Seminar (4): Complex integrals. Cauchy's integral formula

Note Title 10/21/2020

1. <u>Integrals</u> $f(z)$ be defined and continuous on a curve C

$$\int_C f(z)\,dz = \int_{(x_1,y_1)}^{(x_2,y_2)} (u+iv)(dx+i\,dy)$$

$z = x+iy$

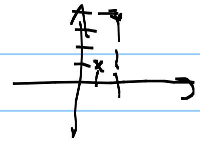
(1) Evaluate $\displaystyle\int_{1+i}^{2+4i} z^2\,dz$

a) along the parabola $x=t,\ y=t^2,\ 1\le t\le 2$

b) along the straight line $1+i$ and $2+4i$

We have: $$\int_{1+i}^{2+4i} z^2\,dz = \int_{(1,1)}^{(2,4)} (x+iy)^2(dx+i\,dy) = \int_{(1,1)}^{(2,4)} (x^2-y^2+2xyi)(dx+i\,dy) =$$

$$= \int_{(1,1)}^{(2,4)} (x^2-y^2)\,dx - 2xy\,dy + i\int_{(1,1)}^{(2,4)} 2xy\,dx + (x^2-y^2)\,dy$$



a) $t=1,\ t=2$

$x=t,\ y=t^2,\ dx=dt$

$$\Rightarrow I = \int_1^2 \left[(t^2-t^4)\,dt - 2t^3\cdot 2t\,dt\right] + i\int_1^2 \left[2\cdot t^3\,dt + (t^2-t^4)\cdot 2t\,dt\right]$$

$dy = 2t\,dt$

$$I = \int_1^2 (t^2 - 5t^4)\,dt + i\int_1^2 (4\cdot t^3 - 2t^5)\,dt = \frac{t^3}{3}\Big|_1^2 - 5\frac{t^5}{5}\Big|_1^2 + i\left(4\frac{t^4}{4}\Big|_1^2 - 2\frac{t^6}{6}\Big|_1^2\right) =$$

$$= \frac{1}{3}(8-1) - (32-1) + i\cdot\left(16 - 1 - \frac{1}{3}(64-1)\right) = \frac{7}{3} - 31 + i\left(15 - \frac{63}{3}\right) = -\frac{86}{3} - 6i$$

b) The eq. of the line $(1,1)$ and $(2,4)$: $\frac{x-1}{2-1} = \frac{y-1}{4-1} \Leftrightarrow x-1 = \frac{y-1}{3} \Rightarrow y = 3x-2$

$dx = dx,\ dy = 3dx$

$$I = \int_1^2 \left\{[x^2 - (3x-2)^2]\,dx - 2x(3x-2)3\,dx\right\} + i\int_1^2 \left\{2x(3x-2)\,dx + [x^2-(3x-2)^2]\cdot 3\,dx\right\} = \dots = -\frac{86}{3} - 6i$$

Remark: The line integrals are independent of the path.

Met II: $I = \int_{1+i}^{2+4i} z^2\,dz = \frac{z^3}{3}\Big|_{1+i}^{2+4i} = \frac{(2+4i)^3}{3} - \frac{(1+i)^3}{3} = \dots = -\frac{86}{3} - 6i$

2) $\int_C (z-a)^n\,dz$, where $C: |z-a| = r$, $n \in \mathbb{Z}$

the circle with center in a, of radius r

Remark 1.6

We make the change of variable $z = a + re^{i\theta}$, $\theta \in [0, 2\pi)$

$dz = ire^{i\theta}\,d\theta$

$$I = \int_0^{2\pi} r^n e^{i\theta n} \cdot ire^{i\theta}\,d\theta$$

$$I = i r^{n+1} \int_0^{2\pi} e^{i\theta(n+1)} \, d\theta$$

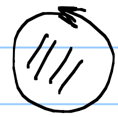
for $n = -1$ $\Rightarrow I = \int_C \frac{dz}{z-a} = i \int_0^{2\pi} d\theta = 2\pi i$

for $n \neq -1$ $\Rightarrow I = \int_C (z-a)^n dz = i r^{n+1} \frac{1}{i(n+1)} e^{i\theta(n+1)} \Big|_0^{2\pi} = 0$

$e^{i2\pi(n+1)} - e^0 = \cos 2\pi(n+1) + i \sin 2\pi(n+1) - 1 = 0$

1.31. Homework

2. Cauchy's integral formula



C - curve: closed, simple, positively oriented w.r.t its interior $\text{int}\, C$

$f: D \to \mathbb{C}$, $\text{int}\, C \subset D$, $z_0 \in \text{int}\, C$

holomorphic function

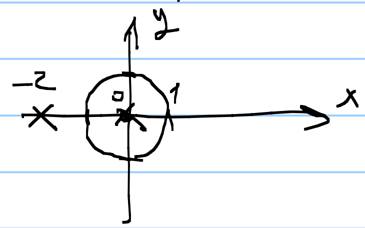
Then $\int_C f(z)\, dz = 0$; $\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$; $\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

Remark: 1) $\int_C \frac{f(z)}{z-z_0}\,dz$ is defined $\forall z_0 \in \mathbb{C}$ which is not on the boundary of the dom.

