

Graphs - Introduction

Fundamental Algorithms

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Agenda

- 1 Graph Representation
- 2 Minimum Spanning Tree (MST)
 - Kruskal's Algorithm
 - Prim's Algorithm
- 3 Breadth-First Search (BFS)



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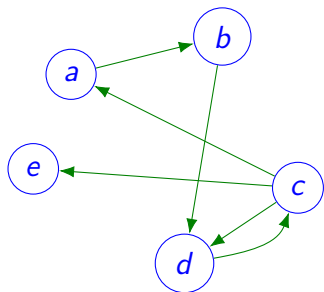


Graph Representation

$$G = (V, E)$$

- V - node/vertex set
- E - edge set, $E \subseteq V \times V$

Representation:





Graph Representation

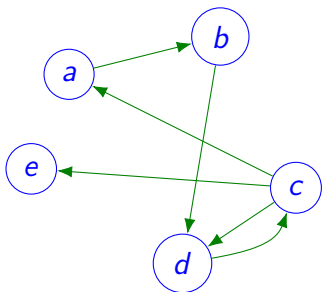
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Representation:

- adjacency matrix

	a	b	c	d	e
a	0	1	0	0	0
b	0	0	0	1	0
c	1	0	0	1	1
d	0	0	1	0	0
e	0	0	0	0	0





Graph Representation

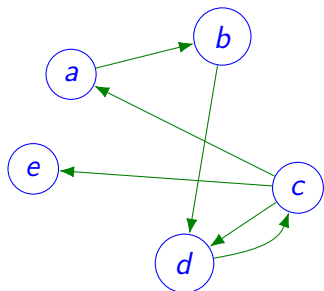
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Representation:

- adjacency matrix
- adjacency lists

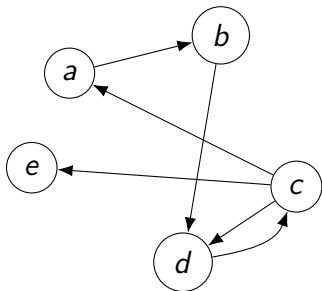
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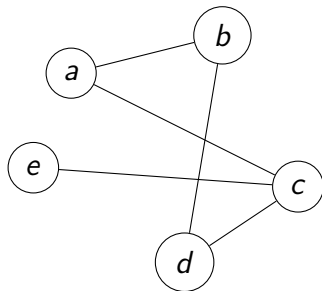
- $a \rightarrow \{b\}$
- $b \rightarrow \{d\}$
- $c \rightarrow \{a, d, e\}$
- $d \rightarrow \{c\}$
- $e \rightarrow \emptyset$



Concepts (1)



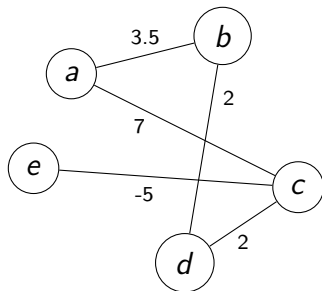
Directed graph



Undirected graph



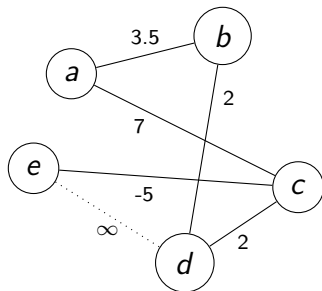
Concepts (2)



Weighted graph



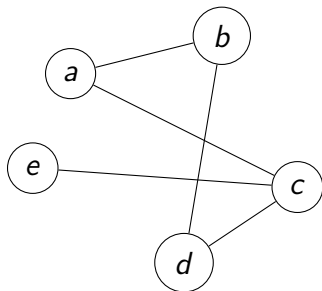
Concepts (2)



Weighted graph



Concepts (3)



Node degree

$$a.deg = 2$$

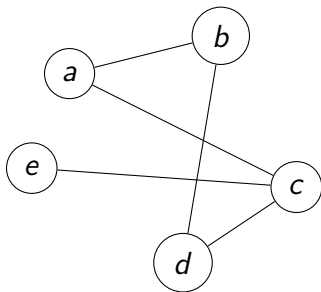
$$b.deg = 2$$

$$c.deg = 3$$

...



Concepts (3)



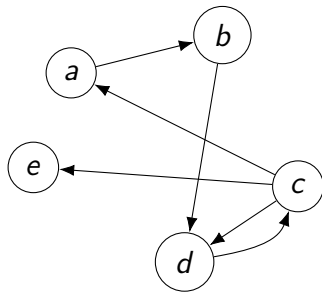
Node degree

$$a.deg = 2$$

$$b.deg = 2$$

$$c.deg = 3$$

...



In-degree and out-degree

$$a.deg_{in} = 1, a.deg_{out} = 1$$

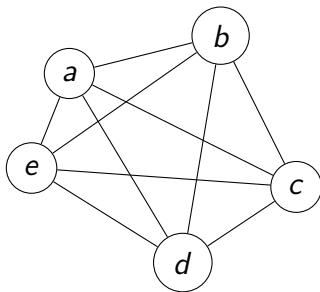
$$c.deg_{in} = 1, c.deg_{out} = 3$$

$$e.deg_{in} = 1, e.deg_{out} = 0$$

...



Concepts (4)



Complete graph

$$E = \{(u, v) \in V \times V, u \neq v\}$$



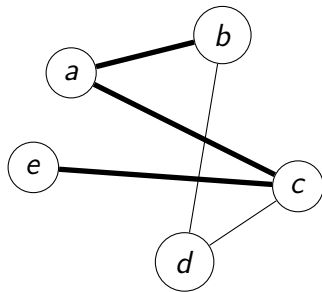
Concepts (5)

The distance between two nodes

The shortest path (in number of edges) between the two nodes.

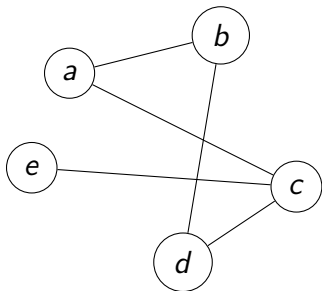
Graph diameter

The longest distance between (any) two nodes in the graph.

