Mechanics Today

Volume 1

1972

Edited by

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Published by Pergamon Press
on behalf of the
AMERICAN ACADEMY OF MECHANICS

PERGAMON PRESS INC.

New York · Toronto · Oxford · Sydney · Braunschweig

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Qualitative Theory of the Ordinary Differential Equations of Nonlinear Elasticity

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1 INTRODUCTION

The elastica is the simplest nonlinear model for the static deformation of flexible rods. Its theory, developed by Jas. Bernoulli, D. Bernoulli, and Euler, rests on three types of assumptions:

(i) *Kinematic restrictions*. The configuration of the elastica is described by an inextensible curve in a plane, which we take to be the x, y-plane. If S denotes the arc length parameter of this curve, $S_1 \leqslant S \leqslant S_2$, and if $\theta(S)$ denotes the angle between the tangent to the curve at S and the x-axis, then the coordinates $x(S)$, $y(S)$, of the particle S satisfy

$$x'(S) = \cos\theta(S), \qquad y'(S) = \sin\theta(S). \tag{1.1}$$

Here the primes denote differentiation with respect to S.

(ii) *Balance of moments*. Since the configuration of the elastica is determined from the single variable θ via Eq. (1.1), only one equilibrium equation is needed. This is obtained from the requirement that the resultant moment on any segment of the elastica be zero. If the external loads applied to the elastica consist only of a force with components $(\Lambda\cos\alpha, \Lambda\sin\alpha, 0)$ and moment with components $(0, 0, M_0)$ applied at the end S_1, and a balancing set of loads at the other end, then the bending moment M satisfies

$$M' + \Lambda\sin(\theta - \alpha) = 0. \tag{1.2}$$

(iii) *Constitutive assumptions*. The bending moment M depends linearly on the change in curvature from the undeformed to the deformed configuration:

$$M(S) = B(S)[\theta'(S) - \Theta'(S)], \tag{1.3}$$

where Θ is the given tangent angle to the undeformed curve at S and $B(S)$ is the given bending stiffness at S. Equation (1.3) is the Bernoulli–Euler law. If we substitute Eq. (1.3) into Eq. (1.2), we obtain the following nonlinear ordinary differential equation for θ:

$$[B(S)(\theta' - \Theta')]' + \Lambda \sin(\theta - \alpha) = 0. \tag{1.4}$$

In 1744, Euler[1] studied the system (1.1), (1.4) when $B = \text{const.}$, $\Theta = 0$, $\alpha = 0$. By a penetrating analysis of the geometric implications of the equations, he was able to classify and describe fully all possible solutions of the equations. This work, which relied on a minimum of computation, was a triumph of nonlinear physics and nonlinear mathematics. (Cf. Ref. [2] for the historical development of the elastica.)

In the second half of the nineteenth century it was recognized that elastica problems could be solved in terms of elliptic functions. (Euler, in his study of the elastica, had effectively determined many important properties of these functions.) From then on, the whole formal mathematical apparatus, consisting of identities, integrals, and tables of values, that had grown up about these functions was exploited to treat special elastica problems. (Cf. Refs. [3–5].) This work has continued even till today. It is now clear, however, that numerical values for the solutions are more directly obtained by numerical solutions of the differential equations.

In recent years the formulation of general theories of elastic rods and shells (cf. Refs. [6, 7]) has led to nonlinear systems of equations with a far richer structure than Eqs. (1.1)–(1.3). For example, the simple Bernoulli–Euler law (1.3), is replaced by a general system of nonlinear constitutive relations. (The nonlinearity of the constitutive relations in geometrically exact theories is essential to prevent such unrealistic situations as the compression of a fiber of positive length to zero length by a bounded force system.)

There is scarcely any likelihood of solving such problems in closed form by means of special functions. To determine the behavior of solutions, one is therefore compelled either to resort to numerical computation or to study the qualitative behavior of the solutions just as Euler did. The latter course has the advantage of treating a whole class of problems at one time and thereby determining the general consequences of the physics.

