Seminar 12

Wednesday, December 16, 2020 4:05 PM

Applications to the Laplace transform

- Differential equations with nonconstant coefficients

$t x''(t) + 2x'(t) = t-1$, $x(0)=3$, $x'(0) = -\frac{1}{2}$

$\mathcal{L}[t x''(t)](p) + 2\mathcal{L}[x'(t)](p) = \mathcal{L}[t-1](p)$

$\mathcal{L}[t x''(t)](p) = (-1)\left(\mathcal{L}[x''(t)](p)\right)' = (-1)\left(p^2 X(p) - p\underbrace{x(0)}_{3} - \underbrace{x'(0)}_{-\frac{1}{2}}\right)' =$

$= -\left(p^2 X'(p) + 2pX(p) - 3\right) = -p^2 X'(p) - 2pX(p) + 3$

$\mathcal{L}[x'(t)](p) = pX(p) - \underbrace{x(0)}_{3} = pX(p) - 3$

$\Rightarrow -p^2 X'(p) - \cancel{2pX(p)} + 3 + \cancel{2pX(p)} - 6 = \frac{1}{p^2} - \frac{1}{p}$

$-p^2 X'(p) = 3 + \frac{1}{p^2} - \frac{1}{p} \quad / \cdot (-p^2)$

$X'(p) = \frac{-3}{p^2} - \frac{1}{p^4} + \frac{1}{p^3} \Big/ \int \quad \Rightarrow X(p) = \int (-3p^{-2} - p^{-4} + p^{-3})\,dp$

$\Rightarrow X(p) = -3\frac{p^{-1}}{-1} - \frac{p^{-3}}{-3} + \frac{p^{-2}}{-2} + C$

$X(p) = \frac{3}{p} + \frac{1}{3p^3} - \frac{1}{2p^2} + C$

We use the property of $X(p)$
$\lim_{p\to\infty} X(p) = 0$

$\xrightarrow{\text{property}} C = 0$

$\Rightarrow x(t) = -\frac{1}{2}t + \frac{1}{6}t^2 + 3$

② Solve the equation

$x(t) = 2\sin 4t + \int_0^t \sin 4(t-u)\,x(u)\,du \quad / \mathcal{L}$

$\mathcal{L}[x(t)](p) = X(p)$

$\Rightarrow X(p) = 2\,\frac{4}{p^2+16} + \mathcal{L}[\sin 4t](p)\cdot\mathcal{L}[x(t)](p)$

$X(p) = \frac{8}{\ldots} + \frac{4}{\ldots}\cdot X(p) \Rightarrow X(p)\cdot\left(1 - \frac{4}{p^2+16}\right) = \frac{8}{\ldots}$ p^2+16

$$X(p) = \frac{8}{p^2+16} + \frac{4}{p^2+16} \cdot X(p) \Rightarrow X(p)\cdot\left(1 - \frac{4}{p^2+16}\right) = \frac{8}{p^2+16}$$

$$\Rightarrow X(p) \cdot \frac{p^2+12}{p^2+16} = \frac{8}{p^2+16} \Rightarrow X(p) = \frac{8}{p^2+12} \quad \Big/ \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{8}{p^2+12}\right] = \frac{8}{2\sqrt{3}}\mathcal{L}^{-1}\left[\frac{2\sqrt{3}}{p^2+(2\sqrt{3})^2}\right] = \frac{4}{\sqrt{3}}\sin(2\sqrt{3}t)$$

$$\Rightarrow x(t) = \frac{4}{\sqrt{3}}\sin(2\sqrt{3}t)$$

is not in the table

(3) $x''(t) + x(t) = \frac{1}{\cos t} \Big/ \mathcal{L}$, $x(0) = 0$, $x'(0) = 2$; $\mathcal{L}[x(t)](p) = X(p)$

$$p^2X(p) - p\underset{0}{x(0)} - \underset{2}{x'(0)} + X(p) = \mathcal{L}\left[\frac{1}{\cos t}\right](p)$$

$$X(p)(p^2+1) = 2 + \mathcal{L}\left[\frac{1}{\cos t}\right](p) \quad \Big/ \cdot \frac{1}{p^2+1}$$

$$X(p) = \frac{2}{p^2+1} + \frac{1}{p^2+1}\cdot\mathcal{L}\left[\frac{1}{\cos t}\right](p)$$

$$X(p) = 2\mathcal{L}[\sin t](p) + \mathcal{L}[\sin t](p)\cdot\mathcal{L}\left[\frac{1}{\cos t}\right] \Big/ \mathcal{L}^{-1}$$

