Seminar 9 grupa 1

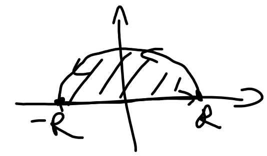
Wednesday, November 25, 2020 4:04 PM

Using Residue Theorem to evaluate real integrals

IV $\int_0^\infty f(x)dx$, $f(x) = \frac{P(x)}{Q(x)}$

$Q(x) \neq 0, \forall x \in \mathbb{R}$
P, Q are polynomials m, n, $n \geq m+2$
f is even function

$$\int_0^\infty f(z)dz = \frac{1}{2}\int_{-\infty}^{+\infty} f(z)dz = \frac{1}{2}\cdot 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$$



ex 1 $\int_0^\infty \frac{dx}{(x^2+4)(x^2+16)}$

$f(z) = \frac{1}{(z^2+4)(z^2+16)}$ even function

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dz}{(z^2+4)(z^2+16)} = \frac{1}{\cancel{2}}\cancel{2}\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$$

$z^2+4 = 0 \Rightarrow z_{1,2} = \pm 2i$
$z^2+16 = 0 \Rightarrow z_{3,4} = \pm 4i$

singularities $\Rightarrow$ $z_1 = 2i$, $z_3 = 4i$ poles of order 1 with $\operatorname{Im} z_k > 0$

$$\operatorname{Res}_{z=2i} f(z) = \frac{1}{(z^2+16)(z+2i)}\Big|_{z=2i} = \frac{1}{(4i^2+16)\cdot 4i} = \frac{1}{48i}$$

$$\operatorname{Res}_{z=4i} f(z) = \frac{1}{(z^2+4)(z+4i)}\Big|_{z=4i} = \frac{1}{(16i^2+4)\cdot 8i} = -\frac{1}{96i}$$

$$I = \pi i \left(\frac{1}{48i} - \frac{1}{96i}\right) = \frac{^{2/}\pi}{48} - \frac{\pi}{96} = \frac{\pi}{96}$$

ex2 $\int_0^\infty \frac{x^2}{(x^2+1)^2}dx$

$f(z) = \frac{z^2}{(z^2+1)^2}$ even function

$$I = \frac{1}{\cancel{2}}\cdot \cancel{2}\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$$

$(z^2+1)^2=0 \Rightarrow z_{1,2}=\pm i$ poles of order 2

only $z_1=i$, $\operatorname{Im} z_1>0$

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{(2-1)!}\lim_{z\to i}\Big((z-i)^2\,\frac{z^2}{\underbrace{(z^2+1)^2}_{(z-i)^2(z+i)^2}}\Big)' = \lim_{z\to i}\Big(\frac{z^2}{(z+i)^2}\Big)' =$$

$$=\lim_{z\to i}\frac{2z(z+i)^2-2(z+i)z^2}{(z+i)^4} = \lim_{z\to i}\frac{2z(z+i)-2z^2}{(z+i)^3}=$$

$$=\frac{2i\cdot 2i-2i^2}{(2i)^3}=\frac{-4+2}{-8i}=\frac{1}{4i}$$

$$I=\pi i\cdot\frac{1}{4i}=\frac{\pi}{4}$$

V $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$, $f(x)=\frac{P(x)}{Q(x)}e^{\lambda i x}$ $Q(x)\neq 0$, degree(P) < degree(Q)

$$\int_{-\infty}^{+\infty}\frac{P(z)}{Q(z)}e^{i\lambda z}\,dz=\begin{cases}2\pi i\displaystyle\sum_{\operatorname{Im} z_k>0}\operatorname*{Res}_{z=z_k} f(z), & \lambda>0\\[2ex] -2\pi i\displaystyle\sum_{\operatorname{Im} z_k<0}\operatorname*{Res}_{z=z_k} f(z), & \lambda<0\end{cases}$$

ex3

$$\int_{-\infty}^{+\infty}\frac{e^{-ix}}{x^2-2x+5}\,dx$$

$$f(z)=\frac{e^{-iz}}{z^2-2z+5}$$ $\boxed{\lambda=-1<0}$ $\Rightarrow I=-2\pi i\displaystyle\sum_{\operatorname{Im} z_k<0}\operatorname*{Res}_{z_k} f(z)$