In this article we study the qualitative behavior of the ordinary differential equations arising from general one-dimensional models of nonlinear elasticity. We concentrate on autonomous systems (i.e. systems in which the dependent variable does not appear explicitly). We first formulate the governing equations for the models and then obtain some simple consequences of these. Then we analyze the behavior of rods under terminal loadings, paying special attention to the compressive buckling of straight rods. Next we treat the buckling of rings under hydrostatic pressure. After this we examine the problem of necking for bars in tension. In each of these problems we just examine those properties that can be analyzed by simple computations. We accordingly sacrifice the richness of detail that would follow from careful estimates. Our work should nevertheless provide a convenient entrée to a more intensive analysis. We conclude with a brief survey of related work and of open problems.

Throughout this work we make the blanket smoothness assumption that the functions constituting the data of a problem have as many derivatives as are needed in the analysis. Under these conditions, the problems we treat will have classical solutions.

2 FORMULATION OF THE BOUNDARY-VALUE PROBLEMS

We employ a general one-dimensional model, consisting of a system of ordinary differential equations, for the static plane strain of thin nonlinearly elastic bodies. This theory is refined enough to describe flexure, longitudinal extension, transverse extension, and shear. It is easily modified to treat the planar deformation of rods and the axisymmetric deformation of axisymmetric shells. (The spatial deformation of rods is governed by a more complicated, but no doubt accessible, system of ordinary differential equations; other shell problems lead to systems of partial differential equations.)

In this theory the configuration of a body is determined by two 2-vector functions $\mathbf{r}$ and $\mathbf{p}$ of a material coordinate S with values

$$\mathbf{r}(S), \mathbf{p}(S), \qquad S_1 \leqslant S \leqslant S_2. \tag{2.1}$$

We denote differentiation with respect to S by a prime. We call the material curve with position $\mathbf{r}$ the *axis* and we call $\mathbf{p}$ the *director*.

We take S to be the arc length parameter of the axis in its reference (undeformed) configuration and we let $\mathbf{R}(S)$, $\mathbf{P}(S)$ be the values of $\mathbf{r}(S)$ and $\mathbf{p}(S)$ in this configuration. These vectors are shown in Fig. 1a. Note that $\mathbf{R}'$ is the unit tangent vector to the axis in the reference configuration.

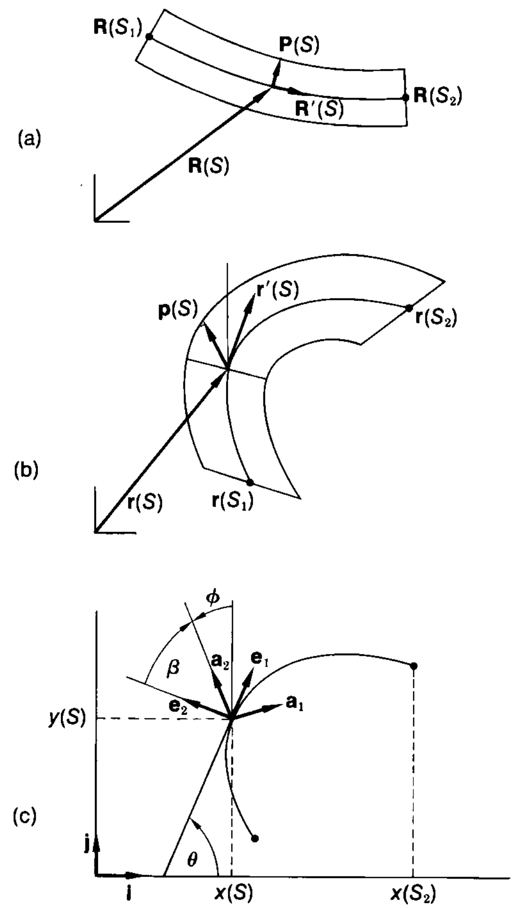

Fig. 1.

