Systems Theory Laboratory Assignment 2: Transfer functions. State space models. System response

Objective: At the end of this lab the students must be able to:

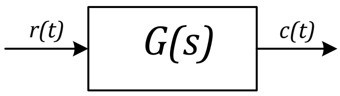
- Obtain the transfer function model for a linear system
- Obtain a state-space model for a linear system
- Compute and plot the step response and the impulse response of a linear system.

1 Solved exercises

Exercise 1.1 *Transfer function from input and output signals*

An input

$$r(t) = t$$



is applied to a black box system with a transfer function $G(s)$. The resulting output, when the initial conditions are zero, is:

$$c(t) = 1 - e^{-t},\ t \geq 0$$

Determine $G(s)$ for this system.

Solution.

Use Table 1 to compute the Laplace transform for the input and output signals ($R(s)$ and $C(s)$), then determine the transfer function $G(s) = \dfrac{C(s)}{R(s)}$.

$f(t)$	$\mathcal{L}\{f(t)\}$	$f(t)$	$\mathcal{L}\{f(t)\}$
$\delta(t)$	1	$\sin at$	$\dfrac{a}{s^2+a^2}$
1	$\dfrac{1}{s}$	$\cos at$	$\dfrac{s}{s^2+a^2}$
t	$\dfrac{1}{s^2}$	$e^{-at}\sin bt$	$\dfrac{b}{(s+a)^2+b^2}$
e^{-at}	$\dfrac{1}{s+a}$	$e^{-at}\cos bt$	$\dfrac{s+a}{(s+a)^2+b^2}$

Table 1: Table of Laplace transforms

$$r(t) = t \quad \Rightarrow \quad R(s) = \frac{1}{s^2}$$

$$c(t) = 1 - e^{-t} \quad \Rightarrow \quad C(s) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{\dfrac{1}{s(s+1)}}{\dfrac{1}{s^2}} = \frac{s}{s+1}$$

Exercise 1.2 *Transfer function from a differential equation. Impulse response and step response.*

A system having the input $r(t)$ and the output $y(t)$ is described by the following linear differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = r(t) \tag{1}$$

1. Determine the transfer function, $H(s)$.
2. Determine the impulse response for this system, $y_i(t)$.
3. Determine the step response for this system, $y_s(t)$
4. Use the Matlab functions *impulse* and *step* to plot the impulse and step response for this system. Compare the results to the plot of $y_i(t)$ and $y_s(t)$, over a time interval $t \in [0, 10]$.

Solution

1. Use the following property of the Laplace transform: $\mathcal{L}\{\dfrac{d^n f(t)}{dt^n}\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(n-k)}(0)$.

   Apply the Laplace transform to differential equation (1) when the initial conditions are zero, and obtain:

   $$s^2Y(s) + 3sY(s) + 2 = R(s) \quad \Rightarrow \quad (s^2+3s+2)Y(s) = R(s)$$

   and the transfer function is:

   $$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{s^2+3s+2}$$

2. For an ideal impulse input $r(t) = \delta(t)$, the output signal is $Y_i(s) = H(s)R(s)$ and $y_i(t) = \mathcal{L}^{-1}\{H(s)R(s)\}$, where $R(s) = \mathcal{L}\{\delta(t)\} = 1$:

   $$y_i(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+3s+2}\cdot 1\} = \mathcal{L}^{-1}\{\frac{1}{(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{1}{s+1} - \frac{1}{s+2}\} = e^{-t} - e^{-2t}$$

3. For a unit step input $r(t) = 1$, the output signal is $Y_s(s) = H(s)R(s)$ and $y_s(t) = \mathcal{L}^{-1}\{H(s)R(s)\}$, where $R(s) = \mathcal{L}\{1\} = \frac{1}{s}$:

   $$y_s(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+3s+2}\cdot\frac{1}{s}\} = \mathcal{L}^{-1}\{\frac{1}{s(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

4. Impulse response and step response using Matlab:
   - Create the system transfer function in Matlab, using the function *tf*, with the general form:

