Seminar 7 gr. 2

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The residue of a complex function

$$f(z)=\sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n = \underbrace{\dots + \frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{z-z_0}}_{\text{principal part}} + \underbrace{a_0 + a_1(z-z_0)+a_2(z-z_0)^2+\dots}_{\text{analytical part}}$$

a_{-1} = RESIDUE

Met I – to compute the Laurent series of $f(z)$ around z_0

ex1 Find the residue of $\frac{\sin z}{z^2}$ at $z_0=0$

$$\frac{\sin z}{z^2}=\frac{1}{z^2}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots\right)=\frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}-\dots$$

$$\operatorname{Res}_{z=0} f(z)=1$$

$\left(\lim_{z\to 0}\left(z\cdot\frac{\sin z}{z^2}\right)=1 \Rightarrow z=0 \text{ pole of order } 1\right)$

Met II – the function $f(z)$ has a simple pole (pole of order 1)

$$\left.\begin{aligned} & f(z)=\frac{g(z)}{h(z)} \\ & g,h \text{ holom. functions at } z_0 \\ & h(z_0)=0,\ h'(z_0)\neq 0 \end{aligned}\right\} \Rightarrow \boxed{\operatorname{Res}_{z=z_0} f(z)=\frac{g(z_0)}{h'(z_0)}} = \left.\frac{g(z)}{h'(z)}\right|_{z=z_0}$$

ex2 $f(z)=\frac{\cos z}{z^4-1}$ at $z_0=i$

Factor the denominator: $z^4-1=(z^2-1)(z-i)(z+i)$

$$f(z)=\frac{\cos z}{(z^2-1)(z-i)(z+i)}$$

$\cos z$ holomorphic at $z=i$

$z-i$ – appears once at the denominator

$\Rightarrow z=i$ is a simple pole

$$\operatorname{Res}_{z=i} f(z)=\left.\frac{\cos z}{(z^4-1)'}\right|_{z=i}=\left.\frac{\cos z}{4z^3}\right|_{z=i}=\frac{\cos i}{4i^3}=-\frac{1}{4i}\cdot\frac{e^{i\cdot i}+e^{-i\cdot i}}{2}=-\frac{e^{i\cdot i}+e^{-i\cdot i}}{8i}$$

$g(z)=\cos z$

$h(z)=z^4-1$

How do you know z_0 is a simple pole?

- if $\lim_{z\to z_0}(z-z_0)f(z)=0 \Rightarrow f(z)$ is analytic at z_0
- if $\lim_{z\to z_0}(z-z_0)f(z)=\infty \Rightarrow z_0$ is a higher order pole of $f(z)$
- if $\lim_{z\to z_0}(z-z_0)f(z)$ is not zero and finite $\Rightarrow z_0$ is simple pole.

Met III z_0 pole of order $n\geq 1$.

$$\operatorname{Res}_{z=z_0} f(z)=\frac{1}{(n-1)!}\lim_{z\to z_0}\left[(z-z_0)^n f(z)\right]^{(n-1)}$$

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!}\lim_{z\to z_0}\left[(z-z_0)^n f(z)\right]^{(n-1)}}$$

ex 3 Find the residue of $f(z)=\dfrac{z\cos z}{(z-\pi)^3}$ at $z_0=\pi$

$\Rightarrow z=\pi$ is a pole of order 3

$$\Rightarrow \operatorname*{Res}_{z=\pi} f(z) = \frac{1}{(3-1)!}\lim_{z\to\pi}\left[\cancel{(z-\pi)^3}\cdot\frac{z\cos z}{\cancel{(z-\pi)^3}}\right]^{(3-1)} = \frac{1}{2!}\lim_{z\to\pi}(z\cos z)'' =$$

$$(z\cos z)'' = (\cos z - z\sin z)' = -\sin z - \sin z - z\cos z$$

$$= \frac{1}{2}\lim_{z\to\pi}\left(-2\sin z - z\cos z\right) = \frac{1}{2}\left(-2\underbrace{\sin\pi}_{0} - \pi\underbrace{\cos\pi}_{-1}\right) = \frac{\pi}{2}$$

Residue Theorem for evaluate Integrals

$$\boxed{\int_C f(z)\,dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z)}$$

1.49 Find the residues of the singular points of the following functions.

a) $f(z)=\dfrac{z^2}{(1+z)^3}$

$(1+z)^3=0 \Rightarrow z=-1$ pole of order $\underline{3}$ $\Rightarrow$ ($n=3$ in the formula)

