Contents

Chapter 2

Introduction to System Modeling

2.1 Introduction

A **mathematical model** is an equation or set of equations which adequately describes the behavior of a system.

System modeling is a subject in its own right. Essentially there are two approaches to finding the model, (Dutton et al., 1997).

- In the first, the system is broken down into smaller elements. For each element a mathematical description is then established by working from the physical laws which describe the system's behavior. The simplest such technique is lumped-parameter modeling, which is considered in this course.
- The second approach is known as system identification in which it is assumed that an experiment can be carried out on the system, and that a mathematical model can be found from the results. This approach can clearly only be applied to existing plants, whereas lumped-parameter modeling can additionally be applied to a plant yet to be built, working purely from the physics of the proposed plant components.

System modeling is a specialization of the more general area of mathematical modeling. In a system model, the important relationship is that between the manipulated inputs and measurable outputs. Ideally this relationship should be linear and capable of being described by an expression of low-order (that is, an equation or set of equations containing as few terms as possible). An usual graphical representation of single input, single output (SISO) dynamical system is shown in Figure 2.1. A model of such a system is a mathematical relationship between the input $u(t)$ and the output $y(t)$.

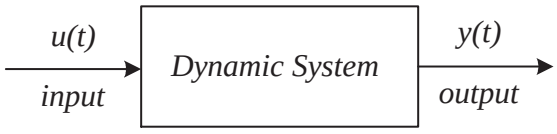

Figure 2.1: SISO dynamic system

Most commonly, a low-order linear differential equation model is used.

When fundamental physical laws are applied to the lumped-parameter model, the resulting equations may be nonlinear in which case further assumptions may have to be made in order to produce an ordinary linear differential equation model which is soluble. In such cases, it is not unusual to assume that system operation will be restricted to small perturbations about a given operating point. If the assumed operating region is small enough, most nonlinear plants may be described by a set of linear equations. If these assumptions cannot be made, nonlinear control techniques are needed.

2.2 Lumped-parameter models

The systems studied in this course are:

Linear - They must obey the principle of superposition that is, if an input $u_1(t)$ causes an output $y_1(t)$ and an input $u_2(t)$ causes an output $y_2(t)$, then an input $u_1(t) + u_2(t)$ causes an output $y_1(t)+y_2(t)$ (see next section **Linear Approximations of Physical systems**)

In the case of the resistor, if an input of 1A produces an output of 3V, and an input of 2A produces an output of 6V then an input of 3A produces an output of 9V.

Stationary (or time invariant) - The parameters inside the element must not vary with time. In other words, an input applied today must give the same result as the same input applied yesterday or tomorrow.

A vehicle which burns large masses of fuel, such as racing cars or space vehicles, is an example of system which is **not** stationary (or it is time-varying). Its dynamic behavior will alter significantly as its mass decreases, (Dutton et al., 1997).

Deterministic - The outputs of the system at any time can be determined from a knowledge of the system's inputs up to that time. In other words there is no random (or stochastic) behavior in the system, since its outputs are always a specific function of the inputs. The term *causal* is also used for such systems, (Dutton et al., 1997).

Example 2.2.1 *Consider an element representing an idealized component such as:*

- *The resistor. If the resistor is linear then its model is a relationship between the voltage $v(t)$ and the current $i(t)$:*
$$i(t) = \frac{1}{R}v(t)$$
- *The inductor is described by the equation:*
$$i(t) = \frac{1}{L}\int v(t)dt \quad \text{or} \quad v(t) = L\frac{di(t)}{dt}$$
- *The capacitor is described by:*
$$i(t) = C\frac{dv(t)}{dt}$$

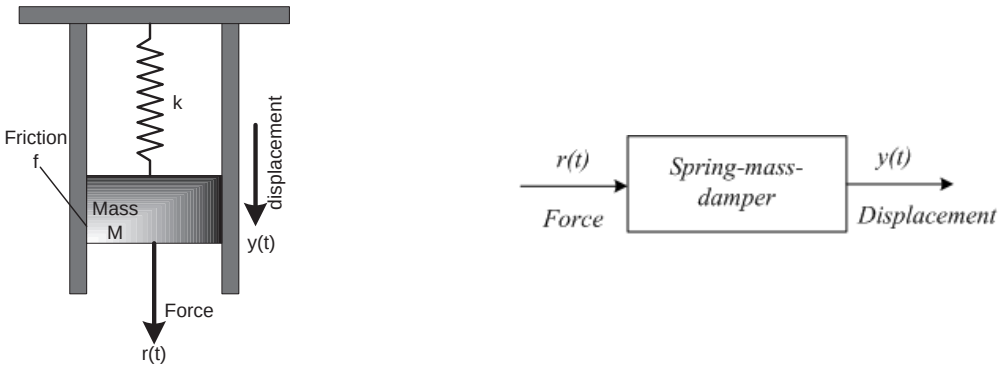

Figure 2.2: Spring-mass-damper system

- *The spring-mass-damper mechanical system (Figure 2.2). The system could represent for example an automobile shock absorber. We model the wall friction as a viscous damper. Assuming the system is linear, a model of the system can be derived from the Newton's second law of motion.*

  The force equation of this system is:

$$M\frac{d^2y(t)}{dt^2} = -f\frac{dy(t)}{dt} - ky(t) + r(t)$$

  where f is the friction coefficient, M - the mass, k - the stiffness of the linear spring. The derivatives $\frac{d^2y(t)}{dt^2}$ and $\frac{dy(t)}{dt}$ represent the acceleration and velocity of the mass, respectively.

  Alternatively, the former equation may be rewritten into an input-output form:

$$M\frac{d^2y(t)}{dt^2} + f\frac{dy(t)}{dt} + ky(t) = r(t)$$

  It is a relationship between the (input) force $r(t)$ and the (output) displacement $y(t)$.

2.3 Linear Approximation of Physical Systems

A great majority of physical systems are linear within some domain of variables. However, all systems ultimately become nonlinear as the variables are increased without limit.

A system is defined as linear in terms of the system variables. In the case of an electrical network, the excitation is the input current and the response is the voltage. In general a *necessary condition* for a linear system can be determined in terms of and excitation $x(t)$ and a response $y(t)$. When a system at rest is subjected to an excitation $x_1(t)$, it provides a response $y_1(t)$. Furthermore, when the system is subjected to an excitation $x_2(t)$, it provides a corresponding response $y_2(t)$. For a linear system, it is *necessary* that the excitation $x_1(t) + x_2(t)$ result in a response $y_1(t) + y_2(t)$. This is usually called the *principle of superposition*.

Furthermore, it is necessary that the magnitude scale factor be preserved in a linear system. Again, consider a system with an input x and an output y. Then it is necessary that the response of a linear system to a constant multiple β of an input x be equal to the response to the input multiplied by the same constant so that the output is equal to βy. This is called the property of *homogeneity*.

Example 2.3.1 , *(Dorf and Bishop, 2008). A system characterized by the relation* $y = x^2$ *is not linear because the superposition property is not satisfied. A system represented by the relation* $y = mx + b$ *is not linear because it does not satisfy the homogeneity property. However, this device may be considered linear about an operating point* x_0, y_0 *for small changes* Δx *and* Δy*. When* $x = x_0 + \Delta x$ *and* $y = y_0 + \Delta y$*, we have*

$$y = mx + b$$

or

$$y_0 + \Delta y = mx_0 + m\Delta x + b$$

and because

$$y_0 = mx_0 + b$$

then:

$$\Delta y = m\Delta x$$

which satisfies the necessary conditions (Figure 2.3).

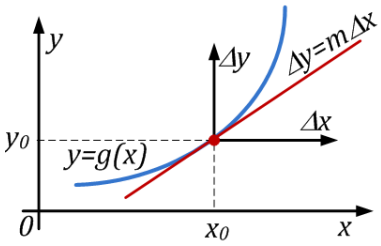

Figure 2.3: Linear approximation

Consider a general element with an input $x(t)$ and a response $y(t)$. The relationship of the two variables is written as:

$$y(t) = g(x(t))$$

where $g(x(t))$ is a continuous function of $x(t)$.

In general, Taylor series may be used to expand a nonlinear function $g(x(t))$ about an operating value x_0. An operating value could be the equilibrium position in a spring-mass-damper, a fixed voltage in an electrical system, steady-state pressure in a fluid system, and so on.

Because the function is continuous over the range of interest, a Taylor series expansion about the operating point x_0 may be utilized. Then we have:

$$y = g(x) = g(x_0) + \frac{dg}{dx}|_{x=x_0}\frac{x - x_0}{1!} + \text{higher order terms} \tag{2.1}$$

The slope at the operating point,

$$m = \frac{dg}{dx}|_{x=x_0},$$

is a good approximation to the curve over a small range of $(x - x_0)$, the deviation from the operating point. Then, as a reasonable approximation, the expression from example 1 becomes:

$$y = g(x_0) + \frac{dg}{dx}|_{x=x_0}(x - x_0) = y_0 + m(x - x_0),$$

where m is the slope at the operating point. Finally, this equation can be rewritten as the linear equation

$$(y - y_0) = m(x - x_0)$$

or

$$\Delta y = m\Delta x.$$

Example 2.3.2 *Pendulum oscillator model, (Dorf and Bishop, 2008).*

Consider the pendulum oscillator shown in Figure 2.4. The torque on the mass is

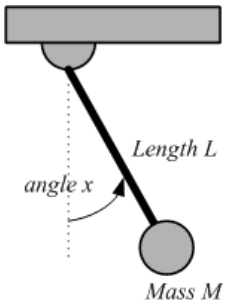

Figure 2.4: Pendulum oscillator

$$T = MgLsin(x)$$

where g is the gravity constant. The equilibrium condition for the mass is $x_0 = 0^o$. The first derivative evaluated at equilibrium provides the linear approximation, which is:

$$T - T_0 \cong MgL\frac{\partial sinx}{\partial x}|_{x=x_0}(x - x_0),$$

where $T_0 = 0$. Then, we have

$$T = MgL(cos0^o)(x - 0^o) = MgLx$$

The approximation is reasonably accurate for $-\pi/4 \leq x \leq \pi/4$. For example, the response of the linear model for the swing through $\pm 30^o$ is within 2% of the actual nonlinear pendulum response (check in simulations).

If the variable y depends upon several excitation variables $x_1, x_2, ..., x_n$, then the functional relationship is written as:

$$y = g(x_1, x_2, ..., x_n).$$

The Taylor series expansion about the operating point $x_{10}, x_{20}, ..., x_{n0}$ is useful for a linear approximation to a nonlinear function or a nonlinear differential equation. When the higher-order terms are neglected, the linear approximation is written as

$$y = g(x_{10}, x_{20}, ..., x_{n0}) + \frac{\partial g}{\partial x_1}|_{x=x_0}(x_1 - x_{10}) + \frac{\partial g}{\partial x_2}|_{x=x_0}(x_2 - x_{20}) + ... + \frac{\partial g}{\partial x_n}|_{x=x_0}(x_n - x_{n0}) \quad (2.2)$$

where $x_0 = [x_{10}, x_{20}, ..., x_{n0}]$ is the operating point. An example will illustrate this method.

Example 2.3.3 Magnetic levitation, *(Xie, 2009).*

The device shown in Figure 2.5 is an actively controlled magnetic levitator. Magnetic levitation is a technique widely used to create noncontact bearings, to eliminate friction for operation at high speeds or in special environments such as vacuum systems. In a simplified approach, the system consists of an iron-core electromagnet and the steel ball levitated by the electromagnet.

