Seminar 6

Wednesday, November 4, 2020 5:59 PM

Laurent series and singularities

Th If $f(z)$ is analytic on $R_1 < |z-z_0| < R_2$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}, \quad \forall z \text{ on } R_1 < |z-z_0| < R_2$$

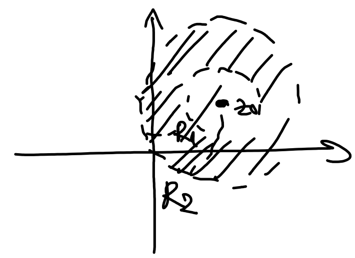
or

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \underbrace{\dots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}}_{\text{principal part}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}_{\text{analytic part (regular part)}}$$

principal part
- conv. outside a circle centered at z_0

analytic part (regular part)
- conv. inside a circle centered at z_0

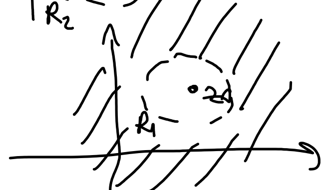
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}\,ds, \quad n = 0, \pm 1, \pm 2, \dots$$



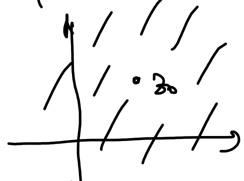
$\boxed{R_1 < |z-z_0| < R_2}$ an annulus domain

→ $R_1 = 0$

→ $R_2 = \infty$



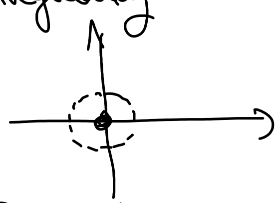
→ $R_1 = 0$ & $R_2 = \infty$



- The coefficient a_{-1} is called the Residue of the function $f(z)$ at $z = z_0$

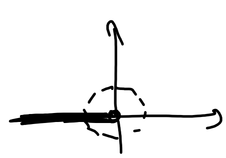
Singularities

- singularity : a point z_0 at which the function f is not analytic
- isolated singularity a singularity z_0 of f for which there exists a neighborhood of z_0 in which f is analytic



$f(z) = \frac{1}{z}$, $z_0 = 0$

- nonisolated singularity

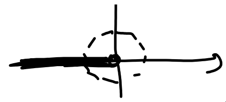


$f(z) = \ln z$, $z_0 = 0$

any neighborhood of z_0 contains a ...

$f(z) = \ln z \;,\; z_0 = 0$

- nonisolated singularity



– every neighborhood of z_0 contains a singularity of f

- removable singularity

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$

$$f(z) = \frac{\cos z - 1}{z^2} \;,\; z_0 = 0$$

$$f(z) = \frac{1}{z^2}\left(\cancel{1} - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \cancel{-1}\right) =$$

$$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots$$

no negative powers of z

we don't have a principal part

$\Rightarrow z_0$ removable singularity

- pole

$$f(z) = \frac{\cos z}{z^4} = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) =$$

$z_0 = 0$

$$= \underbrace{\frac{1}{z^4} - \frac{1}{2!\,z^2}}_{\text{the principal part}} + \underbrace{\frac{1}{4!} - \frac{z^2}{6!} + \frac{z^4}{8!} - \dots}_{\text{the analytic part}}$$

we have a finite no of negative powers of z

$\Rightarrow z_0 =$ pole of order 4

$$f(z) = \underbrace{a_{-N}(z-z_0)^{-N} + \dots + a_{-1}(z-z_0)^{-1}}_{\text{a finite no}} + a_0 + a_1(z-z_0) + \dots$$

$a_{-N} \neq 0$

$z_0 =$ pole of order N

- essential singularity

$$f(z) = \cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!\,z^4} - \frac{1}{6!\,z^6} + \dots$$

$z_0 = 0$

infinite no. of terms with negative powers

$\Rightarrow z_0 =$ essential singularity

(1) Determine all possible Laurent series for the functions:

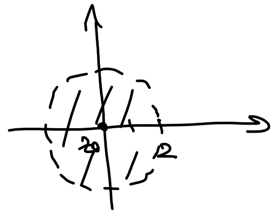
a) $f(z) = \frac{1}{z-2}$, $z_0 = 0$; b) $f(z) = \frac{1}{z-2}$, $z_0 = 1$

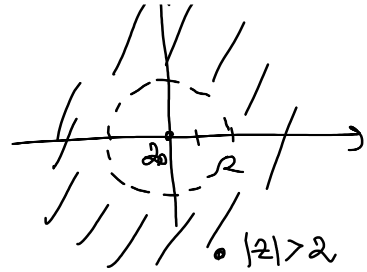
Solutions

a)

Solutions

a)





• $|z|<2$ • $|z|>2$

• $|z|<2 \Rightarrow f(z) = \frac{1}{z-2} = \frac{-1}{2-z} = \frac{-1}{2\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} =$

$\Downarrow$

$\frac{|z|}{2}<1$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}(-1)\frac{z^n}{2^{n+1}}$$, valid for $|z|<2$.