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{(3-1)!}\lim_{z\to -1}\left[\cancel{(z+1)^3}\cdot\frac{z^2}{\cancel{(1+z)^3}}\right]^{(3-1)} = \frac{1}{2!}\lim_{z\to-1}(z^2)'' = \frac{1}{2}\lim_{z\to-1}2 = 1$$

$$(z^2)''=(2z)'=2$$

b) $f(z)=\dfrac{1}{1+e^z}$

$1+e^z=0 \Rightarrow e^z=-1 \mid \log \Rightarrow z=\log(-1) \Rightarrow z=\ln|-1|+i(\pi+2k\pi),\ k\in\mathbb{Z}$

$\Rightarrow z=i\pi(1+2k),\ k\in\mathbb{Z}$ pole of order $\underline{1}$

$$\operatorname*{Res}_{z=i\pi(1+2k)} f(z) \underset{(\text{met I})}{=\!=} \frac{g(z_0)}{h'(z_0)} = \frac{1}{e^{\pi i(1+2k)}} = \frac{1}{\cos(\pi+2k\pi)+i\sin(\pi+2k\pi)} = -1$$

$h(z)=1+e^z$

$g(z)=1$

$$\underset{(\text{met II})}{=\!=}\ \frac{1}{(1-1)!}\lim_{z\to i\pi(1+2k)}\left[(z-i\pi(1+2k))\cdot\frac{1}{1+e^z}\right]^{(1-1)} =$$

$$=\lim_{z\to i\pi(1+2k)}\frac{z-i\pi(1+2k)}{1+e^z}\ \underset{\frac{0}{0}}{\overset{l'H}{=}}\ \lim_{z\to i\pi(1+2k)}\frac{1}{e^z} = \frac{1}{e^{i\pi(1+2k)}} = -1$$

c) $f(z) = \dfrac{1}{\sin \pi z}$

$\sin \pi z = 0 \Rightarrow \pi z = k\pi, k \in \mathbb{Z} \Rightarrow z = k, k \in \mathbb{Z}, \quad k = 0, \pm 1, \pm 2, \ldots$

$z = k$ pole of order 1

$$\operatorname*{Res}_{z=k} f(z) = \frac{g(z_0)}{h'(z_0)} \underset{\substack{g(z)=1 \\ h(z)=\sin \pi z}}{=} \left.\frac{1}{\pi \cos \pi z}\right|_{z=k} = \frac{1}{\pi \cos \pi k} = \frac{1}{\pi (-1)^k} = \frac{(-1)^k}{\pi}, \quad k \in \mathbb{Z}$$

d) $f(z) = \dfrac{\cos z}{(z-1)^2}$

$(\cos z)' = (-\sin z)$

$z = 1$ pole of order 2

$\operatorname*{Res}_{z=1} f(z) = -\sin 1$ ✓

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!} \lim_{z \to 1} \left(\cancel{(z-1)^2} \cdot \frac{\cos z}{\cancel{(z-1)^2}} \right)' = \lim_{z \to 1} (-\sin z) = -\sin 1$$

1) Evaluate the following integrals.

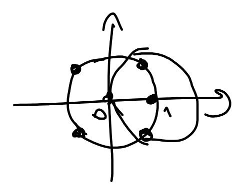
a) $\displaystyle\int \frac{dz}{z^4+1}$

$C: |z-1| = 1$

Met I:

$z^4 + 1 = 0 \Rightarrow z^4 = -1 \qquad -1 = \cos \pi + i \sin \pi$

$z_k = \cos \dfrac{\pi + 2k\pi}{4} + i \sin \dfrac{\pi + 2k\pi}{4}, \quad k = \overline{0,3}$



$z_0 = \cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} + i \dfrac{\sqrt{2}}{2}$

$z_1 = \cos \dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4} = -\dfrac{\sqrt{2}}{2} + i \dfrac{\sqrt{2}}{2}$

$z_2 = \cos \dfrac{5\pi}{4} + i \sin \dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2} - i \dfrac{\sqrt{2}}{2}$

$z_3 = \cos \dfrac{7\pi}{4} + i \sin \dfrac{7\pi}{4} = \dfrac{\sqrt{2}}{2} - i \dfrac{\sqrt{2}}{2}$