     ```
     sys = tf(numerator,denominator)
     ```

     where `numerator` and `denominator` are the polynomials from the numerator and the denominator of the transfer function. Introduce the polynomials as row vectors containing the coefficients ordered by descending powers of s, between square brackets. `sys` is the name of the system and can be chosen by the user.
     A system with the transfer function:

     $$H(s) = \frac{1}{s^2+3s+2}$$

     is created as:

     ```
     sys = tf(1, [1 3 2])
     ```

   - Plot the system response for an ideal impulse input $r(t) = \delta(t)$ using the Matlab function *impulse*. The function can be used as follows:

     ```
     impulse(sys)
     ```

     The function computes and plots the response of the system `sys`, if the input is an ideal impulse signal: $y(t) = \mathcal{L}^{-1}\{H(s)\cdot R(s)\}$ where $R(s) = \mathcal{L}\{\delta(t)\} = 1$. The function *impulse* will compute:

     $$y(t) = \mathcal{L}^{-1}\{H(s)\cdot 1\}$$

     and will plot the result.

   - Plot the unit step response of the system. The function is:

     ```
     step(sys)
     ```

For a unit step input $r(t) = 1$ with $R(s) = \frac{1}{s}$, the function *step* will compute:

$$y(t) = \mathcal{L}^{-1}\{H(s) \cdot \frac{1}{s}\}$$

and will plot the result.

- To simulate the system impulse or step response from $t = 0$ to the final time t=10, write:

```
impulse(sys, 10)
```

or

```
step(sys, 10)
```

- Compare the results of the simulation with the plots of $y_i(t)$ and $y_s(t)$ obtained at step 2 and 3, over a time interval $t \in [0,\ 10]$. The plot should be similar to Figure 1.

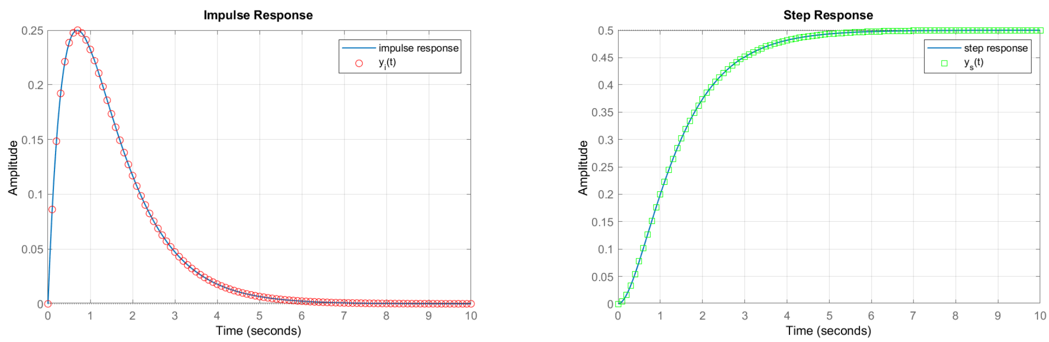


Figure 1: Impulse and step response

Exercise 1.3 *The state-space model from a differential equation. Impulse response and step response.*

A system having the input $r(t)$ and the output $y(t)$ is described by the following linear differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = r(t)$$

or

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = r(t) \tag{2}$$

1. Determine a state space model for this system.
2. Use the Matlab functions *impulse* and *step* to plot the impulse and step response for this system.

Solution

1. **The state space model**
   - Transform the linear second-order differential equation (2) into a system of two first-order differential equations. Denote:

$$\begin{aligned} x_1(t) &= y(t) \quad (3) \\ x_2(t) &= \dot{y}(t) = \dot{x}_1(t) \quad (4) \end{aligned}$$

   By differentiating (4) with respect to time we obtain:

$$\dot{x}_2(t) = \ddot{y}(t) \tag{5}$$

   - Replace the relations (3), (4) and (5) into (2) and obtain the system of first-order diferential equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -2x_1(t) - 3x_2(t) + u(t) \end{cases} \tag{6}$$

   The output equation is: $y(t) = x_1(t)$. The input is denoted: $u(t) = r(t)$.