• $|z|>2$ $(2<|z|<\infty)$

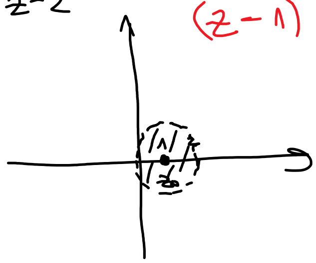
$\Downarrow$

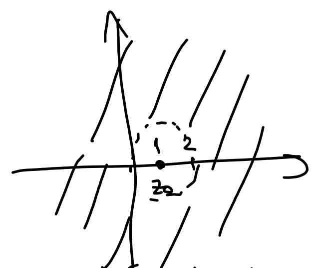
$\frac{2}{|z|}<1$

$$\Rightarrow f(z) = \frac{1}{z-2} = \frac{1}{z}\cdot\frac{1}{1-\left(\frac{2}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$, valid for $|z|>2$

b) $f(z) = \frac{1}{z-2}$, $z_0 = 1$

$(z-1)$





• $|z-1|<1$ • $|z-1|>1$

• $|z-1|<1 \Rightarrow f(z) = \frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{-1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n$, valid for $|z-1|<1$

• $|z-1|>1$

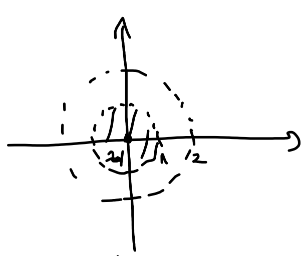
$\Downarrow$

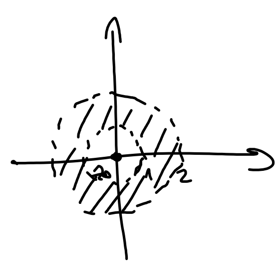
$\frac{1}{|z-1|}<1$

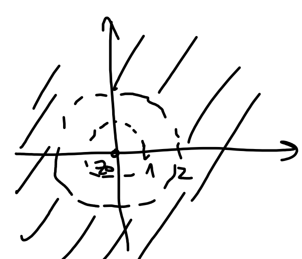
$$\Rightarrow f(z) = \frac{1}{z-2} = \frac{1}{\underline{(z-1)}-1} = \frac{1}{z-1}\cdot\frac{1}{1-\left(\frac{1}{z-1}\right)} = \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n =$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{(z-1)^n}$$, valid for $|z-1|>1$

c) $f(z) = \frac{1}{(z-1)(z-2)}$, $z_0 = 0$







• $|z|<1$ • $1<|z|<2$ • $|z|>2$

• we use partial fraction expansion

$$\frac{1}{(z-1)(z-2)} = \frac{(z-1)-(z-2)}{(z-1)\cdot(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

• $|z|<1 \Rightarrow f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} - \frac{-1}{1-z} =$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n, \text{ valid } |z|<1$$

• $1<|z|<2$

$\Downarrow$ $\frac{1}{|z|}<1$ $\quad$ $\Downarrow$ $\frac{|z|}{2}<1$

$$\Rightarrow f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\cdot\frac{1}{1-\left(\frac{z}{2}\right)} - \frac{1}{z}\cdot\frac{1}{1-\left(\frac{1}{z}\right)} =$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)z^n}{2^{n+1}} - \sum\frac{1}{z^{n+1}}, \text{ valid for } 1<|z|<2$$

• $|z|>2$ $\quad (2<|z|<\infty)$

$\Downarrow$

$\frac{2}{|z|}<1, \ \frac{1}{|z|}<1$

$$\Rightarrow f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} =$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

