Seminar 13

Wednesday, January 6, 2021 4:04 PM

The Fourier Transform

$f : \mathbb{R} \to \mathbb{C}$ a Fourier original, $f \in \mathcal{L}^1(\mathbb{R})$ the space of Lebesgue measurable functions

$\int_{-\infty}^{+\infty} |f(x)|\, dx$ is convergent

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-isx}\, dx, \ s \in \mathbb{R}$$ the image of f under the Fourier transform

Notation: $\mathcal{F}[f(x)](s) = F(s)$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s)\, e^{-isx}\, ds$$

- $f : \mathbb{R} \to \mathbb{C}$ is even function $\Rightarrow$ $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$ the cosine Fourier transform of f

  $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(s) \cos sx\, ds$$

- $f : \mathbb{R} \to \mathbb{C}$ an odd function $\Rightarrow$ $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, d$ the sine FT of f

  $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(s) \sin sx\, ds$$

Problems - $a > 0$, $b \in \mathbb{R}$. Find

2.17 i) $$\mathcal{F}\left[\frac{x}{(x^2+a^2)^2}\right](s) = -\frac{1}{2}\mathcal{F}\left[\frac{-2x}{(x^2+a^2)^2}\right](s) = -\frac{1}{2}\mathcal{F}\left[\left(\frac{1}{x^2+a^2}\right)'\right](s) =$$

$$\boxed{\mathcal{F}[f'(x)](s) = is\,\mathcal{F}[f(x)](s)}$$

$$= -\frac{1}{2}\cdot is\,\mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = -\frac{1}{2}\cdot is \cdot \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a|s|} = \frac{-is}{2a}\sqrt{\frac{\pi}{2}}\, e^{-a|s|}$$

$$\boxed{\mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a|s|}}$$

ii) $$\mathcal{F}\left[\frac{1}{(x^2+a^2)^2}\right](s) = ?$$

$$\boxed{\mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = \frac{1}{a}\cdot\sqrt{\frac{\pi}{2}}\cdot e^{-a|s|}} \quad / (\cdot)'_a$$

$$\mathcal{F}\left[\frac{-2a}{(x^2+a^2)^2}\right](s) = -\frac{1}{a^2}\sqrt{\frac{\pi}{2}}\, e^{-a|s|} + \frac{1}{a}\sqrt{\frac{\pi}{2}}\cdot(-|s|)\cdot e^{-a|s|}$$

$$-2a\,\mathcal{F}\left[\frac{1}{(x^2+a^2)^2}\right](\rho) = -\frac{1}{a^2}\sqrt{\frac{\pi}{2}}\,e^{-a|\rho|}\left(1+a|\rho|\right) \quad \Big/ \cdot\left(-\frac{1}{2a}\right)$$

$$\mathcal{F}\left[\frac{1}{(x^2+a^2)^2}\right](\rho) = \frac{1}{2a^3}\sqrt{\frac{\pi}{2}}\,e^{-a|\rho|}\left(1+a\cdot|\rho|\right)$$

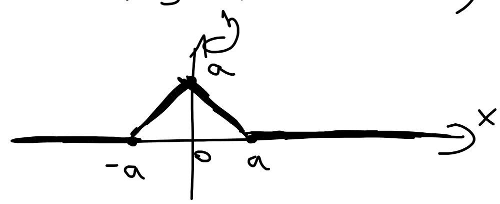
iii) $\mathcal{F}\left[\dfrac{e^{-ibx}}{x^2+a^2}\right](\rho) = ?$

$$\boxed{\mathcal{F}\left[\frac{1}{x^2+a^2}\right](\rho) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\,e^{-a|\rho|}}$$

$$\boxed{\mathcal{F}\left[f(x)e^{-iax}\right](\rho) = \mathcal{F}[f(x)](\rho+a)}$$ Translation of the image

$$\mathcal{F}\left[\frac{e^{-ibx}}{x^2+a^2}\right](\rho) = \mathcal{F}\left[\frac{1}{x^2+a^2}\right](\rho+b) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\,e^{-a|\rho+b|}.$$