$$x(t) = 2\sin t + \sin t * \frac{1}{\cos t}$$

$$x(t) = 2\sin t + \int_0^t \sin(t-z)\cdot\frac{1}{\cos z}\,dz$$

$$x(t) = 2\sin t + \int_0^t \frac{\sin t\cos z - \cos t\sin z}{\cos z}\,dz$$

$$x(t) = 2\sin t + \int_0^t \left(\sin t - \cos t\,\frac{\sin z}{\cos z}\right)dz$$

$$x(t) = 2\sin t + \sin t\cdot z\Big|_0^t + \cos t\cdot\ln(\cos z)\Big|_0^t$$

$$x(t) = 2\sin t + t\sin t + \cos t\cdot\ln(\cos t)$$

(4) $\displaystyle\int_0^{2\pi}\frac{dx}{13+5\cos x} \longrightarrow \int_{C:\,|z|=1}$

$$e^{ix} = z, \quad \cos x = \frac{z^2+1}{2z}, \quad dx = \frac{dz}{iz}$$

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$$I = \int_C \frac{\frac{dz}{iz}}{13+5\cdot\frac{z^2+1}{2z}} = \int_C \frac{dz}{i\cancel{z}\cdot\frac{26z+5z^2+5}{2\cancel{z}}} = \frac{2}{i}\int_C \frac{dz}{5z^2+26z+5}$$

$$5z^2+26z+5=0 \ , \ \Delta = 676-100 = 576 \Rightarrow z_{1,2} = \frac{-26 \pm 24}{10} \begin{cases} -5 \notin \operatorname{int} C \\ -\frac{1}{5} \in \operatorname{int} C \end{cases}$$

$$5z^2+26z+5 = 5(z+5)(z+\tfrac{1}{5})$$

$z = -\frac{1}{5}$ pole of order 1

$$\operatorname{Res}_{z=-\frac{1}{5}} f(z) = \frac{\frac{1}{5(z+5)}}{1}\Bigg|_{z=-\frac{1}{5}} = \frac{1}{5(-\frac{1}{5}+5)} = \frac{1}{24}$$

$$I = 2\pi \cancel{i}\cdot\frac{2}{\cancel{i}}\cdot\frac{1}{24} = \frac{\pi}{6}$$

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$$x''(t) + 2x'(t) + x(t) = \left[e^{t}\cdot(t+1)\right]^{-1} \ , \ x'(0) = x(0) = 0$$

$$\mathcal{L}[x(t)](p) = X(p)$$

$$\underline{p^2X(p)} - p\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + \underline{2pX(p)} - 2\underbrace{x(0)}_{0} + \underline{X(p)} = \underbrace{\mathcal{L}\left[\left[e^{t}\cdot(t+1)\right]^{-1}\right](p)}_{F(p)}$$

$$X(p)\cdot(p^2+2p+1) = F(p)$$

$$\Rightarrow X(p) = \frac{F(p)}{(p+1)^2} \Big/ \mathcal{L}^{-1}$$

$$\mathcal{L}\left[t\cdot e^{-t}\right](p) = \frac{1}{p^2}\bigg|_{p=p+1} = \frac{1}{(p+1)^2}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{F(p)}{(p+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(p+1)^2}\right] * \mathcal{L}^{-1}\left[F(p)\right]$$

$$x(t) = t\cdot e^{-t} * \left[e^{t}\cdot(t+1)\right]^{-1} = \int_0^t (t-u)\cdot e^{-t+u}\cdot\left[e^{u}(u+1)\right]^{-1} du$$

$$= \int_0^t (t-u)\underbrace{e^{-t+u}\cdot e^{-u}}\cdot(u+1)^{-1} du = \int_0^t (t-u)\,e^{-t}\,(u+1)^{-1} du$$

$$= e^{-t}\int_0^t \frac{t-u}{u+1} du = e^{-t}\int_0^t \left(\frac{t}{u+1} - \frac{u+1-1}{u+1}\right) du$$

$$= e^{-t}\left[t\ln(u+1) - u + \ln(u+1)\right]\Big|_0^t$$

$$= e^{-t}\left[t\ln(t+1) - t + \ln(t+1)\right] =$$

$$= e^{-t}\left[t\ln(t+1) - t + \ln(t+1)\right] =$$

$$\Rightarrow x(t) = e^{-t}\left[(t+1)\ln(t+1) - t\right]$$

⑥ $\sin t = \dfrac{t^3}{e^t} + \displaystyle\int_0^t x''(u)(t-u)^2\,du \Big/ \mathcal{L}$, $x(0)=0,\ x'(0)=2$

$$\Rightarrow \frac{1}{p^2+1} = \frac{3!}{(p+1)^4} + \mathcal{L}[x''(t)](p)\cdot\mathcal{L}[t^2](p)$$