- Write the state-space model in the standard form and compute the matrices **A**, **B**, **C** and **D**:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned}$$

  or:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}u(t)\end{aligned}$$

2. **System response in Matlab using a state-space model**
   - Create the model in Matlab using the function *ss*:

   ```
   A = [0 1; -2 -3];
   B = [0; 1];
   C = [1 0];
   D = 0;
   sys=ss(A,B,C,D);
   ```

   - Plot the impulse response and the step response:

   ```
   figure, step(sys)
   figure, impulse(sys)
   ```

   The plots should be similar to Figure 1.

2 Proposed exercises

Exercise 2.1 ***Transfer function from input and output signals***

An input $r(t) = e^{-t}$ is applied to a black box system with a transfer function $G(s)$. The resulting output, when the initial conditions are zero, is:

$$c(t) = 2 - 3e^{-t} + 3e^{-2t}cos2t,\ t \geq 0$$

Determine $G(s)$ for this system.

Exercise 2.2 ***Transfer function from a differential equation. Impulse response and step response.***

A system having the input $r(t)$ and the output $y(t)$ is described by the following linear differential equation:

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = r(t)$$

1. Determine the transfer function, $H(s)$.
2. Determine the poles and the zeros.
3. Determine (on paper) the impulse response of the system, $r(t) = \delta(t)$.
4. Determine (on paper) the step response of the system, $r(t) = 1$.
5. Use the Matlab functions *impulse* and *step* to plot the impulse and step response for this system. Compare the results to the plot of $y_i(t)$ and $y_s(t)$, over a time interval $t \in [0, 10]$.

Exercise 2.3 ***System response and the location of the poles***

Consider the following transfer functions:

$$G_1(s) = \frac{1}{s+1},\quad G_2(s) = \frac{1}{s-1},\quad G_3(s) = \frac{1}{s^2-1},\quad G_4(s) = \frac{1}{s^2+1}$$

$$G_5(s) = \frac{1}{s^2+2s+17},\quad G_6(s) = \frac{1}{s^2-2s+17}$$

- Plot the impulse response and the step response for each system over a time interval $t \in [0, 10]$.

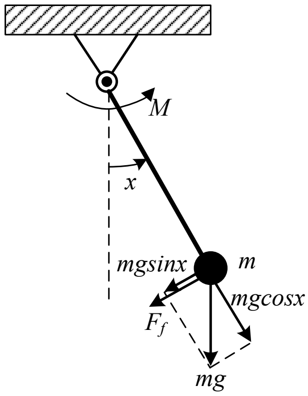


Figure 2: Pendulum

- Determine the poles of each transfer function and comment on the form of the response and the location of the poles (use Table 1).

Exercise 2.4 ***A pendulum model***

Consider a mechanical pendulum as shown in Figure 2.

The pendulum consists of a heavy, small-diameter ball with mass m suspended on a rigid and very light rod of length l. The rod can rotate around the horizontal axis. It follows that the ball can move along a circle in a vertical plane, and its position is determined by a single coordinate, for instance, by the angular displacement denoted as x in Figure 2. The motion of the ball is ruled by the gravity force mg, the damping friction force F_f, and the moment of external periodic forces applied to the axis of rotation, $M(t)$.

The pendulum motion in the vertical plane is governed by the differential equation:

$$ml^2\ddot{x}(t) = M(t) - mgl\sin x(t) - bl\dot{x}(t) \tag{7}$$

Notice that $\sin x \approx x$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and the equation can be written as:

$$ml^2\ddot{x}(t) = M(t) - mglx(t) - bl\dot{x}(t) \tag{8}$$

where

- $x(t)$ is the angle position of the pendulum
- $M(t)$ is the moment of force (torque) at the pivot point
- m is the mass of the ball, $m = 0.5$ kg
- l is the length of the rod, $l = 1$ m
- g is the acceleration of gravity, $g = 9.8\ m/s^2$
- b is the viscous friction coefficient, $b = 0.5$

Consider the pendulum as a system with the input $M(t)$ and the output $x(t)$ and solve the following problems:

1. **The transfer function**. Consider that the pendulum dynamical system has the input signal $M(t)$ and the output $x(t)$, with the Laplace transforms $M(s)$ and $X(s)$, respectively, as shown in Figure 3. The initial conditions are zero, i.e. at the initial time the pendulum is in the equilibrium position with the velocity equal to zero.

