Seminar 3

Elementary functions of a complex variable

The **exponential function**, for every $z = x + iy,\ z \in \mathbb{C}$

$$e^z = e^x(\cos y + i \sin y)$$

$z = iy \Rightarrow$ $\cancel{x = 0} \Rightarrow e^z = e^{iy} = \cos y + i \sin y$
$z = x - iy \Rightarrow e^z = e^x(\cos y - i \sin y)$
$z = -iy \Rightarrow e^{-iy} = \cos y - i \sin y$

The **trigonometric functions**:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

The **hyperbolic functions**

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$
$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

The trigonometric and hyperbolic functions are related through the following relations

$$\cosh iz = \cos z$$
$$\sinh iz = i \sin z$$
$$\cos iz = \cosh z$$
$$\sinh z = -i \sin iz$$

All the formulas of trigonometry remain valid for trigonometric functions of a complex variable.

Example 1)

$$\begin{aligned}\sin a \cos b &= \frac{e^{ia} - e^{-ia}}{2i} \cdot \frac{e^{ib} + e^{-ib}}{2} \\ &= \frac{e^{i(a+b)} - e^{-i(a+b)}}{4i} + \frac{e^{i(a-b)} - e^{-i(a-b)}}{4i} \\ &= \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right].\end{aligned}$$

2) $(\cosh z)^2 - (\sinh z)^2 = 1$

$$\begin{aligned}(\cosh z)^2 - (\sinh z)^2 &= \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 \\ &= \frac{2+2}{4} = 1\end{aligned}$$

The **logarithmic function** (is a multi-valued function) $z \in \mathbb{C},\ z \neq 0$
The complex number w is called logarithm of z if

$$e^w = z. \ / \mathcal{L}og$$

The set of all these logarithms is

$$\boxed{Log z = \{\ln |z| + i(\arg z + 2k\pi), k \in \mathbb{Z}\}.}$$

The general power function (multi-valued function) is defined by

$$\boxed{z^\alpha = e^{\alpha Log z}}, \forall z \neq 0, \alpha \in \mathbb{C}.$$

1) (2.5) Write in algebraic form
a) $\cos(\pi + i)$; b) $Log(1 + i\sqrt{3})$;c) $Log(-1)$; d) 1^i; e) i^i; f) $\sin(1-i)$.

a) $\cos(\pi+i) = \dfrac{e^{i(\pi+i)} + e^{-i(\pi+i)}}{2} = \dfrac{e^{-1+i\pi} + e^{1-i\pi}}{2} = \dfrac{e^{-1}(\cos\pi + i\underbrace{\sin\pi}_{0}) + e^{1}(\cos\pi - i\underbrace{\sin\pi}_{0})}{2}$

$= \underbrace{\cos\pi}_{-1} \cdot \underbrace{\dfrac{e^{1}+e^{-1}}{2}}_{} = -\cosh 1.$

$\dfrac{1}{i} = -i$

f) $\sin(1-i) = \dfrac{e^{i(1-i)} - e^{-i(1-i)}}{2i} = \dfrac{e^{1+i} - e^{-1-i}}{2i} = \dfrac{1}{i}\left[\dfrac{e^{1}(\cos 1 + i\sin 1) - e^{-1}(\cos 1 - i\sin 1)}{2}\right] =$

$= \dfrac{1}{i}\left[\cos 1 \cdot \underbrace{\dfrac{e^{1}-e^{-1}}{2}}_{} + i\sin 1 \cdot \underbrace{\dfrac{e^{1}+e^{-1}}{2}}_{\cosh 1}\right] = \dfrac{1}{i}\left[\cos 1 \cdot \sinh 1 + i \sin 1 \cosh 1\right]$

$= \sin 1 \cosh 1 - i \cos 1 \cdot \sinh 1$

b) $Log(1+i\sqrt{3}) = \ln|1+i\sqrt{3}| + i(\arg(1+i\sqrt{3}) + 2k\pi) = \ln 2 + i\left(\dfrac{\pi}{3} + 2k\pi\right), k \in \mathbb{Z}.$

$|1+i\sqrt{3}| = \sqrt{1+3} = 2$

$\arg(1+i\sqrt{3}) \underset{Q\,I}{=} \arctan\sqrt{3} = \dfrac{\pi}{3}$

c) $Log(-1) = \ln|-1| + i(\arg(-1) + 2k\pi) = 0 + i(\pi + 2k\pi) = i(\pi + 2k\pi), k \in \mathbb{Z}.$

d) $1^i = e^{i Log 1} = e^{i[\ln|1| + i(\arg 1 + 2k\pi)]} = e^{i \cdot i \cdot 2k\pi} = e^{-2k\pi}, k \in \mathbb{Z}.$

e) $i^i = e^{i Log i} = e^{i\cdot[\ln|i| + i(\underbrace{\arg i}_{\frac{\pi}{2}} + 2k\pi)]} = e^{i \cdot i\left(\frac{\pi}{2} + 2k\pi\right)} = e^{-\left(\frac{\pi}{2} + 2k\pi\right)}, k \in \mathbb{Z}$

