Seminar 6

Wednesday, November 4, 2020 3:37 PM

Laurent series. Singularities

Th. If $f(z)$ is analytic in $R_1 < |z-z_0| < R_2$ then

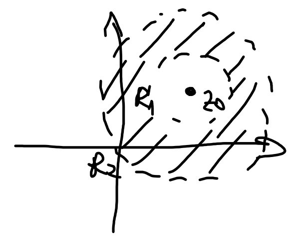
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n} \quad \forall z \text{ on } R_1<|z-z_0|<R_2$$

or

$$f(z) = \underbrace{\cdots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0}}_{\text{principal part}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{\substack{\text{analytic part} \\ \text{(regular part)}}}$$

- principal part: conv. outside a circle centered at z_0
- analytic part: conv. inside a circle centered at z_0

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}\, ds, \quad n = 0, \pm 1, \pm 2, \ldots$$

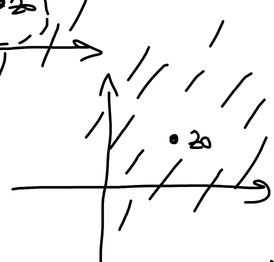


$R_1 < |z-z_0| < R_2$ is an annulus domain

- $R_1 = 0$

- $R_2 = \infty$

- $R_1 = 0$ & $R_2 = \infty$



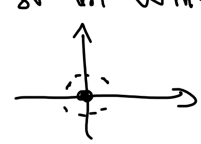
* Laurent series still works if z_0 is an isolated singularity

* The coefficient a_{-1} is called the Residue of the function $f(z)$ at $z = z_0$

Singularities

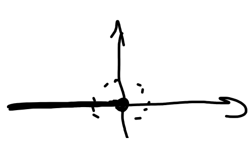
- singularity: a point z_0 at which the function f is not analytic
- isolated singularity: a singularity z_0 of f for which there exists a deleted neighborhood about z_0 in which f is analytic

$f(z) = \frac{1}{z}$



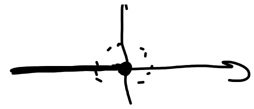
$z_0 = 0$

- nonisolated singularity:



$f(z) = \ln z$, $z_0 = 0$

• nonisolated singularity :



$f(z) = \ln z$, $z_0 = 0$

• removable singularity

$$f(z) = \frac{\cos z - 1}{z^2} , \quad z_0 = 0$$

$$f(z) = \frac{1}{z^2}\left(\cancel{1} - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots - \cancel{1}\right) =$$

$$= -\frac{1}{2!} + \frac{z^2}{4!} - z^4$$

! no negative powers of z
the principal part does not exists $\Rightarrow$ z_0 removable singularity

• pole

$$f(z) = \frac{\cos z}{z^4} = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) =$$

$$= \underbrace{\frac{1}{z^4} - \frac{1}{2!\, z^2}}_{\text{the principal part}} + \underbrace{\frac{1}{4!} - \frac{z^2}{6!} + \dots}_{\text{the analytic part}}$$

$\exists$ a finite number of negative powers of z
$\Rightarrow$ z_0 is a pole of order 4

$$f(z) = \underbrace{a_{-N}(z-z_0)^{-N} + \dots}_{a_{-N} \neq 0} + a_0 + a_1(z-z_0) + \dots$$

$a_{-N} \neq 0$ $\Rightarrow$ z_0 pole of order N

• essential singularity

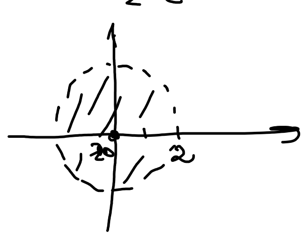
$$f(z) = \cos\left(\frac{1}{z}\right) = 1 - \underbrace{\frac{1}{2!\, z^2} + \frac{1}{4!\, z^4} - \frac{1}{6!\, z^6} + }_{\text{principal part}}$$

- infinite numbers of negative powers of z
z_0 - essential singularity

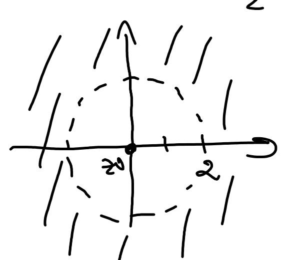
(1) Determine all possible Laurent series for the functions

a) $f(z) = \frac{1}{z-2}$, $z_0 = 0$; b) $f(z) = \frac{1}{z-2}$, $z_0 = 1$

a)



