Table of Contents

CHAPTER 1

Complex functions

1.1 Complex Numbers. The complex plane

The *complex numbers set* (denoted by $\mathbb{C}$) consists of numbers of the form $z = x + iy$, where x and y are real numbers and i is the *imaginary unit*, having the property $i^2 = -1$.

This is the *algebraic form* of a complex number.

The *real part* of z is x, denoted $x = \mathrm{Re}z$ and the imaginary part of z is y, denoted $y = \mathrm{Im}z$. Complex numbers can also be represented as a pair of real numbers, $z = (x, y)$.

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if both their corresponding real and imaginary parts are equal, that is $x_1 = x_2$ and $y_1 = y_2$.

The *conjugate* of z is $\bar{z} = x - iy$. Some properties of the conjugate are: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$ and $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$.

The real part and the imaginary part of a complex number z can be expressed in terms of z and $\bar{z}$ by $\mathrm{Re}z = \dfrac{1}{2}(z + \bar{z})$ and $\mathrm{Im}z = \dfrac{1}{2i}(z - \bar{z})$.

The *modulus* (*absolute value*, *magnitude*) of the complex number $z = x+iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

Some properties of the modulus are:

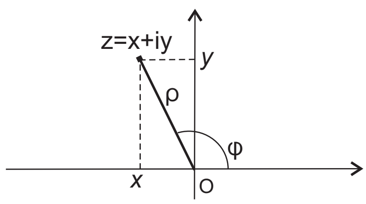
$$\begin{aligned}
&|z_1 z_2| = |z_1||z_2|,\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ |z_1+z_2| \le |z_1|+|z_2|,\\
&z\cdot\bar{z} = |z|^2,\ |\bar{z}| = |z|,\ |z^n| = |z|^n,\\
&|\mathrm{Re}z| \le |z|,\ |\mathrm{Im}z| \le |z|,\ ||z_1|-|z_2|| \le |z_1-z_2|.
\end{aligned}$$

The *trigonometric (polar) form* of a complex number is

$$z = \rho(\cos\varphi + i\sin\varphi)$$

with $\rho = |z| = \sqrt{x^2+y^2}$ and $\varphi = \arctan\dfrac{y}{x} + k\pi$ ($k=0$ if z is in the first quadrant, $k=1$ if z is in the second or third quadrant and $k=2$ if z is in the fourth quadrant). The real number ρ is called the polar radius and φ is the reduced argument of z. We have $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$.

Complex numbers can be displayed on the complex plane. Also, they are often identified with vectors pointing from the origin to the point (x, y).



If $z_1 = \rho_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = \rho_2(\cos\varphi_2 + i\sin\varphi_2)$, then

$$\begin{aligned}
z_1 z_2 &= \rho_1\rho_2[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)] \text{ and}\\
\frac{z_1}{z_2} &= \frac{\rho_1}{\rho_2}[\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)].
\end{aligned}$$

In particular, $z^n = (\rho(\cos\varphi + i\sin\varphi))^n = \rho^n(\cos n\varphi + i\sin n\varphi)$.

We have the formula

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi \quad \text{(De Moivre)}$$

The rots of order n of the complex number $z = \rho(\cos\varphi + i\sin\varphi)$ are given by

$$\sqrt[n]{z} = \sqrt[n]{\rho}\Big(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\Big),$$

where $k \in \{0, 1, \ldots, n-1\}$. The points corresponding to the values of $\sqrt[n]{z}$ are the vertices of the regular n-gon inscribed in the circle of radius $\rho^{\frac{1}{n}}$ centered at the origin of coordinates.

We mention that in electrical engineering the notation i is frequently used for current, so the complex unit $\sqrt{-1}$ is denoted j $(z = x + jy)$.

Solved problems

1.1 Write the complex number $z = -1 + i\sqrt{3}$ in trigonometric (polar) form and find z^{207}.

Solution. $|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$. Since the image of the complex number is in the second quadrant, we have

$$\varphi = \arctan(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$$

So the trigonometric form is $z = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$. Then

$$z^{207} = 2^{207}(\cos 207 \cdot \frac{2\pi}{3} + i\sin 207 \cdot \frac{2\pi}{3}) = 2^{207}(\cos 138\pi + i\sin 138\pi) = 2^{207}.$$

1.2 Specify the curve represented by the equation $|z - 2 + 3i| = 2$.

Solution. Writing z in algebraic form $z = x + iy$, $x, y \in \mathbb{R}$ we get $|x - 2 + (y+3)i| = 2$, that is $(x-2)^2 + (y+3)^2 = 4$. This is the equation of the circle with the center at $C(2, -3)$ and radius 2.

Generally, if z_0 is a complex number and R is a real number, the equation

$$|z - z_0| = R$$

represents the equation of the circle with center at z_0 and radius R.

1.3 Write the following equations in complex form:

i). $Ax + By + C = 0, \quad A, B, C \in \mathbb{R}$

ii). $(x-a)^2 + (y-b)^2 = r^2, \quad a, b \in \mathbb{R}, r > 0.$

Solution. i). Using $x = \dfrac{\bar{z}+z}{2}$ and $y = i\dfrac{\bar{z}-z}{2}$ in the equation of the straight line, we obtain

$$A(\bar{z}+z) + Bi(\bar{z}-z) + 2C = 0, \quad \text{or}$$

$$(A+iB)\bar{z} + (A-iB)z + 2C = 0.$$

Finally we can write $a\bar{z} + \bar{a}z + 2C = 0$, where $a = A + iB$.
ii). We have $x^2 + y^2 = |z|^2 = z\bar{z}$, $2x = \bar{z} + z$, $2y = i(\bar{z} - z)$, hence the equation of the circle becomes

$$\bar{z}z - (a-bi)z - (a+bi)\bar{z} + a^2 + b^2 - r^2 = 0.$$

Another form for the equation of the circle can be obtained if we notice that for $z = x+iy$, we have $(x-a)^2 + (y-b)^2 = |z-a-bi|^2$. The equation becomes

$$|z - (a+bi)| = r.$$

Proposed problems

1.4 Represent the complex numbers in trigonometric form:

i). $-1-i$

ii). $\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$

iii). 1

iv). i

v). -1

vi). $-i$

vii). $\cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}$

viii). $-\sin\dfrac{\pi}{8} - i\cos\dfrac{\pi}{8}$

ix). $(-1+i)^{105}$

1.5 Solve in $\mathbb{C}$ the following equations:

i). $z^4 = 1 - i$

ii). $z^6 = 1$

iii). $z^4 + (1-i)z^2 - i = 0$

iv). $z^4 - z^3 + z^2 - z + 1 = 0$

1.6 Specify the curves represented by the following equations:

i). $\mathrm{Im}\, z^2 = 1$

ii). $z^2 + \bar{z}^2 = 2$

iii). $|z+c| + |z-c| = 2a$, $a > c > 0$

iv). $\mathrm{Re} z = |z|$

v). $\mathrm{Im}(z+i) = |z|$

vi). $\mathrm{Re}(\bar{z})^{-1} = 1$

vii). $(1+i)z + (1-i)\bar{z} + 4 = 0$

1.7 Let $x \in \mathbb{R}$ such that $x \neq 2k\pi$, $k \in \mathbb{Z}$ and let $n \in \mathbb{N}$. Calculate the sum $S = \sin x + \sin 2x + \cdots + \sin nx$.

Solutions and answers

$\boxed{\textbf{1.4}}$ **i)** $\sqrt{2}(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4})$; **ii)** $\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$; **iii)** $\cos 0 + i \sin 0$;
iv) $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$; **v)** $\cos \pi + i \sin \pi$; **vi)** $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$;
vii) $\cos \frac{\pi}{8} - i \sin \frac{\pi}{8} = \cos\left(-\frac{\pi}{8}\right) + i \sin\left(-\frac{\pi}{8}\right) = \cos\left(-\frac{\pi}{8} + 2\pi\right) + i \sin\left(-\frac{\pi}{8} + 2\pi\right) = \cos \frac{15\pi}{8} + i \sin \frac{15\pi}{8}$;
viii) Using various trigonometric formulas we get that $-\sin \frac{\pi}{8} - i \cos \frac{\pi}{8} = \sin\left(\frac{\pi}{8} + \pi\right) + i \cos\left(\frac{\pi}{8} + \pi\right) = \cos\left(\frac{\pi}{2} - \frac{9\pi}{8}\right) + i \sin\left(\frac{\pi}{2} - \frac{9\pi}{8}\right) = \cos\left(-\frac{5\pi}{8}\right) + i \sin\left(-\frac{5\pi}{8}\right) = \cos \frac{11\pi}{8} + i \sin \frac{11\pi}{8}$;
ix) $2^{52}\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$.

$\boxed{\mathbf{1.5}}$ **i)** $z_k = \sqrt[8]{2}\big(\cos\dfrac{7\pi+8k\pi}{16} + i\sin\dfrac{7\pi+8k\pi}{16}\big)$, $k \in \{0,1,2,3\}$;
ii) $z^6 = \cos 0 + i\sin 0$, so $z_k = \cos\dfrac{k\pi}{3} + i\sin\dfrac{k\pi}{3}$, $k \in \{0,\ldots,5\}$. Explicitly, $z_0 = 1$, $z_1 = \dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}$, $z_2 = -\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}$, $z_3 = -1$, $z_4 = -\dfrac{1}{2}-i\dfrac{\sqrt{3}}{2}$, $z_5 = \dfrac{1}{2}-i\dfrac{\sqrt{3}}{2}$;
iii) $z^2 = t$, $t_1 = i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$, $t_2 = -1 = \cos\pi + i\sin\pi$. From here follow the four solutions: $\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$, $-\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$, i, $-i$;
iv) We multiply the equation by $z+1$ and get $z^5+1 = 0$. From $z^5 = -1$ follow the roots $z_k = \cos\dfrac{\pi+2k\pi}{5} + i\sin\dfrac{\pi+2k\pi}{5}$, with $k \in \{0,1,3,4\}$ ($z_2 = -1$ is not a solution of the initial equation).

$\boxed{\mathbf{1.6}}$ **i)** Writing $z = x+iy$, we get $xy = \dfrac{1}{2}$, which is a hyperbola;
ii) The hyperbola $x^2 - y^2 = 1$;
iii) $|z-c|$ represents the distance between the points z and c, $|z+c|$ the distance between z and $-c$. The equation tells that the sum of the distances from z to c and $-c$ is the constant $2a$. This means that z lies on an ellipse, having the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $b^2 = a^2 - c^2$. Another method to find the same result: write $z = x+iy$, then $\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$. Raising to square two times one gets the equation of the ellipse;
iv) The axis $0x$, $y^2 = 0$; **v)** The parabola $x^2 = 2y+1$;
vi) The circle $\left(x-\dfrac{1}{2}\right)^2 + y^2 = \dfrac{1}{4}$;
vii) The straight line $x - y + 2 = 0$.

$\boxed{\mathbf{1.7}}$ Consider also the sum $T = 1 + \cos x + \cos 2x + \cdots + \cos nx$. Denoting $z = \cos x + i\sin x$ we have

$$T + iS = 1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z} = \frac{1-\cos(n+1)x - i\sin(n+1)x}{1-\cos x - i\sin x}.$$

$$T + iS = \frac{2\sin^2\frac{(n+1)x}{2} - 2i\sin\frac{(n+1)x}{2}\cos\frac{(n+1)x}{2}}{2\sin^2\frac{x}{2} - 2i\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\cdot[\cos\frac{nx}{2} + i\sin\frac{nx}{2}].$$

The imaginary part of this number is the sum S.

1.2 Differentiating functions of a complex variable. The Cauchy-Riemann conditions.

Consider a function $f : D \to \mathbb{C}$, where D is an open set in the complex plane. For $z \in D$ we often denote $w = f(z)$. Let $z = x+iy$ and $w = u+iv$. The dependence of w on z can be described by specifying two real-valued functions u and v of the real variables x and y:

$$u = u(x, y), \quad v = v(x, y).$$

Example 1.2.1 Let $w = f(z) = z^3+i\bar{z}-1$. Denoting $z = x+iy$ and $w = u+iv$, we find that

$$u + iv = (x + iy)^3 + i(x - iy) - 1 = (x^3 - 3xy^2 + y - 1) + i(3x^2y - y^3 + x).$$

Identifying the corresponding real and imaginary parts, we get that the equation $w = z^3 + i\bar{z} - 1$ is equivalent to the following two equalities:

$$u = x^3 - 3xy^2 + y - 1, \quad v = 3x^2y - y^3 + x.$$

Remark 1.1 If a function f is known as $w = u(x, y) + iv(x, y)$ and we want to express it as a function of the variable $z = x + iy$, we can replace the variables x and y by the expressions $x = \dfrac{z + \bar{z}}{2}$ and $y = \dfrac{z - \bar{z}}{2i}$.

The problem of differentiating a complex function is a very important one.

Definition 1.2 Let $D \subset \mathbb{C}$ be an open set, $f : D \to \mathbb{C}$ and $z_0 \in D$. If the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.2.1}$$

exists in $\mathbb{C}$, we say that f has a *derivative* at z_0 (or that f is *monogenic* at z_0). We denote the limit (1.2.1) by $f'(z_0)$.

If f is monogenic at every point of D, we say that f is *holomorphic* in D.

The above definition of the derivative is similar to that used in the real case. Hence for the evaluation of derivatives of a complex variable the same rules hold as for functions of a real variable; in particular,

$$(f + g)' = f' + g', \ (fg)' = f'g + fg', \ (z^n)' = nz^{n-1}, \text{ etc.}$$

Theorem 1.3 *a) If f is monogenic at $z_0 = x_0 + iy_0$, then u and v have partial derivatives at (x_0, y_0) and*

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) &= -\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned} \tag{1.2.3}$$

b) If u and v are differentiable at (x_0, y_0) as functions of two real variables, and if conditions (1.2.3) are satisfied, then f is monogenic at z_0.

Conditions (1.2.3) are called *the Cauchy-Riemann conditions.*

Example 1.2.2 Let $f(z) = z^2 - iz$, $z \in \mathbb{C}$. Then $f'(z) = 2z - i$, hence f is holomorphic in $\mathbb{C}$.

We have also

$$f = u + iv = (x + iy)^2 - i(x + iy) = x^2 + 2xyi - y^2 - ix + y,$$

hence $u(x, y) = x^2 - y^2 + y$ and $v(x, y) = 2xy - x$.

Let's verify the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y + 1 = -\frac{\partial v}{\partial x}.$$

Example 1.2.3 Let $f(z) = z\bar{z}$. Then

$$u + iv = (x + iy)(x - iy) = x^2 + y^2,$$

hence $u = x^2 + y^2$ and $v = 0$. The Cauchy-Riemann conditions give us $2x = 0$ and $2y = 0$. Hence the function f is monogenic only at $z = 0$.

Example 1.2.4 Let $f(z) = \bar{z}$. Now $u + iv = x - iy$, that is, $u = x$, $v = -y$. It follows that $\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial v}{\partial y} = -1$. The first Cauchy-Riemann condition holds nowhere, hence f is nowhere monogenic.

The next theorem shows that not any function can be the real or imaginary part of a holomorphic function.

Theorem 1.4 *Suppose that $f = u + iv$ is holomorphic on D and u, v have continuous second-order partial derivatives in D. Then u and v are harmonic functions, i.e.,*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Proof. We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. It follows that

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y}.
\end{aligned}$$

Hence $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Similarly $\Delta v = 0$. □

Remark 1.5 A holomorphic function $w = u(x, y) + iv(x, y)$ can be written in the form $w = f(z)$ noticing that $f(z) = u(z, 0) + iv(z, 0)$.

Solved problems

1.8 Find the holomorphic function $f = u + iv$ from its known real part $u(x, y) = x^3 + 6x^2y - 3xy^2 - 2y^3$ and the additional condition that $f(0) = 0$.

Solution. We have $\frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2$, hence $\frac{\partial v}{\partial y} = 3x^2 + 12xy - 3y^2$.

It follows that

$$v(x, y) = 3x^2y + 6xy^2 - y^3 + \varphi(x),$$

where $\varphi(x)$ is yet to be found. From $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ we get

$$6x^2 - 6xy - 6y^2 = -6xy - 6y^2 - \varphi'(x),$$

hence $\varphi'(x) = -6x^2$, or $\varphi(x) = -2x^3 + C$. Now

$$f = x^3 + 6x^2y - 3xy^2 - 2y^3 + i(3x^2y + 6xy^2 - y^3 - 2x^3) + Ci.$$

From $f(0) = 0$ we deduce that $C = 0$. Finally,

$$f = (x^3 + 3x^2yi - 3xy^2 - y^3i)(1 - 2i) = (1 - 2i)(x + iy)^3 = (1 - 2i)z^3.$$

1.9 Prove that any holomorphic function $f : D \to \mathbb{C}$ satisfies the relation

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2,$$

for every $z \in D$.

Solution. Let $f = u + iv$. Then $|f| = \sqrt{u^2 + v^2}$ and

$$f' = u'_x + iv'_x = v'_y - iu'_y.$$

We have

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = \left(\frac{uu'_x + vv'_x}{\sqrt{u^2 + v^2}}\right)^2 + \left(\frac{uu'_y + vv'_y}{\sqrt{u^2 + v^2}}\right)^2 =$$
$$= \frac{u^2u'^2_x + v^2v'^2_x + 2uvu'_xv'_x + u^2u'^2_y + v^2v'^2_y + 2uvu'_yv'_y}{u^2 + v^2} =$$
$$= \frac{u^2(u'^2_x + u'^2_y) + v^2(v'^2_x + v'^2_y)}{u^2 + v^2} = \frac{u^2(u'^2_x + v'^2_x) + v^2(v'^2_x + u'^2_x)}{u^2 + v^2} = u'^2_x + v'^2_x.$$

1.10 Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Prove that the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = f(z) - \overline{f(-\bar{z})}$ is also holomorphic.

Solution. Denote $f(z) = u(x, y) + iv(x, y)$. Then

$$g(z) = u(x, y) + iv(x, y) - \overline{u(-x, y) + iv(-x, y)}.$$

Denoting $g(z) = u_1(x, y) + iv_1(x, y)$ it follows that

$$u_1(x, y) = u(x, y) - u(-x, y) \text{ and } v_1(x, y) = v(x, y) + v(-x, y).$$

We have, using the Cauchy-Riemann conditions for f:

$$\frac{\partial u_1}{\partial x}(x, y) = \frac{\partial u}{\partial x}(x, y) - \frac{\partial u}{\partial x}(-x, y) \cdot (-1) = \frac{\partial v}{\partial y}(x, y) + \frac{\partial v}{\partial y}(-x, y) = \frac{\partial v_1}{\partial y}(x, y).$$

The relation $\dfrac{\partial u_1}{\partial y}(x, y) = -\dfrac{\partial v_1}{\partial x}(x, y)$ is obtained in a similar way.

1.11 Write the Cauchy-Riemann conditions for a function given in polar coordinates.

Solution. The connection between the cartesian and the polar coordinates is given by:

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta. \end{cases}$$

A function $f(x, y) = u(x, y) + iv(x, y)$ becomes

$$f(\rho\cos\theta, \rho\sin\theta) = u(\rho\cos\theta, \rho\sin\theta) + iv(\rho\cos\theta, \rho\sin\theta),$$

or $g(\rho, \theta) = U(\rho, \theta) + iV(\rho, \theta)$.

We have

$$U'_\rho(\rho, \theta) = u'_x(\rho\cos\theta, \rho\sin\theta)\cos\theta + u'_y(\rho\cos\theta, \rho\sin\theta)\sin\theta$$

and using the conditions (1.2.3),

$$U'_\rho(\rho, \theta) = v'_y(\rho\cos\theta, \rho\sin\theta)\cdot\cos\theta - v'_x(\rho\cos\theta, \rho\sin\theta)\cdot\sin\theta.$$

On the other hand,

$$V'_\theta(\rho, \theta) = v'_x(\rho\cos\theta, \rho\sin\theta)\cdot(-\rho\sin\theta) + v'_y(\rho\cos\theta, \rho\sin\theta)\cdot\rho\cos\theta.$$

From these follows the first condition: $U'_\rho(\rho, \theta) = \dfrac{1}{\rho}\cdot V'_\theta(\rho, \theta)$.

In the same way can be obtained the second Cauchy-Riemann condition: $U'_\theta(\rho, \theta) = -\rho\cdot V'_\rho(\rho, \theta)$.

Proposed problems

1.12 Determine the points where the following functions are monogenic.

i). $f(z) = ze^z$;

ii). $f(z) = z^2 + 3iz$;

iii). $f(z) = \bar{z}\mathrm{Im}z$;

iv). $f(z) = |z|\mathrm{Re}z$;

v). $f(z) = \dfrac{1}{z-1}$, $z \neq 1$;

vi). $f(z) = \mathrm{Im}z + 2i\mathrm{Re}z$;

vii). $f(z) = (z^2 + \bar{z}^2)(1-i) + 2z\bar{z}(1+i) - 4iz$.

1.13 Can the following functions be the real or imaginary part of an analytic function $f = u + iv$?

i). $u(x, y) = x^2 - y^2 + 2xy$;

ii). $v(x, y) = x^2 + y^2$;

iii). $v(x, y) = \operatorname{arctg} \dfrac{y}{x}$.

1.14 Reconstruct the holomorphic function $f = u + iv$ from its known real part $u(x, y)$ or imaginary part $v(x, y)$.

i). $u(x, y) = 3xy$;

ii). $v(x, y) = e^x(x \sin y + y \cos y)$

1.15 Reconstruct the holomorphic function $f = u + iv$ from its known real part $u(x, y)$ or imaginary part $v(x, y)$ and the value $f(z_0)$.

i). $u(x, y) = x^2 - y^2 + 3x$, $f(i) = 3i - 1$;

ii). $v(x, y) = \dfrac{y}{x^2 + y^2}$, $f(1) = 1$;

iii). $u(x, y) = \dfrac{x}{x^2 + y^2}$, $f(i\sqrt{2}) = -\dfrac{i}{\sqrt{2}}$;

iv). $v(x, y) = 2e^x \sin y$, $f(0) = 2$;

v). $u(x, y) = \dfrac{1 - x^2 - y^2}{1 + 2x + x^2 + y^2}$, $f(1) = 0$;

vi). $u(x, y) = \ln(x^2 + y^2)$, $f(1) = 0$, $x \neq 0$;

vii). $v(x, y) = \cosh x \sin y$, $f(0) = 1$.

1.16 Find a holomorphic function $f = u + iv$ such that $u(x, y) - v(x, y) = x^2 - y^2 - 2xy - 2x - 2y$ and $f(i) = -3$.

1.17 Determine the harmonic functions of the form $u(x, y) = g(x^2+y^2)$, with g a function of class C^2 on $\mathbb{R}$. Reconstruct the holomorphic function f that has u as the real part.

1.18 In what conditions can both the functions $f : D \to \mathbb{C}$ and $\bar{f} : D \to \mathbb{C}$ be holomorphic?

Solutions and answers

$\boxed{\mathbf{1.12}}$ **i)**, **ii)** f is holomorphic on $\mathbb{C}$;
iii), **iv)** f is monogenic only at the origin;
v) $u(x, y) = \dfrac{x-1}{(x-1)^2 + y^2}$, $v(x, y) = \dfrac{-y}{(x-1)^2 + y^2}$, f is holomorphic on the set $\mathbb{C} \setminus \{1\}$;
vi) f has no points of monogenicity;
vii) $u(x, y) = 4x^2 + 4y$, $v(x, y) = 4y^2 - 4x$, f is monogenic only at the points where $\mathrm{Re}z = \mathrm{Im}z$.

$\boxed{\mathbf{1.13}}$ We verify if the laplacean $\Delta u = u''_{x^2} + u''_{y^2}$ is equal to 0. **i)** Yes; **ii)** No; **iii)** Yes.

$\boxed{\mathbf{1.14}}$ **i)** $f(x, y) = 3xy + \left(\dfrac{3}{2}(y^2 - x^2) + c\right)i$, which can also be written as $f(z) = -\dfrac{3i}{2}z^2 + ci$, $c \in \mathbb{R}$;
ii) $u'_x(x, y) = v'_y(x, y) = e^x(x\cos y + \cos y - y\sin y)$, so integrating with respect to x we get $u(x, y) = e^x(x\cos y - y\sin y) + c(y)$. To find $c(y)$ we use the equality between $u'_y(x, y) = e^x(-x\sin y - \sin y - y\cos y) + c'(y)$ and $-v'_x(x, y) = -e^x(x\sin y + y\cos y + \sin y)$. This gives $c'(y) = 0$, that is $c(y) = c$, $c \in \mathbb{R}$. The requested function is $f(x, y) = e^x(x\cos y - y\sin y) + c + ie^x(x\sin y + y\cos y) = e^x(\cos y + i\sin y)(x + iy) + c = e^{x+iy}(x + iy) + c$. Finally, we can write $f(z) = ze^z + c$.

