Fundamental Algorithms

Lecture #7

Cluj-Napoca

November 13, 2019

Agenda

- **Red-Black Trees**
- **Disjoint Sets**

Red-Black trees

- Balanced trees
- Both insert/delete operations take O(lgn), with constant time for rebalancing

Def: is a BST with the following properties:

P_0: the root is **black**

P_1: each node is colored either **black** or **red**

P_2: each leaf (nil!) is **black**

P_3: both children of a **red** node are **black**

P_4: every path from any node to a leaf has the same **number of black nodes** (black height)

Red-Black trees

Theorem: A RB tree with n internal nodes has height at most $2\lg(n+1)$

Proof: Denote **bh(x)** – the black height (without x) of node x

Step 1: We will prove the following statement, denoted P(bh):

P(bh): $\forall x \in RBT$, the tree rooted by x has at least $2^{bh(x)}-1$ nodes

Induction:

P(0) $2^0-1=1$

Assume P(bh) true =>P(bh+1) true?

x has 2 children; each has the black height bh(x) (if x is red) or
bh(x)-1 (if x is black)

nb of internal nodes of x= nb of internal nodes of children(x) +1 (itself) =>

=> at least $(2^{bh(x)-1}-1)+(2^{bh(x)-1}-1)+1= 2^{bh(x)}-1$ nodes *q.e.d* (end of step 1)

Red-Black trees

Step 2:

We know P(bh) is true, ie

P(bh): $\forall x \in RBT$, the tree rooted by x has at least $2^{bh(x)}-1$ nodes (1)

By P_3 of RBT def (use contradiction to prove) $bh(x) \geq h/2$ (2)

//since after each red node comes a black one

$$\Rightarrow n \geq 2^{bh(x)}-1 \quad \text{(from (1))}$$

$$\geq 2^{h/2}-1 \quad \text{(from (2))}$$

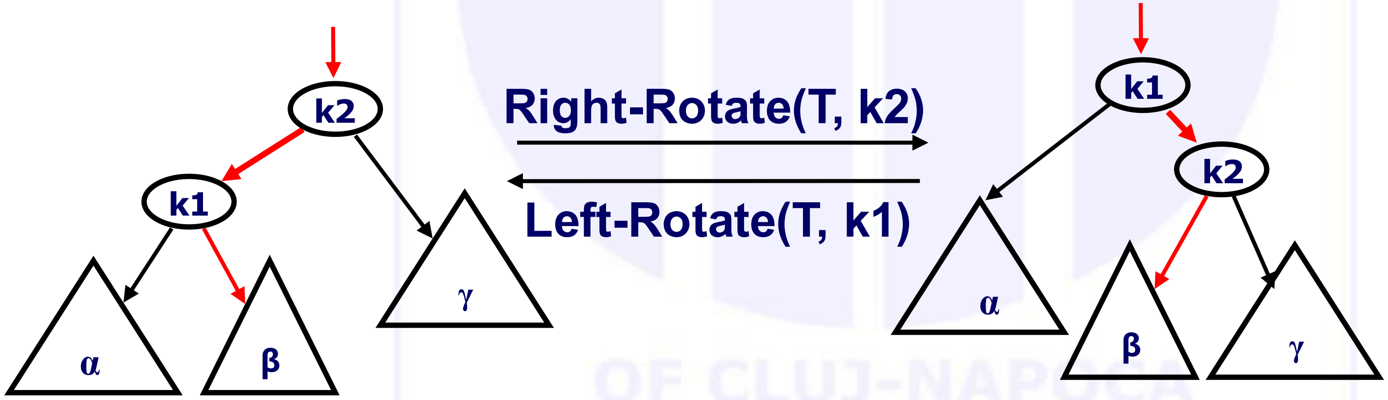
$$n \geq 2^{h/2}-1 \Leftrightarrow n+1 \geq 2^{h/2} \Leftrightarrow h/2 \leq \lg(n+1) \Leftrightarrow h \leq 2\lg(n+1)$$

(q.e.d., end of Theorem proof)

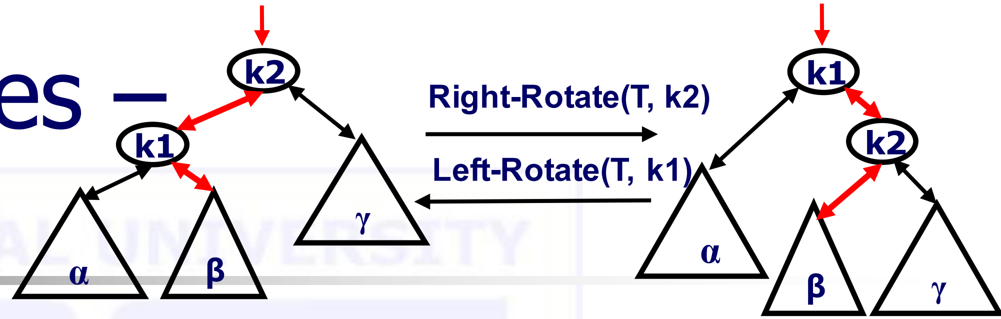
Red-Black trees - rotations

Similar to single rotations for AVL

They are symmetric



Red-Black trees – rotations




```
Left-Rotate(T,x)
//x root of rotation (points on k1)
y<-right[x]            //y saves k2
right[x]<-left[y]      //right of k1 goes on β
if left[y]<>nil        //if β exists = is not nil
      then p[left[y]]<-x  // β's parent becomes k1
p[y]<-p[x]             //k2's parent what was k1's parent
if p[y]=nil //k1 used to be the root of the tree
      then root[T]<-y
      else if x=left[p[x]]      // the parent of k1 becomes the parent of k2
                then  left[p[x]]<-y
                else  right[p[x]]<-y
left[y]<-x             //k1 goes the left child of k2
p[x]<-y                //k2 becomes the parent of k1
```

