Seminar 11

Wednesday, December 9, 2020 6:01 PM

Applications to the Laplace transform

Differentiation of the original

$f(t), f'(t), f''(t), \dots$ are originals and $\boxed{\mathcal{L}[f(t)](p) = F(p)}$

$$\boxed{\begin{aligned}&\mathcal{L}[f'(t)](p) = pF(p) - f(0)\\ &\mathcal{L}[f''(t)](p) = p^2F(p) - pf(0) - f'(0)\end{aligned}}$$

Solving ordinary differential equations

- we convert ODE to algebraic equations

we solve the algebraic equations for the unknown function $(F(p), X(p), Y(p), \dots)$

- we use partial fraction expansion to express the unknown function
- we use Inverse of Laplace Transform to obtain the solution

1 Differential equations with constant coefficients

1) $x''(t) - 5x'(t) + 6x(t) = 0, \;/\mathcal{L}$ $\quad x(0) = 1, \; x'(0) = -1$

$$\mathcal{L}[x''(t)](p) - 5\mathcal{L}[x'(t)](p) + 6\mathcal{L}[x(t)](p) = 0$$

$$\mathcal{L}[x(t)](p) = X(p)$$

$$p^2X(p) - p\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=-1} - 5(pX(p) - \underbrace{x(0)}_{=1}) + 6X(p) = 0$$

$$p^2X(p) - p + 1 - 5pX(p) + 5 + 6X(p) = 0$$

$$X(p)(\underbrace{p^2 - 5p + 6}_{(p-2)(p-3)}) = p - 6 \Rightarrow X(p) = \frac{p-6}{(p-2)(p-3)}$$

$$\frac{p-6}{(p-2)(p-3)} = \frac{A}{p-2} + \frac{B}{p-3} \Rightarrow p - 6 = A(p-3) + B(p-2)$$

$p = 3 \Rightarrow \boxed{B = -3}$

$p = 2 \Rightarrow -A = -4 \Rightarrow \boxed{A = 4}$

$$X(p) = \frac{4}{p-2} - \frac{3}{p-3} \;/\mathcal{L}^{-1}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{4}{p-2}\right] - \mathcal{L}^{-1}\left[\frac{3}{p-3}\right] \Rightarrow x(t) = 4e^{2t} - 3e^{3t}$$

$X(p) = \frac{4}{p-2} - \frac{3}{p-3}$ $/\mathcal{L}^{-1}$

$$x(t) = \mathcal{L}^{-1}\left[\frac{4}{p-2}\right] - \mathcal{L}^{-1}\left[\frac{3}{p-3}\right] \Rightarrow x(t) = 4e^{2t} - 3e^{3t}$$

2) $\begin{cases} x''(t) + x(t) = 2\cos t & /\mathcal{L} \\ x(0) = 0,\ x'(0) = -1 \end{cases}$ $\qquad \mathcal{L}[x(t)](p) = X(p)$

$$\mathcal{L}[x''(t)](p) + \mathcal{L}[x(t)](p) = \mathcal{L}[2\cos t](p)$$

$$p^2X(p) - p\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=-1} + X(p) = 2\frac{p}{p^2+1} \Rightarrow X(p)\cdot(p^2+1) = \frac{2p}{p^2+1} - 1 \quad /:(p^2+1)$$

$$\Rightarrow X(p) = \frac{2p}{(p^2+1)^2} - \frac{1}{p^2+1} \quad /\mathcal{L}^{-1}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{2p}{(p^2+1)^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right] \Rightarrow x(t) = t\sin t - \sin t$$

