2D and 3D Transformations

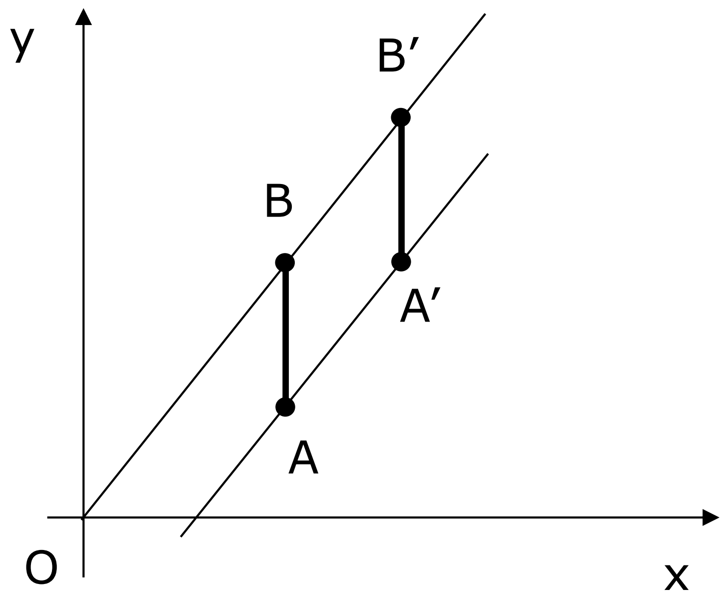
Contents

2D Transformations

- Translation
- Scaling
- Rotation
- Reflection
- Shearing

Translation

$$x' = x + T_x$$
$$y' = y + T_y$$

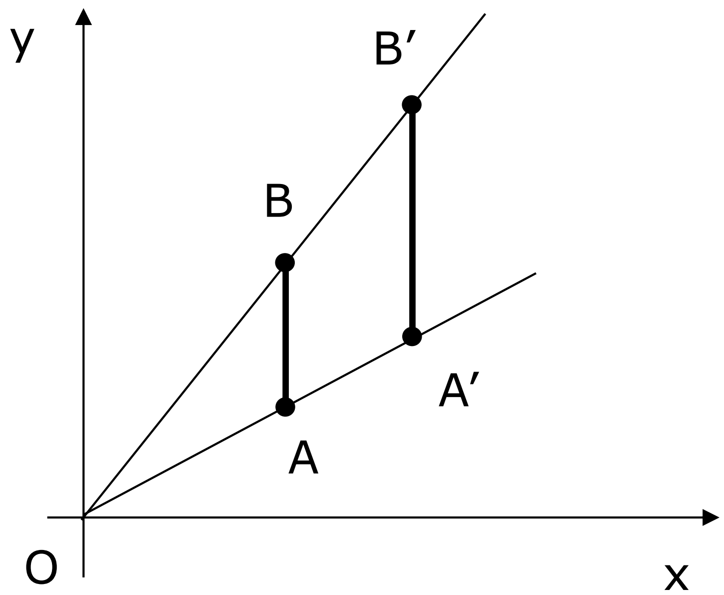


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling

$$x' = S_x x$$
$$y' = S_y y$$

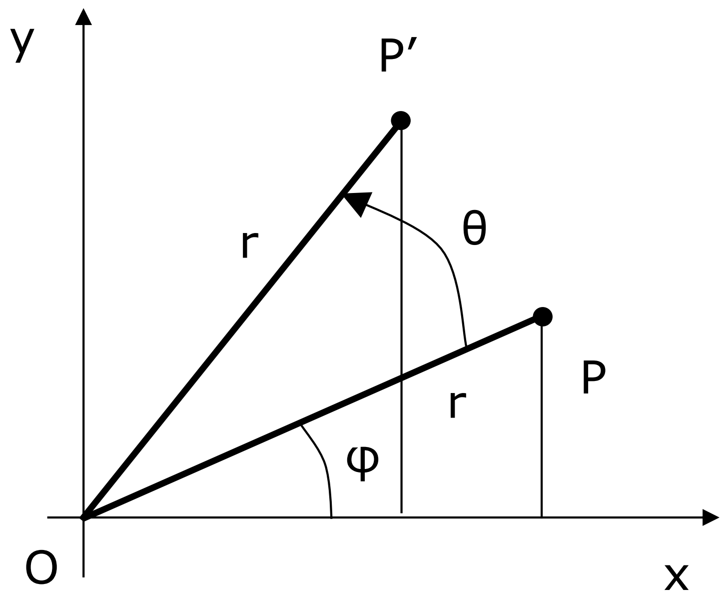
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotation

$$x' = x\cos\theta - y\sin\theta$$
$$y' = y\cos\theta + x\sin\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Uniform transformations

- Scaling and rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Translation

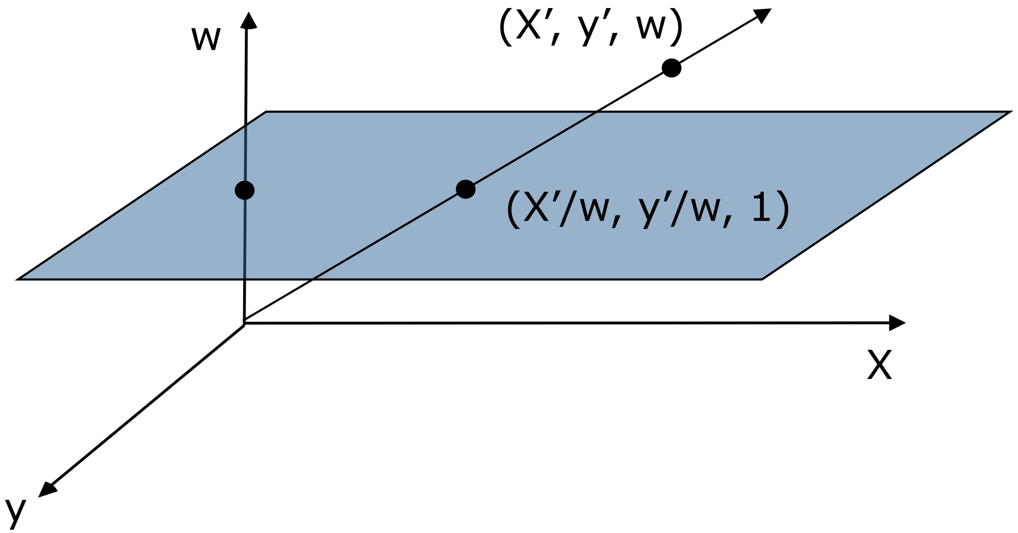
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinate System

- Uniformly matrix transformations
  P′ = H P

$$P' = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} x'/w \\ y'/w \\ 1 \end{bmatrix}$$



- No unique homogeneous representation of a cartesian point
  (x, y) ≡ (xw, yw, w), w ≠ 0
- Normalization
  (x, y, 1), w = 1

