Seminar 14

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The Z transform

$f : \mathbb{N} \to \mathbb{C}$

$$\mathcal{Z}[f](z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots + \frac{f(n)}{z^n} + \dots$$

$$\mathcal{Z}[f](z) = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$$

$$\mathcal{Z}[f](z) = F(z)$$

$\mathcal{Z}[1] = \frac{z}{z-1}$, $\mathcal{Z}[a^n] = \frac{z}{z-a}$, $\mathcal{Z}[n] = \frac{z}{(z-1)^2}$, $\mathcal{Z}[n^2] = \frac{z(z+1)}{(z-1)^3}$

$\mathcal{Z}[\cos an] = \frac{z(z-\cos a)}{z^2-2z\cos a+1}$, $\mathcal{Z}[\sin an] = \frac{z\sin a}{z^2-2z\cos a+1}$

$\mathcal{Z}[\cosh an] = \frac{z(z-\cosh a)}{z^2-2z\cosh a+1}$, $\mathcal{Z}[\sinh an] = \frac{z\sinh a}{z^2-2z\sinh a+1}$

$$\mathcal{Z}[f(n+p)](z) = z^p \cdot F(z) - z^p f(0) - z^{p-1} f(1) - \dots - z f(p-1)$$

$$\mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right] = \frac{z}{z-1} F(z)$$

$$\mathcal{Z}[n \cdot f(n)] = -z \cdot F'(z)$$

The inverse of Z transform

Method I: $F(z) = \mathcal{Z}[f] \Rightarrow f(n) = \frac{1}{2\pi i} \int_C z^{n-1} \cdot F(z)\, dz$

where C contains the singular points in the interior

- we apply the Residue Theorem

Method II: $F(z) = \mathcal{Z}[f]$ (is similar with $\mathcal{L}$ transform)

we use partial fraction decomposition

$\frac{F(z)}{z}$ - we split into partial fraction

- we multiply by z

(1) (2.27. i))

Find the general term of the sequence given by.

$x_{n+1} - 2x_n = n \,/\, \mathcal{Z}$, $x_0 = 0$.

$x_n = x(n)$, $x : \mathbb{N} \to \mathbb{C}$

$x(n+1) - 2x(n) = n$

we use the formula:

$$\mathcal{Z}[f(n+p)](z) = z^p \cdot F(z) - z^p f(0) - z^{p-1} f(1) - \dots - z f(p-1)$$

$\mathcal{Z}[x_{n+1}] - 2\mathcal{Z}[x_n] = \mathcal{Z}[n]$

$$\mathcal{Z}[f(n+p)](z) = \underline{z^p} \cdot F(z) - \underline{z^p} f(0) - z^{p-1} f(1) - \dots - z f(p-1)$$

$$\mathcal{Z}[x_{n+1}] - 2\mathcal{Z}[x_n] = \mathcal{Z}[n]$$

$$\mathcal{Z}[x_{n+1}] = \mathcal{Z}[x(n+1)] \underset{p=1}{=} zF(z) - z \cdot \underset{0}{x_0} = 2F(z) \qquad f(0) = x_0$$

we denote by $F(z) = \mathcal{Z}[x_n]$

$$zF(z) - 2F(z) = \frac{z}{(z-1)^2} \Rightarrow F(z)\cdot(z-2) = \frac{z}{(z-1)^2} \Rightarrow F(z) = \frac{z}{(z-2)(z-1)^2}$$

method I : $x_n = \mathcal{Z}^{-1}[F(z)]$

$$x_n = \mathcal{Z}^{-1}\left[\frac{z}{(z-2)(z-1)^2}\right]$$

$$\Rightarrow x_n = \frac{1}{2\pi i}\int_C \underbrace{z^{n-1} \cdot \frac{z}{(z-2)(z-1)^2}}_{f(z)} dz \underset{\text{Res-Th.}}{=\!=} \frac{1}{\cancel{2\pi i}} \cdot \cancel{2\pi i} \sum_{k=1}^{n} \underset{z=k}{\mathrm{Res}} f(z) =$$

$$= \underset{z=2}{\mathrm{Res}} f(z) + \underset{z=1}{\mathrm{Res}} f(z)$$

$z=2$ pole of order 1
$z=2 \in \mathrm{int}\, C$
$z=1$ pole of order 2 $\in \mathrm{int}\, C$