In Fig. 1b we show an arbitrary configuration of this body. Here $\mathbf{r}'$ is tangent to the axis but not necessarily a unit vector.

We interpret $\mathbf{r}$ as characterizing the gross deformation of the body and $\mathbf{p}$ as characterizing the local deformation of a section. In particular, a contraction of a section S corresponds to a decrease in the length of $\mathbf{p}(S)$ and a shear of a section S corresponds to a change in the angle between $\mathbf{r}'(S)$ and $\mathbf{p}(S)$. On the basis of the geometrical significance of $\mathbf{r}'$ and $\mathbf{p}$, we assume that they are independent.

Let the plane of deformation be the x, y-plane with unit base vectors

$\mathbf{i}, \mathbf{j}$. Let $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. We define two additional pairs of orthonormal vectors:

$$\mathbf{e}_1(S) = \mathbf{r}'(S)/|\mathbf{r}'(S)| = \cos\theta(S)\mathbf{i} + \sin\theta(S)\mathbf{j},$$
$$\mathbf{e}_2(S) = \mathbf{k} \times \mathbf{e}_1(S) = -\sin\theta(S)\mathbf{i} + \cos\theta(S)\mathbf{j}, \tag{2.2}$$

$$\mathbf{a}_1(S) = -\mathbf{k} \times \mathbf{p}(S)/|\mathbf{p}(S)| = \cos\phi(S)\mathbf{i} + \sin\phi(S)\mathbf{j},$$
$$\mathbf{a}_2(S) = \mathbf{p}(S)/|\mathbf{p}(S)| = -\sin\phi(S)\mathbf{i} + \cos\phi(S)\mathbf{j}. \tag{2.3}$$

We set

$$\theta = \beta + \phi. \tag{2.4}$$

We introduce component representations of our geometrical variables by

$$\mathbf{r}(S) = x(S)\mathbf{i} + y(S)\mathbf{j}, \tag{2.5}$$

$$\mathbf{r}'(S) = [1 + \epsilon(S)]\mathbf{e}_1(S), \tag{2.6}$$

$$\mathbf{p}(S) = [1 + \rho(S)]\mathbf{a}_2(S). \tag{2.7}$$

The independence of $\mathbf{r}'$ and $\mathbf{p}$ then implies that

$$1 + \epsilon > 0, \quad -\frac{\pi}{2} < \beta < \frac{\pi}{2}, \quad 1 + \rho > 0. \tag{2.8}$$

We assume that in the reference configuration,

$$\epsilon = 0, \quad \beta = 0, \quad \rho = 0. \tag{2.9}$$

Thus ϵ is the extension of the axis, ρ is the extension of the director, and β is the angle of shear between the axis and the director. Also, θ is the angle from the x-axis to the tangent to the axis and ϕ is the angle from the y-axis to the director. These variables are shown in Fig. 1c.

In treating problems for which there is shear, (i.e. for which β is not constrained to vanish) it proves convenient to replace the strain variables ϵ, β with a new pair ξ, η by

$$1 + \xi = (1 + \epsilon)\cos\beta, \quad \eta = (1 + \epsilon)\sin\beta. \tag{2.10}$$

In terms of these variables, the scalar version of Eq. (2.6) becomes

$$x' = (1 + \xi)\cos\phi - \eta\sin\phi, \quad y' = (1 + \xi)\sin\phi + \eta\cos\phi. \tag{2.11}$$

The first two of inequalities (2.8) require

$$1 + \xi > 0. \tag{2.12}$$

This inequality, when supplemented with the requirements that the second of inequalities (2.8) hold at one point and that ϵ and β be continuous, implies that the first two of inequalities (2.8) hold.