$\boxed{\mathbf{1.15}}$ **i)** $f(x,y) = x^2 - y^2 + 3x + (2xy + 3y + c)i$, from the condition $f(0,1) = 3i - 1$ follows $c = 0$, so $f(z) = z^2 + 3z$;

ii) We have $u'_y(x,y) = -v'_x(x,y) = \dfrac{2xy}{(x^2+y^2)^2}$. Integrating with respect to y, we get $u(x,y) = \dfrac{-x}{x^2+y^2} + c(x)$. Then, from $u'_x(x,y) = \dfrac{x^2-y^2}{(x^2+y^2)^2} + c'(x)$ and $v'_y(x,y) = \dfrac{x^2-y^2}{(x^2+y^2)^2}$ follows $c'(x) = 0$, so $c(x) = c$, $c \in \mathbb{R}$. The function becomes $f(x,y) = -\dfrac{x}{x^2+y^2} + c + \dfrac{yi}{x^2+y^2}$ and from $f(1) = 1$ follows $c = 2$. Finally, we get $f(z) = -\dfrac{\overline{z}}{z \cdot \overline{z}} + 2 = -\dfrac{1}{z} + 2$;

iii) $f(z) = \dfrac{1}{z}$; **iv)** $f(x,y) = 2e^x \cos y + 2e^x i \sin y$ or $f(z) = 2e^z$;

v) We have $v'_x(x,y) = -u'_y(x,y) = \dfrac{4y(x+1)}{((x+1)^2+y^2)^2}$. To calculate the integral $\displaystyle\int \frac{4y(x+1)}{((x+1)^2+y^2)^2}\mathrm{d}x$ we make the change of variable $t = (1+x)^2 + y^2$ and we get $v(x,y) = -\dfrac{2y}{(x+1)^2+y^2} + c(y)$. On the other hand $u'_x(x,y) = \dfrac{2y^2 - 2(1+x)^2}{((x+1)^2+y^2)^2}$, so it follows that $c'(y) = 0$, or $c(y) = c$. The constant c is obtained to be 0 from the condition $f(1) = 0$. Then the requested function is $f(x,y) = \dfrac{1 - x^2 - y^2 - 2yi}{(x+1)^2+y^2}$. To write f as a function of the complex variable $z = x + yi$, we notice that $(x+1)^2 + y^2 = |z+1|^2 = (z+1)(\overline{z+1})$ and $1 - x^2 - y^2 - 2yi = (1-yi)^2 - x^2 = (1 - yi - x)(1 - yi + x) = (1-z)(\overline{z+1})$. Finally, $f(z) = \dfrac{1-z}{1+z}$. (the expression of f as a function of z can be also obtained using one of the Remarks 1.1 or 1.5);

vi) $f(x,y) = \ln(x^2+y^2) + 2i \operatorname{arctg} \dfrac{y}{x}$;

vii) $f(x,y) = \sinh x \cos y + 1 + i \cosh x \sin y$, or $f(z) = \sinh z + 1$.

$\boxed{\mathbf{1.16}}$ We have that $u'_x(x,y) - v'_x(x,y) = 2x - 2y - 2$ and $u'_y(x,y) - v'_y(x,y) = -2y - 2x - 2$. Using these equalities and the Cauchy-Riemann conditions we get the system

$$\begin{cases} u'_x(x,y) - v'_x(x,y) = 2x - 2y - 2 \\ -u'_x(x,y) - v'_x(x,y) = -2y - 2x - 2 \end{cases}$$

with the solutions $u'_x(x,y) = 2x$ and $u'_y(x,y) = -2y-2$. From here, like in the previous exercise, $u(x,y) = x^2 + \varphi(y)$, $\varphi'(y) = -2y-2$, so $\varphi(y) = -y^2-2y+c$. We get then $v(x,y) = 2xy + 2x + c$ and from $f(i) = -3$ follows $c = 0$.

Finally $f(x,y) = x^2 - y^2 - 2y + i(2xy + 2y)$, or $f(z) = z^2 + 2iz$.

1.17 We have $u''_{x^2}(x,y) = g''(x^2+y^2)\cdot(2x)^2 + g'(x^2+y^2)\cdot 2$ and $u''_{y^2}(x,y) = g''(x^2+y^2)\cdot(2y)^2 + g'(x^2+y^2)\cdot 2$, so the equation $\Delta u = 0$ becomes $4tg''(t) + 4g'(t) = 0$, where we denoted $t = x^2 + y^2$. This can be written $\dfrac{g''(t)}{g'(t)} = -\dfrac{1}{t}$, which gives $\ln g'(t) = -\ln t + \ln c_1$ or $g'(t) = \dfrac{c_1}{t}$. Integrating once again we get $g(t) = c_1 \ln t + c_2$, with c_1, c_2 real constants.

The requested function is $u(x,y) = c_1 \ln(x^2+y^2) + c_2$.

1.18 Let $f = u + iv$. From the Cauchy-Riemann conditions applied both to f and $\bar{f} = u - iv$ we get $u'_x(x,y) = u'_y(x,y) = 0$ and $v'_x(x,y) = v'_y(x,y) = 0$ which implies that f must be a constant function.

1.3 Elementary functions of a complex variable

1. The **exponential function** e^z is defined on the whole set of complex numbers $\mathbb{C}$; for $z = x + iy$,

$$e^z = e^x(\cos y + i \sin y).$$

This function is holomorphic in $\mathbb{C}$ and $(e^z)' = e^z$.

In particular, for $\theta \in \mathbb{R}$ we have $e^{i\theta} = \cos\theta + i \sin\theta$.

Any complex number can be written in *exponential form*

$$z = \rho e^{i\varphi}$$

where ρ and φ are the same as in the polar form of the complex number, $\rho = |z|$ and $\tan\varphi = \dfrac{y}{x}$. The exponential form is the most frequently used form in science applications.

It is easy to verify that $e^{z_1+z_2} = e^{z_1}e^{z_2}$, for every $z_1, z_2 \in \mathbb{C}$.

Let us also remark that for each $z \in \mathbb{C}$ and $k \in \mathbb{Z}$,

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z(\cos 2k\pi + i \sin 2k\pi) = e^z.$$

Hence e^z is a periodic function with period $2\pi i$.

2. The **polynomial function** is defined by

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

where a_i, $i = 1, \ldots, n$ are given complex numbers.

3. The **rational function** is defined by

$$w = \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}$$

where a_i and b_j are given complex numbers. The polynomial function is a particular case of the rational function.

Each rational function is holomorphic in its natural domain of definition.

4. The **trigonometric functions** $\sin z$ and $\cos z$ are defined by:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \text{ for every } z \in \mathbb{C}.$$

The functions $\tan z$ and $\cot z$ are defined by

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

All the formulas of trigonometry remain valid for trigonometric functions of a complex variable.

5. The **hyperbolic functions** $\sinh z$, $\cosh z$, $\tanh z$ and $\coth z$ are defined by:

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \quad \cosh z = \frac{e^{z} + e^{-z}}{2},$$
$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}.$$

The trigonometric and hyperbolic functions are related through the following relations:

$$\sin z = -i \sinh iz; \quad \sinh z = -i \sin iz; \quad \cos z = \cosh iz; \quad \cosh z = \cos iz;$$
$$\tan z = -i \tanh iz; \ \tanh z = -i \tan iz; \ \cot z = i \coth iz; \ \coth z = i \cot iz.$$

6. The logarithmic function. Let $z \in \mathbb{C}$, $z \neq 0$. The complex number w is called a **logarithm** of z if $e^w = z$.

We write z in trigonometric form

$$z = \rho(\cos\theta + i \sin\theta)$$

and $w = u + iv$. Then $e^w = e^{u+iv} = e^u(\cos v + i \sin v)$ so that $e^u = \rho$ and $v = \theta + 2k\pi$, with $k \in \mathbb{Z}$.

Since $u = \ln\rho$, we get $w = \ln\rho + i(\theta + 2k\pi)$.

Thus z has infinitely many logarithms $w = \ln|z| + i(\arg z + 2k\pi)$, $k \in \mathbb{Z}$.

The set of all these logarithms is denoted by

$$\operatorname{Log} z = \{\ln|z| + i(\arg z + 2k\pi) \mid k \in \mathbb{Z}\}.$$

We have a **multiple-valued** function Log, defined on $\mathbb{C}^*$ and associating to each $z \in \mathbb{C}^*$ infinitely many complex numbers.

The function $\log : \mathbb{C}^* \to \mathbb{C}$, $\log z = \ln|z| + i \arg z$ is called the **principal value** of Log.

For instance, we have

$$\log(-1) = \ln|-1| + i\arg(-1) = i\pi; \quad \mathrm{Log}\,(-1) = i(\pi + 2k\pi)$$

$$\log i = i\frac{\pi}{2}; \quad \mathrm{Log}\, i = i(\frac{\pi}{2} + 2k\pi).$$

7. The **general power function** $w = z^a$, where $a \in \mathbb{C}$, is defined for $z \neq 0$ by

$$z^a = e^{a\mathrm{Log}\, z}.$$

Generally, this function is multiple-valued, its principal value is $z^a = e^{a \log z}$.

7. The **homographic function** has the form

$$w = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

It is easy to prove that the mapping composed of two homographic mapping is a homographic mapping, and the mapping inverse to a homographic mapping is also homographic.

Solved problems

1.19 Find $\mathrm{Log}\,(-1-i)$ and i^{1-i}.

Solution. We have $|-1-i| = \sqrt{2}$ and $\arg(-1-i) = \dfrac{5\pi}{4}$ so the complex logarithm is the set of numbers

$$\mathrm{Log}\,(-1-i) = \ln\sqrt{2} + i(\frac{5\pi}{4} + 2k\pi), \quad k \in \mathbb{Z}.$$

Using the definition of the power function we write $i^{1-i} = e^{(1-i)\mathrm{Log}\, i}$. Since $\mathrm{Log}\, i = \ln 1 + i(\dfrac{\pi}{2} + 2k\pi) = i(\dfrac{\pi}{2} + 2k\pi)$ we get

$$i^{1-i} = e^{(1-i)i(\frac{\pi}{2}+2k\pi)} = e^{(1+i)(\frac{\pi}{2}+2k\pi)}$$

and next, according to the definition of the exponential,

$$i^{1-i} = e^{\frac{\pi}{2}+2k\pi}(\cos(\frac{\pi}{2} + 2k\pi) + i\sin(\frac{\pi}{2} + 2k\pi)) = ie^{\frac{\pi}{2}+2k\pi}, \quad k \in \mathbb{C}.$$

1.20 Solve the equation $\cos z = \dfrac{3+i}{4}$ in the set of complex numbers.

Solution. Using the definition of the function cosine, we write

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{3+i}{4}.$$

Denoting now $t = e^{iz}$ we get the equation

$$2t^2 - (3+i)t + 2 = 0,$$

with $\Delta = 6i - 8 = (1+3i)^2$ and the solutions $t_1 = 1+i$ and $t_2 = \dfrac{1}{2} - \dfrac{1}{2}i$.

From here

$$iz = \mathrm{Log}\,(1+i) = \ln\sqrt{2} + i(\frac{\pi}{4} + 2k\pi),\ k \in \mathbb{Z}.$$

This gives the solution

$$z = \frac{\pi}{4} - \frac{i}{2}\ln 2 + 2k\pi, \quad k \in \mathbb{Z}.$$

The other possibility is $iz = \mathrm{Log}\,(\dfrac{1}{2} - \dfrac{1}{2}i) = \ln\dfrac{1}{\sqrt{2}} + i(\dfrac{7\pi}{4} + 2k\pi)$, $k \in \mathbb{Z}$, which gives

$$z = \frac{7\pi}{4} + \frac{i}{2}\ln 2 + 2k\pi = -\frac{\pi}{4} + 2\pi + \frac{i}{2}\ln 2 + 2k\pi, \quad k \in \mathbb{Z}.$$

The two types of solutions can be written together in the form

$$z = \pm\big(\frac{\pi}{4} - \frac{i}{2}\ln 2\big) + 2k\pi, \quad k \in \mathbb{Z}.$$

1.21 Find the points from the complex plane where the function $\cos z$ takes real values.

Solution. We separate the real and imaginary parts of the function:

$$\begin{aligned}\cos z &= \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2} = \frac{e^{-y+ix}+e^{y-ix}}{2} = \\ &= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2} = \\ &= \cos x\,\frac{e^{y}+e^{-y}}{2} + i\sin x\,\frac{e^{-y}-e^{y}}{2} = \cos x\cosh y - i\sin x\sinh y.\end{aligned}$$

The condition that this should be a real number gives the equation

$$\sin x\sinh y = 0, \quad \text{with } x, y \in \mathbb{R}.$$

From $\sin x = 0$ we get $x_k = k\pi$, $k \in \mathbb{Z}$ (straight lines parallel to Oy).
From $\sinh y = 0$ we get $e^y = 1$, so $y = 0$ (the axis Ox).

1.22 Find the image of the region $D = \{z \in \mathbb{C} \mid |z| < 1\}$ of the complex plane through the homographic functions $w_1 = \dfrac{z}{z+1}$ and $w_2 = \dfrac{z}{z+2}$.

Solution. From $w_1 = \dfrac{z}{z+1}$ we get the inverse $z = \dfrac{w_1}{-w_1+1}$, and the condition $|z| < 1$ implies that $|w_1| < |-w_1+1|$.

If $w_1 = u_1 + iv_1$ this inequality can be written in the form

$$u_1^2 + v_1^2 < (1-u_1)^2 + v_1^2,$$

that is $u_1 < \dfrac{1}{2}$.

The set D (the interior of a circle) is transformed by w_1 in the half-plane $w_1(D) = \{w \in \mathbb{C} \mid \operatorname{Re} w < \dfrac{1}{2}\}$.

Applying the same reasoning for the function w_2 we get the inequality $|2w_2| < |-w_2+1|$ which gives $(u_2 + \dfrac{1}{3})^2 + v_2^2 < \dfrac{4}{9}$. So the image $w_2(D)$ is the interior of the circle with the center at $(-\dfrac{1}{3}, 0)$ of radius $\dfrac{2}{3}$.

1.23 Consider a series RC circuit for which the voltage across the resistor is given by $V_R(t) = A\cos\omega t$ and the voltage across the capacitor by $V_C(t) = B\sin\omega t$. Find the amplitude and the phase of the voltage across the combination.

Solution. An important use of complex numbers in science is in the representation of sinusoidal quantities. By Kirchhoff's law, the voltage across the combination is

$$V_R(t) + V_C(t) = A\cos\omega t + B\sin\omega t.$$

The sum of sinusoidal quantities is also sinusoidal, with new amplitude and phase.

Considering the exponential form of a complex number, notice that V_R is the real part of $Ae^{i\omega t}$,

$$V_R(t) = \mathrm{Re}(Ae^{i\omega t}).$$

On the other hand,

$$-iBe^{i\omega t} = -i(B\cos\omega t + i\sin\omega t) = B\sin\omega t - iB\cos\omega t$$

so $V_C(t) = \mathrm{Re}(-iBe^{i\omega t})$.

The sum becomes

$$\begin{aligned} V_R(t) + V_C(t) &= \mathrm{Re}(Ae^{i\omega t}) + \mathrm{Re}(-iBe^{i\omega t}) = \\ &= \mathrm{Re}(Ae^{i\omega t} - iBe^{i\omega t}) = \mathrm{Re}((A - iB)e^{i\omega t}). \end{aligned}$$

We have $|A - iB| = \sqrt{A^2 + B^2}$ and denote by ϕ the argument such that $\tan\phi = \dfrac{B}{A}$. Then

$$A - iB = \sqrt{A^2 + B^2}(\cos(-\phi) + i\sin(-\phi)) = \sqrt{A^2 + B^2}e^{-i\phi}.$$

The product is

$$(A - iB)e^{i\omega t} = \sqrt{A^2 + B^2}e^{-i\phi + i\omega t} = \sqrt{A^2 + B^2}e^{i(\omega t - \phi)}$$

and finally

$$V_R(t) + V_C(t) = Re(\sqrt{A^2 + B^2}e^{i(\omega t - \phi)}) = \sqrt{A^2 + B^2}\cos(\omega t - \phi).$$

The new amplitude is $\sqrt{A^2 + B^2}$ and the new phase $\omega t - \phi$.

The method can be used to study more complicated linear circuits. It works also when dealing with harmonic oscillations of mechanical systems.

Proposed problems

1.24 Separate the real and imaginary parts of the functions $w = u + iv$:

i). $w = e^{-z}$;

ii). $w = \sin z$;

iii). $w = \cosh(z - i)$;

iv). $w = \sinh z$;

v). $w = \tan z$;

vi). $w = \dfrac{z + \bar{z}}{z}$.

1.25 Write the following numbers in algebraic form:

i). $\sin(1 - i)$;

ii). $\cos(\pi + i)$;

iii). $\cosh(2 - i)$;

iv). $\sinh i$;

v). $\tan(i\pi)$;

vi). $\operatorname{Log}(1 + i\sqrt{3})$;

vii). $\operatorname{Log}(-1)$;

viii). $\operatorname{Log} e$;

ix). $\operatorname{Log}(-i)$;

x). 1^i;

xi). i^i;

xii). $(-\sqrt{3} + i)^i$.

1.26 Verify that $\cos^2 z + \sin^2 z = 1$ and $\cosh^2 z - \sinh^2 z = 1$.

1.27 Solve the equations:

i). $\sin z = \dfrac{4i}{3}$;

ii). $\cos z = \dfrac{3i}{4}$;

iii). $\sin z = \dfrac{5}{3}$;

iv). $\tan z = \dfrac{5i}{3}$;

v). $\sinh z = \dfrac{1}{2}i$;

vi). $\cosh z = \dfrac{1}{2}$;

vii). $\cot z = 2 + i$;

viii). $\sin z + \cos z = 1$;

ix). $\sin z - 2\cos z = 3$;

1.28 Determine the image through the function $f(z) = z^2$ of the sets $A_1 = \{z \ : \ \operatorname{Re} z = a\}$ and $A_2 = \{z \ : \ \operatorname{Im} z = b\}$, with $a, b \in \mathbb{R}$.

1.29 For $a, b \in \mathbb{R}$ prove the trigonometric formulas for $\cos(a+b)$ and $\sin(a+b)$ using complex numbers.

Solutions and answers

$\boxed{\textbf{1.24}}$ **i)** $u = e^{-x}\cos y$, $v = -e^{-x}\sin y$;
ii) $u = \sin x \cosh y$, $v = \cos x \sinh y$;
iii) $u = \cosh x \cos(y-1)$, $v = \sinh x \sin(y-1)$;
iv) $u = \sinh x \cos y$, $v = \cosh x \sin y$;
v) $u = \dfrac{\sin x \cos x}{\cosh^2 y - \sin^2 x}$, $v = \dfrac{\sinh y \cosh y}{\cosh^2 y - \sin^2 x}$
vi) $u = \dfrac{2x^2}{x^2+y^2}$, $v = -\dfrac{2xy}{x^2+y^2}$.

$\boxed{\mathbf{1.25}}$ **i)** $\sin(1-i) = \dfrac{e^{i+1}-e^{-i-1}}{2i} = \dfrac{e(\cos 1 + i\sin 1) - e^{-1}(\cos 1 - i\sin 1)}{2i}$ $= \sin 1 \cosh 1 - i\cos 1 \sinh 1$;

ii) $-\cosh 1$; **iii)** $\cos 1 \cosh 2 - i\sin 1 \sinh 2$; **iv)** $i\sin 1$; **v)** $i\,\dfrac{e^{2\pi}-1}{e^{2\pi}+1}$;

vi) $\mathrm{Log}\,(1+i\sqrt{3}) = \ln 2 + i(\frac{\pi}{3} + 2k\pi)$, $k \in \mathbb{Z}$;

vii) $(2k+1)\pi i$, $k \in \mathbb{Z}$; **viii)** $1 + 2k\pi i$, $k \in \mathbb{Z}$; **ix)** $(2k - \frac{1}{2})\pi i$, $k \in \mathbb{Z}$;

x) $1^i = e^{i\mathrm{Log}1} = e^{i(\ln 1 + 2k\pi i)} = e^{-2k\pi}$, $k \in \mathbb{Z}$;

xi) $e^{-\frac{\pi}{2}+2k\pi}$, $k \in \mathbb{Z}$; **xii)** $e^{-\frac{5\pi}{6}+2k\pi}(\cos\ln 2 + i\sin\ln 2)$, $k \in \mathbb{Z}$.

$\boxed{\mathbf{1.27}}$ **i)** From $\dfrac{e^{iz}-e^{-iz}}{2i} = \dfrac{4i}{3}$ we get $e^{iz} = -3$ or $e^{iz} = \dfrac{1}{3}$ and next $iz = \mathrm{Log}\,(-3)$ or $iz = \mathrm{Log}\,\dfrac{1}{3}$. The solution is $z = k\pi + i(-1)^k \ln 3$, $k \in \mathbb{Z}$;

ii) $e^{iz} = 2i$ or $e^{iz} = -\dfrac{i}{2}$, $z = (\dfrac{\pi}{2} + 2k\pi) - i\ln 2$ or $z = (\dfrac{3\pi}{2} + 2k\pi) + i\ln 2$. This can be also written as $z = \pm(\dfrac{\pi}{2} - i\ln 2) + 2k\pi$, $k \in \mathbb{Z}$;

iii) $\dfrac{\pi}{2} + 2k\pi \pm i\ln 3$, $k \in \mathbb{Z}$; **iv)** $\pm\dfrac{\pi}{2} + 2k\pi + i\ln 2$, $k \in \mathbb{Z}$;

v) $e^z = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ or $e^z = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ which gives $z = i(\dfrac{\pi}{6} + 2k\pi)$ or $z = i(\dfrac{5\pi}{6} + 2k\pi) = i(-\dfrac{\pi}{6} + \pi + 2k\pi)$. The general solution can be written then $(-1)^k\dfrac{\pi i}{6} + k\pi i$, $k \in \mathbb{Z}$; **vi)** $\pm\dfrac{\pi i}{3} + 2k\pi i$, $k \in \mathbb{Z}$;

vii) Denoting $e^{iz} = t$ we get $t^2 = 1 + i$ which gives $e^{iz} = \sqrt[4]{2}(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8})$ or $e^{iz} = \sqrt[4]{2}(\cos\dfrac{9\pi}{8} + i\sin\dfrac{\pi}{8})$, so the solution can be written in the form $z = \dfrac{\pi}{8} + k\pi - \dfrac{i}{4}\ln 2$, $k \in \mathbb{Z}$;

viii) Denoting $e^{iz} = t$ we get the equation $(1+i)t^2 - 2it - 1 + i = 0$ with the solutions $t_1 = 1$ and $t_2 = i$. From here $iz = \mathrm{Log}1$ or $iz = \mathrm{Log}i$ and finally the solution $z = 2k\pi$ or $z = \dfrac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$;

ix) $-\arctan\dfrac{1}{2} + (2k+1)\pi \pm \dfrac{i}{2}\ln 5$, $k \in \mathbb{Z}$.

$\boxed{\mathbf{1.28}}$ Let $z = x + iy$ and $w = u + iv$. From $w = z^2$ we get $u = x^2 - y^2$, $v = 2xy$. If $z \in A_1$ then $x = a$ so $u = a^2 - y^2$, $v = 2ay$. Eliminating y we obtain the equation of a parabola $u = a^2 - \dfrac{v^2}{4a^2}$. The image of A_2 is

$\{w = u + iv \ : \ u = \dfrac{v^2}{4b^2} - b^2\}.$

$\boxed{\mathbf{1.29}}$ We have $\cos(a+b) + i\sin(a+b) = e^{(a+b)i} = e^{ai}e^{bi}$. On the other hand, $e^{ai}e^{bi} = (\cos a + i\sin a)(\cos b + i\sin b) = \cos a\cos b - \sin a\sin b + i(\cos a\sin b + \sin a\cos b)$. Equating the corresponding real and imaginary parts we get the well-known trigonometric formulas.

1.4 Integral of a function of a complex variable. Cauchy's integral theorem

Let $D \subset \mathbb{C}$ be a domain, i.e., an open, connected set.

Suppose that the function $f : D \to \mathbb{C}$ is continuous on D. Denote $z = x+iy$ and $f(z) = u+iv$, where $u = u(x, y)$ and $v = v(x, y)$ are real-valued functions.

Let C be a piecewise smooth, oriented curve in D, closed or open.

Calculating the integral $\int_C f(z)\mathrm{d}z$ reduces to calculating two ordinary line integrals:

$$\int_C f(z)\mathrm{d}z = \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y) = \int_C (u\mathrm{d}x - v\mathrm{d}y) + i\int_C (v\mathrm{d}x + u\mathrm{d}y).$$

The complex integral has properties similar to those of line integrals. In particular

1) $\int_C [k_1 f_1(z) + k_2 f_2(z)]\mathrm{d}z = k_1 \int_C f_1(z)\mathrm{d}z + k_2 \int_C f_2(z)\mathrm{d}z,\ k_1, k_2 \in \mathbb{C}$.