Support for uniform transformations

- Scaling

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Rotation

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Translation

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

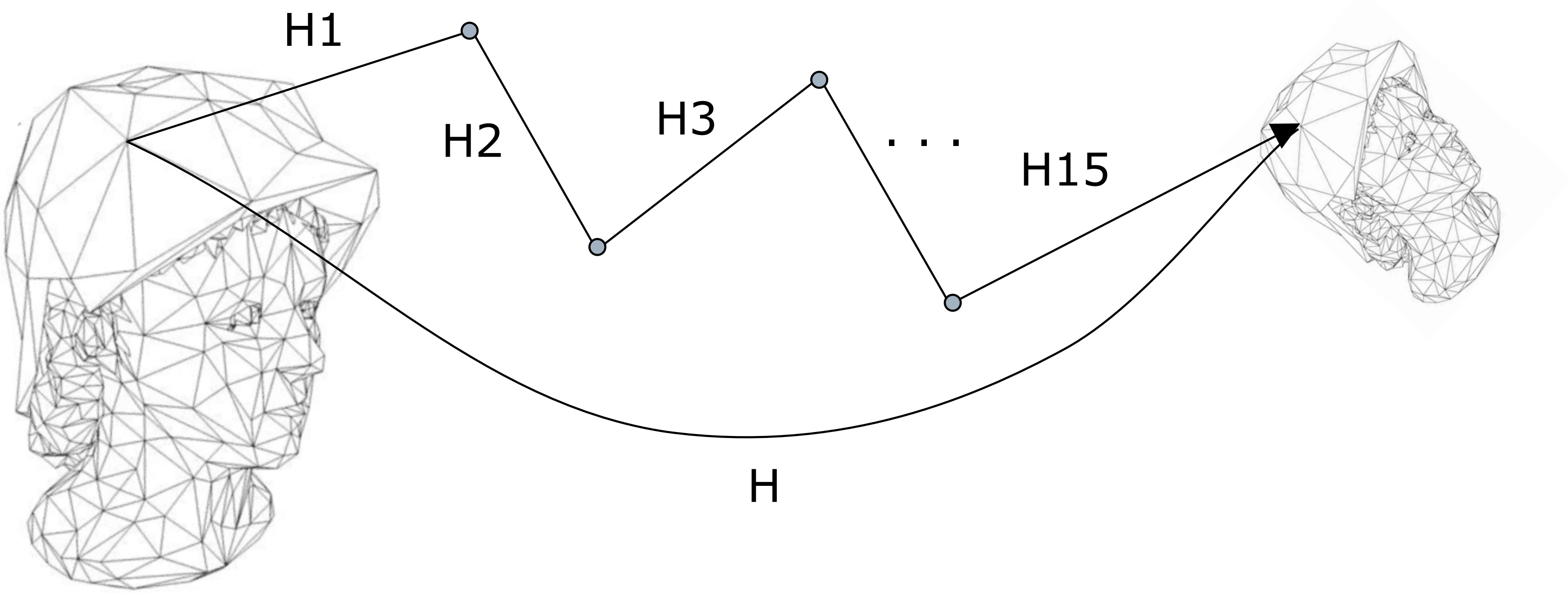
Composite transformations

- $P' = H_n \cdot \cdot \cdot H_2 \cdot H_1 \cdot P$
- $P' = H \cdot P$

Example: n=15,

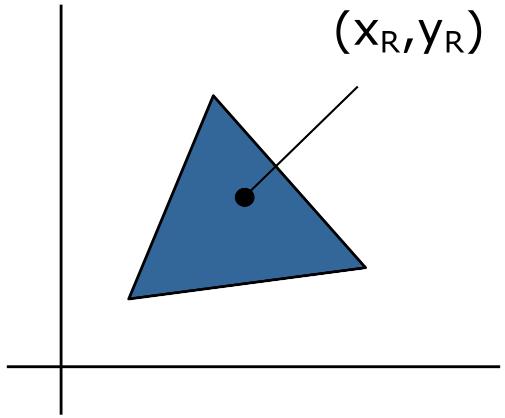
Individual matrix: 1.000 vertices

Global matrix: 1.000 vertices x 1 op = 1.000 matrix op

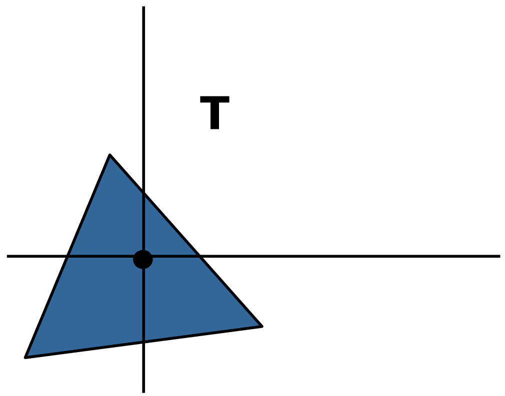


Example: scaling relative to a position

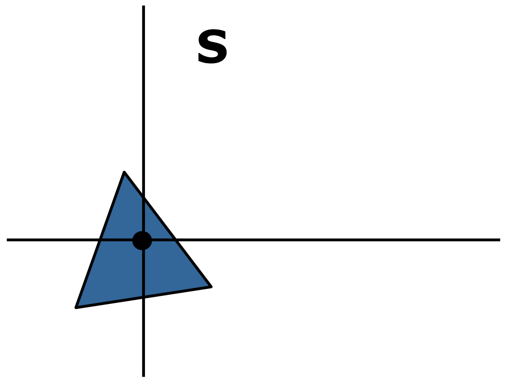
$$H = T^{-1}\,S\,T$$
$$P' = T^{-1}\,S\,T\,P$$



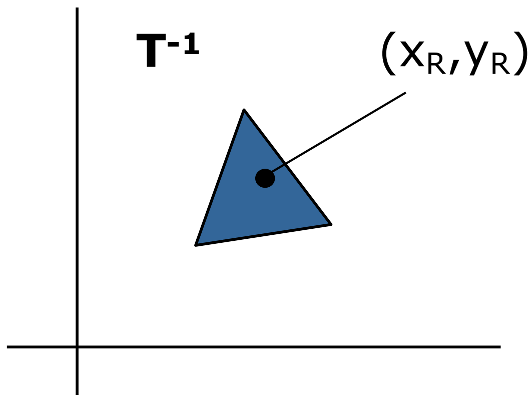
(a) Original position of object and fixed point



