Seminar 8

Wednesday, November 18, 2020 3:48 PM

Using the Residue Theorem to evaluate real integrals

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}_{z=z_k} f(z)$$

(I) $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$ — rational function

(II) $\int_{-\infty}^{+\infty} f(x)\,dx$, $f(x) = \frac{P(x)}{Q(x)}$

(III) $\int_{-\infty}^{+\infty} f(x)\cos\alpha x\,dx$ or $\int_{-\infty}^{+\infty} f(x)\sin\alpha x\,dx$

(IV) $\int_0^{\infty} f(x)\,dx$

(I) $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$

$\boxed{z = e^{i\theta}}$

$\boxed{\cos\theta = \frac{z^2+1}{2z}\ ;\ \sin\theta = \frac{z^2-1}{2iz}}$

$$I = \int_{C:|z|=1} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz} = 2\pi i \sum_{z_k \in \operatorname{int} C} \operatorname{Res}_{z=z_k} f(z)$$

$\boxed{d\theta = \frac{dz}{iz}}$

(ex 1)

$$\int_0^{2\pi} \frac{d\theta}{\frac{5}{4}+\sin\theta}$$

$$I = \int_{C:|z|=1} \frac{dz}{iz}\cdot\frac{1}{\frac{5}{4}+\frac{z^2-1}{2iz}} = \int_C \frac{dz}{\cancel{iz}\cdot\frac{10iz+4z^2-4}{8\cancel{iz}}} = \int_C \frac{4}{2z^2+5iz-2}\,dz =$$

$2z^2+5iz-2=0$

$\Delta = 25i^2+16 = -9 \Rightarrow z_{1,2} = \frac{-5i \pm 3i}{4}$

$z_1 = -2i \notin \operatorname{int} C$

$z_2 = -\frac{i}{2} \in \operatorname{int} C$ pole of order 1

$2(z+2i)(z+\frac{i}{2}) = (z+2i)(2z+i)$

$$= \int_C \frac{4\,dz}{(2z+i)(z+2i)}$$

$$I = 2\pi i \operatorname{Res}_{z=-\frac{i}{2}} f(z) = 2\pi \cancel{i}\cdot\frac{4}{3\cancel{i}} = \frac{8}{3}\pi$$

$$\operatorname{Res}_{z=-\frac{i}{2}} f(z) = \frac{g(z)}{h'(z)}\Big|_{z=-\frac{i}{2}} = \frac{\cancel{4}^{\,2}}{(z+2i)\cancel{2}}\Big|_{z=-\frac{i}{2}} = \frac{2}{-\frac{i}{2}+\frac{4}{2}i} = \frac{4}{3i}$$

$g(z) = \frac{4}{z+2i}$, $h(z) = 2z+i$

II

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\,dx$$

P, Q are polynomials of degrees m and n
$n \geq m+2$

$$I = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$$

z_k are all the poles of $f(z) = \frac{P(z)}{Q(z)}$ that lie in the upper half plane

! We are interested only with the roots with positive imaginary part

1.57 i) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)(x^2+9)}\,dx$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+9)} = \frac{z^2}{(z+i)(z-i)(z+3i)(z-3i)}$$

poles of order 1

$z^2+1=0 \Rightarrow z_{1,2} = \pm i$ $\Rightarrow$ $z_1 = i$, $z_3 = 3i$ (lie in the upper half plane) (z_1 and z_3 have pos. im part)

$z^2+9=0 \Rightarrow z_{3,4} = \pm 3i$

$$I = 2\pi i \left(\operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_3} f(z) \right) \quad (*)$$

$$\operatorname{Res}_{z=z_1} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=i} = \frac{i^{\cancel{2}}}{(i^2+9)\cdot 2\cancel{i}} = \frac{i}{16}$$