Current through the coils of the electromagnet induces a magnetic field in the iron core. The electromagnet exerts a force on the steel ball so as to keep it floating at a given height (in conjunction with a sensor and control electronics).

A mathematical model for this system can be derived from basic physics. As shown in the free body diagram (Figure 2.5), two forces act on the steel ball: gravity and the electro-magnetic force F_m. In the simplest model, F_m is proportional to the square of the current passing through the inductor and inversely proportional to the square of the distance between the magnet and the steel ball:

$$F_m = C\frac{i^2(t)}{z^2(t)}$$

where the constant C depends on the physical construction of the electromagnet.

Figure 2.5: Magnetic levitation and the free body diagram

We will determine a nonlinear model for this system and a linear approximation when the ball is in equilibrium for a distance $z_0 = 10^{-2}m$ from the electromagnet. Consider the following values for the constants: the gravitational acceleration $g = 10m/s^2$, mass of the ball $m = 0.1kg$ and the constant $C = 10^{-2}Nm^2/A^2$.

The equation of motion for the ball is:

$$m\ddot{z}(t) = mg - F_m$$

or

$$m\ddot{z}(t) = mg - C\frac{i^2(t)}{z^2(t)}$$

where $\ddot{z}(t) = \frac{d^2z(t)}{dt^2}$. From the description of this system, the input signal (the cause) is the current through the coils of the electromagnet $i(t)$ and the output (the effect) is the displacement of the ball $z(t)$ (Figure 2.6).

The differential equation obtained above is a nonlinear relationship between $i(t)$ and $z(t)$, thus it is a mathematical model for this system.

The equation can also be written as a function of 3 variables: the acceleration $\ddot{z}(t)$, the

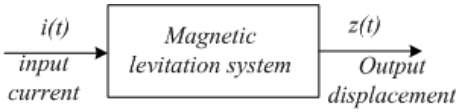

Figure 2.6: Magnetic levitation system. Input and output

position $z(t)$ and the current $i(t)$, that equals zero:

$$g(\ddot{z}(t), z(t), i(t)) = m\ddot{z}(t) - mg + C\frac{i^2(t)}{z^2(t)} = 0 \tag{2.3}$$

The first step in linearizing this equation is setting the desired operating point. If the ball is at equilibrium for z_0, then the speed and acceleration of the ball are zero for this distance: $\ddot{z}_0 = 0$. The current that keeps the ball floating at the distance z_0 is obtained from the nonlinear equation for $z(t) = z_0$ and $\ddot{z}(t) = \ddot{z}_0$:

$$m\ddot{z}_0 = mg - C\frac{i_0^2}{z_0^2}$$

or

$$0 = mg - C\frac{i_0^2}{z_0^2}, \;\; \Rightarrow i_0 = \sqrt{\frac{mg}{C}}z_0 = 0.1A$$

The operating point for linearization is then: $(\ddot{z}_0,\ z_0,\ i_0) = (0, 0.01, 0.1)$

According to (2.2),the Taylor series for the approximation of the nonlinear function (2.3) is written as

$$\begin{aligned} 0 = g(\ddot{z}(t), z(t), i(t)) \;\; &\approx \;\; g(\ddot{z}_0, z_0, i_0) + \frac{\partial g}{\partial \ddot{z}}|_{(\ddot{z}_0,\ z_0,\ i_0)} \cdot (\ddot{z}(t) - \ddot{z}_0) + \\ &+ \;\; \frac{\partial g}{\partial z}|_{(\ddot{z}_0,\ z_0,\ i_0)} \cdot (z(t) - z_0) + \frac{\partial g}{\partial i}|_{(\ddot{z}_0,\ z_0,\ i_0)} \cdot (i(t) - i_0) \end{aligned}$$

or:

$$0 \approx 0 + m \cdot (\ddot{z}(t) - \ddot{z}_0) - 2C\frac{i_0^2}{z_0^3} \cdot (z(t) - z_0) + 2C\frac{i_0}{z_0^2} \cdot (i(t) - i_0)$$

We denote by $\Delta\ddot{z}(t) = \ddot{z}(t) - \ddot{z}_0$, $\Delta z(t) = z(t) - z_0$ and $\Delta i(t) = i(t) - i_0$ the variations of variables around the operating point, and re-arranging the equation above, we obtain:

$$m\Delta\ddot{z}(t) = 2C\frac{i_0^2}{z_0^3}\Delta z(t) - 2C\frac{i_0}{z_0^2}\Delta i(t) \tag{2.4}$$

All coefficients are constant and the relation (2.4) is a linear and homogeneous differential equation. By replacing the constant values, the equation (2.4) is:

$$0.1\Delta\ddot{z}(t) = 200\Delta z(t) - 20\Delta i(t)$$

2.4 The Laplace Transform

The ability to obtain linear approximations of physical systems allows the analyst to consider the use of Laplace transformation. The Laplace transform method substitutes the more difficult

differential equations with the relatively easily solved algebraic equations. The basic Laplace transform of a time signal $f(t)$ is defined as:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

and written symbolically as: $F(s) \equiv \pounds[f(t)]$

It is common practice to use a capital letter F which is a function of the new variable s for the transform of the time signal $f(t)$. Also it is assumed that $f(t)$ is zero for all times before t=0. In the definition of Laplace transform the exponent st must be dimensionless otherwise the expression e^{-st} is meaningless. Thus the variable s has dimension of 1/time which is the dimension of frequency. Since s is also a complex quantity it is often referred to as the *the complex frequency*.

Table 2.1: Laplace transform operations

1	Transform integral	f(t)	$\int_0^\infty f(t)e^{-st}dt$ or $\pounds[f(t)]$
2	Linearity	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
3	Constant multiplication	af(t)	aF(s)
4	Complex shift theorem	$e^{\pm at}f(t)$	F(s±a)
5	Real shift theorem	f(t-T)	$e^{-Ts}F(s)$, T≥0
6	Scaling theorem	$f(\frac{t}{a})$	aF(as)
7	First derivative	$\frac{d}{dt}$f(t)	sF(s) - f(0)
8	n-th derivative	$\frac{d^n}{dt^n}f(t)$	$s^nF(s) - \sum_{r=1}^{n} \frac{d^{r-1}}{dt^{r-1}}f(0)s^{n-r}$
9	First integral	$\int_0^t f(t)dt$	$\frac{1}{s}F(s)$

Table 2.2: Laplace transforms of common functions

	f(t)	F(s)
1	Unit impulse (Dirac) δ (t)	1
2	Unit step u(t)=1	$\frac{1}{s}$
3	Unit ramp v(t)=t	$\frac{1}{s^2}$
4	e^{at}	$\frac{1}{s-a}$
5	$cos\omega t$	$\frac{s}{s^2+\omega^2}$
6	$sin\omega t$	$\frac{\omega}{s^2+\omega^2}$

2.5 Signals

A few commonly used signals will be introduced in this section:

1. The unit step is defined by (Figure 2.7):

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{2.5}$$

The Laplace transform of the step function, according to Table 2.2, is:

$$\pounds[u(t)] = \frac{1}{s}, \tag{2.6}$$

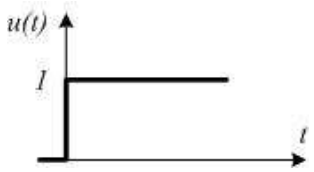

Figure 2.7: Unit step signal

2. The unit ramp is shown in Figure 2.8. It is defined by:

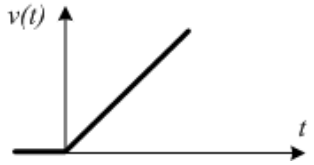

Figure 2.8: The ramp signal

$$v(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases} \tag{2.7}$$

The Laplace transform of the ramp signal is:

$$\pounds[v(t)] = \frac{1}{s^2} \tag{2.8}$$

3. The ideal impulse also called Dirac impulse is shown in Figure 2.9 It is defined by:

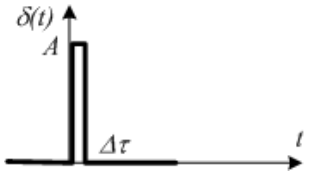

Figure 2.9: The impulse signal

$$\delta(t) = \begin{cases} 0, & t < 0 \ \ and \ \ t > \Delta\tau \\ A, & 0 \leq t \leq \Delta\tau \end{cases} \tag{2.9}$$

where ideally

$$\lim_{\Delta\tau \to 0} \int_0^{\Delta\tau} \delta(t)dt = 1 \ \text{ and } \ A \to \infty \tag{2.10}$$

The Dirac impulse is a signal with unit area and infinite amplitude.

The Laplace transform of the unit impulse is:

$$\pounds[\delta(t)] = 1 \tag{2.11}$$

2.6 Input/output models

2.6.1 The transfer function

The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all the initial conditions assumed to be zero.

A transfer function may be defined only for a linear stationary system. A transfer function is an input-output description of the behavior of a system.

A linear differential equation that describes such a system can be:

$$a_0 r(t) + a_1 \frac{dr(t)}{dt} + ... + a_m \frac{d^m r(t)}{dt^m} = b_0 y(t) + b_1 \frac{dy(t)}{dt} + ... + b_n \frac{d^n y(t)}{dt^n}$$

where $r(t)$ and $y(t)$ are the input and output variables, as shown in Figure 2.10 where $R(s) = \mathcal{L}[r(t)]$ and $Y(s) = \mathcal{L}[y(t)]$.

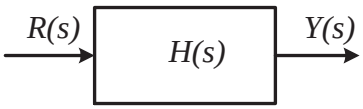

Figure 2.10: Block diagram of a system system

Applying the Laplace transform for the initial conditions zero, the differential equation becomes:

$$(a_0 + a_1 s + ... + a_m s^m) R(s) = (b_0 + b_1 s + ... + b_n s^n) Y(s)$$

and the transfer function will then be:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{a_0 + a_1 s + ... + a_m s^m}{b_0 + b_1 s + ... + b_n s^n}$$

Example 2.6.1 Spring-mass-damper

The transfer function of the spring-mass-damper system, shown in Figure 2.2, is obtained from the original equation model by applying the Laplace transform:

$$M \frac{d^2 y(t)}{dt} + f \frac{dy(t)}{dt} + k y(t) = r(t)$$

$$M s^2 Y(s) + f s Y(s) + k Y(s) = R(s)$$

$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{M s^2 + f s + k}$$

The block diagram now contains all the information given in the transfer function model, that is, "output = contents × input". $H(s)$ is called a transfer function. It shows how the input is transferred to the output.