(b) Translate object so that fixed point (x_R, y_R) is at origin

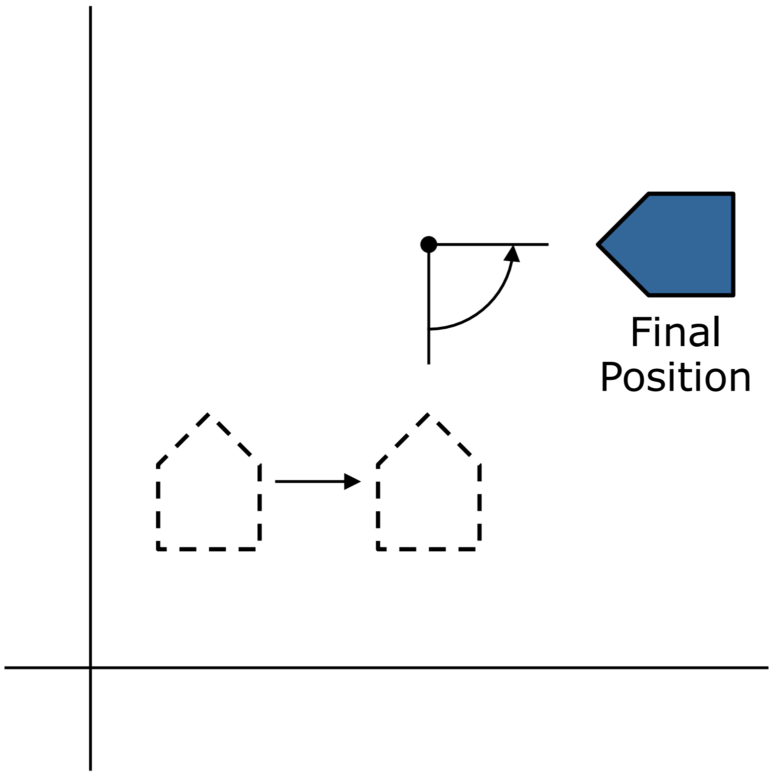


(c) Scale object with respect to origin

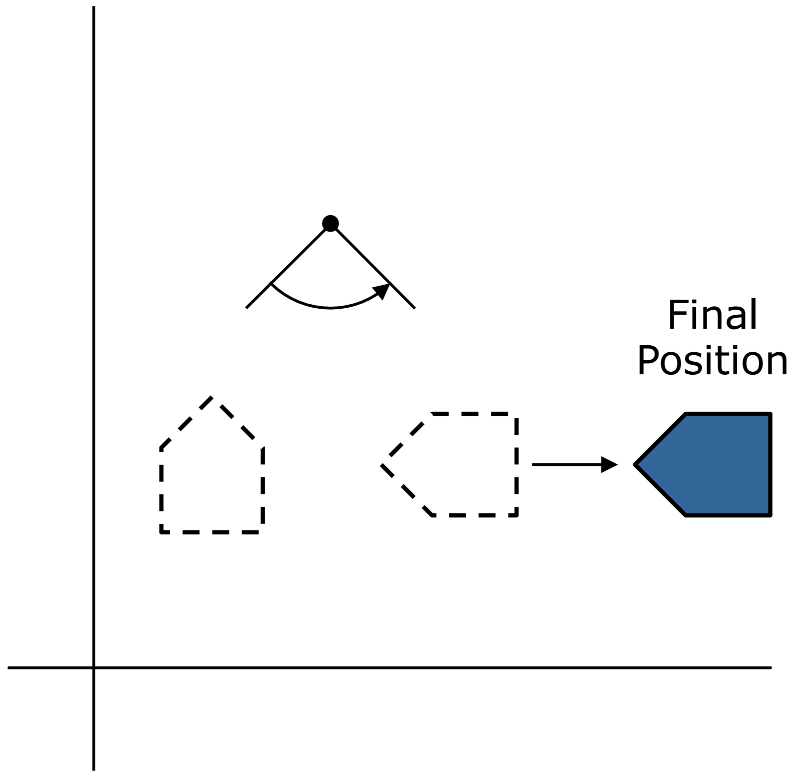


(d) Translate object so that fixed point is returned to position (x_R, y_R)

