# Seminar 10

Wednesday, December 2, 2020 6:02 PM

## The Laplace Transform

- it is a mathematical TOOL

$$\underset{\text{time domain}}{\underbrace{x(t)}} \underset{iLT}{\overset{LT}{\rightleftarrows}} X(p)$$

a function in the complex frequency domain

ex1) signals
- continuous time → Laplace transform
- discrete time → Z transform, Fourier transform

ex2) solving differential equations with boundary conditions

$y(t) = c_1 e^{2t} + c_2 \cos(3t)$

using the L.T. we get the answer directly

$f : \mathbb{R} \to \mathbb{C}$ the original

$$F(p) = \int_0^\infty f(t) e^{-pt} \, dt \overset{\text{not.}}{=} \mathcal{L}[f(t)](p) \quad \text{the image}$$

$$\mathcal{L}[\underbrace{f(t)}_{\text{original}}](p) = \underbrace{F(p)}_{\text{image}}$$

$$\mathcal{L}[a](p) = \frac{a}{p}$$

$$\boxed{\mathcal{L}[e^{\lambda t}](p) = \frac{1}{p-\lambda}}$$

$$\mathcal{L}[t^n](p) = \frac{n!}{p^{n+1}}, \quad n \in \mathbb{N}$$

$$\mathcal{L}[t^\alpha](p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}, \quad \alpha \in \mathbb{C}, \ \operatorname{Re} \lambda > -1$$

$$\mathcal{L}[f(at)](p) = \frac{1}{a} \mathcal{L}[f(t)]\left(\frac{p}{a}\right)$$

$$\boxed{\mathcal{L}[e^{at} \cdot f(t)](p) = \mathcal{L}[f(t)](p-a)}$$

translation of the image

$$\boxed{\begin{aligned} &\mathcal{L}[t^1 \cdot f(t)](p) = (-1)^1 \left(\mathcal{L}[f(t)](p)\right)' \\ &\mathcal{L}[t^n \cdot f(t)](p) = (-1)^n \left(\mathcal{L}[f(t)](p)\right)^{(n)} \end{aligned}}$$

differentiation of the image

$$\mathcal{L}[\sin(at)](p) = \frac{a}{p^2+a^2}$$

$$\mathcal{L}[\cos(at)](p) = \frac{p}{p^2+a^2}$$

$$\mathcal{L}[\operatorname{sh}(at)](p) = \frac{a}{p^2-a^2}$$

$$\mathcal{L}[\operatorname{ch}(at)](p) = \frac{p}{p^2-a^2}$$

$$\mathcal{L}\left[\int_0^t f(s)\, ds\right](p) = \frac{1}{p} \cdot \mathcal{L}[f(t)](p) \quad \text{integration of the original}$$

$$\mathcal{L}\left[\frac{f(t)}{t}\right](p) = \int_p^\infty \mathcal{L}[f(t)](q)\, dq \quad \text{integration of the image}$$

1) Find the images by the Laplace transform of the originals

a) $\mathcal{L}\left[sh(at)\right](p) = \mathcal{L}\left[\frac{e^{at}-e^{-at}}{2}\right](p) = \frac{1}{2}\left(\mathcal{L}[e^{at}](p) - \mathcal{L}[e^{-at}](p)\right)$

$= \frac{1}{2}\left(\frac{1}{p-a} - \frac{1}{p+a}\right) = \frac{1}{2}\cdot\frac{\cancel{p}+a-\cancel{p}+a}{p^2-a^2} = \frac{a}{p^2-a^2}$

b) $\mathcal{L}\left[\cos(\underline{\underline{3}}t)\right](p) = \frac{p}{p^2+9}$

$= \frac{1}{3}\mathcal{L}[\cos t]\left(\frac{p}{3}\right) = \frac{1}{3}\frac{p}{p^2+1}\Big/_{p=\frac{p}{3}} = \frac{1}{3}\frac{\frac{p}{3}}{\left(\frac{p}{3}\right)^2+1} = \frac{p}{p^2+9}$

$\mathcal{L}[f(at)](p) = \frac{1}{a}\mathcal{L}[f(t)]\left(\frac{p}{a}\right)$

c) $\mathcal{L}\left[\underbrace{e^{2t}\cos(3t)} + \underbrace{e^{3t}\sin(2t)}\right](p) = \mathcal{L}\left[\underline{\underline{e}}^{2t}\cos(3t)\right](p) + \mathcal{L}\left[\underline{\underline{e}}^{3t}\sin(2t)\right](p)$