• $|z| < 2$



• $|z| > 2$

• $|z| < 2$ $\quad f(z) = \frac{1}{z-2} = \frac{-1}{2-z} = -\frac{1}{2} \cdot \frac{1}{1 - \frac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} (-1) \frac{z^n}{2^{n+1}}$

valid for $|z| < 2$

$\Downarrow$

$\frac{|z|}{2} < 1$

$$z-2 \qquad \overline{2-z} = \frac{\ }{2}\cdot\frac{\ }{1-\frac{z}{2}} = \frac{\ }{2}\sum_{n=0}(\ )= \sum_{n=0}\frac{\ }{2^{\ }}$$

valid for $|z|<2$

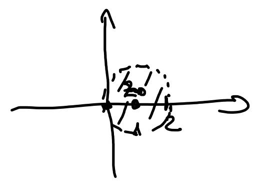
$\frac{|z|}{2}<1$

- $|z|>2$ $\qquad f(z)=\dfrac{1}{z-2}=\dfrac{1}{z}\cdot\dfrac{1}{1-\frac{2}{z}}=\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n=\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$ valid for $|z|>2$

$\Downarrow$

$\frac{2}{|z|}<1$

b) $f(z)=\dfrac{1}{z-2}$, $z_0=1$



- $|z-1|<1$

- $|z-1|>1$

- $|z-1|<1$ $\Rightarrow f(z)=\dfrac{1}{z-2}=\dfrac{1}{(z-1)-1}=\dfrac{-1}{1-(z-1)}=-\displaystyle\sum_{n=0}^{\infty}(z-1)^n$ valid for $|z-1|<1$

- $|z-1|>1$ $\Rightarrow f(z)=\dfrac{1}{z-2}=\dfrac{1}{(z-1)-1}=\dfrac{1}{z-1}\cdot\dfrac{1}{1-\frac{1}{z-1}}=\dfrac{1}{z-1}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n=\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}$

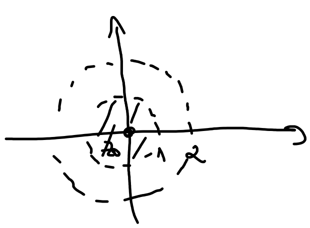
$\Downarrow$

$\frac{1}{|z-1|}<1$

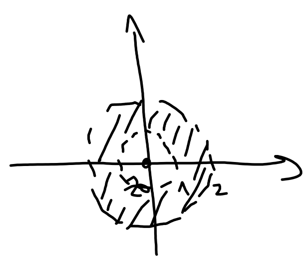
valid for $|z-1|>1$

*

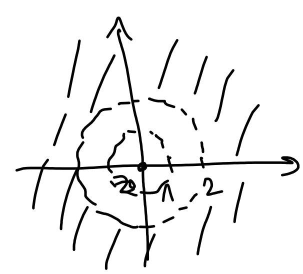
c) $f(z)=\dfrac{1}{(z-1)(z-2)}$, $z_0=0$



- $|z|<1$



- $1<|z|<2$



- $|z|>2$

- we use partial fraction expansion

$$f(z)=\frac{1}{(z-1)(z-2)}=\frac{(z-1)-(z-2)}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$$

- $|z|<1$ $\Rightarrow f(z)=-\dfrac{1}{2}\cdot\dfrac{1}{1-\frac{z}{2}}-\dfrac{-1}{1-z}=-\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n$

valid for $|z|<1$

- $1<|z|<2$ $\Rightarrow f(z)=\dfrac{1}{z-2}-\dfrac{1}{z-1}=\dfrac{-1}{2}\cdot\dfrac{1}{1-\frac{z}{2}}-\dfrac{1}{z}\cdot\dfrac{1}{1-\frac{1}{z}}=$

$\Downarrow \qquad \Downarrow$

$\frac{1}{|z|}<1 \qquad \frac{|z|}{2}<1$

$$=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=$$

$$=\sum_{n=0}^{\infty}(-1)\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

valid for $1<|z|<2$

- $|z|>2$, $(2<|z|<+\infty)$

$$\sum_{n=0} \quad 2^{n+1} \quad \sum_{n=0}$$

- $|z|>2$ $(2<|z|<+\infty)$ $\Downarrow$ $\frac{2}{|z|}<1$; $\frac{1}{|z|}<1$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}-\frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)}=$$

$$=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n,\ \text{valid for } 2<|z|<\infty$$