2) $\int_C f(z)\mathrm{d}z = -\int_{C'} f(z)\mathrm{d}z$, where C' is the curve C described in the opposite direction.

3) $\left|\int_C f(z)\mathrm{d}z\right| \leq \sup_{z\in C} |f(z)| \cdot \mathrm{length}(C)$

4) Let F be a primitive of f in D (i.e., $F' = f$ in D). Let C be a curve lying in D, with initial point z_1 and end point z_2. Then

$$\int_C f(z)\mathrm{d}z = F(z_2) - F(z_1).$$

The fact that the value of the integral does not depend on the path of integration in this case (but only on the initial and end points of the curve C) is often expressed by writing it as $\int_{z_1}^{z_2} f(z)\mathrm{d}z$.

Example 1.4.1 The primitive function of $f(z) = z^2$ is $F(z) = \frac{1}{3}z^3 + \mathcal{C}$ in the whole plane. Hence (for every curve)

$$\int_{z_1}^{z_2} z^2\mathrm{d}z = \frac{z_2^3}{3} - \frac{z_1^3}{3}.$$

Remark 1.6 If the path of integration is a ray starting at z_0 or a circle centered at z_0, it is expedient to introduce the change of variables

$$z - z_0 = \rho e^{i\theta}.$$

In the first case we have $\theta =$ constant and ρ is the real-valued integration variable, while in the second $\rho =$ constant and θ is the real-valued integration variable.

Theorem 1.7 (Cauchy's Integral Theorem) *Let C be a closed oriented simple curve lying in D and such that its interior G lies in D. If f is holomorphic in D then $\int_C f(z)\,dz = 0$.*

Proof: Let $f = u + iv$. We consider only the case when the partial derivatives of u and v exist and are continuous. Then we have:

$$\int_C f(z)\mathrm{d}z = \int_C (u\mathrm{d}x - v\mathrm{d}y) + i\int_C (v\mathrm{d}x + u\mathrm{d}y).$$

By using Green's formula we deduce

$$\int_C f(z)\mathrm{d}z = \iint_G \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y.$$

Since f is supposed to be holomorphic, the Cauchy-Riemann conditions are satisfied, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

It follows that $\int_C f(z)\mathrm{d}z = 0$.

Theorem 1.8 (Cauchy's Integral Formula) *Let a closed simple curve C, positively oriented with respect to its interior G, lie in a domain D in which the function f is holomorphic. Let $z_0 \in G \subset D$. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz \tag{1.4.1}$$

and, more generally,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz. \tag{1.4.2}$$

Solved problems

1.30 Evaluate the integral $I = \int_C (3\bar{z} - z + i)\mathrm{d}z$, where C is one of the following curves connecting the points $z_1 = 0$ and $z_2 = 1 + i$:

a) the parabola $y = x^2$;

b) the straight line connecting z_1 and z_2.

Solution. We have

$$I = \int_C 2x\mathrm{d}x - (1-4y)\mathrm{d}y + i\int_C (1-4y)\mathrm{d}x + 2x\mathrm{d}y$$

a) We have $(C) : y = x^2$, with $x \in [0,1]$ and $\mathrm{d}y = 2x\mathrm{d}x$ so

$$\begin{aligned} I &= \int_0^1 2x\mathrm{d}x - (1-4x^2)\cdot 2x\mathrm{d}x + i\int_0^1 (1-4x^2)\mathrm{d}x + 2x\cdot 2x\mathrm{d}x \\ &= \int_0^1 8x^3\mathrm{d}x + i\int_0^1 \mathrm{d}x = 2+i. \end{aligned}$$

b) We have $(C) : y = x$, with $x \in [0,1]$ and $\mathrm{d}y = \mathrm{d}x$ so

$$I = \int_0^1 (-1+6x)\mathrm{d}x + i\int_0^1 (1-2x)\mathrm{d}x = 2.$$

The same integral can be calculated also using Remark 1.6. Since the path of integration is a ray at the angle $\frac{\pi}{4}$ relative to $0x$, we make the change of variable

$$z = \rho e^{i\frac{\pi}{4}},\ \ \rho \in [0,\sqrt{2}].$$

Then $\mathrm{d}z = e^{i\frac{\pi}{4}}\mathrm{d}\rho$ and

$$\begin{aligned} I &= \int_0^{\sqrt{2}} (3\rho e^{-i\frac{\pi}{4}} - \rho e^{i\frac{\pi}{4}} + i)e^{i\frac{\pi}{4}}\mathrm{d}\rho = (3 - e^{i\frac{\pi}{2}})\frac{\rho^2}{2}\Big|_0^{\sqrt{2}} + ie^{i\frac{\pi}{4}}\rho\Big|_0^{\sqrt{2}} = \\ &= 3 - i + i\Big(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)\sqrt{2} = 2. \end{aligned}$$

Notice that the integral of a non-analytic function generally depends on the shape of the integration path.

1.31 Evaluate the integral $I = \int_C (\bar{z} + z^3)\mathrm{d}z$, where C is the arc of the circle $|z| = 2$ situated in the first quadrant.

Solution. According to the Remark 1.6, we make the change of variable $z = 2e^{i\theta}$, with $\theta \in [0, \frac{\pi}{2}]$. Then $\mathrm{d}z = 2ie^{i\theta}\mathrm{d}\theta$, $\bar{z} = 2e^{-i\theta}$ and

$$I = \int_0^{\frac{\pi}{2}} (2e^{-i\theta} + 8e^{3i\theta}) \cdot 2ie^{i\theta}\mathrm{d}\theta = 4i \int_0^{\frac{\pi}{2}} (1 + 4e^{4i\theta})\mathrm{d}\theta =$$
$$= 4i\Big[\theta + 4 \cdot \frac{e^{4i\theta}}{4i}\Big]\Big|_0^{\frac{\pi}{2}} = 2\pi i.$$

1.32 Calculate the integral $I = \int_C \frac{e^z + 1}{z^2 - 4z}\mathrm{d}z$, where

a) $C: \ |z - 1| = \frac{1}{2}$;
b) $C: \ |z - 1| = 2$;
c) $C: \ |z - 1| = 4$.

Solution. Let us evaluate I by using Cauchy's integral formula.
a) In the domain bounded by the circle $|z-1| = \frac{1}{2}$ the integrand is an analytic function, whence in view of Cauchy's integral Theorem

$$\int_{|z-1|=\frac{1}{2}} \frac{e^z + 1}{z^2 - 4z}\mathrm{d}z = 0.$$

b) In the domain bounded by the circle $|z - 1| = 2$ there is only one point at which the denominator vanishes, $z_0 = 0$. We write the integral in the form

$$\int_{|z-1|=2} \frac{e^z + 1}{z^2 - 4z}\mathrm{d}z = \int_{|z-1|=2} \frac{\frac{e^z+1}{z-4}}{z}\mathrm{d}z$$

The function $f(z) = \frac{e^z + 1}{z - 4}$ is analytic in the given domain. Then

$$\int_{|z-1|=2} \frac{e^z + 1}{z^2 - 4z}\mathrm{d}z = 2\pi i \frac{e^z + 1}{z - 4}\Big|_{z=0} = 2\pi i \frac{2}{-4} = -\pi i.$$

c) There are two points, $z_1 = 0$ and $z_2 = 4$, where the denominator vanishes in the domain bounded by the circle $|z - 1| = 4$. We expand $\frac{1}{z^2 - 4z}$ in partial fractions:

$$\frac{1}{z^2 - 4z} = \frac{1}{4} \cdot \frac{1}{z - 4} - \frac{1}{4} \cdot \frac{1}{z}$$

Now we have

$$\int_{|z-1|=4} \frac{e^z+1}{z^2-4z}\mathrm{d}z = \frac{1}{4}\int_{|z-1|=4} \frac{e^z+1}{z-4}\mathrm{d}z - \frac{1}{4}\int_{|z-1|=4} \frac{e^z+1}{z}\mathrm{d}z =$$
$$= \frac{1}{4}2\pi i(e^z+1)|_{z=4} - 2\pi i\frac{1}{4}(e^z+1)|_{z=0} = \frac{\pi i}{2}(e^4-1).$$

1.33 Consider the integral $\displaystyle\int_{|z-1|=1} \frac{\cos \pi z}{(z^2-1)^2}\mathrm{d}z$.

Solution. The integrand is analytic in the domain $|z-1| \leq 1$ except at $z_0 = 1$. We write it in the form

$$\frac{\cos \pi z}{(z^2-1)^2} = \frac{\frac{\cos \pi z}{(z+1)^2}}{(z-1)^2}$$

and take $f(z) = \dfrac{\cos \pi z}{(z+1)^2}$. Putting $n = 1$ in Cauchy's formula, we obtain

$$\int_{|z-1|=1} \frac{\frac{\cos \pi z}{(z+1)^2}}{(z-1)^2}\mathrm{d}z = 2\pi i f'(1).$$

Next we find the derivative $f'(z) = \dfrac{-\pi(\sin \pi z)(z+1) - 2\cos \pi z}{(z+1)^3}$. Consequently $f'(1) = \dfrac{1}{4}$ and

$$\int_{|z-1|=1} \frac{\cos \pi z}{(z^2-1)^2}\mathrm{d}z = \frac{\pi i}{2}.$$

Proposed problems

1.34 Evaluate the integral $\displaystyle\int_C (1-2\bar{z})\mathrm{d}z$ where C is:

i). the broken line $z_1z_3z_2$, with $z_1 = 0$, $z_2 = 1+i$, $z_3 = i$;

ii). the broken line $z_1z_4z_2$, with $z_1 = 0$, $z_2 = 1+i$, $z_4 = 1$;

1.35 Evaluate the following integrals by using Cauchy's Integral Formula (all curves are circuited counterclockwise):

i). $\displaystyle\int_C \frac{e^z \sinh z^2}{z^3 - z}\mathrm{d}z$ where C is $|z - 2i| = 1$;

ii). $\displaystyle\int_C z^2 \cos z \sin 2iz\mathrm{d}z$ where C is $|z| = r$, $r > 0$;

iii). $\displaystyle\int_C \frac{ze^{2z}}{z - 1}\mathrm{d}z$ where C is $|z - 1| + |z + 1| = 4$;

iv). $\displaystyle\int_C \frac{e^z}{z^2 + 2z}\mathrm{d}z$ where C is $|z| = 1$;

v). $\displaystyle\int_C \frac{\sin z}{z^2(z - 4)}\mathrm{d}z$ where C is $|z| = 2$;

vi). $\displaystyle\int_C \frac{\mathrm{d}z}{z^2 + 25}, \quad C : |z| = 6$;

vii). $\displaystyle\int_C \frac{z \sinh z}{(z^2 - 1)^2}\mathrm{d}z$ where C is $|z| = 2$;

viii). $\displaystyle\int_C \frac{\cos(z + \pi i)}{z(e^z + 2)}\mathrm{d}z$ where C is $|z| = 3$;

ix). $\displaystyle\int_C \frac{\cosh^2 iz}{z^3}\mathrm{d}z$ where C is $|z| = r$, $r > 0$;

x). $\displaystyle\int_C \frac{z\mathrm{d}z}{(z - 2)^3(z + 4)}$ where C is $|z - 3| = 6$;

xi). $\displaystyle\int_C \frac{z^{100}e^{i\pi z}}{z^2 + 1}\mathrm{d}z$, where $C : x^2 + \dfrac{y^2}{4} = 1$;

xii). $\displaystyle\int_C \frac{\sinh \frac{\pi z}{2}}{(z + i)^{101}}\mathrm{d}z$, where C is $|z + 2i| = 2$;

Solutions and answers

$\boxed{\mathbf{1.34}}$ **i)** We have $I = \displaystyle\int_C (1 - 2x)\mathrm{d}x - 2y\mathrm{d}y + i\int_C 2y\mathrm{d}x + (1 - 2x)\mathrm{d}y$ and we split the path in two: z_1z_3 and z_3z_2. On the first segment we have $x = 0$, $\mathrm{d}x = 0$ and $y \in [0, 1]$ so $\displaystyle\int_{z_1z_2} f(z)\mathrm{d}z = \int_0^1 -2y\mathrm{d}y + i\int_0^1 \mathrm{d}y = -1 + i$. On the segment $[z_3z_2]$, $y=2$, $\mathrm{d}y=0$, $x\in[0,1]$ so

$$\int_{z_3z_2} f(z)\mathrm{d}z = \int_0^1 (1-2x)\mathrm{d}x + i\int_0^1 2\mathrm{d}x = 2i.$$

Finally (by the additivity relative to the curve) we obtain $I=-1+3i$;
ii) $I=-1-i$.

$\boxed{\mathbf{1.35}}$ **i)** 0 (since none of the points where the denominator vanishes belongs to the interior of the contour C, it means the integrand is an analytic function and we can use Theorem 1.7);
ii) 0; **iii)** $2\pi e^2 i$; **iv)** πi;
v) The only point from the interior of the circle C where the denominator vanishes is $z=0$, so choosing $f(z)=\dfrac{\sin z}{z-4}$ we can write the integral in the form
$\displaystyle\int_C \frac{\frac{\sin z}{z-4}}{z^2}\mathrm{d}z = \frac{2\pi i}{1!}f'(0) = 2\pi i\Big[-\frac{\sin z}{(z-4)^2}+\frac{\cos z}{z-4}\Big]\Big|_{z=0} = 2\pi i\frac{1}{-4} = -\frac{\pi i}{2}$;
vi) 0; **vii)** 0; **viii)** $\dfrac{2}{3}\pi i\cosh\pi$;
ix) $I=\dfrac{2\pi i}{2!}(\cosh^2 iz)''\big|_{z=0} = -2\pi i(\sinh^2 iz+\cosh^2 iz)\big|_{z=0} = -2\pi i$;
x) $-\dfrac{\pi i}{27}$;
xi) Decomposing $\dfrac{1}{z^2+1} = \dfrac{1}{2i}\Big(\dfrac{1}{z-i}-\dfrac{1}{z+i}\Big)$ the integral can be written $\displaystyle\frac{1}{2i}\int_C\frac{z^{100}e^{i\pi z}}{z-i}\mathrm{d}z-\frac{1}{2i}\int_C\frac{z^{100}e^{i\pi z}}{z+i}\mathrm{d}z = \frac{2\pi i}{2i}\big(z^{100}e^{i\pi z}\big|_{z=i}-z^{100}e^{i\pi z}\big|_{z=-i}\big) = \pi(e^{-\pi}-e^{\pi}) = -2\pi\sinh\pi$;
xii) We have $I=\dfrac{2\pi i}{100!}f^{(100)}(-i)$, where $f(z)=\sinh\dfrac{\pi z}{2}$. The derivative of order 100 is $f^{(100)}(z)=\Big(\dfrac{\pi}{2}\Big)^{100}\sinh\dfrac{\pi z}{2}$ and $\sinh\dfrac{\pi(-i)}{2}=-i$ so we finally get the value of the integral $I=\dfrac{2\pi i}{100!}\Big(\dfrac{\pi}{2}\Big)^{100}(-i)=\dfrac{\pi^{101}}{2^{99}100!}$.

1.5 Series of complex numbers

A series in the complex domain has the form

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty}(x_n + iy_n) \tag{1.5.1}$$

with x_n and y_n real numbers.

The series (1.5.1) is *convergent* if and only if both series with real terms $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent.

The series (1.5.1) is said to be *absolutely convergent* if $\sum_{n=1}^{\infty} |z_n|$ is convergent.

If a series is absolutely convergent, then it is also convergent. The converse is not always true.

A series of the form

$$\sum_{n=0}^{\infty} c_n z^n = c_0 = c_1 z + c_2 z^2 + \cdots + c_n z^n + \ldots \tag{1.5.2}$$

where $c_j \in \mathbb{C}$ and z is the complex variable, is called a *power series* in the complex domain.

A question arising naturally is: For which values of the variable z the corresponding series of complex numbers is convergent and for which it is divergent?

Theorem 1.9 (Abel's Theorem) *a) If the power series (1.5.2) is convergent for some value of z equal to z_0, then it is absolutely convergent for all values of z such that $|z| < |z_0|$.*

b) If the power series (1.5.2) is divergent for some value of z equal to z_1, then it is divergent for all values of z such that $|z| > |z_1|$.

Consequently, to every series (1.5.2) there corresponds a real number $r \geq 0$ ($r = +\infty$ is also admitted) such that (1.5.2) is convergent for all z satisfying $|z| < r$ and divergent for all z satisfying $|z| > r$.

The number r is called the *radius of convergence* of the power series. It can be found (if the limits exist) using one of the formulas

$$r = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \quad \text{or} \quad r = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

The disk $\{z \in \mathbb{C} \mid |z| < r\}$ is called the *disk of convergence* of the series (1.5.2).

Now consider the series

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots = s(z)$$

where z_0, a_0, a_1, $\ldots$ are complex numbers. It converges absolutely in its disk of convergence

$$K = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

In addition, it converges uniformly in every closed region contained in K. The sum $s(z)$ is a holomorphic function in K. Its derivatives can be calculated by differentiating the given series term by term. All these series have the same radius of convergence.

Solved problems

1.36 Study the convergence of the series:

i). $\displaystyle\sum_{n=1}^{\infty} \frac{e^{i2n}}{n\sqrt{n}}$

ii). $\displaystyle\sum_{n=1}^{\infty} \frac{e^{i\frac{\pi}{n}}}{n}$.

Solution. i). Since $e^{i2n} = \cos 2n + i \sin 2n$, we have to study the convergence of two series with real terms:

$$\sum_{n=1}^{\infty} \frac{\cos 2n}{n\sqrt{n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin 2n}{n\sqrt{n}}.$$

It can be proved, by comparison with a generalized harmonic series, that they are both absolutely convergent. Consequently, the series of complex numbers is absolutely convergent and is also convergent.

ii). We have to study the series with real terms $\sum_{n=1}^{\infty} \frac{\cos \frac{\pi}{n}}{n}$ and $\sum_{n=1}^{\infty} \frac{\sin \frac{\pi}{n}}{n}$. The first one has positive terms. We can compare it to the harmonic series:

$$\lim_{n\to\infty} \frac{\frac{\cos \frac{\pi}{n}}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \cos \frac{\pi}{n} = \cos 0 = 1,$$

so it is a divergent series. Hence, the given series is divergent.

1.37 Find the disk of convergence of the power series

i). $\sum_{n=0}^{\infty} \frac{z^n}{\sin in}$

ii). $\sum_{n=0}^{\infty} (1+i)^n (z-i)^n$.

Solution. i) We have $c_n = \frac{1}{\sin in} = \frac{2i}{e^{-n} - e^n}$ so

$$\left|\frac{c_n}{c_{n+1}}\right| = \left|\frac{2i}{e^{-n} - e^n} \cdot \frac{e^{-n-1} - e^{n+1}}{2i}\right| = \frac{e^{n+1}|1 - e^{-2n-2}|}{e^n|1 - e^{-2n}|}.$$

The radius of convergence is $r = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = e$ and the disk of convergence $K = \{z \in \mathbb{C} \mid |z| < e\}$.

ii) We have $\sqrt[n]{|c_n|} = \sqrt[n]{|(1+i)^n|} = |1+i| = \sqrt{2}$, so $r = \frac{1}{\sqrt{2}}$. The disk of convergence has the center at i, $K = \{z \in \mathbb{C} \mid |z - i| < \frac{1}{\sqrt{2}}\}$.

Proposed problems

1.38 Find the radius of convergence of each of the power series:

i). $\sum_{n=0}^{\infty} (\cos in) z^n$

ii). $\sum_{n=0}^{\infty} i^n z^n$

iii). $\displaystyle\sum_{n=0}^{\infty}\Big(\frac{z}{n+i}\Big)^n$

iv). $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{(1+i)^n}$

v). $\displaystyle\sum_{n=0}^{\infty}e^{in\pi}z^n$

vi). $\displaystyle\sum_{n=0}^{\infty}(\cos\frac{i}{n})^n z^n.$

1.39 Find the disk of convergence of each of the power series:

i). $\displaystyle\sum_{n=1}^{\infty}(\frac{1}{n}+in)(z+1+i)^n$

ii). $\displaystyle\sum_{n=1}^{\infty}\frac{(z-i)^n}{2^n}.$

1.40 Find the sum of the series $1+2z+3z^2+\dots+z^n+\dots$.

Solutions and answers

$\boxed{\mathbf{1.38}}$ **i)** $r=\displaystyle\lim_{n\to\infty}\frac{e^n+e^{-n}}{2}\cdot\frac{2}{e^{n+1}+e^{-n-1}}=\lim_{n\to\infty}\frac{e^n(1+e^{-2n})}{e^{n+1}(1+e^{-2n-2})}=\frac{1}{e}$;
ii) $r=\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt[n]{|i^n|}}=1$; **iii)** $r=\displaystyle\lim_{n\to\infty}|n+i|=\infty$; **iv)** $r=\sqrt{2}$;
v) $r=\displaystyle\lim_{n\to\infty}\frac{|e^{in\pi}|}{|e^{(n+1)\pi}|}=\frac{|\cos n\pi+i\sin n\pi|}{|\cos(n+1)\pi+i\sin(n+1)\pi|}=1$; **vi)** $r=1$.

$\boxed{\mathbf{1.39}}$ **i)** $|z+1+i|<1$; **ii)** $|z-i|<2$.

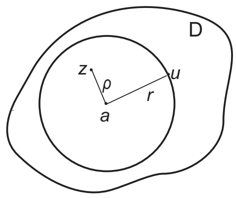
$\boxed{\mathbf{1.40}}$ For $|z|<1$ we have $1+z+z^2+z^3+\dots+z^n+\dots=\dfrac{1}{1-z}$. Hence
$1+2z+3z^2+\dots=(1+z+z^2+z^3+\dots)'=\Big(\dfrac{1}{1-z}\Big)'=\dfrac{1}{(z-1)^2}.$

1.6 Taylor and Laurent series

Let $f : D \to \mathbb{C}$ be holomorphic in the domain D and let $a \in D$. Let $C = \{u \in \mathbb{C} \mid |u-a| = r\} \subset D$, let $\rho < r$. By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(u)}{u-z}\mathrm{d}u,$$

for each $z \in \mathbb{C}$ such that $|z-a| = \rho < r = |u-a|$.



On the other hand,

$$\frac{1}{u-z} = \frac{1}{u-a-(z-a)} = \frac{1}{u-a}\cdot\frac{1}{1-\frac{z-a}{u-a}} = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(u-a)^{n+1}},$$

since $\left|\dfrac{z-a}{u-a}\right| = \dfrac{\rho}{r} < 1$. It follows that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C\left(\sum_{n=0}^{\infty}\frac{(z-a)^n}{(u-a)^{n+1}}\right)f(u)\mathrm{d}u = \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-a)^n\int_C\frac{f(u)}{(u-a)^{n+1}}\mathrm{d}u = \\ &= \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}\cdot\frac{n!}{2\pi i}\int_C\frac{f(u)}{(u-a)^{n+1}}\mathrm{d}u. \end{aligned}$$

Since $\dfrac{n!}{2\pi i}\displaystyle\int_C\frac{f(u)}{(u-a)^{n+1}}\mathrm{d}u = f^{(n)}(a)$ (see 1.4.2) we have finally

$$f(z) = \sum_{n-0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n. \tag{1.6.1}$$

This is the *Taylor series* of the holomorphic function f in the neighborhood of the point a.

Example 1.6.1 Let $f(z) = e^z$ and $a = 0$.
Then $f^{(n)}(z) = e^z$, hence $f^{(n)}(0) = 1$. We have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \ldots \tag{1.6.2}$$

The radius of convergence is

$$r = \lim_{n\to\infty} \frac{1}{n!} \frac{(n+1)!}{1} = \lim_{n\to\infty} (n+1) = \infty.$$

Hence the formula (1.6.2) holds for all $z \in \mathbb{C}$.

Example 1.6.2

$$\begin{aligned}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} = \\
&= \frac{1}{2}\left[1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots + 1 - \frac{iz}{1!} + \frac{(iz)^2}{2!} - \frac{(iz)^3}{3!} + \ldots\right] = \\
&= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots, \quad z \in \mathbb{C}.
\end{aligned}$$

Example 1.6.3 Similarly,

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots, \quad z \in \mathbb{C}.$$

Example 1.6.4 $\cosh z = \dfrac{e^z + e^{-z}}{2} = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \dfrac{z^6}{6!} + \ldots, z \in \mathbb{C}.$

Example 1.6.5 $\sinh z = \dfrac{z}{1!} + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \dfrac{z^7}{7!} + \ldots, z \in \mathbb{C}.$

Remark 1.10 Formula (1.6.1) often leads to complicated calculations, hence for practical purposes it is usually avoided. The common practice is to use, when possible, the known expansions into Taylor series of the elementary functions.

Example 1.6.6 Expand in powers of of $z - 2$ the function $f(z) = \dfrac{1}{z}$.

