Seminar 4 : Complex integrals. Cauchy's integral formula

Note Title 10/21/2020

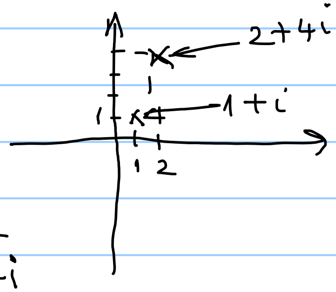
1. Integrals $f$ defined and continuous on a curve $C$

We define the integral along some path

$$\int_C f(z)\,dz = \int_{(x_1,y_1)}^{(x_2,y_2)} (u+iv)(dx+idy)$$

$z = x+iy$

$dz = dx + idy$




(1) Evaluate $\displaystyle\int_{1+i}^{2+4i} z^2\,dz$

a) along the parabola $x=t,\ y=t^2,\ 1\le t\le 2$

b) along the straight line joining $1+i$ and $2+4i$

We have : $$\int_{1+i}^{2+4i} z^2\,dz = \int_{(1,1)}^{(2,4)} (x+iy)^2(dx+idy) = \int_{(1,1)}^{(2,4)} (x^2-y^2+2xyi)(dx+idy) =$$

$$= \int_{(1,1)}^{(2,4)} (x^2-y^2)dx - 2xy\,dy + i\int_{(1,1)}^{(2,4)} 2xy\,dx + (x^2-y^2)\,dy$$

a) The points (1,1) and (2,4) correspond to $t=1$ and $t=2$ $\begin{cases} x=t \\ y=t^2 \end{cases} \Rightarrow \begin{matrix} dx = dt \\ dy = 2t\,dt \end{matrix}$

$$I = \int_1^2 \left[(t^2 - t^4)\,dt - 2t^3 \cdot 2t\,dt\right] + i\int_1^2 \left[2t^3\,dt + (t^2 - t^4)\,2t\,dt\right]$$

$$= \int_1^2 (t^2 - 5t^4)\,dt + i\int_1^2 (4t^3 - 2t^5)\,dt = \frac{t^3}{3}\Big|_1^2 - 5\frac{t^5}{5}\Big|_1^2 + i\left(4\frac{t^4}{4}\Big|_1^2 - 2\frac{t^6}{6}\Big|_1^2\right) =$$

45-63

$$= \frac{1}{3}(8-1) - (32-1) + i\left[16 - 1 - \frac{1}{3}(64-1)\right] = \frac{7}{3} - 31 + i\left(15 - \frac{63}{3}\right) = -\frac{86}{3} - 6i$$

b) The eq. of the line (1,1) and (2,4) : $\frac{x-1}{2-1} = \frac{y-1}{4-1} \Rightarrow x - 1 = \frac{y-1}{3} \Rightarrow y = 3x - 2$

$dx = dx$, $dy = 3dx$

$$I = \int_1^2 \left\{\left[x^2 - (3x-2)^2\right]dx - 2x(3x-2)\,3dx\right\} + i\int_1^2 \left\{2x(3x-2)\,dx + \left[x^2 - (3x-2)^2\right]\cdot 3\,dx\right\}$$

$$= -\frac{86}{3} - 6i$$

Remark: The line integrals are independent of the path.

Met II

$$I = \int_{1+i}^{2+4i} z^2\,dz = \frac{z^3}{3}\Big|_{1+i}^{2+4i} = \frac{(2+4i)^3}{3} - \frac{(1+i)^3}{3} = -\frac{86}{3} - 6i$$

We apply Remark 1.6 / 28

(2) $\int_C (z-a)^n dz$ $\quad C: |z-a| = r$, $n \in \mathbb{Z}$

↑ the circle centered in $a$, of radius $r$

We make the following change of variables:

$$z = a + re^{i\theta}, \quad \theta \in [0, 2\pi]$$
$$dz = rie^{i\theta} d\theta$$

$$I = \int_0^{2\pi} (re^{i\theta})^n rie^{i\theta} d\theta = \int_0^{2\pi} r^n e^{ni\theta} rie^{i\theta} d\theta = ir^{n+1} \int_0^{2\pi} e^{i\theta(n+1)} d\theta$$

- for $n = -1 \Rightarrow I = \int_C \frac{dz}{z-a} = i \int_0^{2\pi} d\theta = i\theta \Big|_0^{2\pi} = 2\pi i$
- for $n \neq -1 \Rightarrow I = \int_C (z-a)^n dz = ir^{n+1} \cdot \frac{1}{i(n+1)} \underbrace{e^{i\theta(n+1)} \Big|_{\theta=0}^{\theta=2\pi}}_{0} = 0$

$$e^{i2\pi(n+1)} - e^0 = \underbrace{\cos(2\pi(n+1))}_{1} + i\sin(2\pi(n+1)) - 1 = 0$$

Homework: 1.31 / 31

2. Cauchy's integral formula

$f: D \to \mathbb{C}$ holomorphic on $D$, $z_0 \in \operatorname{int} C$

$C$ – curve: closed, simple, positively oriented w.r.t. its interior $\operatorname{int} C$

Then:

$$\int_C f(z)\,dz = 0 \ ; \quad \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) \ ; \quad \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

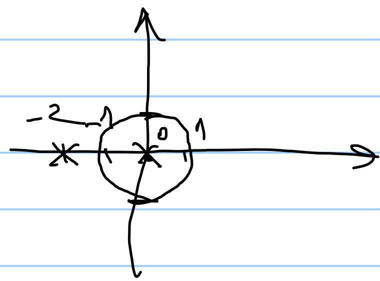
Remarks: 1) $\int \frac{f(z)}{z-z_0}\,dz$ — it is defined $\forall z \in \mathbb{C}$ which is not on the boundary of the domain