1.45 Expand the function f in Laurent series and determine the type of singularity

a) $f(z)=z^3e^{\frac{1}{z}}$, $0<|z|<+\infty$, $z_0=0$ $\qquad e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$

$$f(z)=z^3\left(1+\frac{1}{1!\,z}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z^3}+\frac{1}{4!\,z^4}+\dots\right)=$$

$$=\underbrace{z^3+\frac{z^2}{1!}+\frac{z}{2!}+\frac{1}{3!}}_{\text{analytical part}}+\underbrace{\frac{1}{4!\,z}+\frac{1}{5!\,z^2}+\dots}_{\text{principal part}}$$

(there are infinite no. of negative powers of z)

$\Rightarrow z_0$ is essential singularity

b) $f(z)=\frac{2\sin^2 z}{z^5}$, $0<|z|<+\infty$, $z_0=0$ $\qquad \cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+$

$$f(z)=\frac{1-\cos 2z}{z^5}=\frac{1}{z^5}\left[1-\left(1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}+\dots\right)\right]=$$

$$=\frac{1}{z^5}\left(\frac{4z^2}{2!}-\frac{2^4z^4}{4!}+\frac{2^6z^6}{6!}+\dots\right)=\underbrace{2\frac{1}{z^3}-\frac{2^4}{4!}\frac{1}{z}}+\underbrace{\frac{2^6}{6!}z-\dots}$$

$z_0=0$ pole of order 3

c) $f(z)=ze^{\frac{1}{z+i}}$, $z_0=-i$, $0<|z-i|<+\infty$, $\underline{z+i}$

$$f(z)=(z+i-i)e^{\frac{1}{z+i}}=(z+i)e^{\frac{1}{z+i}}-ie^{\frac{1}{z+i}}=$$

$$=(z+i)\left(1+\frac{1}{1!(z+i)}+\frac{1}{2!(z+i)^2}+\frac{1}{3!(z+i)^3}+\dots\right)-i\left(1+\frac{1}{1!(z+i)}+\frac{1}{2!(z+i)^2}+\frac{1}{3!(z+i)^3}+\dots\right)$$

$$=z+i+1-i+\underbrace{\left(\frac{1}{2!}-\frac{i}{1!}\right)\frac{1}{(z+i)}+\left(\frac{1}{3!}-\frac{i}{2!}\right)\frac{1}{(z+i)^2}+\dots}_{\text{inf no. of terms in the principal part}}$$

$\Rightarrow z_0=-i$ essential singularity

(3) $f(z)=\frac{1}{(z-1)(z-4)}$ about $z_0=1$ when $0<|z-1|<3$ $\Downarrow$ $\frac{|z-1|}{3}<1$

$\underline{z-1}$

$$f(z)=\frac{1}{z-1}\cdot\frac{1}{z-4}=\frac{1}{z-1}\cdot\frac{1}{(z-1)-3}=$$

$$\infty \quad n$$

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-4} = \frac{1}{z-1} \cdot \frac{1}{(z-1)-3} =$$

$$= \frac{1}{z-1} \cdot (-1) \cdot \frac{1}{3-(z-1)} = \frac{1}{z-1} \cdot \left(-\frac{1}{3}\right) \cdot \frac{1}{1-\frac{z-1}{3}} = -\frac{1}{3} \cdot \frac{1}{z-1} \cdot \sum_{n=0}^{\infty} \left(\frac{z-1}{3}\right)^n =$$

$$= \sum_{n=0}^{\infty} (-1) \frac{(z-1)^{n-1}}{3^{n+1}} . \quad \text{valid for } 0 < |z-1| < 3$$

$$= \sum_{m=-1}^{\infty} -\frac{1}{3^{m+2}} (z-1)^m . \qquad \begin{array}{l} l = n-1 \\ n = l+1 \end{array}$$

$$f(z) = \frac{1}{(z-1)(z-4)} \quad , z_0 = 1 \quad , \text{when } |z-1| > 3$$

$$\underline{z-1} \qquad \Downarrow$$

$$\frac{3}{|z-1|} < 1$$

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{(z-1)-3} = \frac{1}{z-1} \cdot \frac{1}{z-1} \cdot \frac{1}{1-\frac{3}{z-1}} = \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} \left(\frac{3}{z-1}\right)^n =$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{(z-1)^{n+2}} . \quad , \text{valid for } |z-1| > 3$$

Homework

1) $f(z) = z \cos \frac{3}{z}$, $z_0 = 0$, the type of z_0

2) $f(z) = \sin \frac{1}{z}$, $z_0 = 0$

3) $f(z) = \frac{1+2z^2}{z^3+z^5}$, $z_0 = 0$, $0 < |z| < 1$

4) $f(z) = \frac{z}{z^2+1}$, $z_0 = i$, $0 < |z-i| < 2$