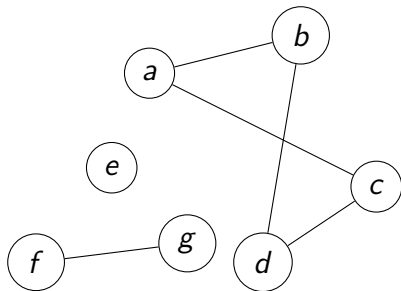




Concepts (6)



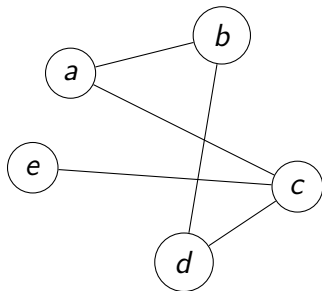
Connected graph



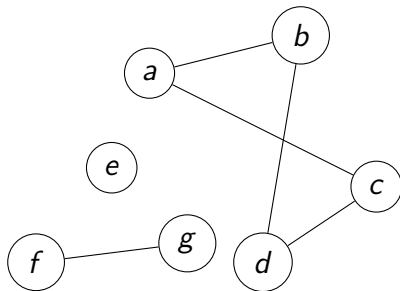
Unconnected graph



Concepts (6)



Connected graph



Unconnected graph

Strongly connected graph

Directed graph, in which we have a path from any node to any other node, considering the edge direction.



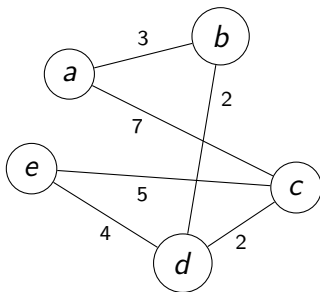
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Minimum Spanning Tree (MST)

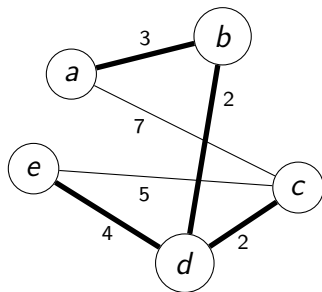
- Given a weighted, undirected, graph $G = (V, E)$,
 $w : V \times V \rightarrow \mathbb{R}$
 - $w(u, v)$ - weight/cost of the edge between u and v
- Find $G' = (V, T)$
 - $T \subseteq E$, $|T| = |V| - 1$, T contains no cycle
 - $\sum_{(u,v) \in T} w(u, v)$ is minimum





Minimum Spanning Tree (MST)

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MST Applications

- network design
 - phone, electrical, hidraulic, computer, road



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- approximation algorithms for NP-hard problems
 - TSP (*traveling salesman problem*)
 - Steiner trees



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 - Steiner trees
- indirect applications
 - max bottleneck path
 - LDPC codes for transmission error correction
 - feature learning in face recognition
 - reducing storage space in protein amino-acid sequencing
 - Ethernet auto-configuration for cycle avoidance



Kruskal's Algorithm - Approach

- *greedy* approach
- build the spanning tree by repeatedly linking partial trees
- initially, each node is a tree
- in each step, link the two closest sub-trees
 - by an edge which will be added to the MST
 - the sub-trees should be distinct (the edge must not close a cycle)
- stop when we have selected $|V| - 1$ edges
- how can we know that an edge links two different trees, and not two different branches of the same tree? (i.e. cycle)
 - use a disjoint set forest



