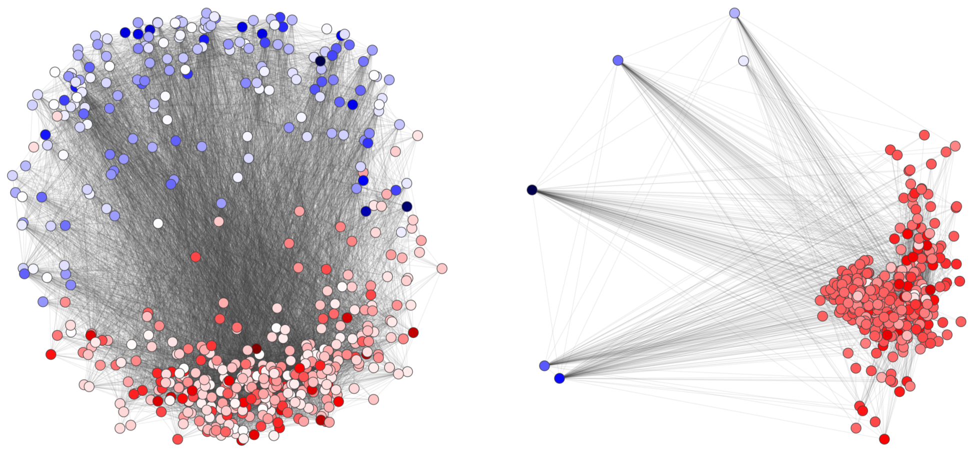
Nonetheless, it is precisely this last section that may be mostly attacked by criticisms. The strong assumptions made on the opinion dynamics models might even be reasonably realistic, but the way in which the authors quantify engagement on the platform seems to be too simple to deeply explain how this algorithmic shift will impact the companies' profits. This is, I think, the major shortcoming of the study, which leaves lots of space for future improvements and massive large-scale ad-hoc experiments which should be carried out on the topic to assess the quality of the proposed solution.

Part 2

Figure 1 displays the two graphs obtained from the reddit and the twitter datasets. The nodes' innate opinions s are shown using a linear colormap, where blue stands for the lowest opinion value and red for the highest one, and have been computed as the result of the linear system:

$$s = (L + I)z$$

where I is the identity matrix, $L = D - A$ is the laplacian of the graph - with A being its adjacency matrix and D a diagonal matrix containing the nodes' degrees -, and z is the expressed opinion of each node. In particular, given that the original data contained more than one expressed opinion for



(a) The reddit graph. (b) The twitter graph.

Figure 1: Innate opinions of users in the social networks. Blue nodes have low opinion values, red nodes have high one.

each node, z has been obtained by averaging all the opinions expressed by a node, similarly to what has been done in the paper by Chitra and Musco.

Part 3

Recalling the definition of average neighborhood opinion which was used in **Question 4** of the previous section to quantify polarization, we might as well adopt it to assess the level of homophily of each single node as:

$$\mathcal{H}(v, G) = 1 - (\frac{z_v - \mathcal{Z}(v, G)}{2})^2$$

or rather, one minus the squared difference between the node's opinion and the average opinion of its neighbors divided by two - this factor is used to normalized the result in $[0, 1]$. Reasonably, whenever $z_v = \mathcal{Z}(v, G)$ we get that $\mathcal{H}(v, G) = 1$, while the quantity decreases as the distance between these two values increases. Same as before, the overall homophily of the graph may be computed as the nodes' average, i.e.:

$$\mathcal{H}(G) = \frac{1}{\mid V \mid} \sum_{v \in V} \mathcal{H}(v, G)$$

	REDDIT		TWITTER	
	Expressed	**Innate**	**Expressed**	**Innate**
Avg	0.9999	0.9841	0.9999	0.9909
Std	0.0001	0.0306	0.0014	0.0263
Min	0.9991	0.7014	0.9705	0.5532
Max	1.0000	1.0000	1.0000	1.0000

Table 1: Homophily statistics for the two datasets.

Table 1 lists some statistics about users' homophily computed respectively to their innate and - average - expressed opinion for both the reddit and the twitter datasets. The *average* column represents the overall homophily of each graph, but *minimum*, *maximum*, and *standard deviation* are included as well to guarantee a broader view. As expected, the homophily levels are very high, and in particular they are on average higher when considering the expressed opinions even though, in that case, their range is lower. This is reasonable as well, since the expressed opinions come from the Friedkin-Johnsen model, thus the beliefs of the nodes are subjected to the influence of their neighbors, leading to an overall stronger homophily and a lower standard deviation.