2.18 i) $f(x) = \begin{cases} 0, & |x| \geq a \\ x+a, & -a < x \leq 0 \\ -x+a, & 0 < x < a \end{cases}$ $\qquad f(x) = a - |x|,\ |x| \leq a \ (a>0)$



f is an even function $(f(-x) = f(x))$

$\Downarrow$

$\mathcal{F}[f](\rho) = \mathcal{F}_c[f](\rho)$

$\boxed{\rho \neq 0}$

$$\mathcal{F}_c[f](\rho) = \sqrt{\frac{2}{\pi}}\int_0^a f(x)\cos\rho x\,dx = \sqrt{\frac{2}{\pi}}\int_0^a (a-x)\cos\rho x\,dx =$$

$$= \sqrt{\frac{2}{\pi}}\left(\int_0^a a\cos\rho x\,dx - \int_0^a x\cos\rho x\,dx\right) =$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{\rho}\sin\rho x\Big|_0^a - \int_0^a x\left(\frac{1}{\rho}\sin\rho x\right)'dx\right) =$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{\rho}\sin\rho a - \frac{x}{\rho}\sin\rho x\Big|_0^a + \int_0^a 1\cdot\frac{1}{\rho}\sin\rho x\,dx\right) =$$

$$= \sqrt{\frac{2}{\pi}}\left(\cancel{\frac{a}{\rho}\sin\rho a} - \cancel{\frac{a}{\rho}\sin\rho a} - \frac{1}{\rho^2}\cos\rho x\Big|_0^a\right) =$$

$$= \sqrt{\frac{2}{\pi}}\left(-\frac{1}{\rho^2}\cos\rho a + \frac{1}{\rho^2}\right) = \frac{1}{\rho^2}\sqrt{\frac{2}{\pi}}\left(1-\cos\rho a\right) = \frac{1}{\rho^2}\sqrt{\frac{2}{\pi}}\,2\sin^2\frac{\rho a}{2} =$$

$1 - \cos x = 2\sin^2\frac{x}{2}$

$$= \sqrt{\frac{2}{\pi}}\cdot\left(\frac{\sin\frac{\rho a}{2}}{\frac{\rho a}{2}}\right)^2\cdot\frac{a^2}{4}\cdot 2 = \sqrt{\frac{2}{\pi}}\cdot\frac{a^2}{2}\cdot\left(\frac{\sin\frac{\rho a}{2}}{\frac{\rho a}{2}}\right)^2$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\quad}{\frac{\omega a}{2}} \right) \cdot \frac{a}{4} \cdot 2 = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{2} \cdot \left(\frac{\sin \frac{\omega a}{2}}{\frac{\omega a}{2}} \right)$$

• $\boxed{\omega = 0}$

$$\mathcal{F}[f(x)](\omega) = \sqrt{\frac{2}{\pi}} \int_0^a (a-x) \underbrace{\cos 0}_{=1} \, dx = \sqrt{\frac{2}{\pi}} \left[ax \Big|_0^a - \frac{x^2}{2} \Big|_0^a \right] =$$

$$= \sqrt{\frac{2}{\pi}} \left(a^2 - \frac{a^2}{2} \right) = \frac{a^2}{2} \cdot \sqrt{\frac{2}{\pi}} = \frac{a^2}{\sqrt{2\pi}}$$

$$\mathcal{F}[f(x)](\omega) = \begin{cases} \frac{a^2}{\sqrt{2\pi}} \cdot \left(\frac{\sin \frac{\omega a}{2}}{\frac{\omega a}{2}} \right)^2 , & \omega \neq 0 \\ \frac{a^2}{\sqrt{2\pi}} , & \omega = 0 \end{cases}$$

ii) $f(x) = \begin{cases} e^{ax} , & x < 0 \\ \frac{1}{2} , & x = 0 \\ 0 , & x > 0 \end{cases} \quad , a > 0$

$$\mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ax} \cdot e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-i\omega)x} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{a-i\omega} e^{(a-i\omega)x} \Big|_{-\infty}^{0} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a-i\omega}^{a+i\omega} \left[e^0 - \lim_{x \to -\infty} e^{(a-i\omega)x} \right] =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{a+i\omega}{a^2+\omega^2} \left[1 - 0 \right] = \frac{a+i\omega}{\sqrt{2\pi}\,(a^2+\omega^2)} \quad , a > 0, \ \omega \in \mathbb{R}$$

