Seminar 7 gr. 1

Wednesday, November 11, 2020 4:00 PM

The residue of a complex function

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\dots + \frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{z-z_0}}_{\text{principal part}} + \underbrace{a_0 + a_1(z-z_0)+a_2(z-z_0)^2+}_{\text{analytical part}}$$

a_{-1} = RESIDUE

Met I

- to compute the Laurent series $\Rightarrow$ $\boxed{a_{-1} = \operatorname{Res}_{z=z_0} f(z)}$

ex Find the residue of $\frac{\sin z}{z^2}$ at $z_0=0$

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}- \dots\right) = \frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!} - \dots$$

$$\Rightarrow \operatorname{Res}_{z=0} f(z) = 1$$

$\lim_{z\to 0}(z-0)\cdot\frac{\sin z}{z^2} = 1 \neq 0$
$\Rightarrow z=0$ simple pole

Met II

$f(z)$ has a simple pole (pole of order 1)

$f(z)=\frac{g(z)}{h(z)}$ $\Rightarrow$ $\boxed{\operatorname{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)}} = \frac{g(z)}{h'(z)}\Big|_{z=z_0}$

g, h holom. function at z_0

$h(z_0)=0$, $h'(z_0)\neq 0$

ex $f(z)=\frac{\cos z}{z^4-1}$ at $z_0=i$

Factor the denominator $z^4-1=(z^2-1)(z-i)(z+i)$

$$\Rightarrow f(z) = \frac{\cos z}{(z^2-1)\underline{(z-i)}(z+i)}$$

$(z-i)$ appears once at the denominator; $\cos z$ holom. function at $z=i$

$\Rightarrow z=i$ pole of order 1

$$\operatorname{Res}_{z=i} f(z) = \frac{\cos z}{4z^3}\Big|_{z=i} = \frac{\cos i}{4i^3} = -\frac{1}{4i}\cdot\left(\frac{e^i+e^{-i}}{2}\right) = -\frac{1}{8i}(e^i+e^{-i})$$

How do you know z_0 is a simple pole?

- if $\lim_{z\to z_0}(z-z_0)f(z)=0$ $\Rightarrow$ $f(z)$ is analytic at z_0
- if $\lim_{z\to z_0}(z-z_0)f(z)=+\infty$ $\Rightarrow$ z_0 is a higher order pole of $f(z)$
- if $\lim_{z\to z_0}(z-z_0)f(z)$ is not zero and finite $\Rightarrow$ z_0 simple pole

Met III

• if $\lim_{z \to z_0} (z-z_0) f(z)$ ...

Met III

$$\boxed{z_0 \text{ pole of order } n \geq 1 \Rightarrow \operatorname{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[(z-z_0)^n f(z)\right]^{(n-1)}}$$

ex $\quad f(z) = \dfrac{z \cos z}{(z-\pi)^3} \qquad z_0 = \pi \qquad \left(\lim_{z \to \pi} \cancel{(z-\pi)} \dfrac{z \cos z}{(z-\pi)^{\cancel{3}2}} = \infty \right)$

• $z = \pi$ pole of order 3

$$\operatorname{Res}_{z=\pi} f(z) = \frac{1}{(3-1)!} \lim_{z \to \pi} \left[\cancel{(z-\pi)^3} \cdot \frac{z \cos z}{\cancel{(z-\pi)^3}} \right]^{(3-1)} = \frac{1}{2!} \lim_{z \to \pi} (z \cos z)^{(2)} = \frac{1}{2!} \lim_{z \to \pi} (-2 \sin z - z \cos z) =$$

$$(z \cos z)^{(2)} = (\cos z - z \sin z)' = -\sin z - \sin z - z \cos z$$

$$= \frac{1}{2}\Big(-2 \underbrace{\sin \pi}_{=0} - \pi \underbrace{\cos \pi}_{-1}\Big) = \frac{\pi}{2}$$

