Seminar 10

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The Laplace Transform

– it is a mathematical TOOL

$$x(t) \underset{\text{iLT}}{\overset{\text{LT}}{\rightleftarrows}} X(p)$$

time domain — Laplace domain

ex1) continuous time signal → Laplace Transform
discrete time signal → Z Transform

ex2) in solving differential equation more easily

$$y(t) = c_1 e^{2t} + c_2 \sin 3t$$

$f : \mathbb{R} \to \mathbb{C}$ the original

$$F(p) = \int_0^\infty f(t) e^{-pt} dt = \mathcal{L}[f(t)](p) \quad \text{the image}$$

$$F(p) = \mathcal{L}[f(t)](p)$$

$$\mathcal{L}[a](p) = \frac{a}{p}$$

$$\mathcal{L}[e^{\lambda t}](p) = \frac{1}{p-\lambda}$$

$$\mathcal{L}[t^n](p) = \frac{n!}{p^{n+1}}, \; n \in \mathbb{N}$$

$$\mathcal{L}[t^\alpha](p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}, \; \alpha \in \mathbb{C}, \operatorname{Re}\alpha > -1$$

$$\mathcal{L}[f(at)](p) = \frac{1}{a}\mathcal{L}[f(t)]\left(\frac{p}{a}\right)$$

$$\mathcal{L}[e^{at} \cdot f(t)](p) = \mathcal{L}[f(t)](p-a)$$

translation of the image

$$\mathcal{L}[t^1 \cdot f(t)](p) = (-1)^1 \cdot \left(\mathcal{L}[f(t)](p)\right)'$$

$$\mathcal{L}[t^n f(t)](p) = (-1)^n \left(\mathcal{L}[f(t)](p)\right)^{(n)}$$

$$\mathcal{L}[\sin(at)](p) = \frac{a}{p^2+a^2}$$

$$\mathcal{L}[\cos(at)](p) = \frac{p}{p^2+a^2}$$

$$\mathcal{L}[\operatorname{sh}(at)](p) = \frac{a}{p^2-a^2}$$

$$\mathcal{L}[\operatorname{ch}(at)](p) = \frac{p}{p^2-a^2}$$

$$\mathcal{L}\left[\int_0^t f(s)\, ds\right](p) = \frac{1}{p}\mathcal{L}[f(t)](p) \quad \text{integration of the original}$$

$$\boxed{\mathcal{L}\left[\frac{f(t)}{t}\right](p) = \int_p^\infty \mathcal{L}[f(t)](q)\, dq} \quad \text{integration of the image}$$

① Find the images by the Laplace transform of the originals

a) $\mathcal{L}[\operatorname{sh}(at)](p) = \mathcal{L}\left[\frac{e^{at} - e^{-at}}{2}\right](p) = \mathcal{L}\left[\frac{e^{at}}{2}\right](p) - \mathcal{L}\left[\frac{e^{-at}}{2}\right](p) =$

1) Find the images ...

a) $\mathcal{L}[sh(at)](p) = \mathcal{L}\left[\frac{e^{at}-e^{-at}}{2}\right](p) = \mathcal{L}\left[\frac{e^{at}}{2}\right](p) - \mathcal{L}\left[\frac{e^{-at}}{2}\right](p) =$

$= \frac{1}{2}\left(\mathcal{L}[e^{at}](p) - \mathcal{L}[e^{-at}](p)\right) \overset{\uparrow}{=} \frac{1}{2}\left(\frac{1}{p-a} - \frac{1}{p+a}\right) =$

$= \frac{1}{2} \cdot \frac{\cancel{p}+a-\cancel{p}+a}{p^2-a^2} = \frac{a}{p^2-a^2}$.

$\mathcal{L}[e^{\lambda t}](p) = \frac{1}{p-\lambda}$

b) $\mathcal{L}[\cos(3t)](p) = \frac{p}{p^2+9}$

$\overset{\uparrow}{=} \frac{1}{3}\mathcal{L}[\cos t]\left(\frac{p}{3}\right) = \frac{1}{3}\left.\frac{p}{p^2+1}\right|_{p=\frac{p}{3}} = \frac{1}{3}\frac{\frac{p}{3}}{\left(\frac{p}{3}\right)^2+1} = \frac{p}{p^2+9}$

$\mathcal{L}[f(at)](p) = \frac{1}{a}\mathcal{L}[f(t)]\left(\frac{p}{a}\right)$

c) $\mathcal{L}[\underbrace{e^{2t}\cos(3t)} + e^{3t}\sin(2t)](p) = \mathcal{L}[e^{2t}\cos(3t)](p) + \mathcal{L}[e^{3t}\sin(2t)](p) =$

$= \mathcal{L}[\cos(3t)](p-2) + \mathcal{L}[\sin(2t)](p-3) =$

$\overset{\uparrow}{=} \left.\frac{p}{p^2+9}\right|_{p=p-2} + \left.\frac{2}{p^2+4}\right|_{p=p-3} = \frac{p-2}{(p-2)^2+9} + \frac{2}{(p-3)^2+4}$.