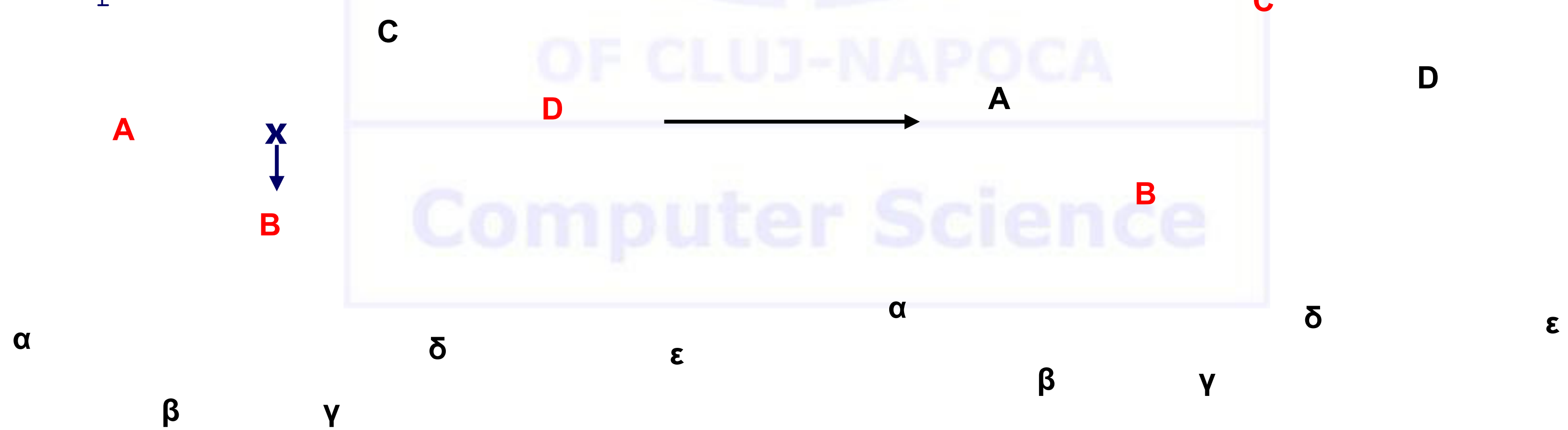
RB-INSERT

- **Insert** like in ANY other BST
  - As a LEAF, as for any other BST
- **Assign** it a color
  - RED (why?)
- Check the properties
- Re-balance if needed (**RB-INSERT-FIXUP** – check the textbook for the complete code)
- P_3 both children of a red node are black
  - True for the children (*nils*) of the inserted node
  - Not true for the inserted node in case its parent is RED
- 3 cases to analyze and remove inconsistencies

RB-INSERT-FIXUP - Case#1

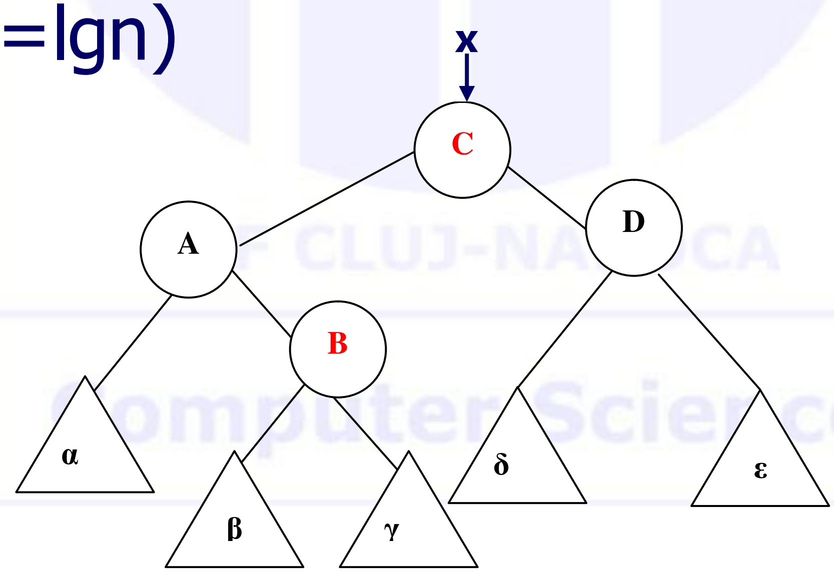
- **B – inserted node (pointed by x)**
- Parent (A)=RED, uncle (D)=RED, grandparent (C)=BLACK
- α, β, γ, δ, ε are RB trees (β, γ empty at first)
- Swap colors between grandparent (C) and parent/uncle

```
parent<-black          //no more P3 conflict A-B
uncle<-black           //to preserve P4
grandparent<-red       //to preserve P4
x<-grandparent
```



RB-INSERT-FIXUP - Case#1 - eval

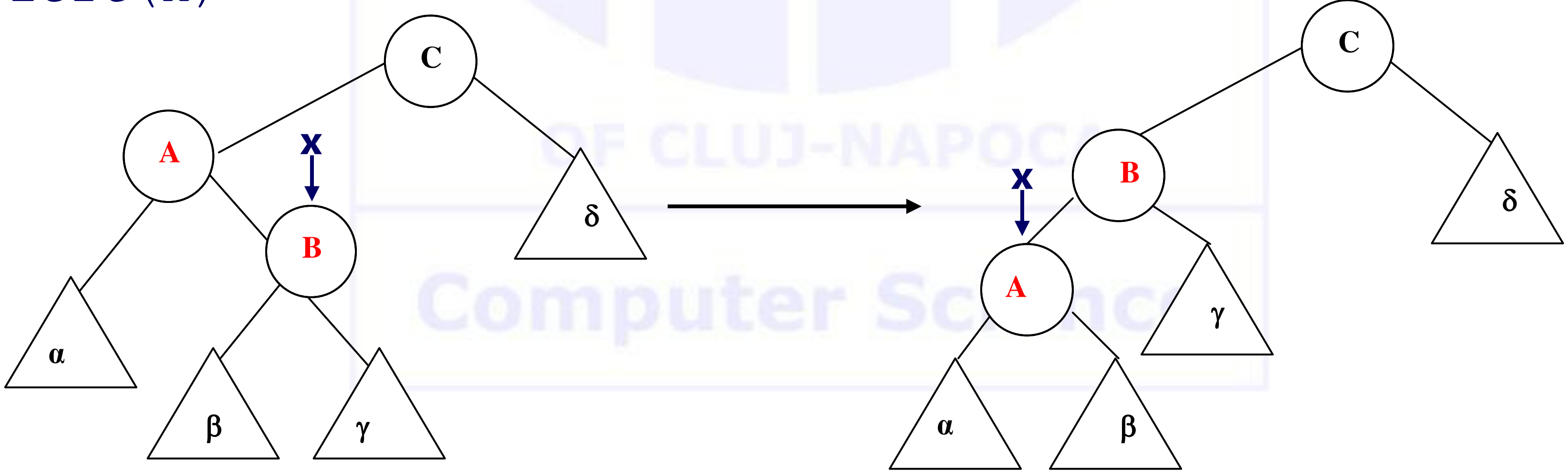
- P_3 may still be invalid, for the new x (i.e. C)
- Problem transferred 2 levels up in the tree (now β, γ not empty any longer)
- It takes (in the worst case) O(h) to rebalance (2lg(n+1)/2=lgn)