1.49 Find the residues of the singular points of the following functions:

a) $f(z) = \dfrac{z^2}{(1+z)^3} \qquad\qquad (z^2)^{(2)} = (2z)' = 2$

$z = -1$ pole of order 3

$$\operatorname{Res}_{z=-1} f(z) = \frac{1}{(3-1)!} \lim_{z \to -1} \left[\cancel{(z+1)^3} \cdot \frac{z^2}{\cancel{(1+z)^3}} \right]^{(2)} = \frac{1}{2!} \lim_{z \to -1} 2 = 1$$

b) $f(z) = \dfrac{1}{1+e^z}$

$1 + e^z = 0 \Rightarrow e^z = -1 \;\Big|\, \operatorname{Log} \Rightarrow z = \operatorname{Log}(-1) \Rightarrow z = \ln|-1| + i(\pi + 2k\pi)$

$\Rightarrow z = \pi i (1+2k),\ k \in \mathbb{Z}$ pole of order 1

$$\operatorname{Res}_{z=\pi i(1+2k)} f(z) \underset{(Met\,II)}{=} \frac{g(z)}{h'(z)}\bigg|_{z=\pi i(1+2k)} \qquad \frac{1}{e^{\pi i(1+2k)}} = \frac{1}{\cos(\pi+2k\pi) + i \sin(\pi + 2k\pi)} = -1$$

$g(z) = 1$

$h(z) = 1 + e^z \Rightarrow h'(z) = e^z$

$$\underset{(Met\,III)}{} \lim_{z \to \pi i(2k+1)} \frac{1}{(1-1)!} \left[(z - \pi i(2k+1)) \cdot \frac{1}{1+e^z} \right]^{(0)} = \lim_{z \to \pi i(2k+1)} \frac{z - \pi i(2k+1)}{e^z + 1} \overset{l'H}{=}$$

$$= \lim_{z \to \pi i(2k+1)} \frac{1}{e^z} = \frac{1}{e^{\pi i(2k+1)}} = -1$$

... $\Rightarrow \pi z = k\pi,\ k \in \mathbb{Z} \Rightarrow z = k,\ k \in \mathbb{Z},\ k = 0, \pm 1, \pm 2, \ldots$

$z \to \pi i(2k+1)$

c) $f(z)=\dfrac{1}{\sin \pi z}$ $\qquad \sin \pi z = 0 \Rightarrow \pi z = k\pi,\ k\in\mathbb{Z} \Rightarrow z=k,\ k\in\mathbb{Z},\ k=0,\pm1,\pm2,.$

$z=k$ poles of order 1, $k\in\mathbb{Z}$

$$\operatorname*{Res}_{z=k} f(z) = \frac{g(z)}{h'(z)}\Big|_{z=k} = \frac{1}{\pi\cos \pi z}\Big|_{z=k} = \frac{1}{\pi \cos \pi k} = \frac{1}{\pi(-1)^k} = \frac{(-1)^k}{\pi},\ k\in\mathbb{Z}$$

d) $f(z) = \dfrac{\cos z}{(z-1)^2}$

$z=1$ - pole of order 2

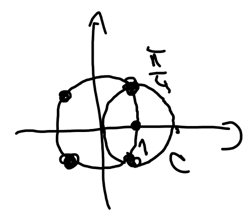
$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!}\lim_{z\to 1}\left[(z-1)^2\cdot\frac{\cos z}{(z-1)^2}\right]' = \lim_{z\to1}(-\sin z) = \underline{\underline{-\sin 1}}$$

<u>Residue Theorem for evaluating integrals</u>

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)}$$

(1) Evaluate the following integral.

$$\int_{|z-1|=1} \frac{dz}{z^4+1}$$



met I

$z^4+1=0 \Rightarrow z^4=-1$, $\quad -1 = \cos\pi + i\sin\pi$

$z_k = \cos\dfrac{\pi+2k\pi}{4} + i\sin\dfrac{\pi+2k\pi}{4}$, $k=\overline{0,3}$

$z_0 = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \boxed{\dfrac{\sqrt2}{2} + i\dfrac{\sqrt2}{2}}$

$z_1 = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4} = -\dfrac{\sqrt2}{2} + i\dfrac{\sqrt2}{2}$

$z_2 = \cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4} = -\dfrac{\sqrt2}{2} - i\dfrac{\sqrt2}{2}$

$z_3 = \cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4} = \boxed{\dfrac{\sqrt2}{2} - i\dfrac{\sqrt2}{2}}$

$\Rightarrow z_0, z_3 \in \operatorname{int} C$

z_0, z_3 are poles of order 1

$f(z) = \dfrac{1}{z^4+1}$

$$I = 2\pi i\left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_3} f(z)\right)$$