We know that $1 - u + u^2 - \cdots = \dfrac{1}{1+u}$ for $|u| < 1$. Hence we have

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{2 + (z-2)} = \frac{1}{2} \cdot \frac{1}{1 + \frac{z-2}{2}} = \\
&= \frac{1}{2}\Big[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \ldots\Big] = \\
&= \frac{1}{2} - \frac{1}{2^2}(z-2) + \frac{1}{2^3}(z-2)^3 - \frac{1}{2^4}(z-2)^3 + \ldots
\end{aligned}$$

for $\left|\dfrac{z-2}{2}\right| < 1$, i.e. $|z-2| < 2$.

Now let f be holomorphic in the annulus M with the center z_0, inner radius r_1 and outer radius r_2, i.e. $M = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$. Then for $z \in M$ we have

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n \tag{1.6.3}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z, \;\; n = 0, \pm 1, \pm 2, \ldots, \tag{1.6.4}$$

C being an arbitrary circle with center at z_0, lying in M and positively oriented with respect to its interior.

Formula (1.6.3) gives us the expansion of f into its *Laurent series*. By the convergence of this series we mean the convergence of both series

$$\sum_{n=0}^{+\infty} a_n (z - z_0)^n, \qquad \sum_{n=1}^{+\infty} \frac{a_{-n}}{(z-z_0)^n}. \tag{1.6.5}$$

The first series is called the *regular part*, the second one the *principal part* of the Laurent series.

The regular part converges *inside* a certain circle with center z_0, the principal part *outside* a certain circle. The series (1.6.3) then converges in the common annulus.

Remark 1.11 It is not always necessary, if we expand a function into its Laurent series, to calculate the coefficients according to (1.6.4). We can often - especially in the case of simple rational functions - use a similar method as in the case of the Taylor series, i.e., write the given function in a form involving fractions each having the form of the sum of a geometric series.

Example 1.6.7 Let us expand the function $f(z) = \dfrac{1}{(z-1)(z-3)}$ in a Laurent series with center at $z = 0$ (i.e., in powers of z) and converging in the annulus $1 < |z| < 3$. In this annulus we have

$$\left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{3}\right| < 1$$

hence

$$\begin{aligned} f(z) &= \frac{1}{(z-1)(z-3)} = -\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z-3}\right) = \\ &= -\frac{1}{2z} \cdot \frac{1}{1-\dfrac{1}{z}} - \frac{1}{6} \cdot \frac{1}{1-\dfrac{z}{3}} = -\frac{1}{2z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{z^n}{3^n}. \end{aligned}$$

Finally, $f(z) = -\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \frac{1}{6}\sum_{n-0}^{\infty}\frac{1}{3^n}z^n$

We say that a point z_0 is a *regular point (ordinary point)* of a function f if there exists a neighborhood for z_0 such that f is holomorphic in this neighborhood. A point which is not regular is called a *singular point* of f.

For example, the function $f(z) = \dfrac{1}{z}$ has only one singular point, namely $z = 0$.

A singular point such that there is no other singular point in a sufficiently small neighborhood of it is called an *isolated singular point.*

Consider the Laurent expansion (1.6.3) of a function f, converging in a neighborhood of an isolated singular point z_0 (naturally, the point z_0 is excluded from this neighborhood). Exactly three cases can then arise:

I) If the Laurent series has only a regular part, we say that f has a *removable singularity* at z_0. For example, the function

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

has a removable singularity at $z = 0$.

II) If the principal part of the Laurent series has an infinite number of terms, we say that f has an *essential singularity* at z_0.

III) If there exists a $k > 0$ such that in (1.6.3) we have $a_{-k} \neq 0$ but $a_{-j} = 0$ for all $j > k$ (hence the principal part of the Laurent series has only a finite number of terms), we say that f has a *pole of order* k at z_0.

It is easy to see that z_0 is a pole of order k if and only if we can write

$$f(z) = \frac{\varphi(z)}{(z - z_0)^k}$$

with φ holomorphic in a neighborhood of z_0 and $\varphi(z_0) \neq 0$.

Finally, suppose that equality (1.6.3) holds in the neighborhood of the infinity $|z - z_0| > r_1$. By the substitution

$$z - z_0 = \frac{1}{t}$$

we reduce the investigation of a Laurent series in the neighborhood of infinity to the investigation of another Laurent series at $t = 0$. If this series has a pole, an essential singularity or a removable singularity at the point $t = 0$, we say that the original series (1.6.3) has a pole, an essential singularity or a removable singularity at infinity, respectively.

A holomorphic function given by a power series with an infinite number of terms and radius of convergence $r = +\infty$ is called an *entire function*. As examples we have the functions e^z, $\sin z$, $\cos z$.

A holomorphic function which has no singular points in the complex plane other than poles is called a *meromorphic function*. For example, the rational functions are meromorphic.

Solved problems

1.41 Expand the functions given below in Taylor series in a neighborhood of z_0 and find the radii of convergence for these series:

i). $f(z) = e^z \sin z, \;\; z_0 = 0$

ii). $f(z) = \dfrac{z-1}{z-2}, \ \ z_0 = 0$

iii). $f(z) = \dfrac{z-1}{z-2}, \ \ z_0 = i$

Solution. i) We have

$$f(z) = e^z \cdot \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}[e^{(1+i)z} - e^{(1-i)z}]$$

and using the series for the exponential function

$$\begin{aligned} f(z) &= \frac{1}{2i}\Big(\sum_{n=0}^{\infty} \frac{z^n(1+i)^n}{n!} - \sum_{n=0}^{\infty} \frac{z^n(1-i)^n}{n!}\Big) = \\ &= \frac{1}{2i}\sum_{n=0}^{\infty} \frac{z^n}{n!}(\sqrt{2})^n\Big(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} - \cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\Big) = \\ &= \sum_{n=0}^{\infty}(\sqrt{2})^n \sin\frac{n\pi}{4} \cdot \frac{z^n}{n!}. \end{aligned}$$

The radius of convergence is $r = \infty$.

ii) $f(z) = 1 + \dfrac{1}{z-2} = 1 - \dfrac{1}{2} \cdot \dfrac{1}{1-\frac{z}{2}}$. Using the geometric series, we get

$$f(z) = 1 - \frac{1}{2}\Big(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \dots\Big) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{z^n}{2^{n+1}}.$$

The radius of convergence is determined from the condition $\left|\dfrac{z}{2}\right| < 1$, so $r = 2$.

iii) Since we wish to expand the function in a series of powers of $z - i$, we write

$$f(z) = 1 + \frac{1}{z-i+i-2} = 1 + \frac{1}{i-2} \cdot \frac{1}{1+\frac{z-i}{i-2}} = \frac{i-1}{i-2} + \sum_{n=1}^{\infty}(-1)^n \frac{(z-i)^n}{(i-2)^{n+1}}.$$

From $\left|\dfrac{z-i}{i-2}\right| < 1$ follows that $r = \sqrt{5}$.

1.42 Determine the various Laurent expansions of the function
$f(z) = \dfrac{2z-1}{z^2-z-6}$, with $z_0 = 0$.

Solution. Decomposing into partial fractions, we write

$$f(z) = \frac{1}{z+2} + \frac{1}{z-3}.$$

If $|z| < 2$, we have in fact an expansion in a Taylor series (only a regular part):

$$f(z) = \frac{1}{2} \cdot \frac{1}{1+\frac{z}{2}} - \frac{1}{3} \cdot \frac{1}{1-\frac{z}{3}} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

For $2 < |z| < 3$, the first series above is not convergent, so we write the function in another form:

$$f(z) = \frac{1}{z} \cdot \frac{1}{1+\frac{2}{z}} - \frac{1}{3} \cdot \frac{1}{1-\frac{z}{3}} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

For $|z| > 3$ we have

$$f(z) = \frac{1}{z} \cdot \frac{1}{1+\frac{2}{z}} - \frac{1}{z} \cdot \frac{1}{1-\frac{3}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n.$$

1.43 Expand in Laurent series and determine the type of the corresponding singular points:

i). $f(z) = \dfrac{e^{z^2}-1}{z^2}$, $z_0 = 0$, $0 < |z| < +\infty$

ii). $f(z) = (z-1)\cos\dfrac{1}{z-2}$, $z_0 = 2$, $0 < |z-2| < +\infty$

iii). $f(z) = \dfrac{1-\cos z}{z^7}$, $z_0 = 0$, $0 < |z| < +\infty$

Solution. i) We have

$$f(z) = \frac{1}{z^2}\left(1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \cdots - 1\right) = \sum_{n=1}^{\infty}\frac{z^{2n-2}}{n!}.$$

Since the series has only a regular part, the point $z_0 = 0$ is a removable singular point.

ii) We write

$$f(z) = (z-2+1)\cos\frac{1}{z-2} = (z-2)\cos\frac{1}{z-2} + \cos\frac{1}{z-2} =$$
$$= (z-2)\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\frac{1}{(z-2)^{2n}} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\frac{1}{(z-2)^{2n}} =$$
$$= (z-2) + 1 - \frac{1}{2!(z-2)} - \frac{1}{2!(z-2)^2} + \frac{1}{4!(z-2)^3} + \frac{1}{4!(z-2)^4} - \dots.$$

Having an infinite number of terms in the principal part of the series, $z_0 = 2$ is an essential singularity.

iii) $f(z) = \frac{1}{2!z^5} - \frac{1}{4!z^3} + \frac{1}{6!z} - \frac{z}{8!} + \frac{z^3}{10!} - \dots$, so $z_0 = 0$ is a pole of order 5.

Proposed problems

1.44 Expand the function f in Taylor series in a neighborhood of z_0 and find the radius of convergence of the series:

i). $f(z) = \cos^2 z, \;\; z_0 = 0$

ii). $f(z) = \cosh z \cos z, \;\; z_0 = 0$

iii). $f(z) = \dfrac{2z+7}{3z^2-17z-6}, \; z_0 = 0$

iv). $f(z) = \dfrac{1}{z^2+i}, \;\; z_0 = 0$

v). $f(z) = \dfrac{1}{3z+5}, \;\; z_0 = -1$

vi). $f(z) = \dfrac{1}{z^2-2z+2}, \;\; z_0 = 0$

1.45 Expand the function f in Laurent series and determine the type of the corresponding singular points:

i). $f(z) = z^3 e^{\frac{1}{z}}, \;\; z_0 = 0, \;\; 0 < |z| < +\infty$

ii). $f(z) = \dfrac{2\sin^2 z}{z^5}, \;\; z_0 = 0, \;\; 0 < |z| < +\infty$

iii). $f(z) = ze^{\frac{1}{z+i}}$, $z_0 = -i$, $0 < |z-i| < +\infty$

iv). $f(z) = \dfrac{1-e^{-z}}{z}$, $z_0 = 0$, $0 < |z| < +\infty$

Solutions and answers

$\boxed{\mathbf{1.44}}$ **i)** $f(z) = \dfrac{1+\cos 2z}{2} = 1 + \sum_{n=1}^{\infty}(-1)^n \dfrac{2^{2n-1}}{(2n)!} z^{2n}, \;\; r = \infty;$

ii) $f(z) = \sum_{n=0}^{\infty}(-1)^n 2^{2n} \dfrac{z^{4n}}{(4n)!}, \;\; r = \infty;$

iii) $f(z) = -\dfrac{1}{6} \cdot \dfrac{1}{1-\frac{z}{6}} - \dfrac{1}{3z+1} = -\dfrac{1}{6}\sum_{n=0}^{\infty}\left(\dfrac{z}{6}\right)^n - \sum_{n=0}^{\infty}(-1)^n(3z)^n =$

$= \sum_{n=0}^{\infty}\left((-1)^{n+1}3^n - \dfrac{1}{6^{n+1}}\right)z^n, \;\; r = \dfrac{1}{3};$

iv) $f(z) = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{i^{n+1}} z^{2n} = -i + z^2 + iz^4 - z^6 - iz^8 + z^{10} + \ldots, \;\; r = 1;$

v) $f(z) = \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{2}\right)^n (z+1)^n, \;\; r = \dfrac{2}{3}.$

vi) $f(z) = \dfrac{1}{2i}\left(\dfrac{1}{z-1-i} - \dfrac{1}{z-1+i}\right) = \dfrac{1}{2i}\sum_{n=0}^{\infty}\left(\dfrac{1}{(1-i)^{n+1}} - \dfrac{1}{(1+i)^{n+1}}\right)z^n =$

$= \sum_{n=0}^{\infty}\dfrac{1}{\sqrt{2}^{n+1}} \sin\dfrac{(n+1)\pi}{4} z^n, \;\; r = \sqrt{2}.$

$\boxed{\mathbf{1.45}}$ **i)** $f(z) = z^3 + z^2 + \dfrac{1}{2}z + \dfrac{1}{6} + \sum_{n=1}^{\infty}\dfrac{1}{(n+3)!} \cdot \dfrac{1}{z^n}$, 0 is an essential singularity;

ii) 0 is a pole of order 3;

iii) $f(z) = (z+i) + 1 - i + \left(\dfrac{1}{2!} - \dfrac{i}{1!}\right)\dfrac{1}{z+i} - \left(\dfrac{1}{3!} - \dfrac{i}{2!}\right)\dfrac{1}{(z+i)^2} + \ldots$, i is an essential singular point;

iv) $f(z) = 1 - \dfrac{z}{2!} + \dfrac{z^2}{3!} - \ldots$, $z_0 = 0$ is a removable singular point.

1.7 The residue of a function. The Residue Theorem

Let $f(z)$ be a holomorphic function and z_0 an isolated singular point of f. Then we can develop $f(z)$ in the neighborhood of z_0 (for $z \neq z_0$) in its Laurent series:

$$
\begin{aligned}
f(z) = &\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \\
&= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots
\end{aligned}
$$

The number a_{-1} is called *the residue of the function* $f(z)$ at the point z_0. It is denoted by $\operatorname*{Res}_{z=z_0} f(z)$.

Theorem 1.12 *Let C be a circle centered at z_0 and so small that it does not exceed the region of analyticity of $f(z)$ and has no other singular points in its interior. Then*

$$
\frac{1}{2\pi i}\int_C f(z)dz = \mathop{Res}_{z=z_0} f(z). \tag{1.7.1}
$$

Proof: Using $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$ with $n=-1$ we deduce

$$
\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i}\int_C f(z)\mathrm{d}z.
$$

Theorem 1.13 *If z_0 is a pole of order $n \geq 1$, then*

$$
\mathop{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\big((z-z_0)^n f(z)\big) \tag{1.7.2}
$$

Proof: We know that $f(z) = \dfrac{\varphi(z)}{(z-z_0)^n}$ where z_0 is an ordinary point for $\varphi(z)$ and $\varphi(z_0) \neq 0$. By using Theorem 1.12 we have

$$
\begin{aligned}
\operatorname*{Res}_{z=z_0} f(z) &= \frac{1}{2\pi i}\int_C \frac{\varphi(z)}{(z-z_0)^n}\mathrm{d}z = \frac{1}{(n-1)!}\cdot\frac{(n-1)!}{2\pi i}\int_C \frac{\varphi(z)}{(z-z_0)^n}\mathrm{d}z \\
&= \frac{1}{(n-1)!}\varphi^{(n-1)}(z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\big((z-z_0)^n f(z)\big).
\end{aligned}
$$

Remark 1.14 Let $f(z) = \dfrac{g(z)}{h(z)}$, with g and h holomorphic functions. If z_0 is a pole of order one, $h(z_0) = 0$ and $h'(z_0) \neq 0$ then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)}. \tag{1.7.3}$$

The formula (1.7.1) can be written as

$$\int_C f(z)\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

This result can be extended. We have the following fundamental result.

Theorem 1.15 *Let C be a simple piecewise smooth closed curve, positively oriented with respect to its interior D.*

Let $f(z)$ be a function, holomorphic in D with the exception of a finite number of singular points $z_1, \ldots, z_n$, and continuously extensible on C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathop{Res}_{z=z_k} f(z). \tag{1.7.4}$$

Solved problems

1.46 Find the residues of $f(z) = \dfrac{e^{\frac{1}{z}}}{1-z}$ at its singular points.

Solution. The singular points are $z = 1$ and $z = 0$. Since $z = 1$ is a pole of order one, we have

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)f(z) = -e.$$

To establish the type of the singular point $z = 0$ we expand $f(z)$ in a Laurent series. We have

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots \quad \text{and} \quad \frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$$

Multiplying these series, we obtain

$$\frac{e^{\frac{1}{z}}}{1-z} = \cdots + c_{-2}\frac{1}{z^2} + \left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)\frac{1}{z} + c_0 + c_1 z + \ldots.$$

Hence $\operatorname*{Res}_{z=0} f(z) = 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots = e - 1.$

1.47 Evaluate the integral $\displaystyle\int_{|z|=4} \frac{e^z-1}{z^2+z}\mathrm{d}z$.

Solution. In the disk $|z|<4$ the function $f(z)=\dfrac{e^z-1}{z^2+z}$ is holomorphic everywhere except at points $z_1=0$ and $z_2=-1$. According to the residue theorem,

$$\int_{|z|=4} \frac{e^z-1}{z^2+z}\mathrm{d}z = 2\pi i\Big(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z)\Big).$$

We have

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} z f(z) = \lim_{z\to 0}\frac{e^z-1}{z+1} = 0$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}\frac{e^z-1}{z} = 1-\frac{1}{e}.$$

Hence $\displaystyle\int_{|z|=4}\frac{e^z-1}{z^2+z}\mathrm{d}z = 2\pi i\big(1-\frac{1}{e}\big)$.

1.48 Evaluate the integral $I=\displaystyle\int_{|z|=2}\frac{e^{\frac{\pi}{z-1}}}{z^2-(1-i)z-i}\mathrm{d}z$.

Solution. The zeroes of $z^2-(1-i)z-i$ are $z_1=-i$, $z_2=1$.

The function can be written $f(z)=e^{\frac{\pi}{z-1}}\cdot\dfrac{1}{(z-1)(z+i)}$, it is holomorphic in the disk $|z|=4$, with the exception of the points z_1 and z_2. According to the residue theorem,

$$I = 2\pi i\Big(\operatorname*{Res}_{z=-i} f(z) + \operatorname*{Res}_{z=1} f(z)\Big).$$

The point $z_1=-i$ is a pole of order one, so using (1.7.3),

$$\operatorname*{Res}_{z=-i} f(z) = \frac{e^{\frac{\pi}{z-1}}}{(z^2-(1-i)z-i)'}\Big|_{z=-i} = \frac{e^{\frac{\pi}{z-1}}}{2z-1+i}\Big|_{z=-i} = -\frac{1}{1+i}e^{-\frac{\pi}{1+i}}.$$

Since the point $z_2=1$ is an essential singularity, we have to expand $f(z)$ in Laurent series (of powers of $z-1$) to find the residue at 1.

$$f(z) = \frac{1}{z-1}\cdot\frac{1}{z-1+1+i}\cdot e^{\frac{\pi}{z-1}} = \frac{1}{z-1}\cdot\frac{1}{1+i}\cdot\frac{1}{1+\frac{z-1}{1+i}}\cdot e^{\frac{\pi}{z-1}}$$

$$= \frac{1}{1+i}\cdot\frac{1}{z-1}\Big[1-\frac{z-1}{1+i}+\frac{(z-1)^2}{(1+i)^2}-\dots\Big]\Big[1+\frac{\pi}{1!(z-1)}+\frac{\pi^2}{2!(z-1)^2}+\dots\Big]$$

$$= \frac{1}{1+i}\cdot\Big[\frac{1}{z-1}-\frac{1}{1+i}+\frac{(z-1)}{(1+i)^2}-\dots\Big]\Big[1+\frac{\pi}{1!(z-1)}+\frac{\pi^2}{2!(z-1)^2}+\dots\Big].$$

The residue at 1 is the coefficient of $(z-1)^{-1}$:

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{1+i}\Big(1 - \frac{\pi}{1!(1+i)} + \frac{\pi^2}{2!(1+i)^2} - \dots\Big) = \frac{1}{1+i}e^{-\frac{\pi}{1+i}}.$$

Hence $I = 0$.

Proposed problems

1.49 Find the residues at the singular points of the following functions:

i). $f(z) = \dfrac{z^2}{(1+z)^3}$

ii). $f(z) = \dfrac{1}{1+e^z}$

iii). $f(z) = \dfrac{1}{\sin \pi z}$

iv). $f(z) = \dfrac{\cos z}{(z-1)^2}$.

1.50 Evaluate the following integrals:

i). $\displaystyle\int_{|z-1|=1} \frac{\mathrm{d}z}{z^4+1}$

ii). $\displaystyle\int_{|z|=2} \frac{\sin z}{(z+1)^3}\mathrm{d}z$

iii). $\displaystyle\int_C \frac{\mathrm{d}z}{z\cos z^2}$, $C : x^2 + \dfrac{y^2}{4} = 1$

iv). $\displaystyle\int_{|z|=1} (z+1)e^{\frac{1}{z}}\mathrm{d}z$

v). $\displaystyle\int_{|z-i|=3} \frac{z^2\mathrm{d}z}{(z+3)(z-2)^2}$

vi). $\displaystyle\int_{|z|=3} z^2 \sin\frac{1}{z-i}\mathrm{d}z$

Solutions and answers

$\boxed{\mathbf{1.49}}$ **i)** $\operatorname*{Res}_{z=-1} f(z) = 1$; **ii)** $\operatorname*{Res}_{z=(2k+1)\pi i} f(z) = -1,\ k \in \mathbf{Z}$;
iii) $\operatorname*{Res}_{z=k} f(z) = \dfrac{(-1)^k}{\pi},\ k \in \mathbf{Z}$; **iv)** $\operatorname*{Res}_{z=1} f(z) = -\sin 1$.

$\boxed{\mathbf{1.50}}$ **i)** $z_1 = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$ and $z_2 = \cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}$ are the poles of order one situated in the interior of the circle $|z-1| = 1$. The value of the integral is $-\dfrac{\pi i}{\sqrt{2}}$;
ii) $\pi i \sin 1$;
iii) The poles inside the given curve are 0, $i\sqrt{\dfrac{\pi}{2}}$, $-i\sqrt{\dfrac{\pi}{2}}$ and the value of the integral is $2\pi i\big(1 - \dfrac{2}{\pi}\big)$;
iv) $3\pi i$;
v) The only singular point inside the given circle is $z_0 = 2$, the value of the integral is $\dfrac{32\pi i}{25}$;
vi) We have $\sin\dfrac{1}{z-i} = \dfrac{1}{z-i} - \dfrac{1}{3!(z-i)^3} + \dfrac{1}{5!(z-i)^5} + \dots$ and
$z^2 = -1 + 2i(z-i) + (z-i)^2$.
Multiplying these, we get the coefficient of $\dfrac{1}{z-i}$, that is the residue at $z = i$ to be $-\dfrac{7}{6}$. The integral is $-\dfrac{7\pi i}{3}$.

1.8 Applications of the residue theorem

We present in this section some situations when real integrals can be computed using the residue theorem.

1) Consider the integral

$$\int_0^{2\pi} R(\cos x, \sin x)\mathrm{d}x$$

where R is a rational function of $\cos x$ and $\sin x$ that is bounded in the interval of integration.

We denote $e^{ix} = z$. Then $\mathrm{d}z = ie^{ix}\mathrm{d}x$, hence $\mathrm{d}x = \dfrac{\mathrm{d}z}{iz}$. By the definitions of the functions sine and cosine in the complex plane we get

$$\cos x = \frac{z^2+1}{2z}, \quad \sin x = \frac{z^2-1}{2iz}.$$

Here, obviously, $|z| = 1$ and $0 \le x \le 2\pi$. This means that

$$\int_0^{2\pi} R(\cos x, \sin x)\mathrm{d}x = \int_{|z|=1} R\Big(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\Big)\frac{1}{iz}\mathrm{d}z.$$

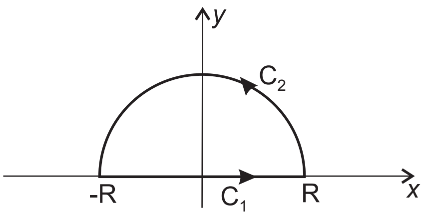
According to the residue theorem, the value of the last integral is $2\pi i\sigma$, where σ stands for the sum of the residues at the poles that lie inside the circle $|z| = 1$.

2) Consider the integral

$$I = \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\mathrm{d}x,$$

where P and Q are polynomials of degrees m and n respectively, $n \ge m+2$ and $Q(x) \ne 0$ for all $x \in \mathbf{R}$. Clearly this integral is convergent.

Let us draw a semi-circle C_2 with center at the origin and radius R and write $C = C_1 \cup C_2$ where C_1 is the segment $[-R, R]$ of the x-axis.



