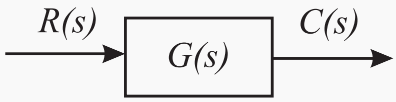
Systems Theory Laboratory Assignment 5. Solution to selected problems



For a system described by the transfer function $G(s)$, with a sinusoidal input: $r(t) = A \sin \omega t$ the output signal **at steady state** is:

$$c_{ss}(t) = A \underbrace{|G(j\omega)|}_{M} \sin(\omega t + \underbrace{\angle G(j\omega)}_{\varphi})$$

- $M > 1$: the magnitude of the output > the magnitude of the input
- $M < 1$: the magnitude of the output < the magnitude of the input
- $\varphi > 0$: phase lead
- $\varphi < 0$: phase lag

Exercise 1. Consider system with the following transfer functions:

$$G_4(s) = \frac{s^2 + s + 1}{s^2 + s + 10}$$

Solution

1. Plot the system response for a sinusoidal input, $r(t) = sin(t)$, using the Matlab function *lsim* for a time interval $t \in [0,\ 30]$ sec.
   The Matlab function *lsim* simulates and plots the time response of a system to a given input signal. The general form is:

   `lsim(SYS,U,T)`

   The function will plot the time response of the dynamic system SYS to the input signal given in the vector U, for a time interval T. A sample code for the transfer function G_4 is given below.

Listing 1: sin_plots.m

```
close all
clear all
clc
% plot the system response to a sin input
t = 0:0.01:30;                          % create the time vector
input = sin(t);                         % create the sin input

G4 = tf([1 1 1], [1 1 10]);             % create the transfer function G4
lsim(G4,input,t), grid on               % simulate the response of G4 to a sin input
```

2. For each system, analyze the magnitude and phase angle of the output signal and compare it with the input signal. Determine if the systems have phase lead or phase lag.

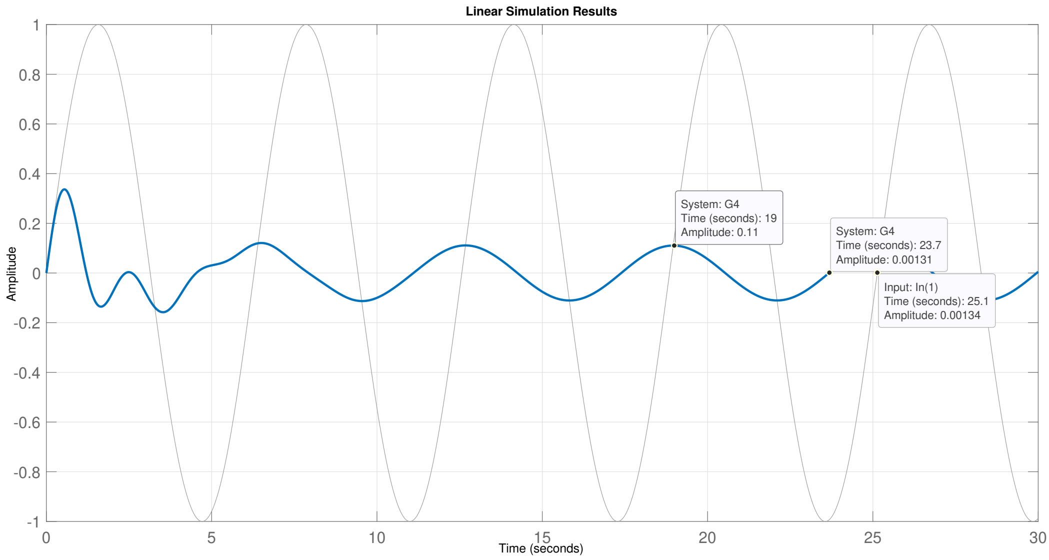


Figure 1: Response to a sin input for the system G_4 (blue line). Input sin(t) (grey line)

- Compute the magnitude $M = |G(j\omega)|$ and $\varphi = \angle G(j\omega)$, where ω is the frequency of the input.
  A sample code to compute the magnitude and the phase angle of the sinusoidal transfer function $G(j\omega)$, where $\omega = 1rad/s$ is given below.

