Seminar 12

Wednesday, December 16, 2020 6:02 PM

Applications to the Laplace transform

(1) Differential equations with nonconstant coefficients

$$\underbrace{t}_{\text{non constant}} x''(t) + 2x'(t) = t - 1 \Big/ \mathcal{L} \quad ; \quad x(0) = 3, \; x'(0) = -\frac{1}{2} \; ; \; \mathcal{L}[x(t)](p) = X(p)$$

$$\mathcal{L}[t\,x''(t)] = (-1)\left(\mathcal{L}[x''(t)](p)\right)' = -\left(p^2 X(p) - p\underbrace{x(0)}_{=3} - \underbrace{x'(0)}_{=-\frac{1}{2}}\right)' =$$

$$= -\left(p^2 X'(p) + 2pX(p) - 3\right) = -p^2 X'(p) - 2pX(p) + 3$$

$$\mathcal{L}[x'(t)](p) = pX(p) - \underbrace{x(0)}_{=3} = pX(p) - 3$$

$$\Rightarrow -p^2X'(p) \cancel{-2pX(p)} + 3 + \cancel{2pX(p)} - 6 = \frac{1}{p^2} - \frac{1}{p}$$

$$-p^2X'(p) = 3 + \frac{1}{p^2} - \frac{1}{p} \Big/ :(-p^2) \longrightarrow X'(p) = -\frac{3}{p^2} - \frac{1}{p^4} + \frac{1}{p^3} \Big/ \int$$

$$X(p) = \int\left(-3p^{-2} - p^{-4} + p^{-3}\right)dp = -3\frac{p^{-1}}{-1} - \frac{p^{-3}}{-3} + \frac{p^{-2}}{-2} + C$$

$$\Rightarrow X(p) = \frac{3}{p} + \frac{1}{3p^3} - \frac{1}{2p^2} + C$$

We use the property $\lim_{p\to\infty} X(p) = 0$

$$\Rightarrow C = 0$$

$$\Rightarrow X(p) = \frac{3}{p} + \frac{1}{3p^3} - \frac{1}{2p^2} \Big/ \mathcal{L}^{-1}$$

$$\Rightarrow \boxed{x(t) = 3 + \frac{1}{6}t^2 - \frac{1}{2}t}$$

(2) Solve the equation: $x(t) = 2\sin 4t + \int_0^t \sin 4(t-u)\,x(u)\,du \Big/ \mathcal{L}$, $\mathcal{L}[x(t)](p) = X(p)$

$$X(p) = 2\cdot\frac{4}{p^2+16} + \mathcal{L}[\sin(4t)](p)\cdot\mathcal{L}[x(t)](p)$$

$$X(p) = \frac{8}{p^2+16} + \frac{4}{p^2+16}\cdot X(p) \rightarrow X(p)\cdot\frac{p^2+16-4}{p^2+16} = \frac{8}{p^2+16}$$

$$\Rightarrow X(p) = \frac{8}{p^2+12} \Big/ \mathcal{L}^{-1}$$

$2\sqrt{3}$

$$\Rightarrow X(p) = \frac{8}{p^2+12} \Big/ \mathcal{L}$$

$$\Rightarrow x(t) = \frac{8}{2\sqrt{3}}\mathcal{L}^{-1}\left[\frac{2\sqrt{3}}{p^2+(2\sqrt{3})^2}\right] \Rightarrow x(t) = \frac{8}{2\sqrt{3}}\sin(2\sqrt{3}t) \Rightarrow \boxed{x(t) = \frac{4}{\sqrt{3}}\sin(2\sqrt{3}t)}$$

(3) $x''(t) + x(t) = \dfrac{1}{\cos t} \Big/ \mathcal{L}$; $x(0) = 0$, $x'(0) = 2$

$\mathcal{L}[x(t)](p) = X(p)$

$$p^2X(p) - p\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{2} + X(p) = \mathcal{L}\left[\frac{1}{\cos t}\right](p)$$

$$X(p)(p^2+1) = 2 + \mathcal{L}\left[\frac{1}{\cos t}\right](p) \Big/ \cdot \frac{1}{p^2+1}$$

$$\Rightarrow X(p) = \frac{2}{p^2+1} + \frac{1}{p^2+1}\cdot\mathcal{L}\left[\frac{1}{\cos t}\right](p)$$

$$X(p) = 2\mathcal{L}[\sin t](p) + \mathcal{L}[\sin t](p)\cdot\mathcal{L}\left[\frac{1}{\cos t}\right](p) \Big/ \mathcal{L}^{-1}$$

$$x(t) = 2\sin t + \sin t * \frac{1}{\cos t} \Rightarrow x(t) = 2\sin t + \int_0^t \sin(t-\tau)\cdot\frac{1}{\cos\tau}\,d\tau$$

$$x(t) = 2\sin t + \int_0^t \frac{\sin t\cos\tau - \cos t\sin\tau}{\cos\tau}\,d\tau$$

