Seminar 13

Wednesday, January 6, 2021 6:04 PM

Fourier Transform

$f : \mathbb{R} \to \mathbb{C}$ a Fourier original, $f \in \mathcal{L}^1(\mathbb{R})$
the space of Lebesgue measurable functions

$\int_{-\infty}^{+\infty} |f(x)| dx$ is convergent

$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-isx} dx$, $s \in \mathbb{R}$ the image of f under Fourier transform

Not. $\mathcal{F}[f(x)](s) = F(s)$

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s) e^{-isx} ds$

- $f : \mathbb{R} \to \mathbb{C}$ is even function $\Rightarrow \begin{cases} F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \\ f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(s) \cos sx \, ds \end{cases}$ the cosine Fourier transform of f
- $f : \mathbb{R} \to \mathbb{C}$ is an odd function $\Rightarrow \begin{cases} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx \\ f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(s) \sin sx \, ds \end{cases}$ the sine FT of f

Problems

2 17 $a > 0$, $b \in \mathbb{R}$

i) $\mathcal{F}\left[\frac{x}{(x^2+a^2)^2}\right](s) = -\frac{1}{2}\mathcal{F}\left[\frac{-2x}{(x^2+a^2)^2}\right](s) = -\frac{1}{2}\mathcal{F}\left[\left(\frac{1}{x^2+a^2}\right)'\right](s) =$

$\boxed{\mathcal{F}[f'(x)](s) = is\,\mathcal{F}[f(x)](s)}$

$= -\frac{1}{2} is\, \mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = -\frac{1}{2} is \cdot \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a|s|} = \frac{-is}{2a}\sqrt{\frac{\pi}{2}}\, e^{-a|s|}$, $s \in \mathbb{R}$.

$\boxed{\mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a|s|}}$

ii) $\mathcal{F}\left[\frac{1}{(x^2+a^2)^2}\right](s) = ?$

$\boxed{\mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = \frac{1}{a}\cdot\sqrt{\frac{\pi}{2}} \cdot e^{-a|s|}, s \in \mathbb{R}}$ $\Big| (--)'_a$

$$\mathcal{F}\left[\frac{-2a}{(x^2+a^2)^2}\right](s) = -\frac{1}{a^2}\cdot(\ )'' \qquad \overline{a} \quad \overline{2}\,(-|s|)\cdot e$$

$$-2a\cdot\mathcal{F}\left[\frac{1}{(x^2+a^2)^2}\right](s) = -\frac{1}{a^2}\sqrt{\frac{\pi}{2}}\cdot e^{-a|s|}\left(1+a|s|\right) \Big/ \cdot\left(-\frac{1}{2a}\right)$$

$$\mathcal{F}\left[\frac{1}{(x^2+a^2)^2}\right](s) = \frac{1}{2a^3}\sqrt{\frac{\pi}{2}}\, e^{-a|s|}\left(1+a|s|\right).$$

iii) $\mathcal{F}\left[\frac{e^{-ibx}}{x^2+a^2}\right](s) = ?$

$$\boxed{\mathcal{F}\left[e^{-iax}f(x)\right](s) = \mathcal{F}[f(x)](s+a)}$$ Translation of the image

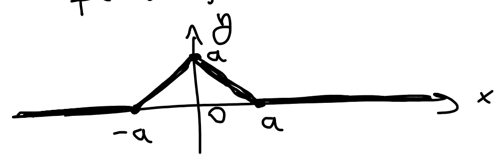
$$\mathcal{F}\left[\frac{e^{-ibx}}{x^2+a^2}\right](s) = \mathcal{F}\left[\frac{1}{x^2+a^2}\right](s+b) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a|s+b|}.$$

$$\boxed{\mathcal{F}\left[\frac{1}{x^2+a^2}\right](s) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a|s|}}$$

2.18

i) $f(x) = \begin{cases} 0, & |x| \geq a \\ x+a, & -a < x \leq 0 \\ -x+a, & 0 < x < a \end{cases}$ $\Rightarrow f(x) = a-|x|,\ |x| \leq a\ (a>0)$