Transformations ordering



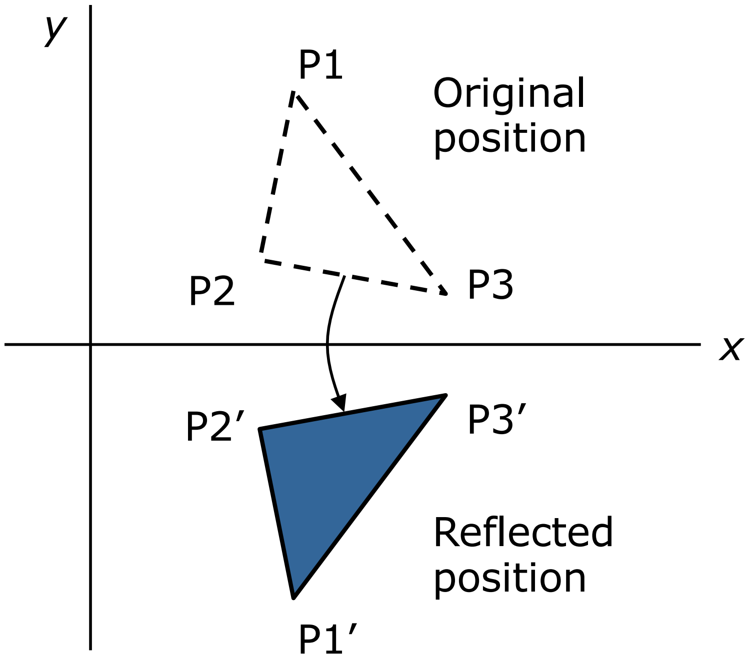
(a) RT



(b) TR

Particular reflections

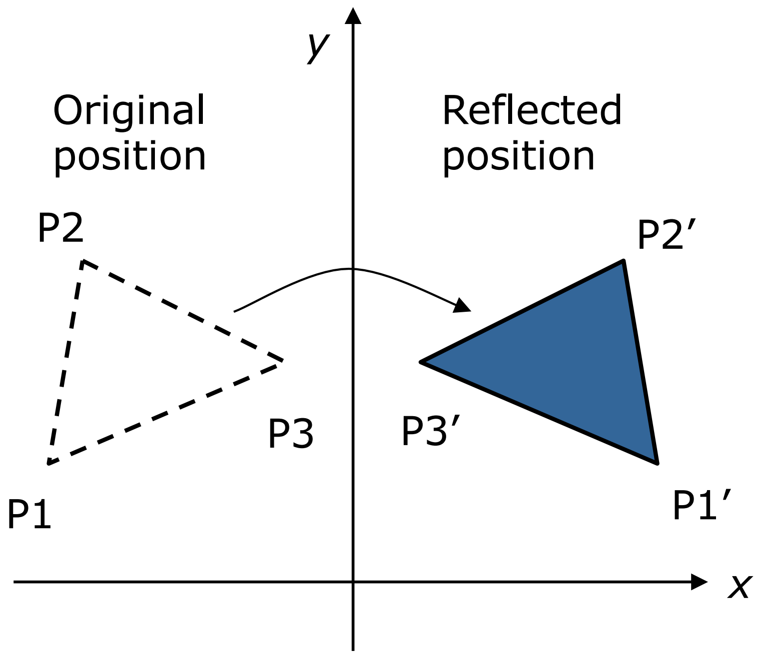
- Reflection against x axis



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Particular reflections

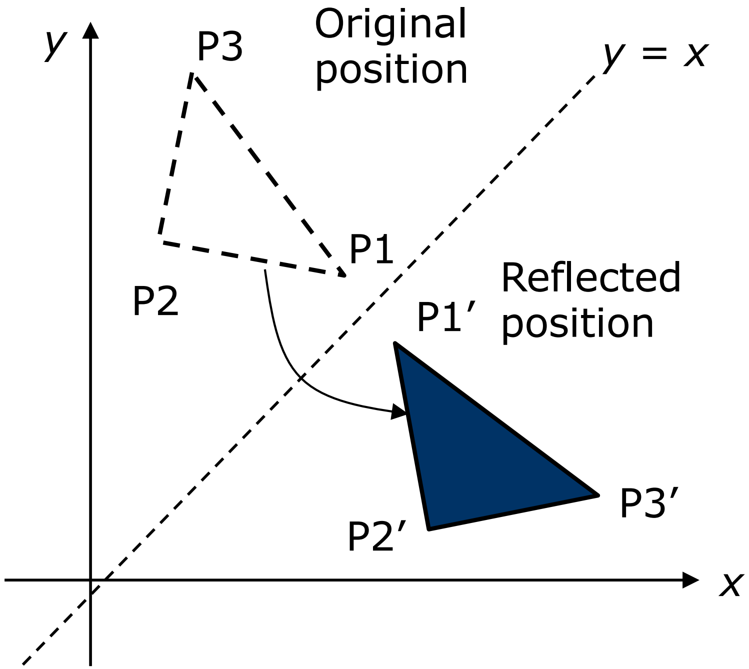
- Reflection against y axis



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Particular reflections

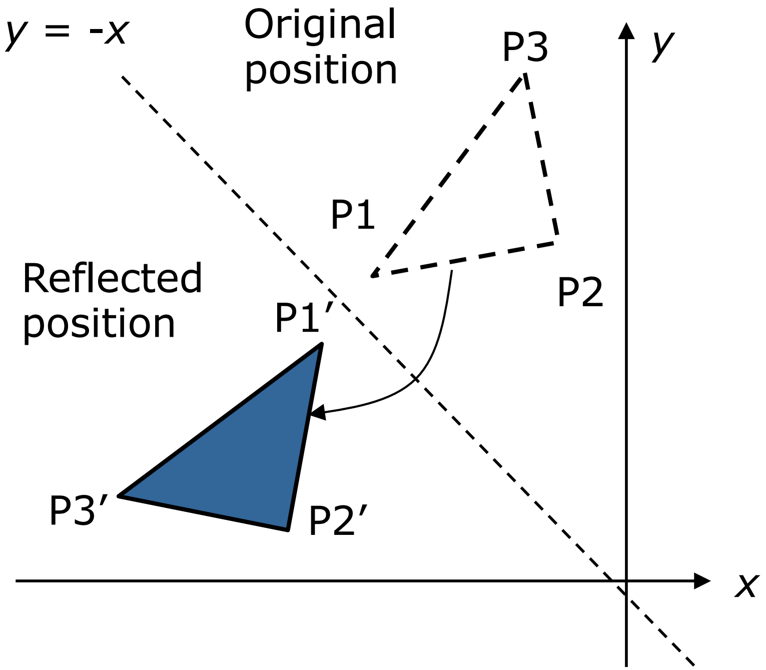
- Reflection against first bisection



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Particular reflections

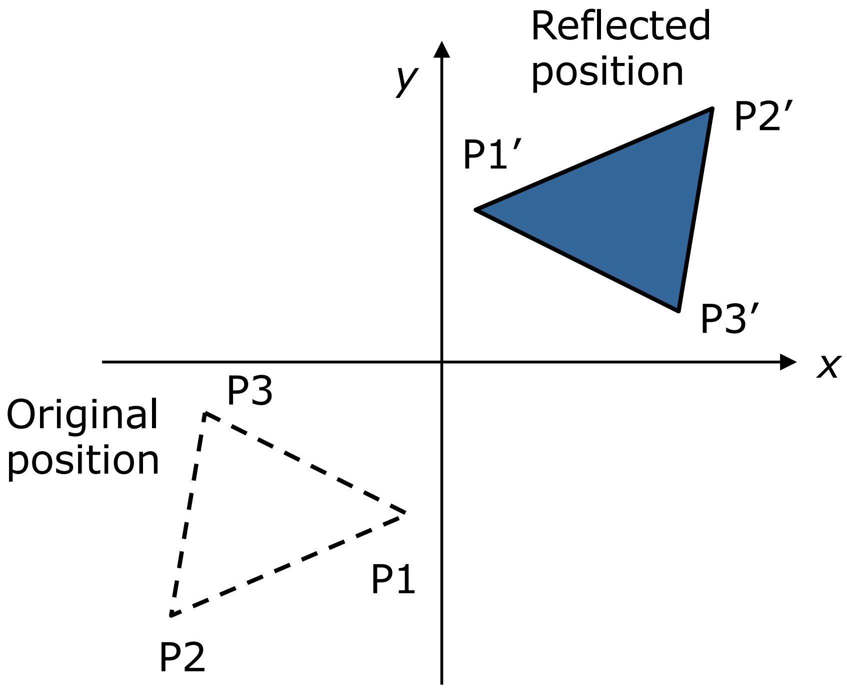
- Reflection against second bisection



$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

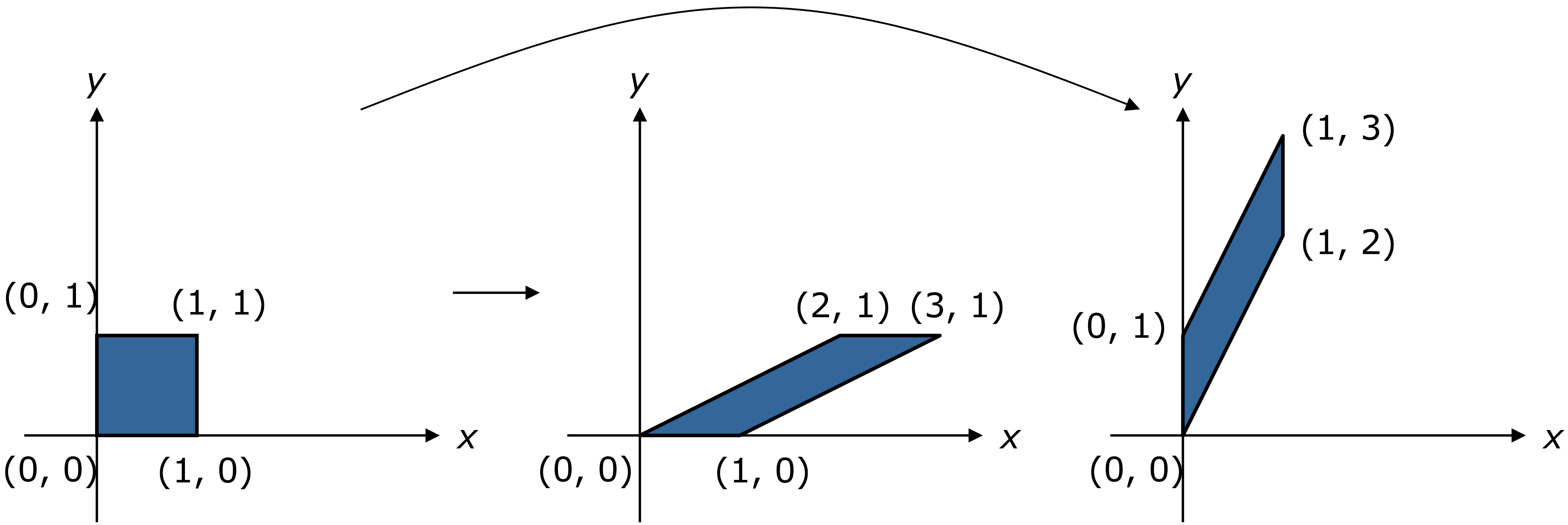
Particular reflections

- Reflection against coordinate system origin



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shearing



(a) (b) (c)

$$\begin{bmatrix} 1 & SH_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ SH_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D Transformations

