Seminar 14

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The Z transform

$f : \mathbb{N} \to \mathbb{C}$ the original

$$\mathcal{Z}[f](z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots + \frac{f(n)}{z^n} + \dots = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$$

↑ the Z transform of f

$F(z) = \mathcal{Z}[f](z)$
↑ the image

Z transform of some functions

$$\mathcal{Z}[1] = \frac{z}{z-1}, \quad \mathcal{Z}[a^n] = \frac{z}{z-a}, \quad \mathcal{Z}[n] = \frac{z}{(z-1)^2}, \quad \mathcal{Z}[n^2] = \frac{z(z+1)}{(z-1)^3}$$

$$\mathcal{Z}[\cos an] = \frac{z(z-\cos a)}{z^2 - 2z\cos a + 1}, \quad \mathcal{Z}[\sin an] = \frac{z \sin a}{z^2 - 2z\cos a + 1}$$

$$\mathcal{Z}[\cosh an] = \frac{z(z-\cosh a)}{z^2 - 2z\cosh a + 1}, \quad \mathcal{Z}[\sinh an] = \frac{z \sinh a}{z^2 - 2z\sinh a + 1}$$

$$\mathcal{Z}[f(n+p)](z) = z^p F(z) - z^p f(0) - z^{p-1} f(1) - \dots - z f(p-1)$$

$$\mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right] = \frac{z}{z-1} F(z) = \frac{z}{z-1} \mathcal{Z}[f]$$

$$\mathcal{Z}[n \cdot f(n)] = -z F'(z)$$

The inverse of Z transform

Method I: $F(z) = \mathcal{Z}[f] \Rightarrow f(n) = \frac{1}{2\pi i} \int_C z^{n-1} F(z)\, dz$

where C contains the singular points in the interior

we apply the Residue Theorem

Method II: $F(z) = \mathcal{Z}[f]$

We use partial fraction decomposition

$\frac{F(z)}{z}$ – we split into partial fraction

– we multiply it back with z

(1) 2.27 i)

Find the general term of the sequence given by

$x_{n+1} - 2x_n = n$, $x_0 = 0$. $\qquad x_n = x(n)$, $x : \mathbb{N} \to \mathbb{C}$

Find the general term of the sequence given by

$$x_{n+1} - 2x_n = n \,\Big/\, \mathcal{Z}, \quad x_0 = 0.$$

$x_n = x(n)$, $x: \mathbb{N} \to \mathbb{C}$

$x(n+1) - 2x(n) = n$

$$\mathcal{Z}[f(n+p)](z) = z^p F(z) - z^p f(0) - z^{p-1} f(1) - \dots - z f(p-1)$$

We denote $\mathcal{Z}[x_n] = F(z)$

$$\mathcal{Z}[x_{n+1}] - 2\mathcal{Z}[x_n] = \mathcal{Z}[n]$$

$$\mathcal{Z}[x_{n+1}] = \mathcal{Z}[x(n+1)] \underset{p=1}{=} zF(z) - z\underbrace{x_0}_{=0} = zF(z)$$

$f(0) = x_0$

$$zF(z) - 2F(z) = \frac{z}{(z-1)^2} \Rightarrow F(z)\cdot(z-2) = \frac{z}{(z-1)^2} \Rightarrow F(z) = \frac{z}{(z-1)^2\cdot(z-2)} \,\Big/\, \mathcal{Z}^{-1}$$

$$\Rightarrow x_n = \mathcal{Z}^{-1}\left[\frac{z}{(z-1)^2(z-2)}\right]$$

Method I. $x_n = \frac{1}{2\pi i}\int_C z^{n-1}\cdot\frac{z}{(z-1)^2(z-2)}\,dz = \frac{1}{2\pi i}\int_C \underbrace{\frac{z^n}{(z-1)^2(z-2)}}_{f(z)}\,dz \underset{\text{Res. th}}{=}$

$$= \frac{1}{\cancel{2\pi i}}\cdot\cancel{2\pi i}\sum_{k=1}^{m}\operatorname{Res}_{z=z_k} f(z) = \operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=2} f(z)$$

$$\operatorname{Res}_{z=1} f(z) = \frac{1}{1!}\cdot\lim_{z\to1}\left[\cancel{(z-1)^2}\,\frac{z^n}{\cancel{(z-1)^2}\cdot(z-2)}\right]' = \lim_{z\to1}\frac{nz^{n-1}\cdot(z-2) - z^n}{(z-2)^2} = \frac{n(-1)-1}{1} =$$