$\mathcal{L}[\cos(at)](p) = \frac{p}{p^2+a^2}$; $\mathcal{L}[\sin(at)](p) = \frac{a}{p^2+a^2}$

d) $\mathcal{L}[t^3 \cdot \underline{e^{-t}}](p) \xlongequal{\text{I method}} \mathcal{L}[t^3](p+1) = \left.\frac{3!}{p^4}\right|_{p=p+1} = \frac{3!}{(p+1)^4}$

$\xlongequal{\text{II method}} (-1)^3\left(\mathcal{L}[e^{-t}](p)\right)^{(3)} = -\left(\frac{1}{p+1}\right)''' = -\left(\frac{-1}{(p+1)^2}\right)'' =$

$= \left(\frac{1}{(p+1)^2}\right)'' = \left(\frac{-2}{(p+1)^3}\right)' = \frac{6}{(p+1)^4}$

e) $\mathcal{L}[t \cdot \underline{\underline{e^{2t}}}\cos t](p) = \mathcal{L}[\underline{\underline{t}}\cos t](p-2) = (-1)^1\left(\mathcal{L}[\cos t]\right)'(p-2) =$

$= -\left.\left(\frac{p}{p^2+1}\right)'\right|_{p=p-2} = -\left.\frac{p^2+1-2p^2}{(p^2+1)^2}\right|_{p=p-2} = \left.\frac{p^2-1}{(p^2+1)^2}\right|_{p=p-2} =$

$(p-2)^2 - 1$

$$= \frac{(p-2)^2-1}{\left[(p-2)^2+1\right]^2}$$

f) $\mathcal{L}\left[\int_0^t \sin 3u\,du\right](p) = \frac{1}{p}\mathcal{L}\left[\sin(3t)\right](p) = \frac{1}{p}\cdot\frac{3}{p^2+9}$

g) $\mathcal{L}\left[\int_0^t u^2 e^{-3u}\,du\right](p) = \frac{1}{p}\mathcal{L}\left[t^2 e^{-3t}\right](p) = \frac{1}{p}\mathcal{L}\left[t^2\right](p+3) =$

$$= \frac{1}{p}\left(\frac{2!}{p^3}\right)\Big|_{p\to p+3} = \frac{1}{p}\cdot\frac{2!}{(p+3)^3}$$

h*) $\mathcal{L}\left[\int_0^t \frac{\sin u}{u}\,du\right](p) \underset{\uparrow}{=} \frac{1}{p}\mathcal{L}\left[\frac{\sin t}{t}\right](p) \underset{\uparrow}{=} \frac{1}{p}\int_p^\infty \mathcal{L}\left[\sin t\right](y)\,dy =$

int. of the original

integration of the image

$$= \frac{1}{p}\int_p^\infty \frac{1}{y^2+1}\,dy = \frac{1}{p}\cdot\arctan y\Big|_p^\infty = \frac{1}{p}\left(\frac{\pi}{2}-\arctan p\right) \underset{\uparrow}{=} \frac{1}{p}\arctan\frac{1}{p}$$

$\arctan p + \arctan\frac{1}{p} = \frac{\pi}{2}$

i) $\mathcal{L}\left[t^2\cos t\right](p) = (-1)^2\left(\mathcal{L}\left[\cos t\right](p)\right)'' = \left(\frac{p}{p^2+1}\right)'' =$

$$= \left(\frac{p^2+1-2p^2}{(p^2+1)^2}\right)' = \left(\frac{-p^2+1}{(p^2+1)^2}\right)' = \frac{-2p\,(p^2+1)^2-2(p^2+1)2p(1-p^2)}{(p^2+1)^4} =$$

$$= \frac{-2p^3-2p-4p+4p^3}{(p^2+1)^3} = \frac{2p^3-6p}{(p^2+1)^3} \quad ✓$$

j) $\mathcal{L}\left[\frac{e^{-2t}-e^{-3t}}{t}\right](p) = \int_p^\infty \mathcal{L}\left[e^{-2t}-e^{-3t}\right](q)\,dq =$

$$= \int_p^\infty\left(\frac{1}{q+2}-\frac{1}{q+3}\right)dq = \left[\ln(q+2)-\ln(q+3)\right]\Big|_p^\infty = \ln\frac{q+2}{q+3}\Big|_p^\infty$$

$$= \lim_{x\to\infty}\ln\frac{x+2}{x+3} - \ln\frac{p+2}{p+3} = -\ln\frac{p+2}{p+3} = \ln\frac{p+3}{p+2}$$

2) $X(p) \xrightarrow{I.L.T.} f(t)$

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Find the originals $f(t)$ corresponding to the following images

— we have to use partial fractions decomposition

$\mathcal{L}[e^{\lambda t}](p) = \frac{1}{p-\lambda} \Big/ \mathcal{L}^{-1}$

$\mathcal{L}^{-1}\left[\frac{1}{p-\lambda}\right] = e^{\lambda t}$

a) $\mathcal{L}^{-1}\left[\frac{1}{p-3}\right] = e^{3t}$

b) $\mathcal{L}^{-1}\left[\frac{1}{2p-3}\right] = \mathcal{L}^{-1}\left[\frac{1}{2(p-\frac{3}{2})}\right] =$