2) $z_0 \in \operatorname{int} C \Rightarrow I = 2\pi i f(z_0)$

$z_0 \notin \operatorname{int} C \Rightarrow$ the integrand is holomorphic function $\Rightarrow I = 0$

(3) $I = \int_C \frac{e^z}{z^2+2z}\,dz$, where $C: |z| = 1$

↑ a circle radius 1 center 0




$z^2+2z = 0$

$z(z+2)=0 \Rightarrow z_1 = 0 \in \text{int}\, C$

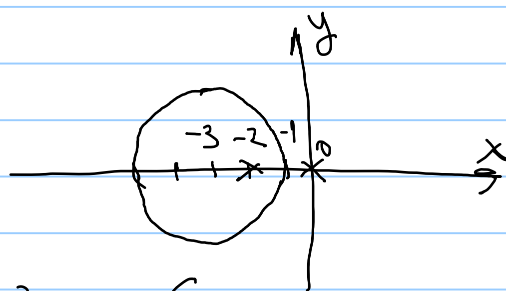
$z_2 = -2 \notin \text{int}\, C$

$$I = \int_C \frac{\overset{f(z)}{\frac{e^z}{z+2}}}{z-0} = 2\pi i\, f(0) = 2\pi i \frac{e^0}{0+2} = 2\pi i \cdot \frac{1}{2} = \pi i$$

$f(z) = \frac{e^z}{z+2}$ holom. function, $z_0 = 0$

(4) $\displaystyle\int_C \frac{e^z}{z^2+2z}\, dz$

$C: |z+3| = 2$




$\left( \begin{array}{l} |0+3| = 3 > 2 \Rightarrow 0 \notin \text{int}\, C \\ |-2+3| = 1 < 2 \Rightarrow -2 \in \text{int}\, C \end{array} \right)$

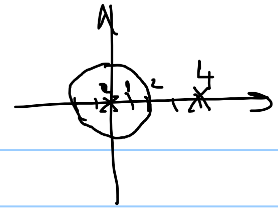
$z_1 = 0 \notin \text{int}\, C$

$z_2 = -2 \in \text{int}\, C$

$f(z) = \frac{e^z}{z}$ holom. function, $z_0 = -2$

$$I = \int_C \frac{\frac{e^z}{z}}{z+2} = 2\pi i\, f(-2) = 2\pi i \cdot \frac{e^{-2}}{-2} = -\frac{\pi i}{e^2}$$

(5) $I = \int_{C : |z| = 2} \frac{\sin z}{z^2 (z-4)} dz$



$z^2 = 0 \Rightarrow z_1 = z_2 = 0 \in \text{int} C$

$z - 4 = 0 \Rightarrow z_3 = 4 \notin \text{int} C$ $\quad f(z) = \frac{\sin z}{z-4}$ holom. function, $z_0 = 0$, $n+1 = 2 \Rightarrow n = 1$

$$I = \int_C \frac{\frac{\sin z}{z-4}}{z^2} dz = \frac{2\pi i}{1!} f'(0) = 2\pi i \left(-\frac{1}{4}\right) = \frac{-\pi i}{2}$$

$$f(z) = \frac{\sin z}{z-4} \Rightarrow f'(z) = \frac{\cos z (z-4) - \sin z}{(z-4)^2} \Rightarrow f'(0) = \frac{\cos 0 (-4) - \sin 0}{(-4)^2} = \frac{-4}{16} = -\frac{1}{4}$$

(6) $I = \int_{C : |z| = 5} \frac{dz}{z^2 + 16}$

$z^2 + 16 = 0 \Rightarrow (z + 4i)(z - 4i) = 0$

$z_1 = 4i \in \text{int} C$ $\quad (|4i| = 4 < 5)$

$z_2 = -4i \in \operatorname{int} C \qquad (|-4i| = 4 < 5)$

– we have to expand by partial fraction

$$I = \frac{1}{8i}\int_C \frac{z+4i-(z-4i)}{(z+4i)(z-4i)}\,dz = \frac{1}{8i}\int_C \frac{dz}{z-4i} - \frac{1}{8i}\int_C \frac{dz}{z+4i} = \qquad f(z)=1$$

$$= \frac{1}{8i}\,2\pi i \cdot f(4i) - \frac{1}{8i}\,2\pi i\, f(-4i) = 0$$

(7) $$\int_{C:|z|=2} \frac{\cosh^2(iz)}{z^3}\,dz = \frac{2\pi i}{2!}\cdot f''(0)$$

$z_{1,2,3} = 0 \in \operatorname{int} C$ ; $n+1=3 \Rightarrow n=2$ ; $f(z) = \cosh^2(iz)$ holom. function

$$\left(\cosh(z)\right)' = \sinh(z)$$
$$\left(\sinh(z)\right)' = \cosh(z)$$

$f'(z) = 2\cosh(iz)\cdot i\sinh(iz) = 2i\cosh(iz)\sinh(iz)$

$f''(z) = 2i\left[\sinh^2(iz) + \cosh^2(iz)\right]\cdot i = -2\left[\cosh^2(iz)+\sinh^2(iz)\right]$

$f''(0) = -2\cdot\left[\cosh^2(0)+\sinh^2(0)\right] =$

$$= -2\left[\left(\frac{e^{0}+e^{0}}{2}\right)^2 + \left(\frac{e^{0}-e^{0}}{2}\right)^2\right] = -2(1+0) = -2$$

$$I = \frac{2\pi i}{2!}(-2) = -2\pi i$$

(8) $\displaystyle\int \frac{e^{z}}{z(z-1)^3}\,dz$

$C: |z-1| = \frac{1}{4}$

homework: 1.35 / 33