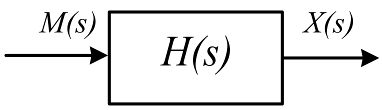


Figure 3: Dynamical system

From the linear differential equation (8) obtain the transfer function from the input $M(s)$ to the output $X(s)$. Apply the Laplace transform to equation (8), taking all the initial conditions equal to zero, then determine the transfer function:

$$H(s) = \frac{X(s)}{M(s)} = ...$$

2. **The state-space model.** From the linearized equation (8), determine a state-space model if:
   - the state variables are: $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$
   - the input signal is: $u(t) = M(t)$ and
   - the output signal is: $y(t) = x_1(t)$.

   Write the state-space model in the standard form and compute the matrices **A**, **B**, **C** and **D**:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned}$$

   or:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}d_1\end{bmatrix}u(t)\end{aligned}$$

3. **The system response using the transfer function**. Obtain a graphical representation of the system response (output), using Matlab and the transfer function:
   - Create the pendulum transfer function in Matlab, using the function *tf*
   - Plot the system response for an (ideal) impulse input $M(t) = \delta(t)$ with the Matlab function *impulse*.
   - Plot the pendulum response for a unit step input $M(t) = 1$ with the Matlab function *step*.
   - Analyze and comment the results.
4. **The system response using the state-space model**
   - Create the state-space model in Matlab with the function *ss*.
   - Plot the system response for a step input and for an impulse input.

Exercise 2.5 *Maglev trains*

Magnetic levitation (MagLev), trains are nowadays a promising solution for transportation. They get propulsion force from linear motors and use electromagnets for the suspension system. Two main types of levitation technologies, [2] will be discussed in this problem:

(I) *electromagnetic suspension* (EMS), that uses magnetic *attractive* force to levitate (Figure 4),
(II) *electrodynamic suspension* (EDS), that uses magnetic *repulsive* force for levitation (Figure 5).

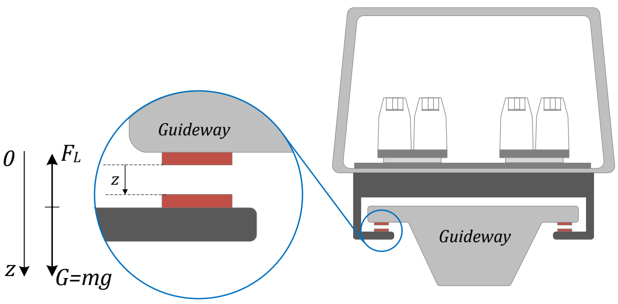


Figure 4: Electromagnetic suspension (magnetic attraction)

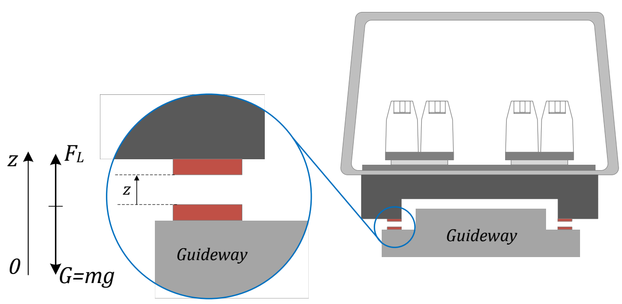


Figure 5: Electrodynamic suspension (magnetic repulsion)

The levitation force F_L depends on the current $i(t)$ in the levitation coils and the air gap $z(t)$ and may be approximated by, [1]:

$$F_L(t) = k \cdot \frac{i^2(t)}{z^2(t)}$$

where k is a constant. This force is opposed by the gravitational force $G = mg$, where m is the mass of the train and g - the acceleration of gravity.