RB-INSERT-FIXUP - Case#2

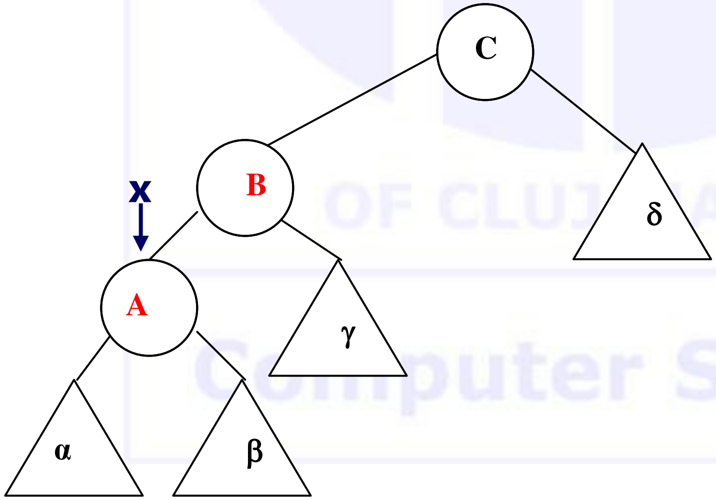
- **B – inserted node (pointed by x)**
- Parent(A)=RED, uncle (δ's root)=BLACK (here is the difference compared to case #1), grandparent (C)=BLACK
- α, β, γ, δ are RB trees; δ's root is BLACK

```
left_rotate(p(x))          //no more P3 conflict B-parent conflict
x<-left(x)
```



RB-INSERT-FIXUP - Case#2-eval

- Case #2 takes just O(1) to apply, but
- P_3 is still invalid, for the new x denoted node (i.e. A-B conflict)
- => case #3 to the rescue



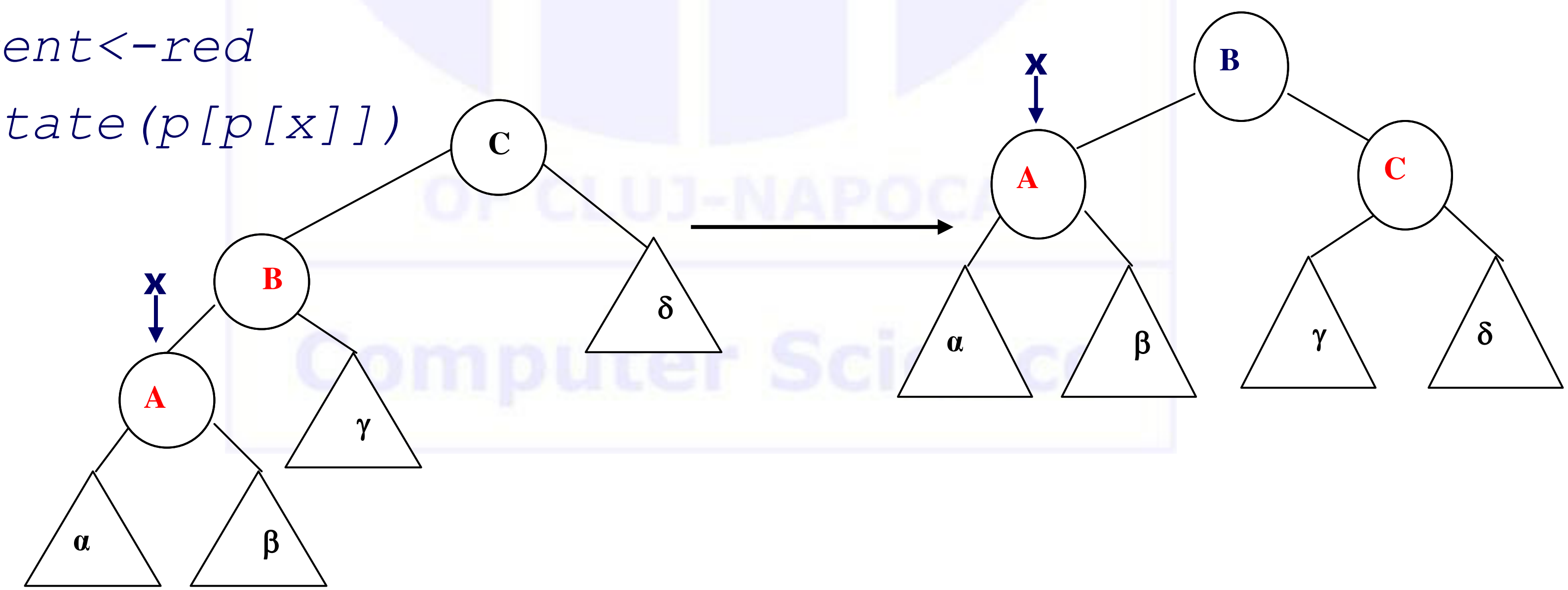
RB-INSERT-FIXUP - Case#3

- **Inserted A** /coming from #2 (node pointed by x)
- Parent (B)=RED, uncle (γ's root)= BLACK, grandparent (C)=BLACK
- α, β, γ, δ are RB trees

parent<-black

grandparent<-red

right_rotate(p[p[x]])



RB-INSERT-FIXUP - Case#3 - eval

- Problem solved
- Each individual case takes O(1)
- Case #1 may repeat (up in the tree)
- Case #2 is followed by #3
- Case #3 solves the problem

RB-insert – Rebalancing eval

- Case #1 repeats up to the root O(h)
- Case #2+#3 => problem fixed O(1)
- Case #3 => problem fixed O(1)
- Insert O(lgn) + rebalancing
  - Worst case: #1 repeats O(lgn)
  - Best case: #3 => 1 rotation O(1)
  - Other case: #2+3 =>2 rotations O(1)
- **O(lgn) overall worst time** (case 1 repeats), **at most 2 rotations** (case 2)

RB-DELETE

- Del as in regular BST + properties check to rebalance, if needed (RB-DELETE-FIXUP – check the textbook for the code)

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```
rb_delete(T,z)
   tree_delete(T,z)
   if color[y]=black
      then RB-DELETE-FIXUP(T,x)
```

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P4 (black height) is an issue!!