- $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{g(z_0)}{h'(z_0)} = \dfrac{1}{4z_0^3} = \dfrac{1}{4\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)^3} = \dfrac{1}{4\left(\frac{\sqrt2}{2}\right)^3\left(1+3i+\underbrace{3i^2}_{-3}+\underbrace{i^3}_{=-i}\right)} = \dfrac{1}{4\frac{2\sqrt2}{8}(2i-2)} = \dfrac{1}{2\sqrt2(i-1)}$

- $\operatorname*{Res}_{z=z_3} f(z) = \dfrac{g(z_0)}{h'(z_0)} = \dfrac{1}{4z_3^3} = \dfrac{1}{4\left(\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\right)^3} = \dfrac{1}{4\left(\frac{\sqrt2}{2}\right)^3\left(1-3i+3i^2-i^3\right)} = \dfrac{1}{2\sqrt2(-i-1)}$

$$\Rightarrow I = 2\pi i\left(\frac{1}{2\sqrt2(i-1)} + \frac{1}{2\sqrt2(-i-1)}\right) = \frac{\cancel{2}\pi i}{\cancel{2}\sqrt2}\cdot\frac{\cancel{-i}-1+\cancel{i}-1}{2} = -\frac{\pi i}{\sqrt2}$$

$$\Rightarrow I = 2\pi i\left(\frac{1}{2\sqrt{2}(i-1)} + \frac{1}{2\sqrt{2}(-i-1)}\right) = \frac{\cancel{2}\pi i}{\cancel{2}\sqrt{2}}\,\frac{\cancel{i}-1\cancel{+i}-1}{2} = -\frac{\pi i}{\sqrt{2}}$$

method II

$z_0, z_3 \in \operatorname{int} C$ poles of order 1

$z_k^4 + 1 = 0$

$\boxed{z_k^4 = -1}$

$f(z) = \dfrac{1}{z^4+1}$

$$\operatorname*{Res}_{z=z_k} f(z) = \frac{\cancel{z_k}\,1}{4z_k^3} = \frac{z_k}{4\underbrace{z_k^4}_{=-1}} = -\frac{1}{4}z_k$$

$k=0, k=3$

$$I = 2\pi i \sum \operatorname*{Res}_{z=z_k} f(z) = 2\pi i \sum_{\substack{k=0\\k=3}} -\frac{1}{4}z_k = \frac{2\pi i}{-4}(z_0+z_3) =$$

$$= -\frac{\pi i}{2}\left(\frac{\sqrt{2}}{2} + i\cancel{\frac{\sqrt{2}}{2}} + \frac{\sqrt{2}}{2} - i\cancel{\frac{\sqrt{2}}{2}}\right) = -\frac{\pi i}{2}\cdot\frac{\cancel{2}\sqrt{2}}{\cancel{2}} = -\frac{\pi i}{\sqrt{2}}$$

② Evaluate the integral: $I = \displaystyle\int_{|z|=3} \frac{e^{\frac{\pi}{z-2i}}}{(z-1)(z-2i)}\,dz$

$z_1 = 1,\ z_2 = 2i \in \operatorname{int} C$

$z_1 = 1$ pole of order 1

$$\operatorname*{Res}_{z=1} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=1} = \left.\frac{e^{\frac{\pi}{z-2i}}}{1\cdot(z-2i)+z-1}\right|_{z=1} = \frac{e^{\frac{\pi}{1-2i}}}{1-2i+\cancel{1}-\cancel{1}} = \frac{1}{1-2i}e^{\frac{\pi}{1-2i}}$$

