Seminar 1
Complex numbers

The number

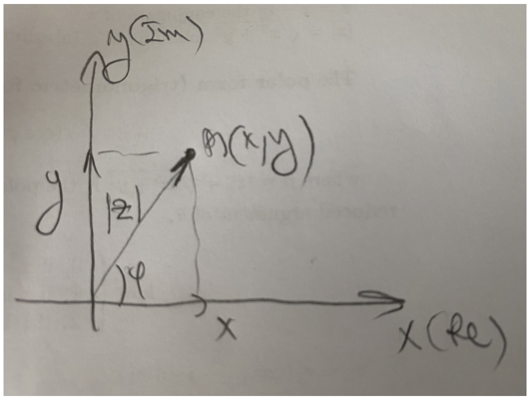
$$z = x + iy$$

is called a complex number, where $x, y \in \mathbb{R}$, $x = \operatorname{Re} z,\ y = \operatorname{Im} z$, and i is the imaginary unit ($i^2 = -1$)

$z = x + iy$ is called the **algebraic form of a complex number.**

$\overline{z} = x - iy$ the conjugate of z

$|z| = \sqrt{x^2 + y^2}$ modulus of z (absolute value, magnitude)



The **polar form (trigonometric form)** of a complex number

$$z = r(\cos\varphi + i\sin\varphi)$$

where $\rho = |z| = \sqrt{x^2 + y^2}$ is the polar radius and $\varphi = \arctan\dfrac{y}{x} + k\pi$ is the reduced argument of z,

$$k = \begin{cases} 0, & \text{if } z \text{ is in Q1} \\ 1, & \text{if } z \text{ is in Q2 or Q3} \\ 2, & \text{if } z \text{ is in Q4} \end{cases}$$

$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$

$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 \left[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}\left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right] \\ z^n &= r^n(\cos n\varphi + i\sin n\varphi) \end{aligned}$$

The roots of order n of a complex number

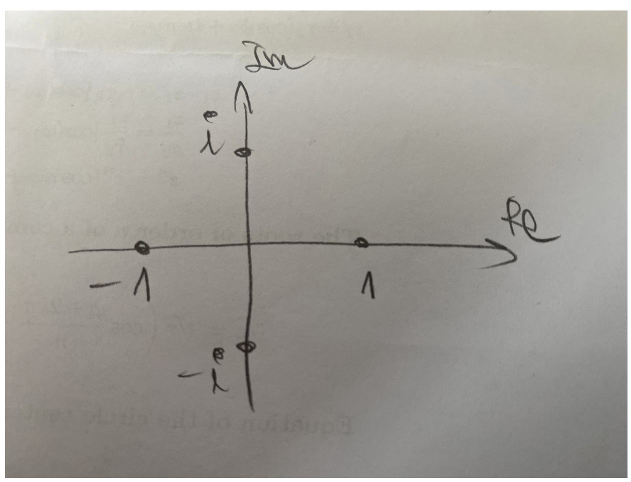
$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right),\ k=\overline{0,n-1}.$$

Equation of the circle centered at z_0 of radius R

$$|z-z_0| = R$$

1. Represent the complex numbers in trigonometric form:
a) $z_1 = 1, z_2 = -1, z_3 = i, z_4 = -i$; b) $z_5 = \dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$; $c) z_6 = (-1+i)^{105}$.
a) $|z_1| = |z_2| = |z_3| = |z_4| = 1$

$$\begin{aligned} 1 &= \cos 0 + i\sin 0 \\ -1 &= \cos\pi + i\sin\pi \\ i &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ -i &= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} \end{aligned}$$



b) $|z_5| = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$, $\arg z_5 = \arctan\dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} + 0 = \dfrac{\pi}{6}$

$$\frac{\sqrt{3}}{2} + i\frac{1}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$

c) $|z_6| = \sqrt{1+1} = \sqrt{2}$, $\arg z_6 = \arctan \frac{1}{-1} + \pi = -\dfrac{\pi}{4} + \pi = \dfrac{3\pi}{4}$

$$\begin{aligned}
(-1+i)^{105} &= \left(\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right)^{105} \\
&= \sqrt{2}^{105}\left(\cos\frac{315\pi}{4} + i\sin\frac{315\pi}{4}\right) \\
&= \sqrt{2}^{105}\left(\cos\left(78\pi + \frac{3\pi}{4}\right) + i\sin\left(78\pi + \frac{3\pi}{4}\right)\right) \\
&= \sqrt{2}^{105}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)
\end{aligned}$$

2. Solve the following equations
a) $z^4 = 1 - i$; $b) z^6 = 1; c) z^3 - 8 = 0$; d) $z^4 + (1-i)z^2 - i = 0$.
a)
$|1-i| = \sqrt{2}$, $\arg\varphi = \arctan(-1) + 2\pi = \frac{-\pi}{4} + 2\pi = \dfrac{7\pi}{4}$
$1 - i = \sqrt{2}\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$

$$z_k = \sqrt[4]{\sqrt{2}}\left(\cos\frac{\frac{7\pi}{4} + 2k\pi}{4} + i\sin\frac{\frac{7\pi}{4} + 2k\pi}{4}\right),\ k = 0, 1, 2, 3$$

$$\begin{aligned}
z_0 &= \sqrt[8]{2}\left(\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}\right), \\
z_1 &= \sqrt[8]{2}\left(\cos\frac{15\pi}{16} + i\sin\frac{15\pi}{16}\right), \\
z_2 &= \sqrt[8]{2}\left(\cos\frac{23\pi}{16} + i\sin\frac{23\pi}{16}\right), \\
z_3 &= \sqrt[8]{2}\left(\cos\frac{31\pi}{16} + i\sin\frac{31\pi}{16}\right).
\end{aligned}$$