$= \mathcal{L}[\cos(3t)](p-2) + \mathcal{L}[\sin(2t)](p-3) = \frac{p}{p^2+9}\Big|_{p=p-2} + \frac{2}{p^2+4}\Big|_{p=p-3} =$

$= \frac{p-2}{(p-2)^2+9} + \frac{2}{(p-3)^2+4}.$

$\mathcal{L}[t^n](p) = \frac{n!}{p^{n+1}}$

d) $\mathcal{L}\left[\underline{\underline{t}}^3\cdot e^{-t}\right](p) \xlongequal[\text{transl. of the image}]{\text{I method}} \mathcal{L}[t^3](p+1) = \frac{3!}{p^4}\Big|_{p=p+1} = \frac{3!}{(p+1)^4}$

$\xlongequal[\text{differentiation of the image}]{\text{II method}} (-1)^3\left(\mathcal{L}[e^{-t}](p)\right)^{(3)} = -\left(\frac{1}{p+1}\right)^{(3)} = -\left(\frac{-1}{(p+1)^2}\right)''$

$= \left(\frac{1}{(p+1)^2}\right)'' = \left(\frac{-2}{(p+1)^3}\right)' = 6\frac{1}{(p+1)^4}$

e) $\mathcal{L}\left[t\cdot e^{2t}\cdot\cos t\right] \xlongequal[\text{transl. of the image}]{} \mathcal{L}[t\cos t](p-2) = (-1)^1\left(\mathcal{L}[\cos t]\right)'(p-2) =$

$= -\left(\frac{p}{p^2+1}\right)'\Big/_{p=p-2} = -\frac{p^2+1-2p^2}{(p^2+1)^2}\Big/_{p=p-2} =$

$= \frac{p^2-1}{(p^2+1)^2}\Big/_{p=p-2} = \frac{(p-2)^2-1}{\left[(p-2)^2+1\right]^2}$

$$\xlongequal[\text{diff the image}]{} (-1)^1\left(\mathcal{L}[e^{2t}\cos t](p)\right)' = \dots$$

f) $\mathcal{L}\left[\int_0^t \sin(3u)\,du\right](p) \xlongequal[\text{integr. of the original}]{} \frac{1}{p}\mathcal{L}[\sin(3t)](p) = \frac{1}{p}\cdot\frac{3}{p^2+9}$

$$\mathcal{L}\left[\int_0^t f(s)\,ds\right](p) = \frac{1}{p}\mathcal{L}[f(t)](p)$$

g) $\mathcal{L}\left[\int_0^t u^2\cdot e^{-3u}\,du\right](p) = \frac{1}{p}\mathcal{L}[t^2\cdot e^{-3t}](p) = \frac{1}{p}\mathcal{L}[t^2](p+3) =$

$$= \frac{1}{p}\cdot\frac{2!}{p^3}\Big|_{p=p+3} = \frac{1}{p}\cdot\frac{2!}{(p+3)^3} = \frac{2!}{p(p+3)^3}$$

$$\mathcal{L}[t^2](p) = \frac{2!}{p^{2+1}} = \frac{2!}{p^3}$$

h) $\mathcal{L}\left[\int_0^t \frac{\sin u}{u}\,du\right](p) \xlongequal[\text{int. of the original}]{} \frac{1}{p}\mathcal{L}\left[\frac{\sin t}{t}\right](p) \xlongequal[\text{int. of the image}]{}$

$$= \frac{1}{p}\int_p^\infty \mathcal{L}[\sin t](y)\,dy = \frac{1}{p}\int_p^\infty \frac{1}{y^2+1}\,dy = \frac{1}{p}\arctan y\Big|_p^\infty =$$

$$= \frac{1}{p}\left(\frac{\pi}{2} - \arctan p\right) = \frac{1}{p}\cdot\arctan\frac{1}{p}$$

$$\arctan y + \arctan\frac{1}{y} = \frac{\pi}{2}$$

i) $\mathcal{L}[t^2\cdot\cos t](p) \xlongequal[\text{diff of the image}]{} (-1)^2\left(\mathcal{L}[\cos(t)](p)\right)'' = \left(\frac{p}{p^2+1}\right)'' =$

$$= \left(\frac{p^2+1-2p^2}{(p^2+1)^2}\right)' = \left(\frac{-p^2+1}{(p^2+1)^2}\right)' = \frac{-2p(p^2+1)^2 - 2(p^2+1)\cdot 2p\cdot(1-p^2)}{(p^2+1)^4} =$$