  Listing 2: mag_phase.m

```
w = 1;                               % the frequency of the input (rad/s)
s = j*w;                             % replace s by jw
M = abs((s^2+s+1)/(s^2+s+10))        % compute the magnitude of G(jw)
phi = angle((s^2+s+1)/(s^2+s+10))    % compute the phase angle of G(jw)
```

  The results for G_4 should be: $M = 0.1104$ and $\varphi = 1.4601$.
  If the input is $r(t) = sin(t)$, the magnitude of the input is $A = 1$ and the frequency $\omega = 1rad/s$.
  The magnitude of the output is $B = A \cdot M = 1 \cdot 0.11 = 0.11$.
- Determine the magnitude and the phase angle from the simulation plot. See Figure 1 where the magnitude of the output is the same as computed before. The phase shift between the input and output is positive (phase lead). In the plot it can be determined from the difference $\varphi = 25.1 - 23.7 = 1.4$ (the time when the input/output waves are zero).

3. Draw the Bode diagrams, using the Matlab function *bode* and read from the plots the magnitude and the phase angle for each output signal, when the input is $r(t) = sin(t)$.
   A Bode plot (and grid) can be obtained with:

   ```
   bode(G4), grid on
   ```

   See Figure 2 for the plot.
   If the input is $r(t) = sin(t)$, the magnitude of the input is $A = 1$ and the frequency $\omega = 1rad/s$.
   From the magnitude plot, for $\omega = 1rad/s$ read M^{dB} and from the phase plot read φ.

$$M^{dB} = -19 \quad \Rightarrow 20\log_{10} M = -19 \quad \Rightarrow \quad \log_{10} M = -\frac{19}{20} \quad \Rightarrow \quad M = 10^{-19/20} = 0.11$$

$$\varphi^{deg} = 83.9 \quad \Rightarrow \quad \varphi^{rad} = 83.9 \cdot \frac{\pi}{180} = 1.46\,rad.$$

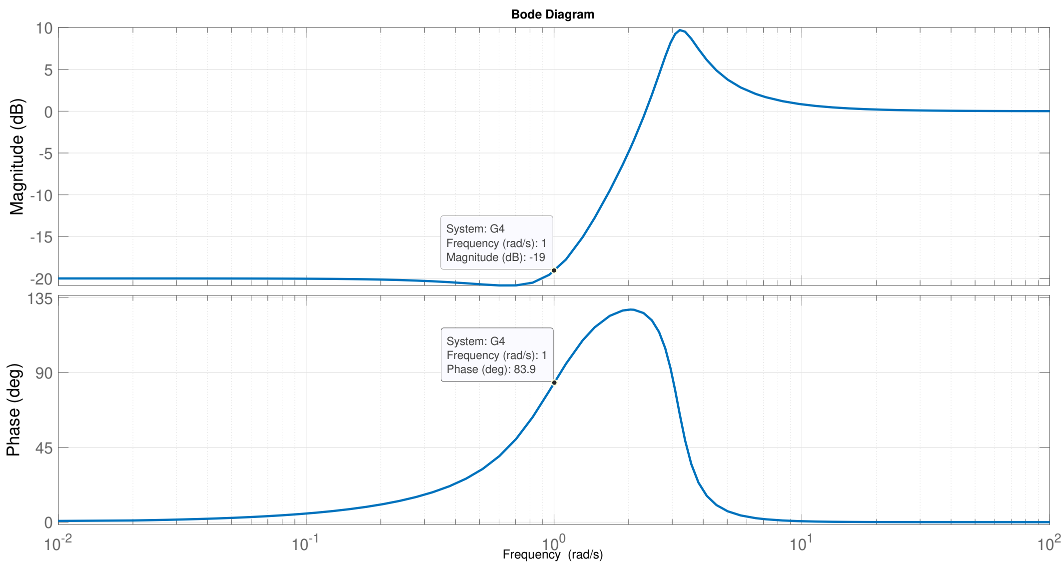


Figure 2: Bode plot for G_4

Exercise 2. 1. Sketch the Bode diagram for the systems with the following transfer functions:

$$G_1(s) = \frac{s^2}{(10s+1)^2}$$

2. Determine the frequencies for which the system amplifies or attenuates the sinusoidal input signals.

3. For each system, use the Bode diagram to determine the magnitude of the output signal if the input is:

$$u_1(t) = sin(t), \qquad u_2(t) = 0.1sin(10^{-3}t), \qquad u_3(t) = 3sin(100t).$$

