Seminar (2) Functions of a complex variable
The Cauchy-Riemann conditions

$f: D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, $\boxed{w = f(z)} \Rightarrow u + iv = f(x+iy)$

$w = u + iv$, $u = u(x,y)$, $v = v(x,y)$

$z = x + iy$

(ex1) $f(z) = z^2 + 2z - 3\bar{z} + 1$ $\quad u(x,y) = ?$ $v(x,y) = ?$

$\Rightarrow u + iv = (x+iy)^2 + 2(x+iy) - 3(x-iy) + 1$

$\Rightarrow u + iv = x^2 + 2xyi - y^2 + 2x + 2yi - 3x + 3yi + 1$

$\Rightarrow u + iv = \underbrace{x^2 - y^2 - x + 1}_{u(x,y)} + i\underbrace{(2xy + 5y)}_{v(x,y)}$

Theorem: Let $f = u + iv$. If f is monogenic at $z_0 = x_0 + iy_0$ then u and v have partial derivatives at (x_0, y_0) and the Cauchy-Riemann conditions hold:

$$\text{the (C-R) cond.} \begin{cases} \dfrac{\partial u}{\partial x}(x_0,y_0) = \dfrac{\partial v}{\partial y}(x_0,y_0) \\ \dfrac{\partial u}{\partial y}(x_0,y_0) = -\dfrac{\partial v}{\partial x}(x_0,y_0) \end{cases}$$

(ex2) Determine the points where the function f is monogenic.

a) $f(z) = 3z^2 - 2iz$; b) $f(z) = 2\,\mathrm{Im}\,z - i\,\mathrm{Re}\,z$

c) $f(z) = (z^2 + \bar{z}^2)(1-i) + 2z\bar{z}(1+i) - 4iz$

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f has a derivative at z_0: $\exists \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} \in \overline{\mathbb{C}}$

f is monogenic at z_0: $\exists \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}$

a) $f(z) = 3z^2 - 2iz$

- we have to identify the real part and the imaginary part of the function

$u + iv = 3(x+iy)^2 - 2i(x+iy)$

$u + iv = 3(x^2 + 2xyi - y^2) - 2xi + 2y$

$u + iv = 3x^2 - 3y^2 + 2y + i(6xy - 2x) \to \begin{cases} u(x,y) = 3x^2 - 3y^2 + 2y \\ v(x,y) = 6xy - 2x \end{cases}$

- we have to check the C-R conditions

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} 6x = 6x & \checkmark \\ -6y + 2 = -(6y - 2) & \checkmark \end{cases}$$

The C-R conditions are verified $\forall x, y \in \mathbb{R} \Rightarrow$

$\Rightarrow$ the function is holomorphic.

b) $f(z) = 2\operatorname{Im} z - i\operatorname{Re} z$ $\qquad z = x + iy$

- $u + iv = 2y - ix \Rightarrow \begin{cases} u(x,y) = 2y \\ v(x,y) = -x \end{cases}$

- $$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} 0 = 0 & T \\ 2 = +1 & F \end{cases}$$

$\Rightarrow$ the function f is nowhere monogenic.

c) $f(z) = (z^2 + \bar{z}^2)(1-i) + 2z\bar{z}(1+i) - 4iz$

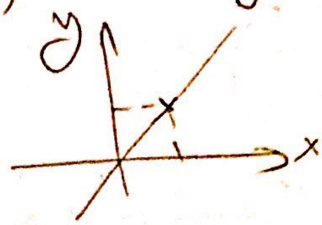
$u + iv = [(x+iy)^2 + (x-iy)^2](1-i) + 2(x+iy)(x-iy)(1+i) - 4i(x+iy)$

$= (x^2 - y^2 + \cancel{2xyi} + x^2 - y^2 - \cancel{2xyi})(1-i) + 2(x^2+y^2)(1+i)$
$- 4xi + 4y$

$= (2x^2 - 2y^2)(1-i) + (2x^2 + 2y^2)(1+i) - 4xi + 4y$

$$= 2x^2 - \cancel{2y^2} - 2x^2 i + 2y^2 i + 2x^2 + \cancel{2y^2} + \cancel{3x^2 i} + 2y^2 i - 4xi + 4y$$

$$\Rightarrow u + iv = 4x^2 + 4y + i(4y^2 - 4x) \rightarrow u(x,y) = 4x^2 + 4y$$
$$v(x,y) = -4x + 4y^2$$