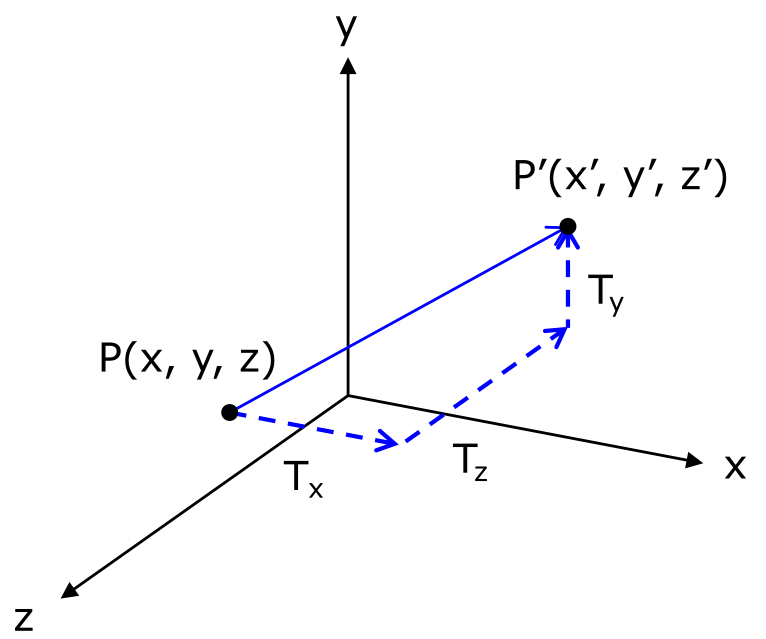
- Translation
- Scaling
- Rotation

Translation

$$x' = x + T_x$$
$$y' = y + T_y$$
$$z' = z + T_z$$

$$P' = TP$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

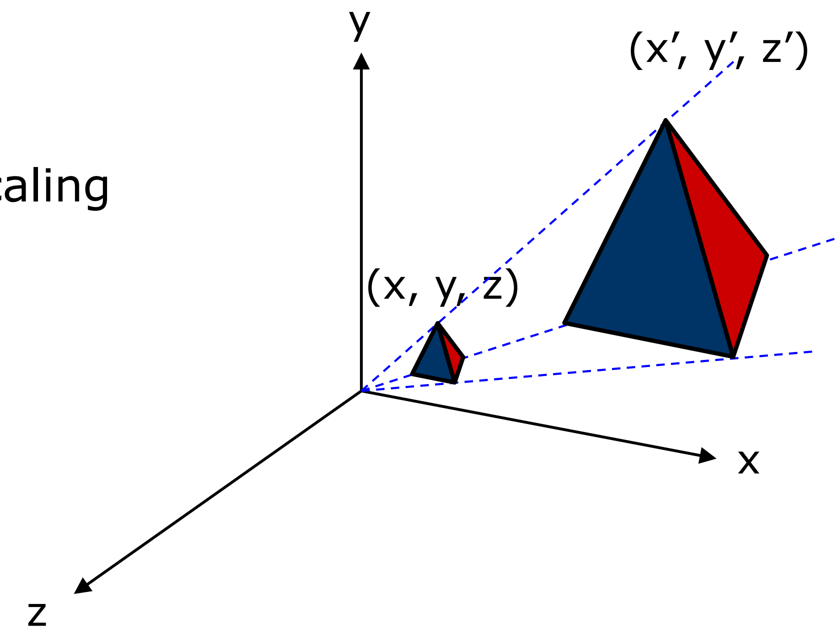


Scaling

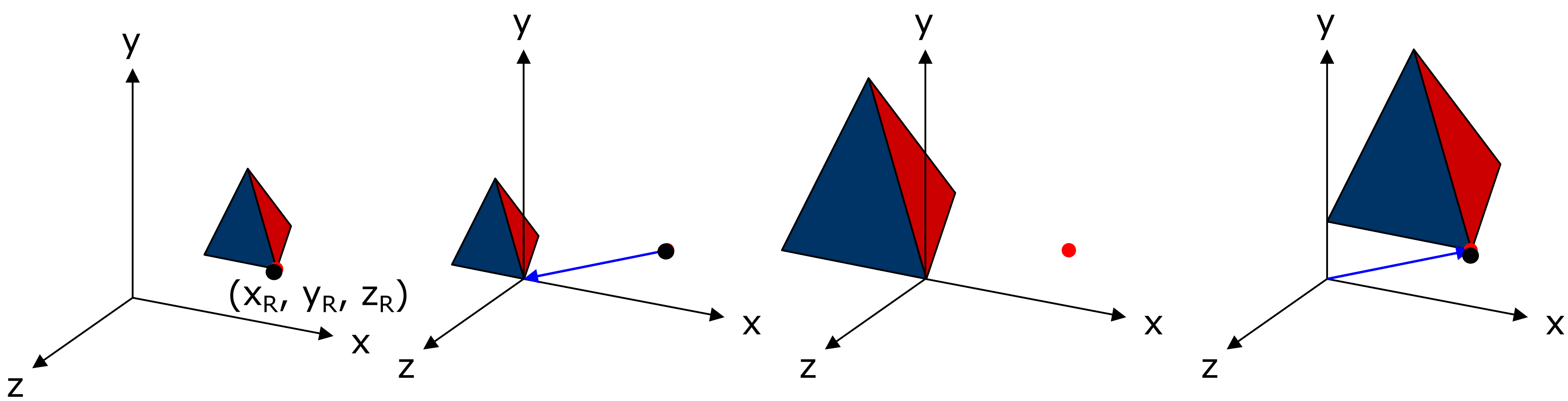
$x' = x\,S_x$
$y' = y\,S_y$
$z' = z\,S_z$
if $S_x = S_y = S_z$ uniform scaling

$P' = SP$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Relative scaling



$$T = \begin{bmatrix} 1 & 0 & 0 & -x_R \\ 0 & 1 & 0 & -y_R \\ 0 & 0 & 1 & -z_R \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_R \\ 0 & 1 & 0 & y_R \\ 0 & 0 & 1 & z_R \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