Solution

1. Sketch the Bode plot for

$$G_1(s) = \frac{s^2}{(10s+1)^2} = s^2 \cdot \frac{1}{10s+1} \cdot \frac{1}{10s+1}$$

- $G_{01}(s) = s^2 = \dfrac{1}{s^{-2}} = \dfrac{k}{s^n} \quad \Rightarrow \quad k = 1, \;\; n = -2$
  - The magnitude plot is a straight line:
    - it has a slope of $-20 \cdot n = -20 \cdot (-2) = 40$ dB/dec
    - for $\omega = 1 = 10^0$ it is equal to $M^{dB}|_{\omega=1} = k^{dB} = 20\log_{10}(1) = 0$ dB.
  - The phase plot is a constant line: $\phi_1 = -90 \cdot n = -90 \cdot (-2) = 180^o$.
  - See Figure 3.
- $G_{02}(s) = G_{03}(s) = \dfrac{1}{10s+1} = \dfrac{1}{T_1 s + 1} \quad \Rightarrow \quad T_1 = 10$
  - The magnitude plot:
    - low frequency asymptote 0 dB
    - high frequency asymptote with a slope of -20 dB/dec.
    - corner frequency $\omega_{c1} = \dfrac{1}{T_1} = \dfrac{1}{10} = 10^{-1}$ rad/s.
  - The phase plot is an arctangent, $\phi \in (0, \; -90^o)$, with an inflection $(\omega_c = 10^{-1}, \; -45^o)$
  - See Figure 4.

