Seminar 5

1 Power series

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots \qquad c_i \in \mathbb{C},\ z \in \mathbb{C}$$

$$\sum_{n=0}^{\infty} c_n z^n$$

R – the radius of convergence

$$\boxed{R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|}$$

or

$$\boxed{R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}}$$

The disk $\{z \in \mathbb{C} \mid |z-z_0| < R\}$ – the disk of convergence

1) Find the radius and the disk of convergence for the series:

a) $\sum_{n=0}^{\infty} (\cos in) z^n$; b) $\sum_{n=0}^{\infty} \left(\frac{z}{n+i}\right)^n$; c) $\sum_{n=1}^{\infty} \left(\cos\frac{i}{n}\right)^n (z+i)^n$

d) $\sum_{n=0}^{\infty} e^{in\pi} (z-1)^n$

Solutions:

a) $$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\cos in}{\cos i(n+1)}\right| = \lim_{n\to\infty} \frac{e^{i^2 n} + e^{-i^2 n}}{\cancel{2}} \cdot \frac{\cancel{2}}{e^{i^2(n+1)} + e^{-i^2(n+1)}}$$

$$= \lim_{n\to\infty} \frac{e^{-n} + e^{n}}{e^{-n-1} + e^{n+1}} = \lim_{n\to\infty} \frac{\cancel{e^n}\left(e^{-2n}+1\right)}{\cancel{e^{n+1}}\,e\left(e^{-2n-2}+1\right)} = \frac{1}{e}$$

$\Rightarrow \{z \in \mathbb{C} \mid |z| < \frac{1}{e}\}$ the disk of convergence

b) $$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\left(\frac{1}{n+i}\right)^n\right|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{1}{n+i}\right|^n}} =$$

$$= \lim_{n\to\infty} |n+i| = \lim_{n\to\infty} \sqrt{n^2+1} = +\infty$$

$\{z \in \mathbb{C} \mid |z| < +\infty\} = \mathbb{C}$ the disk of conv.

c) $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\cos\frac{i}{n}\right|^n}} = \lim_{n\to\infty} \frac{1}{\left|\cos\frac{i}{n}\right|} = \lim_{n\to\infty} \frac{2}{e^{\frac{i\cdot i}{n}} + e^{\frac{-i\cdot i}{n}}} =$

$= \lim_{n\to\infty} \frac{2}{e^{-\frac{1}{n}} + e^{\frac{1}{n}}} = 1$

$\{z \in \mathbb{C} \mid |z+i| < 1\}$ the disk of conv.

d) $R = \lim_{n\to\infty} \left| \frac{e^{in\pi}}{e^{i(n+1)\pi}} \right| = \lim_{n\to\infty} |e^{-i\pi}| = |\cos\pi - i\sin\pi|$

$= \sqrt{\cos^2\pi + i\sin^2\pi} = 1$

$\{z \in \mathbb{C} \mid |z-1| < 1\}$ the disk of conv.

② Taylor series

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \dots$$

Maclaurin series $(z_0 = 0)$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots + (-1)^n \frac{z^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad z \in \mathbb{C}$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^n + \dots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1 \quad z \in \mathbb{C}$$

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = 1 - z + z^2 - \dots + (-1)^n z^n + \dots = \sum_{n=0}^{\infty} (-1)^n z^n \quad |z| < 1$$

→ useful series (from above)

$$f(z)=\sum_{n=0}^{\infty} c_n(z-z_0)^n$$

Then → the power series for f' = the term by term derivative of the series for f

→ $\int_C f(z)\,dz$ = the sum of integrals on C on each of the terms in the series for f

2) Expand the function f in Taylor series in the neighborhood of z_0 and find the radius of convergence.