Kruskal's Algorithm

MST-KRUSKAL(G, w)

```
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2  for each vertex  $v \in G.V$ 
3      MAKE-SET( $v$ )
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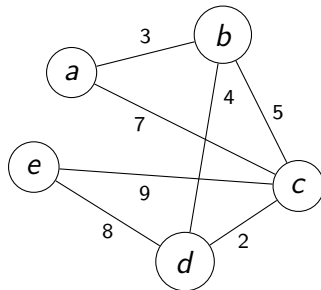


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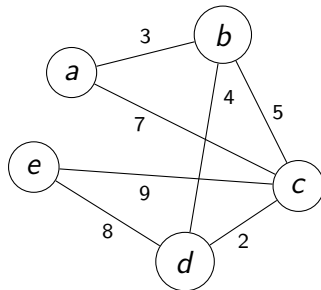


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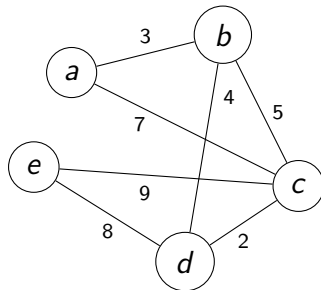


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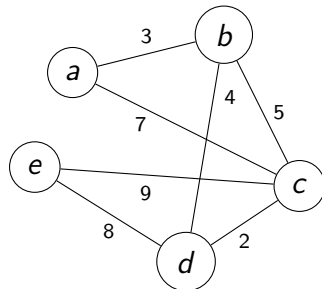


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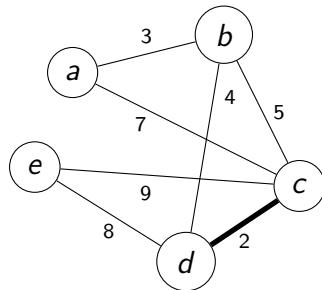
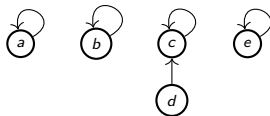


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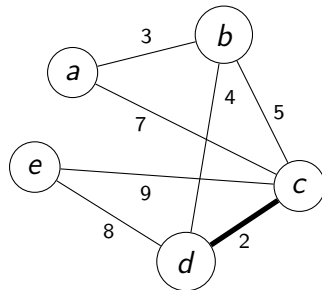
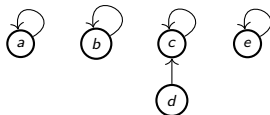


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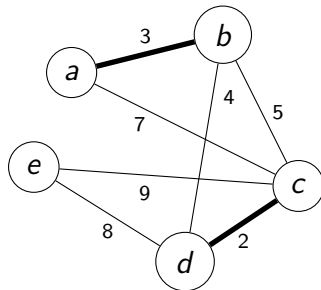
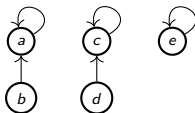


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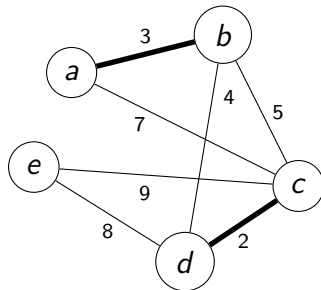
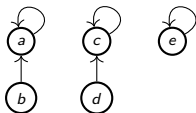


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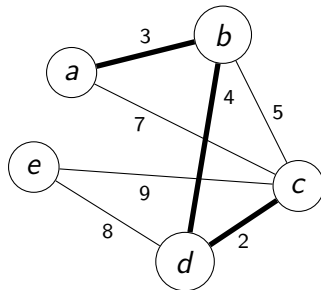
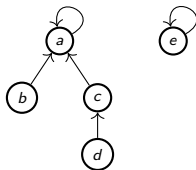


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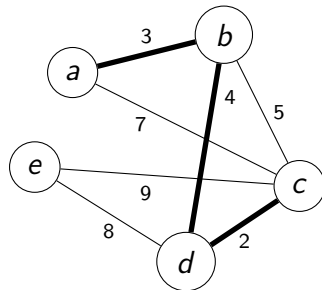
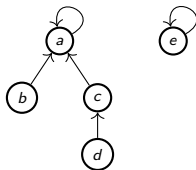


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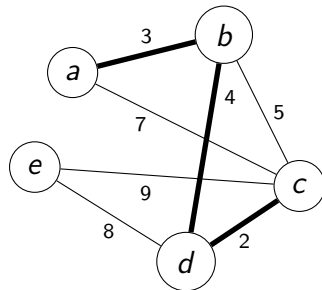
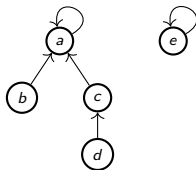


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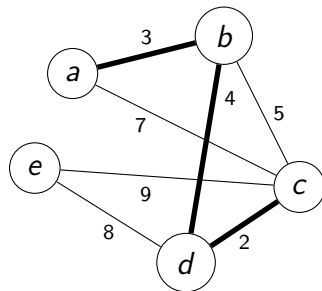
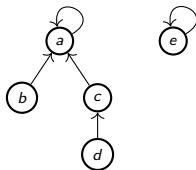


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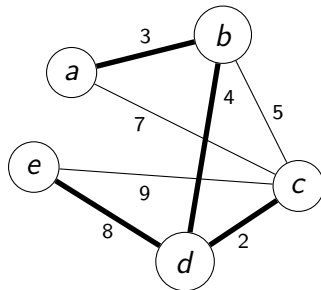
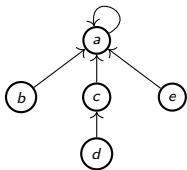


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$$A = \{(c, d), (a, b), (b, d), (d, e)\}$$

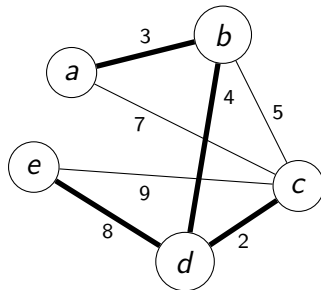
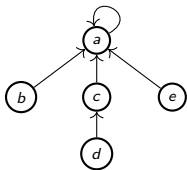


Kruskal's Algorithm

MST-KRUSKAL(G, w)

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2  for each vertex  $v \in G.V$ 
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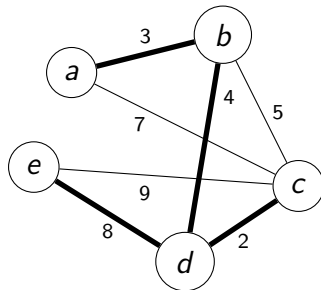
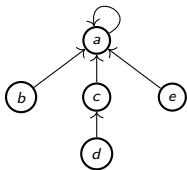


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Kruskal's Algorithm - Complexity

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Kruskal's Algorithm - Complexity

MST-KRUSKAL(G, w)

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$O(V)$

$O(E \log E)$



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6      if FIND-SET( $u$ )  $\neq$  FIND-SET( $v$ ) }
7           $A = A \cup \{(u, v)\}$          }  $\times O(\alpha(V))$ 
8          UNION( $u, v$ )
9  return  $A$ 
  
```



Kruskal's Algorithm - Complexity

MST-KRUSKAL(G, w)

```

1   $A = \emptyset$ 
2  for each vertex  $v \in G.V$  }
3      MAKE-SET( $v$ )           }  $O(V)$ 
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Complexity: $O(E \log E)$



Kruskal's Algorithm - Discussion

- Is the solution unique?
 - on the given example
 - in general



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- How can we obtain other solutions?



Kruskal's Algorithm - Discussion

- Is the solution unique?
 - on the given example
 - in general
- How can we obtain other solutions?
- How can we obtain all solutions?
 - How many solutions are there, in the worst case?



Kruskal's Algorithm - Correctness

- Let T be the MST built by MST-KRUSKAL.
- Consider $G.E = \{e_1, e_2, \dots, e_m\}$ the edges of the graph, ordered ascendingly by weight.



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- Assume that T is not minimal, and that T^* is the MST ($w(T^*) < w(T)$) having the longest common edge prefix with T .



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 - $\Rightarrow e \in T.E$ and $e \notin T^*.E$



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 - Kruskal would not have excluded that edge, unless it closed a cycle
 - $\Rightarrow e \in T.E$ and $e \notin T^*.E$
- If $e = (x, y)$, there is a path in T^* , P , from x to y (which cannot be the direct edge, because, $(x, y) \notin T^*.E$).
- If all edges in P had a weight smaller than e , Kruskal-MST would have selected them prior to e (e is the first edge for which T and T^* "disagree"), and e would close a cycle.
- $\Rightarrow \exists e' \in P, w(e) \leq w(e')$.



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- $\Rightarrow \exists e' \in P, w(e) \leq w(e')$.
- Let $T_1 = T^* \cup \{e\} \setminus \{e'\}$. $\Rightarrow w(T_1) \leq w(T^*)$.
- T_1 is a MST with a longer common prefix with T than T^* - contradiction.



Prim's Algorithm - Approach

- *greedy* approach
- start at any node
- in every step (out of the $|V| - 1$ steps)
 - select the closest node to the current tree
 - add the node and the corresponding edge to the tree



Prim's Algorithm

MST-PRIM(G, w, r)

```
1  for each  $u \in G.V$ 
2       $u.key = \infty$ 
3       $u.\pi = \text{NIL}$ 
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5   $Q = G.V$  // priority queue
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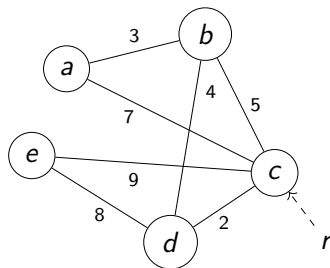
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	π	key