$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \rightarrow \begin{cases} 8x = 8y \\ 4 = +4 \ A \end{cases}$$

$\Rightarrow$ the function is monogenic if and only if $x = y$, that is, only at the points where $\mathrm{Re}\, z = \mathrm{Im}\, z$

(ex3) Can the following function be the real part of an analytic function $f = u + iv$?

$$u(x,y) = \arctan \frac{y}{x}$$

- u has to be a harmonic function ($\Delta u(x,y) = 0$)

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \Delta u \text{ - the laplacian of } u$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = \frac{-\frac{y}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} \ ; \ \frac{\partial^2 u}{\partial x^2} = \frac{+2xy}{(x^2+y^2)^2} \\ \frac{\partial u}{\partial y} = \frac{\frac{1}{x}}{1 + \frac{y^2}{x^2}} = \frac{x}{x^2+y^2} \ ; \ \frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \end{array}\right\} \Rightarrow \Delta u = 0$$

(ex4) Find the holomorphic function $f = u + iv$ from its known real part $u(x,y)$ or imaginary part $v(x,y)$ and the value $f(z_0)$.
(reconstruct)

a) $u(x,y) = 3xy$; $f(0) = i$; b) $v(x,y) = \frac{y}{x^2+y^2}$, $f(1) = 1$

c) $u(x,y) = \ln(x^2+y^2)$, $f(1) = 0$, $x \neq 0$ (Homework)

a) $u(x,y) = 3xy$

Since f is holomorphic $\Rightarrow$ the C-R conditions hold.

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} 3y = \frac{\partial v}{\partial y} \\ 3x = -\frac{\partial v}{\partial x} \end{cases} \Big| \int dy \text{ we integrate relative to } y$$

$v(x,y) = 3\frac{y^2}{2} + \underbrace{\varphi(x)}$ function that depends only on x!

In order to find $\varphi(x)$ we write the other C-R condition:

$$\left.\begin{matrix} \frac{\partial v}{\partial x} = \varphi'(x) \\ \frac{\partial v}{\partial x} = -3x \end{matrix}\right\} \Rightarrow \varphi'(x) = -3x \Big| \int \Rightarrow \varphi(x) = -3\frac{x^2}{2} + c, \ c \in \mathbb{R}$$

So $v(x,y) = 3\frac{y^2}{2} - 3\frac{x^2}{2} + c, \ c \in \mathbb{R}$

$\Rightarrow f(x,y) = u(x,y) + i v(x,y) = 3xy + i\frac{3}{2}(y^2 - x^2) + ci$

We find the constant c from the condition $f(0) = i$.

$f(0,0) = ci \Rightarrow ci = i \Rightarrow c = 1 \Rightarrow f(x,y) = 3xy + \frac{3}{2}i(y^2 - x^2) + i$

$= -\frac{3}{2}i(\underbrace{2xyi + x^2 - y^2}_{z^2}) + i$

$\Rightarrow f(z) = -\frac{3}{2}iz^2 + i$ $\quad \left(\begin{matrix} x \mapsto z \\ y \mapsto 0 \end{matrix} \Rightarrow f(z) = -\frac{3}{2}iz^2 + i \right)$

Remark: A holomorphic function $u(x,y) + iv(x,y)$ can be written in the form $f(z) = u(z,0) + iv(z,0)$.

b) $v(x,y) = \frac{y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

- We write the C-R conditions:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \\ \frac{\partial u}{\partial y} = +\frac{2xy}{(x^2+y^2)^2} \Big| \int \text{ we integrate w.r.t. } y \end{cases}$$

$\Rightarrow u(x,y) = \int \frac{2xy}{(x^2+y^2)^2}\,dy \Rightarrow u(x,y) = \frac{-x}{x^2+y^2} + \underbrace{\varphi(x)}_{\text{depends only on } x}$