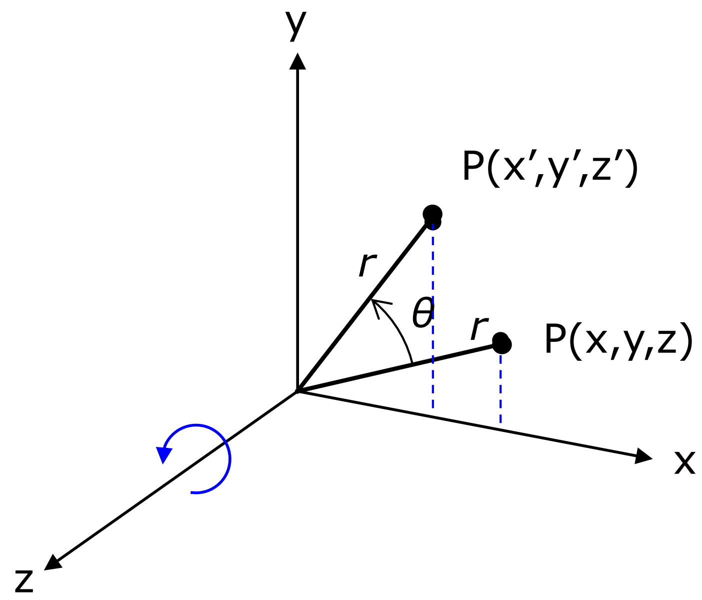
$$P' = T^{-1}STP$$

Rotation around z axis

$x' = x\cos\theta - y\sin\theta$

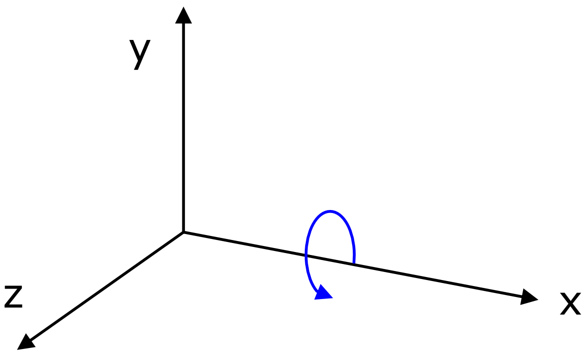
$y' = x\sin\theta + y\cos\theta$

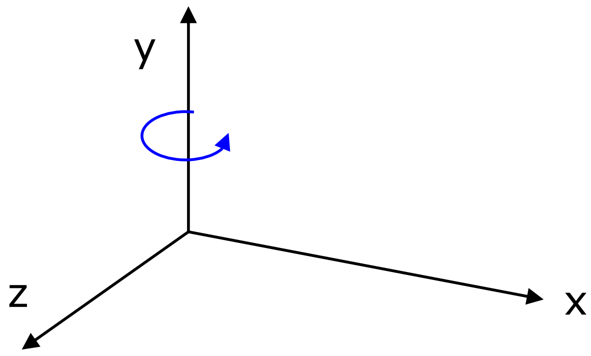
$z' = z$



$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around x and y axes





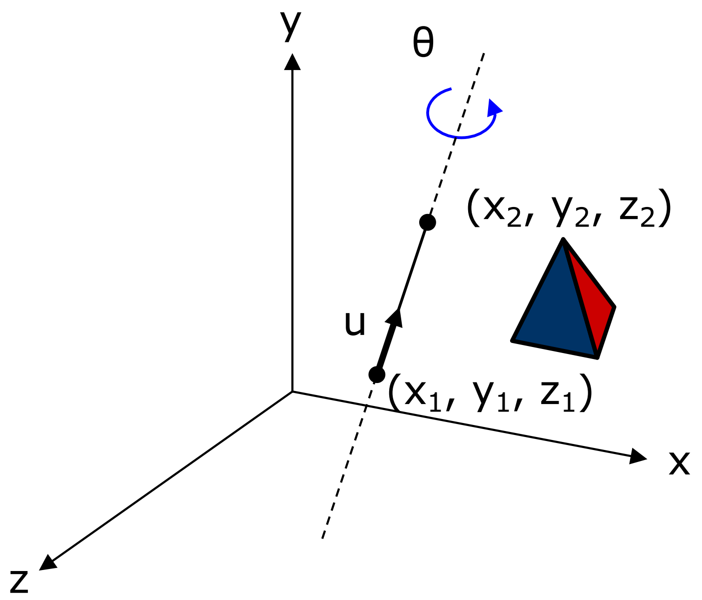
$x' = x$
$y' = y \cos\theta - z \sin\theta$
$z' = y \sin\theta + z \cos\theta$

$x' = x \cos\theta + z \sin\theta$
$y' = y$
$y' = -x \sin\theta + z \cos\theta$

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around an arbitrary axis



Rotation around an arbitrary axis

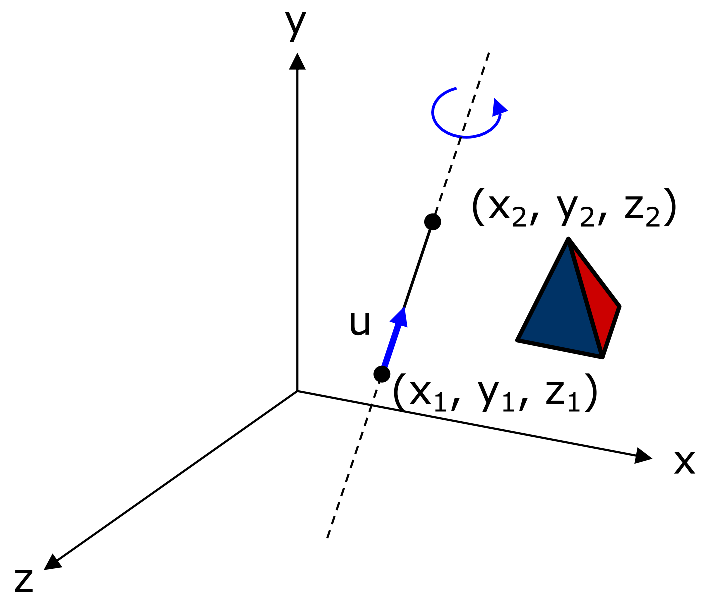
Axis: P1(x1,y1,z1), P2(x2,y2,z2)

V [x2-x1   y2-y1   z2-z1]

Normalized vector u:

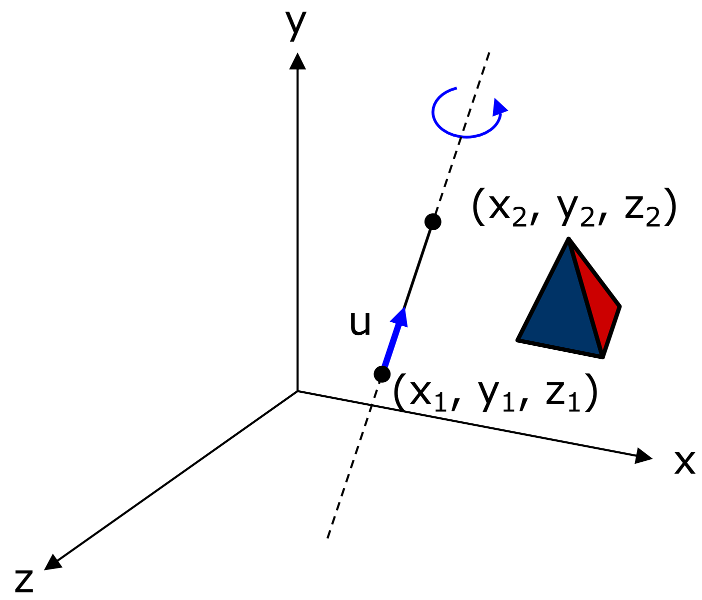
u(a,b,c), |u|=1

a = Δx/|V|, b = Δy/|V|, c = Δz/|V|

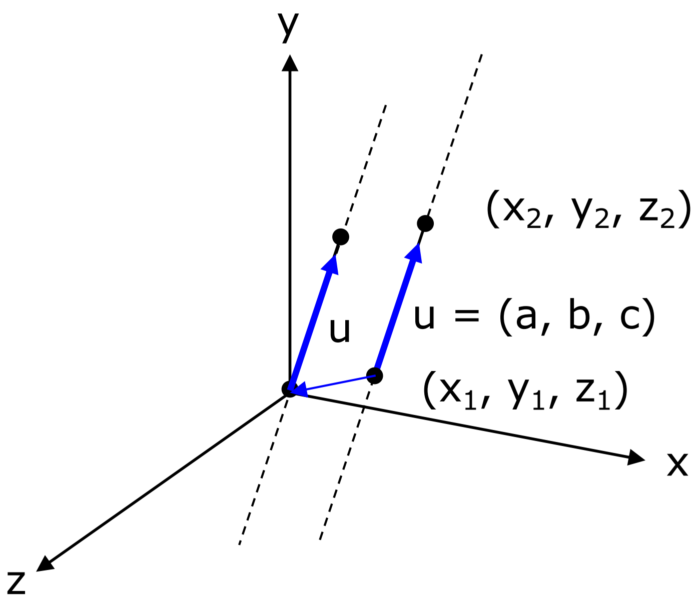


Rotation around an arbitrary axis

1. Translate the object so that the rotation axis passes through the origin (T).
2. Rotate the object so that the rotation axis coincides with one of the coordinate axis (R).
3. Perform the specified rotation.
4. R^{-1}
5. T^{-1}