At equilibrium, the train levitates on an air gap of 1 cm.

Consider the following constants for the model:

- Operating air gap $z_0 = 10^{-2}\ m$

- Mass of the train $m = 10^4\ kg$
- Levitation force constant $k = 10^{-3}\ Nm^2/A^2$
- Acceleration of gravity $g = 10\ m/s^2$

In both cases, the dynamical levitation system has the input $i(t)$ - the current through the levitation coils and the output $z(t)$ - the air gap between the train and the guideway (see Figure 6).

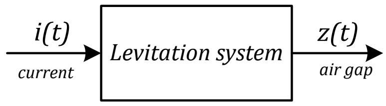


Figure 6: The levitation system: input - current, output - air gap

A model for this system, or a relationship between $i(t)$ and $z(t)$, can be obtained from the differential equations describing the vertical movement of the train as:

$$mass \times acceleration = \sum forces$$

where *acceleration* refers to the acceleration of the vertical movement obtained as the second derivative of $z(t)$. Considering the orientation of forces and the coordinate system for each case as presented in Figures 4 and 5, we obtain the following nonlinear models:

(I) EMS:

$$m\ddot{z}(t) = mg - k\frac{i^2(t)}{z^2(t)} \tag{9}$$

(II) EDS:

$$m\ddot{z}(t) = k\frac{i^2(t)}{z^2(t)} - mg \tag{10}$$

1. A linear approximation of the models around the equilibrium condition.
   The equations (9) and (10) are two nonlinear second-order differential equations. To compute a transfer function for each case, they are approximated to linear differential equations around an equilibrium point. *The extended calculus is given in the Annex (optional).*

   The train is at equilibrium for the nominal levitation distance z_0 = 1cm = 0.01 m. Thus, for $z_0 = 0.01$, the acceleration is zero: $dz^2/dt^2 = \ddot{z}_0 = 0$. The equilibrium current (the current in the levitation coils that keeps the train levitating at 1 cm) is $i_0 = 100A$.

   If we denote by: $\Delta z(t) = z(t) - z_0,\ \Delta i(t) = i(t) - i_0,\ \Delta\ddot{z}(t) = \ddot{z}(t) - \ddot{z}_0$, the variations of the variables around the operating (equilibrium) point ($z_0 = 0.01,\ \ddot{z}_0 = 0,\ i_0 = 100$), the linear approximations of equations (9) and (10) are:
   (I) EMS:

   $$m\Delta\ddot{z}(t) = 2k\frac{i_0^2}{z_0^3}\Delta z(t) - 2k\frac{i_0}{z_0^2}\Delta i(t) \tag{11}$$

   (II) EDS:

   $$m\Delta\ddot{z}(t) = -2k\frac{i_0^2}{z_0^3}\Delta z(t) + 2k\frac{i_0}{z_0^2}\Delta i(t) \tag{12}$$

   The linearized system with the input $\Delta i(t)$ and output $\Delta z(t)$ is shown in Figure 7.

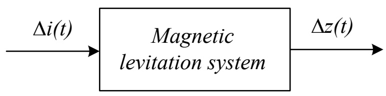


Figure 7: The levitation system: input - $\Delta i(t)$, output - $\Delta z(t)$

2. Transfer function model
   (a) Determine the transfer function from the input current $\Delta i(t)$ to the output position $\Delta z(t)$ in both cases.
   (b) Plot the impulse response of the systems for a period of time of 0.1 seconds (case I) and 1 second (case II). Use the Matlab function *impulse* to simulate the impulse response of the system in both cases.
   Analyze and explain the results.

3 Annex. Linear approximation of differential equations. Maglev trains

A linear approximation of the differential equation that describes the Maglev system (I) is presented below. The second case can be obtained in a similar manner.