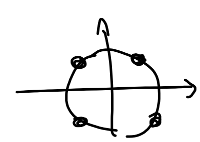
$g(z) = \frac{z^2}{(z^2+9)(z+i)}$; $h'(z) = 1$

$$\operatorname{Res}_{z=z_3} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=3i} = \frac{3\cancel{9}i^{\cancel{2}}}{(9i^2+1)\,\cancel{6}_2\cancel{i}} = \frac{-3i}{16}$$

$g(z) = \frac{z^2}{(z^2+1)(z+3i)}$; $h(z) = z-3i$, $h'(z) = 1$

$$(*) \Rightarrow I = 2\pi i \left(\frac{i}{16} - \frac{3i}{16} \right) = 2\pi i \cdot \frac{(-2i)}{16} = \frac{\pi}{4}$$

ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx$



$$f(z) = \frac{z^2}{z^4+1} \qquad I = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} f(z)$$

$z^4+1=0 \Rightarrow z^4 = -1 \Rightarrow -1 = \cos\pi + i\sin\pi \Rightarrow z_k = \cos\frac{\pi+2k\pi}{4} + i\sin\frac{\pi+2k\pi}{4}$, $k=\overline{0,3}$

$z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt2}{2} + i\frac{\sqrt2}{2}$

$z_2 = -\frac{\sqrt2}{2} + i\frac{\sqrt2}{2}$

$\Big\}$ $\in$ int C, z_1, z_2 poles of order 1

$z_3 = -\frac{\sqrt2}{2} - i\frac{\sqrt2}{2}$

$z_2 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$$\left.\begin{aligned} z_3 &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ z_4 &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \end{aligned}\right\} \text{ are not with pos. im part}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} (z-z_1)\cdot\frac{z^2}{z^4+1} = z_1^2 \lim_{z\to z_1}\frac{z-z_1}{z^4+1} \overset{\frac{0}{0}}{\underset{l'H}{=}} z_1^2\lim_{z\to z_1}\frac{1}{4z^3} = \frac{\cancel{z_1^2}}{4z_1^{\cancel{3}}} = \frac{1}{4z_1}$$

$$= \frac{1}{4\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)} = \frac{\overset{1-i}{\cancel{1}}}{2\sqrt{2}(1+i)} = \frac{1-i}{4\sqrt{2}} = \underline{\frac{\sqrt{2}}{8}(1-i)}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \lim_{z\to z_2}(z-z_2)\frac{z^2}{z^4+1} = z_2^2\lim_{z\to z_2}\frac{z-z_2}{z^4+1} = \dots = \frac{1}{4z_2} =$$

$$= \frac{1}{4\cdot\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)} = \frac{\overset{-1-i}{\cancel{1}}}{2\sqrt{2}(-1+i)} = \frac{-1-i}{4\sqrt{2}} = \underline{\frac{-\sqrt{2}}{8}(1+i)}$$

$$I = 2\pi i\left(\cancel{\frac{\sqrt{2}}{8}} - \frac{\sqrt{2}}{8}i - \cancel{\frac{\sqrt{2}}{8}} - \frac{\sqrt{2}}{8}i\right) = \cancel{2}\pi i\cdot\left(-\frac{\cancel{2}\sqrt{2}\,i}{\cancel{8}_{\,2}}\right) = \frac{\sqrt{2}}{2}\pi = \frac{\pi}{\sqrt{2}}$$

<u>Method II</u>

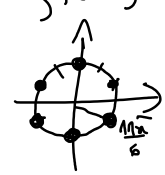
$$I = 2\pi i\sum_{\substack{z=z_k\\k=1,2}}\frac{\overset{z_k^2}{\cancel{z_k}}/z_k}{4z_k^3} = 2\pi i\sum_{z=z_k}\frac{z_k^3}{4\underbrace{(z_k^4)}_{=-1}} = 2\pi i\sum_{\substack{z=z_k\\k=1,2}}\frac{z_k^3}{-4} =$$

$$= \frac{2\pi i}{-4}\sum_{z=z_k} z_k^3 = -\frac{\pi i}{2}\left(\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^3 + \left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^3\right) = \quad = \frac{\pi}{\sqrt{2}}$$