$f(-x) = f(x)$



f is an even function

$\Downarrow$

$\mathcal{F}[f(x)](s) = \mathcal{F}_c[f(x)](s)$

• For $\boxed{s \neq 0}$

$$\mathcal{F}_c(f(x))(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^a (a-x)\cos sx\,dx =$$

$$= \sqrt{\frac{2}{\pi}}\left(\int_0^a a\cos sx\,dx - \int_0^a x\cos sx\,dx\right) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s}\sin sx\Big|_0^a - \int_0^a x\left(\frac{1}{s}\sin sx\right)'dx\right) =$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{s}\sin sa - \frac{x}{s}\sin sx\Big|_0^a + \int_0^a \frac{1}{s}\sin sx\,dx\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\cancel{\frac{a}{s}\sin sa} - \cancel{\frac{a}{s}\sin sa} - \frac{1}{s^2}\cos sx\Big|_0^a\right) =$$

$$= \sqrt{\frac{2}{\pi}}\left(-\frac{1}{s^2}\cos sa + \frac{1}{s^2}\right) = \frac{1}{s^2}\sqrt{\frac{2}{\pi}}\left(1-\cos sa\right) = \frac{1}{s^2}\sqrt{\frac{2}{\pi}}\cdot 2\sin^2\frac{sa}{2} =$$

$$1-\cos x = 2\sin^2\frac{x}{2}$$

$$\sqrt{\ }\ \left(\sin\frac{sa}{2}\right)^2 \cdot \left(\frac{a}{\ }\right)^2 \cdot 2 \qquad \sqrt{2}\cdot a^2\left(\sin\frac{sa}{2}\right)^2 \qquad a^2\left(\sin\frac{sa}{2}\right)^2.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\sin\frac{pa}{2}}{\frac{pa}{2}}\right)^2 \cdot \left(\frac{a}{2}\right)^2 \cdot 2 = \sqrt{\frac{2}{\pi}} \cdot \frac{a^2}{2} \cdot \left(\frac{\sin\frac{pa}{2}}{\frac{pa}{2}}\right)^2 = \frac{a^2}{\sqrt{2\pi}} \left(\frac{\sin\frac{pa}{2}}{\frac{pa}{2}}\right)^2 .$$

• For $\boxed{p=0}$

$$\mathcal{F}_c[f(x)](0) = \sqrt{\frac{2}{\pi}} \int_0^a (a-x)\underbrace{\cos 0}_{1}\, dx = \sqrt{\frac{2}{\pi}} \left(ax\Big|_0^a - \frac{x^2}{2}\Big|_0^a\right) =$$

$$= \sqrt{\frac{2}{\pi}}\left(a^2 - \frac{a^2}{2}\right) = \frac{a^2}{2}\sqrt{\frac{2}{\pi}} = \frac{a^2}{\sqrt{2\pi}}$$

$$\mathcal{F}[f(x)](p) = \begin{cases} \frac{a^2}{\sqrt{2\pi}} \cdot \left(\frac{\sin\frac{pa}{2}}{\frac{pa}{2}}\right)^2, & p \neq 0 \\ \frac{a^2}{\sqrt{2\pi}}, & p = 0 \end{cases}$$

ii) $$f(x) = \begin{cases} e^{ax}, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x > 0 \end{cases}, \quad a > 0$$

$$\mathcal{F}[f(x)](p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underline{\underline{f(x)}}\, e^{-ipx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ax} \cdot e^{-ipx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-ip)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{a-ip} e^{(a-ip)x}\Big|_{-\infty}^{0} = \frac{1}{\sqrt{2\pi}} \cdot {}^{a+ip)}\frac{1}{a-ip} \cdot \left[e^0 - \lim_{x\to-\infty} e^{(a-ip)x}\right] =$$

$$= \frac{a+ip}{\sqrt{2\pi}\,(a^2+p^2)} \cdot [1-0] = \frac{a+ip}{\sqrt{2\pi}\,(a^2+p^2)} .$$

$$\lim_{x\to-\infty} e^{(a-ip)x} = \lim_{x\to-\infty} e^{ax} \cdot e^{-ipx} = \lim_{x\to-\infty} \underset{\downarrow\, 0}{e^{ax}} \cdot \underbrace{(\cos px - i\sin px)}_{\text{bounded}} = 0$$

