Seminar 9

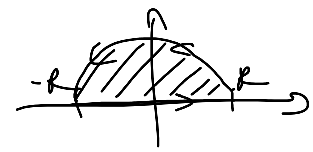
Wednesday, November 25, 2020 5:54 PM

Using Residue Theorem to evaluate real integrals

IV $\int_0^\infty f(x)\,dx$, $f(x) = \frac{P(x)}{Q(x)}$, P, Q are polynomials , $n \geq m+2$
($P \to m$, $Q \to n$)

f is even function

$\Rightarrow \int_0^\infty f(z)\,dz = \frac{1}{2}\int_{-\infty}^{+\infty} f(z)\,dz = \frac{1}{2}\, 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$



ex 1 $\int_0^\infty \frac{dx}{(x^2+4)(x^2+16)}$

- $f(z) = \frac{1}{(z^2+4)(z^2+16)}$ even function

$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dz}{(z^2+4)(z^2+16)} = \frac{1}{\cancel{2}}\cdot \cancel{2}\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$

- $z^2+4=0 \Rightarrow z_{1,2} = \pm 2i$
  $z^2+16=0 \Rightarrow z_{3,4} = \pm 4i$
  singularities $\rightarrow$ $z_1 = 2i$, $z_3 = 4i$ poles of order 1 with $\operatorname{Im} z_k > 0$, $k=1$, $k=3$

$\operatorname{Res}_{z=2i} f(z) = \frac{1}{(z^2+16)(z+2i)}\Big|_{z=2i} = \frac{1}{(4i^2+16)\cdot 4i} = \frac{1}{48i}$

$\operatorname{Res}_{z=4i} f(z) = \frac{1}{(z^2+4)(z+4i)}\Big|_{z=4i} = \frac{1}{(16i^2+4)\cdot 8i} = \frac{-1}{96i}$

$I = \pi i\left(\frac{1}{48i} - \frac{1}{96i}\right) = \frac{\overset{2/}{\pi}}{48} - \frac{\pi}{\ }$

ex 2 $\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx$

$f(z) = \frac{z^2}{(z^2+1)^2}$ even function $\Rightarrow I = \frac{1}{\cancel{2}}\cdot \cancel{2}\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$

$(z^2+1)^2 = 0 \Rightarrow z_{1,2} = \pm i$ poles of order 2 $\Rightarrow z_1 = i$, $\operatorname{Im} z_1 > 0$ pole of order 2

$\operatorname{Res}_{z=i} f(z) = \frac{1}{(2-1)!} \lim_{z\to i}\left((z-i)^2 \frac{z^2}{(z^2+1)^2}\right)' = \lim_{z\to i}\left(\frac{z^2}{(z+i)^2}\right)'$

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{(2-1)!}\lim_{z\to i}\left((z-i)^2 \frac{z^2}{\underbrace{(z^2+1)^2}_{(z-i)^2(z+i)^2}}\right)' = \lim_{z\to i}\left(\frac{z^2}{(z+i)^2}\right)'$$

$$= \lim_{z\to i} \frac{2z\cdot(z+i)^2 - 2(z+i)z^2}{(z+i)^4} = \lim_{z\to i}\frac{2z(z+i)-2z^2}{(z+i)^3} =$$

$$= \frac{2i\cdot 2i - 2\cdot i^2}{(2i)^3} = \frac{-4+2}{-8i} = \frac{1}{4i}$$

$$I = \pi \cancel{i}\cdot\frac{1}{4\cancel{i}} = \frac{\pi}{4}$$

$\overline{\underline{V}}$ $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$, $\underbrace{f(x) = \frac{P(x)}{Q(x)}\cdot e^{\lambda i x}}$ degree $(P) <$ degree (Q), $Q(x)\neq 0,\ \forall x\in\mathbb{R}$

$$I = \int_{-\infty}^{\infty}\frac{P(z)}{Q(z)}e^{i\lambda z}\,dz = \begin{cases} 2\pi i \displaystyle\sum_{\operatorname{Im} z_k>0} \operatorname*{Res}_{z=z_k} f(z), & \lambda>0 \\ -2\pi i \displaystyle\sum_{\operatorname{Im} z_k<0} \operatorname*{Res}_{z=z_k} f(z), & \lambda<0 \end{cases}$$

ex 3 $\displaystyle\int_{-\infty}^{+\infty}\frac{e^{-ix}}{x^2-2x+5}\,dx$

$f(z) = \dfrac{e^{-iz}}{z^2-2z+5}$, $\lambda<0 \Rightarrow I = -2\pi i \displaystyle\sum_{\operatorname{Im} z_k<0}\operatorname*{Res}_{z=z_k} f(z)$

