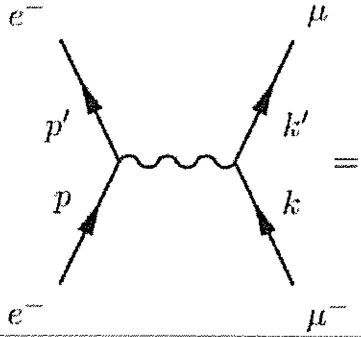
Scattering fra due fermioni

formula di Mott

SCATTERING TRA UN ELETTRONE E UN MUONE, FORMULA DI MOTT.

Consideriamo il caso di un elettrone (particella 1) che viene diffuso da un muone (particella 2) nell'ipotesi che l'energia dell'elettrone sia trascurabile rispetto alla massa del muone.

$i\mathcal{M} =$



$= \frac{ie^2}{q^2}\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu u(k).$

$M \sim (\overline{u}_1(p')\gamma^\mu u_1(p))(\overline{u}_2(k')\gamma_\mu u_2(k))$

$|M|^2 \sim [\overline{u}_1(p')\gamma^\mu u_1(p)][\overline{u}_1(p')\gamma^\nu u_1(p)]^\dagger[\overline{u}_2(k')\gamma_\mu u_2(k)][\overline{u}_2(k')\gamma_\nu u_2(k)]^\dagger$

Ma $[\overline{u}_1(p')\gamma^\mu u_1(p)]^\dagger$

$= \left[u_1^\dagger(p)\gamma^{\mu\dagger}(u_1{}^\dagger(p')\gamma^0)^\dagger\right]$

$= u_1^\dagger(p)\gamma^{\mu\dagger}\gamma^{0\dagger}u_1(p')$

$= u_1^\dagger(p)\gamma^{\mu\dagger}\gamma^0 u_1(p')$

Tenendo conto che

$= u_1^\dagger(p)\gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 u_1(p')$

$= \overline{u}_1(p)\gamma^0\gamma^{\mu\dagger}\gamma^0 u_1(p')$

$= \left[\overline{u}_1(p)\overline{\gamma}^\mu u_1(p')\right]$

Volendo calcolare la media sugli spinori $\langle|M|^2\rangle$ dobbiamo considerare l'espressione

$$\frac{1}{2}\sum_{S,S'=1,2}[\overline{u}_1(p',s')\gamma^\mu u_1(p,s)][\overline{u}_1(p',s')\gamma^\nu u_1(p,s)]^\dagger$$

$$=\frac{1}{2}\sum_{S,S'=1,2}[\overline{u}_1(p',s')\gamma^\mu u_1(p,s)]\left[\overline{u}_1(p,s)\overline{\gamma}^\nu u_1(p',s')\right]$$

$$=\frac{1}{2}\sum_{S,S'=1,2}\overline{u}_1(p',s')_\alpha(\gamma^\mu)_{\alpha\beta}u_1(p,s)_\beta\overline{u}_1(p,s)_\gamma\left(\overline{\gamma}^\nu\right)_{\gamma\delta}u_1(p',s')_\delta$$

$$=\frac{1}{2}\sum_{S'=1,2}u_1(p',s')_\delta\,\overline{u}_1(p',s')_\alpha(\gamma^\mu)_{\alpha\beta}\sum_{s=1,2}u_1(p,s)_\beta\overline{u}_1(p,s)_\gamma\left(\overline{\gamma}^\nu\right)_{\gamma\delta}$$

Ricordando che

$$\sum_{s'=1,2}u_1(p',s')_\delta\,\overline{u}_1(p',s')_\alpha=\frac{(\widehat{p'}+m)_{\delta\alpha}}{2m}$$

$$\frac{1}{2}\sum_{S,S'=1,2}[\overline{u}_1(p',s')\gamma^\mu u_1(p,s)][\overline{u}_1(p',s')\gamma^\nu u_1(p,s)]^\dagger=$$

$$=\frac{1}{2}\frac{(\widehat{p'}+m)_{\delta\alpha}}{2m}(\gamma^\mu)_{\alpha\beta}\frac{(\hat{p}+m)_{\beta\gamma}}{2m}\left(\overline{\gamma}^\nu\right)_{\gamma\delta}=\frac{1}{2}\,Tr\left[\frac{(\widehat{p'}+m)}{2m}\,\gamma^\mu\,\frac{(\hat{p}+m)}{2m}\,\overline{\gamma}^\nu\right]$$

