Seminar 11

Wednesday, December 9, 2020 3:55 PM

Applications to the Laplace transform

Differentiation of the original.

$f(t), f'(t), f''(t), \dots$ are originals

$$\boxed{\mathcal{L}[f(t)](p) = F(p)}$$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{L}[f'(t)](p) &= pF(p) - f(0) \\ \mathcal{L}[f''(t)](p) &= p^2F(p) - pf(0) - f'(0) \end{aligned}}$$

- solving ordinary differential equation
  - we convert ODE to algebraic equation
  - we solve algebraic equations for the unknown function ($F(p)$)
  - we use partial fraction expansion to express the unknown function
  - we use Inverse Laplace Transform to obtain the solution to the original problem

1 Differential equations with constant coefficients

1) $x''(t) - 5x'(t) + 6x(t) = 0 \quad \Big/ \mathcal{L} \Rightarrow \quad x(0) = 1,\ x'(0) = -1$

$$\mathcal{L}[x''(t)](p) - 5\mathcal{L}[x'(t)](p) + 6\mathcal{L}[x(t)](p) = 0 \qquad \mathcal{L}[x(t)](p) = X(p)$$

$$\mathcal{L}[x'(t)](p) = pX(p) - x(0) = pX(p) - 1$$

$$\mathcal{L}[x''(t)](p) = p^2X(p) - p\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=-1} = p^2X(p) - p + 1$$

$$p^2X(p) - p + 1 - 5\left(pX(p) - 1\right) + 6X(p) = 0$$

$$p^2X(p) - p + 1 - 5pX(p) + 5 + 6X(p) = 0$$

$$X(p)\cdot\left(p^2 - 5p + 6\right) = p - 6 \Rightarrow X(p) = \frac{p-6}{p^2-5p+6} \Rightarrow X(p) = \frac{p-6}{(p-2)(p-3)}$$

$$X(p) = \frac{p-6}{(p-2)(p-3)} = \frac{\overset{p-3}{A}}{p-2} + \frac{\overset{p-2}{B}}{p-3} \Rightarrow \quad p - 6 = A(p-3) + B(p-2)$$

$$p = 3 \Rightarrow \boxed{B = -3}$$

$$p = 2 \Rightarrow -A = -4 \Rightarrow \boxed{A = 4}$$

$$\Rightarrow X(p) = \frac{4}{p-2} - \frac{3}{p-3} \quad \Big/ \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{4}{p-2}\right] - \mathcal{L}^{-1}\left[\frac{3}{p-3}\right] \Rightarrow x(t) = 4e^{2t} - 3e^{3t}$$

2) $\dots$ $x(0) = 0,\ x'(0) = -1$

2) $x''(t) + x(t) = 2\cos t \,/\, \mathcal{L}$, $x(0) = 0$, $x'(0) = -1$

$\mathcal{L}[x''(t)](p) + \mathcal{L}[x(t)](p) = \mathcal{L}[2\cos t](p)$

$\mathcal{L}[x(t)](p) = X(p)$

$$p^2X(p) - p\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=-1} + X(p) = 2\frac{p}{p^2+1}$$

$$X(p)\cdot(p^2+1) = \frac{2p}{p^2+1} - 1 \quad \Big| : (p^2+1) \Rightarrow X(p) = \frac{2p}{(p^2+1)^2} - \frac{1}{p^2+1} \,/\, \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{2p}{(p^2+1)^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right]$$

$$\frac{2p}{(p^2+1)^2} = \left(\frac{-1}{p^2+1}\right)' = (-1)\left(\frac{1}{p^2+1}\right)' = \mathcal{L}[t\sin t](p)$$

$$\Rightarrow x(t) = t\cdot\sin t - \sin t$$

Formula:

$$\boxed{F(p) = \frac{p}{(p^2+1)^2} \qquad \mathcal{L}^{-1}\left[\frac{p}{(p^2+1)^2}\right] = \frac{1}{2}t\sin t}$$

Solution: $\mathcal{L}^{-1}\left[\frac{p}{(p^2+1)^2}\right] = \mathcal{L}^{-1}\left[-\frac{1}{2}\left(\frac{1}{p^2+1}\right)'\right] = \frac{1}{2}\mathcal{L}^{-1}\left[(-1)\cdot\underbrace{\left(\frac{1}{p^2+1}\right)'}_{= -\frac{2p}{(p^2+1)^2}}\right] =$

$$= \frac{1}{2}t\sin t$$

② System of differential equations with constant coefficients

3) $\begin{cases} x'(t) = 3x(t) - y(t) & /\mathcal{L} \\ y'(t) = -9x(t) + 3y(t) & /\mathcal{L} \end{cases}$ $\quad x(0) = 1$, $y(0) = 0$

$$\begin{cases} \mathcal{L}[x'(t)](p) = 3\mathcal{L}[x(t)](p) - \mathcal{L}[y(t)](p) \\ \mathcal{L}[y'(t)](p) = -9\mathcal{L}[x(t)](p) + 3\mathcal{L}[y(t)](p) \end{cases}$$