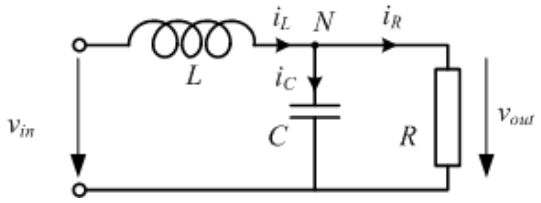

Figure 2.11: Electrical system

Example 2.6.2 Electrical system. *The transfer function for the system shown in Figure 2.11 can be derived as follows: Write the fundamental relation for each element:*

$$\text{Inductor:} \quad \frac{di_L}{dt} = \frac{1}{L} v_L \tag{2.12}$$

$$\text{Capacitor:} \quad \frac{dv_C}{dt} = \frac{1}{C} i_C \tag{2.13}$$

$$\text{Resistor:} \quad v_R = R i_R \tag{2.14}$$

Write the Kirchhoff's current law at node N:

$$i_L = i_C + i_R \tag{2.15}$$

Write the Kirchhoff's voltage law:

$$v_{in} = v_L + v_C \tag{2.16}$$

$$v_C = v_R = v_{out} \tag{2.17}$$

Assume the initial conditions zero, replace (2.17) and apply the Laplace transform to the relations (2.12) - (2.16):

$$sI_L(s) = \frac{1}{L} V_L(s) \tag{2.18}$$

$$sV_{out}(s) = \frac{1}{C} I_C(s) \tag{2.19}$$

$$V_{out}(s) = R I_R(s) \tag{2.20}$$

$$I_L(s) = I_C(s) + I_R(s) \tag{2.21}$$

$$V_{in}(s) = V_L(s) + V_{out}(s) \tag{2.22}$$

Calculate to eliminate I_L, I_C, I_R, V_L from (2.18) - (2.22):

$$V_{in} = sLI_L + V_{out} = sL(I_C + I_R) + V_{out} = sL(sCV_{out} + \frac{1}{R} V_{out}) + V_{out} = (LCs^2 + \frac{L}{R} s + 1) V_{out} \tag{2.23}$$

The transfer function is:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{LCs^2 + \frac{L}{R} s + 1} = \frac{R}{RLCs^2 + Ls + R} \tag{2.24}$$

For a **real physical** system the function $H(s)$ is a ratio of two polynomials such that

$$H(s) = \frac{N(s)}{D(s)}$$

with the degree of the denominator $D(s)$ being greater than or equal to the degree of the numerator polynomial $N(s)$. Such a system is called **proper**. In a proper system, the **system order** is the degree of the denominator polynomial.

If the denominator polynomial is extracted and set equal to zero, that is: $D(s) = 0$, the resulting equation is called **the system's characteristic equation**, since it can be shown to characterize the system's dynamics. It's roots are the **poles** of $H(s)$. The roots of the numerator polynomial are the **zeros** of the transfer function. Both the poles and zeros of H(s) can be complex values, $s = \sigma + j\omega$.

Example 2.6.3 *Consider a system described by the transfer function*

$$H(s) = \frac{s+2}{s^3 + s^2 + s + 1}$$

It is a third order system because the degree of the denominator polynomial is 3. The system is proper since the degree of the denominator is greater than the degree of the numerator. The characteristic polynomial is:

$$D(s) = s^3 + s^2 + s + 1$$

The system's characteristic equation is:

$$s^3 + s^2 + s + 1 = 0$$

and the roots of the characteristic equation are the system poles, that result from:

$$s^3 + s^2 + s + 1 = (s+1)(s^2+1) = 0, \text{ and the poles are: } p_1 = -1,\ p_{2,3} = \pm j$$

The numerator polynomial has one root, that is the zero of the system:

$$s + 2 = 0, \text{ and the zero is: } z_1 = -2$$

Example 2.6.4 *Consider a system described by the transfer function*

$$H(s) = \frac{2}{s^3(s+3)}$$

It is a fourth order system because the degree of the denominator polynomial is 4. The system is proper since the degree of the denominator is greater than the degree of the numerator. The characteristic polynomial is:

$$D(s) = s^3(s+3) = s^4 + 3s^3$$

The system's characteristic equation is:

$$s^4 + 3s^3 = 0$$

and the roots of the characteristic equation are the system poles, that result from:

$$s^4 + 3s^3 = s^3(s+3) = 0, \text{ and the poles are: } p_1 = p_2 = p_3 = 0,\ p_4 = -3$$

The numerator is a zero-order polynomial, i.e. system has no zeros.

2.6.2 System response

The transfer function can be easily used for the calculation of system response to an input signal. Given the input signal $r(t)$ and the transfer function model of a system $H(s)$, we are interested in finding the system behavior, that is the output signal $y(t)$ (see Figure 2.10). From the definition of the transfer function:

$$Y(s) = H(s) \cdot R(s) \tag{2.25}$$

By applying the inverse Laplace transform we obtain:

$$y(t) = \mathcal{L}^{-1}[H(s) \cdot R(s)]. \tag{2.26}$$

Example 2.6.5 Spring-mass-damper response. *Consider the spring-mass-damper system from the previous example and Figure 2.2. The transfer function of this system is given by:*

$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + fs + k}.$$

We shall obtain the system response or output variable, when the input r(t) is a Dirac impulse and the system constants M, f, k are:

1. $M = 1, f = 3, k = 2$
2. $M = 1, f = 1, k = 2$ *(smaller friction coefficient)*

If $r(t)$ is a unit impulse, the Laplace transform will be $R(s) = 1$ and the output variable $y(t)$ is:

$$y(t) = \mathcal{L}^{-1}[H(s) \cdot R(s)] = \mathcal{L}^{-1}[H(s) \cdot 1] = \mathcal{L}^{-1}[H(s)]$$

1. *In the first case the transfer function has two real poles*

$$Y(s) = \frac{1}{(s+1)(s+2)}$$

Expanding $Y(s)$ in a partial fraction expansion, we obtain

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

where k_1 and k_2 are the coefficients of the expansion, or the residues, and are evaluated by multiplying through by the denominator factor corresponding to k_i and setting s equal to the root.

$$k_1 = \frac{1}{s+2}\Big|_{s=-1} = 1$$

$$k_2 = \frac{1}{s+1}\Big|_{s=-2} = -1$$

The inverse Laplace transform of $Y(s)$ is then

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

Using a Laplace transforms table, we find that

$$y(t) = e^{-t} - e^{-2t}$$

The response $y(t)$ is called overdamped (no oscillations) and is shown in Figure 2.12 (left).

2. *In this case $H(s)$ can be written as:*

$$H(s) = \frac{1}{s^2 + s + 2}$$

and has the complex poles:

$$r_{1,2} = \frac{-1 \pm \sqrt{7}j}{2}$$

The system response will be called underdamped (damped oscillations) as the poles of the transfer function are complex and, as in the previous case, is computed from:

$$y(t) = \mathcal{L}^{-1}[H(s)]$$

Using a table with Laplace transform properties, and writing $Y(s)$ as

$$Y(s) = \frac{2}{\sqrt{7}} \frac{\frac{\sqrt{7}}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{7}}{2})^2}$$

the system response is:

$$y(t) = \frac{2}{\sqrt{7}} e^{-t/2} sin(\frac{\sqrt{7}}{2} t)$$

The underdamped response is shown in Figure 2.12 (right).

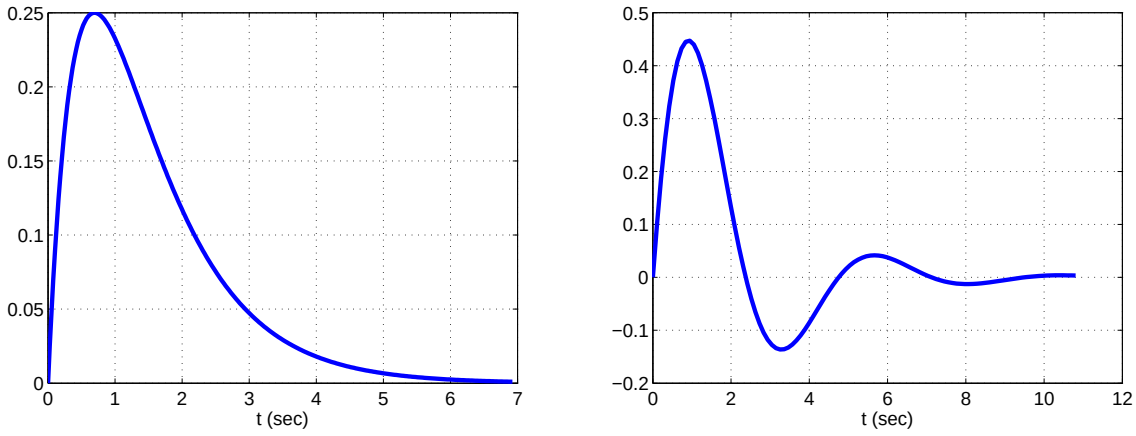

Figure 2.12: Spring-mass-damper response. Overdamped case (left). Underdamped case (right)

2.6.3 The transfer matrix

Because of the great number of systems with multiple inputs and outputs (MIMO systems), calculus techniques specific for this kind of systems were developed. In case of linear systems the mathematical techniques use transfer matrices instead of transfer functions.

Figure 2.13 illustrates a system with m inputs and n outputs.

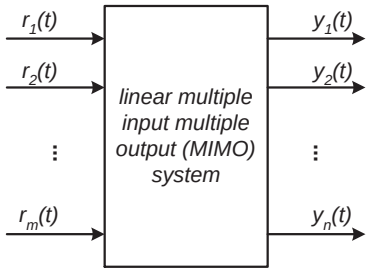

Figure 2.13: MIMO system

If the signal transfer is unidirectional and each input influences each output, as shown in Figure 2.14, we can write the following equation system:

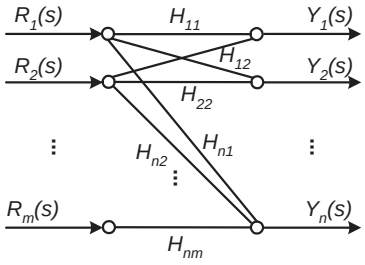

Figure 2.14: Transfer functions between inputs and outputs

$$
\begin{aligned}
Y_1(s) &= H_{11}R_1(s) + H_{12}R_2(s) + \ldots H_{1m}R_m(s) \\
Y_2(s) &= H_{21}R_1(s) + H_{22}R_2(s) + \ldots H_{2m}R_m(s) \\
&\ldots \\
Y_n(s) &= H_{n1}R_1(s) + H_{n2}R_2(s) + \ldots H_{nm}R_m(s)
\end{aligned}
$$

where $R(s)$ and $Y(s)$ are the Laplace transforms of the inputs $r(t)$ and outputs $y(t)$, and

$$H_{jk} = \frac{Y_j(s)}{R_k(s)}$$

is the transfer function from the input k to the output j.

Thus, a multiple linear system is composed by nxm elements having one input and one output. The equation system can be written in a matrix form:

$$\mathbf{Y}(s) = \mathbf{H}(s) \cdot \mathbf{R}(s)$$

where

$$\mathbf{R}(s) = [R_1(s)\ R_2(s)\ ...\ R_m(s)]^T$$

is the $(m \times 1)$ input vector,

$$\mathbf{Y}(s) = [Y_1(s)\ Y_2(s)\ ...\ Y_n(s)]^T$$

is the $(n \times 1)$ output vector , and

$$\mathbf{H} = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \dots & H_{1m}(s) \\ H_{21}(s) & H_{22}(s) & \dots & H_{2m}(s) \\ \dots & \dots & \dots & \dots \\ H_{n1}(s) & H_{n2}(s) & \dots & H_{nm}(s) \end{bmatrix}$$

is called the **transfer matrix**.