Nel nostro caso $\overline{\gamma}^{\mu} = \gamma^{\mu}$ essendo
$(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0$ pertanto

$$\langle |M|^2 \rangle = \frac{1}{4*4*m_e{}^2*4m_{mu}{}^2} Tr\left[\left(\widehat{p'} + m\right)\gamma^{\mu}\,(\hat{p} + m)\,\gamma^{\nu}\right] \mathrm{Tr}\left[\left(\widehat{k'} + m\right)\gamma_{\mu}\,(\hat{k} + m)\,\gamma_{\nu}\right]\frac{e^4}{q^4}$$

Ricordiamo di seguito alcune proprieta' dell'operatore traccia:
$Tr(A + B) = Tr(A) + Tr(B)$
$Tr(\alpha A) = \alpha Tr(A)$
$Tr(AB) = Tr(BA)$
$Tr(ABC) = Tr(CAB) = Tr(BCA)$

La traccia del prodotto di un numenro dispari di matrici gamma è 0
Infatti
$Tr[\hat{a}_1 \dots \hat{a}_n] = Tr[\hat{a}_1 \dots \hat{a}_n\gamma^5\gamma^5] = Tr[\gamma^5\hat{a}_1 \dots \hat{a}_n\gamma^5] = (-1)^n Tr[\hat{a}_1 \dots \hat{a}_n]$
Dove si è usato la proprieta' ciclica della traccia e l'anticommutativita' di γ^5 con γ^{μ}.

$$Tr\left[\hat{a}\hat{b}\right] = Tr\left[a_{\mu}\gamma^{\mu}b_{\nu}\gamma^{\nu}\right] = \frac{1}{2}Tr\left[a_{\mu}b_{\nu}\gamma^{\mu}\gamma^{\nu} + a_{\mu}b_{\nu}\gamma^{\nu}\gamma^{\mu}\right] = \frac{1}{2}a_{\mu}b_{\nu}Tr[\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}]$$

Poiché $Tr[\gamma^{\mu}\gamma^{\nu}] = 4g^{\mu\nu}$
$Tr\left[\hat{a}\hat{b}\right] = 4a_{\mu}b_{\nu}g^{\mu\nu} = 4ab$

Usando poi $\hat{a}\hat{b} = -\hat{b}\hat{a} + 2ab$ si puo' calcolare
$Tr[\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4] = Tr[(-\hat{a}_2\hat{a}_1 + 2a_1a_2)\hat{a}_3\hat{a}_4] = -Tr[\hat{a}_2(-\hat{a}_3\hat{a}_1 + 2a_1a_3)\hat{a}_4] + 8(a_1a_2)(a_3a_4) =$
$= Tr[\hat{a}_2\hat{a}_3(-\hat{a}_4\hat{a}_1 + 2a_1a_4)] - 8(a_1a_3)(a_2a_4) + 8(a_1a_2)(a_3a_4) =$
$= -Tr[\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_1] + 8(a_1a_4)(a_2a_3) - 8(a_1a_3)(a_2a_4) + 8(a_1a_2)(a_3a_4)$
Da cui
$Tr[\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4] = 4(a_1a_2)(a_3a_4) - 4(a_1a_3)(a_2a_4) + 4(a_1a_4)(a_2a_3)$