The model of the Maglev train (I) is a nonlinear second-order differential equation:

$$m\ddot{z}(t) = mg - k\frac{i^2(t)}{z^2(t)}$$

where $\ddot{z}(t) = \frac{d^2z(t)}{dt^2}$. From the description of this system, the input signal is the current through the levitation coils $i(t)$ and the output is $z(t)$ - the distance of levitation (Figure 4, Figure 8).

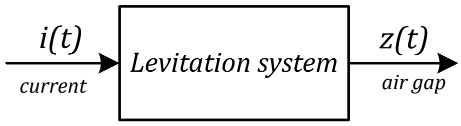


Figure 8: Magnetic levitation system. Input and output

The equation can also be written as a function of 3 variables: the acceleration $\ddot{z}(t)$, the position $z(t)$ and the current $i(t)$, that equals zero:

$$g(\ddot{z}(t), z(t), i(t)) = m\ddot{z}(t) - mg + k\frac{i^2(t)}{z^2(t)} = 0 \tag{13}$$

The first step in linearizing this equation is setting the desired operating point. If the train is at equilibrium for z_0, then the speed and acceleration of the train (vertical movement) are zero for this distance: $\ddot{z}_0 = 0$. The current that keeps the train levitating at the distance z_0 is obtained from the nonlinear equation for $z(t) = z_0$ and $\ddot{z}(t) = \ddot{z}_0$:

$$m\ddot{z}_0 = mg - k\frac{i_0^2}{z_0^2}$$

or

$$0 = mg - k\frac{i_0^2}{z_0^2}, \quad \Rightarrow i_0 = \sqrt{\frac{mg}{k}}z_0$$

The operating point for linearization is then: $(\ddot{z}_0,\ z_0,\ i_0)$

Using a Taylor series approximation for the nonlinear function (13) around the operating point $(\ddot{z}_0,\ z_0,\ i_0)$ we have:

$$\begin{aligned} 0 = g(\ddot{z}(t), z(t), i(t)) \quad &\approx \quad g(\ddot{z}_0, z_0, i_0) + \frac{\partial g}{\partial \ddot{z}}|_{(\ddot{z}_0,\ z_0,\ i_0)} \cdot (\ddot{z}(t) - \ddot{z}_0) + \\ &+ \quad \frac{\partial g}{\partial z}|_{(\ddot{z}_0,\ z_0,\ i_0)} \cdot (z(t) - z_0) + \frac{\partial g}{\partial i}|_{(\ddot{z}_0,\ z_0,\ i_0)} \cdot (i(t) - i_0) \end{aligned}$$

or:

$$0 \approx 0 + m \cdot (\ddot{z}(t) - \ddot{z}_0) - 2k\frac{i_0^2}{z_0^3} \cdot (z(t) - z_0) + 2k\frac{i_0}{z_0^2} \cdot (i(t) - i_0)$$

We denote by $\Delta\ddot{z}(t) = \ddot{z}(t) - \ddot{z}_0$, $\Delta z(t) = z(t) - z_0$ and $\Delta i(t) = i(t) - i_0$ the variations of variables around the operating point, and re-arranging the equation above, we obtain:

$$m\Delta\ddot{z}(t) = 2k\frac{i_0^2}{z_0^3}\Delta z(t) - 2k\frac{i_0}{z_0^2}\Delta i(t) \tag{14}$$

The differential equation (14) is linear in terms of $\Delta i(t)$, $\Delta z(t)$ and $\Delta\ddot{z}(t)$. The approximation is valid only for small perturbations of the variables around the equilibrium values $\ddot{z}_0,\ z_0,\ i_0$.

References

[1] Richard C. Dorf and Robert H. Bishop. *Modern Control Systems*. Pearson, 2011.

[2] Hyung-Woo Lee, Ki-Chan Kim, and Ju Lee. Review of Maglev train technologies. *IEEE Transactions on Magnetics*, 42(7):1917–1926, July 2006.