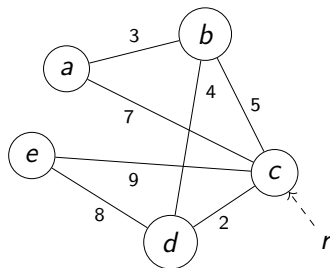


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```



	π	key
a	NIL	∞
b	NIL	∞
c	NIL	∞
d	NIL	∞
e	NIL	∞



Prim's Algorithm

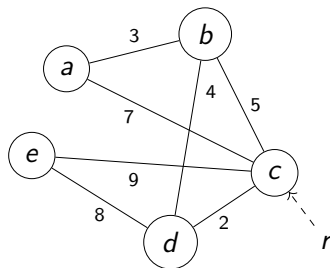
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```

(c)



	π	key
a	NIL	∞
b	NIL	∞
c	NIL	0
d	NIL	∞
e	NIL	∞



Prim's Algorithm

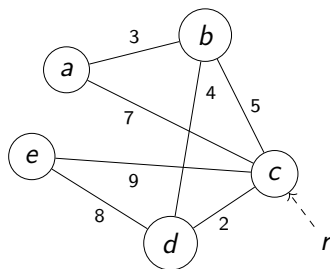
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(c)



	π	key	} Q
c	NIL	0	
a	NIL	∞	
b	NIL	∞	
d	NIL	∞	
e	NIL	∞	



Prim's Algorithm

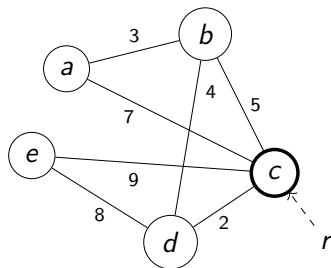
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(c)



	π	key	
c	NIL	0	} $V \setminus Q$
a	NIL	∞	
b	NIL	∞	} Q
d	NIL	∞	
e	NIL	∞	



Prim's Algorithm

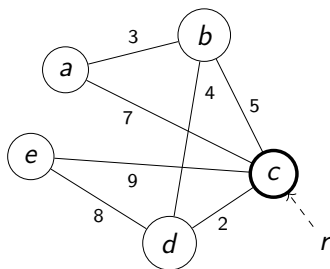
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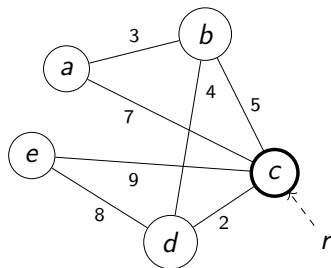
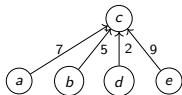
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```



	π	key	
c	NIL	0	} $V \setminus Q$
a	c	7	
b	c	5	} Q
d	c	2	
e	c	9	



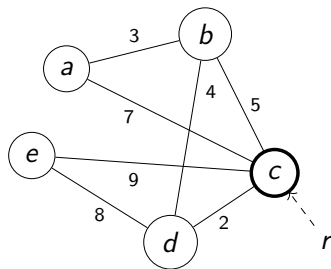
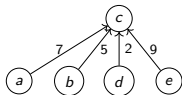
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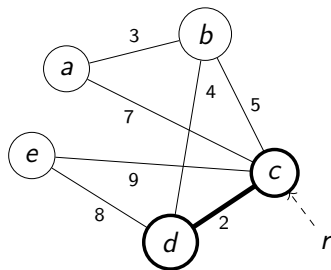
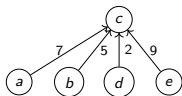
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	π	key	
c	NIL	0	} $V \setminus Q$
d	c	2	
<hr style="border-top: 1px dashed red;"/>			
b	c	5	} Q
a	c	7	
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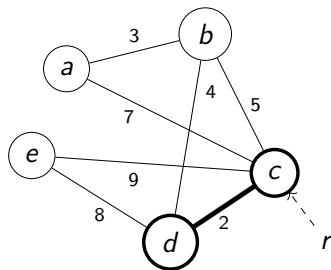
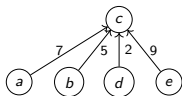
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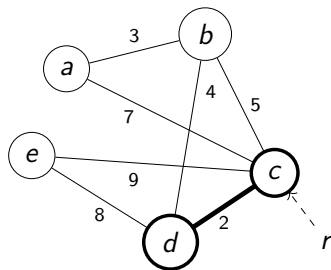
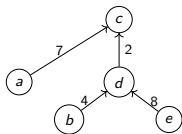
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	π	key	
c	NIL	0	} $V \setminus Q$
d	c	2	
b	d	4	} Q
a	c	7	
e	d	8	



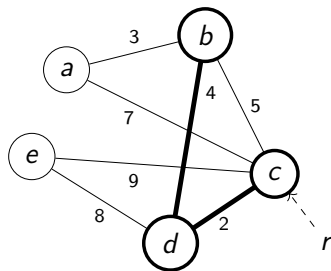
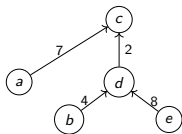
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b	d	4	

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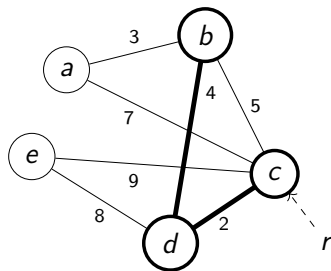
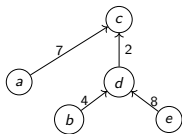
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MST-PRIM(G, w, r)