```
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   tree_delete(T,z)
   if color[y]=black
      then RB-DELETE-FIXUP(T,x)
```

z=node to be removed (see picture on the blackboard)

y=node actually removed (y≡z in case z has at most 1 child); info in y is placed in z's node

x=y's only child before the delete process takes place (could be nil, in case y has no children). After y is deleted, x becomes the child of y's parent (thus, x's parent could have now both children, one being x)

w=x's brother (after delete operation takes place; it's y's brother before the deletion)

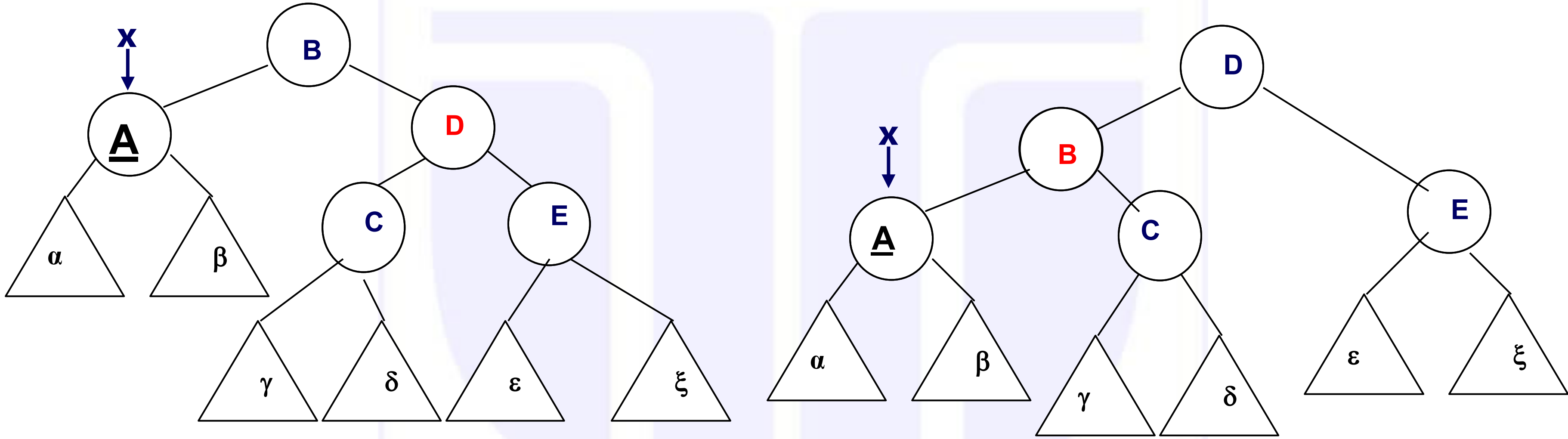
RB-DELETE-FIXUP

On x's branch check P4 property

x is y's (the removed node) only child

```
if color[x]=red
     then color[x]<-black
               //problem fixed; DONE!
               //x brings its former father color
     else color[x]<-double_black
               //P1 property issue!
```

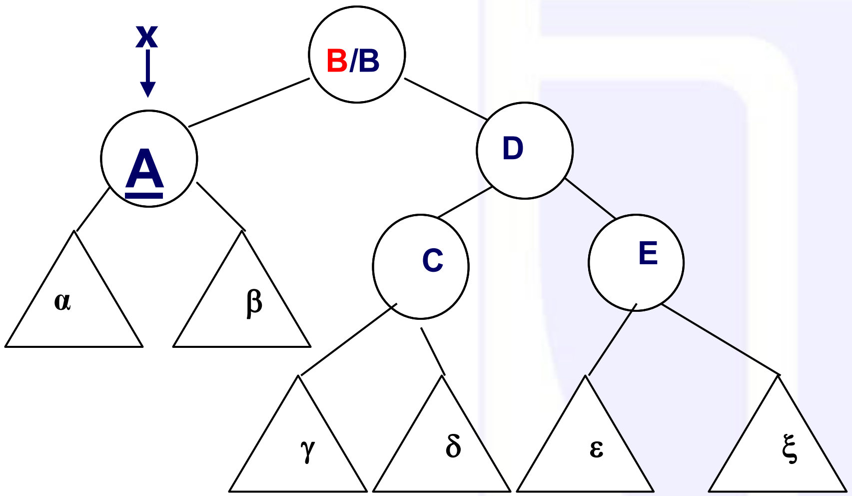
RB-DELETE-FIXUP - Case#1

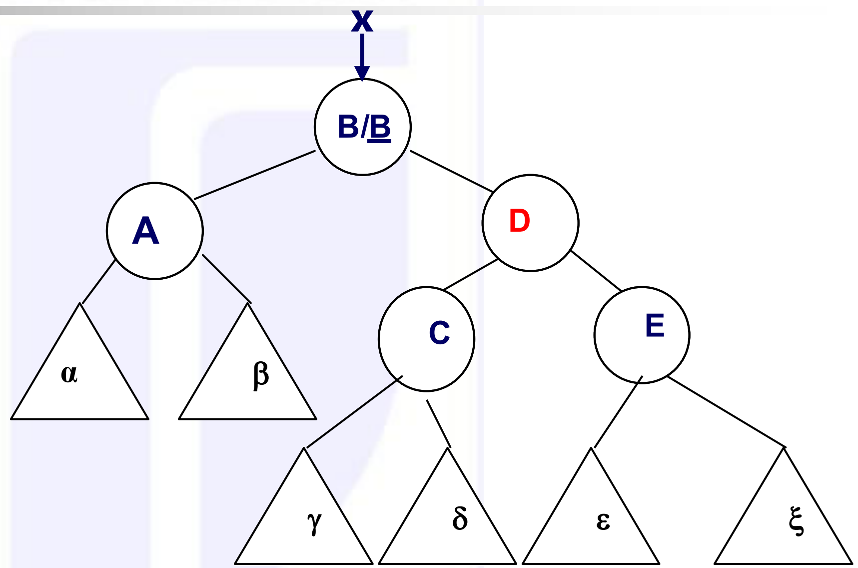


- Issue at node **A** (pointed by x) which is **double black**!
- A = was the only child of the deleted node
- Parent (**B**) = **Black**, brother-w(**D**) =**Red**
- α, β, γ, δ, ε, ξ are RB trees
- B<->D color interchange +left rotate=> case 2 or 3 or 4 (w is black, differ w's children)