$$\frac{1}{p^2+1} = \frac{3!}{(p+1)^4} + \left[p^2X(p) - p\underset{0}{\cancel{x(0)}} - \underset{2}{x'(0)}\right]\cdot\frac{2}{p^3}$$

$$\frac{1}{p^2+1} = \frac{6}{(p+1)^4} + \underline{\underline{\cancel{p^2}\cdot\frac{2}{\cancel{p^3}p}X(p)}} - \frac{4}{p^3} \Rightarrow \frac{2}{p}X(p) = \frac{1}{p^2+1} + \frac{4}{p^3} - \frac{6}{(p+1)^4} \Big| \cdot\frac{p}{2}$$

$$\Rightarrow X(p) = \frac{p}{2(p^2+1)} + \frac{2}{p^2} - \frac{3(p+1-1)}{(p+1)^4}$$

$$X(p) = \frac{p}{2(p^2+1)} + \frac{2}{p^2} - \frac{3}{(p+1)^3} + \frac{3}{(p+1)^4} \Big| \mathcal{L}^{-1}$$

$$\underbrace{x(t) = \frac{1}{2}\cos t + 2t - \frac{3}{2}t^2e^{-t} + \frac{3}{6}t^3e^{-t}}$$

⑦ $\mathcal{L}[\sin\sqrt{t}] = ?$

We expand the function $\sin\sqrt{t} = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(\sqrt{t})^{2n+1}}{(2n+1)!}$

$$\mathcal{L}[t^{\alpha}](p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$$

$$\mathcal{L}[\sin\sqrt{t}](p) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\mathcal{L}\left[t^{\frac{2n+1}{2}}\right](p) =$$

$$\Gamma(p) = \int_0^{\infty} e^{-x}x^{1-p}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{\Gamma\left(\frac{2n+1}{2}+1\right)}{p^{\frac{2n+1}{2}+1}} =$$

$$\Gamma(p+1) = p\Gamma(p),\ \forall p>0$$

$$\Gamma(n+1) = n!\quad n\in\mathbb{N}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{\underset{(2n)!}{\cancel{(2n+1)!}}}\,\frac{\cancel{2n+1}}{2}\,\Gamma\left(\frac{2n+1}{2}\right)\cdot\frac{1}{p^n\cdot p^{3/2}}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n!\cdot 2^{2n}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{2} \Gamma\left(n+\frac{1}{2}\right) \cdot \frac{1}{p^n \cdot p^{3/2}} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{(2n)!}} \cdot \frac{1}{2} \cdot \frac{1}{p^n \cdot p^{3/2}} \cdot \frac{\cancel{(2n)!}\sqrt{\pi}}{n!\, 2^{2n}} =$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{2p^{3/2}} \frac{1}{n!} \left(-\frac{1}{4p}\right)^n = \frac{\sqrt{\pi}}{2p^{3/2}} \sum_{n=0}^{\infty} \left(-\frac{1}{4p}\right)^n \cdot \frac{1}{n!} =$$

$$= \frac{\sqrt{\pi}}{2p\sqrt{p}} e^{-\frac{1}{4p}}.$$

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$$x(t) = 2t + 2\int_0^t \cos u \cdot x(t-u)\,du$$

$$X(p) = \frac{2}{p^2} + 2\mathcal{L}[\cos t](p) \cdot \mathcal{L}[x(t)](p)$$

$$X(p) = \frac{2}{p^2} + \frac{2p}{p^2+1} X(p) \Rightarrow X(p) \cdot \left(\frac{p^2+1-2p}{p^2+1}\right) = \frac{2}{p^2}$$

$$\Rightarrow X(p) = \frac{2(p^2+1)}{p^2(p-1)^2} = \frac{\overset{p(p^2-2p+1)}{A}}{p} + \frac{\overset{p^2-2p+1}{B}}{p^2} + \frac{\overset{p^2(p-1)}{C}}{p-1} + \frac{\overset{p^2}{D}}{(p-1)^2}$$

$$\Rightarrow A(p^3-2p^2+p) + B(p^2-2p+1) + C(p^3-p^2) + D\cdot p^2 = 2p^2+2$$

$$\begin{cases} A + C = 0 \Rightarrow \boxed{C = -4} \\ -2A + B - C + D = 2 \Rightarrow -8 + \cancel{2} + 4 + D = \cancel{2} \quad \boxed{D = 4} \\ A - 2B = 0 \Rightarrow \boxed{A = 4} \\ \boxed{B = 2} \end{cases}$$

$$\boxed{x(t) = 4 + 2t - 4e^t + 4te^t}$$