iii) $\displaystyle\int_{-\infty}^{+\infty}\frac{dx}{x^6+1}$

$f(z) = \frac{1}{z^6+1}$

$z^6+1=0 \Rightarrow z^6=-1 \Rightarrow z^6 = \cos\pi + i\sin\pi \Rightarrow z_k = \cos\frac{\pi+2k\pi}{6} + i\sin\frac{\pi+2k\pi}{6},\ k=\overline{0,5}$



$$\left.\begin{aligned} z_0 &= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} \quad ✓ \\ z_1 &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \quad ✓ \\ z_2 &= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2} \quad ✓ \end{aligned}\right\} \in \operatorname{int} C$$

poles of order 1

$$\left.\begin{aligned} z_3 &= \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - i\frac{1}{2} \\ z_4 &= \underbrace{\cos\frac{9\pi}{6}}_{0} + i\underbrace{\sin\frac{9\pi}{6}}_{-1} = -i \\ z_5 &= \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - i\frac{1}{2} \end{aligned}\right\} \text{ are not with pos. im. part}$$

$I = 2\pi i \sum^{2} \frac{z_k/1}{6 z^5} \dots = 2\pi i\sum^{2}\frac{z_k}{-6} = \frac{2\pi i}{-6}\sum^{2} z_k$

$$z_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

$$I = 2\pi i \sum_{k=0}^{2} \frac{z_k/1}{6 z_k^6} = 2\pi i \sum_{k=0}^{2} \frac{z_k}{6 z_k^6} = 2\pi i \sum_{k=0}^{2} \frac{z_k}{-6} = \frac{2\pi i}{-6} \sum_{k=0}^{2} z_k \qquad z_k^6 = -1$$

$$= -\frac{\pi i}{3}\left(z_0 + z_1 + z_2\right) = -\frac{\pi i}{3}\left(\cancel{\frac{\sqrt{3}}{2}} + \frac{i}{2} + i - \cancel{\frac{\sqrt{3}}{2}} + \frac{i}{2}\right) = -\frac{\pi i}{3} \cdot \frac{\cancel{4} i}{\cancel{2}} = \frac{2\pi}{3}$$

III

$$I = \int_{-\infty}^{+\infty} f(x)\cos\alpha x \, dx \quad ; \quad J = \int_{-\infty}^{+\infty} f(x)\sin\alpha x \, dx$$

$$K = I + iJ = \int_{-\infty}^{+\infty} f(x)(\cos\alpha x + i\sin\alpha x)\,dx = \int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx \Rightarrow \text{II}, \ \text{I}$$

1.56

$$I = \int_{-\pi}^{\pi} \frac{\cos 100x}{5 - 4\sin x}\,dx \quad ; \quad J = \int_{-\pi}^{\pi} \frac{\sin 100x}{5 - 4\sin x}\,dx$$

$$K = I + iJ = \int_{-\pi}^{\pi} \frac{\cos 100x + i\sin 100x}{5 - 4\sin x}\,dx = \int_{-\pi}^{\pi} \frac{e^{i100x}}{5 - 4\sin x}\,dx$$

$$e^{ix} = z \Rightarrow dx = \frac{dz}{iz}, \quad \sin x = \frac{z^2 - 1}{2iz}$$

$$K = I + iJ = \int_{|z|=1} \frac{z^{\cancel{100}\,99}}{5 - 4\,\frac{z^2-1}{2iz}} \cdot \frac{dz}{i\cancel{z}} = \int_{C: |z|=1} \frac{z^{99}\,dz}{\cancel{i}\left(\frac{10iz - 4z^2 + 4}{2\cancel{i}z}\right)} = -\int_C \frac{z^{100}\,dz}{\underbrace{2z^2 - 5iz - 2}_{\substack{= 2(z-2i)(z-\frac{i}{2}) \\ = (z-2i)(2z-i)}}}$$