The mechanical variables entering this theory are the stress resultants $\mathbf{n}, \boldsymbol{\mu}, \boldsymbol{\nu}$. These are the generalized forces corresponding to the generalized deformations $\mathbf{r}', \mathbf{p}', \mathbf{p}$. $\mathbf{n}$ may be interpreted as the resultant force and $\mathbf{m} = \mathbf{p} \times \boldsymbol{\mu}$ as the resultant moment over a section. (For further details and interpretations, *see* Refs. [6, 7] and the references cited therein.) The equilibrium equations, representing the balance of forces, director forces, and moments, have the form

$$\mathbf{n}' + \mathbf{f} = \mathbf{0}, \tag{2.13}$$

$$\boldsymbol{\mu}' - \boldsymbol{\nu} + \mathbf{g} = \mathbf{0}, \tag{2.14}$$

$$(\mathbf{r} \times \mathbf{n} + \mathbf{p} \times \boldsymbol{\mu})' + \mathbf{r} \times \mathbf{f} + \mathbf{p} \times \mathbf{g} = \mathbf{0}. \tag{2.15}$$

Here $\mathbf{f}$ is the prescribed force per unit of S and $\mathbf{g}$ is the prescribed director force per unit of S. From these equations, we obtain

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \mathbf{p} \times \mathbf{g} = \mathbf{0}, \tag{2.16}$$

$$\mathbf{r}' \times \mathbf{n} + \mathbf{p}' \times \boldsymbol{\mu} + \mathbf{p} \times \boldsymbol{\nu} = \mathbf{0}. \tag{2.17}$$

We regard Eq. (2.17) as a constitutive restriction. (It plays the same role in this one-dimensional theory as the symmetry of the stress tensor plays in three-dimensional continuum mechanics.)

We introduce components of the resultants $\mathbf{n}, \boldsymbol{\mu}, \boldsymbol{\nu}, \mathbf{m}$ by

$$\mathbf{n} = N\mathbf{e}_1 + Q\mathbf{e}_2 = T_1\mathbf{a}_1 + T_2\mathbf{a}_2, \tag{2.18}$$

$$\boldsymbol{\mu} = H\mathbf{a}_1 + G\mathbf{a}_2, \tag{2.19}$$

$$\boldsymbol{\nu} = K\mathbf{a}_1 + J\mathbf{a}_2, \tag{2.20}$$

$$\mathbf{m} = M\mathbf{k} = \mathbf{p} \times \boldsymbol{\mu} = -(1+\rho)H\mathbf{k}. \tag{2.21}$$

The component forms of Eq. (2.13) are

$$N' - Q\theta' + \mathbf{f} \cdot \mathbf{e}_1 = 0, \tag{2.22}$$

$$Q' + N\theta' + \mathbf{f} \cdot \mathbf{e}_2 = 0, \tag{2.23}$$

or alternatively,

$$T_1' - T_2\phi' + \mathbf{f} \cdot \mathbf{a}_1 = 0, \tag{2.24}$$

$$T_2' + T_1\phi' + \mathbf{f} \cdot \mathbf{a}_2 = 0. \tag{2.25}$$

The component forms of Eq. (2.14) are

$$H' - G\phi' - K + \mathbf{g} \cdot \mathbf{a}_1 = 0, \tag{2.26}$$

$$G' + H\phi' - J + \mathbf{g} \cdot \mathbf{a}_2 = 0. \tag{2.27}$$

Equation (2.16) has either of the two equivalent forms

$$M' + Q(1+\epsilon) + (\mathbf{p}\times\mathbf{g})\cdot\mathbf{k} = 0, \tag{2.28}$$

$$M' + T_2(1+\xi) - T_1\eta - (1+\rho)\mathbf{g}\cdot\mathbf{a}_1 = 0. \tag{2.29}$$

The constitutive restriction (2.17) reduces to

$$T_2(1+\xi) - T_1\eta - H\rho' - G(1+\rho)\phi' - K(1+\rho) = 0. \tag{2.30}$$

We do not use Eq. (2.26) since it is a consequence of Eqs. (2.21), (2.29), (2.30). Accordingly we can dispense with H and K.