The orientation of C is evident from the above figure. We can write

$$\int_C \frac{P(z)}{Q(z)}\mathrm{d}z = \int_{C_1} \frac{P(z)}{Q(z)}\mathrm{d}z + \int_{C_2} \frac{P(z)}{Q(z)}\mathrm{d}z = \int_{-R}^{R} \frac{P(x)}{Q(x)}\mathrm{d}x + \int_{C_2} \frac{P(z)}{Q(z)}\mathrm{d}z.$$

It follows that

$$I = \lim_{R\to+\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)}\mathrm{d}x = \lim_{R\to+\infty} \int_C \frac{P(z)}{Q(z)}\mathrm{d}z - \lim_{R\to+\infty} \int_{C_2} \frac{P(z)}{Q(z)}\mathrm{d}z.$$

For sufficiently large R, $\displaystyle\int_C \frac{P(z)}{Q(z)}\mathrm{d}z = 2\pi i\sigma$, where σ stands for the sum of the residues of the function $f(z) = \dfrac{P(z)}{Q(z)}$ at all the poles that lie in the upper half-plane.

On the other hand,

$$\Big|\int_{C_2} \frac{P(z)}{Q(z)}\mathrm{d}z\Big| \leq \max_{z\in C_2}\Big|\frac{P(z)}{Q(z)}\Big| \cdot \mathrm{length}(C_2) = \pi R \max_{\theta\in[0,\pi]} \frac{|P(Re^{i\theta})|}{|Q(Re^{i\theta})|}.$$

Since $n > m+1$, it follows that $\displaystyle\lim_{R\to+\infty} \pi \max_{\theta\in[0,\pi]} \frac{|RP(Re^{i\theta})|}{|Q(Re^{i\theta})|} = 0$ and thus $\displaystyle\lim_{R\to+\infty}\int_{C_2} \frac{P(z)}{Q(z)}\mathrm{d}z = 0.$

We conclude that

$$I = \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\mathrm{d}x = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} \frac{P(z)}{Q(z)},$$

where $z_1, z_2, \ldots, z_k$ are all the poles of $\dfrac{P(z)}{Q(z)}$ that lie in the upper half-plane (i.e. $\operatorname{Im} z_j > 0$).

3) Consider the integral

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \cdot e^{i\lambda x}\mathrm{d}x$$

where $\lambda > 0$ and P, Q are polynomial functions with degree$P <$ degree$Q$. If the only real zeroes of Q are simple ones (if $x_0 \in \mathbb{R}$ is such that $Q(x_0) = 0$, then $Q'(x_0) \neq 0$) then the integral can be evaluated by the formula

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \cdot e^{i\lambda x}\mathrm{d}x = 2\pi i \sum_{\mathrm{Im} z_k>0} \operatorname*{Res}_{z=z_k} f(z) + \pi i \sum_{b_k\in\mathbb{R}} \operatorname*{Res}_{z=b_k} f(z)$$

where $f(z) = \dfrac{P(z)}{Q(z)}e^{i\lambda z}$, z_k are nonreal poles with $\mathrm{Im} z_k > 0$ and b_k are real simple poles.

Remark 1.16 If Q does not have real zeroes ($Q(x) \neq 0$ for any $x \in \mathbb{R}$), the above integral can be evaluated by:

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \cdot e^{i\lambda x} \mathrm{d}x = \begin{cases} 2\pi i \sum\limits_{\operatorname{Im} z_k > 0} \operatorname*{Res}\limits_{z=z_j} f(z), & \text{if } \lambda > 0 \\ \\ -2\pi i \sum\limits_{\operatorname{Im} z_k < 0} \operatorname*{Res}\limits_{z=z_j} f(z), & \text{if } \lambda < 0 \end{cases} \tag{1.8.1}$$

Solved problems

1.51 Calculate the integral $I = \displaystyle\int_0^{2\pi} \frac{\cos^2 x}{13 + 12\cos x} \mathrm{d}x$.

Solution. By the change of variable $z = e^{ix}$ we have

$$I = -\frac{i}{4} \int_{|z|=1} \frac{(z^2+1)^2}{(6z^2+13z+6)z^2} \mathrm{d}z.$$

The poles of the function

$$f(z) = \frac{(z^2+1)^2}{(6z^2+13z+6)z^2}$$

are: 0 (pole of order two), $-\dfrac{3}{2}$ and $-\dfrac{2}{3}$ (simple poles).

From these, only 0 and $-\dfrac{2}{3}$ lie inside the circle $|z| = 1$.

The residues of f at these poles are

$$\operatorname*{Res}_{z=-\frac{2}{3}} f(z) = \frac{169}{180} \quad \text{and} \quad \operatorname*{Res}_{z=0} f(z) = -\frac{13}{36}.$$

So $I = -\dfrac{i}{4} \cdot 2\pi i \left(\dfrac{169}{180} - \dfrac{13}{36} \right) = \dfrac{13\pi}{45}$.

1.52 Calculate the integral $I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}x}{(a + b\cos x)^2}$ where $0 < b < a$.

Solution. We have

$$I = \frac{4}{i} \int_{|z|=1} \frac{z}{(bz^2 + 2az + b)^2}$$

with the only pole situated inside the circle $|z| = 1$ being

$$z_0 = -\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b}.$$

This is a pole of order two, and the corresponding residue is

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{a}{4(a^2 - b^2)^{3/2}}.$$

The value of the integral is $I = \dfrac{2\pi a}{(a^2 - b^2)^{3/2}}$.

1.53 Evaluate the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2 + 4)^n}$, $n \geq 2$.

Solution. The poles of the function $f(z) = \dfrac{1}{(z^2 + 4)^n}$ are $z_{1,2} = \pm 2i$.

We consider only the pole situated in the upper half-plane, $z_1 = 2i$, pole of order n.

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{(n-1)!} \lim_{z \to 2i} \left[(z - 2i)^n \frac{1}{(z - 2i)^n (z + 2i)^n} \right]^{(n-1)}.$$

We determine the derivative of order k for the function $g(z) = (z + 2i)^{-n}$,

$$g^{(k)}(z) = (-n)(-n-1)\dots(-n-k+1)(z + 2i)^{-n-k}.$$

In particular,

$$g^{(n-1)}(z) = (-1)^{n-1} n(n+1)\dots(2n-2)(z + 2i)^{-2n+1}$$

and substituting in the previous expression it follows that

$$\operatorname*{Res}_{z=2i} f(z) = \frac{-i}{4^{2n-1}} \binom{2n-2}{n-1}$$

and further $I = \dfrac{2\pi}{4^{2n-1}} \dbinom{2n-2}{n-1}$.

1.54 Evaluate the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{2ix}\mathrm{d}x}{x^3+x^2+x+1}$.

Solution. The zeros of $z^3+z^2+z+1=(z+1)(z^2+1)$ are -1, i, $-i$ so the integral is

$$I = 2\pi i \operatorname*{Res}_{z=i} f(z) + \pi i \operatorname*{Res}_{z=-1} f(z).$$

Since the poles are of order one, we have

$$\operatorname*{Res}_{z=i} f(z) = \frac{e^{2iz}}{3z^2+2z+1}\Big|_{z=i} = -\frac{1}{4}e^{-2}(1+i)$$
$$\operatorname*{Res}_{z=-1} f(z) = \frac{e^{-2i}}{2} = \frac{\cos 2 - i\sin 2}{2}.$$

Finally, the integral will be $I = \dfrac{\pi}{2}(e^{-2}+\sin 2) - \dfrac{\pi i}{2}(e^{-2}-\cos 2)$.

Proposed problems

1.55 Evaluate the following integrals

i). $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}x}{1+3\cos^2 x}$;

ii). $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}x}{13+12\sin x}$;

iii). $\displaystyle\int_0^{2\pi} \frac{\sin^2 x}{a+b\cos x}\mathrm{d}x$, $0<b<a$;

iv). $\displaystyle\int_0^{2\pi} \frac{\cos x+1}{\sin x-2}\mathrm{d}x$.

v). $\displaystyle\int_0^{2\pi} e^{2\cos x}\mathrm{d}x$.

1.56 Evaluate the following integrals

i). $\displaystyle\int_{-\pi}^{\pi} \frac{\cos 100x}{5-4\sin x}\mathrm{d}x$;

ii). $\displaystyle\int_{-\pi}^{\pi} \frac{\sin 7x}{5-4\sin x}\mathrm{d}x$.

1.57 Evaluate the following integrals

i). $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\mathrm{d}x;$

ii). $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\mathrm{d}x;$

iii). $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1};$

iv). $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+9x^2+20}\mathrm{d}x;$

v). $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+x+1)^2};$

vi). $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\mathrm{d}x.$

1.58 Evaluate the following integrals

i). $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{3ix}}{(x^2+1)^2}\mathrm{d}x;$

ii). $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+2x+2}\mathrm{d}x;$

iii). $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4+x^2+1}\mathrm{d}x.$

Solutions and answers

$\boxed{\textbf{1.55}}$ **i)** $\displaystyle I = \frac{4}{i}\int_{|z|=1} \frac{z}{3z^4+10z^2+3}\mathrm{d}z = \frac{4}{i}\cdot 2\pi i\left(\frac{1}{16}+\frac{1}{16}\right) = \pi;$

ii) $\displaystyle I = \int_{|z|=1} \frac{\mathrm{d}z}{6z^2+13iz-6} = 2\pi i\,\mathrm{Res}_{z=-\frac{2i}{3}}\frac{1}{6z^2+13iz-6} = \frac{2\pi}{5};$

iii) $\displaystyle I = -\frac{1}{2i}\int_{|z|=1} \frac{(z^2-1)^2}{z^2(bz^2+2az+b)}\mathrm{d}z = -\frac{1}{2i}\cdot 2\pi i\left(\frac{2\sqrt{a^2-b^2}}{b^2}-\frac{2a}{b^2}\right) = \frac{2\pi}{b^2}(a-\sqrt{a^2-b^2});$

iv) $\displaystyle I = \int_{|z|=1} \frac{z^2+2z+1}{z^3-4iz^2-z}\mathrm{d}z = -\frac{2\pi}{\sqrt{3}}$ (0 and $(2-\sqrt{3})i$ are poles situated inside the circle $|z| = 1$).

v) By the change of variable $z = e^{ix}$ we get

$$I = \int_{|z|=1} e^{\frac{z^2+1}{z}} \cdot \frac{\mathrm{d}z}{iz} = 2\pi i \cdot \frac{1}{i} \mathrm{Res}_{z=0} f(z),$$

where $f(z) = \frac{1}{z} e^{z+\frac{1}{z}}$. To find the residue we expand in Laurent series

$$f(z) = \frac{1}{z}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\right)\left(1 + \frac{1}{z1!} + \frac{1}{z^2 2!} + \dots\right),$$

and get $I = 2\pi \sum_{n=0}^{\infty} \frac{1}{(n!)^2}$. It can be shown that this is $J_0(2i)$, where $J_0(z)$ is the zeroth Bessel function.

$\boxed{\mathbf{1.56}}$ For $n \in \mathbb{N}$, denote $I = \int_{-\pi}^{\pi} \frac{\cos nx}{5 - 4\sin x} \mathrm{d}x$ and $J = \int_{-\pi}^{\pi} \frac{\sin nx}{5 - 4\sin x} \mathrm{d}x$. Then $I + iJ = \int_{-\pi}^{\pi} \frac{e^{inx}}{5 - 4\sin x} \mathrm{d}x = -\int_{|z|=1} \frac{z^n}{2z^2 - 5iz + 2} \mathrm{d}z = \frac{\pi}{3} \cdot \frac{i^n}{2^{n-1}}$. For particular values of n, we get the results **i)** $\frac{\pi}{3 \cdot 2^{99}}$; **ii)** $-\frac{\pi}{3 \cdot 2^6}$.

$\boxed{\mathbf{1.57}}$ **i)** $2\pi i\Big(-\frac{1}{16i} + \frac{3}{16i}\Big) = \frac{\pi}{4}$; **ii)** $\frac{\pi}{\sqrt{2}}$;

iii) $2\pi i\Big(\frac{1}{3(i - \sqrt{3})} + \frac{1}{6i} + \frac{1}{3(i + \sqrt{3})}\Big) = \frac{2\pi}{3}$;

iv) The poles with positive imaginary part are $2i$ and $i\sqrt{5}$ so

$I = 2\pi i \left(\frac{1}{4i} - \frac{1}{2\sqrt{5}i}\right) = \frac{\pi}{10}(5 - 2\sqrt{5})$;

v) $\frac{4\pi}{3\sqrt{3}}$ **vi)** $\frac{3\pi}{8}$.

$\boxed{\mathbf{1.58}}$ **i)** i and $-i$ are poles of order 2, we use only the pole with a positive imaginary part.

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} \left[\frac{z e^{3iz}}{(z + i)^2}\right]' = e^{-3} \cdot \frac{3}{4}$$

and the integral is $\frac{3\pi i}{2} e^{-3}$;

ii) By 1.8.1, the value of the integral is $-2\pi i \operatorname*{Res}_{z=-1-i} f(z) = \pi e^{-1+i}$;

iii) Denoting the given integral by I and $J = \int_{-\infty}^{\infty} \frac{\sin x}{x^4 + x^2 + 1} \mathrm{d}x = 0$ we will evaluate $I = I + iJ = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 + x^2 + 1} \mathrm{d}x$. The poles with positive imaginary part are $z_1 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ and $z_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ having the residues

$$\operatorname*{Res}_{z=z_1} f(z) = -\frac{1}{12} e^{-\frac{\sqrt{3}}{2}} (\cos \frac{1}{2} + i \sin \frac{1}{2})(3 + i\sqrt{3}),$$

$$\operatorname*{Res}_{z=z_2} f(z) = \frac{1}{12} e^{-\frac{\sqrt{3}}{2}} (\cos \frac{1}{2} - i \sin \frac{1}{2})(3 - i\sqrt{3}).$$

Finally, $I = \frac{1}{3} e^{-\frac{\sqrt{3}}{2}} (3 \sin \frac{1}{2} + \sqrt{3} \cos \frac{1}{2})$.

CHAPTER 2

Integral Transforms

2.1 The Laplace transform. Original and image

Let $f : \mathbb{R} \to \mathbb{C}$, be a function that satisfies the following conditions:
a) f is integrable on any finite segment of the real axis;
b) $f(t) = 0$ for all $t < 0$;
c) there exist some constants, $M > 0$ and $s \geq 0$, such that $|f(t)| \leq Me^{st}$, for every $t \in [0, +\infty)$.

Example 2.1.1 The functions
$f_1(t) = \begin{cases} 1, \, t \geq 0 \\ 0, \, t < 0 \end{cases}$ and $f_2(t) = \begin{cases} e^{2t} \sin 3t, \, t \geq 0 \\ 0, \qquad\quad t < 0 \end{cases}$
satisfy the above conditions. f_1 is called the *Heaviside step function*, or the *unit step function*.

Definition 2.1 Each function f that satisfies the conditions a), b), and c) is called **an original**.

Let f be an original. Consider the function

$$F(p) = \int_0^{+\infty} f(t)e^{-pt}\mathrm{d}t \tag{2.1.1}$$

It can be proved that the function F is holomorphic in the half-plane $\{p \in \mathbb{C} \,|\, \mathrm{Re}\, p > s\}$.

Definition 2.2 F is called **the image** of f through the Laplace transformation.

Notation: $F = \mathcal{L}f$ or $F(p) = \mathcal{L}[f(t)]$.

Example 2.1.2 Let $f(t) = \begin{cases} e^{2t}, \ t \geq 0 \\ 0, \quad t < 0 \end{cases}$

In this case $s = 2$. We have, for $p = u + iv \in \mathbb{C}$,

$$F(p) = \int_0^{\infty} e^{2t} e^{-pt} \mathrm{d}t = \int_0^{\infty} e^{-(p-2)t} \mathrm{d}t =$$
$$= -\frac{1}{p-2} e^{-(p-2)t} \Big|_0^{+\infty} = \frac{1}{p-2} [1 - \lim_{t\to\infty} e^{-(u-2)t} e^{-ivt}]$$

For the case $\mathrm{Re}\, p = u > 2$ we have $F(p) = \dfrac{1}{p-2}$.

2.2 Properties of the Laplace transform

It can be proved that if f_1 and f_2 are two originals with the same image, then f_1 and f_2 coincide at all points where they are continuous.

Now, we shall present some properties of the Laplace transform.

1) Linearity. Let f and g be originals and $\alpha, \beta \in \mathbb{C}$. Then

$$\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L} f + \beta \mathcal{L} g$$

Indeed,

$$\mathcal{L}(\alpha f + \beta g)(p) = \int_0^{+\infty} (\alpha f + \beta g)(t) e^{-pt} \mathrm{d}t$$
$$= \alpha \int_0^{+\infty} f(t) e^{-pt} \mathrm{d}t + \beta \int_0^{+\infty} g(t) e^{-pt} \mathrm{d}t = \alpha \mathcal{L} f(p) + \beta \mathcal{L} g(p).$$

Applications

a) Consider the original $f(t) = e^{\lambda t}$, $\lambda \in \mathbb{C}$, $t \geq 0$. Then, as in Example 2.1.2

$$\mathcal{L}e^{\lambda t} = \int_0^{+\infty} e^{\lambda t}e^{-pt}\mathrm{d}t = \int_0^{+\infty} e^{-(p-\lambda)t}\mathrm{d}t = \frac{1}{p-\lambda}$$

Hence, we have

$$(1) \quad \mathcal{L}e^{\lambda t} = \frac{1}{p-\lambda}$$

Particulary, for $\lambda = 0$ we have the original $f(t) = 1$, $t \geq 0$. Thus we can write

$$(2) \quad \mathcal{L}1 = \frac{1}{p}$$

b) For $\omega \in \mathbb{C}$ we have $\cos\omega t = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$.

Hence $\mathcal{L}\cos\omega t = \dfrac{1}{2}(\mathcal{L}e^{i\omega t} + \mathcal{L}e^{-i\omega t}) = \dfrac{1}{2}\left(\dfrac{1}{p-i\omega} + \dfrac{1}{p+i\omega}\right) = \dfrac{p}{p^2+\omega^2}$.

Thus we have

$$(3) \quad \mathcal{L}\cos\omega t = \frac{p}{p^2+\omega^2}$$

Similarly, we have the following transforms:

$$(4) \quad \mathcal{L}\sin\omega t = \frac{\omega}{p^2+\omega^2}$$

$$(5) \quad \mathcal{L}\cosh\omega t = \frac{p}{p^2-\omega^2}$$

$$(6) \quad \mathcal{L}\sinh\omega t = \frac{\omega}{p^2-\omega^2}$$

2) Change of scale. Let f be an original and $\alpha > 0$ a real number. Then $f(\alpha t)$ is again an original. Let $F(p) = \mathcal{L}f(t)$ be the image of $f(t)$.

Then the image of $f(\alpha t)$ is

$$\mathcal{L}f(\alpha t) = \frac{1}{\alpha}F(\frac{p}{\alpha})$$

Indeed, $\mathcal{L}f(\alpha t) = \displaystyle\int_0^{+\infty} f(\alpha t)e^{-pt}\mathrm{d}t$.

By the change of variable $\alpha t = \tau$ we get

$$\mathcal{L}f(\alpha t) = \int_0^{+\infty} f(\tau)e^{-\frac{p}{\alpha}\tau}\frac{1}{\alpha}\mathrm{d}\tau = \frac{1}{\alpha}F(\frac{p}{\alpha}).$$

3) Differentiation of the original. Suppose that $f(t)$, $f^{'}(t)$, ..., $f^n(t)$ are originals and $\mathcal{L}f(t) = F(p)$. Then

$$\begin{aligned}
&\mathcal{L}f^{'}(t) = pF(p) - f(0)\\
&\mathcal{L}f^{''}(t) = p^2F(p) - pf(0) - f'(0)\\
&\cdots\cdots\cdots\\
&\mathcal{L}f^{(n)}(t) = p^nF(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - f^{(n-1)}(0),
\end{aligned}$$

where $f^{(k)}(0)$ is understood as the limit of $f^{(k)}(t)$ as $t \to +0$.

Indeed, integrating by parts we get

$$\begin{aligned}
\mathcal{L}f'(t) &= \int_0^{+\infty} f'(t)e^{-pt}\mathrm{d}t = f(t)e^{-pt}\Big|_0^{+\infty} + p\int_0^{+\infty} f(t)e^{-pt}\mathrm{d}t\\
&= -f(0) + pF(p).
\end{aligned}$$

Now $\mathcal{L}f^{''}(t) = p\mathcal{L}f'(t) - f'(0) = p(pF(p) - f(0)) - f'(0)$.

Hence $\mathcal{L}f^{''}(t) = p^2F(p) - pf(0) - f'(0)$, and so on.

4) Differentiation of the image. It can be proved that, if $\mathcal{L}f(t) = F(p)$, then

$$F'(p) = \mathcal{L}[-tf(t)]$$

and more generally,

$$F^{(n)}(p) = \mathcal{L}[(-t)^n f(t)], \quad n \geq 1.$$

Applications

a) Since $\mathcal{L}e^{\lambda t} = \dfrac{1}{p-\lambda}$, we deduce

$$\mathcal{L}[(-t)^n e^{\lambda t}] = \left(\frac{1}{p-\lambda}\right)^{(n)} = (-1)^n\frac{n!}{(p-\lambda)^{n+1}}$$

It follows that

$$\mathcal{L}(t^n e^{\lambda t}) = \frac{n!}{(p-\lambda)^{n+1}}, n \geq 1$$

Particulary, for $\lambda = 0$ we have

$$(8) \quad \mathcal{L}(t^n) = \frac{n!}{p^{n+1}}$$

A more general formula, for $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > -1$ is

$$(9) \quad \mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}, \ p \in \mathbb{R}, p > 0,$$

where Γ is the well-known Euler's function.

For $\alpha = -\frac{1}{2}$, knowing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get

$$(10) \quad \mathcal{L}\frac{1}{\sqrt{t}} = \sqrt{\frac{\pi}{p}}, \ p \in \mathbb{R}, p > 0.$$

b) Similarly, using *differentiation of the image* property, we can deduce

$$(11) \quad \mathcal{L}(t\cos\omega t) = \frac{p^2-\omega^2}{(p^2+\omega^2)^2}$$

$$(12) \quad \mathcal{L}(t\sin\omega t) = \frac{2p\omega}{(p^2+\omega^2)^2}$$

$$(13) \quad \mathcal{L}(t\cosh\omega t) = \frac{p^2+\omega^2}{(p^2-\omega^2)^2}$$

$$(14) \quad \mathcal{L}(t\sinh\omega t) = \frac{2p\omega}{(p^2-\omega^2)^2}$$

5) Integration of the original. If $\mathcal{L}f(t) = F(p)$, then

$$\mathcal{L}\int_0^t f(\tau)\mathrm{d}\tau = \frac{1}{p}F(p)$$

Indeed, let $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$ and let $G(p) = \mathcal{L}g(t)$.
We have $g'(t) = f(t)$ and $g(0) = 0$. Consequently,

$$F(p) = \mathcal{L}f(t) = \mathcal{L}g'(t) = pG(p) - g(0) = pG(p).$$

It follows that

$$\mathcal{L}\int_0^t f(\tau)\mathrm{d}\tau = \mathcal{L}g(t) = G(p) = \frac{1}{p}F(p).$$

6) Integration of the image. If $\mathcal{L}f(t) = F(p)$, then

$$\mathcal{L}\frac{f(t)}{t} = \int_p^{+\infty} F(q)\mathrm{d}q$$

To verify this equality, set $G(p) = \displaystyle\int_p^{+\infty} F(q)\mathrm{d}q$. Then $G'(p) = -F(p)$. Suppose that $G(p) = \mathcal{L}g(t)$.

We know that $G'(p) = \mathcal{L}[-tg(t)] = -\mathcal{L}[tg(t)]$.

Hence $G'(p) = -\mathcal{L}f(t)$ and $G'(p) = -\mathcal{L}[tg(t)]$.

This means that $g(t) = \dfrac{f(t)}{t}$, that is,

$$\mathcal{L}\frac{f(t)}{t} = \mathcal{L}g(t) = G(p) = \int_p^{+\infty} F(q)\mathrm{d}q.$$

7) Translation of the image. If $\mathcal{L}f(t) = F(p)$, then

$$\mathcal{L}[e^{\lambda t}f(t)] = F(p-\lambda)$$

Indeed,

$$\mathcal{L}[e^{\lambda t}f(t)] = \int_0^{+\infty} e^{\lambda t}f(t)e^{-pt}\mathrm{d}t = \int_0^{+\infty} f(t)e^{-(p-\lambda)t}\mathrm{d}t = F(p-\lambda).$$

8) Convolution of originals. Let $f(t)$ and $g(t)$ be originals with Laplace images $F(p)$ and $G(p)$, respectively.