Example 2.6.6 *Consider a system with one input $r(t)$ and two outputs $y_1(t)$ and $y_2(t)$. If a linear relationship can be determined between the input and each output, then the system can be described by a transfer matrix having the size 2×1. As an example (Figure 2.15) consider the transfer functions between the input and each output as follows:*

$$H_{11}(s) = \frac{s}{s+1}, \quad H_{21}(s) = \frac{1}{s+2}$$

The relation between the input and outputs written in a matrix form is then:

$$\mathbf{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) \\ H_{21}(s) \end{bmatrix} U(s)$$

where the transfer matrix is:

$$\mathbf{H}(s) = \begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s+2} \end{bmatrix}$$

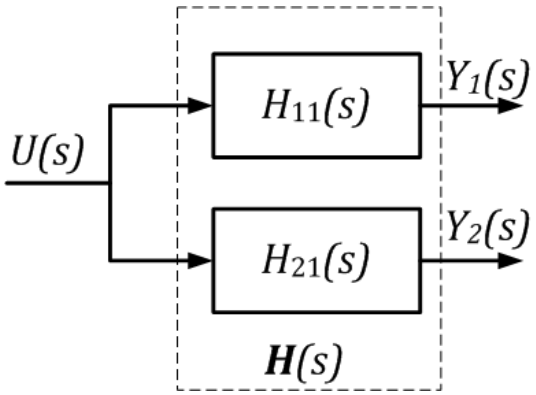

Figure 2.15: Transfer matrix example

2.7 State-space models

A modern complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner. To analyze such a system, it is essential to reduce the complexity of the mathematical expressions, as well as to use computers for most of the complicated computations necessary in the analysis. The state-space approach to system analysis is best suited from this viewpoint.

With the ready availability of computers it is convenient to consider the time-domain formulation of the equations representing dynamical systems. The time-domain formulation can be readily utilized for nonlinear, time-varying and multivariable systems.

The time-domain representation of control systems is an essential basis for modern control theory and system optimization. In this section we shall develop the time-domain representation of control systems and illustrate several methods for the solution of the system time-response, (Dorf and Bishop, 2008).

2.7.1 The state variables of a dynamic system

The time-domain analysis and design of a control system utilize the concept of the state of a system.

The state of a system is a set of numbers such that the knowledge of these numbers and the input functions will, with the equations describing the dynamics, provide the future state and output of the system.

For a dynamic system, the state of a system is described in terms of a set of *state variables* $[x_1(t), x_2(t),..., x_n(t)]$. The state variables are the variables that determine the future behavior of a system when the present state of a system and the excitation signals are known. Consider the system shown in Figure 2.16, where $y_1(t)$ and $y_2(t)$ are the output signals and $u_1(t)$ and $u_2(t)$ are the input signals. A set of state variables $(x_1,x_2,...x_n)$ for the system shown in the

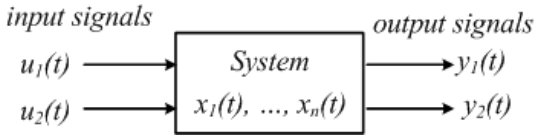

Figure 2.16: System block diagram

figure is a set such that knowledge of the initial values of the state variables $[x_1(t_0), x_2(t_0),..., x_n(t_0)]$ at the initial time t_0 and of the input signal for $t \geq 0$ suffices to determine the future values of the outputs and state variables, (Dorf and Bishop, 2008).

Example 2.7.1 *A simple example of a state variable is the state of an on-off light switch. The switch can be either the on or the off position, and thus the state of the switch can assume one of two possible values. Thus, if we know the present state (position) of the switch at t_0 and if an input is applied, we are able to determine the future value of the state of the element.*

Example 2.7.2 Spring-mass-damper, (Dorf and Bishop, 2008). *The concept of a set of state variables that represent a dynamic system can be illustrated in terms of the spring-mass-damper system (see Figure 2.2). The number of state variables chosen to represent this system should be as few as possible in order to avoid redundant state variables. A set of state variables sufficient to describe this system is the position and the velocity of the mass. Therefore we will define a set of state variables as (x_1, x_2), where*

$$x_1(t) = y(t) \quad and \quad x_2(t) = \frac{dy(t)}{dt}$$

The differential equation describes the behavior of this system and is usually written as

$$M\frac{d^2y(t)}{dt^2} + f\frac{dy(t)}{dt} + ky(t) = r(t)$$

To write this equation in terms of the state variables, we substitute the state variables as already defined and obtain:

$$M\frac{dx_2(t)}{dt} + fx_2(t) + kx_1(t) = r(t)$$

Therefore we can write the differential equations that describe the behavior of the spring-mass-damper system as a set of two first-order differential equations as follows:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= -\frac{f}{M}x_2(t) - \frac{k}{M}x_1(t) + \frac{1}{M}r(t) \end{aligned}$$

This set of differential equations describes the behavior of the state of the system in terms of the rate of change of each state variable.

In this case, the output is connected to the states directly by the relation:

$$y(t) = x_1(t)$$

Example 2.7.3 RLC Circuit. *(Dorf and Bishop, 2008) As another example of the state variable characterization of a system, let us consider the RLC circuit shown in Figure 2.17.*

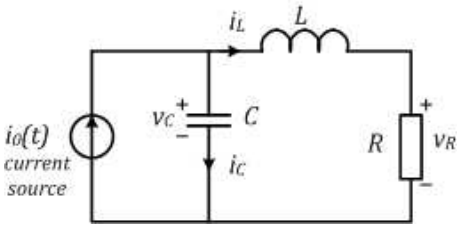

Figure 2.17: RLC circuit

The state of this system can be described in terms of a set of state variables (x_1, x_2) where x_1 is the capacitor voltage $v_c(t)$ and x_2 is equal to the inductor current $i_L(t)$. For a passive RLC network, the number of state variables required is equal to the number of independent energy-storage elements. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables as:

$$E = \frac{1}{2}Li_L^2(t) + \frac{1}{2}Cv_c^2(t)$$

Utilizing Kirchoff's current law at the junction, the source current $i_0(t)$ is the sum of capacitor current $i_C(t)$ and the inductor current $i_L(t)$:

$$i_0(t) = i_C(t) + i_L(t)$$

Using the relation between the capacitor current and voltage we obtain a first-order differential equation by describing the rate of change of the capacitor voltage as:

$$i_0(t) = C\frac{dv_c}{dt} + i_L(t) \tag{2.27}$$

Kirchoff's voltage law for the right-hand loop is:

$$v_C(t) = v_L(t) + v_R(t)$$

and provides the equation describing the rate of change of inductor current as

$$v_C(t) = L\frac{di_L(t)}{dt} + Ri_L(t) \tag{2.28}$$

The output of this system is the voltage drop accross the resistor and is represented by the linear algebraic equation

$$v_R(t) = Ri_L(t)$$

We can rewrite equations (2.27) and (2.28) as a set of first-order differential equations in terms of v_c and i_L:

$$\begin{aligned} \frac{dv_c(t)}{dt} &= -\frac{1}{C}i_L(t) + \frac{1}{C}i_0(t) \\ \frac{di_L(t)}{dt} &= \frac{1}{L}v_C(t) - \frac{R}{L}i_L(t) \end{aligned}$$

or in terms of the state variables $x_1(t)$ and $x_2(t)$, when the input $i_0(t)$ is denoted by $i_0(t) = u(t)$, as follows:

$$\frac{dx_1(t)}{dt} = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t) \tag{2.29}$$

$$\frac{dx_2(t)}{dt} = \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t) \tag{2.30}$$

The output signal is then

$$y(t) = v_R(t) = Rx_2(t) \tag{2.31}$$

At the moment $t_0 = 0$, the capacitor voltage $x_1(t_0) = v_C(t_0)$ and the inductor current $x_2(t_0) = i_L(t_0)$ are considered to be known.

Utilizing equations (2.29), (2.30) and (2.31) the initial conditions of the network represented by $[x_1(t_0), x_2(t_0)]$ we can determine the system's future behavior and its output.

The state variables that describe a system are not a unique set!

Several alternative sets of state variables can be chosen. For example, for the RLC circuit we might choose the set of state variables as the two voltages $v_C(t)$ and $v_L(t)$, where $v_L(t)$ is the voltage drop across the inductor. Then, the state variables x_1^* and x_2^* are related to the old state variables x_1 and x_2, as

$$x_1^* = v_C = x_1$$
$$x_2^* = v_L = v_C - Ri_L = x_1 - Rx_2$$

Thus, in the actual system, there are several choices of a set of variables that specify the energy stored in a system and therefore adequately describe the dynamics of a system.

A widely used choice is a set of state variables that can be readily measured.

Example 2.7.4 *To illustrate the non uniqueness of state variable models, consider again the RLC circuit presented in Example 2.7.3. The input is the source current $i_0(t)$ as before, and the output is the voltage on the resistor $v_R(t)$. As in the previous example, the relations describing the system dynamics are given by Kirchoff's laws for current and voltage together with the equations describing each element separately:*

$$i_0(t) = C\frac{dv_C}{dt} + i_L(t) \tag{2.32}$$

$$v_C(t) = L\frac{di_L(t)}{dt} + Ri_L(t) \tag{2.33}$$

The state variables may be chosen as $x_1(t) = i_L(t)$ and $x_2(t) = \frac{di_L(t)}{dt}$ (the inductor current and its first derivative). A set of two first-order differential equations describing the system in terms of the newly chosen state variables can be obtained as follows:

- *Eliminate all the variables except for the chosen states and the input. Computing the first derivative of (2.33) and combining it with (2.32) gives:*

$$\frac{dv_C(t)}{dt} = L\frac{d^2i_L(t)}{dt^2} + R\frac{di_L(t)}{dt} = \frac{1}{C}\left(u(t) - i_L(t)\right)$$

  or, rearranged:

$$L\frac{d^2i_L(t)}{dt^2} + R\frac{di_L(t)}{dt} + \frac{1}{C}i_L(t) = \frac{1}{C}u(t) \tag{2.34}$$

- *Replace state variables, chosen as: $x_1(t) = i_L(t)$ and $x_2(t) = \frac{di_L(t)}{dt} = \frac{dx_1(t)}{dt}$ in (2.34), denote $u(t) = i_0(t)$ and obtain:*

$$L\frac{dx_2(t)}{dt} + Rx_2(t) + \frac{1}{C}x_1(t) = \frac{1}{C}u(t)$$

- *Write the system of two first-order differential equations resulted from the relations above:*

$$\frac{dx_1(t)}{dt} = x_2(t) \tag{2.35}$$
$$\frac{dx_2(t)}{dt} = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{LC}u(t) \tag{2.36}$$

  and the relation giving the output:

$$y(t) = v_R(t) = Ri_L(t) = Rx_1(t) \tag{2.37}$$

The same RLC circuit as in Example 2.7.3 is described now by the relations (2.35), (2.36) and (2.37). The input and output signals are the same, but the state variables are chosen in a different manner, therefore the model is not unique.

2.7.2 The state-space standard form

The state of a system is described by the set of first-order differential equations written in terms of the state variables $(x_1(t), x_2(t), ..., x_n(t))$ and the inputs $(u_1(t), u_2(t), ..., u_m(t))$. These first order differential equations can be written in the general form:

$$\begin{aligned}
\dot{x_1}(t) &= a_{11}x_1(t) + a_{12}x_2(t) + ...a_{1n}x_n(t) + b_{11}u_1(t) + ... + b_{1m}u_m(t) \\
\dot{x_2}(t) &= a_{21}x_1(t) + a_{22}x_2(t) + ...a_{2n}x_n(t) + b_{21}u_1(t) + ... + b_{2m}u_m(t) \\
&... \\
\dot{x_n}(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + ...a_{nn}x_n(t) + b_{n1}u_1(t) + ... + b_{nm}u_m(t)
\end{aligned}$$

where $\dot{x_i}(t) = \frac{dx_i(t)}{dt}$. Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & ... & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & ... & b_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \tag{2.38}$$

The column matrix consisting of the state variables is called the **state vector** and is written as:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{2.39}$$

where the boldface indicates a matrix. The matrix of input signals is defined as $\mathbf{u}(t)$:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

Then, the system can be represented by the compact notation of the state differential equation as:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2.40}$$

The differential equation (2.40) is also commonly called the **state equation**.