Formula: $\boxed{\begin{array}{l} F(p) = \frac{p}{(p^2+1)^2} \\ \mathcal{L}^{-1}\left[\frac{p}{(p^2+1)^2}\right] = \frac{1}{2}t\sin t \end{array}}$

$$F(p) = \frac{p}{(p^2+1)^2} = -\frac{1}{2}\underbrace{\left(\frac{1}{p^2+1}\right)'}_{=\frac{-2p}{(p^2+1)^2}} = \frac{1}{2}(-1)\left(\mathcal{L}[\sin t](p)\right)' = \frac{1}{2}\left(\mathcal{L}[t\sin t](p)\right)$$

② System of differential equations with constant coefficients

3) $\begin{cases} x'(t) = 3x(t) - y(t) & /\mathcal{L} \\ y'(t) = -9x(t) + 3y(t) & /\mathcal{L} \end{cases}$ $\qquad x(0) = 1,\ y(0) = 0$

$\mathcal{L}[x(t)](p) = X(p)$
$\mathcal{L}[y(t)](p) = Y(p)$

$$\begin{cases} \mathcal{L}[x'(t)](p) = 3\mathcal{L}[x(t)](p) - \mathcal{L}[y(t)](p) \\ \mathcal{L}[y'(t)](p) = -9\mathcal{L}[x(t)](p) + 3\mathcal{L}[y(t)](p) \end{cases}$$

$$\begin{cases} pX(p) - \underbrace{x(0)}_{=1} = 3X(p) - Y(p) \\ pY(p) - \underbrace{y(0)}_{=0} = -9X(p) + 3Y(p) \end{cases} \Rightarrow \begin{cases} X(p)\cdot(p-3) + Y(p) = 1 \\ 9X(p) + Y(p)(p-3) = 0 \end{cases} \longrightarrow$$

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$$\Rightarrow X(p) = -\frac{Y(p)\cdot(p-3)}{9}$$

$$-\frac{Y(p)(p-3)^2}{9} + \frac{9}{4}Y(p) = \frac{9}{1} \Rightarrow Y(p)\left(-p^2+6p-\cancel{9}+\cancel{9}\right) = 9 \Rightarrow \boxed{Y(p) = \frac{-9}{p^2-6p}}$$

$$X(p) = -\frac{-\cancel{9}}{p^2-6p}\cdot\frac{p-3}{\cancel{9}} \Rightarrow \boxed{X(p) = \frac{p-3}{p^2-6p}}$$

$$X(p) = \frac{p-3}{p^2-6p} = \frac{p-3}{p(p-6)} = \frac{p-6+p}{2p(p-6)} = \frac{1}{2p} + \frac{1}{2(p-6)} \quad / \mathcal{L}^{-1} \qquad 2p-6 = 2(p-3)$$

$$\Rightarrow \boxed{x(t) = \frac{1}{2} - \frac{1}{2}e^{6t}}$$

$$Y(p) = \frac{-9}{p^2-6p} = \frac{-9}{p(p-6)} = \frac{-9}{-6}\cdot\frac{p-6-p}{p(p-6)} = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{p-6}\right) \quad / \mathcal{L}^{-1}$$

$$\Rightarrow \boxed{y(t) = \frac{3}{2} - \frac{3}{2}e^{6t}}$$

3 Integral equations

Convolution of the originals

$f(t), g(t)$ originals ———— $F(p), G(p)$ images

The function $\int_0^t f(\tau)g(t-\tau)\,d\tau$ is called the convolution of the functions $f(t)$ and $g(t)$

$$\int_0^t f(\tau)g(t-\tau)\,d\tau = f * g$$

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right] = F(p)\cdot G(p) = \mathcal{L}[f(t)](p)\cdot\mathcal{L}[g(t)](p)$$

