Seminar 8

Wednesday, November 18, 2020 6:02 PM

Using the Residue Theorem to evaluate real integrals

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{\infty} \operatorname{Res}_{z=z_k} f(z)$$

I $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$; f rational function in terms of cos θ and sin θ

II $\int_{-\infty}^{+\infty} f(x)\,dx$; $\frac{P(x)}{Q(x)}$

III $\int_{-\infty}^{+\infty} f(x)\cos\alpha x\,dx$ or $\int_{-\infty}^{+\infty} f(x)\sin\alpha x\,dx$

IV $\int_0^{\infty} f(x)\,dx$

I $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$

$\boxed{z = e^{i\theta}} \Rightarrow dz = ie^{i\theta}\,d\theta \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} \Rightarrow \boxed{d\theta = \frac{dz}{iz}}$

$\boxed{\cos\theta = \frac{z^2+1}{2z}}$; $\boxed{\sin\theta = \frac{z^2-1}{2iz}}$

$$I = \int_{C:|z|=1} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz} = 2\pi i \sum_{\substack{z_k \in \operatorname{int} C \\ z = z_k}} \operatorname{Res} f(z)$$

ex 1 $\int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta}$ $\qquad z(\theta) = e^{i\theta}$; $0 \le \theta \le 2\pi$

$$I = \int_{C:|z|=1} \frac{\frac{dz}{iz}}{\frac{5}{4} + \frac{z^2-1}{2iz}} = \int_C \frac{dz}{i\cancel{z}} \cdot \frac{8\cancel{z}}{10iz + 4z^2 - 4} = \int_C \frac{4\,dz}{2z^2 + 5iz - 2} = \int_C \frac{4\,dz}{(z+2i)(2z+i)}$$

$2z^2 + 5iz - 2 = 0 \qquad \Delta = 25i^2 + 16 = -9 \Rightarrow z_{1,2} = \frac{-5i \pm 3i}{4}$; $z_1 = -2i \notin \operatorname{int} C$, $z_2 = -\frac{i}{2} \in \operatorname{int} C$, pole of order 1

$2z^2 + 5iz - 2 = 2(z+2i)(z+\frac{i}{2}) = (z+2i)(2z+i)$

$$I = 2\pi i \operatorname{Res}_{z=-\frac{i}{2}} f(z) = 2\pi \cancel{i} \cdot \frac{4}{3\cancel{i}} = \frac{8\pi}{3}$$

$$\operatorname{Res}_{z=-\frac{i}{2}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-\frac{i}{2}} = \left.\frac{4}{(z+2i)\cdot 2}\right|_{z=-\frac{i}{2}} = \frac{4}{(-\frac{i}{2}+2i)\cdot 2} = \frac{4}{3i}$$

$g(z) = \frac{4}{z+2i}$, $h(z) = 2z+i$, $h'(z) = 2$

II $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\,dx$, where P, Q are polynomials of degrees m and n, $n \ge m+2$

II) $\displaystyle\int_{-\infty} f(x)\,dx = \int_{-\infty} \frac{\quad}{Q(x)}\,dx$ ... $m \geq n+2$

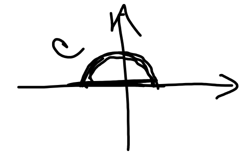
$$I = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} f(z)$$

- z_k are all the poles of $\frac{P(z)}{Q(z)}$ that lie in the upper half plane

! We are interested only with the poles with positive imaginary part.

1.57

i) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)(x^2+9)}\,dx$



$$f(z) = \frac{z^2}{(z^2+1)(z^2+9)} \Rightarrow I = \int_C \frac{z^2}{(z^2+1)\cdot(z^2+9)}\,dz = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} \frac{P(z)}{Q(z)} \quad (*)$$