- we find $\varphi(x)$ from the other C-R condition

$$\frac{\partial u}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow \frac{-(x^2+y^2)-2x(-x)}{(x^2+y^2)^2} + \varphi'(x) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \cancel{\frac{x^2-y^2}{(x^2+y^2)^2}} + \varphi'(x) = \cancel{\frac{x^2-y^2}{(x^2+y^2)^2}} \Rightarrow \varphi'(x) = 0 \Big/ \int \Rightarrow \varphi(x) = C,\ C\in\mathbb{R}$$

$$\Rightarrow u(x,y) = \frac{-x}{x^2+y^2} + C$$

- we write the function $f(x,y) = \frac{-x}{x^2+y^2} + C + \frac{iy}{x^2+y^2}$
- we use the condition $f(1)=1 \Rightarrow \left.\begin{array}{l} f(1,0) = -1+C \\ f(1)=1 \end{array}\right\} \Rightarrow$

$-1+C=1 \Rightarrow C=2$

$$f(x,y) = \frac{-x}{x^2+y^2} + 2 + \frac{yi}{x^2+y^2}$$

$$f(x,y) = \frac{-(x-yi)}{x^2+y^2} + 2 \Rightarrow f(z) = \frac{-\cancel{\bar{z}}}{z\cancel{\bar{z}}} + 2 \Rightarrow f(z) = -\frac{1}{z} + 2$$

(or, in order to express f as a complex function of z:
$x \to z,\ y \to 0 \Rightarrow f(z) = -\frac{\cancel{z}}{z\cancel{z}} + 2 \Rightarrow f(z) = -\frac{1}{z} + 2$)

<u>Remark</u>

If we begin with the other C-R condition:

$$\frac{\partial u}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \Big/ \int \quad \text{we integrate w.r.t } x$$

$$\Rightarrow u(x,y) = \int \frac{x^2-y^2}{(x^2+y^2)^2}\,dx \Rightarrow \dots$$ more complicated than the other way

$$\Rightarrow u(x,y) = \frac{y}{x^2+y^2} + \underbrace{\varphi(y)}\ \text{---}$$

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(ex 5) Find a holomorphic function $f = u + iv$ such that $u(x,y) - v(x,y) = x^2 - y^2 - 2xy - 2x - 2y$ and $f(i) = -3$.

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 2x - 2y - 2 \\ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -2y - 2x - 2 \end{cases}$$

- we write C-R conditions: $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

- we replace in the system $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2x - 2y - 2 \\ -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2x - 2y - 2 \end{cases}$$

$$2\frac{\partial u}{\partial y} = -4y - 4 \;\Big|:2 \Rightarrow \frac{\partial u}{\partial y} = -2y - 2$$

$$\frac{\partial u}{\partial x} = 2x \;\Big/ \int dx$$

$$\Rightarrow u(x,y) = 2\frac{x^2}{2} + \varphi(y)$$

- we find $\varphi(y)$: $\frac{\partial u}{\partial y} = -2y - 2$

$$0 + \varphi'(y) = -2y - 2 \;\Big/ \int \Rightarrow \varphi(y) = -y^2 - 2y + C_{C \in \mathbb{R}}$$

- we write $u(x,y)$: $u(x,y) = x^2 - y^2 - 2y + C$
- we write $v(x,y)$ (from the initial condition): $v(x,y) = u(x,y) - x^2 + y^2 + 2xy + 2x + 2y$

$$\Rightarrow v(x,y) = 2xy + 2x + C$$

- we write $f(x,y)$:

$$f(x,y) = x^2 - y^2 - 2y + C + i(2xy + 2x + C)$$

$$f(i) = -3 \Rightarrow f(0,1) = -3 \Rightarrow \cancel{-1 - 2 + C + Ci = -3}$$

$$C = 0$$

$\Rightarrow f(x,y) = x^2 - y^2 - 2y + (2xy + 2x)i$

$f(x,y) = x^2 - y^2 + 2xyi + 2xi - 2y$

$= (x+iy)^2 + 2i(x+yi)$

$\Rightarrow f(z) = z^2 + 2iz$ $\quad\left(f(z) = z^2 + 2zi \right)$

$x \to z$
$y \to 0$

Homework : 1.8 ; 1.15