The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals.

In general, the outputs of a linear system can be related to the state variables and the input signals by the **output equation**:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{2.41}$$

where $\mathbf{y}(t)$ is the set of output signals expressed in column vector form:

$$\mathbf{y(t)} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

The matrix equation (2.40) together with (2.41):

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{2.42}$$

are the standard form of the **state-space model** or the *state-space representation* of the linear system.

In equations (2.42):

- $\mathbf{x}(t)$ is an $(n \times 1)$ state vector, where n is the number of states or system order
- $\mathbf{u}(t)$ is an $(m \times 1)$ input vector, where m is the number of input functions
- $\mathbf{y}(t)$ is a $(p \times 1)$ output vector where p is the number of outputs
- $\mathbf{A}$ is an $(n \times n)$ square matrix called the **system matrix** or **state matrix**
- $\mathbf{B}$ is an $(n \times m)$ matrix called the **input matrix**
- $\mathbf{C}$ is a $(p \times n)$ matrix called the **output matrix**
- $\mathbf{D}$ is a $(p \times m)$ matrix which represents any direct connection between the input and the output. It as called the *feedthrough* matrix.

Example 2.7.5 RLC Circuit. *Consider again the RLC circuit from Example 2.7.3. We obtained the equations (2.29), (2.30) and (2.31) to describe the system dynamics.*

In a vector-matrix notation the state variable differential equation for the RLC circuit of Figure 2.11 is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} u(t)$$

and the output is

$$y(t) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Using the compact form for state variable vector, input and output vectors, the state-space model is:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} u(t)$$

and

$$y(t) = \begin{bmatrix} 0 & R \end{bmatrix} \boldsymbol{x}(t)$$

The matrices from (2.40) and (2.41) are now:

$$\mathbf{A} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1/C \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & R \end{bmatrix}, \text{ and } \mathbf{D} = [0]$$

Example 2.7.6 *Consider the same RLC circuit as above, but with the model computed in Example 2.7.4 and given by the equations (2.35), (2.36) and (2.37). The state-space model can be written in a matrix form as:*

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1/LC \end{bmatrix} u(t)$$

and

$$y(t) = \begin{bmatrix} R & 0 \end{bmatrix} \boldsymbol{x}(t)$$

The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ *are:*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/LC \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} R & 0 \end{bmatrix}, \text{ and } \mathbf{D} = [0]$$

Example 2.7.7 Inverted pendulum *(Dorf and Bishop, 2008) The problem of balancing a broomstick on a person's hand is similar to the classic problem of the inverted pendulum mounted on a cart as shown in Figure 2.18. The cart must be moved so that the mass m is always in the*

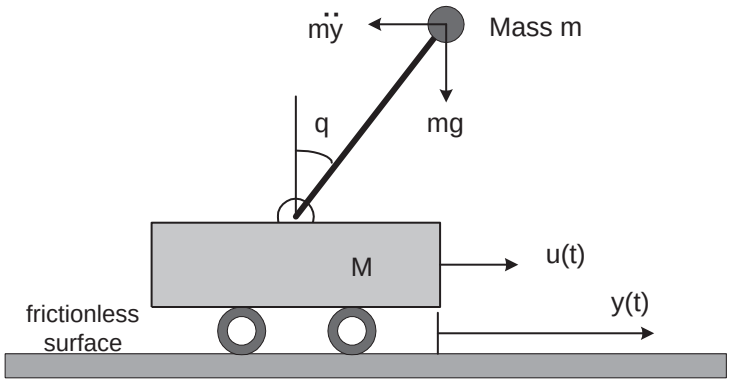

Figure 2.18: Inverted pendulum

upright position. The state variables must be expressed in terms of the angular rotation $q(t)$ and the position of the cart $y(t)$. The differential equations describing the motion of the system can be obtained by writing the sum of forces in the horizontal direction and the sum of moments about the pivot point. We will assume $M \gg m$ and the angle of rotation q is small so that the equations are linear.

The sum of forces in the horizontal direction is

$$M\ddot{y}(t) + ml\ddot{q}(t) - u(t) = 0 \tag{2.43}$$

where $u(t)$ equals the force on the cart and l is the distance from the mass to the pivot point. The sum of the torques about the pivot point is

$$ml\ddot{y}(t) + ml^2\ddot{q}(t) - mlgq(t) = 0 \tag{2.44}$$

The state variables for the equations (2.43) and (2.44) are chosen as $(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) = (y(t), \dot{y}(t), q(t), \dot{q}(t))$, i.e. the states are the position and the velocity of the cart, the angular position and the angular velocity of the pendulum.

Then, the system describing equations are written in terms of the state variables as:

$$M\dot{x_2}(t) + ml\dot{x_4}(t) - u(t) = 0 \tag{2.45}$$

and

$$\dot{x_2}(t) + l\dot{x_4}(t) - gx_3(t) = 0. \tag{2.46}$$

To obtain the necessary first-order differential equations we extract $l\dot{x_4}$ from (2.45) and substitute in (2.46):

$$Ml\dot{x_2}(t) + mgx_3(t) = u(t)$$

since $M \gg m$. Substituting $\dot{x_2}$ we have:

$$Ml\dot{x_4}(t) - Mgx_3(t) + u(t) = 0.$$

The four first order differential equations can be written as:

$$\begin{aligned}
\dot{x_1}(t) &= x_2(t) \\
\dot{x_2}(t) &= -\frac{mg}{M}x_3(t) + \frac{1}{M}u(t) \\
\dot{x_3}(t) &= x_4(t) \\
\dot{x_4}(t) &= \frac{g}{l}x_3(t) - \frac{1}{Ml}u(t)
\end{aligned}$$

The system matrices are:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/Ml \end{bmatrix}$$

The outputs of this system are the position of the cart $x_1(t)$ and the angle of rotation $q(t)$.

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2.7.3 Solution of State Differential Equation

The solution of the state differential equation can be obtain by using the Laplace transform or in a manner similar to the approach we utilize for solving a first-order differential equation.

2.7.3.1 Laplace transform method

In this section we use the Laplace transform to solve the state equations for the state and output vectors. Consider the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2.47}$$

and the output equation:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{2.48}$$

Taking the Laplace transform of both sides of the state and output equations (2.47) and (2.48) for non-zero initial conditions yields:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

In order to separate $\mathbf{X}(s)$ in the state equation replace $s\mathbf{X}(s)$ with $s\mathbf{I}\mathbf{X}(s)$ where $\mathbf{I}$ is an $n \times n$ identity matrix and n is the order of the system. Rearranging we obtain:

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{B}\mathbf{U}(s) + \mathbf{x}(0)) \tag{2.49}$$

Taking the inverse Laplace transform, the state vector, as a function of time is:

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\mathbf{X}(s)]$$

Replacing (2.49) into the output equation, the output vector is:

$$\mathbf{Y}(s) = \mathbf{C}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{x}(0))] + \mathbf{D}\mathbf{U}(s)$$

or

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{2.50}$$

The output vector, as a function of time, is computed by taking the inverse Laplace transform of (2.50):

$$\mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{Y}(s)]$$

Note that, when the initial conditions $\mathbf{x}(0)$ are zero, $\mathbf{C}[(s \cdot \mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$ is the transfer function matrix mapping the input vector $\mathbf{U}(s)$ to the output vector $\mathbf{Y}(s)$. Moreover, the poles of this transfer function are the same as the eigenvalues of the $\mathbf{A}$ matrix of the state space representation of the same system.

From

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s), \quad \mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$$

the transfer matrix will be:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2.51}$$

where

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{det(s\mathbf{I} - \mathbf{A})} adj(s\mathbf{I} - \mathbf{A})$$

The **characteristic equation** of the system is given by:

$$det(s\mathbf{I} - \mathbf{A}) = 0 \tag{2.52}$$

The poles of the system are given by the solution of (2.52). Also, the poles can be determined as the eigenvalues of matrix $\mathbf{A}$.

Example 2.7.8 *Given the system represented in spate-space by:*

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{aligned}$$

with the initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- *Solve the state equation and obtain the output when the input is a unit step $u(t) = 1$, $t \geq 0$.*
- *Find the eigenvalues of the state matrix and the system poles.*

Solution:

- *The system matrices are:*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = [0]$$

We will solve the problem by finding the component parts of (2.50). Compute:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

The adjoint and the determinant of $(s\mathbf{I} - A)$ are:

$$adj(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}, \quad det(s\mathbf{I} - \mathbf{A}) = s(s+3) + 2 = (s+1)(s+2)$$

and the inverse:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

The Laplace transform of the state vector is given by (2.49):

Compute now the state vector from the relation (2.49):

$$\begin{aligned}
\boldsymbol{X}(s) &= (s\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{B}\boldsymbol{U}(s)+\boldsymbol{x}(0)) = \\
&= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \\
&= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{s}+2 \end{bmatrix} = \\
&= \begin{bmatrix} \frac{s+3}{s^2+3s+2} + \frac{1/s+2}{s^2+3s+2} \\ \frac{s(1/s+2)}{s^2+3s+2} - \frac{2}{s^2+3s+2} \end{bmatrix} = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}
\end{aligned} \tag{2.53}$$

The inverse Laplace transform of $X_1(s)$ and $X_2(s)$ will give the time variations of the state variables.

$$\begin{aligned}
x_1(t) &= \mathcal{L}^{-1}(X_1(s)) = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2} \\
x_2(t) &= \mathcal{L}^{-1}(X_2(s)) = 5e^{-2t} - 3e^{-t}
\end{aligned}$$

The output equation from this example is $y(t) = [1 \;\; 0]\mathbf{x}(t) = x_1(t)$. So the system output is already computed.

Otherwise, the output may be obtained from the output equation:

$$\begin{aligned}
\boldsymbol{Y}(s) &= \boldsymbol{C}\mathbf{X}(s) + \boldsymbol{D}\boldsymbol{U}(s) = \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + [0]\,\boldsymbol{U}(s) = X_1(s)
\end{aligned}$$

and

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[X_1(s)] = x_1(t)$$

- *The eigenvalues of the state matrix are given by:*

$$det(s\mathbf{I}-\mathbf{A}) = s(s+3)+2 = (s+1)(s+2) = 0$$

so they are:

$$s_1 = -1, \quad s_2 = -2$$

The system transfer function is computed from the relation (2.51) where $\mathbf{D} = [0]$, thus:

$$\begin{aligned}
\boldsymbol{H}(s) &= \boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \\
&= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s+1)(s+2)}
\end{aligned} \tag{2.54}$$

Because the system has one input and one output signal, the transfer function is a scalar function of s. The poles are -1 and -2 the same as the eigenvalues of the system matrix $\boldsymbol{A}$.