```
parent[x]<-red
brother[x]<-black       //colors interchanged
left_rotate(p[x])
```

RB-DELETE-FIXUP - Case#2

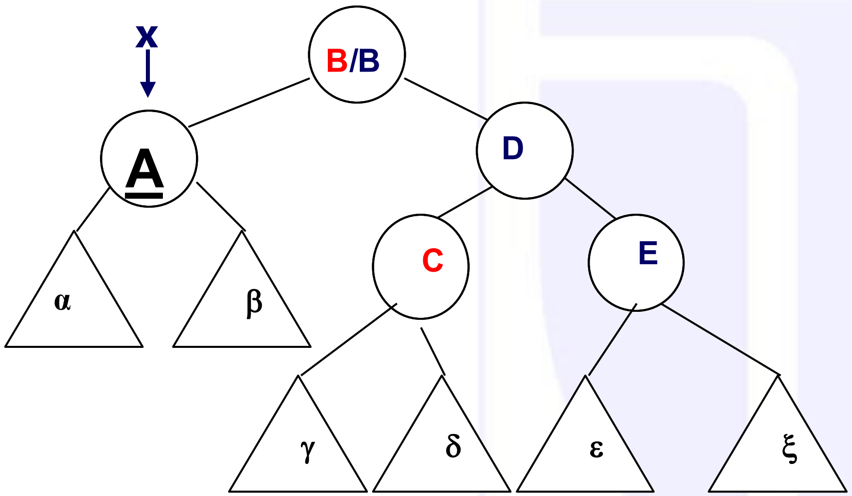


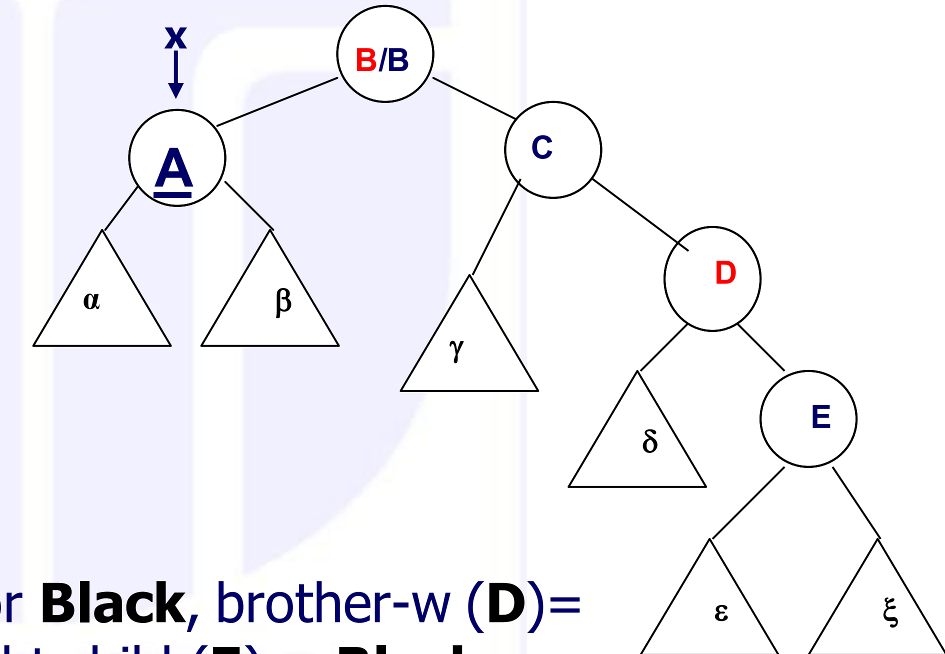


Parent (**B**)=**Red** (if after case #1) or **Black**, brother-w (**D**)=**Black**, both children of w (**C** and **E**) are **Black**

```
brother[x]<-red
if p[x]=red
   then p[x]<-black          //when case#2 comes after case#1; problem solved
   else p[x]<-double_black
        x<-p[x]  //the same problem as at the beginning of case #2, just 1 level
                 above; case 2 repeats; in lgn problem solved
```

RB-DELETE-FIXUP - Case#3

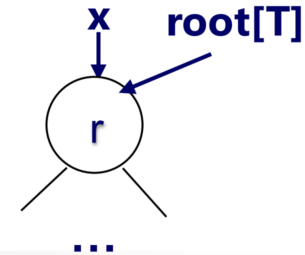


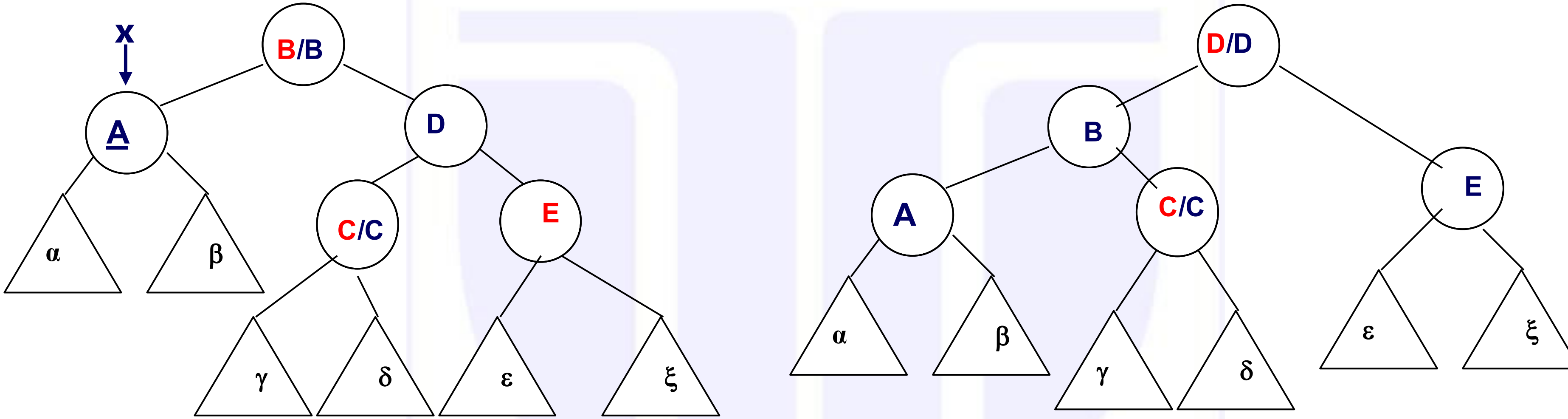


- Parent (**B**)=**Red** (if after case #1) or **Black**, brother-w (**D**)= **Black**, w's left child (**C**) = **Red**, w's right child (**E**) = **Black**
- α, β, γ, δ, ε, ξ are RB trees
- A=child of the deleted node (double Black A (pointed by x))
- C<->D color interchange +right rotate => **case 4**