iii) $$f(x) = \begin{cases} \cos 2x, & |x| < a \\ 0, & |x| > a \end{cases}$$

• For $\boxed{p \neq \pm 2}$

$$\mathcal{F}_c[f(x)](p) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underline{\underline{f(x)}} \cos px\, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos 2x \cos px\, dx =$$

$$\boxed{\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^a [\cos(2x+px) + \cos(2x-px)]\, dx = \frac{1}{\sqrt{2\pi}} \int_0^a [\cos(2+p)x + \cos(2-p)x]\, dx$$

$$\Big[\ \ldots\ \Big|^a \quad \sin(2-p)x\,\Big|^a\ \Big] \qquad \boxed{p \neq \pm 2}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(2+\omega)x}{2+\omega}\Big|_0^a + \frac{\sin(2-\omega)x}{2-\omega}\Big|_0^a\right] =$$

$\boxed{\omega \neq \pm 2}$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(2+\omega)a}{2+\omega} + \frac{\sin(2-\omega)a}{2-\omega}\right].$$

• For $\boxed{\omega = \pm 2}$

$$\mathcal{F}_c[f(x)](2) = \sqrt{\frac{2}{\pi}}\int_0^a \cos 2x \cos 2x\, dx = \sqrt{\frac{2}{\pi}}\int_0^a \cos^2 2x\, dx =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a \frac{1+\cos 4x}{2}\, dx = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\left(x\Big|_0^a + \frac{\sin 4x}{4}\Big|_0^a\right) = \frac{1}{\sqrt{2\pi}}\left(a + \frac{\sin 4a}{4}\right).$$

iv) $\mathcal{F}\left[e^{-\frac{x^2}{2}}\right](\omega) = ?$ $\mathcal{F}\left[\frac{d^n}{dx^n} e^{-\frac{x^2}{2}}\right](\omega) = ?$ $\mathcal{F}\left[e^{-x^2}\right](\omega) = ?$

• $$\mathcal{F}\left[e^{-\frac{x^2}{2}}\right](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} - i\omega x}\, dx =$$

$$\boxed{\int_{-\infty}^{+\infty} e^{-u^2}\, du = \sqrt{\pi}}$$ the Gaussian integral

we construct a perfect square:

$$-\frac{x^2}{2} - i\omega x = -\left(\frac{x^2}{2} + i\omega x\right) = -\left[\left(\frac{x}{\sqrt{2}}\right)^2 + 2\cdot\frac{x}{\sqrt{2}}\cdot\frac{i\omega}{\sqrt{2}} + \frac{i^2\omega^2}{2} - \frac{i^2\omega^2}{2}\right] =$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$= -\left(\frac{x}{\sqrt{2}} + \frac{i\omega}{\sqrt{2}}\right)^2 - \frac{\omega^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}}+\frac{i\omega}{\sqrt{2}}\right)^2 - \frac{\omega^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}}\int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}}+\frac{i\omega}{\sqrt{2}}\right)^2}\, dx =$$

$$u = \frac{x}{\sqrt{2}} + \frac{i\omega}{\sqrt{2}}$$

$$du = \frac{1}{\sqrt{2}}\, dx \Rightarrow dx = \sqrt{2}\, du$$

$$\begin{cases} x = \infty \Rightarrow u = \infty \\ x = -\infty \Rightarrow u = -\infty \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}}\int_{-\infty}^{+\infty} e^{-u^2}\sqrt{2}\, du = \frac{\sqrt{2}\sqrt{\pi}}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} = e^{-\frac{\omega^2}{2}}$$

$\sqrt{2\pi}$ $\int_{-\infty}$ $\sqrt{2\pi}$

$$\Rightarrow \mathcal{F}\left[e^{-\frac{x^2}{2}}\right](\omega) = e^{-\frac{\omega^2}{2}}.$$

- $$\mathcal{F}\left[\frac{d^n}{dx^n} e^{-\frac{x^2}{2}}\right](\omega) = (i\omega)^n \mathcal{F}\left[e^{-\frac{x^2}{2}}\right](\omega) = (i\omega)^n \cdot e^{-\frac{\omega^2}{2}}.$$

$$\boxed{\mathcal{F}\left[f^{(n)}(x)\right](\omega) = (i\omega)^n \mathcal{F}\left[f(x)\right](\omega)}$$