Our one-dimensional material is termed *elastic* if T_1, T_2, G, J, M are given as functions of

$$1+\xi, \eta, 1+\rho, \rho', \phi', S. \tag{2.31}$$

The one-dimensional body is termed *homogeneous* if the argument S does not appear explicitly in these functions. A two-dimensional interpretation of homogeneity is that the combined constitutive contributions of thickness variation and material variation are constant along the axis. It is convenient to let these resultants depend on variables that vanish in the reference state. Toward this end, we let $\Theta(S)$ be the value of $\phi(S)$ (and of $\theta(S)$) in the reference state and we set

$$\psi(S) = \phi(S) - \Theta(S). \tag{2.32}$$

The constitutive equations can now be written explicitly in the form

$$T_1(S) = \hat{T}_1(\mathbf{w}(S), S), \ldots, \quad M(S) = \hat{M}(\mathbf{w}(S), S), \tag{2.33}$$

where $\hat{T}_1, \ldots, \hat{M}$ are functions of

$$(\mathbf{w}, S) \equiv (\xi, \eta, \rho, \rho', \psi', S). \tag{2.34}$$

We frequently denote derivatives with respect to components of $\mathbf{w}$ by subscripts. Thus $\partial M/\partial\xi = M_\xi$, etc. Note that the dependence of Θ on S is now manifested in the explicit dependence of the constitutive functions $\hat{T}_1, \ldots, \hat{M}$ on S. If these functions are to be independent of S, then either (i) the body must be homogeneous and $\Theta' =$ const. (in which case the reference configuration of the axis must be either circular or straight), or (ii) the non-homogeneity must exactly compensate for the non-constancy of Θ'. (This second possibility is rather special.)

For materials with independent constitutive variables $\mathbf{r}, \mathbf{r}', \mathbf{p}, \mathbf{p}'$, these constitutive relations are the most general form of elastic response consistent with frame-indifference and the assumption of planarity [6].

We must impose further conditions on the constitutive functions to ensure physically reasonable behavior. We adopt the requirement that the symmetric part of the matrix

$$\begin{pmatrix} \dfrac{\partial \hat{T}_1}{\partial \xi} & \dfrac{\partial \hat{T}_2}{\partial \xi} & \dfrac{\partial \hat{G}}{\partial \xi} & \dfrac{\partial \hat{M}}{\partial \xi} \\ \dfrac{\partial \hat{T}_1}{\partial \eta} & \dfrac{\partial \hat{T}_2}{\partial \eta} & \dfrac{\partial \hat{G}}{\partial \eta} & \dfrac{\partial \hat{M}}{\partial \eta} \\ \dfrac{\partial \hat{T}_1}{\partial \rho'} & \dfrac{\partial \hat{T}_2}{\partial \rho'} & \dfrac{\partial \hat{G}}{\partial \rho'} & \dfrac{\partial \hat{M}}{\partial \rho'} \\ \dfrac{\partial \hat{T}_1}{\partial \psi'} & \dfrac{\partial \hat{T}_2}{\partial \psi'} & \dfrac{\partial \hat{G}}{\partial \psi'} & \dfrac{\partial \hat{M}}{\partial \psi'} \end{pmatrix} \tag{2.35}$$

be positive-definite. Among the immediate consequences of this assumption are that N is an increasing function of ϵ, G is an increasing function of ρ', and M is an increasing function of ψ'. That Q be an increasing function of β is not implied (nor is it excluded). If the model is given a two-dimensional interpretation, it can be shown that this requirement is a direct consequence of the strong ellipticity condition of the two-dimensional theory [8].