The function $\displaystyle\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$ is called the convolution of the functions $f(t)$ and $g(t)$ and is denoted $f * g$. It can be proved that

$$\mathcal{L}\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau = F(p)G(p).$$

Solved problems

2.1 Find the images by the Laplace transform of the functions:

i). $f(t) = e^{-t}t^3$,

ii). $f(t) = te^{2t}\cos t$,

iii). $f(t) = \displaystyle\int_0^t \frac{\sin u}{u}\mathrm{d}u$

Solution. i). We apply the translation of the image and formula (8) for the transform of t^n:

$$\mathcal{L}[e^{-t}t^3](p) = \mathcal{L}[t^3](p+1) = \frac{3!}{(p+1)^4}.$$

ii). We have

$$\mathcal{L}[t\cos t](p) = -\left(\mathcal{L}[\cos t]\right)'(p) = -\left(\frac{p}{p^2+1}\right)' = \frac{p^2-1}{(p^2+1)^2}.$$

Finally, translating the image $\mathcal{L}[f(t)](p) = \dfrac{(p-2)^2-1}{((p-2)^2+1)^2}$.

iii). Apply first *integration of the original* property and then *integration of the image* property:

$$\begin{aligned}&\mathcal{L}[\int_0^t \frac{\sin u}{u}\mathrm{d}u](p) = \frac{1}{p}\mathcal{L}[\frac{\sin t}{t}](p) = \frac{1}{p}\int_p^\infty \mathcal{L}[\sin t](y)\mathrm{d}y\\ &= \frac{1}{p}\int_p^\infty \frac{1}{y^2+1}\mathrm{d}y = \frac{1}{p}\arctan y\Big|_p^\infty = \frac{1}{p}(\frac{\pi}{2} - \arctan p) = \frac{1}{p}\arctan\frac{1}{p}.\end{aligned}$$

2.2 Let f be a periodic function, $f(t+T) = f(t)$ for any $t > 0$. Prove that, for s with $\operatorname{Re} s > 0$, its Laplace transform is given by

$$\mathcal{L}f(t) = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,\mathrm{d}t.$$

Solution. We have

$$\mathcal{L}f(t) = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t = \lim_{n\to\infty}\int_0^{nT} e^{-st}f(t)\,\mathrm{d}t = \sum_{k=0}^\infty \int_{kT}^{(k+1)T} e^{-st}f(t)\,\mathrm{d}t.$$

By the change of variable $x = t - kT$ it follows that

$$\int_{kT}^{(k+1)T} e^{-st} f(t)\,\mathrm{d}t = e^{-skT} \int_0^T e^{-sx} f(x)\,\mathrm{d}x.$$

Using the geometric series, since $|e^{-sT}| = e^{-T\mathrm{Re}\,s} < 1$ we get

$$\mathcal{L}f(t) = \int_0^T e^{-st} f(t)\,\mathrm{d}t \sum_{k=0}^{\infty} e^{-skT} = \frac{1}{1-e^{-sT}} \int_0^T f(t) e^{-st}\,\mathrm{d}t.$$

Proposed problems

2.3 Find the images of the following originals:

i). $f(t) = t^2 \cos t$

ii). $f(t) = t(e^t + \cosh t)$

iii). $f(t) = (t+1)\sin 2t$

iv). $f(t) = e^{2t} \sin t$

v). $f(t) = e^t \cos nt$

vi). $f(t) = e^t \sinh t$

vii). $f(t) = e^{3t} \sin^2 t$

viii). $f(t) = t^{-1} \sin^2 2t.$

ix). $f(t) = t - n$, for $n < t \le n+1$, $n = 0, 1, 2, \ldots.$

x). $f(t) = |\sin 3t|.$

2.4 Find the originals $f(t)$ corresponding to the following images:

i). $F(p) = \dfrac{p}{p^2 - 5p + 6}$

ii). $F(p) = \dfrac{1}{p^2 - 9p + 25}$

iii). $F(p) = \dfrac{p}{(p^2+1)^2}$

iv). $F(p) = \dfrac{1}{(p^2+4)^2}$

v). $F(p) = (p+4)^{-7}$

Solutions and answers

$\boxed{2.3}$ **i)** $\dfrac{2p^3-6p}{(p^2+1)^3}$; **ii)** $\dfrac{2(p^2+p+1)}{(p^2-1)^2}$; **iii)** $\dfrac{2p^2+4p+8}{(p^2+4)^2}$;
iv) $\dfrac{1}{(p-2)^2+1}$; **v)** $\dfrac{p-1}{(p^2-1)^2+n^2}$; **vi)** $\dfrac{1}{(p-1)^2}$;
vii) $\dfrac{1}{2(p-3)} - \dfrac{1}{2}\dfrac{p-3}{(p-3)^2+4}$;
viii) $\dfrac{1}{2}\displaystyle\int_p^\infty \left(\frac{1}{y} - \frac{y}{y^2+16}\right) \mathrm{d}y = \frac{1}{4}\ln\frac{p^2+16}{p^2}$;
ix) f is a periodic function with $T = 1$, using the solved problem 2.2 we get $F(p) = \dfrac{1}{s^2(1-e^{-s})}[1 - e^{-s}(1+s)]$;
x) $T = \dfrac{\pi}{3}$, $F(p) = \dfrac{3}{p^2+9}\coth\dfrac{\pi p}{6}$.

$\boxed{2.4}$ **i)** $F(p) = \dfrac{3}{p-3} - \dfrac{2}{p-2}$, $f(t) = \mathcal{L}^{-1}[F(p)](t) = 3e^{3t} - 2e^{2t}$,
ii) $F(p) = \dfrac{1}{(p-3)^2+16}$; the corresponding original is $f(t) = \dfrac{1}{4}e^{3t}\sin t$,
iii) Notice that $F(p) = -\dfrac{1}{2}\left(\dfrac{1}{p^2+1}\right)'$ so $f(t) = \dfrac{1}{2}t\sin t$ (see the *differentiation of the image* property)
iv) $\mathcal{L}^{-1}\left[\dfrac{1}{p^2+4}\cdot\dfrac{1}{p^2+4}\right](t) = \mathcal{L}^{-1}\left[\dfrac{1}{p^2+4}\right] * \mathcal{L}^{-1}\left[\dfrac{1}{p^2+4}\right]$ and further on $f(t) = \dfrac{1}{2}\sin 2t * \dfrac{1}{2}\sin 2t = \dfrac{1}{4}\displaystyle\int_0^t \sin 2(t-u)\sin 2u\mathrm{d}u = \frac{1}{16}\sin 2t - \frac{1}{8}t\cos 2t$.
v) $\mathcal{L}^{-1}[F(p)](t) = \dfrac{1}{6!}t^6e^{-4t}$.

2.3 Applications of the Laplace transform

The Laplace transform is very useful for solving certain types of ordinary differential equations, particularly those of constant coefficients, and certain types of partial differential equations.

Example 2.3.1 Solve the Cauchy problem

$$\begin{cases} x'' + x = 2\cos t \\ x(0) = 0 \\ x'(0) = -1 \end{cases}$$

Denote $\mathcal{L}x(t) = X(p)$. Then

$$\mathcal{L}x'(t) = pX(p) - x(0) = pX(p) \quad \text{and}$$

$$\mathcal{L}x''(t) = p^2X(p) - px(0) - x'(0) = p^2X(p) + 1.$$

From

$$\mathcal{L}x''(t) + \mathcal{L}x(t) = 2\mathcal{L}\cos t$$

we obtain

$$p^2X(p) + 1 + X(p) = \frac{2p}{p^2+1}$$

and consequently

$$X(p) = \frac{2p}{(p^2+1)^2} - \frac{1}{p^2+1}$$

Now, $\dfrac{1}{p^2+1} = \mathcal{L}\sin t$ and $\dfrac{2p}{(p^2+1)^2} = (-\dfrac{1}{p^2+1})' = \mathcal{L}(t\sin t)$.
Hence $\mathcal{L}x(t) = X(p) = \mathcal{L}(t\sin t - \sin t)$.
Finally we obtain the solution

$$x(t) = (t-1)\sin t.$$

The Laplace transform can also be used to evaluate some **improper integrals**. Suppose that $\mathcal{L}f(t) = F(p)$.

It can be proved that, if the integral $\displaystyle\int_0^{+\infty} \frac{f(t)}{t}\mathrm{d}t$ is convergent, then

$$\int_0^{+\infty} \frac{f(t)}{t}\mathrm{d}t = \int_0^{+\infty} F(p)\mathrm{d}p.$$

Example 2.3.2 We know that $\mathcal{L}\sin t = \dfrac{1}{p^2+1}$. Hence

$$\int_0^{+\infty} \frac{\sin t}{t}\mathrm{d}t = \int_0^{+\infty} \frac{1}{p^2+1}\mathrm{d}p = \arctan p\Big|_0^{+\infty} = \frac{\pi}{2}.$$

Thus we have the important formula

$$\int_0^{+\infty} \frac{\sin t}{t}\mathrm{d}t = \frac{\pi}{2}.$$

Example 2.3.3 Let $a>0, b>0$. We know that

$$\mathcal{L}(e^{-at} - e^{-bt}) = \frac{1}{p+a} - \frac{1}{p+b}.$$

Hence

$$\int_0^{+\infty} \frac{e^{-at} - e^{-bt}}{t}\mathrm{d}t = \int_0^{+\infty} (\frac{1}{p+a} - \frac{1}{p+b})\mathrm{d}p = \ln\frac{p+a}{p+b}\Big|_0^{+\infty} = -\ln\frac{a}{b} = \ln\frac{b}{a}.$$

Solved problems

2.5 Solve the Cauchy problem

$$\begin{cases} x'' - 3x' + 2x = 4e^{2t} \\ x(0) = -3 \\ x'(0) = 5 \end{cases}$$

Solution.

Let $\mathcal{L}x(t) = X(p)$. Then

$$\mathcal{L}x'(t) = pX(p) - x(0) = pX(p) + 3,$$

$$\mathcal{L}x''(t) = p^2X(p) - px(0) - x'(0) = p^2X(p) + 3p - 5$$

$$\mathcal{L}e^{2t} = \frac{1}{p-2}.$$

Thus $p^2X(p) + 3p - 5 - 3pX(p) - 9 + 2X(p) = \dfrac{4}{p-2}$ and

$$X(p) = -7\frac{1}{p-1} + 4\frac{1}{p-2} + 4\frac{1}{(p-2)^2}$$

Since $\dfrac{1}{(p-2)^2} = -(\dfrac{1}{p-2})' = \mathcal{L}(te^{2t})$, we obtain

$$x(t) = -7e^t + 4e^{2t} + 4te^{2t}.$$

2.6 Solve the Cauchy problem

$$\begin{cases} tx'' - 2tx' - 2x = 0 \\ x(0) = 0 \\ x'(0) = 1 \end{cases}$$

Solution.

Let $\mathcal{L}x(t) = X(p)$. Then

$$\mathcal{L}x'(t) = pX(p), \quad \mathcal{L}x''(t) = p^2X(p) - 1.$$

By *differentiation of the image* property we get

$$\mathcal{L}[tx'] = -X(p) - pX'(p),\ \mathcal{L}[tx''] = -2pX(p) - p^2X'(p)$$

and we obtain the differential equation with the unknown $X(p)$:

$$X'(p)(-p^2 + 2p) = 2pX(p).$$

This can be written

$$\frac{X'(p)}{X(p)} = \frac{-2}{p-2}$$

and has the general solution $X(p) = \dfrac{c}{(p-2)^2}$, $c \in \mathbb{R}$.

Applying the inverse Laplace transform and using the condition $x'(0) = 1$ to determine c, it follows that the solution of the Cauchy problem is

$$x(t) = te^{2t}.$$

2.7 Find the function $x(t)$, $t \geq 0$ which satisfies the equation

$$x(t) - 2\int_0^t x(\tau)\mathrm{d}\tau = \frac{1}{9}(1 - \cos 3t)$$

Solution. Let $\mathcal{L}x(t) = X(p)$. Then $\mathcal{L}\int_0^t x(\tau)\mathrm{d}\tau = \frac{1}{p}X(p)$.

It follows that

$$X(p) = \frac{1}{13}\frac{1}{p-2} - \frac{1}{13}\frac{p}{p^2+9} - \frac{2}{13}\frac{1}{p^2+9}$$

and hence $x(t) = \frac{1}{13}e^{2t} - \frac{1}{13}\cos 3t - \frac{2}{39}\sin 3t$.

2.8 Find the function $x(t)$, $t \geq 0$ which satisfies the equation

$$x(t) + \int_0^t x(\tau)\cos(t-\tau)\mathrm{d}\tau = e^t$$

Solution.

Let $\mathcal{L}x(t) = X(p)$. The function $\int_0^t x(\tau)\cos(t-\tau)\mathrm{d}\tau$ is the convolution of the functions $x(t)$ and $\cos t$. Hence

$$\mathcal{L}(\int_0^t x(\tau)\cos(t-\tau)\mathrm{d}\tau) = \mathcal{L}(x(t))\mathcal{L}(\cos t) = X(p)\frac{p}{p^2+1}.$$

It follows that

$$X(p) = \frac{2}{3}\frac{1}{p-1} + \frac{1}{3}\frac{p-1}{p^2+p+1}$$

$$= \frac{2}{3}\frac{1}{p-1} + \frac{1}{3}\left[\frac{p+\frac{1}{2}}{\left(p+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \sqrt{3}\frac{\frac{\sqrt{3}}{2}}{\left(p+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right],$$

$$x(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{3}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t.$$

2.9 Evaluate

$$f(t) = \int_0^\infty \frac{\cos tu}{a^2+u^2}\mathrm{d}u, \quad t > 0.$$

Solution.

$$\mathcal{L}f(t) = \int_0^\infty f(t)e^{-pt}\mathrm{d}t = \int_0^\infty (\int_0^\infty \frac{\cos tu}{a^2+u^2}\mathrm{d}u)e^{-pt}\mathrm{d}t =$$

$$= \int_0^\infty \int_0^\infty \frac{1}{a^2+u^2}\cos tue^{-pt}\mathrm{d}u\mathrm{d}t = \int_0^\infty \frac{1}{a^2+u^2}(\int_0^\infty \cos tue^{-pt}\mathrm{d}t)\mathrm{d}u =$$

$$= \int_0^\infty \frac{1}{a^2+u^2}\frac{p}{p^2+u^2}\mathrm{d}u = \int_0^\infty \frac{p}{p^2-a^2}(\frac{1}{a^2+u^2} - \frac{1}{p^2+u^2})\mathrm{d}u =$$

$$= \frac{p}{p^2-a^2}[\frac{1}{a}\arctan\frac{u}{a} - \frac{1}{p}\arctan\frac{u}{p}]\Big|_0^\infty = \frac{\pi}{2a}\frac{1}{p+a}.$$

We conclude that $f(t) = \mathcal{L}^{-1}(\frac{\pi}{2a}\frac{1}{p+a}) = \frac{\pi}{2a}e^{-at}$.

Hence

$$\int_0^\infty \frac{\cos tu}{a^2+u^2}\mathrm{d}u = \frac{\pi}{2a}e^{-at}, \quad a, t > 0.$$

Proposed problems

2.10 Solve the following Cauchy problems:

i). $\begin{cases} x'' + 4x = \frac{1}{2}(\cos t - \cos 2t) \\ x(0) = 1 \\ x'(0) = 0 \end{cases}$

ii). $\begin{cases} x''' - 3x'' + 3x' - x = t^2e^t \\ x(0) = 1, x'(0) = 0, x''(0) = -2 \end{cases}$

iii). $\begin{cases} x' + 2x = \sin t \\ x(0) = 0 \end{cases}$

iv). $\begin{cases} x'' - 2ax' + a^2x = e^{at}(t+1)^{-1}, \ a \in \mathbb{R} \\ x(0) = x'(0) = 0 \end{cases}$

v). $\begin{cases} x''' + x' = 1 \\ x(0) = x'(0) = x''(0) = 0 \end{cases}$

vi). $\begin{cases} x''' - x'' = \sin t \\ x(0) = x'(0) = x''(0) = 0 \end{cases}$

vii). $\begin{cases} x'' + 2x' + x = t^2 \\ x(0) = 1 \\ x'(0) = 0 \end{cases}$

viii). $\begin{cases} x'' + x = te^t + 4\sin t \\ x(0) = x'(0) = 0 \end{cases}$

ix). $\begin{cases} x''' + x = \dfrac{1}{2}t^2e^t \\ x(0) = x'(0) = x''(0) = 0 \end{cases}$

x). $\begin{cases} x'' - 2\alpha x' + (\alpha^2 + \beta^2)x = 0, \;\; \alpha, \beta \in \mathbb{R} \\ x(0) = 0 \\ x'(0) = 1 \end{cases}$

xi). $\begin{cases} x' + 2x(t) + \displaystyle\int_0^t x(\tau)\mathrm{d}\tau = \sin t \\ x'(0) = 0 \end{cases}$

2.11 Solve the following systems of Cauchy problems:

i). $\begin{cases} x'(t) = 2x - 3y \\ y'(t) = -2x + y \\ x(0) = 8 \\ y(0) = 3 \end{cases}$

ii). $\begin{cases} x' - y' - 2x + 2y = \sin t \\ x'' + 2y' + x = 0 \\ x(0) = x'(0) = y(0) = 0 \end{cases}$

iii). $\begin{cases} x'' = 3(-x + y + z) \\ y'' = x - y \\ z'' = -z \\ x(0) = x'(0) = 0 \\ y(0) = 0, y'(0) = -1 \\ z(0) = 1, z'(0) = 0 \end{cases}$

iv). $\begin{cases} x' = -y - z \\ y' = -x - z \\ z' = -x - y \\ x(0) = -1 \\ y(0) = 0 \\ z(0) = 1 \end{cases}$

2.12 Evaluate the integrals:

i). $\displaystyle\int_0^\infty \frac{e^{-\alpha t}\sin at}{t}\mathrm{d}t,\ a>0,\ \alpha>0$

ii). $\displaystyle\int_0^\infty \frac{\cos at-\cos bt}{t}\mathrm{d}t,\ a>0,\ b>0$

Solutions and answers

$\boxed{\textbf{2.10}}$ **i)** $x(t)=\frac{1}{6}\cos t+\frac{5}{6}\cos 2t-\frac{1}{8}t\sin 2t;$

ii) $x(t)=e^t-te^t-\frac{1}{2}t^2e^t+\frac{1}{60}t^5e^t;$

iii) $x(t)=\frac{1}{5}(e^{-2t}-\cos t+2\sin t);$

iv) Denote $F(p)=\mathcal{L}[e^{at}(t+1)^{-1}]$ and obtain $X(p)=\dfrac{F(p)}{(p-a)^2}$. Then

$$\begin{aligned}x(t)&=\mathcal{L}^{-1}[F(p)]*\mathcal{L}^{-1}[(p-a)^{-2}]=e^{at}(t+1)^{-1}*e^{at}t\\&=\int_0^t e^{au}(u+1)^{-1}e^{a(t-u)}(t-u)\mathrm{d}u=e^{at}[(t+1)\ln(t+1)-t].\end{aligned}$$

v) $x(t)=t-\sin t;$

vi) $x(t)=\frac{1}{2}e^t-t-1+\frac{1}{2}(\cos t+\sin t);$

vii) $x(t)=t^2-4t+6-5e^{-t}-te^{-t};$

viii) $x(t)=\frac{1}{2}(t-1)e^t+\frac{1}{2}\cos t+2\sin t-2t\cos t;$

ix) $x(t)=\frac{1}{4}(t^2-3t+\frac{3}{2})e^t+\frac{1}{3}e^{\frac{t}{2}}(\sqrt{3}\sin\frac{\sqrt{3}}{2}t-\cos\frac{\sqrt{3}}{2}t)-\frac{1}{24}e^{-t};$

x) $x(t)=\frac{1}{\beta}e^{\alpha t}\sin\beta t;$

xi) $x(t)=\frac{1}{2}\sin t-\frac{1}{2}te^{-t};$

$\boxed{\mathbf{2.11}}$ **i)** Denoting $\mathcal{L}x(t) = X(p)$ and $\mathcal{L}y(t) = Y(p)$ we obtain the system

$$\begin{cases} (p-2)X(p) + 3Y(p) = 8 \\ 2X(p) + (p-1)Y(p) = 3 \end{cases}$$

with the solution $X(p) = \dfrac{8p-17}{p^2-3p-4} = \dfrac{3}{p-4} + \dfrac{5}{p+1}$, $Y(p) = \dfrac{3p-22}{p^2-3p-4} = \dfrac{-2}{p-4} + \dfrac{5}{p+1}$.
By the inverse transform this gives $x(t) = 5e^{-t} + 3e^{4t}$ and $y(t) = 5e^{-t} - 2e^{4t}$;

ii) $x(t) = \dfrac{1}{9}e^{-t} + \dfrac{1}{3}te^{-t} + \dfrac{4}{45}e^{2t} - \dfrac{1}{5}\cos t - \dfrac{2}{5}\sin t$, $y(t) = \dfrac{1}{9}e^{-t} + \dfrac{1}{3}te^{-t} - \dfrac{1}{9}e^{2t}$;

iii) $x(t) = \dfrac{3}{4}(1-t) - \dfrac{3}{4}\cos 2t + \dfrac{3}{8}\sin 2t$, $y(t) = \dfrac{3}{4}(1-t) + \dfrac{1}{4}\cos 2t - \dfrac{1}{8}\sin 2t - \cos t$, $z(t) = \cos t$;

iv) $x(t) = -e^t$, $y(t) = 0$, $z(t) = e^t$.

$\boxed{\mathbf{2.12}}$ **i)** $\displaystyle\int_0^\infty \frac{e^{-\alpha t}\sin at}{t}\mathrm{d}t = \int_0^\infty \frac{a}{(p+\alpha)^2 + a^2}\mathrm{d}p = \arctan \frac{p+\alpha}{a}\bigg|_0^\infty = \frac{\pi}{2} - \arctan\frac{\alpha}{a} = \arctan\frac{a}{\alpha}$.

ii) $\displaystyle\int_0^\infty \frac{\cos at - \cos bt}{t}\mathrm{d}t = \int_0^\infty \frac{p}{p^2+a^2} - \frac{p}{p^2+b^2}\mathrm{d}p = \frac{1}{2}\ln\frac{p^2+a^2}{p^2+b^2}\bigg|_0^\infty = \ln\frac{b}{a}$.

2.4 The Fourier Transform

1. Let $f : \mathbb{R} \to \mathbb{C}$; it is called a *Fourier original* if:
a) f is piecewise differentiable;
b) at every point of discontinuity x_0 we have

$$f(x_0) = \frac{1}{2}(f(x_0 - 0) + f(x_0 + 0);$$

(where by $f(x_0 - 0)$ and $f(x_0 + 0)$ are denoted the limit to the left and to the right of f at x_0)
c) the integral $\displaystyle\int_{-\infty}^{+\infty} |f(x)|\mathrm{d}x$, is convergent.

Let f be a Fourier original. The function defined by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-isx}\mathrm{d}x, \quad s \in \mathbb{R} \tag{2.4.1}$$

is called the *image* of f under the Fourier transform. Under these hypotheses we have also

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s)e^{isx}\mathrm{d}s, \quad x \in \mathbb{R}. \tag{2.4.2}$$

We shall use the notations $F = \mathcal{F}[f]$ or $\mathcal{F}[f(x)] = F(s)$.

2. Suppose that f is *even*. Then

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\cos sx\mathrm{d}x - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\sin sx\mathrm{d}x.$$

The function $x \to f(x)\sin sx$ is odd, so that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\cos sx\mathrm{d}x.$$

Since the function $x \to f(x)\cos sx$ is even, we have finally

$$F(s) = 2\frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} f(x)\cos sx\mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} f(x)\cos sx\mathrm{d}x.$$

The function

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos sx \mathrm{d}x, \quad s \in \mathbb{R} \tag{2.4.3}$$

is called the *Fourier cosine transform* of the even function f, and we have

$$F(s) = F_c(s), \quad s \in \mathbb{R}$$

Obviously F_c is an even function, which means that F is even. From (2.4.2) we infer that

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s) \cos sx \mathrm{d}s + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s) \sin sx \mathrm{d}s = \\ &= 2\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} F(s) \cos sx \mathrm{d}s \end{aligned}$$

so

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} F_c(s) \cos sx \mathrm{d}s, \quad x \in \mathbb{R}. \tag{2.4.4}$$

3. Suppose that f is *odd*. Then

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cos sx \mathrm{d}x - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \sin sx \mathrm{d}x = \\ &= -2\frac{i}{\sqrt{2\pi}} \int_0^{+\infty} f(x) \sin sx \mathrm{d}x = \\ &= -i\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \sin sx \mathrm{d}x. \end{aligned}$$

The function

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \sin sx \mathrm{d}x, \quad s \in \mathbb{R}. \tag{2.4.5}$$

is called the *Fourier sine transform* of the odd function f, and we have

$$F(s) = -iF_s(s), \quad s \in \mathbb{R}$$

Since F_s is odd, F is also odd, consequently

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s) \cos sx\mathrm{d}s + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s) \sin sx ds = \\
&= 2\frac{i}{\sqrt{2\pi}} \int_{0}^{+\infty} F(s) \sin sx\mathrm{d}s
\end{aligned}
$$

so that

$$
f(x) = i\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} (-iF(s)) \sin sx\mathrm{d}s
$$

and finally

$$
f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} F_s(s) \sin sx\mathrm{d}s, x \in \mathbb{R}. \tag{2.4.6}
$$

To resume:

- If f is an arbitrary Fourier original, then the direct and inverse Fourier transforms are described by (2.4.1) and (2.4.2);
- If f is an even original, then the direct and inverse cosine Fourier transforms are given by (2.4.3) and (2.4.4);
- If f is an odd original, then the direct and inverse sine Fourier transforms are given by (2.4.5) and (2.4.6).