$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{p-\frac{3}{2}}\right] = \frac{1}{2}e^{\frac{3}{2}t}$

c) $\mathcal{L}^{-1}\left[\frac{1}{p^2+9}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{p^2+3^2}\right] = \frac{1}{3}\sin(3t)$

$\mathcal{L}[\cos(at)](p) = \frac{p}{p^2+a^2}$

$\mathcal{L}[\sin(at)](p) = \frac{a}{p^2+a^2}$

d) $\mathcal{L}^{-1}\left[\frac{1}{p^3}\right] = \frac{1}{2!}\mathcal{L}^{-1}\left[\frac{2!}{p^3}\right] = \frac{1}{2}t^2$

$\mathcal{L}[t^2](p) = \frac{2!}{p^3}$

e) $\mathcal{L}^{-1}\left[\frac{3!}{(p-2)^4}\right] = t^3e^{2t} \quad = \mathcal{L}^{-1}\left[\frac{3!}{p^4}\Big/_{p=p-2}\right] = t^3e^{2t}$

$\mathcal{L}[t^3](p) = \frac{3!}{p^4}$

the image is translated => we have in the original e^{2t}

f) $\mathcal{L}^{-1}\left[\frac{1}{p(p-3)}\right] =$

$\frac{1}{p-a} \quad \frac{1}{p-b}$

$\frac{1}{p\cdot(p-3)} = \frac{A}{p} + \frac{B}{p-3} \longrightarrow \frac{1}{p(p-3)} = \frac{\frac{1}{3}}{p-3} - \frac{\frac{1}{3}}{p}$

$1 = A(p-3) + Bp$

$p = 0 \Rightarrow -3A = 1 \Rightarrow A = -\frac{1}{3}$

$p = 3 \Rightarrow 3B = 1 \Rightarrow B = \frac{1}{3}$

$\mathcal{L}[a](p) = \frac{a}{p}$

$\mathcal{L}[1](p) = \frac{1}{p}$

$= \mathcal{L}^{-1}\left[\frac{\frac{1}{3}}{p-3}\right] - \mathcal{L}^{-1}\left[\frac{\frac{1}{3}}{p}\right] = \frac{1}{3}e^{3t} - \frac{1}{3}$

g) $\mathcal{L}^{-1}\left[\frac{p}{p^2-4p+11}\right] = \mathcal{L}^{-1}\left[\frac{p}{p^2-4p+4+7}\right] = \mathcal{L}^{-1}\left[\frac{p}{(p-2)^2+7}\right] =$

$\Delta = 16-44 < 0 \Rightarrow$ no real solutions

g) $\mathcal{L}^{-1}\left[\dfrac{\;\;}{p^2-4p+11}\right] = \mathcal{L}^{-1}\left[\dfrac{\;\;}{p^2-4p+4+7}\right] = \mathcal{L}^{-1}\left[\dfrac{\;\;}{(p-2)^2+7}\right]$

$p^2-4p+11=0 \qquad \Delta = 16-44<0 \Rightarrow$ no real solutions

$\mathcal{L}[\cos t](p) = \dfrac{p}{p^2+a^2}$

$$= \mathcal{L}^{-1}\left[\frac{p-2+2}{(p-2)^2+(\sqrt{7})^2}\right] = \mathcal{L}^{-1}\left[\frac{p-2}{(p-2)^2+(\sqrt{7})^2}\right] + \frac{2}{\sqrt{7}}\mathcal{L}^{-1}\left[\frac{\sqrt{7}}{(p-2)^2+(\sqrt{7})^2}\right] =$$

$\mathcal{L}[\sin t](p) = \dfrac{a}{p^2+a^2}$

$$= e^{2t}\cdot\cos(\sqrt{7}t) + \frac{2}{\sqrt{7}}e^{2t}\cdot\sin(\sqrt{7}t)$$

h) $$\mathcal{L}^{-1}\left[\frac{1}{p^2-6p+25}\right] = \mathcal{L}^{-1}\left[\frac{1}{\underbrace{p^2-6p+9}+16}\right] = \mathcal{L}^{-1}\left[\frac{1}{(p-3)^2+4^2}\right] =$$

$\Delta<0$

$$= \frac{1}{4}\mathcal{L}^{-1}\left[\frac{4}{(p-3)^2+4^2}\right] = \frac{1}{4}e^{3t}\sin(4t)$$

Homework.

1) $\mathcal{L}[(t+1)\sin 2t](p)$

2) $\mathcal{L}^{-1}\left[\dfrac{3}{(p+1)^3}\right]$ f

3) $\mathcal{L}^{-1}\left[\dfrac{4}{(p+1)^2}\right]$ f

4) $\mathcal{L}^{-1}\left[\dfrac{3p+6}{p^2+3p}\right]$ p.f

5) $\mathcal{L}^{-1}\left[\dfrac{p}{p^2-5p+6}\right]$ p.f

6) $\mathcal{L}^{-1}\left[\dfrac{p}{p^2+3p+7}\right]$ $\Delta<0$

7) $\mathcal{L}^{-1}\left[\dfrac{1}{(p+4)^7}\right]$ f.