$z_0, z_3 \in \operatorname{int} C, \quad z_1, z_2 \notin \operatorname{int} C$

$$I = 2\pi i \left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_3} f(z) \right)$$

$f(z) = \dfrac{1}{z^4+1}$

z_0, z_3 poles of order 1

• $$\operatorname*{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)} \underset{h(z)=z^4+1}{=} \left.\frac{1}{4z^3}\right|_{z=z_0} = \frac{1}{4z_0^3} = \frac{1}{4\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^3} = \frac{1}{4\left(\frac{\sqrt{2}}{2}\right)^3 (1 + 3i + \underbrace{3i^2}_{-3} + \underbrace{i^3}_{-i})} =$$

$$= \frac{1}{4 \cdot \frac{2\sqrt{2}}{8}(2i-2)} = \frac{1}{2\sqrt{2}(i-1)}$$

• $$\operatorname*{Res}_{z=z_3} f(z) = \frac{g(z_3)}{h'(z_3)} = \left.\frac{1}{4z_3^3}\right|_{z=z_3} = \frac{1}{4\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^3} = \frac{1}{4\left(\frac{\sqrt{2}}{2}\right)^3 (1 - 3i + \underbrace{3i^2}_{-3} - \underbrace{i^3}_{-i})} =$$

$$\bullet \operatorname*{Res}_{z=z_3} f(z) = \frac{g(z_3)}{h'(z_3)} = \left(\frac{1}{4z^3}\right)\Bigg|_{z=z_3} = \frac{1}{4\left(\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\right)^3} = \frac{1}{4\left(\frac{\sqrt2}{2}\right)^3(1-3i+3\underbrace{i^2}_{-3}-i^3)} =$$

$$= \frac{1}{4\,\frac{2\sqrt2}{8}(-2i-2)} = \frac{1}{2\sqrt2(-i-1)}$$

$$I = 2\pi i\left(\frac{1}{2\sqrt2(i-1)} + \frac{1}{2\sqrt2(-i-1)}\right) = \frac{\cancel{2}\pi i}{\cancel{2}\sqrt2}\,\frac{-\cancel{i}-1+\cancel{i}-1}{2} = -\frac{\pi i}{\sqrt2}$$

Met II.

$z^4=-1 \Rightarrow z_0, z_3 \in \operatorname{int} C,\ z_1, z_2 \notin \operatorname{int} C$

$$\operatorname*{Res}_{\substack{z=z_k\\k=0,k=3}} f(z) = \frac{\cancel{z_k}\,1}{4(z_k^3)} = \frac{z_k}{4z_k^4} \underset{z_k^4=-1}{=} \frac{z_k}{4(-1)} = -\frac{1}{4}z_k$$

$$I = 2\pi i \sum_{\substack{k=0\\k=3}} \operatorname*{Res}_{z=z_k} f(z) = 2\pi i \sum_{\substack{k=0\\k=3}}\left(-\frac14 z_k\right) = \frac{2\pi i}{-4}(z_0+z_3) =$$

$$= -\frac{\pi i}{2}\left(\frac{\sqrt2}{2}+\cancel{i\frac{\sqrt2}{2}}+\frac{\sqrt2}{2}-\cancel{i\frac{\sqrt2}{2}}\right) = -\frac{\pi i}{\cancel{2}}\cdot\frac{\cancel{2}\sqrt2}{2} = -\frac{\pi i}{\sqrt2}$$

b) $$\int_{|z|=3} \frac{e^{\frac{\pi}{z-2i}}}{(z-1)(z-2i)}\,dz$$

$z-1=0 \Rightarrow z_1=1$
$z-2i=0 \Rightarrow z_2=2i$ $\in \operatorname{int} C$

$$z_1 = 1 \text{ - pole of order 1} \Rightarrow \operatorname*{Res}_{z=1} f(z) = \frac{g(z)}{h'(z)}\Bigg|_{z=1} = \frac{e^{\frac{\pi}{z-2i}}}{1(z-2i)+(z-1)1}\Bigg|_{z=1} = \frac{e^{\frac{\pi}{1-2i}}}{1-2i}$$