4. Properties of the Fourier transform

In what follows we shall suppose that the functions subject to the Fourier transform are originals, i.e. they have the properties a), b), c) above.

a) *Linearity* If $\alpha, \beta \in \mathbb{C}$, then

$$
\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[g]
$$

b) *Differentiation of the image*

$$
\frac{d}{ds} \mathcal{F}[f(x)](s) = -i\mathcal{F}[xf(x)](s)
$$

Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[f(x)](s) &= \frac{\mathrm{d}}{\mathrm{d}s}(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-isx}\mathrm{d}x) = \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-isx}(-ix)\mathrm{d}x = \\ &= -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} xf(x)e^{-isx}\mathrm{d}x = \\ &= -i\mathcal{F}[xf(x)](s)\end{aligned}$$

c) *Differentiation of the original*

If

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$$

then

$$\mathcal{F}[f'(x)](s) = is\mathcal{F}[f(x)](s) \tag{2.4.7}$$

Proof.

$$\begin{aligned}\mathcal{F}[f'(x)](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f'(x)e^{-isx}\mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi}}\left(f(x)e^{-isx}|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(x)e^{-isx}(-is)\mathrm{d}x\right) = \\ &= is\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-isx}\mathrm{d}x = is\mathcal{F}[f(x)](s)\end{aligned}$$

d) *Integration of the original*

If $\displaystyle\int_{-\infty}^{+\infty} f(t)\mathrm{d}t = 0$, then

$$\mathcal{F}\left[\int_{-\infty}^{x} f(t)\mathrm{d}t\right](s) = \frac{1}{is}\mathcal{F}[f(x)](s). \tag{2.4.8}$$

Proof. Consider the function $g(x) = \displaystyle\int_{-\infty}^{x} f(t)dt$, then

$$\lim_{x\to-\infty} g(x) = \lim_{x\to+\infty} g(x) = 0$$

According to (2.4.7),

$$\mathcal{F}[g'(x)](s) = is\mathcal{F}[g(x)](s)$$

Since $g'(x) = f(x)$, we get $\mathcal{F}[f(x)](s) = is\mathcal{F}\left[\int_{-\infty}^{x} f(t)\mathrm{d}t\right](s)$ which is equivalent to (2.4.8).

e) *Change of scale*

If $a \in \mathbb{R}$, $a > 0$, then

$$\mathcal{F}\left[f\left(\frac{x}{a}\right)\right](s) = a\mathcal{F}[f(x)](as). \tag{2.4.9}$$

Proof.

$$\mathcal{F}\left[f\left(\frac{x}{a}\right)\right](s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f\left(\frac{x}{a}\right)e^{-isx}\mathrm{d}x$$

Denoting $\dfrac{x}{a} = y$, we can write

$$\begin{aligned}\mathcal{F}\left[f\left(\frac{x}{a}\right)\right](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-isay}a\mathrm{d}y = \\ &= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-i(sa)y}\mathrm{d}y = a\mathcal{F}[f(x)](as).\end{aligned}$$

f) *Translation of the original*

$$\mathcal{F}[f(x-a)](s) = e^{-ias}\mathcal{F}[f(x)](s), \quad a \in \mathbb{R} \tag{2.4.10}$$

Indeed, $\mathcal{F}[f(x-a)](s) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} f(x-a)e^{-isx}\mathrm{d}x$

Denoting $x - a = y$, we can write

$$\begin{aligned}\mathcal{F}[f(x-a)](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-is(y+a)}\mathrm{d}y = \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-isy}e^{-isa}\mathrm{d}y.\end{aligned}$$

g) *Translation of the image*

$$\mathcal{F}[f(x)](s+a) = \mathcal{F}[e^{-iax}f(x)](s), \quad a \in \mathbb{R} \tag{2.4.11}$$

Indeed,

$$\begin{aligned}\mathcal{F}[f(x)](s+a) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i(s+a)x}\mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-iax}f(x)e^{-isx}\mathrm{d}x = \mathcal{F}[e^{-iax}f(x)](s).\end{aligned}$$

h) *Convolution of originals*

The convolution of the Fourier originals f and g is by definition the function

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

We have

$$\mathcal{F}[f*g] = \sqrt{2\pi}\mathcal{F}[f]\mathcal{F}[g].$$

Indeed,

$$\begin{aligned}\mathcal{F}[f*g](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (f*g)(x)e^{-isx}\mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(t)g(x-t)\mathrm{d}t\right)e^{-isx}\mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(t)g(x-t)e^{-isx}\mathrm{d}x\right)\mathrm{d}t = \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\left(\int_{-\infty}^{+\infty} g(x-t)e^{-isx}\mathrm{d}x\right)\mathrm{d}t.\end{aligned}$$

Denoting $x - t = y$, we get

$$\begin{aligned}\mathcal{F}[f*g](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\left(\int_{-\infty}^{+\infty} g(y)e^{-ist}e^{-isy}\mathrm{d}y\right)\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{-ist}\mathrm{d}t\int_{-\infty}^{+\infty} g(y)e^{-isy}\mathrm{d}y = \sqrt{2\pi}\,\mathcal{F}[f](s)\mathcal{F}[g](s).\end{aligned}$$

i) *Parseval's formula* Let $F = \mathcal{F}[f]$, $G = \mathcal{F}[g]$. Then

$$\int_{-\infty}^{+\infty} F(s)\overline{G(s)}\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\mathrm{d}x$$

Proof.

$$
\begin{aligned}
\int_{-\infty}^{\infty} F(s)\overline{G(s)}\mathrm{d}s &= \int_{-\infty}^{+\infty} F(s)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\overline{g(x)e^{-isx}}\mathrm{d}x\right)\mathrm{d}s = \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} F(s)\overline{g(x)}e^{isx}\mathrm{d}s\right)\mathrm{d}x = \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\overline{g(x)}\left(\int_{-\infty}^{+\infty} F(s)e^{isx}\mathrm{d}s\right)\mathrm{d}x = \\
&= \int_{-\infty}^{+\infty}\overline{g(x)}f(x)\mathrm{d}x.
\end{aligned}
$$

Particular case. Taking $f = g$ in Parseval's formula, we get

$$\int_{-\infty}^{+\infty} |F(s)|^2\mathrm{d}s = \int_{-\infty}^{+\infty} |f(x)|^2\mathrm{d}x$$

Solved problems

2.13 Find the Fourier transform of the following originals:

(i) $f(x) = \dfrac{1}{x^2+1}$;
(ii) $f(x) = \dfrac{1}{x^2+a^2}$, $a > 0$;
(iii) $f(x) = \dfrac{1}{(x-b)^2+a^2}$, $a > 0, b \in \mathbb{R}$;

Solution.
(i) $F(s) = \displaystyle\int_{-\infty}^{+\infty} e^{-isx}\frac{1}{x^2+1}\mathrm{d}x.$
Let $s < 0$. Then $-s > 0$ and

$$
\begin{aligned}
F(s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i(-s)x}\frac{1}{x^2+1}\mathrm{d}x = \\
&= \frac{1}{\sqrt{2\pi}}2\pi i \operatorname*{Res}_{z=i}\frac{e^{i(-s)z}}{z^2+1} = \sqrt{2\pi}i\frac{e^s}{2i} = \sqrt{\frac{\pi}{2}}e^s
\end{aligned}
$$

.

Let now $s > 0$. Then $-s < 0$ and

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i(-s)x} \frac{1}{x^2+1} \mathrm{d}x =$$
$$= -\frac{1}{\sqrt{2\pi}} 2\pi i \operatorname*{Res}_{z=-i} \frac{e^{i(-s)z}}{z^2+1} = -\sqrt{2\pi} i \frac{e^{-s}}{-2i} = \sqrt{\frac{\pi}{2}} e^{-s}$$

.

If $s = 0$, then $F(0) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \mathrm{d}x = \sqrt{\frac{\pi}{2}}$.

Summing up, we get

$$\mathcal{F}\left[\frac{1}{x^2+1}\right](s) = \sqrt{\frac{\pi}{2}} e^{-|s|}, \; s \in \mathbb{R}. \tag{2.4.12}$$

(ii) A solution can be given as in the previous example, but we prefer to use the "change of scale" formula 2.4.9.

Let $f(x) = \dfrac{1}{x^2+1}$. According to 2.4.9,

$$\mathcal{F}\left[f\left(\frac{x}{a}\right)\right](s) = a\mathcal{F}\left[f(x)\right](as), a > 0$$

.

Combined with the result from (i), this yields

$$\mathcal{F}\left[\frac{1}{\left(\frac{x}{a}\right)^2+1}\right](s) = a\sqrt{\frac{\pi}{2}} e^{-|as|}, \text{ that is,}$$

$$\mathcal{F}\left[\frac{a^2}{x^2+a^2}\right](s) = a\sqrt{\frac{\pi}{2}} e^{-a|s|}.$$

Finally we get

$$\mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = \frac{1}{a}\sqrt{\frac{\pi}{2}} e^{-a|s|}, a > 0. \tag{2.4.13}$$

(iii) According to 2.4.10 we have

$$\mathcal{F}\left[f(x-b)\right](s) = e^{-ibs}\mathcal{F}\left[f(x)\right](s), b \in \mathbb{R}$$

Let $f(x) = \dfrac{1}{x^2+a^2}$, $a > 0$. Then we can use 2.4.13 in order to obtain

$$\mathcal{F}\left[\frac{1}{(x-b)^2+a^2}\right](s) = e^{-ibs}\frac{1}{a}\sqrt{\frac{\pi}{2}}e^{-a|s|},$$

and finally

$$\mathcal{F}\left[\frac{1}{(x-b)^2+a^2}\right](s) = \frac{1}{a}\sqrt{\frac{\pi}{2}}e^{-a|s|-ibs}, a > 0, b \in \mathbb{R} \tag{2.4.14}$$

2.14 Find $\mathcal{F}\left[\frac{x^2+1}{x^4+x^2+1}\right](s)$

Solution.

$$\frac{x^2+1}{x^4+x^2+1} = \frac{x^2+1}{(x^2+1)^2-x^2} = \frac{1}{2}\left(\frac{1}{x^2+x+1}+\frac{1}{x^2-x+1}\right).$$

$$\mathcal{F}\left[\frac{x^2+1}{x^4+x^2+1}\right](s) = \frac{1}{2}\mathcal{F}\left[\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}+\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\right](s).$$

Now using 2.4.14 we get

$$\mathcal{F}\left[\frac{x^2+1}{x^4+x^2+1}\right](s) = \frac{1}{2}\frac{2}{\sqrt{3}}\sqrt{\frac{\pi}{2}}e^{-\frac{\sqrt{3}}{2}|s|}\left(e^{i\frac{s}{2}}+e^{-i\frac{s}{2}}\right) = 2\sqrt{\frac{\pi}{6}}e^{-\frac{\sqrt{3}}{2}|s|}\cos\frac{s}{2}.$$

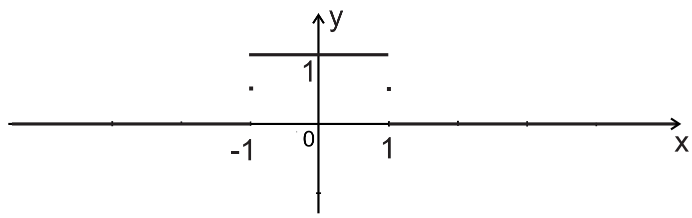
2.15 Find the cosine Fourier transform of the function
$f(x) = \begin{cases} 1, |x| < 1 \\ \frac{1}{2}, |x| = 1 \\ 0, |x| > 1 \end{cases}$ and prove that

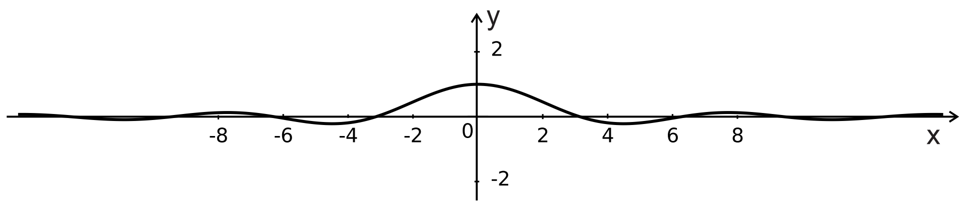
$$\int_0^{+\infty}\frac{\cos sx\sin s}{s}\mathrm{d}s = \begin{cases} \frac{\pi}{2},\ |x| < 1 \\ 0,\ |x| > 1 \\ \frac{\pi}{4},\ |x| = 1 \end{cases}$$

Solution. The function f is even. According to 2.4.3,

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} f(x)\cos sx\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_0^1 \cos sx\mathrm{d}x = \sqrt{\frac{2}{\pi}}\frac{\sin s}{s}, s \in \mathbb{R}.$$

Notice that the rectangular function f is transformed in $\frac{\sin x}{x}$ (see the graphic representation bellow). This is the *unnormalized sinc function*, defined by $\frac{\sin x}{x}$, for $x \neq 0$ and by sinc(0)=1.





Now 2.4.4 gives

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} F_c(s) \cos sx \mathrm{d}s = \frac{2}{\pi} \int_0^{+\infty} \frac{\cos sx \sin s}{s} \mathrm{d}s.$$

Consequently,

$$\int_0^{+\infty} \frac{\cos sx \sin s}{s} \mathrm{d}s = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \\ \frac{\pi}{4}, & |x| = 1. \end{cases}$$

In particular, for $x = 0$ we obtain

$$\int_0^{+\infty} \frac{\sin s}{s} \mathrm{d}s = \frac{\pi}{2}.$$

2.16 Find the sine Fourier transform of the function
$f(x) = \begin{cases} \sin x, & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}$ and prove that

$$\int_0^{+\infty} \frac{\sin sx \sin \pi s}{1 - s^2} ds = \begin{cases} \frac{\pi}{2} \sin x, & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}$$

Solution. f is an odd function and 2.4.5 shows that

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \sin sx \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin x \sin sx \mathrm{d}x =$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\pi} (\cos x(1-s) - \cos x(1+s))\mathrm{d}x = \frac{1}{2}\sqrt{\frac{2}{\pi}} \left(\frac{\sin \pi s}{1-s} + \frac{\sin \pi s}{1+s} \right) =$$

$= \sqrt{\frac{2}{\pi}} \frac{\sin \pi s}{1-s^2}$. Briefly, $F_s(s) = \sqrt{\frac{2}{\pi}} \frac{\sin \pi s}{1-s^2}$ and now, 2.4.6 leads to

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} F_s(s) \sin sx \mathrm{d}s = \frac{2}{\pi} \int_0^{+\infty} \frac{\sin sx \sin \pi s}{1-s^2} \mathrm{d}s.$$

This entails

$$\int_0^{+\infty} \frac{\sin sx \sin \pi s}{1-s^2} \mathrm{d}s = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} \sin x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$$

Proposed problems

2.17 Let $a > 0$, $b \in \mathbb{R}$. Find:

i). $\mathcal{F}\left[\frac{x}{(x^2+a^2)^2}\right](s)$;

ii). $\mathcal{F}\left[\frac{1}{(x^2+a^2)^2}\right](s)$;

iii). $\mathcal{F}\left[\frac{e^{-ibx}}{x^2+a^2}\right](s)$.

2.18 Find the Fourier transforms of the following functions:

i). $f(x) = \begin{cases} 0, & |x| \ge a \\ x+a, & -a < x \le 0 \\ -x+a, & 0 < x < a \end{cases}$

ii). $f(x) = \begin{cases} e^{ax}, \;\; x < 0 \\ \dfrac{1}{2}, \;\; x = 0 \\ 0, x > 0 \end{cases}, \; a > 0;$

iii). $f(x) = \begin{cases} \cos 2x, \;\; |x| < a \\ 0, \;\; |x| > a \end{cases};$

iv). $f(x) = \dfrac{d^n}{dx^n}(e^{-\frac{x^2}{2}});$

v). $f(x) = \begin{cases} 1, \;\; |x| < \pi \\ \dfrac{1}{2}, \; |x| = \pi \\ 0, \;\; |x| > \pi. \end{cases}$

Solutions and answers

$\boxed{\mathbf{2.17}}$ **i)** Take $f(x) = \dfrac{1}{x^2 + a^2}$; use 2.4.13 and $\mathcal{F}[f'(x)](s) = is\mathcal{F}[f(x)](s)$.

$\mathcal{F}\left[\dfrac{x}{(x^2+a^2)^2}\right](s) = -\dfrac{is}{2a}\sqrt{\dfrac{\pi}{2}}e^{-a|s|}$

ii) From 2.4.13 we know that $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} \dfrac{e^{-isx}}{x^2+a^2}dx = \sqrt{\dfrac{\pi}{2}}\dfrac{e^{-a|s|}}{a}.$

Take the derivative with respect to a.

$\mathcal{F}\left[\dfrac{1}{(x^2+a^2)^2}\right](s) = \sqrt{\dfrac{\pi}{2}}\dfrac{a|s|+1}{2a^3}e^{-a|s|}$

iii) Take $f(x) = \dfrac{1}{x^2+a^2}$; use 2.4.13 and 2.4.11.

$\mathcal{F}\left[\dfrac{e^{-ibx}}{x^2+a^2}\right](s) = \dfrac{1}{a}\sqrt{\dfrac{\pi}{2}}e^{-a|s+b|}.$

$\boxed{\mathbf{2.18}}$ **i)** Since f is an even function, we have $F(s) = F_c(s)$. For $s \neq 0$, $F_c(s) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^a (a-x)\cos sx\mathrm{d}x = \sqrt{\dfrac{2}{\pi}}\dfrac{1-\cos sa}{s^2}$

Consequently, $\displaystyle\int_0^\infty \dfrac{1-\cos sa}{s^2}\cos sx\mathrm{d}x = \dfrac{\pi}{2}f(x).$

ii) $F(s) = \dfrac{1}{\sqrt{2\pi}} \dfrac{a+is}{a^2+s^2}$

iii) $F(s) = F_c(s) = \dfrac{1}{\sqrt{2\pi}} \left(\dfrac{\sin a(2+s)}{2+s} + \dfrac{\sin a(2-s)}{2-s} \right)$

iv) We have $\mathcal{F}[e^{-\frac{x^2}{2}}](s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} e^{-isx} dx =$
$= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} e^{-(\frac{x}{\sqrt{2}}+\frac{is}{\sqrt{2}})^2-\frac{s^2}{2}} dx = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \displaystyle\int_{-\infty}^{+\infty} e^{-u^2} \sqrt{2} du =$
$= e^{-\frac{s^2}{2}}$, *i.e.*, $\mathcal{F}[e^{-\frac{x^2}{2}}](s) = e^{-\frac{s^2}{2}}$. It follows that

$$\mathcal{F}\left[\frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})\right](s) = (is)^n e^{-\frac{s^2}{2}}.$$

v) By 2.4.1 we obtain $\mathcal{F}[f(x)](s) = \sqrt{2\pi}\dfrac{\sin \pi s}{\pi s}$, for $s \neq 0$ and $\sqrt{2\pi}$ for $s = 0$. ($\dfrac{\sin \pi s}{\pi s}$ is the *normalized sinc function*, an important function in electrical engineering).

2.5 The z transform

The z *transform* of the function $f : \mathbb{N} \to \mathbb{C}$ is the function defined by

$$F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots + \frac{f(n)}{z^n} + \ldots,$$

for every $z \in \mathbb{C}$ with $|z| > R$, for a certain $R > 0$.

We use the notation $F(z) = \mathcal{Z}[f(n)]$. The Z-transform is a Laurent series representation of a discrete-time sequence.

Example 2.5.1 For $a \in \mathbb{C}$ consider the function $f(n) = a^n$, $n \geq 0$. Then the z transform is

$$F(z) = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a},$$

where $\left|\frac{a}{z}\right| < 1$, that is $|z| > |a|$. Thus

$$(1) \quad \mathcal{Z}[a^n] = \frac{z}{z-a}.$$

For the particular case $a = 1$, we get the *unit step* $u(n) = 1$, for all $n \in \mathbb{N}$ and

$$(2) \quad \mathcal{Z}[u(n)] = \mathcal{Z}[1] = \frac{z}{z-1}.$$

Example 2.5.2 Let $\delta(n) = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{in rest} \end{cases}$ be the *unit impulse*. It is very easy to see that

$$\mathcal{Z}[\delta(n)] = 1.$$

Example 2.5.3 $f(n) = n$, $n \geq 0$. Then

$$\begin{aligned} F(z) &= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots = z\Big(\frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \ldots\Big) \\ &= z\Big(-1 - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} + \ldots\Big)' = -z\left(\frac{1}{1-\frac{1}{z}}\right)' \\ &= -z\left(\frac{z}{z-1}\right)' = \frac{z}{(z-1)^2} \end{aligned}$$

Thus

$$(3)\quad \mathcal{Z}[n] = \frac{z}{(z-1)^2}.$$

Example 2.5.4 $f(n) = \sin an$, $n \geq 0$. Using

$$\sin an = \frac{e^{ian} - e^{-ian}}{2i}$$

and formula (2) we obtain

$$\begin{aligned} F(z) &= \frac{1}{2i}\big(\mathcal{Z}[(e^{ia})^n] - \mathcal{Z}[(e^{-ia})^n]\big) = \frac{1}{2i}\left(\frac{z}{z-e^{ia}} - \frac{z}{z-e^{-ia}}\right) \\ &= \frac{z}{2i} \cdot \frac{z - \cos a + i\sin a - z + \cos a + i \sin a}{(z-\cos a - i\sin a)(z - \cos a + i \sin a)} = \frac{z \sin a}{(z-\cos a)^2 + \sin^2 a} \end{aligned}$$

Thus

$$(4)\quad \mathcal{Z}[\sin an] = \frac{z \sin a}{z^2 - 2z\cos a + 1}$$

In the same way can be obtained:

$$(5)\quad \mathcal{Z}[\cos an] = \frac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$$

$$(6)\quad \mathcal{Z}[\sinh an] = \frac{z \sinh a}{z^2 - 2z\sinh a + 1}$$

$$(7)\quad \mathcal{Z}[\cosh an] = \frac{z(z-\cosh a)}{z^2 - 2z\cosh a + 1}$$

Example 2.5.5 $f(n) = \dfrac{1}{n+1}$, $n \geq 0$. Then

$$F(z) = 1 + \frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \dots.$$

We calculate

$$\frac{F(z)}{z} = \frac{\frac{1}{z}}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots = \log \frac{1}{1-\frac{1}{z}} = \log\frac{z}{z-1}.$$

Thus

$$(8)\quad \mathcal{Z}\left[\frac{1}{n+1}\right] = z \log\frac{z}{z-1}.$$

In what follows we present several properties of the z transform, useful in applications.

Theorem 2.3 *If* $\mathcal{Z}[f(n)] = F(z)$, *then* $\mathcal{Z}[a^n f(n)] = F\left(\frac{z}{a}\right)$.

Proof.

$$\begin{aligned}\mathcal{Z}[a^n f(n)] &= f(0) + \frac{af(1)}{z} + \frac{a^2 f(2)}{z^2} + \dots \\ &= f(0) + \frac{f(1)}{\frac{z}{a}} + \frac{f(2)}{\left(\frac{z}{a}\right)^2} + \dots = F\left(\frac{z}{a}\right).\end{aligned}$$

Theorem 2.4 *If* $\mathcal{Z}[f(n)] = F(z)$, *then* $\mathcal{Z}[f(n+1)] = zF(z) - zf(0)$.

Proof. Using the definition we get

$$\begin{aligned}\mathcal{Z}[f(n+1)] &= f(1) + \frac{f(2)}{z} + \frac{f(3)}{z^2} + \dots + \frac{f(n)}{z^{n+1}} + \dots \\ &= z\Big(f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots\Big) - zf(0) = zF(z) - zf(0).\end{aligned}$$

Generally, for $p \in \mathbb{N}^*$ we have

$$\mathcal{Z}[f(n+p)] = z^p F(z) - z^p f(0) - z^{p-1} f(1) - \dots - zf(p-1). \tag{2.5.1}$$

Consider the function of Heaviside $u : \mathbb{Z} \to \mathbb{R}$, $u(k) = \begin{cases} 0, \text{ if } k < 0 \\ 1, \text{ if } k \geq 0. \end{cases}$

Let $f : \mathbb{N} \to \mathbb{C}$ and consider an arbitrary extension of it, $f : \mathbb{Z} \to \mathbb{C}$. Then it is possible to define for an arbitrary $q \in \mathbb{N}^*$ the function

$$u(n-q)f(n-q) = \begin{cases} 0, & \text{if } n \in \{0, 1, \dots, q-1\} \\ f(n-q), & \text{if } n \in \{q, q+1, \dots\} \end{cases}$$

and we have

Theorem 2.5 $\mathcal{Z}[u(n-q)f(n-q)] = z^{-q} F(z)$.