$$\lim_{x \to -\infty} e^{(a-i\omega)x} = \lim_{x \to -\infty} e^{ax} \cdot e^{-i\omega x} = \lim_{x \to -\infty} \underset{\downarrow 0}{e^{ax}} \underbrace{(\cos \omega x - i \sin \omega x)}_{\text{bounded}} = 0$$

iii) $f(x) = \begin{cases} \cos 2x , & |x| < a \\ 0 , & |x| > a \end{cases}$

• For $\boxed{\omega \neq \pm 2}$

$$\mathcal{F}_c[f](\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underline{\underline{f(x)}} \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos 2x \cos \omega x \, dx =$$

$$\boxed{\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^a [\cos(2x+\omega x) + \cos(2x - \omega x)] \, dx =$$

$$\int^a [\quad (\quad)x + \cos(2-\omega)x] \, dx = \qquad \omega \neq \pm 2$$

$$\pi \quad 2 \int_0$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^a [\cos(2+p)x + \cos(2-p)x]\, dx = \qquad p \neq \pm 2$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(2+p)x}{2+p} \Big/_0^a + \frac{\sin(2-p)x}{2-p} \Big/_0^a \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(2+p)a}{2+p} + \frac{\sin(2-p)a}{2-p} \right]$$

• For $\boxed{p = \pm 2}$ $\Rightarrow \mathcal{F}_c[f(x)](2) = \sqrt{\frac{2}{\pi}} \int_0^a \cos 2x \cos 2x\, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos^2 2x\, dx =$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1+\cos 4x}{2}\, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left(x \Big/_0^a + \frac{\sin 4x}{4} \Big/_0^a \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left(a + \frac{\sin 4a}{4} \right).$$

iv) $\mathcal{F}[e^{-\frac{x^2}{2}}](p) = ?$ $\quad \mathcal{F}\left[\frac{d^n}{dx^n} \left(e^{-\frac{x^2}{2}} \right) \right] = ?$ $\quad \mathcal{F}[e^{-x^2}](p)$

• $\mathcal{F}[e^{-\frac{x^2}{2}}](p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \cdot e^{-ipx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} - ipx}\, dx =$

$$\boxed{\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}}$$ the Gaussian integral

$$-\frac{x^2}{2} - ipx = -\left(\frac{x^2}{2} + ipx \right) = -\left(\left(\frac{x}{\sqrt{2}} \right)^2 + 2 \cdot \frac{x}{\sqrt{2}} \cdot \frac{ip}{\sqrt{2}} + \frac{i^2 p^2}{2} - \frac{i^2 p^2}{2} \right) =$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$= -\left(\frac{x}{\sqrt{2}} + \frac{ip}{\sqrt{2}} \right)^2 - \frac{p^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}} + \frac{ip}{\sqrt{2}} \right)^2 - \frac{p^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}} + \frac{ip}{\sqrt{2}} \right)^2} dx$$

$$u = \frac{x}{\sqrt{2}} + \frac{ip}{\sqrt{2}} \qquad x = +\infty \Rightarrow u = +\infty$$
$$x = -\infty \Rightarrow u = -\infty$$

$$du = \frac{1}{\sqrt{2}}\, dx \Rightarrow dx = \sqrt{2}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \int_{-\infty}^{+\infty} e^{-u^2} \cdot \sqrt{2}\, du = \frac{\sqrt{2}\sqrt{\pi}}{\sqrt{2\pi}} e^{-\frac{p^2}{2}} = e^{-\frac{p^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{m^2}{2}} \cdot \sqrt{2}\, du = \frac{\sqrt{2}\sqrt{\pi}}{\sqrt{2\pi}} e^{} = e^{}$$

$$\Rightarrow \mathcal{F}\left[e^{-\frac{x^2}{2}}\right](p) = e^{-\frac{p^2}{2}}.$$