$g(z) = e^{\frac{\pi}{z-2i}}$

$h(z) = (z-1)(z-2i)$

$\boxed{z_2 = 2i}$ is not a pole, it is an essential singularity

- We expand by Laurent series around $z = 2i$ (with the powers of $(z-2i)$)

$$f(z) = \frac{e^{\frac{\pi}{z-2i}}}{(z-1)(z-2i)} = \frac{1}{z-2i}\cdot e^{\frac{\pi}{z-2i}}\cdot\frac{1}{(z-2i)+2i-1} = \frac{1}{z-2i}e^{\frac{\pi}{z-2i}}\cdot\frac{1}{2i-1}\cdot\frac{1}{1+\frac{z-2i}{2i-1}} =$$

$$= \frac{1}{2i-1}\cdot\frac{1}{1-\frac{z-2i}{1-2i}}\cdot\frac{1}{z-2i}\cdot e^{\frac{\pi}{z-2i}} =$$

$$= \frac{1}{2i-1}\cdot\left(1+\frac{z-2i}{1-2i}+\left(\frac{z-2i}{1-2i}\right)^2+\left(\frac{z-2i}{1-2i}\right)^3+\dots\right)\cdot\frac{1}{z-2i}\cdot e^{\frac{\pi}{z-2i}} =$$

$$= \frac{1}{2i-1}\left(\frac{1}{z-2i}+\frac{1}{1-2i}+\frac{z-2i}{(1-2i)^2}+\frac{(z-2i)^2}{(1-2i)^3}+\dots\right)\left(1+\frac{\pi}{1!(z-2i)}+\frac{\pi^2}{2!(z-2i)^2}+\frac{\pi^3}{3!(z-2i)^3}+\dots\right)$$

$$\operatorname*{Res}_{z=2i} f(z) = a_{-1} = \frac{1}{2i-1}\left(1 + \frac{\pi}{1!(1-2i)} + \frac{\pi^2}{2!(1-2i)^2} + \frac{\pi^3}{3!(1-2i)^3} + \dots\right) =$$

(the coeff of $\frac{1}{z-2i}$)

$$= \frac{1}{2i-1}\left(1 + \frac{\frac{\pi}{1-2i}}{1!} + \frac{\left(\frac{\pi}{1-2i}\right)^2}{2!} + \frac{\left(\frac{\pi}{1-2i}\right)^3}{3!} + \dots\right) =$$

$$= \frac{1}{2i-1} e^{\frac{\pi}{1-2i}}$$

$$I = 2\pi i\left(\frac{1}{1-2i} e^{\frac{\pi}{1-2i}} + \frac{1}{2i-1} e^{\frac{\pi}{1-2i}}\right) = 0$$

(3) $I = \displaystyle\int_{C:|z|=3} \frac{e^{\frac{\pi}{z-1}}}{z^2-3z+2}\,dz = \int \frac{e^{\frac{\pi}{z-1}}\,dz}{(z-1)(z-2)}$ $\quad z^2-3z+2=0 \Rightarrow (z-1)(z-2)=0 \Rightarrow \underline{\underline{z_1=2}}, z_2=1 \in \operatorname{int} C$

$z_1 = 2$ pole of order 1 $\quad \displaystyle\operatorname*{Res}_{z=2} f(z) = \frac{g(z_0)}{h'(z_0)} = \left.\frac{e^{\frac{\pi}{z-1}}}{2z-3}\right|_{z=2} = \frac{e^{\pi}}{1} = e^{\pi}$

$z_2 = 1$ essential singularity

we expand $f(z)$ by Laurent series around $z_2 = 1$

$$f(z) = \frac{e^{\frac{\pi}{z-1}}}{(z-1)(z-2)} = \frac{1}{z-1} e^{\frac{\pi}{z-1}} \cdot \frac{1}{(z-1)-1} = (-1)\underbrace{\frac{1}{1-(z-1)}}\cdot\frac{1}{z-1}\cdot e^{\frac{\pi}{z-1}} =$$

$$= (-1)\left(1 + (z-1) + (z-1)^2 + (z-1)^3 + \dots\right)\frac{1}{z-1}\cdot e^{\frac{\pi}{z-1}} =$$

$$= \underline{(-1)}\left[\frac{1}{z-1} + 1 + (z-1) + (z-1)^2 + \dots\right]\left[1 + \frac{\pi}{1!(z-1)} + \frac{\pi^2}{2!(z-1)^2} + \frac{\pi^3}{3!(z-1)^3} + \dots\right]$$

$$\operatorname*{Res}_{z=1} f(z) = a_{-1} = (-1)\left(1 + \frac{\pi}{1!} + \frac{\pi^2}{2!} + \dots\right) = -e^{\pi}$$

the coeff of $\frac{1}{z-1}$

$\Rightarrow I = 0$

Homework: $\boxed{1.50}$