```
brother[x]<-red
left[brother[x]]<-black
right_rotate(brother[x])
```

RB-DELETE-FIXUP - Case#4





- Parent(**B**)=**Red** (if after case #1) or **Black**, brother(**D**)=**Black**
- Node C is either **Red** or **Black**; node **E** is **Red**
- α, β, γ, δ, ε, ξ are RB trees
- A=child of the deleted node (double Black <u>A</u> (pointed by x))
- B<->D color interchange +left rotate, color E **Black** => probl. solved

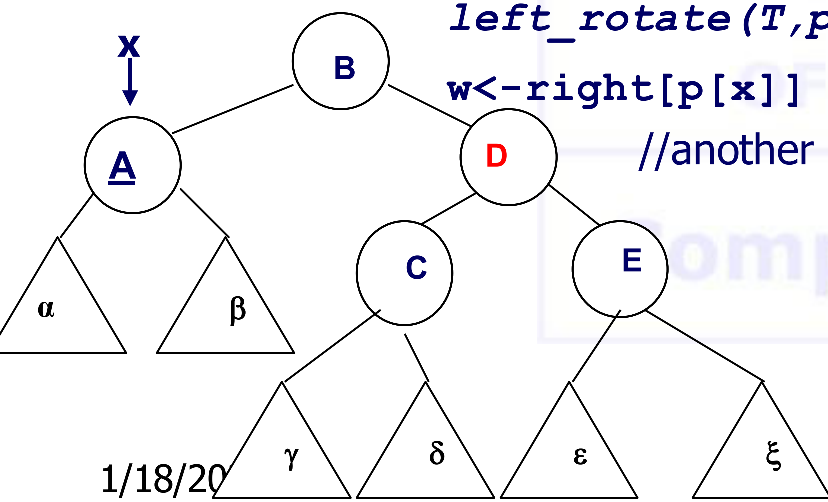
```
brother[x]<-color[parent[x]]
parent[x]<-black
left_rotate(p[x])      //1 more black node on x's branch
color[right[[w]]<-black
x<-root[T]
```

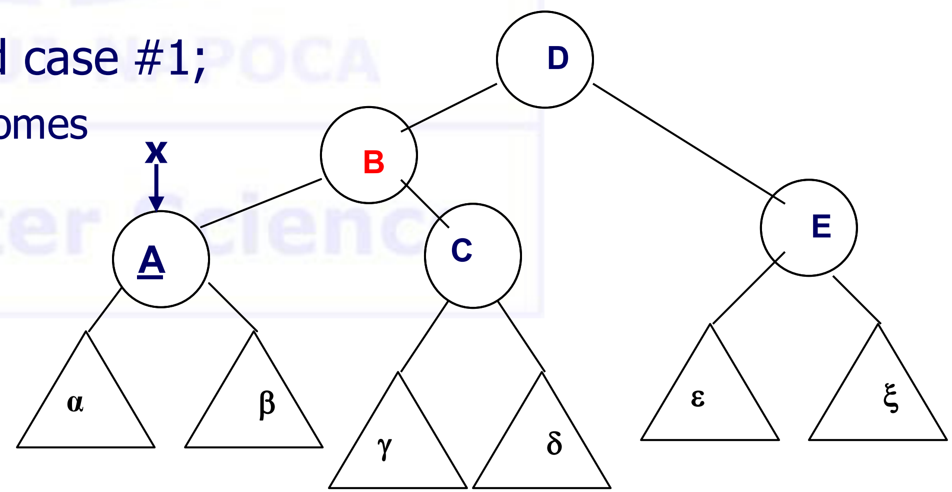
RB-del – Rebalancing eval

- Case #1 rotation followed by any other case
  - 1+2 => problem solved O(1)
  - 1+3+4=> problem solved O(1)
  - 1+4=> problem solved O(1)
- Case #2 (**no rotation,** only recoloring) repeats 1 level up in the tree
  - Worst case O(lgn)
  - Best case O(1)
- Case #3 rotation followed by case #4 O(1)
- Case #4 rotation; solves the problem O(1)
- Delete O(lgn) + rebalancing
  - Worst case: #2 repeats (recoloring only) O(lgn)
  - Best case: #4=> 1 rotation O(1)
  - Other cases: #1+2 or 1+ 3+4=>2 or 3 rotations O(1)
- O(lgn) overall, at most 3 rotations

RB-DELETE-FIXUP - procedure

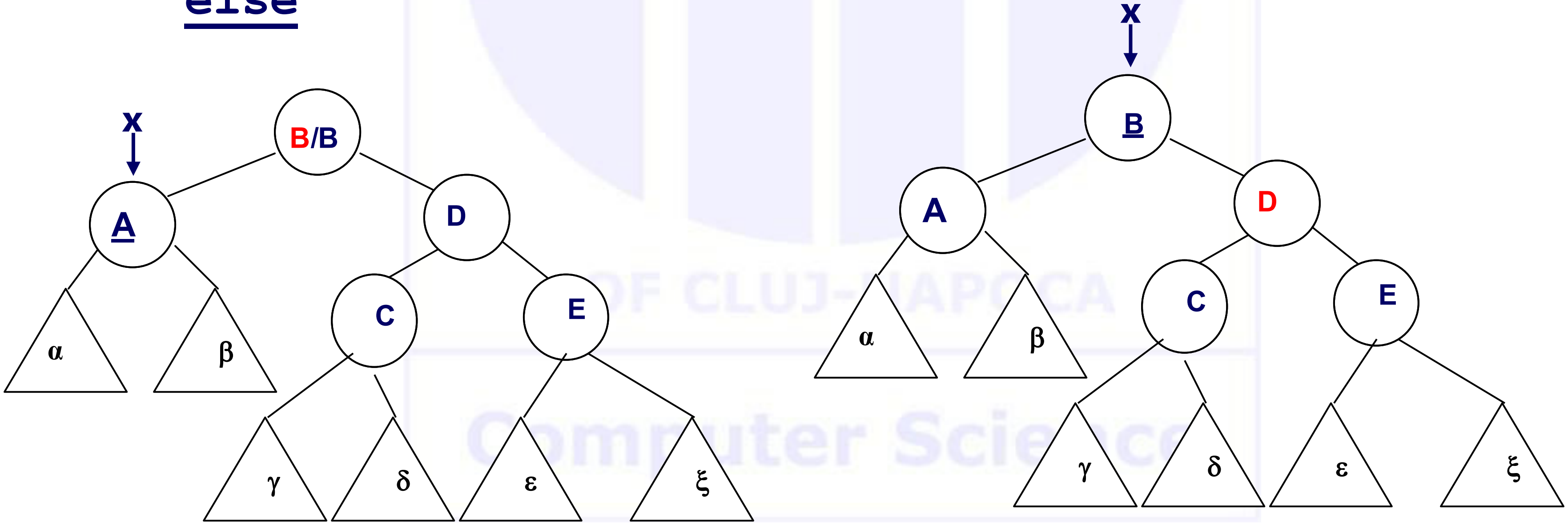
```
RB-DELETE-FIXUP(T,x)
while x<>root[T] and color[x]=black
do
   if x=left[p[x]]                 //cases on the left
   then                            //else case symmetric on the right; not discussed
        w<-right[p[x]]             //w=x's brother
        if color[w]=red
        then    color[w]<-black            //case #1 APPLY ; coloring+rotation
                color[p[x]]<-red
                left_rotate(T,p[x])
                w<-right[p[x]]  //end case #1;
                          //another case comes
```





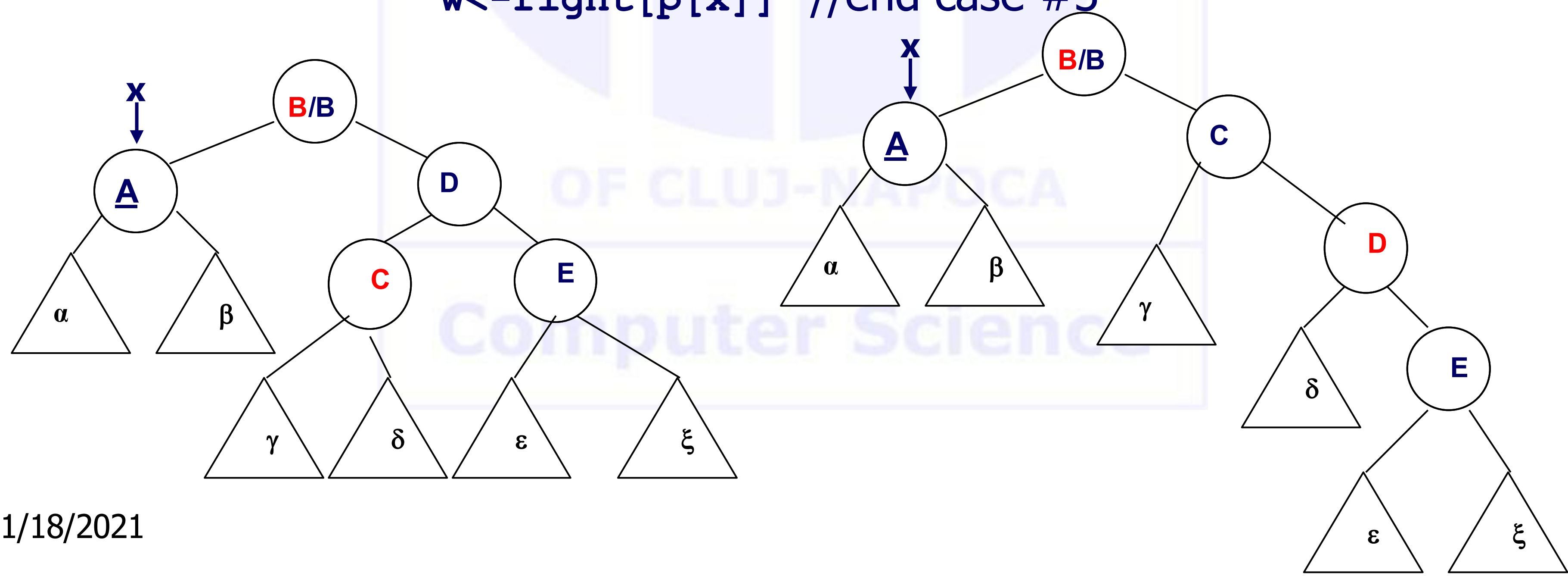
RB-DELETE-FIXUP – contd.