$$= \frac{-2p^3 - 2p - 4p + 4p^3}{(p^2+1)^3} = \frac{2p^3 - 6p}{(p^2+1)^3}.$$

j) $\mathcal{L}\left[\frac{e^{-2t} - e^{-3t}}{t}\right](p) \xlongequal[\text{integration of the image}]{} \int_p^\infty \mathcal{L}[e^{-2t} - e^{-3t}](y)\,dy =$

$$= \int^\infty \left(\frac{1}{\quad} - \frac{1}{\quad}\right)dy = \left[\ln(y+2) - \ln(y+3)\right]\Big|_p^\infty = \ln\frac{y+2}{y+3}\Big|_p^\infty$$

$$= \int_p^\infty \left( \frac{1}{y+2} - \frac{1}{y+3} \right) dy = \left[ \ln(y+2) - \ln(y+3) \right] \Big/_p^\infty = \ln \frac{y+2}{y+3} \Big/_p^\infty$$

$$= \underbrace{\lim_{x \to \infty} \ln \frac{x+2}{x+3}}_{=0} - \ln \frac{p+2}{p+3} = -\ln \frac{p+2}{p+3} = \ln \left( \frac{p+2}{p+3} \right)^{-1} = \underline{\underline{\ln \frac{p+3}{p+2}}}$$

(2) $X(p) \xrightarrow[\mathcal{L}^{-1}]{ILT} x(t)$

Find the originals $f(t)$ corresponding to the following images

$\mathcal{L}[e^{\lambda t}](p) = \frac{1}{p-\lambda} \Big/ \mathcal{L}^{-1}$

$\mathcal{L}^{-1}\left[\frac{1}{p-\lambda}\right] = e^{\lambda t}$

a) $\mathcal{L}^{-1}\left[\frac{1}{p-\underbrace{3}_{\lambda}}\right] = e^{\overset{\lambda}{3}t}$

b) $\mathcal{L}^{-1}\left[\frac{1}{2p-3}\right] = \mathcal{L}^{-1}\left[\frac{1}{2\cdot(p-\frac{3}{2})}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{p-\underbrace{\frac{3}{2}}_{\lambda}}\right] = \frac{1}{2}e^{\frac{3}{2}t}$

c) $\mathcal{L}^{-1}\left[\frac{1}{p^2+9}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{p^2+9}\right] = \frac{1}{3}\sin(3t)$

$\mathcal{L}[\sin(3t)](p) = \frac{3}{p^2+9}$

$\mathcal{L}[\cos(3t)](p) = \frac{p}{p^2+9}$

d) $\mathcal{L}^{-1}\left[\frac{1}{p^3}\right] = \frac{1}{2!}\mathcal{L}^{-1}\left[\frac{2!}{p^3}\right] = \frac{1}{2}t^2$

$\mathcal{L}[t^2](p) = \frac{2!}{p^3}$

e) $\mathcal{L}^{-1}\left[\frac{3!}{(p-2)^4}\right] = \mathcal{L}^{-1}\left[\frac{3!}{p^4}\Big/_{p=p-2}\right] = t^3 \cdot e^{2t}$

$\mathcal{L}[t^3](p) = \frac{3!}{p^4}$

f) $\mathcal{L}^{-1}\left[\frac{1}{p(p-3)}\right] = (*)$

$\mathcal{L}^{-1}\left[\frac{1}{p-a}\right] \quad \mathcal{L}^{-1}\left[\frac{1}{p+b}\right]$

we use partial function decomposition

$$\frac{1}{p(p-3)} = \frac{A}{p} + \frac{B}{p-3} \Rightarrow 1 = A\cdot(p-3) + Bp$$

$p = 0 \Rightarrow -3A = 1 \Rightarrow A = -\frac{1}{3}$

$p = 3 \Rightarrow 3B = 1 \Rightarrow B = \frac{1}{3}$

$\mathcal{L}^{-1}\left[\frac{1}{\dots}\right] = e^{\lambda t} \quad \mathcal{L}^{-1}\left[\frac{a}{\dots}\right] = a$

$p = 3 \Rightarrow 3B = 1 \rightarrow B = \frac{1}{3}$

$\boxed{\mathcal{L}^{-1}\left[\frac{1}{p-\lambda}\right] = e^{\lambda t} \quad ; \mathcal{L}^{-1}\left[\frac{a}{p}\right] = a}$