$z^2-2z+5=0$ $\Delta=4-20=-16$ $\Rightarrow z_{1,2}=\frac{2\pm 4i}{2}=1\pm 2i$

$z_1=1-2i$, $\operatorname{Im} z_1<0$ pole of order 1

$$\operatorname*{Res}_{z=1-2i} f(z)=\frac{e^{-iz}}{2z-2}\Big|_{z=1-2i}=\frac{e^{-i(1-2i)}}{\cancel{2}-4i-\cancel{2}}=\frac{e^{2-i}}{-4i}$$

$$I=-\cancel{2}\pi\cancel{i}\Big(-\frac{1}{\cancel{4i}_{\,2}}\Big)e^2(\cos 1-i\sin 1)=\frac{\pi e^2}{2}(\cos 1-i\sin 1)$$

ex4 $+\infty$ e^{2ix}

ex 4 $\displaystyle\int_{-\infty}^{+\infty} \frac{x e^{2ix}}{(x^2+4)^2}\,dx$

$$f(z)=\frac{z e^{2iz}}{(z^2+4)^2}$$

$\lambda = 2 > 0 \Rightarrow I = 2\pi i \sum\limits_{\substack{\operatorname{Im} z_k > 0}} \operatorname{Res}_{z=z_k} f(z)$

$(z^2+4)^2=0 \Rightarrow z_{1,2} = \pm 2i$ poles of order 2

$\Rightarrow z_1 = 2i,\ \operatorname{Im} z_1 > 0$

$$\operatorname{Res}_{z=2i} f(z) = \frac{1}{(2-1)!}\lim_{z\to 2i}\left(\cancel{(z-2i)^2}\cdot\frac{z e^{2iz}}{\cancel{(z-2i)^2}(z+2i)^2}\right)' =$$

$$= \lim_{z\to 2i}\frac{(e^{2iz}+2ize^{2iz})(z+2i)^{\cancel{2}} - z e^{2iz}\cdot 2\cancel{(z+2i)}}{(z+2i)^{\cancel{4}\,3}} =$$

$$= \frac{(e^{-4}-4e^{-4})\cdot 4i - 4ie^{-4}}{(4i)^3} = \frac{-12ie^{-4}-4ie^{-4}}{-64i}$$

$$= \frac{1}{4}e^{-4}$$

$$\Rightarrow I = 2\pi i\,\frac{1}{4e^4} = \frac{\pi i}{2e^4}$$

ex 5 $\displaystyle\int_0^{2\pi} \frac{\cos 2x}{5-3\cos x}\,dx$

$z = e^{ix},\ 0 \le \theta \le 2\pi$

$\cos x = \frac{z^2+1}{2z}$; $dx = \frac{dz}{iz}$

$\cos 2x = \underline{\cos^2 x} - \sin^2 x$

$$I = \int_{C:\,|z|=1} \frac{\left(\frac{z^2+1}{2z}\right)^2 - \left(\frac{z^2-1}{2iz}\right)^2}{5-3\,\frac{z^2+1}{2z}}\cdot\frac{dz}{iz} = \int_C \frac{\frac{(z^2+1)^2}{4z^2}+\frac{(z^2-1)^2}{4z^2}}{\frac{10z-3z^2-3}{2z}}\cdot\frac{dz}{iz} =$$

$$= \int_C \frac{z^4+\cancel{2z^2}+1+z^4-\cancel{2z^2}+1}{\cancel{4}_{2}\,z^{\cancel{2}}}\cdot\frac{\cancel{2z}}{-3z^2+10z-3}\cdot\frac{dz}{iz} = \frac{-1}{i}\int_C \frac{z^4+1}{z^2(3z^2-10z+3)}\,dz$$

$z^2=0 \Rightarrow z_{1,2}=0 \in \text{int}\, C$ pole of order 2 $\quad (C: |z|=1)$

$3z^2-10z+3=0 \Rightarrow \Delta = 100-36=64 \to z_{3,4}=\frac{10\pm 8}{6}$ $\begin{cases} z_3 = 3 \notin \text{int}\, C \\ z_4=\frac{1}{3} \in \text{int}\, C \text{ pole of order 1} \end{cases}$

$\operatorname{Res}_{z=0} f(z) = \frac{10}{9}$ ✓

$\operatorname{Res}_{z=\frac{1}{3}} f(z) = -\frac{41}{12}$

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z\to 0}\left(\cancel{z^2}\,\frac{z^4+1}{\cancel{z^2}(3z^2-10z+3)}\right)' =$$