We also demand that appropriate stress resultants become unbounded as components of $\mathbf{w}$ become unbounded, as $\xi \to -1$, and as $\rho \to -1$. These requirements mean that an infinite force is needed to produce an infinite strain. In particular, the last two conditions mean that an infinite force is required to squeeze a fiber down to zero length. We term this set of conditions the *growth* conditions.

Our one-dimensional elastic material is termed *hyperelastic* if there is a scalar valued strain-energy function W of $\mathbf{w}, S$ such that

$$\hat{T}_1 = W_\xi, \quad \hat{T}_2 = W_\eta, \quad \hat{G} = W_{\rho'}, \quad \hat{M} = W_{\psi'}, \quad \hat{J} + \frac{\hat{M}\phi'}{1+\rho} = W_\rho. \tag{2.36}$$

In this case the positive-definiteness of the symmetric part of matrix (2.35) reduces to the positive-definiteness of the Hessian matrix of second partial derivatives of W with respect to ξ, η, ρ', ψ'. The unboundedness of some resultants as either $\xi \to -1$ or $\rho \to -1$ is ensured by the stronger requirement that $W \to \infty$ as either $\xi \to -1$ or $\rho \to -1$.

The governing equations consist of the geometric relations (2.4), (2.11), the equilibrium Eqs. (2.24), (2.25), (2.27), (2.29), and the constitutive Eqs. (2.33) or (2.36). When the constitutive equations are substituted into the equilibrium equations, and these are supplemented by the geometric relations, we get a system of seven equations for the seven geometric unknowns $\xi, \eta, \rho, \psi, x, y, \theta$. These equations are to be supplemented with a suitable set of boundary conditions.

Before proceeding with our development, we note that a full existence and regularity theory for these boundary-value problems in the hyperelastic case has been obtained by use of the direct methods of the calculus of variations[6, 9]. This theory ensures the existence of at least one solution for bodies under dead loading and gives conditions on the material response for which there is at least one solution for bodies under other kinds of loading, such as hydrostatic pressure. For problems that admit "trivial" solutions, there are global multiplicity results[10, 11] based on variational arguments and there are local results[11–14] based on bifurcation theory. We do not discuss such results, but we use them implicitly, since they prevent our conclusions from being vacuous. (For static problems, existence theory is not a mere mathematical formality, but may have considerable physical significance. Indeed, in nonlinear elasticity there are innocuous looking problems that do not possess equilibrium solutions because the forces acting on the system overwhelm the resistance of the material and cause the body to respond dynamically. This happens in rubber balloons: There is a maximum internal pressure beyond which there are no equilibrium solutions.)

3 INTEGRALS OF THE GOVERNING EQUATIONS

If in the elastica Eq. (1.4) we take B and Θ' to be constant, (so that the elastica is homogeneous and has a circular or straight reference state), then we may multiply the equation by θ' and integrate to obtain the first integral

$$(\theta')^2 - (2\Lambda/B)\cos(\theta - \alpha) = \text{const.} \tag{3.1}$$

This is the explicit equation for the trajectories of the phase plane diagram for Eq. (1.4), (*see* Fig. 2), from which one can determine the qualitative behavior of any boundary-value problem for Eq. (3.1). In this section we obtain a variety of analogous, but certainly more complicated, results for the equations described in Section 2. These results help us to analyze specific problems in the remainder of this article.

If $\mathbf{f} = \mathbf{0}$, then Eq. (2.13) implies that

$$\mathbf{n} = \mathbf{const.} \tag{3.2}$$

If the resultant force applied to the end S_1 is $\Lambda(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$ with $\Lambda \geqslant 0$, then

$$\mathbf{n}(S_1) = -\Lambda(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}), \tag{3.3}$$