4) $$y(t) - 2\int_0^t y(t-u)\sin u\,du = \cos t \quad / \mathcal{L}$$

$$\mathcal{L}[y(t)](p) - 2\mathcal{L}\Big[\underbrace{\int_0^t y(t-u)\sin u\,du}_{y(t)*\sin t}\Big](p) = \mathcal{L}[\cos t](p)$$

$$\mathcal{L}[y(t)](p) = Y(p)$$

$$Y(p) - 2\mathcal{L}[y(t)](p)\cdot\mathcal{L}[\sin t](p) = \frac{p}{p^2+1}$$

$$Y(p) - 2Y(p)\cdot\frac{1}{p^2+1} = \frac{p}{p^2+1} \Rightarrow Y(p)\left(p^2+1-2\right) = p$$

$$\Rightarrow Y(p) = \frac{p}{p^2-1} \quad / \mathcal{L}^{-1} \quad \Rightarrow \boxed{y(t) = \operatorname{ch} t}$$

or

$$Y(p) = \frac{p}{(p-1)(p+1)} = \frac{p+1+p-1}{2(p-1)(p+1)} = \frac{1}{2}\left[\frac{1}{p-1} + \frac{1}{p+1}\right] \quad / \mathcal{L}^{-1}$$

$$\Rightarrow \boxed{y(p) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} = \operatorname{ch} t}$$

4 Integral - differential equations

5) $y'(t) + \int_0^t u\cdot y(t-u)\,du = t \;/ \mathcal{L}, \quad y(0) = -1$

$$\mathcal{L}[y'(t)](p) + \mathcal{L}\Big[\underbrace{\int_0^t u\cdot y(t-u)\,du}_{t * y(t)}\Big](p) = \mathcal{L}[t](p)$$

$$\mathcal{L}[y(t)](p) = Y(p) \qquad\qquad Y(p) = ?$$

$$pY(p) - \underbrace{y(0)}_{=-1} + \mathcal{L}[t](p)\cdot\mathcal{L}[y(t)](p) = \frac{1}{p^2}$$

$$pY(p) + 1 + \frac{1}{p^2}\cdot Y(p) = \frac{1}{p^2} \Rightarrow Y(p)\cdot\left(\frac{p^{2}}{}p + \frac{1}{p^2}\right) = \frac{1}{p^2} - \frac{p^2}{1}$$

$$\Rightarrow Y(p)\cdot(p^3+1) = 1-p^2 \Rightarrow Y(p) = \frac{1-p^2}{p^3+1} = \frac{(1-p)\cancel{(p+1)}}{\cancel{(p+1)}(p^2-p+1)}$$

$$\Rightarrow Y(p) = \frac{1-p}{p^2-p+1}$$

$$Y(p) = \frac{1}{p^2-p+1} - \frac{p}{p^2-p+1} = \frac{1}{p^2-2\cdot p\cdot\frac{1}{2}+\frac{1}{4}+\frac{4}{4}-\frac{1}{4}} - \frac{p}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{p-\frac{1}{2}+\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{2}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{2}{\sqrt{3}}\cdot\frac{\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{2}{\sqrt{3}} \cdot \frac{\frac{1}{2}\cdot 2}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \quad / \mathcal{L}^{-1}$$

$$\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y(t) = \frac{1}{\sqrt{3}} e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right) - e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right).$$

6) $$\int_0^t \sin(t-\tau)\, x(\tau)\, d\tau = \sin^2 t \quad /, \mathcal{L} \qquad t \geq 0$$

$$\cos 2t = \cos^2 t - \sin^2 t$$
$$= 2\cos^2 t - 1$$
$$= 1 - 2\sin^2 t$$

$$\mathcal{L}\Big[\underbrace{\int_0^t \sin(t-\tau)\, x(\tau)\, d\tau}_{\sin t * x(t)}\Big](p) = \mathcal{L}\left[\sin^2 t\right](p)$$

$$\mathcal{L}\left[\sin^2 t\right](p) = \mathcal{L}\left[\frac{1-\cos 2t}{2}\right](p)$$

$$\mathcal{L}[\sin t](p) \cdot \mathcal{L}[x(t)](p) = \mathcal{L}\left[\frac{1-\cos 2t}{2}\right](p) \qquad \mathcal{L}[x(t)](p) = X(p)$$