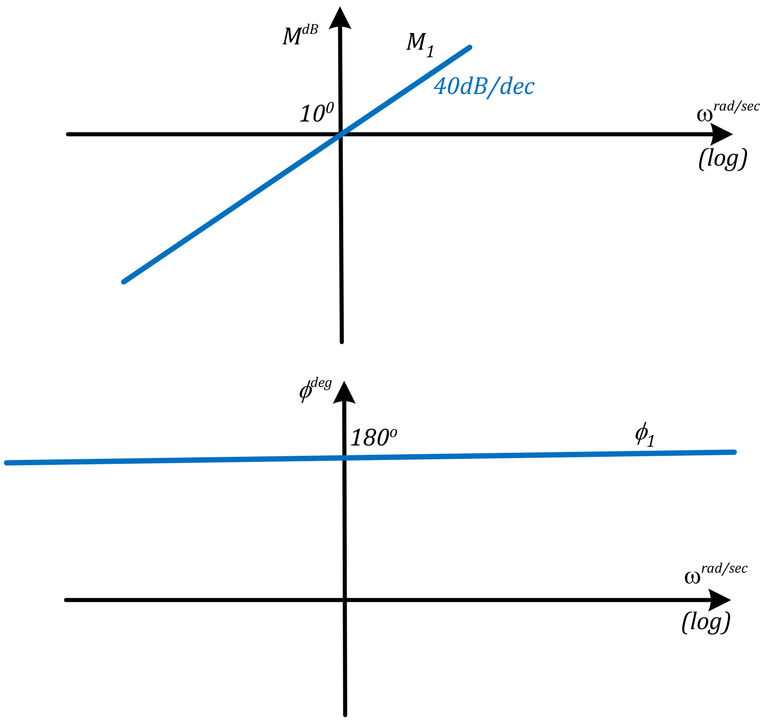


Figure 3: Bode plot G_{01}

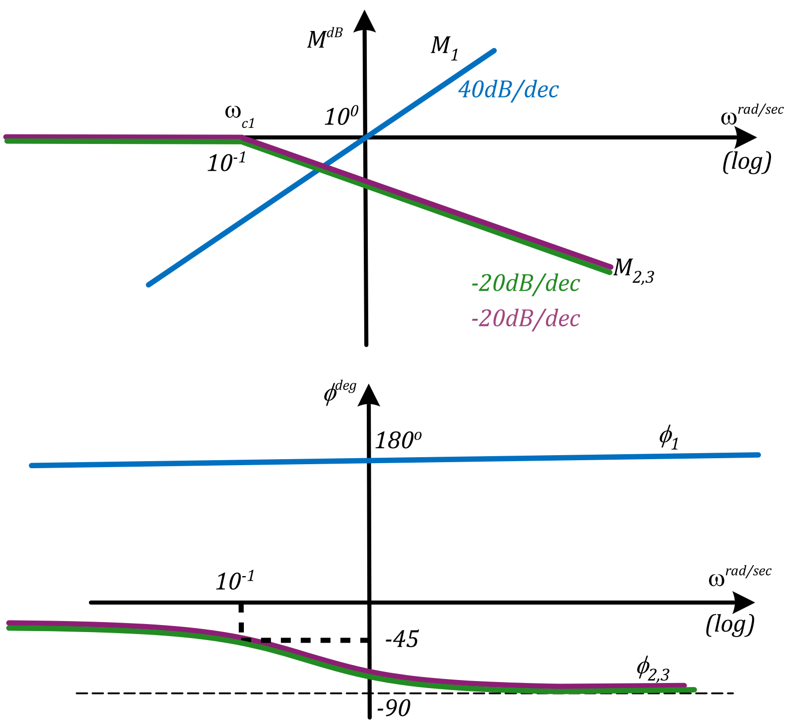


Figure 4: Bode plot G_{01}, G_{02}, G_{03}

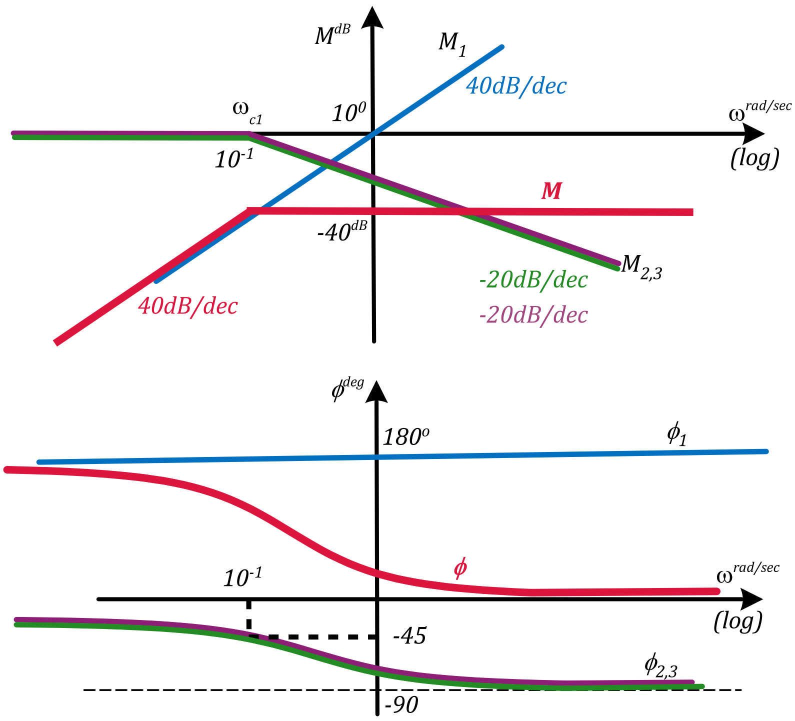


Figure 5: The overall Bode plot $G_1(s)$

The overall plot is given in Figure 5.