$$x(t) = 2\sin t + \int_0^t (\sin t - \cos t\cdot\tan\tau)\,d\tau$$

$$x(t) = 2\sin t + \sin t\cdot\tau\Big|_0^t + \cos t\ln(\cos\tau)\Big|_0^t$$

$$\boxed{x(t) = 2\sin t + t\sin t + \cos t\ln(\cos t)}$$

(4) $$\int_0^{2\pi}\frac{dx}{13+5\cos x} \longrightarrow \int_{C:|z|=1} f(z)\,dz \qquad e^{ix} = z,\ x\in[0,2\pi]$$

$$\cos x = \frac{z^2+1}{2z};\quad dx = \frac{dz}{iz}$$

$$I = \int_{C:|z|=1}\frac{\frac{dz}{iz}}{13+5\cdot\frac{z^2+1}{2z}} = \int_C \frac{dz}{i\cancel{z}\cdot\frac{26z+5z^2+5}{2\cancel{z}}} = \frac{2}{i}\int\frac{dz}{5z^2+26z+5}$$

$5z^2+26z+5 = 0 \qquad \Delta = 676 - 100 = 576 = 24^2 \Rightarrow z_{1,2} = \dfrac{-26\pm 24}{10}$, $z_1 = -5$, $z_2 = -\dfrac{1}{5}$

$z_1 = -5 \notin \operatorname{int} C$

$5z^2+26z+5 = 5(z+5)(z+\frac{1}{5})$

$z_2 = -\frac{1}{5} \in \operatorname{int} C$ pole of order 1

$$\operatorname{Res}_{z=-\frac{1}{5}} f(z) = \frac{\frac{1}{5(z+5)}}{1}\Big|_{z=-\frac{1}{5}} = \frac{1}{5(-\frac{1}{5}+\frac{25}{5})} = \frac{1}{24}$$

$$I = 2\pi \cancel{i} \cdot \frac{2}{\cancel{i}} \cdot \frac{1}{24} = \frac{\pi}{6}$$

(5) $x''(t) + 2x'(t) + x = \left[e^t \cdot (t+1)\right]^{-1} \Big/ \mathcal{L}$, $x'(0) = x(0) = 0$

$\mathcal{L}[x(t)](p) = X(p)$

$$\underline{p^2 X(p)} - p\underset{0}{\underset{\shortparallel}{x(0)}} - \underset{0}{\underset{\shortparallel}{x'(0)}} + \underline{2pX(p)} - 2\underset{0}{\underset{\shortparallel}{x(0)}} + \underline{X(p)} = \underbrace{\mathcal{L}\left[e^t \cdot (t+1)\right]^{-1}}_{F(p)}$$

$$X(p) \cdot (p^2+2p+1) = F(p)$$

$$X(p) = \frac{F(p)}{(p+1)^2} = \frac{1}{(p+1)^2} \cdot F(p) = \mathcal{L}[te^{-t}] \cdot \mathcal{L}\left[e^t \cdot (t+1)\right]^{-1}(p) \Rightarrow$$

$$\mathcal{L}[te^{-t}] = \mathcal{L}[t](p+1) = \frac{1}{p^2}\Big|_{p=p+1} = \frac{1}{(p+1)^2}$$

$$\Rightarrow X(p) = \mathcal{L}\left[te^{-t} * \left[e^t \cdot (t+1)\right]^{-1}\right] \Big/ \mathcal{L}^{-1}$$

$$\Rightarrow x(t) = te^{-t} * \left[e^t \cdot (t+1)\right]^{-1} = \int_0^t (t-u)e^{-t+u} \cdot \left[e^u (u+1)\right]^{-1} du =$$

$$f(t) * g(t) = \int_0^t f(t-u) g(u) du$$

$$= \int_0^t (t-u) e^{-t+u} \cdot e^{-u} \cdot \frac{1}{u+1} du = \int_0^t \frac{(t-u)e^{-t}}{u+1} du = e^{-t} \int_0^t \frac{t-u}{u+1} du$$

$$= e^{-t} \int_0^t \left(\frac{t}{u+1} - \frac{u+1-1}{u+1}\right) du = e^{-t}\left[t \ln(u+1) - u + \ln(u+1)\right]\Big|_0^t =$$

$$= e^{-t}\left[t\ln(t+1) - t + \ln(t+1)\right].$$

$$\boxed{x(t) = e^{-t}\left[(t+1)\ln(t+1) - t\right]}$$

(6) $\quad t^3 \quad \int^t x''(u)(t-u)^2 du \;/\quad x(0) = 0,\ x'(0) = 2$

⑥ $\sin t = \frac{t^3}{e^t} + \int_0^t x''(u)(t-u)^2 du \Big/ \mathcal{L}$, $x(0)=0,\ x'(0)=2$

$\mathcal{L}[x(t)](p) = X(p)$

$\frac{1}{p^2+1} = \mathcal{L}[t^3 e^{-t}](p) + \mathcal{L}[x''(t)](p) \cdot \mathcal{L}[t^2](p)$

$\mathcal{L}[t^3 e^{-t}](p) = \mathcal{L}[t^3](p+1) = \frac{3!}{p^4}\Big|_{p \to p+1} = \frac{3!}{(p+1)^4}$