$$\frac{1}{p^2+1} \cdot X(p) = \frac{\overset{p^2+4}{1}}{2p} - \frac{1}{2} \frac{\overset{p}{p}}{p^2+4} \Rightarrow \frac{X(p)}{p^2+1} = \frac{\cancel{p^2}+4-\cancel{p^2}}{2p(p^2+4)} \quad / \cdot (p^2+1)$$

$$\Rightarrow X(p) = \frac{2(p^2+1)}{p(p^2+4)} \qquad x(t) = ?$$

$$\frac{2p^2+2}{p(p^2+4)} = \frac{\overset{p^2+4}{A}}{p} + \frac{\overset{p}{Bp+C}}{p^2+4} \longrightarrow 2p^2+2 = Ap^2+4A + Bp^2 + Cp$$

$$\Rightarrow \begin{cases} A+B=2 \longrightarrow B = 2 - \frac{1}{2} \Rightarrow \boxed{B = \frac{3}{2}} \\ \boxed{C=0} \\ 4A = 2 \Rightarrow \boxed{A = \frac{1}{2}} \end{cases}$$

$$X(p) = \frac{1}{2p} + \frac{3}{2} \cdot \frac{p}{p^2+4} \quad / \mathcal{L}^{-1} \longrightarrow \boxed{x(t) = \frac{1}{2} + \frac{3}{2}\cos 2t}$$

7) $$\begin{cases} x''(t) + 3x'(t) + 2x(t) = e^{-t} \\ x(0) = x'(0) = 0 \end{cases} \qquad \mathcal{L}[x(t)](p) = X(p)$$

$$p^2 X(p) - p \underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + 3pX(p) - 3\underbrace{x(0)}_{0} + 2X(p) = \frac{1}{p+1}$$

$$p^2X(p) - p\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=0} + 3pX(p) - 3\underbrace{x(0)}_{=0} + 2X(p) = \frac{1}{p+1}$$

$$X(p)\underbrace{(p^2+3p+2)}_{(p+1)(p+2)} = \frac{1}{p+1} \Rightarrow X(p) = \frac{1}{(p+1)^2(p+2)}$$

$$\frac{1}{(p+1)^2(p+2)} = \frac{\overset{p^2+2p+1}{A}}{p+2} + \frac{\overset{p^2+3p+2}{B}}{p+1} + \frac{\overset{p+2}{C}}{(p+1)^2} \Rightarrow 1 = p^2(A+B) + p(2A+3B+C) + A+2B+2C$$

$$\Rightarrow \begin{cases} A+B=0 \longrightarrow B=-A \Rightarrow \boxed{B=-1} \\ 2A+3B+C=0 \\ A+2B+2C=1 \end{cases} \longrightarrow \begin{cases} 2A-3A+C=0 \\ A-2A+2C=1 \end{cases} \Rightarrow \begin{cases} A=C \\ -A+2C=1 \end{cases} \Rightarrow \boxed{A=1=C}$$

$$X(p) = \frac{1}{p+2} - \frac{1}{p+1} + \frac{1}{(p+1)^2} \quad / \mathcal{L}^{-1}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{p+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{p+1}\right] + \mathcal{L}\left[\frac{1}{(p+1)^2}\right] \Rightarrow \boxed{x(t) = e^{-2t} - e^{-t} + te^{-t}}$$

Formula:

$$\boxed{\begin{gathered} F(p) = \frac{1}{(p+1)^2} \\ \mathcal{L}^{-1}\left[\frac{1}{(p+1)^2}\right] = te^{-t} \end{gathered}}$$

$$F(p) = \frac{1}{(p+1)^2} = \left(-\frac{1}{p+1}\right)' = (-1)\cdot\left(\frac{1}{p+1}\right)' = (-1)\left(\mathcal{L}[e^{-t}](p)\right)' = \mathcal{L}[te^{-t}](p)$$