2) $z_0 \in \operatorname{int} C \Rightarrow I = 2\pi i f(z_0)$

$z_0 \notin \operatorname{int} C \Rightarrow$ the integrant is an holom. function $\Rightarrow I = 0$

(3) $I = \int_C \frac{e^z}{z^2+2z}\,dz$; $C: |z| = 1$



$z^2 + 2z = 0$

$z(z+2) = 0$

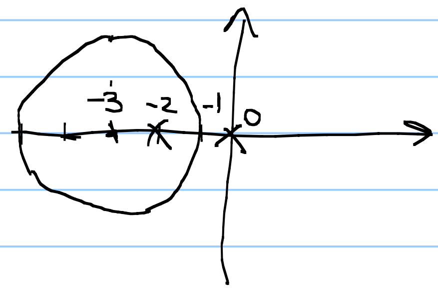
$z_1 = 0 \in \operatorname{int} C$

$z_2 = -2 \notin \operatorname{int} C$

$f(z) = \frac{e^z}{z+2}$ holom. function, $z_0 = 0$

$$I = \int_C \frac{\frac{e^z}{z+2}}{z}\,dz = 2\pi i f(0) = 2\pi i \cdot \frac{e^0}{0+2} = 2\pi i \cdot \frac{1}{2} = \pi i$$

(3) $I = \int_C \frac{e^z}{z^2+2z}\,dz$, $C: |z+3| = 2$



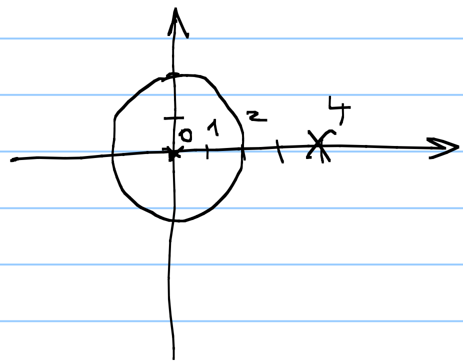
$z_1 = 0 \notin \operatorname{int} C$

$z_2 = -2 \in \operatorname{int} C$

$f(z) = \dfrac{e^z}{z}$ holom. function, $z_0 = -2$

$$I = \int_C \frac{\frac{e^z}{z}}{z+2}\,dz = 2\pi i f(-2) = \cancel{2}\pi i \frac{e^{-2}}{-\cancel{2}} = -\frac{\pi i}{e^2}$$

(4) $\displaystyle\int_C \frac{\sin z}{z^2(z-4)}\,dz$, $C: |z| = 2$



$z^2 = 0 \Rightarrow z_1 = z_2 = 0 \in \operatorname{int} C$ $\quad (|0| = 0 < 2)$

$z - 4 = 0 \Rightarrow z_3 = 4 \notin \operatorname{int} C$ $\quad (|4| = 4 > 2)$

$$I = \int_C \frac{\frac{\sin z}{z-4}}{z^2}\,dz = \frac{2\pi i}{1!} \cdot f'(0) = 2\pi i \cdot \left(-\frac{1}{4}\right) = -\frac{\pi i}{2}$$

$f(z) = \dfrac{\sin z}{z-4}$ holom. function, $z_0 = 0$, $n+1 = 2 \Rightarrow n = 1$

$$f'(z) = \frac{\cos z(z-4) - \sin z}{(z-4)^2} \Rightarrow f'(0) = \frac{\cos 0(0-4) - \sin 0}{(0-4)^2} = \frac{-4}{16} = -\frac{1}{4}$$

⑤ $I=\int\limits_{C:|z|=5} \frac{dz}{z^2+16} = \frac{1}{8i}\int\limits_C \frac{z+4i-(z-4i)}{(z+4i)(z-4i)}\,dz = \frac{1}{8i}\int\limits_C \frac{dz}{z-4i} - \frac{1}{8i}\int\limits_C \frac{dz}{z+4i} =$

$z^2+16=0$

$(z+4i)(z-4i)=0$

$z_1=4i \in \text{int}\, C$ $\quad (|4i|=4<5)$

$z_2=-4i \in \text{int}\, C$

- we expand by partial fractions

$= \frac{1}{8i}\, 2\pi i\, \underbrace{f(4i)}_{1} - \frac{1}{8i}\, 2\pi i\, \underbrace{f(-4i)}_{1} = $ $f(z)=1$ holom. f.

$=0$

⑥ $\int\limits_{C:|z|=1} \frac{\cosh^2(iz)}{z^3}\,dz = \frac{2\pi i}{2!}\, f''(0)$

$z^3=0 \Rightarrow z_{1,2,3}=0 \in \text{int}\, C$ $\quad f(z)=\cosh^2(iz)$ holom. fract., $z_0=0$

$3=n+1 \Rightarrow n=2$

$f(z)=\cosh^2(iz)$

$f'(z)=2\cosh(iz)\cdot i\sinh(iz) = 2i\cosh(iz)\sinh(iz)$

$(\cosh(z))'=\sinh(z)$

$(\sinh(z))'=\cosh(z)$

$$f''(z) = 2i\left[\sinh^2(iz)\cdot i + \cosh^2(iz)\cdot i\right] = -2\left[\sinh^2(iz) + \cosh^2(iz)\right]$$

$$f''(0) = -2\left[\sinh^2(0) + \cosh^2(0)\right] = -2\left[\left(\frac{e^0-e^0}{2}\right)^2 + \left(\frac{e^0+e^0}{2}\right)^2\right] = -2$$

$$\Rightarrow I = \frac{2\pi i}{2!}(-2) = -2\pi i$$

7) $I = \int \frac{e^z}{z(z-1)^3}\,dz$

$C: |z-1| = \frac{1}{4}$

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