$b) z^6 = \cos 0 + i\sin 0$

$z_k = \cos\dfrac{2k\pi}{6} + i\sin\dfrac{2k\pi}{6} = \cos\dfrac{k\pi}{3} + i\sin\dfrac{k\pi}{3}$, $k = 0, 1, ..., 5$

We can write the algebraic form

$$z_0 = \cos 0 + i \sin 0 = 1$$
$$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$z_3 = \cos\pi + i\sin\pi = -1$$
$$z_4 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$
$$z_5 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

c) $z^3 - 8 = 0 \Rightarrow z^3 = 8$
$8 = r(\cos\varphi + i\sin\varphi)$
$r = 8$
$\varphi = 0$
$8 = 8(\cos 0 + i \sin 0)$
$z_k = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \ k = \overline{0, n-1}.$
$z_k = \sqrt[3]{8}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right), \ k = \overline{0,2}.$
$z_k = 2\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right), \ k = \overline{0,2}.$
$z_0 = 2$
$z_1 = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$
$z_2 = 2(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$

d) $z^4 + (1-i)z^2 - i = 0$

$z^2 = t \Rightarrow t^2 + (1-i)t - i = 0$
$\Delta = (1+i)^2$
$t_1 = i, \ t_2 = -1$
We have to solve: $z^2 = i, \ z^2 = -1.$

3)Specify and draw the curves represented by the following equations, with $z \in \mathbb{C}:$
a) $|z+1-2i| = 3;$ *b*) $|z+c| + |z-c| = 2a, \ a > c > 0;$
c)$(2-i)z + (2+i)\overline{z} + 3 = 0;$ *d*) $z + \overline{z} = 2;$ *e*) $\operatorname{Im}(z+i) = |z|.$

a) Let $z = x + iy, \ x, y \in \mathbb{R}.$

$$|x + iy + 1 - 2i| = 3$$
$$|x + 1 + i(y-2)| = 3$$
$$\sqrt{(x+1)^2 + (y-2)^2} = 3$$
$$(x+1)^2 + (y-2)^2 = 9$$

which is a circle centered at $(-1, 2)$, of radius 3.

b)

$$\begin{aligned}
\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a-\sqrt{(x-c)^2+y^2} \\
x^2+2cx+c^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \\
4xc &= 4a^2-4a\sqrt{(x-c)^2+y^2} | :4 \\
a\sqrt{(x-c)^2+y^2} &= a^2-xc|^2 \\
a^2\left(x^2-2cx+c^2+y^2\right) &= a^4-2xca^2+x^2c^2 \\
a^2x^2-2a^2cx+a^2c^2+a^2y^2 &= a^4-2xca^2+x^2c^2 \\
(a^2-c^2)x^2+a^2y^2 &= a^2(a^2-c^2)
\end{aligned}$$

We denote $b^2 := a^2-c^2$ and we divide the above equality by a^2b^2, so we obtain an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

c)

$$\begin{aligned}
(2-i)z+(2+i)\overline{z}+3 &= 0 \\
(2-i)(x+iy)+(2+i)(x-iy)+3 &= 0 \\
2x+2iy-ix+y+2x-2iy+ix+y+3 &= 0 \\
4x+2y+3 &= 0
\end{aligned}$$

The above equation represents a straight line.

4) Represent in the complex plane the numbers z:

a) $\mathrm{Re}(z-3i+2)=0$; b) $\mathrm{Re}\,\dfrac{z-i}{z+i}=1$; c) $|z-2+3i|=2$.

a) $z=x+iy \Rightarrow z-3i+2=x+iy-3i+2$

$z-3i+2=x+2+i(y-3)$

$\mathrm{Re}(z-3i+2)=0 \Rightarrow x+2=0 \Rightarrow x=-2$ - straight line

b) $z=x+iy \Rightarrow \dfrac{z-i}{z+i}=\dfrac{x+iy-i}{x+iy+i} \Rightarrow \dfrac{z-i}{z+i}=\dfrac{x+i(y-1)}{x+i(y+1)}$

$\Rightarrow \dfrac{z-i}{z+i}=\dfrac{[x+i(y-1)]\,[x-i(y+1)]}{[x+i(y+1)]\,[x-i(y+1)]} \Rightarrow \dfrac{z-i}{z+i}=\frac{x^2+xi(y-1)-xi(y+1)+y^2-1}{x^2+(y+1)^2}$

$\Rightarrow \dfrac{z-i}{z+i}=\frac{x^2+y^2-1-2xi}{x^2+(y+1)^2}$

$\mathrm{Re}\,\dfrac{z-i}{z+i}=1 \Rightarrow \frac{x^2+y^2-1}{x^2+(y+1)^2}=1 \Rightarrow x^2+y^2-1=x^2+(y+1)^2$

$y=-1$