2. Determine the frequencies for which the system amplifies or attenuates the sinusoidal input signals.
   In Figure 5, notice that for all frequencies $\omega \in (0,\ \infty)$, the magnitude M^{dB} is negative (red plot).

$$M^{dB} < 0 \quad \Rightarrow \quad 20\log_{10} M < 0 \quad \Rightarrow \quad \log_{10} M < 0 \quad \Rightarrow \quad M < 1$$

   All sinusoidal inputs will be attenuated because the magnitude of the output is smaller than the input: $A \cdot M < A$.

3. For each system, use the Bode diagram (See Figure 6) to determine the magnitude of the output signal if the input is:
   - $u_1(t) = sin(t)$: $A = 1$, $\omega = 1$ rad/s.
     From the magnitude plot read $M^{dB}|_{\omega=10^0} = -40$ dB.

$$20\log_{10} M = -40 \quad \Rightarrow \quad \log_{10} M = -\frac{40}{20} = -2 \quad \Rightarrow \quad M = 10^{-2}$$

     and the magnitude of the output is: $A \cdot M = 1 \cdot 10^{-2} = 10^{-2}$.
   - $u_2(t) = 0.1sin(10^{-3}t)$: $A = 0.1$, $\omega = 10^{-3}$ rad/s.
     From the magnitude plot read $M^{dB}|_{\omega=10^{-3}} = -120$ dB.

$$20\log_{10} M = -120 \quad \Rightarrow \quad \log_{10} M = -\frac{120}{20} = -6 \quad \Rightarrow \quad M = 10^{-6}$$

     and the magnitude of the output is: $A \cdot M = 0.1 \cdot 10^{-6} = 10^{-7}$.
   - $u_3(t) = 3sin(100t)$: $A = 3$, $\omega = 10^2$ rad/s.
     From the magnitude plot read $M^{dB}|_{\omega=10^2} = -40$ dB.

$$20\log_{10} M = -40 \quad \Rightarrow \quad \log_{10} M = -\frac{40}{20} = -2 \quad \Rightarrow \quad M = 10^{-2}$$

     and the magnitude of the output is: $A \cdot M = 3 \cdot 10^{-2} = 0.003$.

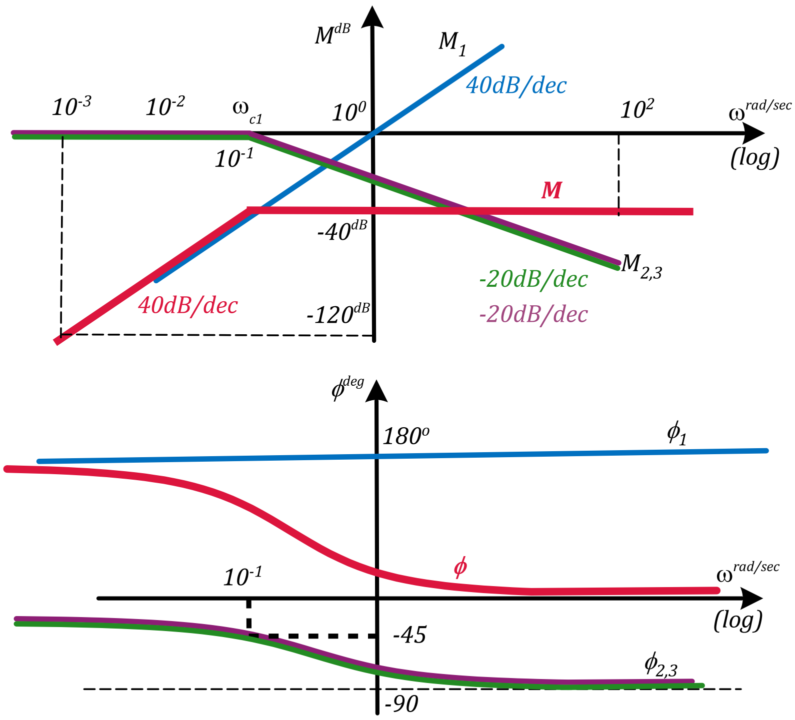


Figure 6: Bode plot