$$2z^2 - 5iz - 2 = 0$$

$$\Delta = 25i^2 + 16 = -9 \Rightarrow z_{1,2} = \frac{5i \pm 3i}{4} \begin{cases} z_1 = 2i \notin \text{int } C \\ z_2 = \frac{i}{2} \in \text{int } C, \text{ pole of order } 1 \end{cases}$$

$$\operatorname*{Res}_{z=\frac{i}{2}} f(z) = \left.\frac{z^{100}}{2(z-2i)}\right|_{z=\frac{i}{2}} = \frac{\left(\frac{i}{2}\right)^{100}}{2\left(\frac{i}{2} - \frac{2}{2}2i\right)} = \frac{1}{2^{100}\cdot(-3i)}$$

$$h(z) = 2z - i, \quad g(z) = \frac{z}{z - 2i}$$

$$K = -2\pi i \operatorname*{Res}_{z=\frac{i}{2}} f(z) = -\cancel{2\pi i}\,\frac{1}{2^{\cancel{100}\,99}(-3\cancel{i})} = \frac{\pi}{2^{99}\cdot 3}$$

$$K = I + iJ = \underbrace{\frac{\pi}{2^{99}\cdot 3}}_{I} + i\cdot\underbrace{0}_{J} \Rightarrow I = \frac{\pi}{2^{99}\cdot 3}$$

Ex 1 $\displaystyle\int^{2\pi} \frac{d\theta}{}$

ex1 $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}$

$$I = \int_{(C)\cdot|z|=1} \frac{\frac{dz}{iz}}{\sqrt{2}-\frac{z^2+1}{2z}} = \int_C \frac{dz}{i\cancel{z}\cdot\frac{2\sqrt{2}z-z^2-1}{2\cancel{z}}} = \int_C \frac{2}{-i(z^2-2\sqrt{2}z+1)}\,dz =$$

$$= -\frac{2}{i}\int_C \frac{dz}{z^2-2\sqrt{2}z+1}$$

$\Delta = 8-4=4 \Rightarrow z_{1,2} = \frac{2\sqrt{2}\pm 2}{2}$ — $z_1=\sqrt{2}+1 \notin \operatorname{int} C$; $z_2=\sqrt{2}-1 \in \operatorname{int} C$, pole of order 1

$$I = -\frac{2}{\cancel{i}}\cdot 2\pi\cancel{i}\operatorname*{Res}_{z=z_2} f(z) \qquad z^2-2\sqrt{2}z+1 = (z-\sqrt{2}-1)\underbrace{(z-\sqrt{2}+1)}_{R}$$

$$\operatorname*{Res}_{z=\sqrt{2}-1} f(z) = \frac{1}{z-\sqrt{2}-1}\Big|_{z=\sqrt{2}-1} = \frac{1}{\cancel{\sqrt{2}}-1-\cancel{\sqrt{2}}-1} = -\frac{1}{2}$$

$$I = -4\pi\left(-\frac{1}{2}\right) = 2\pi$$

1.57 (v) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^4+9x^2+20}\,dx$

$$f(z) = \frac{1}{z^4+9z^2+20}\ ; \quad I = \int_C \frac{1}{z^4+9z^2+20}\,dz = 2\pi i \sum_{\operatorname{Im} z_k>0} \operatorname*{Res}_{z=z_k} f(z)$$

$z^2=t$

$t^2+9z+20=0$

$t^2=-4 \Rightarrow z_{1,2}=\pm 2i \Rightarrow z_1=2i$, $z_3=\sqrt{5}\,i$ poles of order I with pos im-part

$t^2=-5 \qquad z_{3,4}=\pm\sqrt{5}\,i$

$$I = 2\pi i\left(\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=\sqrt{5}i} f(z)\right) = 2\pi i\left(\frac{1}{4i} - \frac{1}{2\sqrt{5}}i\right) = \boxed{\frac{\pi}{2}-\frac{\pi}{\sqrt{5}}}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{1}{4i}$$

$$\operatorname*{Res}_{z=z_3} f(z) = \frac{-1}{2\sqrt{5}}i$$