Consideriamo ora il termine $Tr\left[\left(\widehat{p'} + m\right)\gamma^{\mu}\,(\hat{p} + m)\,\gamma^{\nu}\right]$
$Tr\left[\left(\widehat{p'} + m\right)\gamma^{\mu}\,(\hat{p} + m)\,\gamma^{\nu}\right] = Tr\left[\left(\widehat{p'}\gamma^{\mu} + m\gamma^{\mu}\right)(\hat{p}\gamma^{\nu} + m\gamma^{\nu})\right] =$
$= Tr\left[\widehat{p'}\gamma^{\mu}\hat{p}\gamma^{\nu} + m^2\gamma^{\mu}\gamma^{\nu}\right]$
I termimi lineari in m sono nulli perché prodotti di un numero dispari di matrici gamma.
$Tr\left[\left(\widehat{p'} + m\right)\gamma^{\mu}\,(\hat{p} + m)\,\gamma^{\nu}\right] = Tr\left[\widehat{p'}\gamma^{\mu}\hat{p}\gamma^{\nu} + m^2\gamma^{\mu}\gamma^{\nu}\right] = 4\left[p'_{\mu}p_{\nu} + p'_{\nu}p_{\mu} - g_{\mu\nu}p'p + m^2g_{\mu\nu}\right]$

Consideriamo ora il termine
$Tr\left[\left(\widehat{p'} + m\right)\gamma^{\mu}\,(\hat{p} + m)\gamma^{\nu}\right]\mathrm{Tr}\left[\left(\widehat{k'} + m\right)\gamma_{\mu}\,\left(\hat{k} + m\right)\gamma_{\nu}\right] =$
$= 16\left[p'_{\mu}p_{\nu} + p'_{\nu}p_{\mu} - g_{\mu\nu}p'p + m_{el}{}^2g_{\mu\nu}\right]\left[k'^{\mu}k^{\nu} + k'^{\nu}k^{\mu} - g^{\mu\nu}k'k + m_{mu}{}^2g^{\mu\nu}\right] =$
$= 16[(k'p')(kp) + (k'p)(kp') - (p'p)(k'k) + m_{mu}{}^2(p'p) +$
$+(p'k)(pk') + (p'k')(pk) - (p'p)(k'k) + m_{mu}{}^2(p'p) +$
$-(p'p)(k'k) - (p'p)(k'k) - 4(p'p)(k'k) - 4m_{mu}{}^2(p'p) +$
$+m_{el}{}^2(k'k) + m_{el}{}^2(k'k) - 4m_{el}{}^2(k'k) + 4m_{el}{}^2m_{mu}{}^2]$

$= 16[2(k'p')(kp) + 2(k'p)(kp') - 2m_{mu}{}^2(p'p) - 2m_{el}{}^2(k'k) + 4m_{el}{}^2m_{mu}{}^2]$
Ipotizzando la massa del muone molto grande possiamo considerare il centro di massa del sistema coincidente con il muone pertanto $k = k' = (m_{mu}, 0)$
$Tr\left[\left(\widehat{p'} + m\right)\gamma^{\mu}\,(\hat{p} + m)\gamma^{\nu}\right]\mathrm{Tr}\left[\left(\widehat{k'} + m\right)\gamma_{\mu}\,\left(\hat{k} + m\right)\gamma_{\nu}\right] =$

$$= 16[2m_{mu}{}^2E'_eE_e + 2m_{mu}{}^2E'E - 2m_{mu}{}^2(p'p) - 2m_{el}{}^2m_{mu}{}^2 + 4m_{el}{}^2m_{mu}{}^2]$$
$$= 16[4m_{mu}{}^2E'_eE_e - 2m_{mu}{}^2(p'p) + 2m_{el}{}^2m_{mu}{}^2] =$$
$$= 16 * 2[2m_{mu}{}^2E'_eE_e - m_{mu}{}^2(p'p) + m_{el}{}^2m_{mu}{}^2]$$

$$\langle |M|^2\rangle = \frac{16*2}{4*4*m_e{}^2*4m_{mu}{}^2} Tr\left[\left(\widehat{p'} + m\right)\gamma^\mu\,(\hat{p} + m)\,\gamma^\nu\right] \mathrm{Tr}\,\left[\left(\widehat{k'} + m\right)\gamma_\mu\,(\hat{k} + m)\,\gamma_\nu\right]\frac{e^4}{q^4}$$