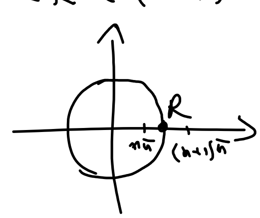
$$= \lim_{z\to 0} \frac{4z^3(3z^2-10z+3)-(z^4+1)(6z-10)}{(3z^2-10z+3)^2} = \frac{10}{9} \checkmark$$

$$\operatorname{Res}_{z=\frac{1}{3}} f(z) = \frac{z^4+1}{z^2\, 3(z-3)\cdot 1}\Bigg|_{z=\frac{1}{3}} = \frac{\frac{1}{81}+1}{\frac{1}{9}\cdot 3\left(\frac{1}{3}-3\right)} = \frac{82}{81}\cdot\frac{9}{-8} = \frac{-41}{36}$$

$$\left(\begin{array}{l} 3z^2-10z+3 = 3\underline{\left(z-\frac{1}{3}\right)}(z-3) \\ \operatorname{Res}_{z=\frac{1}{3}} f(z) = \frac{g(z)}{h'(z)}\Big|_{z=\frac{1}{3}} \qquad g(z) = \frac{z^4+1}{z^2\cdot 3(z-3)} \\ \qquad\qquad h(z) = z-\frac{1}{3} \quad h'(z)=1 \end{array} \right)$$

$$I = 2\pi \cancel{i}\cdot\left(-\frac{1}{\cancel{i}}\right)\cdot\left(\frac{10}{9}-\frac{41}{36}\right) = 2\pi\cdot\frac{1}{36} = \frac{\pi}{18}$$

ex6 $\quad \displaystyle\int_C \frac{dz}{z^2 \sin z} \qquad C: |z| = R \qquad n\pi < R < (n+1)\pi$



$z=0$ triple pole $\in \text{int}\, C$

$\sin z = 0 \Rightarrow z_k = k\pi,\ k\in \mathbb{Z}^*,\ k = \pm 1, \pm 2, \dots, \pm n \in \text{int}\, C$

simple poles

$$\operatorname*{Res}_{z=k\pi} f(z) = \frac{g(z)}{h'(z)} = \frac{\frac{1}{z^2}}{\cos z}\Bigg|_{z=k\pi} = \frac{\frac{1}{k^2\pi^2}}{(-1)^k} = \frac{(-1)^k}{k^2\pi^2}$$

$$\operatorname*{Res}_{z=k\pi} f(z) + \operatorname*{Res}_{z=-k\pi} f(z) = \frac{(-1)^k}{k^2\pi^2} + \frac{(-1)^k}{k^2\pi^2}$$

method I

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(3-1)!} \lim_{z\to 0}\left(z^3 \cdot \frac{1}{z^2 \sin z}\right)'' = \frac{1}{2}\lim_{z\to 0}\left(\frac{z}{\sin z}\right)'' = \dots = \frac{1}{6}$$

method II

$$\frac{1}{z^2 \sin z} = \frac{1}{z^2}\cdot\frac{1}{\sin z} = \frac{1}{z^2}\cdot\frac{1}{\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots} =$$

$$= \frac{1}{z^2}\cdot\frac{1}{z}\cdot\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots} =$$

$$\frac{1}{1-u} = 1 + u + u^2 + \dots$$

$$= \frac{1}{z^3}\;\frac{1}{1 - \underbrace{\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)}_{u}} =$$

$$= \frac{1}{z^3}\left[1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)^2 + \left(\dots\right)^3 + \dots\right]$$

We have to find the coeff. of $\frac{1}{z}$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{3!} = \frac{1}{6}$$

$$\Rightarrow I = 2\pi i\left[\frac{1}{6} + 2\cdot\sum_{k=1}^{n}\frac{(-1)^k}{k^2\pi^2}\right]$$

Homework 1) $\displaystyle\int_0^\infty \frac{dx}{(x^2+4)(x^2+25)}$ $\frac{\pi}{140}$; 2) $\displaystyle\int_{-\pi}^{\pi}\frac{\sin 7x}{5-4\sin x}\,dx$

3) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+2)^2}\,dx$ $\frac{\pi}{4\sqrt{2}}$; 4) $\displaystyle\int_0^{2\pi}\frac{\sin^2 x}{5+4\cos x}\,dx$

$\displaystyle\int^{+\infty} \frac{dx}{\dots}$

$4\sqrt{2}$

5) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+2)}$ $\frac{\pi}{6}$