$z^2-2z+5=0$, $\Delta = 4-20=-16 \Rightarrow z_{1,2} = \dfrac{2\pm 4i}{2} = 1\pm 2i$

$z_1 = 1-2i$, $\operatorname{Im} z_1<0$ pole of order 1

$$\operatorname*{Res}_{z=1-2i} f(z) = \left.\frac{e^{-iz}}{2z-2}\right|_{z=1-2i} = \frac{e^{-i(1-2i)}}{\cancel{2}-4i-\cancel{2}} = \frac{e^{-i-2}}{-4i}$$

$$I = -\cancel{2}\pi\cancel{i}\cdot\left(-\frac{1}{\cancel{4}_2\cancel{i}}\right)e^{-2-i} = \frac{\pi}{2}e^{-2}(\cos 1 - i\sin 1).$$

ex 4 $\displaystyle\int^{+\infty}\frac{x e^{2ix}}{(x^2+4)^2}\,dx$

(ex 4) $\displaystyle\int_{-\infty}^{+\infty} \frac{x e^{2ix}}{(x^2+4)^2}\,dx$

$f(z) = \dfrac{z e^{2iz}}{(z^2+4)^2}$ $\qquad \lambda = 2 > 0 \Rightarrow I = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$

$(z^2+4)^2 = 0 \Rightarrow z_{1,2} = \pm 2i$

$\Rightarrow$ we will use $\underset{\operatorname{Im} z_1 > 0}{z_1 = 2i}$ pole of order 2

$$\operatorname{Res}_{z=2i} f(z) = \frac{1}{(2-1)!} \lim_{z\to 2i} \left((z-2i)^2 \frac{z e^{2iz}}{(z^2+4)^2} \right)' =$$

$$= \lim_{z\to 2i} \left(\frac{z e^{2iz}}{(z+2i)^2} \right)' = \lim_{z\to 2i} \frac{\left(e^{2iz} + 2iz e^{2iz}\right)\cdot (z+2i)^2 - 2(z+2i) z e^{2iz}}{(z+2i)^4} =$$

$$= \lim_{z\to 2i} \frac{\left(e^{2iz} + 2iz e^{2iz}\right)(z+2i) - 2z e^{2iz}}{(z+2i)^3} = \frac{\left(e^{-4} - 4e^{-4}\right)\cdot 4i - 4i\cdot e^{-4}}{(4i)^3} =$$

$$= \frac{-12 i e^{-4} - 4i e^{-4}}{-64 i} = \frac{1}{4} e^{-4}$$

$$I = \cancel{2}\pi i \cdot \frac{1}{\cancel{4}_{2} e^4} = \frac{\pi i}{e^4}$$

(ex 5) $\displaystyle\int_0^{2\pi} \frac{\cos 2x}{5 - 3\cos x}\,dx$

$z = e^{ix} \Rightarrow dx = \dfrac{dz}{iz}$

$\cos x = \dfrac{z^2+1}{2z}$, $\sin x = \dfrac{z^2-1}{2iz}$

$\cos 2x = \cos^2 x - \sin^2 x$

$$I = \int_{C:|z|=1} \frac{\left(\frac{z^2+1}{2z}\right)^2 - \left(\frac{z^2-1}{2iz}\right)^2}{5 - 3\,\frac{z^2+1}{2z}} \cdot \frac{dz}{iz} = \int_C \frac{\frac{(z^2+1)^2}{4z^2} + \frac{(z^2-1)^2}{4z^2}}{\frac{10z - 3z^2 - 3}{2z}} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{z^4 + \cancel{2z^2} + 1 + z^4 - \cancel{2z^2} + 1}{2\cancel{4z^2}} \cdot \frac{\cancel{2z}}{-3z^2 + 10z - 3} \cdot \frac{dz}{iz}$$

$C \qquad 2i$

$$= -\frac{1}{i} \cdot \int_C \frac{z^4+1}{z^2(3z^2-10z+3)}\, dz$$

$z^2 = 0 \Rightarrow z_{1,2} = 0 \in \operatorname{int} C \quad (C: |z| = 1)$
pole of order 2

$3z^2 - 10z + 3 = 0, \ \Delta = 100 - 36 = 64 \Rightarrow z_{3,4} = \frac{10 \pm 8}{6}$; $z_3 = 3 \notin \operatorname{int} C$; $z_4 = \frac{1}{3} \in \operatorname{int} C$ pole of order 1