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	π	key	
c	NIL	0	} $V \setminus Q$
d	c	2	
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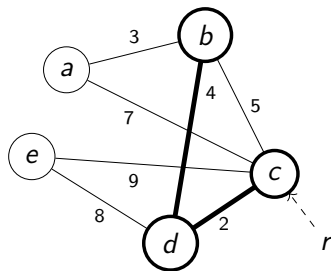
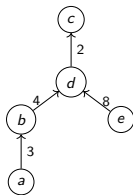
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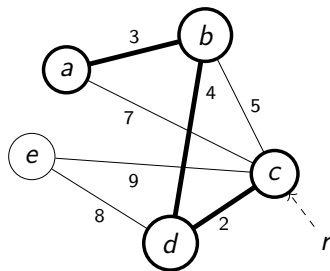
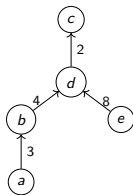
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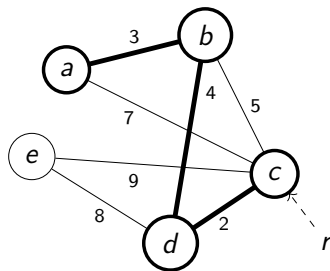
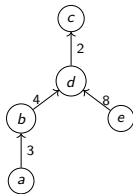
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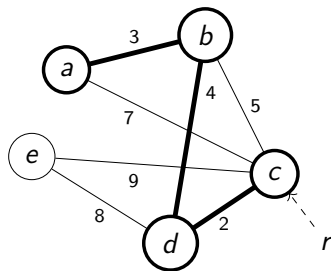
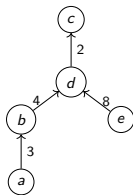
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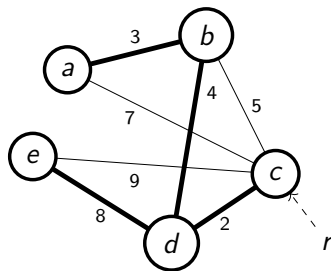
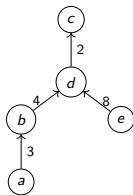
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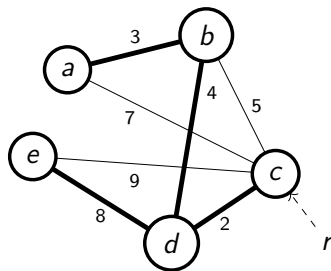
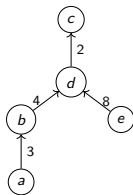
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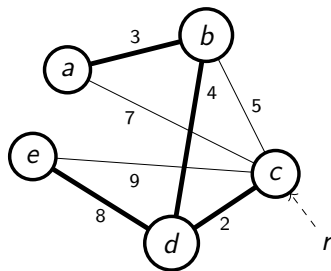
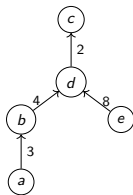
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Prim's Algorithm - Complexity

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$\rightarrow O(V)$



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Annotations:

- Lines 1-3 are grouped by a bracket and an arrow pointing to $O(V)$.
- Line 5 is annotated with an arrow pointing to $O(V)$ - build heap.



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Complexity annotations:

- Lines 1-3: $O(V)$
- Line 5: $O(V)$ - build heap
- Line 6: $|V|$ steps



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Complexity analysis annotations:

- Lines 1-3: $O(V)$
- Line 5: $O(V)$ - build heap
- Line 6: $|V|$ steps
- Line 7: $O(\log V)$
- Lines 8-11: each neighbor for every node, so $|E|$ steps in total



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- Line 9: for every node, so $|E|$ steps in total
- Line 11: binary heap: $O(\log V)$
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binary heap implementation: $O(V + V + V \log V + E \log V) = O(E \log V)$



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Fibonacci heap implementation: $O(V + V + V \log V + E) = O(E + V \log V)$



Prim's Algorithm - Discussion

- Advantage over Kruskal's algorithm
 - builds a rooted tree
 - more efficient in the Fibonacci heap implementation



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- Is the algorithm deterministic?



Prim's Algorithm - Correctness

- Let T be the MST build by Prim-MST.
- Let $T.E = \{e_1, e_2, \dots, e_{n-1}\}$ be the sequence of edges, in the order they are selected by the algorithm.



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- Assume that T is not minimal, and that T^* is the MST, ($w(T^*) < w(T)$), which includes the longest prefix $E' = \{e_1, e_2, \dots, e_{i-1}\}$ from $T.E$, while the edge $e_i = (x, y) \notin T^*.E$. We denote by V' the nodes added to T before the edge e_i .



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- In T^* there is a path from x to y . As $y \notin T'$, there is in the path from x to y an edge (a, b) with $a \in T'$ and $b \notin T'$.
- Let $T_1 = T^* \cup \{(x, y)\} \setminus \{(a, b)\}$ - also a MST.



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- We have 3 cases:
 - if $w(x, y) < w(a, b)$, then $w(T_1) < w(T^*) \Rightarrow T^*$ is not minimal
 - if $w(x, y) > w(a, b)$, then Prim-MST would have selected (a, b) instead of (x, y)
 - if $w(x, y) = w(a, b)$, then $w(T_1) = w(T^*)$, so T_1 is minimal and contains a longer prefix of $T.E$



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- All three cases result in a contradiction, so T^* does not exist $\Rightarrow T$ is minimal.



Agenda

- 1 Graph Representation
- 2 Minimum Spanning Tree (MST)
 - Kruskal's Algorithm
 - Prim's Algorithm
- 3 Breadth-First Search (BFS)



Breadth-First Search (BFS)

- start at a given (source) node s
- produce a tree rooted in s
- all reachable nodes at distance k from the source will be discovered prior to all those at distance $k + 1$
- the resulting tree contains the minimum length paths from s to all nodes reachable from s



BFS Algorithm - Approach

- associate the current attributes to the nodes:
 - *color*: WHITE, GRAY, BLACK
 - *d*: distance from the source *s*
 - π : parent node in BFS tree
- the algorithm starts by adding the source node to a queue and coloring it GRAY
- while the queue still has elements
 - extract the first node from it
 - color with GRAY all neighbors of this node that haven't yet been discovered, and add them to the queue
 - color the extracted node with BLACK



