Laboratory work 3

1 Objectives

The objective of this laboratory is to implement specific C++ classes for handling matrices.

2 Matrix

A matrix is a collection of numbers arranged into rows and columns. In computer graphics we are using matrices to represent spatial transformations. The number of rows and columns are equal in this case, denoting a square matrix. In this laboratory we are using 3x3 and 4x4 matrices.

3 Identity matrix

The identity matrix is the matrix that has 1 on the diagonal and 0 elsewhere. For example, the 3x3 identity matrix is:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4 Matrix operations

4.1 Multiplication with a scalar

By multiplying a matrix **M** with a scalar **k** we multiply each element of the matrix with the scalar value.

$$\mathrm{k}\mathbf{M} = k \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} km_{11} & km_{12} & km_{13} \\ km_{21} & km_{22} & km_{23} \\ km_{31} & km_{32} & km_{33} \end{bmatrix}$$

4.2 Addition of matrices

The matrices addition is done element by element, like in this example:

$$\mathbf{M} + \mathbf{N} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} = \begin{bmatrix} m_{11} + n_{11} & m_{12} + n_{12} & m_{13} + n_{13} \\ m_{21} + n_{21} & m_{22} + n_{22} & m_{23} + n_{23} \\ m_{31} + n_{31} & m_{32} + n_{32} & m_{33} + n_{33} \end{bmatrix}$$

4.3 Multiplication with another matrix

In order to multiply two matrices, the number of columns in the first matrix must be the same with the number of rows of the second matrix.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{im} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1c} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \cdots & b_{mj} & \cdots & b_{mc} \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1c} \\ \vdots & & \vdots & & \vdots \\ p_{i1} & \cdots & p_{ij} & \cdots & p_{ic} \\ \vdots & & \vdots & & \vdots \\ p_{r1} & \cdots & p_{rj} & \cdots & p_{rc} \end{bmatrix}$$

The element p_{ij} is equal to:

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

Matrix multiplication is not commutative ($\mathbf{M_1 M_2} \neq \mathbf{M_2 M_1}$), but is associative and distributive:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

4.4 Multiplication with a column vector

This operation is a particular case of the multiplication of two matrices operation.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{im} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rm} \end{bmatrix} \begin{bmatrix} b_{11} \\ \vdots \\ b_{j1} \\ \vdots \\ b_{m1} \end{bmatrix} = \begin{bmatrix} p_{11} \\ \vdots \\ p_{i1} \\ \vdots \\ p_{r1} \end{bmatrix}$$

The element p_{i1} is equal to:

$$p_{i1} = a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{im}b_{m1}$$

4.5 Transposition

The transpose of a matrix (denoted as $\mathbf{M}^\mathbf{T}$) is the matrix where the columns and rows are switched. For example, for a 3x3 matrix:

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The transpose of two matrices product is $(\mathbf{AB})^\mathrm{T} = \mathbf{B}^\mathrm{T}\mathbf{A}^\mathrm{T}$.

4.6 Determinant of a matrix

For a 2x2 matrix the determinant is:

$$|\mathbf{M}| = \left|\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\right| = m_{11}m_{22} - m_{12}m_{21}$$

In order to compute the determinant, we have to find the cofactors of the matrix elements. The cofactor of each element of a square matrix is the determinant of a matrix (obtained by removing from the original matrix the row and column in which the element is in) multiplied by minus one in some cases. The sign of the cofactor can be determined by the following pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For a 4x4 matrix:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

the cofactor for the first row are

$$m_{11}^c = \left|\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}\right| \qquad m_{12}^c = -\left|\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}\right|$$

$$m_{13}^c = \left|\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}\right| \qquad m_{14}^c = -\left|\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}\right|$$

The determinant equals the sum of the products of the elements (of any row or column) with their cofactors.

For a 3x3 matrix the determinant is:

$$|\mathbf{M}| = \left|\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\right| = m_{11}\left|\begin{bmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{bmatrix}\right| - m_{12}\left|\begin{bmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{bmatrix}\right| + m_{13}\left|\begin{bmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}\right|$$

Similar, the determinant for a 4x4 matrix is:

$$|\mathbf{M}| = \left|\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}\right| = m_{11}\left|\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}\right| - m_{12}\left|\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}\right| + m_{13}\left|\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}\right| - m_{14}\left|\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}\right|$$

4.7 Inverse of a matrix

The inverse matrix (denoted as $\mathbf{A}^{-1}$) is the matrix for which $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. The inverse of two matrices product is $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

For a 4x4 matrix the inverse is:

$$\mathrm{M}^{-\mathbf{1}} = \frac{1}{|\mathbf{M}|}\begin{bmatrix} m_{11}^c & m_{21}^c & m_{31}^c & m_{41}^c \\ m_{12}^c & m_{22}^c & m_{32}^c & m_{42}^c \\ m_{13}^c & m_{23}^c & m_{33}^c & m_{43}^c \\ m_{14}^c & m_{24}^c & m_{34}^c & m_{44}^c \end{bmatrix}$$

5 Assignment

Download the source code from the web repository. You have to implement the methods inside the source files (mat3.cpp and mat4.cpp). The header files contain the definition of classes and the methods that should be implemented.