Sostituendo

$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}[2m_{mu}{}^2E'_eE_e - m_{mu}{}^2(p'p) + m_{el}{}^2m_{mu}{}^2]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}\left[2m_{mu}{}^2E'_eE_e - m_{mu}{}^2\left(E_e{}^2 - |\vec{p}|^2cos\vartheta\right) + m_{el}{}^2m_{mu}{}^2\right]\frac{e^4}{q^4}$$

In un sistema coincidente con il cento di massa $E'_e = E_e$

$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}\left[2m_{mu}{}^2E_e{}^2 - m_{mu}{}^2\left(E_e{}^2 - |\vec{p}|^2cos\vartheta\right) + m_{el}{}^2m_{mu}{}^2\right]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}\left[2m_{mu}{}^2E_e{}^2 - m_{mu}{}^2\left(E_e{}^2 - |\vec{p}|^2cos\vartheta\right) + m_{el}{}^2m_{mu}{}^2\right]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}\left[2m_{mu}{}^2E_e{}^2 - m_{mu}{}^2E_e{}^2 + m_{mu}{}^2|\vec{p}|^2cos\vartheta + (E_e{}^2 - |\vec{p}|^2)m_{mu}{}^2\right]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}\left[2m_{mu}{}^2E_e{}^2 + m_{mu}{}^2|\vec{p}|^2cos\vartheta - m_{mu}{}^2|\vec{p}|^2\right]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}\left[2m_{mu}{}^2E_e{}^2 - 2m_{mu}{}^2|\vec{p}|^2\left(\frac{1-cos\vartheta}{2}\right)\right]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{1}{2*m_e{}^2*m_{mu}{}^2}\left[2m_{mu}{}^2E_e{}^2 - 2m_{mu}{}^2|\vec{p}|^2sen^2\frac{\vartheta}{2}\right]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{m_{mu}{}^2}{m_e{}^2*m_{mu}{}^2}\left[E_e{}^2 - |\vec{p}|^2sen^2\frac{\vartheta}{2}\right]\frac{e^4}{q^4}$$

Ma

$$|\vec{p}|^2 = E_e{}^2 - m_e{}^2 = E_e{}^2(1 - \frac{m_e{}^2}{E_e{}^2})$$
$$|\vec{p}|^2 = E_e{}^2 - m_e{}^2 = E_e{}^2(1 - \frac{m_e{}^2}{m_e{}^2/(1-\beta^2)})$$
$$|\vec{p}|^2 = E_e{}^2 - m_e{}^2 = \beta^2E_e{}^2$$

Con $\beta = \frac{v}{c}$

$$E_e{}^2 = \frac{|\vec{p}|^2}{\beta^2}$$

$$\langle |M|^2\rangle = \frac{1}{m_e{}^2}\left[\frac{|\vec{p}|^2}{\beta^2} - \frac{\beta^2}{\beta^2}|\vec{p}|^2sen^2\frac{\vartheta}{2}\right]\frac{e^4}{q^4}$$
$$\langle |M|^2\rangle = \frac{1}{m_e{}^2}\frac{|\vec{p}|^2}{\beta^2}\left[1 - \beta^2sen^2\frac{\vartheta}{2}\right]\frac{e^4}{q^4}$$

Ma $|q| = 2|\overline{\boldsymbol{p}}|sen\frac{\vartheta}{2}$

$$\langle |M|^2\rangle = \frac{1}{m_e{}^2}\frac{|\vec{p}|^2}{\beta^2}\left[1 - \beta^2sen^2\frac{\vartheta}{2}\right]\frac{e^4}{16|\overline{\boldsymbol{p}}|^4sen^4\frac{\vartheta}{2}}$$
$$\langle |M|^2\rangle = \frac{1}{m_e{}^2}\frac{1}{\beta^2}\left[1 - \beta^2sen^2\frac{\vartheta}{2}\right]\frac{e^4}{16|\overline{\boldsymbol{p}}|^2sen^4\frac{\vartheta}{2}}$$