$$\underset{z=2}{\mathrm{Res}} f(z) \underset{\text{pole of order 1}}{=} \frac{1}{1} \lim_{z\to 2} \cancel{(z-2)} \frac{z^n}{\cancel{(z-2)}(z-1)^2} = 2^n$$

$$\underset{z=1}{\mathrm{Res}} f(z) \underset{\text{pole of order 2}}{=\!=} \frac{1}{1!} \lim_{z\to 1} \left[\cancel{(z-1)^2} \frac{z^n}{(z-2)\cancel{(z-1)^2}}\right]' = \lim_{z\to 1} \frac{nz^{n-1}(z-2) - z^n}{(z-2)^2} = \frac{n(-1)-1}{1}$$

$$= -n-1$$

$$x_n = 2^n - n - 1$$

method II. $F(z) = \dfrac{z}{(z-2)(z-1)^2}$

we take $\frac{F(z)}{z}$ and we split in partial fractions

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z-1)^2} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$1 = A(z-1)^2 + B(z-2)(z-1) + C(z-2)$$

$z=1 \Rightarrow 1 = -C \Rightarrow \boxed{C=-1}$

$z=2 \Rightarrow 1 = A \Rightarrow \boxed{A=1}$

$z=0 \Rightarrow 1 = A + 2B - 2C \Rightarrow \cancel{1} = \cancel{1} + 2B + 2 \Rightarrow \boxed{B=-1}$

$$\frac{F(z)}{z} = \frac{1}{z-2} - \frac{1}{z-1} - \frac{1}{(z-1)^2} \quad \Big/ \cdot z$$

$$F(z) = \frac{z}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^2} \Rightarrow x_n = \mathcal{Z}^{-1}\left[\frac{z}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^2}\right]$$

$$x_n = 2^n - 1 - n$$

$x_n = \mathcal{Z}^{-1} - n$

② 2.27. iv)
Find the sequence x_n such that $x_{n+2} + 4x_{n+1} + 3x_n = 3^n$ / $\mathcal{Z}$, $x_0 = 0$, $x_1 = 1$

We apply $\mathcal{Z}$ transform
We denote $F(z) = \mathcal{Z}[x_n]$

$$\mathcal{Z}[x_{n+2}] = z^2 F(z) - z^2 \underset{0}{\underset{\parallel}{x_0}} - z \underset{1}{\underset{\parallel}{x_1}} = z^2 F(z) - z$$

$$\mathcal{Z}[x_{n+1}] = z F(z) - z \underset{0}{\underset{\parallel}{x_0}} = z F(z)$$

$$\mathcal{Z}[3^n] = \frac{z}{z-3}$$

$$F(z) \cdot \underbrace{(z^2 + 4z + 3)}_{(z+1)(z+3)} = \frac{z}{z-3} + z$$

$$F(z) = \frac{z^2 - 2z}{(z-3)(z+1)(z+3)} \Rightarrow F(z) = \frac{z(z-2)}{(z-3)(z+1)(z+3)}$$

Method II: $\frac{F(z)}{z} = \frac{z-2}{(z-3)(z+1)(z+3)} = \frac{A}{z-3} + \frac{B}{z+1} + \frac{C}{z+3}$

$$z - 2 = A(z+1)(z+3) + B(z-3)(z+3) + C(z-3)(z+1)$$

$z = -3 \Rightarrow -5 = (-6)(-2) \cdot C \Rightarrow C = -\frac{5}{12}$

$z = -1 \Rightarrow -3 = B \cdot (-4) \cdot 2 \Rightarrow B = \frac{3}{8}$

$z = 3 \Rightarrow 1 = A \cdot 4 \cdot 6 \Rightarrow A = \frac{1}{24}$

$$\frac{F(z)}{z} = \frac{1}{24} \cdot \frac{1}{z-3} + \frac{3}{8} \cdot \frac{1}{z+1} - \frac{5}{12} \cdot \frac{1}{z+3} \quad / \cdot z$$

$$F(z) = \frac{1}{24} \frac{z}{z-3} + \frac{3}{8} \frac{z}{z+1} - \frac{5}{12} \frac{z}{z+3} \quad / \mathcal{Z}^{-1}$$

$$x_n = \frac{1}{24} 3^n + \frac{3}{8} \cdot 1 - \frac{5}{12} (-3)^n$$

③ 1.38 Find the general form of the sequence given by:

$$x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n = 1, \quad x_0 = x_1 = x_2 = 0$$