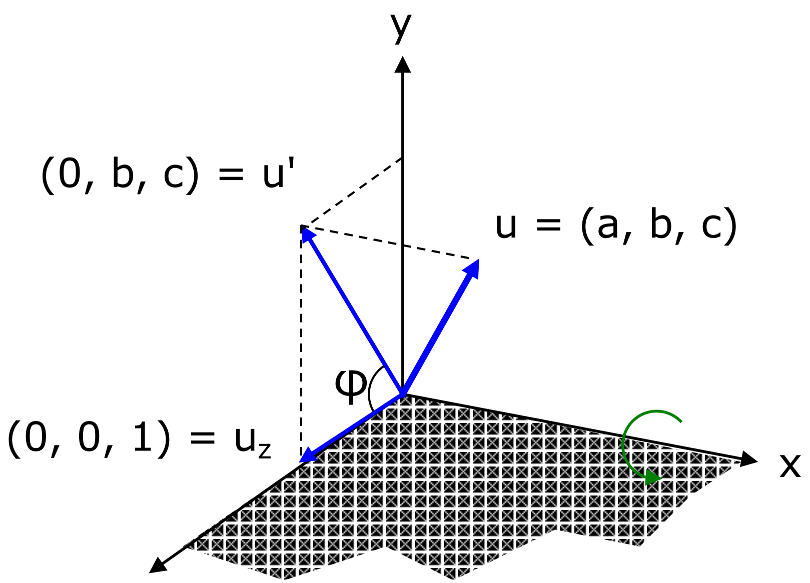


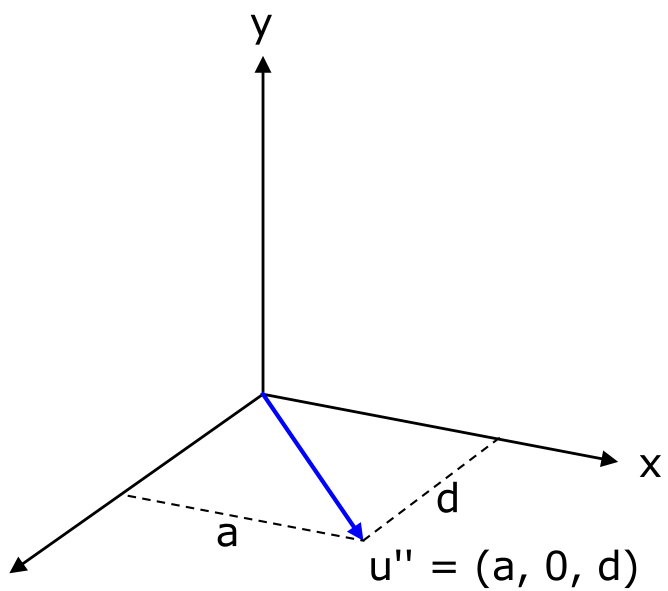
Step1 - translation



$$T = \begin{bmatrix} 1 & 0 & 0 & -x1 \\ 0 & 1 & 0 & -y1 \\ 0 & 0 & 1 & -z1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step2 – align u with z axis