- $$\mathcal{F}\left[e^{-x^2}\right](\omega) = \mathcal{F}\left[e^{-\frac{2x^2}{2}}\right](\omega) = \mathcal{F}\left[e^{-\frac{(\sqrt{2}x)^2}{2}}\right](\omega) = \mathcal{F}\left[f(\sqrt{2}x)\right](\omega) =$$

we use the property "change of scale".

$$\boxed{\mathcal{F}\left[f(ax)\right](\omega) = \frac{1}{|a|}\mathcal{F}\left[f(x)\right]\left(\frac{\omega}{a}\right) \quad \text{or} \quad \mathcal{F}\left[f\left(\frac{x}{a}\right)\right](\omega) = |a|\,\mathcal{F}\left[f(x)\right](a\omega)}$$

$$= \frac{1}{\sqrt{2}}\mathcal{F}\left[f(x)\right]\left(\frac{\omega}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot e^{-\left(\frac{\omega}{\sqrt{2}}\right)^2 \cdot \frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot e^{-\frac{\omega^2}{4}}.$$

ex) $$I = \int_{-\pi}^{\pi} \frac{3+\cos x}{7\sin x + 25}, \quad 7\sin x + 25 \neq 0, \ \forall x \in \mathbb{R}$$

Remark. $$\boxed{\int_a^{a+2\pi} R(\sin x, \cos x)\,dx = \int_0^{2\pi} R(\sin x, \cos x)\,dx}$$

$$I = \int_0^{2\pi} \frac{3+\cos x}{7\sin x + 25}\,dx \ ; \quad J = \int_0^{2\pi} \frac{\sin x}{7\sin x + 25}\,dx$$

$$K = I + iJ = \int_0^{2\pi} \frac{3 + \overbrace{\cos x + i\sin x}^{e^{ix}}}{7\sin x + 25}\,dx = \int_0^{2\pi} \frac{3 + e^{ix}}{7\sin x + 25}\,dx$$

$$e^{ix} = z \Rightarrow i\underbrace{e^{ix}}_{z}\,dx = dz \Rightarrow dx = \frac{dz}{iz}$$

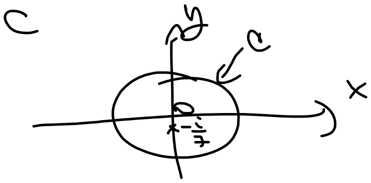
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$$

$$K = \int_{C:|z|=1} \frac{z+3}{7 \cdot \frac{z^2-1}{2iz} + 25} \cdot \frac{dz}{iz} = \int_C \frac{(z+3)\cdot \cancel{2iz}}{7z^2 - 7 + 50iz} \cdot \frac{dz}{\cancel{iz}} = 2\int_C \underbrace{\frac{z+3}{7z^2 + 50iz - 7}}_{f(z)}\,dz$$

$7z^2 + 50iz - 7 = 0$

$\Delta = -2500 + 4\cdot 49 = -2304 =$

$$z_{1,2} = \frac{-50i \pm 48i}{14} = \frac{-25i \pm 24i}{7} \quad \begin{cases} z_1 = -\dfrac{i}{7} \in \text{int}\, C, \text{ pole of order } 1 \\ z_2 = -7i \notin \text{int}\, C \end{cases}$$



$$\operatorname*{Res}_{z=-\frac{i}{7}} f(z) = \left.\frac{z+3}{7(z+7i)}\right|_{z=-\frac{i}{7}} = \frac{-\frac{i}{7}+3}{7\left(-\frac{i}{7}+7i\right)} = \frac{-i+21}{7\cdot(48i)}.$$

or

$$\operatorname*{Res}_{z=-\frac{i}{7}} f(z) = \left.\frac{z+3}{\left(7z^2+50iz-7\right)'}\right|_{z=-\frac{i}{7}}$$

$$K = 2\cdot 2\pi i \cdot \operatorname*{Res}_{z=-\frac{i}{7}} f(z) = \cancel{2\cdot 2\pi i}\cdot \frac{21-i}{7\cdot \cancel{48 i}_{12}} = \frac{\pi}{84}(21-i) = \frac{\pi}{4} - i\frac{\pi}{84}$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$