Inserendo i coefficienti di normalizzazione $\sqrt{\frac{m}{VE}}$

$$S_{fi} = \frac{m_em_{mu}}{\sqrt{E_eE'_eE_{mu}E'_{mu}}}\langle |M|^2\rangle(2\pi)^4\delta^4(\boldsymbol{p'} - \boldsymbol{p} - (\boldsymbol{k'} - \boldsymbol{k}))$$

Abbiamo tenuto conto che

$$|(2\pi)^4\delta^4(\boldsymbol{p}'-\boldsymbol{p}-(\boldsymbol{k}'-\boldsymbol{k})|^2 = TV(2\pi)^4\delta^4(\boldsymbol{p}'-\boldsymbol{p}-(\boldsymbol{k}'-\boldsymbol{k})$$

La sezione d'urto differenziale è data dal prodotto di W per lo spazio delle fasi diviso per il flusso incidente

dove $w=\frac{|S_{fi}|^2}{T}$ rappresenta la probabilità di transizione per unità di tempo

$$d\sigma = w\frac{spazio\ delle\ fasi}{flusso\ incidente}$$

con $flusso\,incidente=\frac{v_{rel}}{V}$ dove v_{rel} è la velocità relativa tra la particella incidente e la particella bersaglio e che $spazio\ delle\ fasi=\frac{d^3\mathbf{p}'d^3\mathbf{k}'}{(2\pi)^3V(2\pi)^3V}$

Per calcolare il fattore di spazio delle fasi occorre effettuare questo calcolo nel sistema del centro di massa dove $\overrightarrow{\boldsymbol{p}'}+\overrightarrow{\boldsymbol{k}'}=0$ e $\overrightarrow{\boldsymbol{p}}+\overrightarrow{\boldsymbol{k}}=0$.

Integrando in $d^3\boldsymbol{k}'$

$$d\sigma = \frac{e^2m_e^2m_{mu}^2}{E_eE_e'E_{mu}E_{mu}'}<|M|^2>(2\pi)^4\delta(E_e'-E_e-E_{mu}+E_{mu}')\frac{d^3\mathbf{p}'}{(2\pi)^3(2\pi)^3v_{rel}}$$

$$d\sigma = \frac{e^2m_e^2m_{\mu}^2}{E_eE_e'E_{mu}E_{mu}'}<|M|^2>(2\pi)^4\delta(E_e'-E_e-E_{mu}+E_{mu}')\frac{|\overrightarrow{\mathbf{p}'}|^2d|\overrightarrow{\mathbf{p}'}|d\Omega}{(2\pi)^3(2\pi)^3v_{rel}}$$

Consideriamo ora le seguenti proprieta' della delta di Dirac

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$
$$f(x)\cong f(x_0)+f'(x_0)(x-x_0)$$
$$\delta(f(x))=\frac{1}{|f'(x_0)|}\delta(x-x_0)$$

Ora considerando $x=|\vec{p}'|$ e $x_0=|\vec{p}|$

Scriviamo

$$E'_e=\sqrt{|\vec{p}'|^2-m_e}$$
$$E'_{mu}=\sqrt{|\vec{p}'|^2-m_{mu}}$$
$$E_e=\sqrt{|\vec{p}|^2-m_e}$$
$$E_{mu}=\sqrt{|\vec{p}|^2-m_{mu}}$$