2.7.3.2 Time-domain method

The solution of equation (2.40) can be also obtained in a similar manner to solving a first order differential equation

$$\dot{x} = ax + bu \tag{2.55}$$

where $x(t)$ and $u(t)$ are scalar functions of time. We expect an exponential of the form e^{at}. Taking the Laplace transform of equation (2.55), we have:

$$sX(s) - x(0) = aX(s) = bU(s)$$

and therefore

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s). \tag{2.56}$$

The inverse Laplace transform of (2.56) results in the solution:

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \tag{2.57}$$

We expect the solution of the state differential equation to be similar to equation (2.57). The matrix exponential function can be calculated by a Taylor series expansion as:

$$e^{\mathbf{A}t} = exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + ... + \frac{\mathbf{A}^kt^k}{k!} + ...$$

which converges for all finite t and any $\mathbf{A}$. Then the solution of the state differential equation is found to be

$$\mathbf{x} = exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t exp(\mathbf{A}(t-\tau))\mathbf{B}\mathbf{u}(\tau)d\tau \tag{2.58}$$

The matrix exponential function describes the unforced response of the system and is called the **fundamental** or **state transition matrix**, $\Phi(t)$. The solution of the unforced system (that is, when $\mathbf{u}$=0)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is simply

$$\mathbf{x}(t) = exp(\mathbf{A}t)\mathbf{x}(0) = \Phi(t)\mathbf{x}(0).$$

2.7.4 Conversion from state-space to transfer function or transfer matrix

The method of conversion from state-space models to transfer function (or transfer matrix) models was already presented in section 2.7.3.1. Summarizing, the method is as follows:

- Write the state-space model in the matrix form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

- Consider the initial conditions equal to zero ($\mathbf{x}(0) = \mathbf{0}$) and taking the Laplace transform of both sides of the state and output equations:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

- Separate $\mathbf{X}(s)$ from the state equation and replace it in the output equation:

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

The output vector is:

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

or

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \tag{2.59}$$

- The transfer matrix is defined by the relation

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) \tag{2.60}$$

From (2.59) and (2.60), the transfer matrix is:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2.61}$$

The size of the transfer matrix will be $p \times m$ where p is the number of output signals and m the number of inputs. For a single-input single-output (SISO) system, the transfer matrix is a scalar transfer function.

Example 2.7.9 *Find the transfer function for a system with the state-space representation:*

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}(t) + 0 \cdot u(t) \end{aligned}$$

The size of the system matrix $\boldsymbol{A}$ *is* 2×2*, therefore we have two state variables. The size of* $\boldsymbol{B}$ *is* 2×1 *so the number of inputs is* 1*. The size of* $\boldsymbol{C}$ *is* 1×2 *so the number of outputs is also* 1*. This is a SISO system with a transfer function that can be computed from (2.61) as follows:*

- *Compute* $s\mathbf{I} - \boldsymbol{A}$ *and its inverse:*

$$s\mathbf{I} - \boldsymbol{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} s+2 & 2 \\ -1 & s \end{bmatrix}$$

$$(s\mathbf{I} - \boldsymbol{A})^{-1} = \frac{det(s\mathbf{I} - \boldsymbol{A})}{adj(s\mathbf{I} - \boldsymbol{A})}$$

$$det(s\mathbf{I} - \boldsymbol{A}) = s^2 + 2s + 2$$

$$adj(s\mathbf{I} - \boldsymbol{A}) = \begin{bmatrix} s & -2 \\ 1 & s+2 \end{bmatrix}$$

$$(s\mathbf{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} \frac{s}{s^2+2s+2} & \frac{-2}{s^2+2s+2} \\ \frac{1}{s^2+2s+2} & \frac{s+2}{s^2+2s+2} \end{bmatrix}$$

- *Replace all the matrices in (2.61). Note that* $\mathbf{D} = 0$*, therefore the relation is:*

$$H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

$$\begin{aligned} H(s) &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{s}{s^2+2s+2} & \frac{-2}{s^2+2s+2} \\ \frac{1}{s^2+2s+2} & \frac{s+2}{s^2+2s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s^2+2s+2} & \frac{2s+2}{s^2+2s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{s+2}{s^2+2s+2} \end{aligned}$$

2.7.5 Conversion from transfer function to state-space

Although the transformation from transfer function to a state-space model is not unique, in this section we will present only one method to obtain the state variables in the form of *phase variables*. The state variables are *phase variables* when each subsequent state is defined to be the derivative of the previous state variable.

At first, we show how to represent a general n-th order linear differential equation with constant coefficients in state space in the phase variable form. This method will be applied then to transfer functions.

Consider a system with the input $u(t)$ and the output $y(t)$ described by the differential equation:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t) \tag{2.62}$$

A convenient way to chose the state variables is to choose the output $y(t)$ and its $n-1$ derivatives as the state variables. They are called *phase variables*:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \frac{dy(t)}{dt} \\ &\vdots \\ x_n(t) &= \frac{d^{n-1}y(t)}{dt^{n-1}} \end{aligned} \tag{2.63}$$

Differentiating both sides of the last defined phase variable yields:

$$\dot{x}_n(t) = \frac{d^n y(t)}{dt^n} \tag{2.64}$$

If we denote $\dot{x}_i(t) = \frac{d^i x(t)}{dt^i}$ the system (2.63) can be written also as:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt} = \dot{x}_1(t) \\ x_3(t) &= \frac{d^2y(t)}{dt^2} = \frac{dx_2(t)}{dt} = \dot{x}_2(t) \\ &\vdots \\ x_n(t) &= \frac{d^{n-1}y(t)}{dt^{n-1}} = \frac{dx_{n-1}(t)}{dt} = \dot{x}_{n-1}(t) \end{aligned} \tag{2.65}$$

Substituting the definitions (2.63) and (2.64) into (2.62) we obtain:

$$\dot{x}_n(t) + a_{n-1}x_n(t) + \cdots + a_1x_2(t) + a_0x_1(t) = b_0u(t) \tag{2.66}$$

The n-th order differential equation (2.62) is equivalent to a system of n first order differential equations obtained from the definitions of the derivatives from (2.65) together with the $\dot{x}_n(t)$ that results from (2.66):

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \\ \dot{x}_n(t) &= -a_0x_1(t) - a_1x_2(t) - \cdots - a_{n-1}x_n(t) + b_0u(t) \end{aligned} \tag{2.67}$$

In a matrix-vector form equations (2.67) become:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u(t) \tag{2.68}$$

Equation (2.68) is the *phase-variable* form of the state equation. This form is easily recognized by the pattern of 1's above the main diagonal and 0's for the rest of the state matrix, except for the last row that contains the coefficients of the differential equation written in reverse order, (Nise, 2004).

The output equation is:

$$y(t) = x_1(t)$$

or, in a vector form:

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + 0 \cdot u(t) \tag{2.69}$$

2.7.5.1 Converting a transfer function with constant term at numerator

For a system with an input $u(t)$ and an output $y(t)$ consider a general transfer function with constant term at numerator:

$$H(s) = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} = \frac{Y(s)}{U(s)}$$

Cross-multiplying the relation above yields:

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = b_0 U(s)$$

and by taking the inverse Laplace transform we get:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t)$$

This is exactly the equation (2.62) for which the phase-variable form was obtained above. The state equation is then (2.68) and the output equation is (2.69). Note that the matrix $\mathbf{D} = 0$.

Example 2.7.10 *Consider a system with the input $u(t)$ and an output $y(t)$ described by the transfer function:*

$$H(s) = \frac{2}{s^3 + 2s^2 + 3s + 4}$$

The transfer function written as the ratio of the Laplace transform of the output to the Laplace transform of the input, with all the initial conditions assumed to be zero is:

$$\frac{Y(s)}{U(s)} = \frac{2}{s^3 + 2s^2 + 3s + 4}$$

or

$$(s^3 + 2s^2 + 3s + 4)Y(s) = 2U(s)$$

Taking the inverse Laplace transform we obtain the differential equation:

$$\frac{d^3 y(t)}{dt^3} + 2\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 4y(t) = 2u(t) \tag{2.70}$$

Choosing the state variables as successive derivatives (phase variable form) we get:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt} = \dot{x}_1(t) \\ x_3(t) &= \frac{d^2 y(t)}{dt^2} = \frac{dx_2(t)}{dt} = \dot{x}_2(t) \end{aligned}$$

and the derivative of the last state variable is:

$$\dot{x}_3(t) = \frac{d^3 y(t)}{dt^3}$$

All the above definitions are now replaced into (2.70):

$$\dot{x}_3(t) + 2x_3(t) + 3x_2(t) + 4x_1(t) = 2u(t)$$

This last relation and the definitions of the phase variables will give the state equation in the phase variable form:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= -4x_1(t) - 3x_2(t) - 2x_3(t) + 2u(t) \end{aligned}$$

and the output equation:

$$y(t) = x_1(t)$$

In the matrix-vector form, the state-space model is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t) \tag{2.71}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{2.72}$$

If the vector of state variables is denoted by

$$\mathbf{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}$$

the state-space model is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t)$$

2.7.5.2 Converting a transfer function with polynomial at numerator

Although the method presented below can be applied for systems of any order, to simplify the demonstration, consider a third-order transfer function with a second-order polynomial in the numerator:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Notice that *the denominator is a monic polynomial* (the leading coefficient or the coefficient of highest degree is equal to 1). If the polynomial in the numerator is of order less than the polynomial in the denominator, the numerator and denominator can be handled separately. First, separate the transfer function into two cascaded transfer functions, as shown in Figure 2.19: the first is the denominator and the second one is just the numerator, (Nise, 2004).

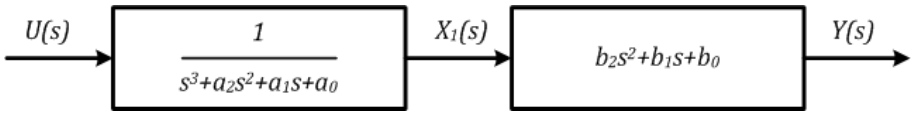

Figure 2.19: Decomposing a transfer function

The first transfer function will be converted to the phase-variable representation in state-space as demonstrated in the previous subsection 2.7.5.1. Hence, phase variable $x_1(t)$ is the

output and the rest of the phase variables are the internal variables of the first block as shown in Figure 2.19.

The first transfer function is:

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{2.73}$$

and the second one:

$$\frac{Y(s)}{X_1(s)} = b_2 s^2 + b_1 s + b_0 \tag{2.74}$$

Following the procedure described in the previous section, the state equation resulting from (2.73) will be:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{2.75}$$

The second transfer function with just the numerator yields:

$$Y(s) = (b_2 s^2 + b_1 s + b_0) X_1(s)$$

where, after taking the inverse Laplace transform with zero initial conditions:

$$y(t) = b_2 \frac{d^2 x_1(t)}{dt^2} + b_1 \frac{dx_1(t)}{dt} + b_0 x_1(t)$$

But the derivative terms are the definitions of the phase variables obtained in the first block. Thus, writing the terms in reverse order, the output equation is:

$$y(t) = b_0 x_1(t) + b_1 x_2(t) + b_2 x_3(t)$$

or, in a matrix form:

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{2.76}$$

Hence, the second block simply forms a specified linear combination of the state variables developed in the first block.

From another perspective, the denominator of the transfer function yields the state equations while the numerator yields the output equation.