$z^2+1=0 \Rightarrow z_{1,2} = \pm i$

$z^2+9=0 \Rightarrow z_{3,4} = \pm 3i$

$\Rightarrow z_1 = i$, $z_3 = 3i$ poles of order 1, lie in the upper half plane

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{g(z)}{h'(z)}\Big|_{z=i} = \frac{z^2}{(z^2+9)(z+i)}\Big|_{z=i} = \frac{i^2}{(i^2+9)(2i)} = \frac{i}{16}$$

$$g(z) = \frac{z^2}{(z^2+9)(z+i)},\ h'(z) = 1$$

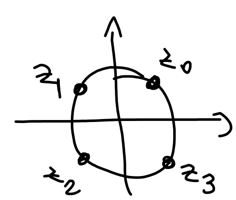
$$f(z) = \frac{z^2}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$\operatorname*{Res}_{z=z_3} f(z) = \frac{g(z)}{h'(z)}\Big|_{z=3i} = \frac{z^2}{(z^2+1)(z+3i)}\Big|_{z=3i} = \frac{9i^2}{(9i^2+1)6i} = \frac{3i}{-16} = \frac{-3i}{16}$$

$$g(z) = \frac{z^2}{(z^2+1)(z+3i)},\ h'(z) = 1$$

$$(*)\ I = 2\pi i\left(\frac{i}{16} - \frac{3i}{16}\right) = 2\pi i \cdot \left(\frac{-2i}{16}\right) = \frac{\pi}{4}$$

ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx$



$$f(z) = \frac{z^2}{z^4+1} \Rightarrow I = \int_C \frac{z^2}{z^4+1}\,dz = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} f(z)$$

$z^4+1=0 \Rightarrow z^4 = -1$, $-1 = \cos\pi + i\sin\pi \Rightarrow z_k = \cos\frac{\pi+2k\pi}{4} + i\sin\frac{\pi+2k\pi}{4}$, $k = \overline{0,3}$

$z^4+1=0 \Rightarrow z^4=-1$, $-1=\cos\pi+i\sin\pi \Rightarrow z_k=\cos\frac{\pi+2k\pi}{4}+i\sin\frac{\pi+2k\pi}{4}$, $k=\overline{0,3}$

$$\left.\begin{array}{l} z_0=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{\sqrt2}{2}+i\frac{\sqrt2}{2} \\ z_1=-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\end{array}\right\} \in \operatorname{int} C\text{, poles of order 1}$$

$$\left.\begin{array}{l} z_2=-\frac{\sqrt2}{2}-i\frac{\sqrt2}{2} \\ z_3=\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\end{array}\right\} \text{ are not with positive im. part}$$

$$\operatorname*{Res}_{z=z_0} f(z)=\lim_{z\to z_0}(z-z_0)\frac{z^2}{z^4+1}=z_0^2\lim_{z\to z_0}\frac{z-z_0}{z^4+1}\overset{l'H}{\underset{\frac00}{=}}z_0^2\lim_{z\to z_0}\frac{1}{4z^3}=\frac{z_0^2}{4z_0^3}=\frac{1}{4z_0}$$

$$=\frac{1}{4\cdot\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)}=\frac{1-i}{2\sqrt2(1+i)(1-i)}=\frac{1-i}{4\sqrt2}=\frac{\sqrt2}{8}(1-i)$$

$$\operatorname*{Res}_{z=z_1} f(z)=\lim_{z\to z_1}(z-z_1)\frac{z^2}{z^4+1}=z_1^2\lim_{z\to z_1}\frac{z-z_1}{z^4+1}\overset{l'H}{\underset{\frac00}{=}}z_1^2\lim_{z\to z_1}\frac{1}{4z^3}=\frac{z_1^2}{4z_1^3}=\frac{1}{4z_1}=$$

$$=\frac{1}{4\left(-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)}=\frac{-1-i}{2\sqrt2(-1+i)(-1-i)}=\frac{-1-i}{4\sqrt2}=-\frac{\sqrt2}{8}(i+1)$$

$$I=2\pi i\left(\cancel{\frac{\sqrt2}{8}}-\frac{\sqrt2}{8}i-\frac{\sqrt2}{8}i-\cancel{\frac{\sqrt2}{8}}\right)=2\pi i\left(\frac{-2\sqrt2 i}{8}\right)=\frac{\pi\sqrt2}{2}=\frac{\pi}{\sqrt2}$$