BFS Algorithm

BFS(G, s)

```

1  for each  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
4       $u.\pi = \text{NIL}$ 
5   $s.color = \text{GRAY}$ 
6   $s.d = 0$ 
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8   $Q = \emptyset$  // regular queue
9  ENQUEUE( $Q, s$ )
10 while  $Q \neq \emptyset$ 
11      $u = \text{DEQUEUE}(Q)$ 
12     for each  $v \in G.Adj[u]$ 
13         if  $v.color == \text{WHITE}$ 
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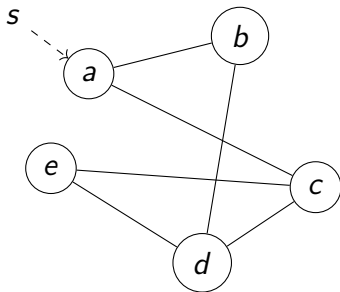


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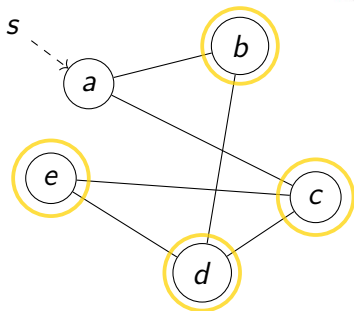


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13         if  $v.color == \text{WHITE}$ 
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15              $v.d = u.d + 1$ 
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17             ENQUEUE( $Q, v$ )
18      $u.color = \text{BLACK}$ 
  
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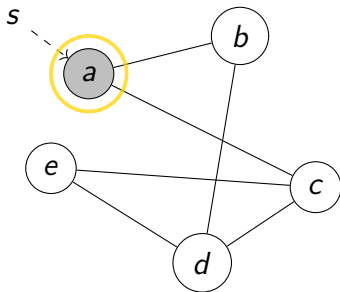


BFS Algorithm

BFS(G, s)

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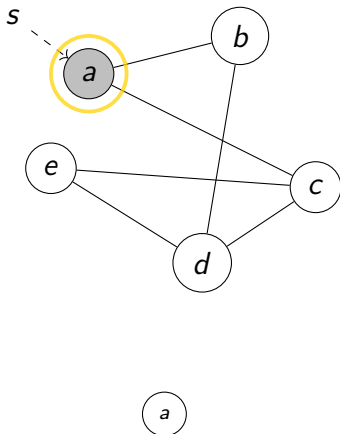


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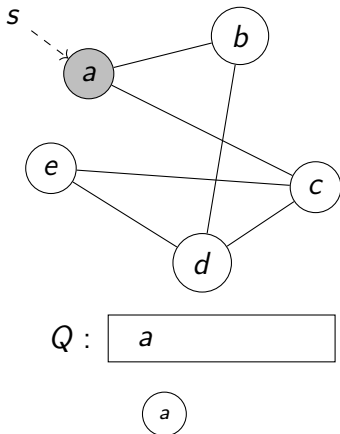


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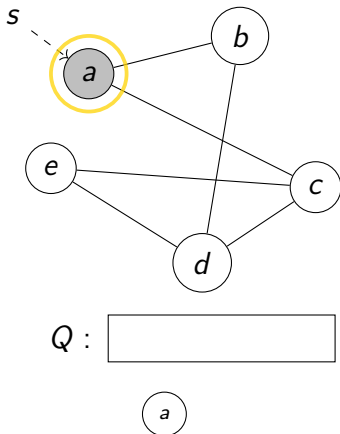


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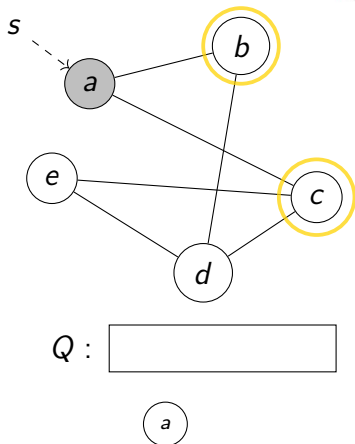


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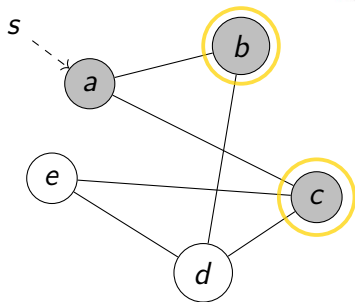


BFS Algorithm

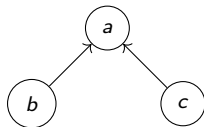
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$Q :$ $b \ c$



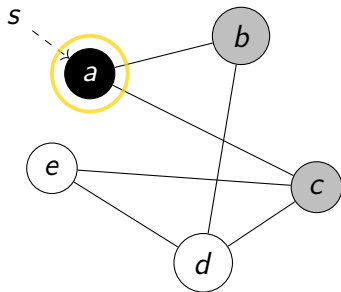


BFS Algorithm

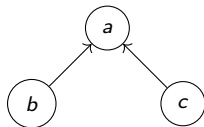
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$Q :$ b c



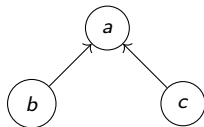
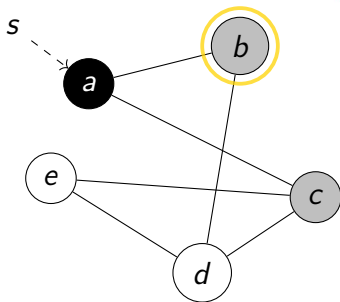


BFS Algorithm

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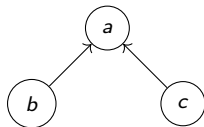
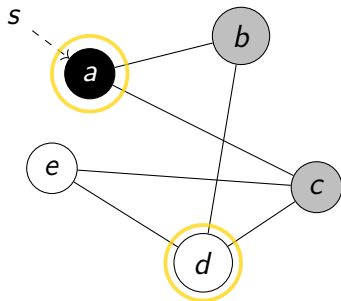


BFS Algorithm

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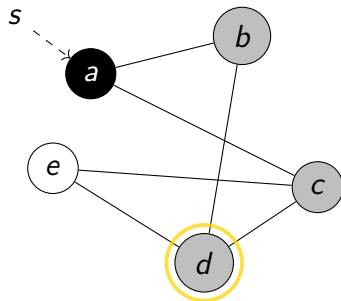


BFS Algorithm

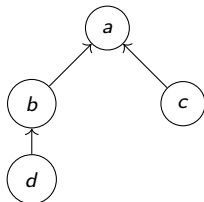
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$Q :$ $c \ d$



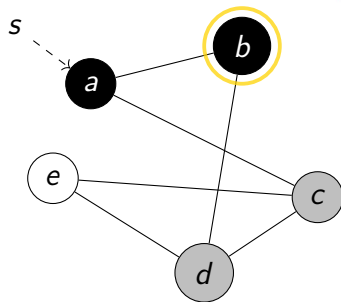


BFS Algorithm

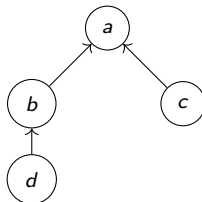
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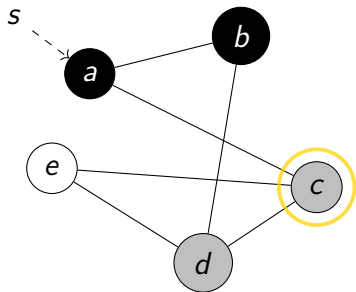


BFS Algorithm

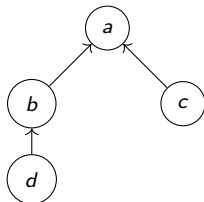
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$Q :$ d



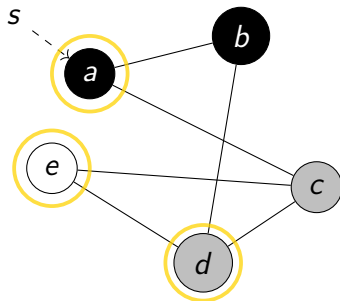


BFS Algorithm

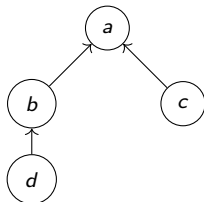
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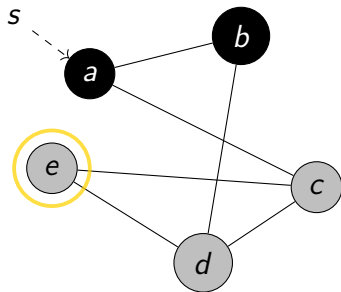


BFS Algorithm

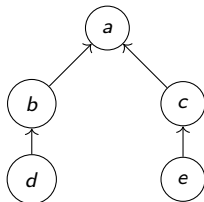