$z=1$ pole of order 2

$$= -n-1$$

$$\operatorname{Res}_{z=2} f(z) = \frac{1}{1}\lim_{z\to2}\left(\cancel{(z-2)}\cdot\frac{z^n}{(z-1)^2\cancel{(z-2)}}\right) = \frac{2^n}{1} = 2^n$$

$z=2$ pole of order 1

$$\Rightarrow x_n = -n-1+2^n$$

Method II. $F(z) = \frac{z}{(z-1)^2(z-2)}$

– We use partial fraction decomposition for $\frac{F(z)}{z}$

$$\frac{F(z)}{z} = \frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

$z=0 \Rightarrow 1 = 2A - 2B + C \Longrightarrow 1 = 2A + 2 + 1 \Rightarrow A = -1$

$z=1 \Rightarrow 1 = -B \Rightarrow B = -1$

$z=2 \Rightarrow 1 = C \Rightarrow C = 1$

$$\frac{F(z)}{z} = -\frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{z-2} \Big/ \cdot z \quad \Rightarrow F(z) = \frac{-z}{z-1} - \frac{z}{(z-1)^2} + \frac{z}{z-2}$$

$$\Rightarrow x_n = \mathcal{Z}^{-1}\left[-\frac{z}{z-1} - \frac{z}{(z-1)^2} + \frac{z}{z-2}\right] \Rightarrow \underline{\underline{x_n = -1 - n + 2^n}}$$

② (2.27 iv))

Find the sequence x_n such that $x_{n+2} + 4x_{n+1} + 3x_n = 3^n \big/ \mathcal{Z}$, $x_0 = 0$, $x_1 = 1$

We denote by : $F(z) = \mathcal{Z}[x_n]$

$$\mathcal{Z}[x_{n+2}] \underset{\underset{p=2}{\uparrow}}{=} z^2 F(z) - z^2 \underset{\underset{0}{\shortparallel}}{x_0} - z \underset{\underset{1}{\shortparallel}}{x_1} = z^2 F(z) - z \quad ; \quad \mathcal{Z}[3^n] = \frac{z}{z-3}$$

$$\mathcal{Z}[x_{n+1}] \underset{\underset{p=1}{\uparrow}}{=} z F(z) - z \underset{\underset{0}{\shortparallel}}{x_0} = zF(z)$$

$$\Rightarrow z^2 F(z) - z + 4zF(z) + 3F(z) = \frac{z}{z-3}$$

$$F(z)(z^2 + 4z + 3) = \frac{z}{z-3} + \overset{\frac{z-3}{}}{z} \Rightarrow F(z)\underbrace{(z^2+4z+3)}_{(z+1)(z+3)} = \frac{z^2-2z}{z-3}$$

$$\Rightarrow F(z) = \frac{z(z-2)}{(z-3)(z+3)(z+1)}$$

$$\frac{F(z)}{z} = \frac{z-2}{(z-3)(z+3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+3} + \frac{C}{z+1}$$

$$z-2 = A(z+3)(z+1) + B(z-3)(z+1) + C(z-3)(z+3)$$

$z=-3 \Rightarrow -5 = B(-6)(-2) \Rightarrow B = -\frac{5}{12}$

$z=-1 \Rightarrow -3 = C(-4)\cdot 2 \Rightarrow C = \frac{3}{8}$

$z=3 \Rightarrow 1 = A\cdot 6\cdot 4 \Rightarrow A = \frac{1}{24}$

$$\frac{F(z)}{z} = \frac{1}{24}\cdot\frac{1}{z-3} - \frac{5}{12}\cdot\frac{1}{z+3} + \frac{3}{8}\cdot\frac{1}{z+1} \Big/ \cdot z$$

$$F(z) = \frac{1}{24}\,\frac{z}{z-3} - \frac{5}{12}\,\frac{z}{z+3} + \frac{3}{8}\cdot\frac{z}{z+1} \quad \Big/ \mathcal{Z}^{-1}$$

$$x_n = \frac{1}{24}3^n - \frac{5}{12}(-3)^n + \frac{3}{8}\cdot 1$$

(3) Find the general form of the sequence given by

$$x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n = 1 \;\Big/ \mathcal{Z}, \quad x_0 = x_1 = x_2 = 0$$