$\mathcal{L}[x(t)](p) = X(p)$

$\mathcal{L}[y(t)](p) = Y(p)$

$$\begin{cases} \ldots \\ \mathcal{L}[y'(t)](p) = -9\mathcal{L}[x(t)](p) + 3\mathcal{L}[y(t)](p) \end{cases} \qquad \mathcal{L}[y(t)](p) = Y(p)$$

$$\Rightarrow \begin{cases} pX(p) - \underbrace{x(0)}_{=1} = 3X(p) - Y(p) \\ pY(p) - \underbrace{y(0)}_{=0} = -9X(p) + 3Y(p) \end{cases} \Rightarrow \begin{cases} X(p)(p-3) + Y(p) = 1 \\ 9X(p) + Y(p)\cdot(p-3) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow X(p) = -\frac{Y(p)(p-3)}{9}$$

$$-\frac{Y(p)(p-3)}{9}\cdot(p-3) + Y(p) = 1 \Rightarrow Y(p)\left(-(p-3)^2 + 9\right) = 9 \Rightarrow Y(p) = \frac{9}{-p^2+6p-9+9}$$

$$\Rightarrow \boxed{Y(p) = \frac{-9}{p^2-6p}}$$

$$X(p) = \frac{\cancel{9}}{p^2-6p}\cdot\frac{p-3}{\cancel{9}} \Rightarrow \boxed{X(p) = \frac{p-3}{p(p-6)}}$$

$$X(p) = \frac{p-3}{p\cdot(p-6)} = \frac{A}{p} + \frac{B}{p-6} \Rightarrow p-3 = A(p-6) + Bp$$

$$p = 0 \Rightarrow -6A = -3 \Rightarrow A = \frac{1}{2}$$

$$p = 6 \Rightarrow 6B = 3 \Rightarrow B = \frac{1}{2}$$

$$X(p) = \frac{1}{2}p + \frac{1}{2}\cdot\frac{1}{p-6} \quad \Big|\, \mathcal{L}^{-1}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{2} + \frac{1}{2}e^{6t}}$$

$$Y(p) = \frac{-9}{p(p-6)} = -9\,\frac{1}{p(p-6)} = \frac{-9}{-6}\cdot\frac{p-6-p}{p(p-6)} = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{p-6}\right)$$

$$\Rightarrow \boxed{y(t) = \frac{3}{2}\left(1 - e^{6t}\right)}$$

3 Integral equations

Convolution of originals

$f(t), g(t)$ originals $\longrightarrow$ Laplace images $F(p), G(p)$

The function $\int_0^t f(\tau)g(t-\tau)\,d\tau$ is called the convolution of the functions $f(t)$ and $g(t)$

|| not

$f * g$

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right] = F(p)\cdot G(p) = \mathcal{L}[f(t)](p)\cdot\mathcal{L}[g(t)](p)$$

$\mathcal{L}[\int_0 \ldots$

4) $y(t) - 2\int_0^t y(t-u)\sin u\,du = \cos t \;/\mathcal{L}$

$$\mathcal{L}[y(t)](p) - 2\mathcal{L}\Big[\underbrace{\int_0^t y(t-u)\sin u\,du}_{y(t)*\sin t}\Big](p) = \mathcal{L}[\cos t](p)$$

$\mathcal{L}[y(t)](p) = Y(p)$

$$\Rightarrow Y(p) - 2\mathcal{L}[y(t)](p)\cdot\mathcal{L}[\sin t](p) = \frac{p}{p^2+1}$$

$$^{p^2+1}\big/\; Y(p) - 2Y(p)\cdot\frac{1}{p^2+1} = \frac{p}{p^2+1} \quad \Rightarrow Y(p)(p^2+1-2) = p$$

$$\Rightarrow Y(p) = \frac{p}{p^2-1} \;\Big/\mathcal{L}^{-1} \quad \Rightarrow \boxed{y(t) = \operatorname{ch} t}$$

or

$$Y(p) = \frac{p}{(p-1)(p+1)} = \frac{1}{2}\left[\frac{1}{p-1} + \frac{1}{p+1}\right] \;\Big/\mathcal{L}^{-1}$$

$p + \cancel{1} + p - \cancel{1}$

$$\Rightarrow \boxed{y(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}}$$

4 Integral-differential equations

5) $y'(t) + \int_0^t u\cdot y(t-u)\,du = t \;/\mathcal{L}$, $y(0) = -1$

$$\mathcal{L}[y'(t)](p) + \mathcal{L}\Big[\underbrace{\int_0^t u\cdot y(t-u)\,du}_{t*y(t)}\Big](p) = \mathcal{L}[t](p)$$