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Q : d e



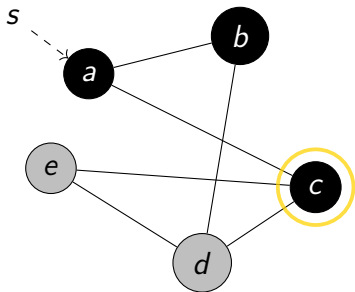


BFS Algorithm

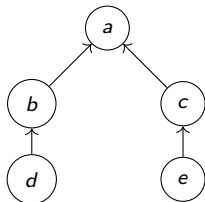
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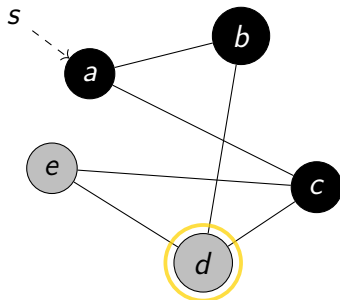


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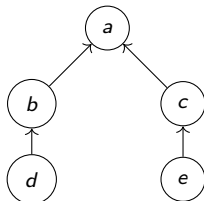
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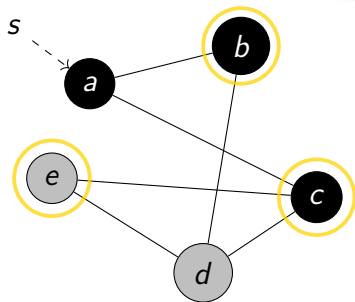


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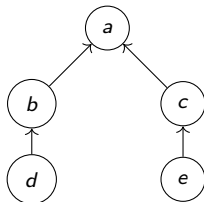
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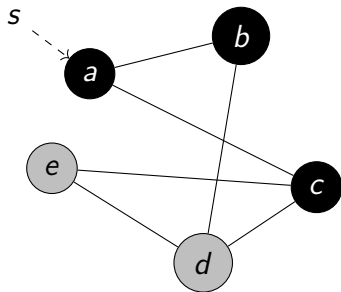


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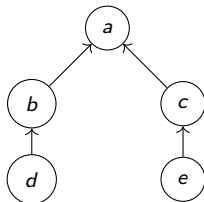
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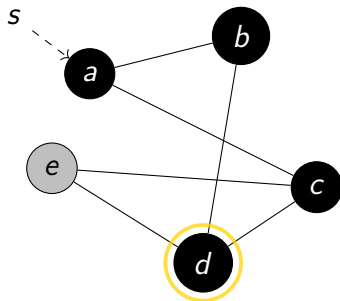


BFS Algorithm

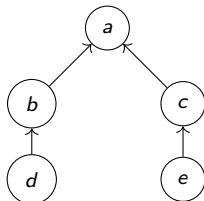
BFS(G, s)

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1  for each  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
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16              $v.\pi = u$ 
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```



Q : e



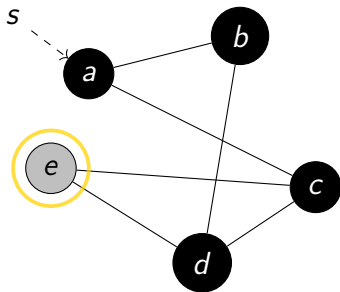


BFS Algorithm

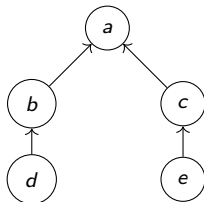
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Q :



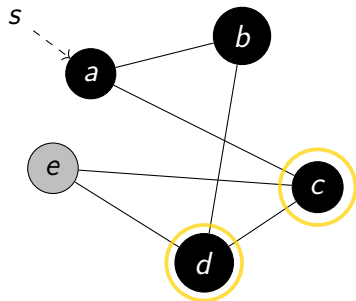


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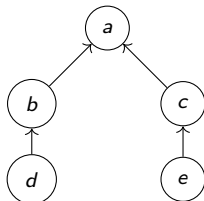
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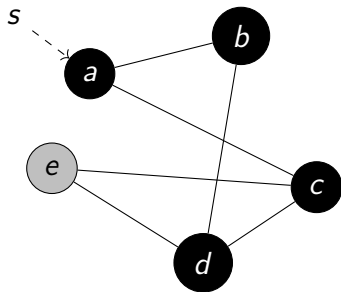


BFS Algorithm

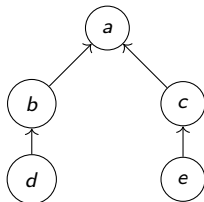
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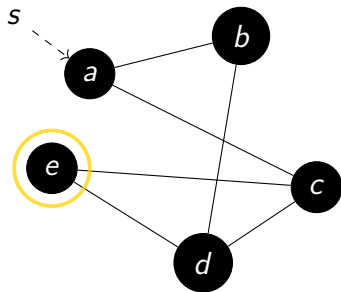


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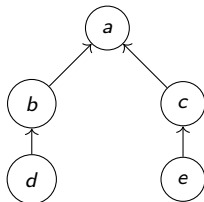
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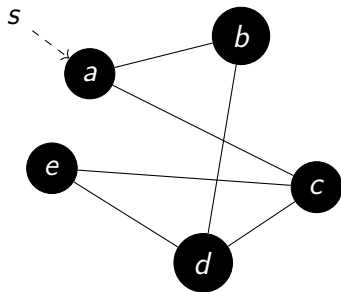


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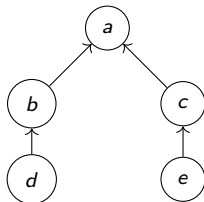
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BFS Algorithm - Complexity

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```

A large curly brace groups lines 1 through 4, with an arrow pointing from it to $O(V)$, indicating that this initialization phase has a time complexity of $O(V)$.



BFS Algorithm - Complexity

BFS(G, s)

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```

Complexity analysis annotations:

- Lines 1-4 are grouped by a bracket and labeled $O(V)$.
- Lines 5-9 are grouped by a bracket and labeled $O(1)$.



BFS Algorithm - Complexity

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Complexity analysis annotations:

- Lines 1-4: $O(V)$
- Lines 5-7: $O(1)$
- Line 9: $|V|$ steps
- Line 10: $|V|$ steps



BFS Algorithm - Complexity