a) $f(z)=\dfrac{1}{3+6z}$, $z_0=0$

$$f(z)=\frac{1}{3(1+2z)}=\frac{1}{3}\cdot\frac{1}{1+2z}=\frac{1}{3}\cdot\frac{1}{1-(-2z)}=\frac{1}{3}\sum_{n=0}^{\infty}(-2z)^n$$

$$=\sum_{n=0}^{\infty}\frac{(-2)^n}{3}\cdot z^n$$

$|-2z|<1 \Rightarrow |z|<\frac{1}{2}$

b) $f(z)=\dfrac{1}{(3+6z)^2}$, $z_0=0$

$$\frac{1}{3+6z}=\sum_{n=0}^{\infty}\frac{(-2)^n}{3}\cdot z^n$$

Differentiating both sides yields:

$$\frac{-6}{(3+6z)^2}=\sum_{n=1}^{\infty}\frac{(-2)^n}{3}\,n z^{n-1}=\sum_{n=0}^{\infty}\frac{(-2)^{n+1}}{3}(n+1)z^n \quad \Big/ :(-6)$$

↑ we shift for z^n

l = new index

$l=n-1$

$n=l+1$ $\quad n=1\to l=0$ $\Rightarrow \displaystyle\sum_{l=0}^{\infty}\frac{(-2)^{l+1}}{3}(l+1)z^l$

$$\frac{1}{(3+6z)^2} = \sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{-18}(n+1)z^n = \sum_{n=0}^{\infty} \frac{(-2)^n}{9}(n+1)z^n$$

Radius of conv : $R = \frac{1}{2}$

c) $f(z) = \frac{z-1}{3-z}$ $\quad z_0 = 1$

$$f(z) = -\frac{-z+3-3+1}{3-z} = -1 + \frac{2}{3-z}$$

$$-1 + \frac{2}{3-z} = -1 + \frac{2}{2-(z-1)} = -1 + \frac{2}{2\left(1-\frac{z-1}{2}\right)} =$$

$$= -1 + \frac{1}{1-\frac{z-1}{2}} = -1 + \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = 0 + \sum_{n=1}^{\infty}\frac{1}{2^n}(z-1)^n$$

$$\left|\frac{z-1}{2}\right| < 1 \Rightarrow |z-1| < 2$$

d) $f(z) = e^z \cos 2z$, $z_0 = 0$

$$f(z) = e^z \cdot \frac{e^{i\cdot 2z} + e^{-i\cdot 2z}}{2} = \frac{e^{z(1+2i)} + e^{z(1-2i)}}{2} =$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(1+2i)^n}{n!}z^n + \sum_{n=0}^{\infty}\frac{(1-2i)^n}{n!}z^n\right] =$$

$$= \sum_{n=0}^{\infty}\frac{(1+2i)^n + (1-2i)^n}{2n!}\cdot z^n \quad , z \in \mathbb{C}$$

$$1+2i = \sqrt{5}(\cos t + i\sin t) \quad , t = \arctan 2$$

$$1-2i = \sqrt{5}(\cos t - i\sin t)$$

$$f(z) = \sum_{n=0}^{\infty}(\sqrt{5})^n \frac{\cos nt + \cancel{i\sin nt} + \cos nt - \cancel{i\sin nt}}{2n!} z^n$$

$$= \sum_{n=0}^{\infty} (\sqrt{5})^n \frac{\cos nt}{n!} \cdot z^n, \quad z \in \mathbb{C}$$

e) $f(z) = \dfrac{1}{z^2-4z+5}, \quad z_0 = 0$

$$\Delta = 16 - 20 = -4 \Rightarrow z_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$f(z) = \frac{1}{(z-2-i)(z-2+i)} \underset{\uparrow}{=} \frac{z-2+i-(z-2-i)}{(z-2-i)(z-2+i)} \cdot \frac{1}{2i}$$

partial fractions

$$= \frac{1}{2i}\left[\frac{1}{z-2-i} - \frac{1}{z-2+i}\right] = \frac{1}{2i}\left[\frac{1}{-2-i} \cdot \frac{1}{1+\frac{z}{-2-i}} - \frac{1}{-2+i} \cdot \frac{1}{1+\frac{z}{-2+i}}\right]$$

$$= \frac{1}{2i}\left[-\frac{1}{2+i} \cdot \frac{1}{1-\frac{z}{2+i}} + \frac{1}{2-i} \cdot \frac{1}{1-\frac{z}{2-i}}\right] =$$