<u>Method II</u>

$$I=2\pi i\sum_{k=0,1}\operatorname*{Res}_{z=z_k} f(z)=2\pi i\sum_{k=0,1}\frac{z_k^2}{4z_k^3}=2\pi i\sum_{k=0,1}\frac{z_k^3}{4\underbrace{z_k^4}_{=-1}}=$$

$$=2\pi i\sum_{k=0,1}\frac{z_k^3}{-4}=\frac{\pi i}{-2}\sum_{k=0}^{1}z_k^3=$$

$$=-\frac{\pi i}{2}\left(z_0^3+z_1^3\right)=-\frac{\pi i}{2}\left(\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)^3+\left(-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)^3\right)$$

$$=\ \dots\ =\frac{\pi}{\sqrt2}$$

III $\quad I=\int_{-\infty}^{+\infty}f(x)\cos\alpha x\,dx$ or $J=\int_{-\infty}^{+\infty}f(x)\sin\alpha x\,dx$

$$K=I+iJ=\int_{-\infty}^{+\infty}f(x)(\cos\alpha x+i\sin\alpha x)\,dx=\int_{-\infty}^{+\infty}f(x)e^{i\alpha x}\,dx\Longrightarrow I$$
$$\Longrightarrow J$$

ex2 $\quad I=\int^{+\infty}\frac{\cos x}{\dots}\,dx\ ;\ J=\int^{+\infty}\frac{\sin x}{\dots}\,dx$

ex2 $I=\int_{-\infty}^{+\infty}\frac{\cos x}{x^2+1}dx$; $J=\int_{-\infty}^{+\infty}\frac{\sin x}{x^2+1}dx$

$$K=I+iJ=\int_{-\infty}^{+\infty}\frac{\cos x+i\sin x}{x^2+1}dx=\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^2+1}dx \quad (\underline{\text{II}})$$

$f(z)=\frac{e^{iz}}{z^2+1} \Rightarrow K=\int_C\frac{e^{iz}}{z^2+1}dz$, C includes only the poles from the upper half plane

$z^2+1=0 \Rightarrow z_{1,2}=\pm i \Rightarrow z_1=i\in \operatorname{int}C$

$z_2=-i\notin \operatorname{int}C$

$$f(z)=\frac{\overbrace{e^{iz}}^{g(z)}}{(z+i)\underbrace{(z-i)}_{h(z)}}$$

$K=2\pi i\operatorname{Res}_{z=i}f(z) \Rightarrow$ (*)

$$\operatorname{Res}_{z=i}f(z)=\left.\frac{g(z)}{h'(z)}\right|_{z=i}=\left.\frac{e^{iz}}{(z+i)\cdot 1}\right|_{z=i}=\frac{e^{i^2}}{2i}=\frac{e^{-1}}{2i}$$

$$K=\cancel{2}\pi\cancel{i}\,\frac{e^{-1}}{\cancel{2i}}=\frac{\pi}{e}$$

$$K=I+iJ \Rightarrow \frac{\pi}{e}=\frac{\pi}{e}+i\,0 \Rightarrow \begin{cases} I=\frac{\pi}{e}\\ J=0\end{cases}$$

ex3 $\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}$

$$I=\int_{C:|z|=1}\frac{\frac{dz}{iz}}{\sqrt{2}-\frac{z^2+1}{2z}}=\int_C\frac{dz}{i\cancel{z}\cdot\frac{2\sqrt{2}z-z^2-1}{2\cancel{z}}}=\int_C\frac{2}{-i(z^2-2\sqrt{2}z+1)}dz=$$

$$=-\frac{2}{i}\int_C\frac{dz}{z^2-2\sqrt{2}z+1}=-\frac{2}{i}\int_C\frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

$z^2-2\sqrt{2}z+1=0 \qquad \Delta=8-4=4 \Rightarrow z_{1,2}=\frac{2\sqrt{2}\pm 2}{2}$; $z_1=\sqrt{2}+1\notin \operatorname{int}C$; $z_2=\sqrt{2}-1\in\operatorname{int}C$ pole of order 1

$$I=-\frac{2}{i}\cdot 2\pi i\operatorname{Res}_{z=\sqrt{2}-1}f(z)$$

$$\operatorname{Res}_{z=\sqrt{2}-1}f(z)=\left.\frac{g(z)}{h'(z)}\right|_{z=z_2}=\left.\frac{1}{(z-\sqrt{2}-1)\cdot 1}\right|_{z=z_2}=\frac{1}{\cancel{\sqrt{2}}-1-\cancel{\sqrt{2}}-1}=-\frac{1}{2}$$

$z=z_2$

$$I = 2\pi i \left(-\frac{1}{\cancel{2}}\right)\left(-\frac{\cancel{2}}{\cancel{2}}\right) = 2\pi$$