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Annotations for complexity analysis:

- Lines 1-4: $O(V)$
- Lines 5-7: $O(1)$
- Line 9: $|V|$ steps
- Line 10: $|V|$ steps
- Lines 12-17: each neighbor for every node, so $|E|$ steps in total



BFS Algorithm - Complexity

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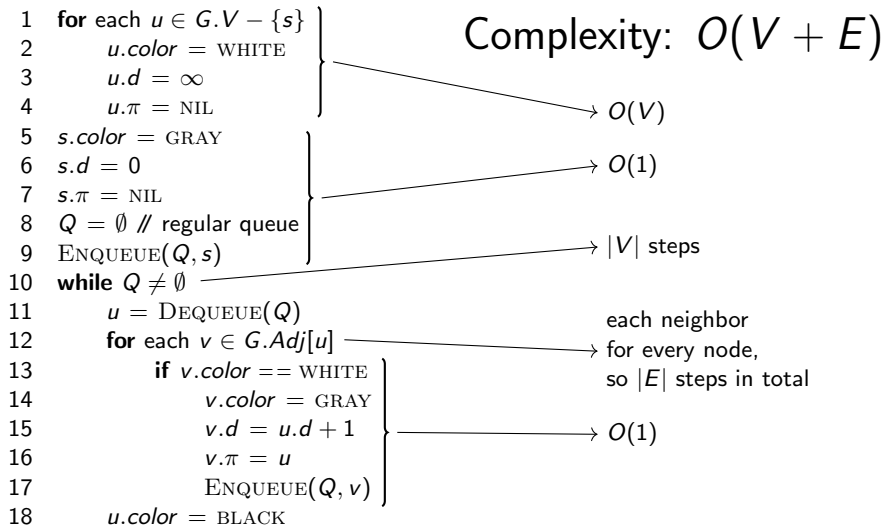
Complexity analysis annotations:

- Lines 1-4: $O(V)$
- Lines 5-7: $O(1)$
- Line 9: $|V|$ steps
- Line 10: $|V|$ steps
- Lines 12-17: $O(1)$ per neighbor, for every node, so $|E|$ steps in total



BFS Algorithm - Complexity

BFS(G, s)





BFS Algorithm - Discussion

- How does the algorithm behave if the graph is not connected?



BFS Algorithm - Discussion

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BFS Algorithm - Discussion

- How does the algorithm behave if the graph is not connected?
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- What complexity would the algorithm have on an adjacency matrix graph representation?



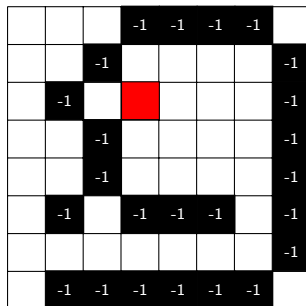
BFS Algorithm - Discussion

- How does the algorithm behave if the graph is not connected?
 - How do we modify the algorithm to completely traverse unconnected graphs?
- What complexity would the algorithm have on an adjacency matrix graph representation?
- How can we use it to determine the graph diameter?



Lee's Algorithm

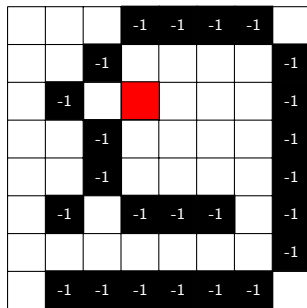
- how do we find the closest exit in a maze?
- represent the maze as a matrix, in which 0 means free pass and -1 means wall
- we move using V4 or V8





Lee's Algorithm

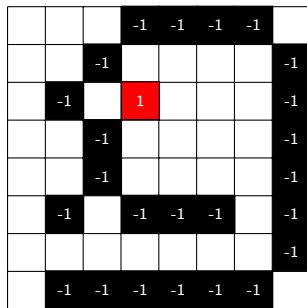
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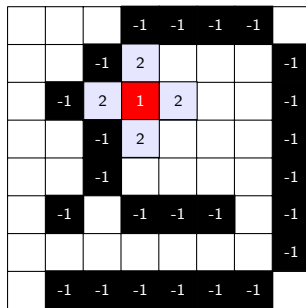
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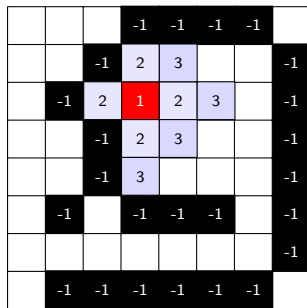
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		-1	2	3	4		-1
	-1	2	1	2	3	4	-1
		-1	2	3	4		-1
		-1	3	4			-1
	-1		-1	-1	-1		-1
							-1
	-1	-1	-1	-1	-1	-1	



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		-1	3	4	5	6	-1
	-1	13	-1	-1	-1	7	-1
	13	12	11	10	9	8	-1
	-1	-1	-1	-1	-1	-1	



Lee's Algorithm

- how do we find the closest exit in a maze?
- represent the maze as a matrix, in which 0 means free pass and -1 means wall
- we move using V4 or V8
- mark starting point with $k = 1$
- while we haven't reached the exit
 - the neighbors of nodes $k \rightarrow k + 1$
 - $k = k + 1$

			-1	-1	-1	-1	
		-1	2	3	4	5	-1
	-1	2	1	2	3	4	-1
		-1	2	3	4	5	-1
		-1	3	4	5	6	-1
	-1	13	-1	-1	-1	7	-1
14	13	12	11	10	9	8	-1
	-1	-1	-1	-1	-1	-1	



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- how do we rebuild the path?

			-1	-1	-1	-1	
		-1	2	3	4	5	-1
	-1	2	1	2	3	4	-1
		-1	2	3	4	5	-1
		-1	3	4	5	6	-1
	-1	13	-1	-1	-1	7	-1
14	13	12	11	10	9	8	-1
	-1	-1	-1	-1	-1	-1	



Bibliography

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