```
if color[left[w]]=black and color[right[w]]=black
    then                                    //case #2
        color[w]<-red
        x<-p[x]
    else
```



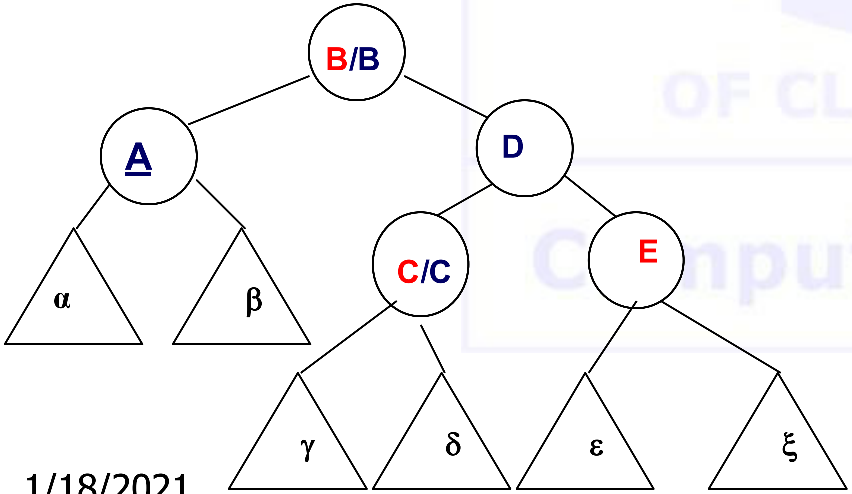
RB-DELETE-FIXUP – contd.

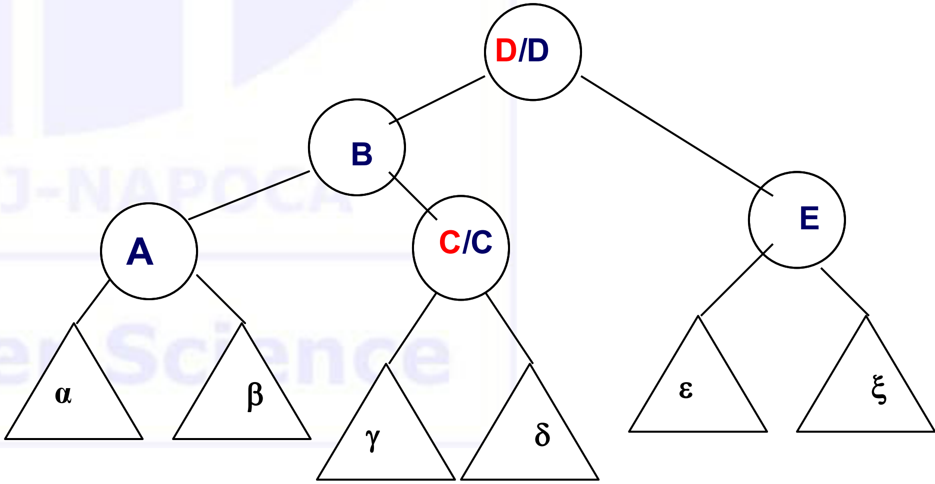
```
else            //color[left[w]]≠ black or color[right[w]] ≠black
    if color[right[w]]=black
    then                //case #3
        color[left[w]]<-black
        color[w]<-red
        right_rotate(T,w)
        w<-right[p[x]]  //end case #3
```



RB-DELETE-FIXUP – contd.