$$= \frac{1}{2i}\left[-\frac{1}{2+i} \sum_{n=0}^{\infty} \left(\frac{z}{2+i}\right)^n + \frac{1}{2-i} \sum_{n=0}^{\infty} \left(\frac{z}{2-i}\right)^n\right] =$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \left[\left(\frac{1}{2-i}\right)^{n+1} - \left(\frac{1}{2+i}\right)^{n+1}\right] \cdot z^n$$

$$\begin{cases} \left|\frac{z}{2+i}\right| < 1 \\ \left|\frac{z}{2-i}\right| < 1 \end{cases} \Rightarrow \begin{array}{l} |z| < |2+i| \\ |z| < |2-i| \end{array} \Rightarrow |z| < \sqrt{5}$$

f) $f(z) = \dfrac{1}{z^2-4z+5}, \quad z_0 = 1$

$$f(z) = \frac{1}{2i}\left[\frac{1}{z-2-i} - \frac{1}{z-2+i}\right]$$

– we have to create the quantity $z-1$!

$$f(z)=\frac{1}{2i}\left[\frac{1}{z-1+1-2-i}-\frac{1}{z-1+1-2+i}\right]=$$

$$=\frac{1}{2i}\left[\frac{1}{-1-i+\underline{\underline{(z-1)}}}-\frac{1}{-1+i+\underline{\underline{(z-1)}}}\right]=$$

$$=\frac{1}{2i}\left[\frac{1}{-1-i}\cdot\frac{1}{1+\frac{z-1}{-1-i}}-\frac{1}{-1+i}\cdot\frac{1}{1+\frac{z-1}{-1+i}}\right]=$$

$= \text{----}$

g) $f(z)=\dfrac{1}{z^2-4z+5}$, $z_0=2$

$$=\frac{1}{(z-2)^2+1}=\frac{1}{1+\underline{\underline{(z-2)^2}}}=\sum_{n=0}^{\infty}(-1)^n(z-2)^{2n}$$

$|z-2|<1$

h) $f(z)=\dfrac{1}{(1-z)^2}$ $z_0=0$

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n \qquad |z|<1$$

$$\left(\begin{array}{l}l=m-1\\ m=l+1\\ \displaystyle\sum_{l=0}^{\infty}(l+1)\cancel{z}^{l}\end{array}\right)$$

$$f'(z)=\sum_{n=1}^{\infty}n z^{n-1}=\sum_{m=0}^{\infty}(m+1)z^m$$

$$\frac{1}{(1-z)^2}=\sum_{n=0}^{\infty}(n+1)z^n$$

i) $f(z)=\dfrac{1}{1+z^2}$ $z_0=1$

$$f(z)=\frac{1}{(z-i)(z+i)}=\frac{z+i-(z-i)}{(z-i)(z+i)}\cdot\frac{1}{2i}=$$

$$=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$$

$z-1$ $z-1$

$$= \frac{1}{2i}\left[\frac{1}{z-1+1-i} - \frac{1}{z-1+1+i}\right] =$$

$$= \frac{1}{2i}\left[\frac{1}{1-i}\cdot\frac{1}{1+\frac{z-1}{1-i}} - \frac{1}{1+i}\cdot\frac{1}{1+\frac{z-1}{1+i}}\right] =$$

$$= \frac{1}{2i}\left[\frac{1}{1-i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1-i}\right)^n - \frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1+i}\right)^n\right] =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2i}\left[\frac{1}{(1-i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right](z-1)^n .$$

$$\begin{cases} \left|\frac{z-1}{1-i}\right| < 1 \\ \left|\frac{z-1}{1+i}\right| < 1 \end{cases} \Rightarrow \begin{cases} |z-1| < |1-i| \\ |z-1| < |1+i| \end{cases} \Rightarrow |z-1| < \sqrt{2}$$

$$1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{-i\frac{\pi}{4}}$$

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{i\frac{\pi}{4}}$$

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2i}\cdot\frac{e^{i(n+1)\frac{\pi}{4}} - e^{-i(n+1)\frac{\pi}{4}}}{(\sqrt{2})^{n+1}}\cdot(z-1)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\sin(n+1)\frac{\pi}{4}}{2^{(n+1)/2}}(z-1)^n .$$

1.38; 1.41; 1.44