1.57 iv)
$$\int_{-\infty}^{+\infty} \frac{1}{x^4+9x^2+20}\, dx$$

$$I = \int_C \frac{1}{z^4+9z^2+20}\, dz = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} f(z)$$

$$z^2 = t \Rightarrow t^2+9t+20=0 \Rightarrow \begin{array}{l} t_1=-4 \\ t_2=-5 \end{array} \Rightarrow \begin{array}{l} z_{1,2} = \pm 2i \\ z_{3,4} = \pm\sqrt{5}\, i \end{array}$$

$\Rightarrow \left.\begin{array}{l} z_1 = 2i \\ z_3 = \sqrt{5}i \end{array}\right\} \in \operatorname{int} C$ (the upper half plane) $\quad z_2, z_4 \notin \operatorname{int} C$

~~poles of order 1~~

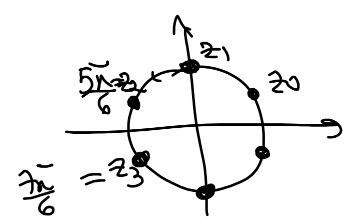
$$I = 2\pi i \left(\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=\sqrt{5}i} f(z) \right) \quad (*)$$

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{1}{1!}\lim_{z\to 2i} \cancel{(z-2i)} \cdot \frac{1}{(z+2i)\cancel{(z-2i)}(z^2+5)} = \frac{1}{4i\cdot 1} = \frac{1}{4i}$$

$$\operatorname*{Res}_{z=z_3} f(z) = \frac{1}{1!}\lim_{z\to\sqrt{5}i} \cancel{(z-\sqrt{5}i)} \frac{1}{\cancel{(z-\sqrt{5}i)}(z+\sqrt{5}i)(z^2+4)} = \frac{1}{2\sqrt{5}i\cdot(-1)} = \frac{-1}{2\sqrt{5}i}$$

$$(*)\ I = 2\pi i \left(\frac{1}{4i} - \frac{1}{2\sqrt{5}i} \right) = \frac{2\pi}{4} - \frac{2\pi}{2\sqrt{5}} = \boxed{\frac{\pi}{2} - \frac{\pi}{\sqrt{5}}}$$

1.57 iii)
$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$$



$$I = \int_C \frac{dz}{z^6+1} = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} f(z)$$

$$z^6+1=0 \Rightarrow z^6 = -1 \Rightarrow z^6 = \cos\pi + i\sin\pi \Rightarrow z_k = \cos\frac{\pi+2k\pi}{6} + i\sin\frac{\pi+2k\pi}{6}, \quad k=\overline{0,5}$$

$$\left.\begin{array}{l} z_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} \\ z_1 = \cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6} = i \\ z_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2} \end{array}\right\} \in \operatorname{int} C, \text{ poles of order 1}$$

$z_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = \ldots \frac{1}{2}$?

$z_2 = \cos\frac{}{6} + i\sin\frac{}{6} = \frac{}{2} \quad \frac{}{2}$

$$\left.\begin{aligned} z_3 &= \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - i\frac{1}{2} \\ z_4 &= \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = -i \\ z_5 &= \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - i\frac{1}{2} \end{aligned}\right\} \notin \text{int}\, C$$

$$\operatorname*{Res}_{z=z_k} f(z) = \lim_{z\to z_k} (z-z_k)\frac{1}{z^6+1} \overset{l'H}{=} \frac{0}{0}$$

$$= \lim_{z\to z_k} \frac{1}{6z^5} = \frac{1}{6z_k^5}$$

$$\Rightarrow I = 2\pi i \sum_{k=0}^{2} \frac{1}{6z_k^5}\cdot\frac{z_k}{z_k} = 2\pi i \sum_{k=0}^{2} \frac{z_k}{6\underbrace{z_k^6}_{=-1}} =$$

$$= \frac{2\pi i}{-6}\sum_{k=0}^{2} z_k = -\frac{\pi i}{3}(z_0 + z_1 + z_2) =$$

$$= -\frac{\pi i}{3}\left(\cancel{\frac{\sqrt{3}}{2}} + i\frac{1}{2} + i - \cancel{\frac{\sqrt{3}}{2}} + i\frac{1}{2}\right) = -\frac{\pi i}{3}\cdot 2i = \frac{2\pi}{3}$$