$\frac{1}{p^2+1} = \frac{3!}{(p+1)^4} + [p^2 X(p) - p\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{2}] \cdot \frac{2}{p^3}$

$\frac{1}{p^2+1} = \frac{6}{(p+1)^4} + \cancel{p^2} \frac{2}{p^{\cancel{3}}} X(p) - \frac{4}{p^3} \Rightarrow \frac{2}{p} X(p) = \frac{1}{p^2+1} + \frac{4}{p^3} - \frac{6}{(p+1)^4} \Big/ \cdot \frac{p}{2}$

$\Rightarrow X(p) = \frac{p}{2(p^2+1)} + \frac{2}{p^2} - \frac{3(p+1-1)}{(p+1)^4}$

$X(p) = \frac{p}{2(p^2+1)} + \frac{2}{p^2} - \frac{1}{2}\frac{3 \cdot 2}{(p+1)^3} + \frac{1}{3!}\frac{3 \cdot 3!}{(p+1)^4} \Big/ \mathcal{L}^{-1}$

$$\boxed{x(t) = \frac{1}{2}\cos t + 2t - \frac{3}{2}t^2 \cdot e^{-t} + \frac{3}{6}t^3 e^{-t}}$$

⑦ $\mathcal{L}[\sin\sqrt{t}] = ?$

we expand $\sin\sqrt{t} = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{t})^{2n+1}}{(2n+1)!}$

$\mathcal{L}[t^\alpha](p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$

$\Gamma(p) = \int_0^\infty e^{-x} \cdot x^{1-p} dx$

$\boxed{\Gamma(p+1) = p\Gamma(p),\ \forall p > 0}$

$\Gamma(n+1) = n!,\ n \in \mathbb{N}$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n! \cdot 2^{2n}}$

$\mathcal{L}[\sin\sqrt{t}](p) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \mathcal{L}[t^{\frac{2n+1}{2}}](p) =$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{\Gamma\left(\frac{2n+1}{2}+1\right)}{p^{\frac{2n+1}{2}+1}} =$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{(2n+1)!}\ (2n)!} \frac{\cancel{2n+1}}{2} \Gamma\left(\frac{2n+1}{2}\right) \cdot \frac{1}{p^n \cdot p^{3/2}} =$

$= \sum \frac{(-1)^n}{(2n)!} \cdot \frac{1}{2} \cdot \frac{1}{p^n \cdot p^{3/2}} \cdot \Gamma\left(n+\frac{1}{2}\right) =$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{2} \cdot \frac{1}{p^n \cdot p^{3/2}} \cdot \Gamma\left(n+\frac{1}{2}\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{(2n)!}} \cdot \frac{1}{2} \cdot \frac{1}{p^n p^{3/2}} \cdot \frac{\cancel{(2n)!}\sqrt{\pi}}{n!\, 2^{2n}} = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{2p^{3/2}} \left(-\frac{1}{4p}\right)^n \cdot \frac{1}{n!} =$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}} \sum_{n=0}^{\infty} \left(-\frac{1}{4p}\right)^n \cdot \frac{1}{n!} \Rightarrow \boxed{\mathcal{L}[\sin\sqrt{t}](p) = \frac{\sqrt{\pi}}{2p\sqrt{p}} \cdot e^{-\frac{1}{4p}}}$$

(8)

$$x(t) = 2t + 2\int_0^t \cos u \cdot x(t-u)\,du \quad / \mathcal{L}$$

$$\mathcal{L}[x(t)](p) = X(p)$$

$$X(p) = 2\frac{1}{p^2} + 2\mathcal{L}[\cos t](p) \cdot \mathcal{L}[x(t)](p)$$

$$X(p) = \frac{2}{p^2} + \frac{2p}{p^2+1} \cdot X(p) \Rightarrow X(p) \cdot \frac{p^2+1-2p}{p^2+1} = \frac{2}{p^2}$$

$$\Rightarrow X(p) = \frac{2(p^2+1)}{p^2(p-1)^2} = \frac{\overset{p(p^2-2p+1)}{A}}{p} + \frac{\overset{p^2-2p+1}{B}}{p^2} + \frac{\overset{p^2(p-1)}{C}}{p-1} + \frac{\overset{p^2}{D}}{(p-1)^2}$$

$$2p^2+2 = A(p^3-2p^2+p) + B(p^2-2p+1) + C(p^3-p^2) + Dp^2$$

$$\begin{cases} A+C=0 \longrightarrow \boxed{C=-4} \\ -2A+B-C+D=2 \longrightarrow -8+\cancel{2}+4+D=\cancel{2} \Rightarrow \boxed{D=4} \\ A-2B=0 \Rightarrow \boxed{A=4} \\ \boxed{B=2} \end{cases}$$

$$X(p) = \frac{4}{p} + \frac{2}{p^2} - \frac{4}{p-1} + \frac{4}{(p-1)^2} \quad / \mathcal{L}^{-1}$$

$$\boxed{x(t) = 4 + 2t - 4e^t + 4te^t}$$