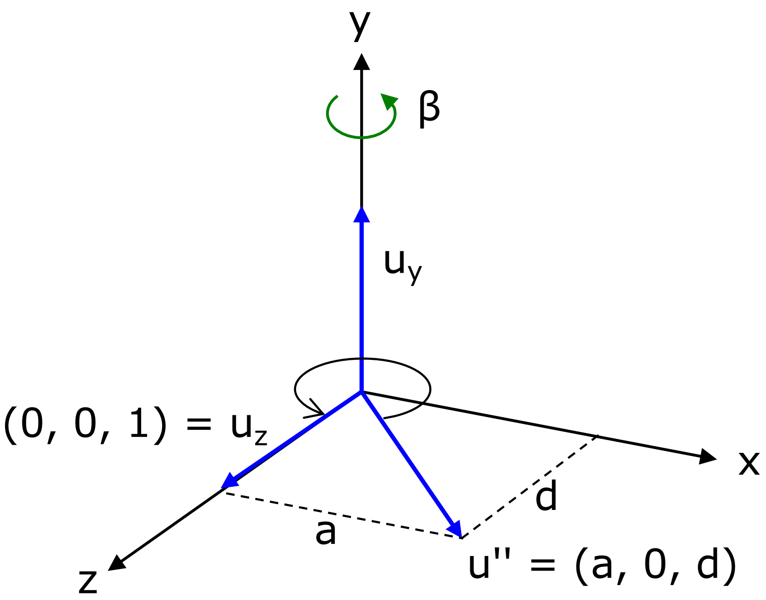
$u' \cdot u_z = |u'|\ |u_z| \cos\Phi$

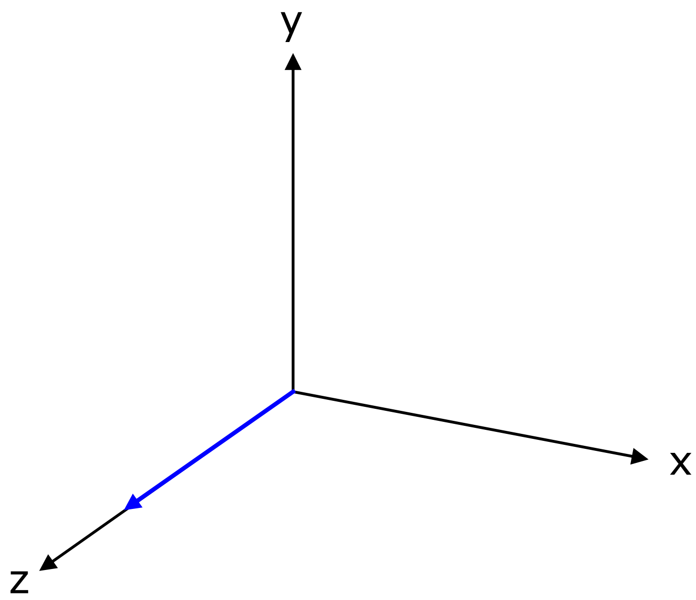
$\cos\Phi = c/d$

where $d = \mathrm{sqrt}(b^2+c^2)$

$\sin\Phi = b/d$

$$R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





$u''\cdot u_z=|u''|\ |u_z|\ \cos\beta$

$\cos\beta = d$

$u'' \times u_z = u_y$

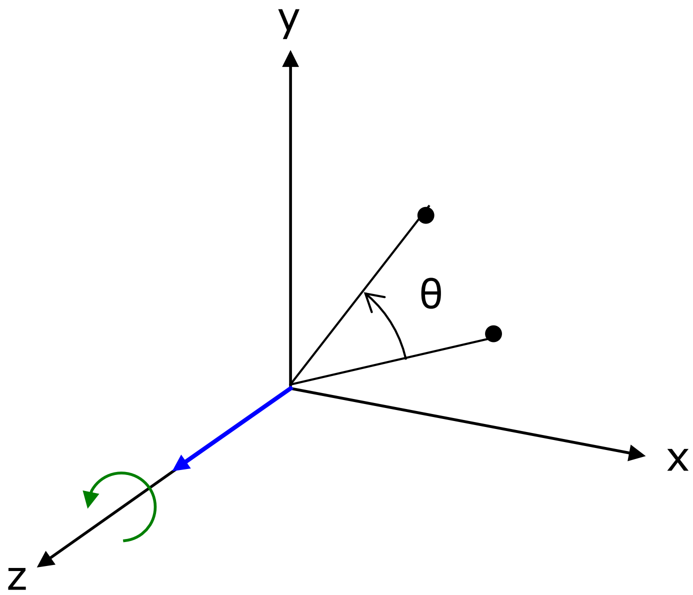
$|u''|\ |\ u_z\ |\ \sin\beta = u_y \sin\beta$

$u'' \times u_z = u_y\ (-a)$

$\sin\beta = -a$

$$R_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Rotate around z axis by θ



$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Steps 4 and 5

- Step 4: $R^{-1} = R_x(\Phi)^{-1}\ R_y(\beta)^{-1}$
- Step 5: T^{-1}

Finally the matrix global operator:

$H = T^{-1}\ R_x(\Phi)^{-1}\ R_y(\beta)^{-1}\ R_z(\theta)\ R_y(\beta)\ R_x(\Phi)\ T$

Other transformations

- Reflection

  xy plane : $(x, y, \mathbf{z}) \rightarrow (x, y, \mathbf{-z})$

  yz plane : $(\mathbf{x}, y, z) \rightarrow (\mathbf{-x}, y, z)$

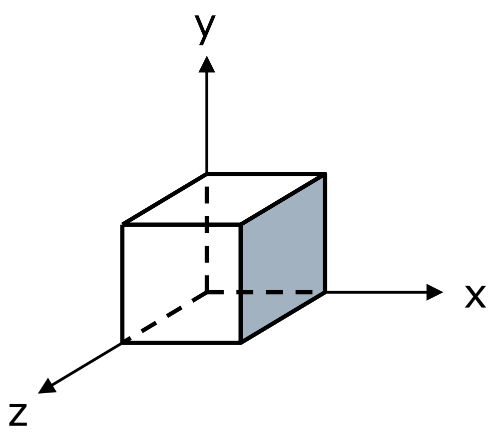
  zx plane : $(x, \mathbf{y}, z) \rightarrow (x, \mathbf{-y}, z)$

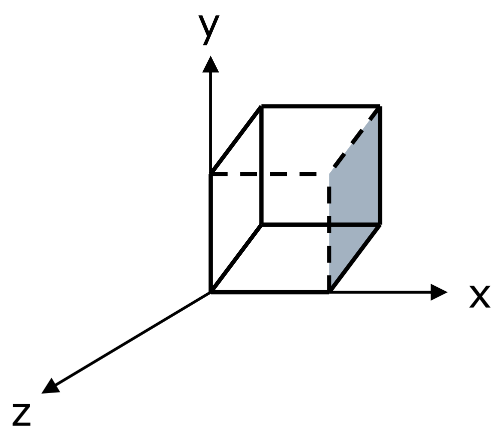
- Shearing

  Z axis shearing

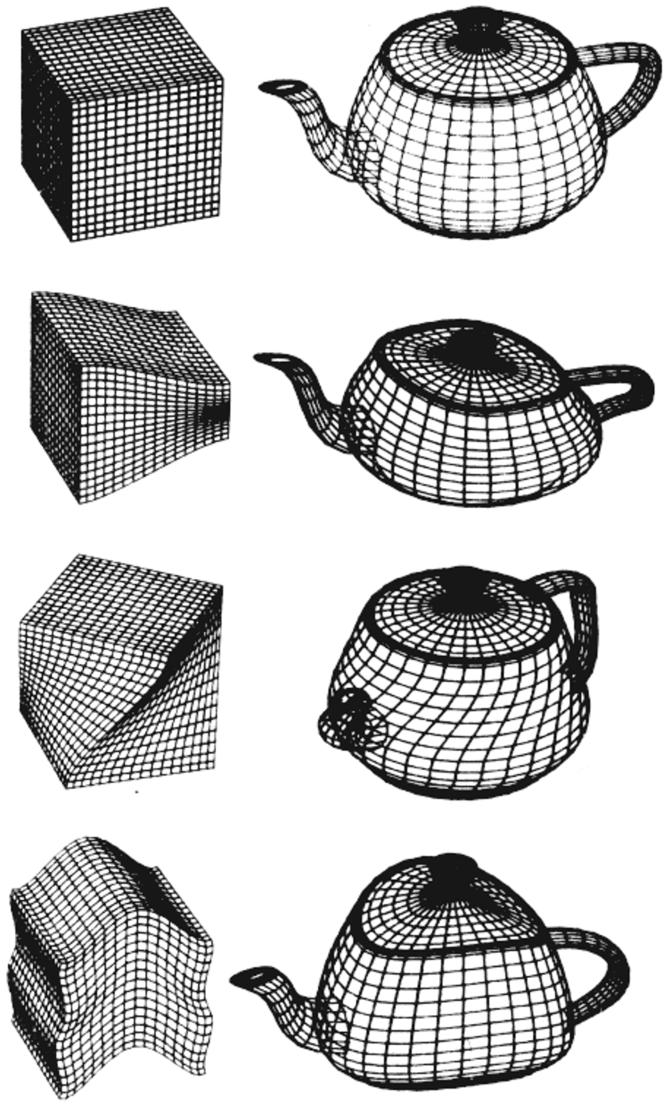
  $x' = x + az$

  $y' = y + bz$





Other transformations



- Tapering

  $x' = r(z) \cdot x$

  $y' = r(z) \cdot y$

  $z' = z$

- Twisting

  $x' = x \cos q\,(z) - y \sin q\,(z)$

  $y' = x \sin q\,(z) + y \cos q\,(z)$

  $z' = z$

- Bending

Questions and proposed problems

1. What means uniformly matrix transformation?
2. Explain the utility of the uniformly matrix transformation.
3. What is the relationship between the real and the homogeneous coordinates?
4. What are the matrix operators (i.e. rotation, translation, scaling) in the following two point transformations: (a) $P' = H_1 P$; (b) $P' = P H_2$, where P and P' are the points in 3D space, and H_1, H_2 are the matrix operators?
5. Explain the rotation of the square ABCD, where A(5,3), C(12, 10) around the point P(-2,3), by 45^0. Explain the computation of the global matrix operator.
6. Explain the rotation of the cube ABCDA'B'C'D', where A(5,3,4), C'(12,10,11), around the line P1P2, where P1(-2,3,4), P2(-2,3,11), by 45^0. Explain the computation of the global matrix operator.