$Y(p) = ?$

$$pY(p) - \underbrace{y(0)}_{-1} + \mathcal{L}[t](p)\cdot\mathcal{L}[y(t)](p) = \frac{1}{p^2}$$

$$\underline{pY(p)} + 1 + \underline{\frac{1}{p^2}Y(p)} = \frac{1}{p^2} \quad \Rightarrow Y(p)\left(p + \frac{1}{p^2}\right) = \frac{1}{p^2} - 1 \quad \Rightarrow Y(p)(p^3+1) = 1 - p^2$$

$$\Rightarrow Y(p) = \frac{1-p^2}{p^3+1} \quad \Rightarrow Y(p) = \frac{(1-p)\cancel{(1+p)}}{\cancel{(p+1)}(p^2-p+1)}$$

$$\Rightarrow Y(p) = \frac{1-p}{p^2-p+1}$$

$y(t) = ?$

$$Y(p) = \frac{1}{p^2-p+1} - \frac{p}{p^2-p+1} = \frac{1}{p^2-2p\frac{1}{2}+\frac{1}{4}+\frac{4}{1}-\frac{1}{4}} - \frac{p}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{p-\frac{1}{2}+\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} \cdot \frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \quad / \mathcal{L}^{-1}$$

$$\Rightarrow y(t) = \frac{1}{\sqrt{3}} e^{\frac{t}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right) - e^{\frac{t}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t$$

c) $\int_0^t \sin(t-z)\, x(z)\, dz = \sin^2 t \quad , t \geq 0$

$$\mathcal{L}[\sin^2 t](p)$$

$$\boxed{\sin^2 t = \frac{1-\cos 2t}{2}}$$

$X(p) = ?$

$$\mathcal{L}\Big[\underbrace{\int_0^t \sin(t-z)\, x(z)\, dz}_{\sin t * x(t)}\Big](p) = \mathcal{L}[\sin^2 t](p)$$

$$\frac{1}{p^2+1} \cdot X(p) = \mathcal{L}\left[\frac{1-\cos 2t}{2}\right](p)$$

$$\frac{1}{p^2+1} X(p) = \frac{1}{2p} - \frac{1}{2} \cdot \frac{p}{p^2+4} \Rightarrow \frac{1}{p^2+1} X(p) = \frac{\cancel{p^2}+4-\cancel{p^2}}{2p(p^2+4)} \quad / (p^2+1)$$

$$\Rightarrow X(p) = \frac{2(p^2+1)}{p(p^2+4)}$$

$$\frac{2(p^2+1)}{p(p^2+4)} = \frac{A}{p} + \frac{Bp+C}{p^2+4} \Rightarrow 2p^2+2 = Ap^2+4A+Bp^2+Cp$$

$$\Rightarrow \begin{cases} A+B=2 \\ \boxed{C=0} \\ 4A=2 \end{cases} \quad \begin{array}{l} \Rightarrow B = 2 - \frac{1}{2} \Rightarrow \boxed{B = \frac{3}{2}} \\ \\ \Rightarrow \boxed{A = \frac{1}{2}} \end{array}$$

$$\Rightarrow X(p) = \frac{1}{2p} + \frac{\frac{3}{2}p}{p^2+4} \quad / \mathcal{L}^{-1}$$

$$\Rightarrow X(p) = \frac{1}{2p} + \frac{\frac{3}{2}p}{p^2+4} \quad / \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = \frac{1}{2} + \frac{3}{2}\cos 2t$$

7) $\begin{cases} x''(t) + 3x'(t) + 2x(t) = e^{-t} \\ x(0) = x'(0) = 0 \end{cases} \quad / \mathcal{L}$

$$\mathcal{L}[x(t)](p) = X(p)$$

$$p^2 X(p) - p\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + 3pX(p) - 3\underbrace{x(0)}_{0} + 2X(p) = \frac{1}{p+1}$$

$$X(p)\underbrace{(p^2+3p+2)}_{(p+1)(p+2)} = \frac{1}{p+1} \Rightarrow X(p) = \frac{1}{(p+1)^2(p+2)}$$

$$\frac{1}{(p+1)^2(p+2)} = \frac{\overset{(p+1)^2}{A}}{p+2} + \frac{\overset{p^2+3p+2}{B}}{p+1} + \frac{\overset{p+2}{C}}{(p+1)^2} \Rightarrow 1 = p^2(A+B) + p(2A+3B+C) + A+2B+2C$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 2A+3B+C = 0 \\ A+2B+2C = 1 \end{cases} \Rightarrow \boxed{B = -1} \Rightarrow \begin{cases} 2A-3A+C = 0 \\ A-2A+2C = 1 \end{cases} \Rightarrow \begin{cases} -A+C = 0 \quad / \cdot(-1) \\ -A+2C = 1 \end{cases}$$

$$\boxed{C = 1} \quad \boxed{A = 1}$$

$$X(p) = \frac{1}{p+2} - \frac{1}{p+1} + \frac{1}{(p+1)^2}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{p+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{p+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(p+1)^2}\right] = e^{-2t} - e^{-t} + te^{-t}$$

Formula:

$$\boxed{\begin{aligned} F(p) &= \frac{1}{(p+1)^2} \\ \mathcal{L}^{-1}\left[\frac{1}{(p+1)^2}\right] &= te^{-t} \end{aligned}}$$

$$F(p) = \frac{1}{(p+1)^2} = \left(-\frac{1}{p+1}\right)' = (-1)\left(\frac{1}{p+1}\right)' = (-1)\left(\mathcal{L}[e^{-t}](p)\right)' = \mathcal{L}[te^{-t}](p)$$