Denote $F(z) = \mathcal{Z}[x_n]$

$$\mathcal{Z}[x_{n+3}] \underset{p=3}{\underset{\uparrow}{=}} z^3 F(z) - z^3 \underset{0}{\underset{\parallel}{x_0}} - z^2 \underset{0}{\underset{\parallel}{x_1}} - z \underset{0}{\underset{\parallel}{x_2}} = z^3 F(z)$$

$$\mathcal{Z}[x_{n+2}] \underset{p=2}{\underset{\uparrow}{=}} z^2 F(z) - z^2 \underset{0}{\underset{\parallel}{x_0}} - z \underset{0}{\underset{\parallel}{x_1}} = z^2 F(z)$$

$$\mathcal{Z}[x_{n+1}] = z F(z) - z x_0 = z F(z)$$

$$z^3F(z)-3z^2F(z)+3zF(z)-F(z)=\frac{z}{z-1}$$

$$F(z)\cdot(\underbrace{z^3-3z^2+3z-1}_{(z-1)^3})=\frac{z}{z-1} \Rightarrow F(z)=\frac{z}{(z-1)^4}$$

we have to use Method I

$$x_n=\mathcal{Z}^{-1}\left[\frac{z}{(z-1)^4}\right]=\frac{1}{2\pi i}\int_C \underbrace{z^{n-1}\cdot\frac{z}{(z-1)^4}}_{f(z)}dz=\frac{1}{\cancel{2\pi i}}\cdot\cancel{2\pi i}\operatorname{Res}_{z=1}f(z)$$

$z=1$ pole of order 4

$$\operatorname{Res}_{z=1}f(z)=\frac{1}{3!}\lim_{z\to1}\left[\cancel{(z-1)^4}\frac{z^n}{\cancel{(z-1)^4}}\right]'''=\frac{1}{6}\lim_{z\to1}n(n-1)(n-2)z^{n-3}=\frac{n(n-1)(n-2)}{6}$$

$z=1$ pole of order 4

$$\Rightarrow x_n=\frac{n(n-1)(n-2)}{6}.$$

(4) Find the sum of the series $s=\sum_{n=0}^{\infty}\frac{n^2-3n+5}{6^n}$

$$\mathcal{Z}[f](z)=f(0)+\frac{f(1)}{z}+\frac{f(2)}{z^2}+\dots+\frac{f(n)}{z^n}+\dots$$

$$\mathcal{Z}[f](z)=\sum_{n=0}^{\infty}\frac{f(n)}{z^n}$$

We use the definition of $\mathcal{Z}$ transform

$$s=\sum_{n=0}^{\infty}\frac{n^2}{6^n}-3\sum_{n=0}^{\infty}\frac{n}{6^n}+5\sum_{n=0}^{\infty}\frac{1}{6^n}$$

$$F(3)=\sum_{n=0}^{\infty}\frac{f(n)}{3^n}$$

Let's identify function $f(n)$

$$s=\frac{z(z+1)}{(z-1)^3}\Big|_{z=6}-3\frac{z}{(z-1)^2}\Big|_{z=6}+5\frac{z}{z-1}\Big|_{z=6}=\frac{6\cdot7}{5^3}-3\cdot\frac{6}{5^2}+5\cdot\frac{6}{5}=$$

$$=\frac{42}{125}-\frac{18}{25}+\frac{30}{5}=\frac{42-90+750}{125}=\frac{750-48}{125}=\frac{702}{125}$$

(5) Find the sum of the series $s=\sum_{n=0}^{\infty}\frac{1}{2^n}\cos\frac{n\pi}{3}$

we use the definition: $F(z)=\sum_{n=0}^{\infty}\frac{f(n)}{z^n}$

$$s=\mathcal{Z}\left[\cos\frac{n\pi}{3}\right](z)=\frac{\left(z-\cos\frac{\pi}{3}\right)z}{z^2-2z\underbrace{\cos\frac{\pi}{3}}_{\frac12}+1}\Bigg|_{z=2}=\frac{\left(z-\frac12\right)z}{z^2-z+1}\Bigg|_{z=2}=\frac{\left(2-\frac12\right)2}{3}=$$