- $$\mathcal{F}\left[\frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)\right](p) = (ip)^n \mathcal{F}\left[e^{-\frac{x^2}{2}}\right](p) = (ip)^n e^{-\frac{p^2}{2}}.$$

$$\mathcal{F}\left[f^{(n)}(x)\right](p) = (ip)^n \mathcal{F}[f(x)](p)$$

- $$\mathcal{F}\left[e^{-x^2}\right](p) = \mathcal{F}\left[e^{-\frac{2x^2}{2}}\right](p) = \mathcal{F}\left[e^{-\frac{(\sqrt{2}x)^2}{2}}\right](p) = \mathcal{F}\left[f(\sqrt{2}x)\right](p) =$$

$$\boxed{\mathcal{F}\left[e^{-\frac{x^2}{2}}\right](p) = e^{-\frac{p^2}{2}}}$$

we use the property "change of scale"

$$\boxed{\mathcal{F}\left[f(ax)\right](p) = \frac{1}{|a|}\mathcal{F}[f(x)]\left(\frac{p}{a}\right) \text{ or } \mathcal{F}\left[f\left(\frac{x}{a}\right)\right](p) = |a|\,\mathcal{F}[f(x)](ap)}$$

$$= \frac{1}{\sqrt{2}}\mathcal{F}[f(x)]\left(\frac{p}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{-\left(\frac{p}{\sqrt{2}}\right)^2 \cdot \frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot e^{-\frac{p^2}{4}}.$$

Ex

$$I = \int_{-\pi}^{\pi} \frac{3+\cos x}{7\sin x + 25}\, dx \quad , \quad 7\sin x + 25 \neq 0, \ \forall x \in \mathbb{R}$$

Remark: $$\boxed{\int_a^{a+2\pi} R(\sin x, \cos x)\, dx = \int_0^{2\pi} R(\sin x, \cos x)\, dx}$$

$$I = \int_0^{2\pi} \frac{3+\cos x}{7\sin x + 25}\, dx \quad ; \text{ we add } J = \int_0^{2\pi} \frac{\sin x}{7\sin x + 25}\, dx$$

$$K = I + iJ = \int_0^{2\pi} \frac{3 + \overbrace{\cos x + i\sin x}^{e^{ix}}}{7\sin x + 25}\, dx = \int_0^{2\pi} \frac{3 + e^{ix}}{7\sin x + 25}\, dx =$$

$$e^{ix} = z \Rightarrow i\underbrace{e^{ix}}_{z} dx = dz \Rightarrow dx = \frac{dz}{iz}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$I = ?$

$$= \int \frac{z+3}{7\frac{z^2-1}{2iz} + 25} \frac{dz}{iz} = \int \frac{(z+3)\,2i\cancel{z}}{7z^2 - 7 + 50iz} \cdot \frac{dz}{i\cancel{z}} = 2\int \frac{z+3}{7z^2+50iz-7}\, dz$$

$$= \int_{C: |z|=1} \frac{z+3}{7 \cdot \frac{z^2-1}{2iz} + 25} \frac{dz}{iz} = \int_C \frac{(z+3)\, 2\cancel{iz}}{7z^2 - 7 + 50iz} \cdot \frac{dz}{\cancel{iz}} = 2 \int_C \underbrace{\frac{z+3}{7z^2+50iz-7}}_{f(z)} dz$$

$$\Delta = -2500 + 4 \cdot 49 = -2304$$

$$z_{1,2} = \frac{-50i \pm 48i}{14} = \frac{-25i \pm 24i}{7}$$

$z_1 = -\frac{i}{7} \in \text{int}\, C$ pole of order 1

$z_2 = -7i \notin \text{int}\, C$

$$\operatorname*{Res}_{z=-\frac{i}{7}} f(z) = \frac{z+3}{7(z+7i)} \Bigg|_{z=-\frac{i}{7}} = \frac{-\frac{i}{7}+3}{7\left(-\frac{i}{7}+7i\right)} = \frac{-i+21}{7 \cdot 48i} = \frac{21-i}{7 \cdot 48i}$$

or

$$\operatorname{Res} f(z) = \frac{z+3}{(7z^2+50iz-7)'} \Bigg|_{z=-\frac{i}{7}}$$

$$I = 2 \cdot 2\pi i \operatorname*{Res}_{z=-\frac{i}{7}} f(z) = \cancel{2 \cdot 2\pi i} \cdot \frac{21-i}{7 \cdot \cancel{48i}_{12}} = \frac{\pi}{84}(21-i) = \frac{\pi}{4} - i\frac{\pi}{84}$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}.$$