$*$ $\qquad z \in \mathbb{C}$

2) (1.27) Solve the equations: a) $\sin z = \frac{5}{3}$, b) $\cosh z = \frac{1}{2}$;

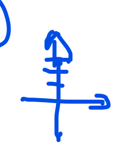
a) $\sin z = \frac{5}{3} \Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = \frac{5}{3} \Rightarrow t - \frac{1}{t} = \frac{10i}{3} \Rightarrow t^2 - \frac{10i}{3}t - 1 = 0 \,/\cdot 3$

we denote $e^{iz} = t$

$3t^2 - 10it - 3 = 0$, $\Delta = -100 + 36 = -64$

$t_{1,2} = \frac{10i \pm 8i}{6}$ $\nearrow t_1 = 3i$, $\searrow t_2 = \frac{1}{3}i$

I $t_1 = 3i \Rightarrow e^{iz} = 3i \,/\, \mathrm{Log} \Rightarrow iz = \mathrm{Log}(3i)$



$iz = \ln|3i| + i(\arg(3i) + 2k\pi)$

$iz = \ln 3 + i\left(\frac{\pi}{2} + 2k\pi\right) \,/\cdot(-i)$

$z_1 = \frac{\pi}{2} + 2k\pi - i\ln 3$, $k \in \mathbb{Z}$

II $t_2 = \frac{1}{3}i \Rightarrow e^{iz} = \frac{1}{3}i \,/\, \mathrm{Log} \Rightarrow iz = \mathrm{Log}\left(\frac{1}{3}i\right)$

$iz = \ln\left|\frac{1}{3}i\right| + i\left(\arg\left(\frac{1}{3}i\right) + 2k\pi\right)$

$iz = \ln\frac{1}{3} + i\left(\frac{\pi}{2} + 2k\pi\right) \,/\cdot(-i)$

$z_2 = \frac{\pi}{2} + 2k\pi + i\ln 3$, $k \in \mathbb{Z}$

The solution is: $z = \left\{\frac{\pi}{2} + 2k\pi - i\ln 3, k \in \mathbb{Z}\right\} \cup \left\{\frac{\pi}{2} + 2k\pi + i\ln 3, k \in \mathbb{Z}\right\}$

or

$z = \frac{\pi}{2} + 2k\pi \pm i\ln 3$, $k \in \mathbb{Z}$.

b) $\cosh z = \frac{1}{2} \Rightarrow \frac{e^{z} + e^{-z}}{2} = \frac{1}{2} \xRightarrow{e^{z} = t} t + \frac{1}{t} = 1 \Rightarrow t^2 - t + 1 = 0$, $\Delta = -3$

$t_{1,2} = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

I $t_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow e^{z} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \,/\, \mathrm{Log} \Rightarrow z = \mathrm{Log}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$\Rightarrow z_1 = \ln\left|\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| + i\left(\arg\underbrace{\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}_{(Q\,I)} + 2k\pi\right) = 0 + i\left(\frac{\pi}{3} + 2k\pi\right) = i\cdot\left(\frac{\pi}{3} + 2k\pi\right)$, $k \in \mathbb{Z}$

II $t_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2} \Rightarrow e^{z} = \frac{1}{2} - i\frac{\sqrt{3}}{2} \,/\, \mathrm{Log} \Rightarrow z = \mathrm{Log}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$z_2 = \ln\left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| + i\left(\arg\underbrace{\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}_{(Q4)} + 2k\pi\right) = 0 + i\left(-\frac{\pi}{3} + 2\pi + 2k\pi\right)$, $k \in \mathbb{Z}$

The solution is: $z = \left\{ i\cdot\left(\frac{\pi}{3}+2k\pi\right), k\in\mathbb{Z} \right\} \cup \left\{ i\left(-\frac{\pi}{3}+2\pi+2k\pi\right), k\in\mathbb{Z} \right\}$

c) $\cot z = 2+i \Rightarrow \frac{\cos z}{\sin z} = 2+i \Rightarrow \frac{e^{iz}+e^{-iz}}{\cancel{2}} \cdot \frac{\cancel{2}i}{e^{iz}-e^{-iz}} = 2+i$

$e^{iz} = t \rightarrow \left(t+\frac{1}{t}\right)i = (2+i)\left(t-\frac{1}{t}\right)$