$$\delta(E'_e-E_e-E_{mu}-E'_{mu})=\delta\left(E_e+E_{mu}-E'_e-E'_{mu}\right)=$$
$$=\delta\left(E_e+E_{mu}-\sqrt{|\vec{p}'|^2-m_e}-\sqrt{|\vec{p}'|^2-m_{mu}}\right)$$
$$\frac{\partial}{\partial|\vec{p}'|}\left(E_e+E_{mu}-E'_e-E'_{mu}\right)=\frac{\partial}{\partial|\vec{p}'|}\left(E_e+E_{mu}-\sqrt{|\vec{p}'|^2-m_e}-\sqrt{|\vec{p}'|^2-m_{mu}}\right)=$$
$$=-\frac{|\vec{p}'|}{E'_e}-\frac{|\vec{p}'|}{E'_{mu}}=-\frac{(E_e+E_{mu})|\vec{p}'|}{E'_eE'_{mu}}$$

$$\delta(E'_e - E_e - E_{mu} - E'_{mu}) = \frac{E'_e E'_{mu}}{(E_e+E_{mu})|\vec{p}'|}\delta(|\vec{p}'| - |\vec{p}|)$$

$$d\sigma = \frac{m_e m_{mu}}{\sqrt{E_e E_e' E_{mu} E_{mu}'}} <|\mathrm{M}|^2> (2\pi)^4 \frac{(E_e' E_{mu}')}{(E_e + E_{mu})\left|\vec{\mathbf{p}}'\right|}\delta(\left|\vec{\mathbf{p}}'\right| - \left|\vec{\mathbf{p}}\right|)\frac{\left|\vec{\mathbf{p}}'\right|^2 d\left|\vec{\mathbf{p}}'\right| d\Omega}{(2\pi)^3(2\pi)^3 v_{rel}}$$

Integrando in $d|\vec{p}'|$

$$d\sigma = \frac{m_e^2 m_{mu}^2}{E_e^2 E_{mu}^2} <|\mathrm{M}|^2> \frac{(E_e E_{mu})}{(E_e + E_{mu})}\frac{\left|\vec{\mathbf{p}}'\right| d\Omega}{(2\pi)^2 v_{rel}}$$

Ma $v_{rel} = \frac{|\vec{p}|_{el}}{E_{el}} - \frac{|\vec{p}|_{mu}}{E_{mu}}$

Nel nostro caso $p_{mu} = (m_{mu}, 0)$

pertanto $v_{rel} = v_{el}$

e $E_e E_{mu} v_{rel} = m_{mu}|\vec{\boldsymbol{p}}|_{el}$

$$d\sigma = \frac{m_e^2 m_{mu}^2}{E_e^2 E_{mu}^2} <|\mathrm{M}|^2> \frac{(E_e E_{mu})^2}{(E_e + E_{mu})}\frac{\left|\vec{\mathbf{p}}'\right| d\Omega}{(2\pi)^2 m_{mu}\left|\vec{\mathbf{p}}\right|}$$

$$d\sigma = m_e^2 m_{mu}^2 <|\mathrm{M}|^2> \frac{1}{(E_e + E_{mu})}\frac{d\Omega}{(2\pi)^2 m_{mu}}$$

sostituendo il valore di

$$d\sigma = m_e^2 m_{mu}^2 \langle |M|^2 \rangle \frac{1}{(E_e + E_{mu})}\frac{d\Omega}{(2\pi)^2 m_{mu}}$$

Poiche' $(E_e + E_{mu}) \sim E_{mu} = m_{mu}$

$$d\sigma = m_e^2 \langle |M|^2 \rangle \frac{d\Omega}{(2\pi)^2}$$

$$d\sigma = m_e^2 \frac{1}{m_e^2}\frac{1}{\beta^2}\left[1 - \beta^2 sen^2\frac{\vartheta}{2}\right]\frac{e^4}{16|\vec{\boldsymbol{p}}|^2 sen^4\frac{\vartheta}{2}}\frac{d\Omega}{(2\pi)^2}$$

$$\sigma = \frac{1}{16\,\beta^2}\left[1 - \beta^2 sen^2\frac{\vartheta}{2}\right]\frac{e^4}{|\vec{\boldsymbol{p}}|^2 sen^4\frac{\vartheta}{2}}\frac{d\Omega}{(2\pi)^2}$$