$\Rightarrow \frac{1}{p \cdot (p-3)} = \frac{\frac{1}{3}}{p-3} - \frac{\frac{1}{3}}{p}$

$\circledast \quad \mathcal{L}^{-1}\left[\frac{\frac{1}{3}}{p-3}\right] - \mathcal{L}^{-1}\left[\frac{\frac{1}{3}}{p}\right] = e^{3t} - \frac{1}{3}$

* g) $\mathcal{L}^{-1}\left[\frac{1}{p^2-4p+11}\right] = \mathcal{L}^{-1}\left[\frac{1}{\underbrace{p^2-4p+4}+7}\right] = \mathcal{L}^{-1}\left[\frac{1}{(p-2)^2+7}\right] =$

$\mathcal{L}^{-1}\left[\frac{a}{p^2+a^2}\right] = \sin(at)$

$p^2 - 4p + 11 = 0$

$\Delta = 16 - 44 < 0 \rightarrow$ no real solutions

$= \frac{1}{\sqrt{7}} \mathcal{L}^{-1}\left[\frac{\sqrt{7}}{(p-2)^2 + (\sqrt{7})^2}\right] = \frac{1}{\sqrt{7}} \mathcal{L}^{-1}\left[\frac{\sqrt{7}}{p^2+(\sqrt{7})^2}\bigg|_{p=p-2}\right] = \frac{1}{\sqrt{7}} \sin(\sqrt{7}t) \cdot e^{2t}$

* h) $\mathcal{L}^{-1}\left[\frac{p}{p^2-6p+25}\right] = \mathcal{L}^{-1}\left[\frac{p}{p^2-6p+9+16}\right] = \mathcal{L}^{-1}\left[\frac{p}{(p-3)^2+16}\right] =$

$p^2 - 6p + 25 = 0 \quad \Delta < 0 \Rightarrow$ no real sol.

$\mathcal{L}^{-1}\left[\frac{p}{p^2+a^2}\right] = \cos(at)$

$= \mathcal{L}^{-1}\left[\frac{p-3+3}{(p-3)^2+4^2}\right] = \mathcal{L}^{-1}\left[\frac{p-3}{(p-3)^2+4^2}\right] + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{4}{(p-3)^2+4^2}\right] =$

$= \mathcal{L}^{-1}\left[\frac{p}{p^2+4^2}\bigg|_{p=p-3}\right] + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{4}{p^2+4^2}\bigg|_{p=p-3}\right] =$

$= e^{3t} \cdot \cos(4t) + \frac{3}{4} e^{3t} \cdot \sin(4t)$

i) $\mathcal{L}^{-1}\left[\frac{p}{p^2+p+1}\right] = \mathcal{L}^{-1}\left[\frac{p}{\underbrace{p^2+p+\frac{1}{4}}+1-\frac{1}{4}}\right] = \mathcal{L}^{-1}\left[\frac{p}{(p+\frac{1}{2})^2+\frac{3}{4}}\right]$

$\Delta = 1 - 4 < 3$

$= \mathcal{L}^{-1}\left[\frac{p+\frac{1}{2}-\frac{1}{2}}{(p+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right] = \mathcal{L}^{-1}\left[\frac{p+\frac{1}{2}}{(p+\frac{1}{2})^2 + \underbrace{(\frac{\sqrt{3}}{2})^2}_{a^2}}\right] - \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \mathcal{L}^{-1}\left[\frac{\frac{\sqrt{3}}{2}}{(p+\frac{1}{2})^2 + \underbrace{(\frac{\sqrt{3}}{2})^2}_{a^2}}\right] =$

$e^{-\frac{1}{2}t} \quad \ldots \quad e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2} \ldots\right)$

$$= e^{-\frac{1}{2}t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right).$$

Homework:

1) $\mathcal{L}\{(t+1)\sin 2t\}(p)$

2) $\mathcal{L}^{-1}\left[\frac{2}{(p+5)^3}\right]$

3) $\mathcal{L}^{-1}\left[\frac{4}{(p-7)^4}\right]$

4) $\mathcal{L}^{-1}\left[\frac{3p+6}{p^2+3p}\right]$

5) $\mathcal{L}^{-1}\left[\frac{p}{p^2-5p+6}\right]$

6) $\mathcal{L}^{-1}\left[\frac{p}{p^2+3p+8}\right]$

7) $\mathcal{L}^{-1}\left[\frac{1}{p^2+2p+9}\right]$