We denote by $F(z) = \mathcal{Z}[x_n]$

$$\mathcal{Z}[x_{n+3}] \underset{p=3}{=} z^3 F(z) - z^3 \underbrace{x_0}_{0} - z^2 \underbrace{x_1}_{0} - z \underbrace{x_2}_{0} = z^3 F(z)$$

$$\mathcal{Z}[x_{n+2}] \underset{p=2}{=} z^2 F(z) - z^2 \underbrace{x_0}_{0} - z \underbrace{x_1}_{0} = z^2 F(z)$$

$$\mathcal{Z}[x_{n+1}] \underset{p=1}{=} z F(z) - z \underbrace{x_0}_{0} = z F(z)$$

We replace in the recurrence relation.

$$z^3 F(z) - 3z^2 F(z) + 3z F(z) - F(z) = \frac{z}{z-1}$$

$$F(z)\cdot\underbrace{(z^3 - 3z^2 + 3z - 1)}_{(z-1)^3} = \frac{z}{z-1} \implies F(z) = \frac{z}{(z-1)^4}$$

↪ we have to use Method I

$$x_n = \mathcal{Z}^{-1}\left[\frac{z}{(z-1)^4}\right] = \frac{1}{2\pi i}\int_C z^{n-1}\frac{z}{(z-1)^4}\,dz = \frac{1}{2\pi i}\int_C \underbrace{\frac{z^n}{(z-1)^4}}_{f(z)}\,dz =$$

$$\overset{\text{Res. Th}}{=} \frac{1}{\cancel{2\pi i}}\,\cancel{2\pi i}\,\operatorname*{Res}_{z=1} f(z)$$

$$\operatorname*{Res}_{z=1} f(z) \underset{\text{pole of order 4}}{=\!=} \frac{1}{(4-1)!}\lim_{z\to 1}\left[\cancel{(z-1)^4}\,\frac{z^n}{\cancel{(z-1)^4}}\right]''' = \frac{1}{6}\lim_{z\to 1} n(n-1)(n-2)z^{n-3} = \frac{n(n-1)(n-2)}{6}$$

(4) Find the sum of the series $s = \sum_{n=0}^{\infty}\frac{n^2-3n+5}{6^n}$

We write the definition of the $\mathcal{Z}$ transform

$$\boxed{F(z) = \sum_{n=0}^{\infty}\frac{f(n)}{z^n}}$$

$$= \sum^{\infty} f(n)$$

$$\boxed{F(z)=\sum_{n=0}^{\infty}\frac{f(n)}{z^n}}$$

$$F(3)=\sum_{n=0}^{\infty}\frac{f(n)}{3^n}$$

$$s=\sum_{n=0}^{\infty}\frac{n^2}{6^n}-3\sum_{n=0}^{\infty}\frac{n}{6^n}+5\sum_{n=0}^{\infty}\frac{1}{6^n}=$$

$$=\frac{z(z+1)}{(z-1)^3}\Bigg|_{z=6}-3\frac{z}{(z-1)^2}\Bigg|_{z=6}+5\frac{z}{z-1}\Bigg|_{z=6}=\frac{6\cdot 7}{\underset{5}{125}}-3\frac{6}{\underset{25}{25}}+5\frac{6}{5}=$$

$$=\frac{42-\cancel{90}+750}{125}=\frac{750-48}{125}=\frac{18\frac{3}{5}}{90}$$

$$=\frac{702}{125}$$

$$f(n)=n^2\Longrightarrow \mathcal{Z}[n^2]=\frac{z(z+1)}{(z-1)^3}$$

$$f(n)=n\longrightarrow \mathcal{Z}[n]=\frac{z}{(z-1)^2}$$

$$f(n)=1\longrightarrow \mathcal{Z}[1]=\frac{z}{z-1}$$

(5) Find the sum of the series $s=\sum_{n=0}^{\infty}\frac{1}{2^n}\cos\frac{\pi n}{3}$

$$F(z)=\sum_{n=0}^{\infty}\frac{f(n)}{z^n}$$

$$s=\mathcal{Z}\left[\cos\frac{n\pi}{\underset{a}{3}}\right](2)=\frac{z\left(z-\cos\frac{\pi}{3}\right)}{z^2-2z\underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}}+1}\Bigg|_{z=2}=\frac{z\left(z-\frac{1}{2}\right)}{z^2-z+1}\Bigg|_{z=2}=\frac{2\cdot\left(\overset{2}{2}-\frac{1}{2}\right)}{4-2+1}=$$

$$=\frac{2\cdot\frac{3}{2}}{3}=1$$

* (6) Find the sum of the series $s=\sum_{n=0}^{\infty}\frac{n}{4^n}\cdot\left(\sum_{k=0}^{n}\frac{1}{k!}\right)$

$$F(z)=\sum_{n=0}^{\infty}\frac{f(n)}{z^n}\Longrightarrow s=\mathcal{Z}\left[n\cdot\sum_{k=0}^{n}\frac{1}{k!}\right](4)$$