If the order of the polynomial in the numerator is equal to the order of the polynomial in the denominator, the third-order transfer function will be written in the general form:

$$H(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

and the numerator resulted from the decomposition is:

$$\frac{Y(s)}{X_1(s)} = b_3s^3 + b_2s^2 + b_1s + b_0 \text{ or } Y(s) = (b_3s^3 + b_2s^2 + b_1s + b_0)X_1(s)$$

After taking the inverse Laplace transform with zero initial conditions:

$$y(t) = b_3\frac{d^3x_1(t)}{dt^3} + b_2\frac{d^2x_1(t)}{dt^2} + b_1\frac{dx_1(t)}{dt} + b_0x_1(t)$$

Using the definitions of the state variables resulted in the first block and re-arranging the terms:

$$y(t) = b_0x_1(t) + b_1x_2(t) + b_2x_3(t) + b_3\dot{x}_3(t)$$

The output equation must not contain any derivatives of the state variables, but we can replace $\dot{x}_3(t)$ with the last equation of the system (2.75):

$$\dot{x}_3(t) = -a_0x_1(t) - a_1x_2(t) - a_2x_3(t) + u(t)$$

so it will become:

$$y(t) = b_0x_1(t) + b_1x_2(t) + b_2x_3(t) + b_3(-a_0x_1(t) - a_1x_2(t) - a_2x_3(t) + u(t))$$

Re-arranging the output equation is:

$$y(t) = (b_0 - b_3a_0)x_1(t) + (b_1 - b_3a_1)x_2(t) + (b_2 - b_3a_2)x_3(t) + b_3u(t)$$

or, in the matrix form:

$$y(t) = \begin{bmatrix} b_0 - b_3a_0 & b_1 - b_3a_1 & b_2 - b_3a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [b_3]\, u(t) \tag{2.77}$$

Note that the output equation (2.76) is the same as (2.77) for $b_3 = 0$. Also, note that the matrix $\mathbf{D}$ is no longer equal to zero. If the system has one input and one output, D is a scalar value and in this case $D = b_3$.

If the system is represented by a transfer function, the minimum number of state variables that have to be chosen is equal to the order of the system (or the order of the polynomial in the denominator of the transfer function).

Example 2.7.11 *Consider a system with the input $u(t)$ and an output $y(t)$ described by the transfer function:*

$$H(s) = \frac{s+2}{s^2+2s+2}$$

Determine the state-space model in the phase variable form.

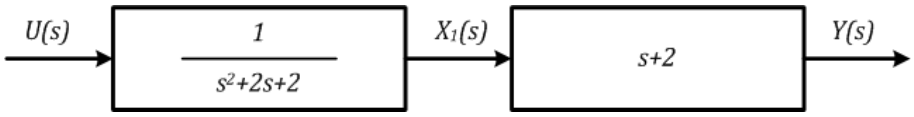

Figure 2.20: Decomposing the transfer function

- *Separate the transfer function into two cascaded blocks: one containing the denominator and the second - the numerator. The output of the first block is the first state variable (see Figure 2.20).*

  The transfer functions written as the ratio of the Laplace transform of the output to the Laplace transform of the input, with all the initial conditions assumed to be zero are:

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^2+2s+2}, \quad \frac{Y(s)}{X_1(s)} = s+2$$

- *Find the state equations for the first block. From the first transfer function we have:*

$$(s^2+2s+2)X_1(s) = U(s)$$

  and the differential equation obtained by taking the inverse Laplace transform is:

$$\ddot{x}_1(t) + 2\dot{x}_1(t) + 2x_1(t) = u(t)$$

  The first state variable was already chosen as the output of the first block $x_1(t)$ and the number of state variables is 2, equal to the order of the system. Therefore we choose the second state in the phase variable form:

$$x_2(t) = \dot{x}_1(t)$$

  Replacing in the differential equation we obtain:

$$\dot{x}_2(t) + 2x_2(t) + 2x_1(t) = u(t)$$

  Re-arranging and taking also the definition of the second state variable we obtain the state equations:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -2x_1(t) - 2x_2(t) + u(t) \end{aligned}$$

  or

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

  or

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.78}$$

- *Introduce now the block with the numerator. From the transfer function we obtain:*

$$Y(s) = (s+2)X_1(s)$$

or, in time-domain:

$$y(t) = \dot{x}_1(t) + 2x_1(t) = x_2(t) + 2x_1(t)$$

The output equation can be written also as:

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

or

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \mathbf{x}(t) \tag{2.79}$$

2.7.6 Why Use State Space

State variable models are a completely generic form of representation in which the system (linear or nonlinear, time varying or time invariant) is decomposed into a set of first-order differential equations.

There are many advantages to modeling systems in state space. The most important advantage for linear systems is the matrix formulation. Computers can easily manipulate matrices. Having the A, B, C, D matrices one can calculate stability, controllability, observability and many other useful attributes of a system. The second most important aspect of state space modeling is that it allows us to model the internal dynamics of the system, as well as the overall input/output relationship as in transfer functions. State space modeling makes the vast, existing, linear system knowledge such as estimation and optimal control theory available to the user.

2.8 Block Diagram Models

Since control systems are concerned with the control of specific signals, the interrelationship of the control variables to the controlling variables is required. This relationship is typically represented by the transfer function of the subsystem relating the input and output variables. Therefore one can correctly assume that the transfer function is an important relation for control engineering.

The importance of the cause-effect relationship of the transfer function is evidenced by the facility to represent the relationship of the system variables by schematic means. The **block diagram** representation of the system relationship is prevalent in control system engineering. Block diagrams consist of *unidirectional*, operational blocks that represent the transfer functions of the variables of interest, (Dorf and Bishop, 2008).

The block diagram representation of a given system can often be reduced, using block diagram reduction techniques. The resulting diagram obeys the law of superposition because of the linear behavior of each subsystem.

In control engineering, the block diagram is a primary tool that together with transfer functions can be used to describe cause-and-effect relationships throughout a dynamic system. The manipulation of block diagrams adheres to a mathematical system of rules often known as block diagram algebra (Mei, 2002).

In general, the block diagram of a linear time invariant system consists of four components, namely signal, block (with transfer function), summing point and pickoff point as shown in Figure 2.21, (Mei, 2002).

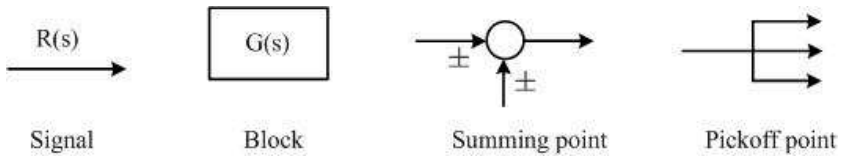

Figure 2.21: The components of a block diagram

2.8.1 Basic connections

In Figure 2.22 are shown the three basic connections: series (cascade), parallel and feedback. The equivalent transfer function from the input $R(s)$ to the output $Y(s)$ for each of these cases can be determined as follows:

1. Series connection.

$$H_1(s) = \frac{Y_1(s)}{R_1(s)};\ H_2(s) = \frac{Y_2(s)}{R_2(s)};\ Y_1(s) = R_2(s)$$

The equivalent transfer function can be calculated from:

$$Y(s) = Y_2(s) = H_2(s)R_2(s) = H_2(s)Y_1(s) = H_2(s)H_1(s)R_1(s) = \underbrace{H_2(s)H_1(s)}_{H(s)}R(s)$$

$$H(s) = \frac{Y(s)}{R(s)} \quad \Rightarrow \quad H(s) = H_1(s)H_2(s)$$

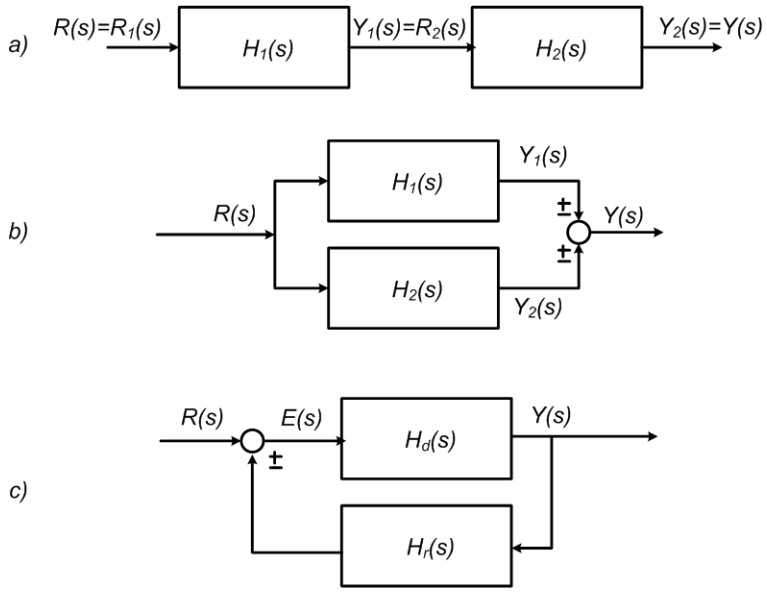

Figure 2.22: Three basic connections. a) series, b) parallel, c) feedback

For n systems connected in cascade the equivalent transfer function will be:

$$H(s) = \prod_{j=1}^{n} H_j(s)$$

2. Parallel connection.

$$H_1(s) = \frac{Y_1(s)}{R(s)};\ H_2(s) = \frac{Y_2(s)}{R(s)};\ Y(s) = \pm Y_1(s) \pm Y_2(s)$$

The equivalent transfer function is:

$$H(s) = \frac{Y(s)}{R(s)} = \pm H_1(s) \pm H_2(s)$$

For n systems connected in parallel the equivalent transfer function is:

$$H(s) = \sum_{j=1}^{n} (\pm H_j(s))$$

3. Feedback connection. The error signal $E(s)$ can be calculated from:

$$E(s) = R(s) \pm H_r(s) \cdot Y(s)$$

The output is:

$$Y(s) = H_d(s) \cdot E(s)$$

If $E(s)$ is eliminated between the previous two relations, the overall transfer function will be:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{H_d(s)}{1 \mp H_d(s) \cdot H_r(s)}$$

Note that in the relation above the sign is $+$ for negative feedback and $-$ for positive feedback.

Example 2.8.1 *Consider a system represented by the block diagram shown in Figure 2.23. Determine the closed-loop transfer function* $H(s) = Y(s)/R(s)$.

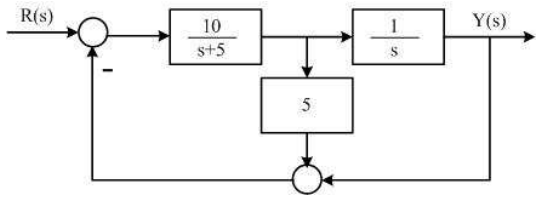

Figure 2.23: Block diagram example exercise.

The block diagram will be transformed so the basic rules of block diagram reduction can be applied. First, the summing points encircled in Figure 2.24 (left) will be considered. Note that the signal Z *can be calculated as:*

$$Z = R - (X + Y) = R - X - Y \tag{2.80}$$

and the two summing points can be reduced to a single one, as shown in Figure 2.24 (right)

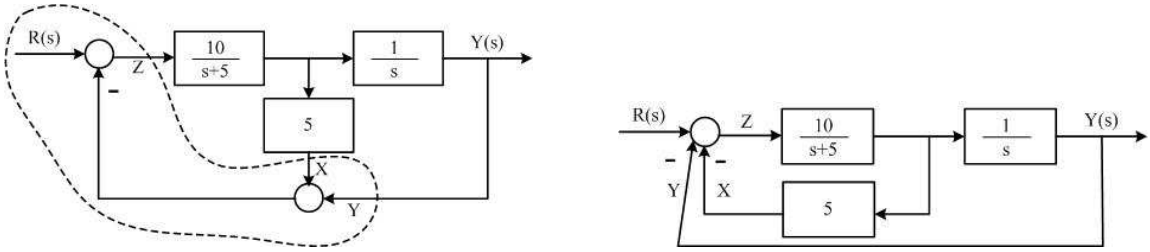

Figure 2.24: Block diagram transformation.