$z_2=2i$ essential singularity

We expand by Laurent series around $z=2i$ (with powers of $z-2i$)

$$f(z) = \frac{e^{\frac{\pi}{z-2i}}}{(z-1)(z-2i)} = \frac{1}{z-2i}\cdot e^{\frac{\pi}{z-2i}}\cdot\frac{1}{(z-2i)+2i-1} = \frac{1}{2i-1}\,\frac{1}{1+\frac{z-2i}{2i-1}}\cdot\frac{1}{z-2i}\cdot e^{\frac{\pi}{z-2i}} =$$

$$= \frac{1}{2i-1}\cdot\frac{1}{1-\frac{z-2i}{1-2i}}\cdot\frac{1}{z-2i}\cdot e^{\frac{\pi}{z-2i}} =$$

$$= \frac{1}{2i-1}\left[1+\frac{z-2i}{1-2i}+\left(\frac{z-2i}{1-2i}\right)^2+\left(\frac{z-2i}{1-2i}\right)^3+\dots\right]\cdot\frac{1}{z-2i}\cdot e^{\frac{\pi}{z-2i}}$$

$$= \frac{1}{2i-1}\left[\frac{1}{z-2i}+\frac{1}{1-2i}+\frac{z-2i}{(1-2i)^2}+\frac{(z-2i)^2}{(1-2i)^3}+\dots\right]\left[1+\frac{\pi}{(z-2i)1!}+\frac{\pi^2}{(z-2i)^2 2!}+\frac{\pi^3}{(z-2i)^3 3!}+\dots\right]$$

$$\operatorname*{Res}_{z=2i} f(z) = a_{-1} = \frac{1}{2i-1}\left[1 + \frac{\pi}{(1-2i)1!} + \frac{\pi^2}{(1-2i)^2 2!} + \dots\right] =$$

a_{-1} = the coefficient of $\frac{1}{z-2i}$

$$= \frac{1}{2i-1}\left[1 + \frac{\frac{\pi}{1-2i}}{1!} + \frac{\frac{\pi^2}{(1-2i)^2}}{2!} + \frac{\frac{\pi^3}{(1-2i)^3}}{3!} + \dots\right] = \frac{1}{2i-1} e^{\frac{\pi}{1-2i}}$$

$$I = 2\pi i\left(\frac{e^{\frac{\pi}{1-2i}}}{1-2i} + \frac{e^{\frac{\pi}{1-2i}}}{2i-1}\right) = 0$$

c) $$\int_{|z|=3} \frac{e^{\frac{\pi}{z-1}}}{z^2-3z+2}\,dz = \int_{|z|=3} \frac{e^{\frac{\pi}{z-1}}}{(z-1)(z-2)}\,dz$$

$z^2-3z+2=0 \Rightarrow (z-1)(z-2)=0$

$z_1=2$
$z_2=1$ $\in \operatorname{int} C$

$z_1=2$ pole of order 1 $\Rightarrow \operatorname*{Res}_{z=2} f(z) = \frac{e^{\frac{\pi}{z-1}}}{2z-3}\Big|_{z=2} = \frac{e^{\pi}}{1} = e^{\pi}$

$z_2=1$ is not a pole
it is an essential singularity

we expand for Laurent series around $z=1$ (with powers of $z-1$)

$$f(z) = \frac{e^{\frac{\pi}{z-1}}}{(z-1)(z-2)} = \frac{1}{z-1}\cdot e^{\frac{\pi}{z-1}}\cdot\frac{1}{z-2} = \frac{1}{z-1}\cdot e^{\frac{\pi}{z-1}}\cdot\frac{1}{(z-1)-1} =$$

$$=(-1)\frac{1}{1-(z-1)}\cdot\frac{1}{z-1}\cdot e^{\frac{\pi}{z-1}} = (-1)\left[1+(z-1)+(z-1)^2+(z-1)^3+\dots\right]\frac{1}{z-1}\,e^{\frac{\pi}{z-1}} =$$

$$=(-1)\left[\frac{1}{z-1}+1+(z-1)+(z-1)^2+\dots\right]\cdot\left[1+\frac{\pi}{(z-1)1!}+\frac{\pi^2}{(z-1)^2 2!}+\frac{\pi^3}{(z-1)^3 3!}+\dots\right]$$

$$\operatorname*{Res}_{z=1} f(z) = a_{-1} = (-1)\left[1+\frac{\pi}{1!}+\frac{\pi^2}{2!}+\frac{\pi^3}{3!}+\dots\right] = (-1)\cdot e^{\pi} = -e^{\pi}$$

a_{-1} = the coefficient of $\frac{1}{z-1}$

$\boxed{\text{Homework: } 1.50}$

$$I = 2\pi i\left(e^{\pi} - e^{\pi}\right) = 0$$