- $\mathcal{Z}\left[\frac{1}{n!}\right](z)\underset{\substack{\uparrow\\ \text{we use the}\\ \text{definition}}}{=}f(0)+\frac{f(1)}{z}+\frac{f(2)}{z^2}+\dots+\frac{f(n)}{z^n}+\dots\underset{\substack{\uparrow\\ f=\frac{1}{n!}}}{=}1+\frac{1}{1!\,z}+\frac{1}{2!\,z^2}+\dots+\frac{1}{n!\,z^n}+\dots$

$$=e^{\frac{1}{z}}$$

- $\mathcal{Z}\left[\sum_{k=0}^{n}\frac{1}{k!}\right]\underset{\uparrow}{=}\frac{z}{z-1}\,\mathcal{Z}\left[\frac{1}{n!}\right]=\frac{z}{z-1}\cdot e^{\frac{1}{z}}$

• $\mathcal{Z}\left[\sum_{k=0} \frac{1}{k!}\right] = \frac{z}{z-1}\, \mathcal{Z}\left[\frac{1}{n!}\right] = \frac{z}{z-1} \cdot e$

$\mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right] = \frac{z}{z-1} \mathcal{Z}[f]$

• $\mathcal{Z}\left[n \cdot \sum_{k=0}^{n} \frac{1}{k!}\right] = -z \cdot \left(\frac{z}{z-1} \cdot e^{\frac{1}{z}}\right)' = -z\left[\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2}\, e^{\frac{1}{z}} + \frac{\cancel{z}}{z-1} \cdot e^{\frac{1}{z}} \cdot \left(-\frac{1}{z^{\cancel{2}}}\right)\right]$

$\mathcal{Z}[n \cdot f(n)] = -z \cdot F'(z)$

$= -z \cdot e^{\frac{1}{z}}\left[\frac{-1}{(z-1)^2} - \frac{1}{z(z-1)}\right]$

$\Rightarrow S = -z e^{\frac{1}{z}}\left[-\frac{1}{(z-1)^2} - \frac{1}{z(z-1)}\right]\Bigg/_{z=4} = 4e^{\frac{1}{4}}\left(\frac{4}{\frac{1}{9}} + \frac{3}{\frac{1}{12}}\right) = \frac{4+3}{3\cancel{6}_{9}} \cdot \cancel{4} e^{\frac{1}{4}} =$

$= \frac{7}{9} e^{\frac{1}{4}}$

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$S = \sum_{n=0}^{\infty} \frac{1}{3^n} \sum_{k=0}^{n} \frac{k}{2^k} = \mathcal{Z}\left[\sum_{k=0}^{\infty} \frac{k}{2^k}\right](3)$

$F(z) = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$

• $\mathcal{Z}\left[n \cdot \left(\frac{1}{2}\right)^n\right] = -z\left(\mathcal{Z}\left(\frac{1}{2}\right)^n\right)' = -z\left(\frac{z}{z-\frac{1}{2}}\right)' = -z\, \frac{\cancel{z}-\frac{1}{2}-\cancel{z}}{\left(z-\frac{1}{2}\right)^2} = \frac{z}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{(2z-1)^2}$

The general form is $n \cdot \left(\frac{1}{2}\right)^n$

$\mathcal{Z}[n \cdot f(n)] = -z F'(z)$

$= \frac{2z}{(2z-1)^2}$

• $\mathcal{Z}\left[\sum_{k=0}^{n} \frac{k}{2^k}\right](z) = \frac{z}{z-1} \cdot \frac{2z}{(2z-1)^2}$

$\mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right] = \frac{z}{z-1} \mathcal{Z}[f]$

we replace $z = 3$

$\Rightarrow S = \frac{3}{\cancel{2}} \cdot \frac{\cancel{2} \cdot 3}{25} \Rightarrow S = \frac{9}{25}.$

Homework. 1) Find the sum $S = \sum_{n=0}^{\infty} \frac{n \cdot \sin \frac{n\pi}{2}}{2^n}$ $R = \frac{6}{25}$

2) Book