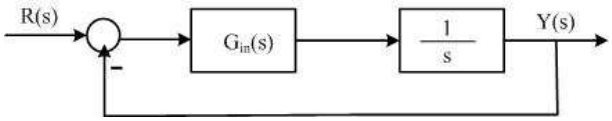

Figure 2.25: Block diagram transformation.

From now on, the basic rules apply. The inner loop is reduced first to one block and the equivalent transfer function is:

$$G_{in}(s) = \frac{\frac{10}{s+5}}{1 + \frac{10}{s+5} \cdot 5} = \frac{10}{s + 55} \tag{2.81}$$

This resulting block, is in a series connection with $1/s$ and the equivalent of these two is the product G_{in}/s. The new block diagram is presented in Figure 2.25.

Note that the feedback path has a transfer function equal to 1, therefore it is a unity feedback loop. The equivalent transfer function from the input $R(s)$ to the output $Y(s)$ is calculated as follows:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{G_{in}(s) \cdot \frac{1}{s}}{1 + G_{in}(s) \cdot \frac{1}{s} \cdot 1} = \frac{\frac{10}{s(s+55)}}{1 + \frac{10}{s(s+55)}} = \frac{10}{s^2 + 55s + 10} \tag{2.82}$$

Diagram transformations

1. Point behind a block - Figure 2.26

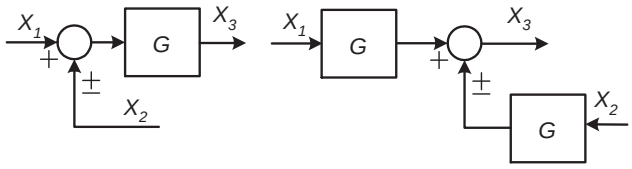

Figure 2.26: Point behind a block

2. Moving a pickoff point ahead of a block - Figure 2.27

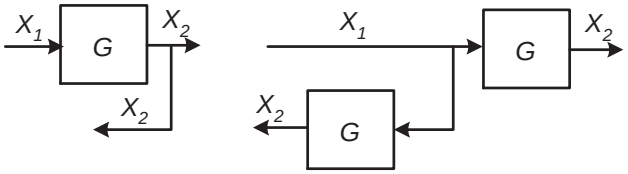

Figure 2.27: Moving a pickoff point ahead of a block

3. Moving a pickoff point behind a block - Figure 2.28

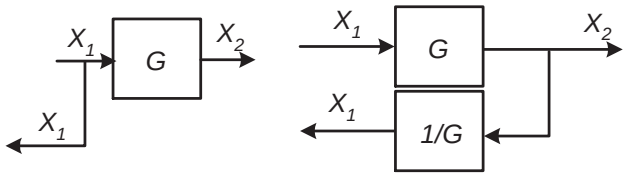

Figure 2.28: Moving a pickoff point behind a block

4. Moving a summing point ahead of a block - Figure 2.29

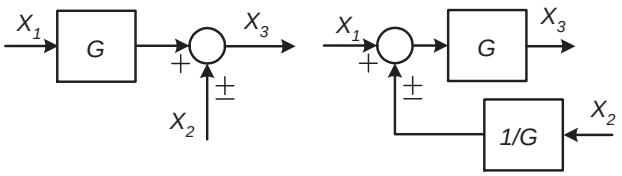

Figure 2.29: Moving a summing point ahead of a block

2.8.2 The overlap of signals

If a system has more than one input, than the output can be considered a result of all effects of these signals.

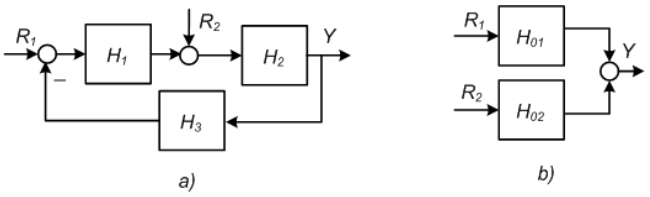

Figure 2.30: Two-input system

Consider the system shown in Figure 2.30 a). Since all linear systems satisfy the principle of superposition, the total output of the system is the algebraic sum (superposition) of the outputs due to each of the inputs. In a general case, all inputs except one are set equal to zero, and then the output due to the non-zero input is determined.

For the system considered above, a transfer function relating each input and the output can be calculated, when the other input is considered zero (Figure 2.30 b)).

$$Y(s) = R_1(s) \cdot H_{01}(s)|_{R_2(s)=0} + R_2(s) \cdot H_{02}(s)|_{R_1(s)=0}$$

The block diagram from Figure 2.30 will be re-arranged for each non-zero input.

When $R_2 = 0$ (see Figure 2.31) the equivalent transfer function between $R_1(s)$ and its effect, or output, $Y_1(s)$ is:

$$H_{01}(s) = \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)H_3(s)} \text{ and the output: } Y_1(s) = H_{01}(s)R_1(s)$$

When $R_1 = 0$ (see Figure 2.32) the equivalent transfer function between $R_2(s)$ and the output $Y_2(s)$ is:

$$H_{02}(s) = \frac{H_2(s)}{1 + H_1(s)H_2(s)H_3(s)} \text{ and the output: } Y_2(s) = H_{02}(s)R_2(s)$$

Thus, the output of the system shown in Figure 2.30 is given by:

$$Y(s) = \frac{H_1H_2}{1 + H_1H_2H_3} \cdot R_1(s) + \frac{H_2}{1 + H_1H_2H_3} \cdot R_2(s)$$

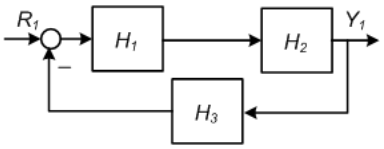

Figure 2.31: Block diagram for $R_2 = 0$

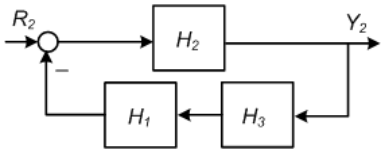

Figure 2.32: Block diagram for $R_1 = 0$

2.8.3 The algebra of MIMO systems

In case of connecting linear MIMO systems, we can use transfer matrices instead of transfer functions, obeying the rules of matrix calculus.

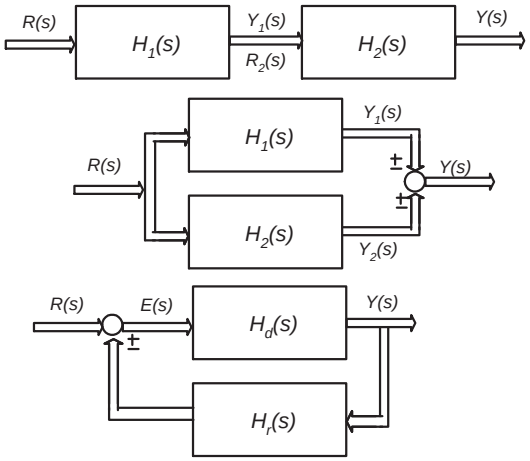

Figure 2.33: MIMO system

For example consider the three basic connections shown in Figure 2.33.

1. Series connection.

$$\mathbf{Y}_1 = \mathbf{H}_1 \cdot \mathbf{R};\ \ \mathbf{Y} = \mathbf{H}_2 \cdot \mathbf{R}_2;\ \ \mathbf{R}_2 = \mathbf{Y}_1;\ \ \Rightarrow\ \ \mathbf{Y} = \mathbf{H}_2 \cdot \mathbf{H}_1 \cdot \mathbf{R}$$

   The equivalent transfer matrix will be:

$$\mathbf{H} = \mathbf{H}_2 \cdot \mathbf{H}_1$$

   For n linear systems connected in series, the equivalent transfer matrix is:

$$\mathbf{H} = \prod_{j=n}^{1} \mathbf{H}_j$$

2. Parallel connection.

$$\mathbf{Y}_1 = \mathbf{H}_1 \cdot \mathbf{R};\quad \mathbf{Y}_2 = \mathbf{H}_2 \cdot \mathbf{R};\quad \mathbf{Y} = \pm\mathbf{Y}_1 \pm \mathbf{Y}_2$$

   The equivalent transfer matrix will be:

$$\mathbf{H} = \pm\mathbf{H}_1 \pm \mathbf{H}_2$$

3. Feedback connection.

$$\mathbf{E} = \mathbf{R} \pm \mathbf{H}_r \cdot \mathbf{Y}; \quad \mathbf{Y} = \mathbf{H}_d \cdot \mathbf{E}; \quad \mathbf{Y} = \mathbf{H}_d \cdot (\mathbf{R} \pm \mathbf{H}_r \cdot \mathbf{Y})$$

$$(\mathbf{1} \mp \mathbf{H}_d \cdot \mathbf{H}_r) \cdot \mathbf{Y} = \mathbf{H}_d \cdot \mathbf{R}; \quad \mathbf{Y} = (\mathbf{1} \mp \mathbf{H}_d \cdot \mathbf{H}_r)^{-1} \cdot \mathbf{H}_d \cdot \mathbf{R}$$

The equivalent transfer matrix will be:

$$\mathbf{H} = (\mathbf{1} \mp \mathbf{H}_d \cdot \mathbf{H}_r)^{-1} \cdot \mathbf{H}_d$$

2.8.3.1 Block diagram representation of state-space models

The matrix-form state equations express the derivatives of the state-variables in terms of the states themselves and the inputs. In this form, the state vector is expressed as the direct result of a vector integration.

The block diagram representation is shown in Figure 2.34 where a vector signal is represented by a thick arrow ($\Rightarrow$). This general block diagram shows the matrix operations from input to output in terms of the **A**, **B**, **C**, **D** matrices, but does not show the path of individual variables. In state-determined systems, the state variables may always be taken as the outputs of integrator blocks. The derivatives of the state variables are the inputs to the integrator blocks, and each state equation expresses a derivative as a sum of weighted state variables and inputs.

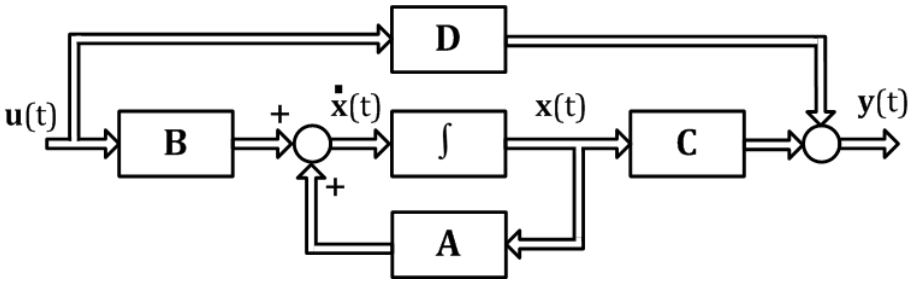

Figure 2.34: Block diagram for state-space models

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