$\cancel{ti} + \frac{i}{t} = 2t - \frac{2}{t} + \cancel{ti} - \frac{i}{t} \Longrightarrow \frac{2i}{t} - 2t + \frac{2}{t} = 0 \,\Big|\, :2$

$\Rightarrow \frac{i}{t} - t + \frac{1}{t} = 0 \Rightarrow i - t^2 + 1 = 0 \Rightarrow t^2 = 1+i$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ $\Big\} \Rightarrow$

$t_k = \sqrt[2]{\sqrt{2}}\left(\cos\frac{\frac{\pi}{4}+2k\pi}{2} + i\sin\frac{\frac{\pi}{4}+2k\pi}{2}\right), k=0,1$

$t_0 = \sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$; $t_1 = \sqrt[4]{2}\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$

I $e^{iz} = t_0 \Rightarrow e^{iz} = \underbrace{\sqrt[4]{2}}\left(\cos\underbrace{\frac{\pi}{8}} + i\sin\frac{\pi}{8}\right) \,\Big|\, \log \Rightarrow iz = \ln\sqrt[4]{2} + i\left(\frac{\pi}{8}+2k\pi\right) \,/(-i)$

$\Rightarrow z_1 = \frac{\pi}{8} + 2k\pi - \frac{i}{4}\ln 2, k\in\mathbb{Z}$

II $e^{iz} = t_1 \Rightarrow e^{iz} = \underbrace{\sqrt[4]{2}}\left(\cos\underbrace{\frac{9\pi}{8}} + i\sin\frac{9\pi}{8}\right) \,\Big|\, \log \Rightarrow iz = \ln\sqrt[4]{2} + i\left(\frac{9\pi}{8}+2k\pi\right) / (-i)$

$\Rightarrow z_2 = \frac{9\pi}{8} + 2k\pi - \frac{i}{4}\ln 2, k\in\mathbb{Z}$

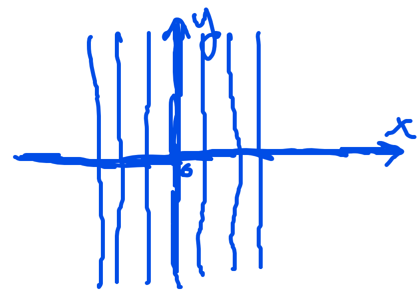
The solution is: $z = \left\{ \frac{\pi}{8} + 2k\pi - \frac{i}{4}\ln 2, k\in\mathbb{Z} \right\} \cup \left\{ \frac{9\pi}{8} + 2k\pi - \frac{i}{4}\ln 2, k\in\mathbb{Z} \right\}$

- we already know the modulus and the argument

3) Find the points from the complex plane where the function $\cos z$ takes real values

$z = x + iy$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$
$$= \frac{e^{-y+ix} + e^{y-ix}}{2}$$
$$= \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)}{2}$$
$$= \cos x \frac{e^{y} + e^{-y}}{2} + i \sin x \frac{e^{-y} - e^{y}}{2}$$
$$= \cos x \cosh y - i \sin x \sinh y$$



the algebraic form of cos z

$$\Rightarrow \sin x \sinh y = 0.$$

$\sin x = 0 \Rightarrow x_k = k\pi,\ k \in \mathbb{Z}$
or
$\sinh y = 0 \Rightarrow \frac{e^{y} - e^{-y}}{2} = 0 \Rightarrow e^{y} = e^{-y} \Rightarrow e^{2y} = 1 \Rightarrow 2y = 0 \Rightarrow y = 0$

4) (1.28) Determine the image through the function $f(z) = z^2$ of the sets $A_1 = \{z : \operatorname{Re} z = a\}$ and $A_2 = \{z : \operatorname{Im} z = b\}$, where $a, b \in \mathbb{R}$.

$f(z) = z^2$

$z = x + iy \rightarrow$ $u + iv = (x + iz)^2 \Rightarrow u + iv = x^2 - y^2 + 2xyi$

$$\begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases}$$

$A_1 = \{z : \operatorname{Re} z = a\}$
$z \in A_1 \Rightarrow x = a$

$$\begin{cases} u = a^2 - y^2 \\ v = 2ay \end{cases}$$

we eliminate y

$y = \dfrac{v}{2a} \Rightarrow u = a^2 - \left(\dfrac{v}{2a}\right)^2$ we obtain a parabola

$u = a^2 - \dfrac{v^2}{4a^2}$

$A_2 = \{z : \operatorname{Im} z = b\}$
$z \in A_2 \Rightarrow y = b$

$$\begin{cases} u = x^2 - b^2 \\ v = 2xb \end{cases}$$

we eliminate x

$x = \dfrac{v}{2b} \Rightarrow u = \left(\dfrac{v}{2b}\right)^2 - b^2$ we obtain a parabola

$u = \dfrac{v^2}{4b^2} - b^2$