$$=\frac{\frac32\cdot2}{3}=1$$

*(6) Find the sum of the series $s=\sum_{n=0}^{\infty}\frac{n}{4^n}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)$

$$\Rightarrow s=\mathcal{Z}\left[n\cdot\sum_{k=0}^{n}\frac{1}{k!}\right](4)$$

$\Rightarrow p = \mathcal{Z}\left[n \cdot \sum_{k=0}^{n} \frac{1}{k!}\right](4)$

- $\mathcal{Z}\left[\frac{1}{n!}\right](z) \underset{\text{definition}}{=} f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots \underset{f:=\frac{1}{n!}}{=} 1 + \frac{1}{1!\,z} + \frac{1}{2!\,z^2} + \dots + \frac{1}{n!\,z^n} + \dots$

$= e^{\frac{1}{z}}$

- $\boxed{\mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right] = \frac{z}{z-1} F(z) = \frac{z}{z-1}\mathcal{Z}[f]}$

$\mathcal{Z}\left[\sum_{k=0}^{n} \frac{1}{k!}\right] = \frac{z}{z-1}\mathcal{Z}\left[\frac{1}{n!}\right] = \frac{z}{z-1} \cdot e^{\frac{1}{z}}$

- $\boxed{\mathcal{Z}\left[n \cdot f(n)\right] = -zF'(z)}$

$\mathcal{Z}\left[n \cdot \sum_{k=0}^{n} \frac{1}{k!}\right](z) = -z \cdot \left(\frac{z}{z-1} \cdot e^{\frac{1}{z}}\right)' = -z \cdot \left[\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2} \cdot e^{\frac{1}{z}} + \frac{\cancel{z}}{z-1} e^{\frac{1}{z}}\left(-\frac{1}{z^{\cancel{2}}}\right)\right]$

$= -z\, e^{\frac{1}{z}}\left[-\frac{1}{(z-1)^2} - \frac{1}{z(z-1)}\right]$

$p = -z e^{\frac{1}{z}}\left[-\frac{1}{(z-1)^2} - \frac{1}{z(z-1)}\right]\Bigg|_{z=4} = -4e^{\frac{1}{4}}\left(-\frac{1}{9} - \frac{1}{12}\right) = \cancel{4}e^{\frac{1}{4}} \frac{7}{\cancel{36}} = \frac{7}{9}e^{\frac{1}{4}}$

(7)* $p = \sum_{n=0}^{\infty} \frac{1}{3^n}\left(\sum_{k=0}^{n} \frac{k}{2^k}\right) = \mathcal{Z}\left[\sum_{k=0}^{n} \frac{k}{2^k}\right](3)$

$F(z) = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$

$\boxed{\mathcal{Z}\left[\sum_{k=0}^{n} f(k)\right] = \frac{z}{z-1}\mathcal{Z}[f]}$

The general term is $n \cdot \left(\frac{1}{2}\right)^n$

$\mathcal{Z}\left[n \cdot \left(\frac{1}{2}\right)^n\right] = -z\left(\mathcal{Z}\left(\frac{1}{2}\right)^n\right)' = -z\left(\frac{z}{z-\frac{1}{2}}\right)' = -z \cdot \frac{\cancel{z}-\frac{1}{2}-\cancel{z}}{\left(z-\frac{1}{2}\right)^2} = \frac{z}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{(2z-1)^2}$

$\mathcal{Z}[n f(n)] = -z F'(z)$ $\qquad$ $\mathcal{Z}[a^n] = \frac{z}{z-a}$

$= \frac{2z}{(2z-1)^2}$

$\mathcal{Z}\left[\sum_{k=0}^{n} \frac{k}{2^k}\right](z) = \frac{z}{z-1} \cdot \frac{2z}{(2z-1)^2}$

we replace z by 3 $\Rightarrow p = \frac{z}{z-1} \cdot \frac{2z}{(2z-1)^2}\Bigg|_{z=3} = \frac{3}{\cancel{2}} \cdot \frac{\cancel{2} \cdot 3}{25} = \frac{9}{25}$.

Homework:

1) Find the sum $\quad p = \sum_{n=0}^{\infty} \frac{n \sin \frac{n\pi}{2}}{2^n}$ $\qquad$ R: $\frac{6}{25}$