1.45 Expand the function f in Laurent series and determine the type of singularity.

a) $f(z) = z^3 e^{\frac{1}{z}}, \ z_0 = 0, \ 0<|z|<+\infty$

$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

$$f(z) = z^3 e^{\frac{1}{z}} = z^3\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \dots\right) =$$

$$= z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \underbrace{\frac{1}{4!z} + \frac{1}{5!z^2} + \dots}_{\text{the principal part}}$$

we have an infinite no of terms

$\Rightarrow z_0 = 0$ essential singularity

b) $f(z) = \frac{2\sin^2 z}{z^5}, \ z_0 = 0, \ 0<|z|<+\infty$

$\boxed{2\sin^2 z = 1-\cos 2z}$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$

$$f(z) = \frac{1-\cos 2z}{z^5} = \frac{1}{z^5}\left[1-\left(1-\frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \dots\right)\right]$$

$$= \frac{1}{z^5}\left(\frac{(2z)^2}{2!} - \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} - \dots\right) =$$

$$= \underbrace{2\frac{1}{z^3} - \frac{2^4}{4!}\frac{1}{z}}_{\text{principal part}} + \underbrace{\frac{2^6}{6!}z - \dots}_{\text{the analytic part}}$$

principal part has a finite no of terms

principal part the analytic part

has a finite no of terms

z_0 = pole of order 3

c) $f(z) = z\, e^{\frac{1}{z+i}}$, $z_0 = -i$, $0 < |z-i| < +\infty$

$z+i$

$$f(z) = ((z+i)-i)e^{\frac{1}{z+i}} = (z+i)e^{\frac{1}{z+i}} - i\, e^{\frac{1}{z+i}} =$$

$$= (z+i)\left(1 + \frac{1}{1!(z+i)} + \frac{1}{2!(z+i)^2} + \frac{1}{3!(z+i)^3} + \right) - i\cdot\left(1 + \frac{1}{1!(z+i)} + \frac{1}{2!(z+i)^2} + \frac{1}{3!(z+i)^3} + \dots\right)$$

$$= \underbrace{z+i+1-i}_{} + \underbrace{\left(\frac{1}{2!} - \frac{i}{1!}\right)\cdot\frac{1}{z+i} + \left(\frac{1}{3!} - \frac{i}{2!}\right)\cdot\frac{1}{(z+i)^2} + \dots}_{\text{principal part}}$$

infinite no of terms in the p.p

$\Rightarrow z_0$ essential singularity

③ Use Laurent series

a) $f(z) = \dfrac{1}{(z-1)(z-4)}$, $z_0 = 1$, $0 < |z-1| < 3 \Rightarrow \dfrac{z-1}{3} < 1$

$z-1$

$$f(z) = \frac{1}{z-1}\cdot\frac{1}{z-4} = \frac{1}{z-1}\cdot\frac{1}{(z-1)-3} = \frac{1}{z-1}(-1)\cdot\frac{1}{3-(z-1)} =$$

$$= \frac{1}{z-1}\cdot\left(-\frac{1}{3}\right)\cdot\frac{1}{1-\frac{z-1}{3}} = -\frac{1}{3}\,\frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{z-1}{3}\right)^n =$$

$$= \sum_{n=0}^{\infty}(-1)\frac{(z-1)^{n-1}}{3^{n+1}} \left(= \sum_{n=-1}^{\infty} \frac{-1}{3^{n+2}}(z-1)^n \right)$$

$l = n-1$

$n = l+1$

b) $f(z) = \dfrac{1}{(z-1)(z-4)}$, $z_0 = 1$, $|z-1| > 3 \Rightarrow \dfrac{3}{|z-1|} < 1$

$z-1$

$$f(z) = \frac{1}{z-1}\cdot\frac{1}{z-4} = \frac{1}{z-1}\cdot\frac{1}{(z-1)-3} = \frac{1}{z-1}\cdot\frac{1}{z-1}\cdot\frac{1}{1-\frac{3}{z-1}} =$$

$$= \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}\left(\frac{3}{z-1}\right)^n = \sum_{n=0}^{\infty}\frac{3^n}{(z-1)^{n+2}}.$$

Homework: Expand using Laurent series around z_0, the type of sing

1) $f(z) = z\cos\dfrac{3}{z}$, $z_0 = 0$

2) $f(z) = \sin \frac{1}{z}$, $z_0 = 0$

3) $f(z) = \frac{1 - e^{-z}}{z}$, $z_0 = 0$

4) $f(z) = \frac{1 + 2z^2}{z^3 + z^5}$, $z_0 = 0$, $0 < |z| < 1$