Proof. With the previous definition it follows that

$$\begin{aligned}\mathcal{Z}[u(n-q)f(n-q)] &= 0 + \frac{0}{z} + \dots + \frac{0}{z^{q-1}} + \frac{f(0)}{z^q} + \frac{f(1)}{z^{q+1}} + \dots \\ &= \frac{1}{z^q}\Big(f(0) + \frac{f(1)}{z} + \dots\Big) = z^{-q} F(z).\end{aligned}$$

Theorem 2.6 $\mathcal{Z}[nf(n)] = -z\dfrac{dF}{dz}$.

Proof. Using the same idea as in Example 2.5.3 we get

$$\begin{aligned}\mathcal{Z}[nf(n)] &= 0 + 1\frac{f(1)}{z} + 2\frac{f(2)}{z^2} + \cdots = z\Big(\frac{f(1)}{z^2} + \frac{2f(2)}{z^3} + \ldots\Big)\\ &= z\Big(-f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} - \ldots\Big)' = -z\frac{\mathrm{d}F}{\mathrm{d}z}.\end{aligned}$$

Theorem 2.7 $\dfrac{d}{dz}\mathcal{Z}[u(n-1)\dfrac{f(n)}{n}] = -\dfrac{F(z)-f(0)}{z}$.

Theorem 2.8 *If* $\mathcal{Z}[f(n)] = F(z)$ *then*

$$\mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right] = \frac{z}{z-1}F(z).$$

Proof. From the definition of the z transform and formula (2) we get

$$\begin{aligned}\frac{z}{z-1}F(z) &= \mathcal{Z}[1]\mathcal{Z}[f(n)]\\ &= \left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)\left(f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots\right)\\ &= f(0) + \frac{f(0)+f(1)}{z} + \frac{f(0)+f(1)+f(2)}{z^2} + \cdots = \mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right].\end{aligned}$$

Let $f, g : \mathbb{N} \to \mathbb{C}$ be two functions. The *convolution* of f and g is the function defined by

$$(f * g)(n) = \sum_{k=0}^{n} f(k)g(n-k), \text{ for every } n \in \mathbb{N}.$$

Theorem 2.9 *If we denote* $\mathcal{Z}[f(n)] = F(z)$ *and* $\mathcal{Z}[g(n)] = G(z)$*, then*

$$\mathcal{Z}[(f * g)(n)] = F(z)G(z).$$

Proof. By the definition of the z transform, we have

$$F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots \text{ and } G(z) = g(0) + \frac{g(1)}{z} + \frac{g(2)}{z^2} + \ldots$$

Multiplying these two series, it follows that

$$
\begin{aligned}
F(z)G(z) = f(0)g(0) &+ \frac{f(0)g(1)+f(1)g(0)}{z} \\
&+ \frac{f(0)g(2)+f(1)g(1)+f(2)g(0)}{z^2} + \dots \\
= (f*g)(0) &+ \frac{(f*g)(1)}{z} + \frac{(f*g)(2)}{z^2} + \dots = \mathcal{Z}[(f*g)(n)].
\end{aligned}
$$

Remark 2.10 It can be seen easily that Theorem 2.8 is a particular case of Theorem 2.9.

Theorem 2.11 *If* $\mathcal{Z}[f(n)] = F(z)$ *then*

$$
\lim_{z\to\infty} F(z) = f(0) \quad \textit{and} \quad \lim_{z\to 1}(z-1)F(z) = \lim_{n\to\infty} f(n).
$$

2.6 The inverse z transform

We have

$$
F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots + \frac{f(n)}{z^n} + \dots, \text{ for } |z| > R.
$$

Let C be a circle with center 0 and radius ρ, where $\rho > R$. Then all the singular points (in $\mathbb{C}$) of the function $F(z)$ lie inside the circle C. We have

$$
\int_C z^{n-1}F(z)\mathrm{d}z = f(0)\int_C z^{n-1}\mathrm{d}z + \dots f(n)\int_C z^{-1}\mathrm{d}z + \dots.
$$

It is easy to verify (using the Residue Theorem and the definition of the residue) that

$$
\int_C z^k\mathrm{d}z = 0 \text{ if } k \neq -1 \text{ and } \int_C z^{-1}\mathrm{d}z = 2\pi i.
$$

Thus $\displaystyle\int_C z^{n-1}F(z)\mathrm{d}z = 2\pi i f(n)$. This gives the *inverse z transform*, denoted by

$$
\mathcal{Z}^{-1}[F(z)] = f(n) = \frac{1}{2\pi i}\int_C z^{n-1}F(z)\mathrm{d}z.
$$

Example 2.6.1 Find the inverse z transform of the function

$$F(z) = \frac{z^2}{z^2 + z - 2}.$$

We give three methods for finding the inverse.

a) Let C : $|z| = 3$ be the circle with center O and radius 3. Then the inverse is

$$\begin{aligned} f(n) &= \frac{1}{2\pi i} \int_C z^{n-1} F(z) \mathrm{d}z \\ &= \frac{1}{2\pi i} \cdot 2\pi i \left(\operatorname*{Res}_{z=1} \frac{z^{n+1}}{(z-1)(z+2)} + \operatorname*{Res}_{z=-2} \frac{z^{n+1}}{(z-1)(z+2)} \right) \end{aligned}$$

$z = 1$ and $z = -2$ are poles of order one, so using the formulas from section 1.7 we get

$$f(n) = \frac{1}{3} + \frac{2}{3}(-2)^n.$$

b) To expand $F(z)$ in a Laurent series we write

$$F(z) = 1 + \frac{1}{3} \cdot \frac{1}{z-1} - \frac{4}{3} \cdot \frac{1}{z+2} = 1 + \frac{1}{3z} \cdot \frac{1}{1 - \frac{1}{z}} - \frac{4}{3z} \cdot \frac{1}{1 + \frac{2}{z}}$$

For $|z| > 2$ we have

$$\begin{aligned} F(z) &= 1 + \frac{1}{3z} \sum_{n=0}^{\infty} \frac{1}{z^n} - \frac{4}{3z} \sum_{n=0}^{\infty} \Big(-\frac{2}{z} \Big)^n \\ &= 1 + \sum_{n=0}^{\infty} \Big(\frac{1}{3z^{n+1}} - \frac{4}{3} \cdot \frac{(-2)^n}{z^{n+1}} \Big) = \sum_{n=0}^{\infty} \Big(\frac{1}{3} + \frac{2}{3}(-2)^n \Big) \frac{1}{z^n}, \end{aligned}$$

so as before $f(n) = \dfrac{1}{3} + \dfrac{2}{3}(-2)^n$.

c) The function can be written

$$F(z) = \frac{1}{3} \cdot \frac{z}{z-1} + \frac{2}{3} \cdot \frac{z}{z+2}.$$

By formula (1) we get

$$f(n) = \frac{1}{3} \mathcal{Z}^{-1} \left[\frac{z}{z-1} \right] + \frac{2}{3} \mathcal{Z}^{-1} \left[\frac{z}{z-(-2)} \right] = \frac{1}{3} \cdot 1 + \frac{2}{3}(-2)^n.$$

Solved problems

2.19 Find the images by the z transform of the functions:

i). $f(n) = n^2$

ii). $f(n) = \dfrac{n(n+1)(2n+1)}{6}$

iii). $f(n) = \dfrac{\cos n}{n+3}$

Solution. i). Using Theorem 2.6 and (3) we get

$$\mathcal{Z}[n^2] = -z\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Z}[n] = -z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{(z-1)^2}\right) = \frac{z(z+1)}{(z-1)^3}.$$

ii). It is well known that $f(n) = 1^2 + 2^2 + \cdots + n^2$. Using Theorem 2.8 and the previous exercise we have

$$\mathcal{Z}[f(n)] = \mathcal{Z}\left[\sum_{k=0}^{n} k^2\right] = \frac{z}{z-1} \cdot \frac{z(z+1)}{(z-1)^3} = \frac{z^2(z+1)}{(z-1)^4}.$$

iii). Using the definition of cosine and Theorem 2.3 we write

$$\mathcal{Z}\left[\frac{\cos n}{n+3}\right](z) = \mathcal{Z}\left[\frac{e^{in}+e^{-in}}{2(n+3)}\right](z) = \frac{1}{2}\left(F\left(\frac{z}{e^i}\right) + F\left(\frac{z}{e^{-i}}\right)\right),$$

where, denoting $g(n) = \dfrac{1}{n+1}$,

$$F(z) = \mathcal{Z}\left[\frac{1}{n+3}\right] = z^2\mathcal{Z}\left[\frac{1}{n+1}\right] - z^2g(0) - zg(1) = z^3\log\frac{z}{z-1} - z^2 - \frac{z}{2}.$$

2.20 Find the images by the z transform of the function

$$f_p(n) = (n+1)(n+2)\dots(n+p)$$

where $p \in \mathbb{N}$ is fixed.

Solution. Denote $F_p(z) = \mathcal{Z}[f_p(n)]$ and by the definition we have

$$\begin{aligned}
F_p(z) &= \sum_{n=0}^{\infty} \frac{(n+1)\dots(n+p)}{z^n} = \frac{1}{p+1}\sum_{n=0}^{\infty} \frac{(n+1)\dots(n+p)[(n+p+1)-n]}{z^n} \\
&= \frac{1}{p+1}\sum_{n=0}^{\infty} \frac{(n+1)\dots(n+p+1)}{z^n} - \frac{1}{p+1}\sum_{n=0}^{\infty} \frac{n(n+1)\dots(n+p)}{z^n} \\
&= \frac{1}{p+1}F_{p+1}(z) - \frac{1}{p+1}\cdot\frac{1}{z}\sum_{n=1}^{\infty} \frac{n(n+1)\dots(n+p)}{z^{n-1}}.
\end{aligned}$$

In the last sum we make the change $m := n-1$ and it follows

$$F_p(z) = \frac{1}{p+1}F_{p+1}(z) - \frac{1}{p+1}\cdot\frac{1}{z}\sum_{m=0}^{\infty} \frac{(m+1)\dots(m+p+1)}{z^m}$$

This gives

$$F_p(z) = \frac{1}{p+1}\left(1-\frac{1}{z}\right)F_{p+1}(z), \text{ or } F_{p+1}(z) = \frac{(p+1)z}{z-1}F_p(z).$$

We write the last recurrence relation from 2 to p

$$\begin{aligned}
F_2(z) &= 2\cdot\frac{z}{z-1}F_1(z) \\
F_3(z) &= 3\cdot\frac{z}{z-1}F_2(z) \\
&\dots\dots\dots \\
F_p(z) &= p\cdot\frac{z}{z-1}F_{p-1}(z)
\end{aligned}$$

and multiplying these equalities we get that

$$F_p(z) = p!\left(\frac{z}{z-1}\right)^{p-1} F_1(z).$$

The first term is

$$F_1(z) = \mathcal{Z}[n+1] = \frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2}{(z-1)^2}$$

so finally

$$F_p(z) = p!\left(\frac{z}{z-1}\right)^{p+1}.$$

Proposed problems

2.21 Prove that $\mathcal{Z}[n^2 f(n)] = z^2 F''(z) + zF'(z)$, where $F(z) = \mathcal{Z}[f(n)]$.

2.22 Find the images of the following functions by the z transform:

i). $f(n) = \dfrac{n}{(n+1)(n+2)}$

ii). $f(n) = \dfrac{5^n n}{(n+1)(n+2)}$

iii). $f(n) = \dfrac{n(n+1)}{2}$

iv). $f(n) = \sin^2 \dfrac{\pi n}{4} + \sin \dfrac{\pi n}{3}$

2.23 Let $a \in \mathbb{C}$ and $f(n) = n + (n-1)a + (n-2)a^2 + \cdots + 2a^{n-2} + a^{n-1}$. Find $\mathcal{Z}[f(n)]$.

Solutions and answers

$\boxed{\textbf{2.21}}$ Use Theorem 2.6 two times.

$\boxed{\textbf{2.22}}$ **i)** $f(n) = n\Big(\dfrac{1}{n+1} - \dfrac{1}{n+2}\Big)$, $\mathcal{Z}\Big[\dfrac{1}{n+2}\Big] = z^2 \log \dfrac{z}{z-1} - z$,

$\mathcal{Z}[f(n)] = -z\dfrac{\mathrm{d}}{\mathrm{d}z}\mathcal{Z}\Big[\dfrac{1}{n+1} - \dfrac{1}{n+2}\Big] = -z\dfrac{\mathrm{d}}{\mathrm{d}z}\Big((z - z^2)\log \dfrac{z}{z-1} + z\Big)$
$= -z\Big(2 + (1-2z)\log \dfrac{z}{z-1}\Big)$;

ii) $F(z) = -\dfrac{z}{5}\Big(2 + (1 - \dfrac{2z}{5})\log \dfrac{z}{z-5}\Big)$ (use Theorem 2.3 and the previous exercise);

iii) $F(z) = \dfrac{z^2}{(z-1)^3}$;

iv) Write $f(n) = \dfrac{1}{2}(1 - \cos \dfrac{\pi n}{2}) + \sin \dfrac{\pi n}{3}$.

Then $\mathcal{Z}[f(n)] = \dfrac{1}{2}(\dfrac{z}{z-1} - \dfrac{z^2}{z^2+1} + \dfrac{\sqrt{3}}{2} \cdot \dfrac{z}{z^2 - z + 1})$.

2.23 We have $f(n) = g * h$, where $g(n) = a^n$ and $h(n) = n$. Thus
$\mathcal{Z}[f(n)] = \mathcal{Z}[a^n] \cdot \mathcal{Z}[n] = \dfrac{z^2}{(z-a)^2(z-1)^2}$.

2.7 Applications of the z transform

1. An important use of the z transform is for solving **difference equations**, that is, equations of the form

$$f(n+p) + a_1 f(n+p-1) + \cdots + a_{p-1} f(n+1) + a_p f(n) = g(n), \text{ for } n \geq 0$$

where $f : \mathbb{N} \to \mathbb{C}$ is the unknown function, $p \geq 0$ is a given integer, g a given function and $a_0, \ldots, a_p \in \mathbb{C}$.

Z-transforms are used in solving difference equations like Laplace transforms are used in solving differential equations.

Taking the z transform of both sides of the equation and denoting $\mathcal{Z}[f(n)] = F(z)$ we obtain an algebraic equation with the unknown $F(z)$. Then it is possible to find the function $F(z)$, and the solution of the problem will be $f(n) = \mathcal{Z}^{-1}[F(z)]$.

Example 2.7.1 Find $f(n)$ such that

$$f(n+2) + 3f(n+1) + 2f(n) = 0 \text{ with } f(0) = 1,\ f(1) = -4.$$

Applying the z transform we have

$$\mathcal{Z}[f(n+2)] + 3\mathcal{Z}[f(n+1)] + 2\mathcal{Z}[f(n)] = 0$$

and by formula (2.5.1) follows

$$z^2 F(z) - z^2 f(0) - z f(1) + 3(zF(z) - zf(0)) + 2F(z) = 0$$

which yields $F(z) = \dfrac{z^2 - z}{z^2 + 3z + 2}$.
We expand $\dfrac{F(z)}{z}$ into partial fractions $\dfrac{F(z)}{z} = \dfrac{-2}{z+1} + \dfrac{3}{z+2}$.
Consequently,

$$F(z) = \frac{-2z}{z+1} + \frac{3z}{z+2}$$

and finally

$$f(n) = \mathcal{Z}^{-1}\left[\frac{-2z}{z+1}\right] + \mathcal{Z}^{-1}\left[\frac{3z}{z+2}\right] = -2(-1)^n + 3(-2)^n.$$

2. The z transform can also be used to find the general terms of sequences given by linear recurrences.

Example 2.7.2 Find the sequence $(u_n)_{n\geq 0}$ such that

$$u_{n+2} = u_{n+1} + u_n, \text{ for } n \geq 0 \text{ and } u_0 = 0, u_1 = 1.$$

Let $\mathcal{Z}[u_n] = F(z)$. Then

$$\mathcal{Z}[u_{n+1}] = zF(z) - u_0,\ \mathcal{Z}[u_{n+2}] = z^2F(z) - z^2u_0 - zu_1.$$

This gives the equation

$$z^2F(z) - z = zF(z) + F(z)$$

and $F(z) = \dfrac{z}{z^2 - z - 1}$. We write

$$\frac{F(z)}{z} = \frac{1}{\sqrt{5}}\left(\frac{1}{z - \dfrac{1+\sqrt{5}}{2}} - \frac{1}{z - \dfrac{1-\sqrt{5}}{2}}\right)$$

and so

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

the famous Fibonacci sequence.

3. Another application is to find the sums of some series of real numbers or power series.

Example 2.7.3 Find the sum of the series $S = \displaystyle\sum_{n=0}^{\infty} \frac{n^2+7}{3^n}$.

We use the formula from Theorem 2.11

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z),$$

with $f(n)$ being the partial sum of the series, $f(n) = \sum_{k=0}^{n} \frac{k^2+7}{3^k}$.

$$F(z) = \mathcal{Z}\Big[\sum_{k=0}^{n} \frac{k^2+7}{3^k}\Big] = \frac{z}{z-1}\mathcal{Z}\Big[\frac{n^2+7}{3^n}\Big].$$

We know (see (2) and Exercise 2.20) that

$$\mathcal{Z}[n^2+7] = \frac{z(z+1)}{(z-1)^3} + \frac{7z}{z-1}.$$

Then using Theorem 2.3 follows that

$$F(z) = \frac{z}{z-1}\left[\frac{3z(3z+1)}{(3z-1)^3} + \frac{21z}{3z-1}\right].$$

Passing to the limit with $z \to 1$ we have finally

$$s = \lim_{z\to 1} z\left[\frac{3z(3z+1)}{(3z-1)^3} + \frac{21z}{3z-1}\right] = 12.$$

Solved problems

2.24 Find the general term for each of the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ given by $u_0 = 0$, $v_0 = 0$ and

$$\begin{cases} u_{n+1} = 2v_n + 2 \\ v_{n+1} = 2u_n - 1 \end{cases}.$$

Solution. Denoting $\mathcal{Z}[u_n] = U(z)$ and $\mathcal{Z}[v_n] = V(z)$ we have the system

$$\begin{cases} zU(z) = 2V(z) + \dfrac{2z}{z-1} \\ zV(z) = 2U(z) - \dfrac{z}{z-1} \end{cases},$$

which gives $U(z) = \dfrac{2z}{z^2-4}$. Expanding in partial fractions $\dfrac{U(z)}{z}$ we can write

$$U(z) = \frac{1}{2}\cdot\frac{z}{z-2} - \frac{1}{2}\cdot\frac{z}{z+2}.$$

Consequently,

$$u_n = \frac{1}{2}\cdot 2^n - \frac{1}{2}\cdot(-2)^n = 2^{n-1}(1+(-1)^{n-1}) \text{ and } v_n = 2^{n-1}(1+(-1)^n) - 1.$$

2.25 Find the general term for the sequence $(u_n)_{n\geq 0}$ given by $u_0 = 1$, $u_1 = 3$ and $u_{n+2}^2 = u_{n+1}u_n$, $u_n > 0$ for all $n \geq 1$.

Solution. Denoting $x_n = \log_3 u_n$ we have $x_0 = 0$, $x_1 = 1$ and the recurrence relation

$$2x_{n+2} = x_{n+1} + x_n.$$

The equation for $F(z) = \mathcal{Z}[x_n]$ is $2z^2F(z) - 2z = zF(z) + F(z)$ with the solution

$$F(z) = \frac{2z}{(z-1)(2z+1)} = \frac{2}{3}\Big(\frac{z}{z-1} - \frac{z}{2\big(z+\frac{1}{2}\big)}\Big).$$

Applying the inverse z transform we get

$$x_n = \frac{2}{3}\left[1 - \Big(-\frac{1}{2}\Big)^n\right] = \frac{2+(-2)^{1-n}}{3}.$$

The requested sequence is then $u_n = 3^{\frac{2+(-2)^{1-n}}{3}}$.

2.26 Find the sum of the series $S = \displaystyle\sum_{n=0}^{\infty} a^n \cos nb$, with $a, b \in \mathbb{R}$.

Solution. Let

$$S_n = 1 + a\cos b + a^2\cos 2b + \cdots + a^n \cos nb = (f * g)(n),$$

where $f(n) = a^n \cos bn$ and $g(n) = 1$. We have

$$\mathcal{Z}[a^n \cos nb] = \mathcal{Z}[\cos nb]\left(\frac{z}{a}\right) = \frac{\frac{z}{a}\left(\frac{z}{a} - \cos b\right)}{\left(\frac{z}{a}\right)^2 - 2\frac{z}{a}\cos b + 1} = \frac{z(z - a\cos b)}{z^2 - 2za\cos b + a^2}$$

and

$$\mathcal{Z}[1] = \frac{z}{z-1}.$$

Then

$$\begin{aligned} S &= \lim_{n\to\infty} S_n = \lim_{n\to\infty}(f*g)(n) = \lim_{z\to 1}(z-1)\mathcal{Z}[(f*g)(n)](z) \\ &= \lim_{z\to 1}(z-1)\frac{z}{z-1}\cdot\frac{z(z-a\cos b)}{z^2-2za\cos b+a^2} = \frac{1-a\cos b}{1-2a\cos b+a^2}. \end{aligned}$$

For instance, taking $b = \pi$ we get

$$S = \sum_{n=0}^{\infty} (-1)^n a^n = \frac{1 - a\cos\pi}{1 - 2a\cos\pi + a^2} = \frac{1+a}{1+2a+a^2} = \frac{1}{1+a}$$

the well-known geometric series.

Proposed problems

2.27 Find the general term of the sequence given by

i). $x_{n+1} - 2x_n = n,\ \ x_0 = 0$

ii). $u_{n+2} - 2u_{n+1} + u_n = 1,\ \ u_0 = u_1 = 0$

iii). $u_{n+p} - \binom{p}{1} u_{n+p-1} + \cdots + (-1)^{p-1}\binom{p}{p-1} u_{n+1} + (-1)^p u_n = 1,$
$u_0 = u_1 = \cdots = u_{p-1} = 0$, with $p \in \mathbb{N}$ fixed

iv). $x_{n+2} + 4x_{n+1} + 3x_n = 3^n,\ \ x_0 = 0,\ x_1 = 1.$

2.28 Find the sum of the series

i). $s = \displaystyle\sum_{n=0}^{\infty} \frac{n^2 - 3n + 5}{6^n}$

ii). $s = \displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n} \sin\frac{n\pi}{2}$

iii). $S = \displaystyle\sum_{n=0}^{\infty} a^n \sin nb,\ \ a, b \in \mathbb{R}.$

Solutions and answers

$\boxed{\mathbf{2.27}}$ **i)** $F(z) = \dfrac{z}{(z-1)^2(z-2)}$,

$$x_n = \frac{1}{2\pi i}\int_C z^{n-1}F(z)\mathrm{d}z = \frac{1}{2\pi i}\int_{|z|=3} \frac{z^n}{(z-1)^2(z-2)}\mathrm{d}z = \operatorname*{Res}_{z=1}\frac{z^n}{(z-1)^2(z-2)}$$
$$+ \operatorname*{Res}_{z=2}\frac{z^n}{(z-1)^2(z-2)} = \frac{1}{1!}\left(\frac{z^n}{(z-1)^2}\right)'\Big|_{z=1} + \frac{1}{0!}\frac{z^n}{z-2}\Big|_{z=2} = 2^n - n - 1.$$

ii) $F(z) = \dfrac{z}{(z-1)^3}$, $u_n = \operatorname*{Res}_{z=1} \dfrac{z^n}{(z-1)^3} = \dfrac{1}{2}n(n-1)$, for $n \geq 2$;

iii) $F(z) = \dfrac{z}{(z-1)^{p+1}}$,

$u_n = \operatorname*{Res}_{z=1} \dfrac{z^n}{(z-1)^{p+1}} = \dfrac{1}{p!} \lim_{z \to 1} (z^n)^{(p)} = \dfrac{1}{p!} n(n-1)\ldots(n-p+1)$, for $n \geq p$;

iv) $x_n = \dfrac{3}{8}(-1)^n + \dfrac{1}{24}3^n - \dfrac{5}{12}(-3)^n$.

$\boxed{\mathbf{2.28}}$ **i)** By the definition of the z transform we have

$$s = \mathcal{Z}[n^2](6) - 3\mathcal{Z}[n](6) + 5\mathcal{Z}[1](6) = \frac{z(z+1)}{(z-1)^3} - \frac{3z}{(z-1)^2} + \frac{5z}{z-1}\Big|_{z=6} = \frac{702}{125};$$

ii) $s = \mathcal{Z}\left[n \sin \dfrac{n\pi}{2}\right](2) = -z \left(\dfrac{z}{z^2+1}\right)' \Big|_{z=2} = -z \cdot \dfrac{1-z^2}{(z^2+1)^2}\Big|_{z=2} = \dfrac{6}{25}$;

iii) $s = \dfrac{a \sin b}{1 - 2a\cos b + a^2}$.

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