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z\to 0} \left(\cancel{z^2} \cdot \frac{z^4+1}{\cancel{z^2}(3z^2-10z+3)} \right)' =$$

$$= \lim_{z\to 0} \left(\frac{z^4+1}{3z^2-10z+3} \right)' = \lim_{z\to 0} \frac{4z^3(3z^2-10z+3) - (z^4+1)(6z-10)}{(3z^2-10z+3)^2}$$

$$= \frac{10}{9}$$

$$\operatorname*{Res}_{z=\frac{1}{3}} f(z) = \frac{g(z)}{h'(z)} = \overbrace{\frac{z^4+1}{z^2(z-3)\cdot 3}}^{g(z)} \cdot \frac{1}{1} \Bigg|_{z=\frac{1}{3}} = \frac{\left(\frac{1}{3}\right)^4 + 1}{\frac{1}{\cancel{9}_3}\left(\frac{1}{3} - \frac{3}{3}\right) \cdot \cancel{3}} =$$

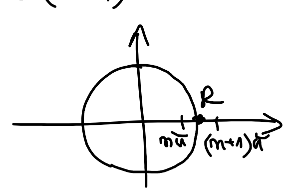
$3z^2 - 10z + 3 = 3\left(z - \frac{1}{3}\right)(z-3)$

$h(z), \ h'(z) = 1$

$$= \frac{\frac{1}{81} + 1}{-\frac{8}{9}} = -\frac{\cancel{82}^{41}}{\cancel{81}_9} \cdot \frac{\cancel{9}}{\cancel{8}_4} = \frac{-41}{36}$$

$$I = -\frac{1}{\cancel{i}} \cdot 2\pi \cancel{i} \left(\frac{\overset{4}{10}}{9} - \frac{41}{36} \right) = \frac{2\pi}{36} = \frac{\pi}{18}$$

ex 6 $\quad \int_C \frac{dz}{z^2 \sin z} \qquad C: |z| = R \qquad n\pi < R < (n+1)\pi$



$z = 0$ triple pole, $\in \operatorname{int} C$

$\sin z = 0 \Rightarrow z_k = k\pi, \ k \in \mathbb{Z}^*, \ k = \pm 1, \pm 2, \dots, \pm n$

simple poles, $\in \operatorname{int} C$

$$\operatorname*{Res}_{z=k\pi} f(z) = \frac{g(z)}{h'(z)}\Big|_{z=k\pi} = \frac{\frac{1}{z^2}}{\cos z}\Big|_{z=k\pi} = \frac{\frac{1}{k^2\pi^2}}{(-1)^k} = \frac{(-1)^k}{k^2\pi^2}$$

$$\underbrace{\operatorname*{Res}_{z=k\pi} f(z) + \operatorname*{Res}_{z=-k\pi} f(z)} = \frac{(-1)^k}{k^2\pi^2} + \frac{(-1)^k}{k^2\pi^2}$$

Method I.

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(3-1)!} \lim_{z\to 0}\left(\cancel{z^3}\ \frac{1}{\cancel{z^2}\sin z}\right)'' = \frac{1}{2}\lim_{z\to 0}\left(\frac{z}{\sin z}\right)'' = \dots = \frac{1}{6}$$

Method II:

$$f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^2}\ \frac{1}{\sin z} = \frac{1}{z^2}\cdot\frac{1}{\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots} =$$

$$\frac{1}{1-u} = 1 + u + u^2 + \dots \quad |u| < 1$$

$$= \frac{1}{z^2}\cdot\frac{1}{z}\cdot\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots} = \frac{1}{z^3}\cdot\frac{1}{1 - \underbrace{\left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)}_{u}} =$$

$$= \frac{1}{z^3}\left[1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)^2 + \left(\ \dots\ \right)^3 + \dots\right]$$

we look for the coefficient of $\frac{1}{z}$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{3!} = \frac{1}{6}$$

$$I = 2\pi i\left(\frac{1}{6} + 2\sum_{k=1}^{m}\frac{(-1)^k}{k^2\pi^2}\right).$$

ex 7 $\displaystyle\int_0^\infty \frac{dx}{(x^2+4)(x^2+25)}$

$$f(z) = \frac{1}{(z^2+4)(z^2+25)} \quad \text{even function}$$

$$I = \frac{1}{\cancel{2}}\cdot\cancel{2}\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} f(z) = \pi i\left(\frac{1}{84i} - \frac{1}{210i}\right) = \frac{\pi}{140}$$

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{84i}$$

$$\operatorname*{Res}_{z=5i} f(z) = \frac{-1}{210i}$$

Homework: ① $\int \frac{x^2}{\ } dx \qquad R: \frac{\pi}{\ }$

$z=5i$

Homework:

(1) $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+2)^2}\,dx \qquad R: \frac{\pi}{4\sqrt{2}}$

(2) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+2)} \qquad R: \frac{\pi}{6}$

(3) $\displaystyle\int_{-\pi}^{\pi} \frac{\cos 100x}{5-4\sin x}\,dx$

(4) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+2x+2)^2} \qquad R: \frac{\pi}{2}$