```
        color[w]<-color[p[x]]          //case #4
        color[p[x]]<-black
        color[right[[w]]<-black
        left_rotate(T,p[x])
        x<-root[T]
else    …          //x=right[p[x], all 4 cases symmetric to the right
color[x]<-black
```





Conclusions on balanced search trees

Tree	Height	Ins	Del
BST	[lgn, n]	O(h)	O(h)
RBT	[lgn, 2lgn]	2 rot	3 rot
AVL	[lgn, 1.45lgn]	1 rot	lgn rot
PBT	lgn	n rot	n rot

For RBT, at most lgn/2 color updates needed

Disjoint Sets

- Collection of dynamic DS $S=\{S_1, \ldots, S_k\}$
- $\exists$n elements (objects) in all k sets ($n \geq k$)
- each set S_i is identified by its *representative* element, $x \in S_i$;
- Basic operations:
  - **MAKE-SET** (x)

    Generates a new set, with a single element => n sets initially, each object has its own set, and it is its own representative el.
  - **UNION** (x, y)

    joins 2 disjoint sets, represented by x and y; builds $S_x \cup S_y$ (and destroys S_x and S_y); the representative becomes any of the 2 representatives;
  - **FIND-SET** (x)

    Returns a pointer to the representative element of the set containing element x.

Disjoint Sets – contd.

n = nb. of objects in the whole S

m = total nb. of operations (MAKE-SET, UNION, FIND-SET)

m>=n (as we have n MAKE-SET operations)

Utility/Applications:

- speeds up execution when we need to find/group items with similar features
- graphs (connected components; MST)
- many other

Disjoint Sets - implementation

- Linked List
- A set = a **linked list**
- representative= the first element (head) of the list
- An object in such a list contains
  - The element from the set;
  - The pointer to the next element in the list (LL)
  - Pointer to the representative (ex: blackboard)
- MAKE-SET(x) – builds a list with a single element O(1)
- FIND-SET(x) – returns the representative O(1)
- UNION(x, y) – adds x's list at the end of y's list;
  - representative = former y's representative
  - all x's elements have to update representative pointer (ex: blackboard)

Disjoint Sets – implementation – contd.

- Worst case: $O(m^2)$ for all operations
  - n MAKE-SET (1 for each element)
  - UNION
    - n times (to get to a single set)
    - $1 + 2 + 3 + \dots + n-1 = O(n^2)$ (show on the blackboard)
  - n~m (actually m>n, yet n is linear in m)
  - On average, O(m) for a call of UNION, m calls => $O(m^2)$

Disjoint Sets – implementation increase efficiency

- Update pointers for the shorter lists
- Keep as knowledge their length (similar to Order Statistic Trees)
- **Theorem**: For n objects in LL with weighted union, for m MAKE-SET, FIND-SET and UNION takes $O(m + n\lg n)$
- **Proof**: (check the textbook – identify an **informal** justification)

Forest of Disjoint Sets

- Set = **tree** with root; keep parent pointer
- 1 node = 1 element (=1 obj) from the set
- 1 tree = a set
- The root = *representative* el.
- Basic Implem. ~ to lists (no improvement)
  - MAKE-SET (x) build the tree with root only
  - FIND-SET (x) goes up and return the representative
  - UNION (x, y) Ex: (blackboard)

Forest of Disjoint Sets – Heuristics
(to increase performance)

- UNION by **rank**
  - Similar to weighted unify on lists
  - The tree with less nodes will point to the tree with many nodes
  - Info kept at root level = rank = max height of the tree
  - rank ≅ lg (dim) (is an approximation, not an exact value; a guarantee that value is never exceeded)
- PATH **compression**
  - Within the **Find-Set**, each node on the *search path* will update the parent node to the *representative* (instead of parent), and leave the **rank unchanged**!
  - **Shrink does NOT change rank!** Why? Ex: blackboard

rank ≅ lg (dim) It is an approximation, ONLY

Forest of Disjoint Sets – Heuristics

- Rank[x]

  = max height of the subtree rooted by x

  = nb. of edges on the longest path from x to a leaf

  rank[leaf] = 0
- Find-Set leave ranks unchanged
- O(m*Ack(n)) -> O(m)

Forest of Disjoint Sets - Implementation

```
MAKE-SET(x)
p[x] <- x
rank[x]<-0
UNION(x, y)
LINK(FIND-SET(x), FIND-SET(y))
LINK(x, y)
if rank [x] > rank [y]
  then p[y] <-x
  else p[x] <-y
if rank [x] = rank [y]
  then rank [y]= rank [y]+1
```

Forest of Disjoint Sets - Implementation

FIND-SET(x)

```
if x!=p[x]
 then p[x] <- FIND-SET(p[x])
 return p[x]
```

Required Bibliography

- From the Bible – Chapter 13 (Red Black Trees), Chapter 21 (Data Structures for Disjoint Sets)