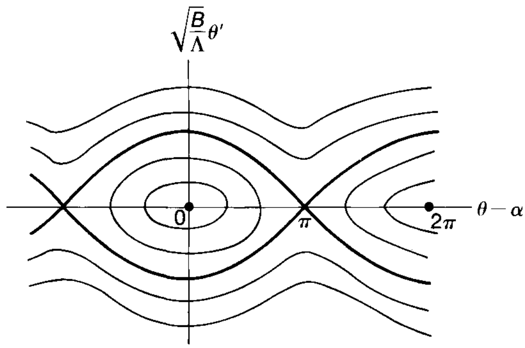

Fig. 2 Phase plane diagram of Eq. (1.4) for B = const., Θ' = const.

and Eqs. (3.2) and (2.33) yield the integrals

$$\hat{T}_1(\mathbf{w}, S) = \mathbf{n}\cdot\mathbf{a}_1 = -\Lambda\cos(\phi-\alpha), \tag{3.4}$$

$$\hat{T}_2(\mathbf{w}, S) = \mathbf{n}\cdot\mathbf{a}_2 = \Lambda\sin(\phi-\alpha). \tag{3.5}$$

If $\mathbf{p}\times\mathbf{g}$ also vanishes, then Eq. (2.16) yields

$$\mathbf{m}+\mathbf{r}\times\mathbf{n} = \mathbf{const.} \tag{3.6}$$

The substitution of Eqs. (2.33) and (3.3) into Eq. (3.6) gives

$$\hat{M}(\mathbf{w}, S) + \Lambda(y\cos\alpha - x\sin\alpha) = \text{const.} \tag{3.7}$$

This integral is of limited utility for our purposes because it contains x and y explicitly.

If $\mathbf{f}$ represents a hydrostatic loading of force intensity q per unit actual length of the axis, then $\mathbf{f}$ is given by

$$\mathbf{f} = q(\mathbf{k}\times\mathbf{r}') = q(1+\epsilon)\mathbf{e}_2 = q[-\eta\mathbf{a}_1 + (1+\xi)\mathbf{a}_2]. \tag{3.8}$$

In this case, Eq. (2.13) implies that

$$\mathbf{n} + q(-y\mathbf{i}+x\mathbf{j}) = \mathbf{c}\,(\text{const.}). \tag{3.9}$$

When Eq. (3.8) holds and $\mathbf{p}\times\mathbf{g} = \mathbf{0}$, we can substitute Eq. (3.9) into Eq. (2.16) to get either of the forms

$$\mathbf{m}+\mathbf{r}\times\mathbf{c} - q(\mathbf{r}\cdot\mathbf{r})\mathbf{k}/2 = \text{const.}\,\mathbf{k}, \tag{3.10}$$

$$\mathbf{m}+\mathbf{r}\times\mathbf{n} + q(\mathbf{r}\cdot\mathbf{r})\mathbf{k}/2 = \text{const.}\,\mathbf{k}. \tag{3.11}$$

We get specific integrals, unfortunately containing x and y, by substituting Eq. (2.33) into Eqs. (3.9)–(3.11). We obtain an integral involving just $\mathbf{w}$

by dotting Eq. (2.13) with $\mathbf{n}$ and observing that $\mathbf{f}\cdot\mathbf{n} = q\mathbf{k}\cdot(\mathbf{r}'\times\mathbf{n}) = -q\mathbf{k}\cdot\mathbf{m}'$ by Eq. (2.16). Thus we obtain

$$\mathbf{n}\cdot\mathbf{n} - 2qM = a\,(\text{const.}), \tag{3.12}$$

which the insertion of Eq. (2.33) converts to the integral

$$[\hat{T}_1(\mathbf{w}, S)]^2 + [\hat{T}_2(\mathbf{w}, S)]^2 - 2q\hat{M}(\mathbf{w}, S) = a. \tag{3.13}$$

Note that when $q = 0$, Eq. (3.13) merely states that the magnitude of $\mathbf{n}$ is constant, in conformity with Eq. (3.2).

To obtain further results we must restrict ourselves to hyperelastic materials. We multiply Eq. (2.24) by $1+\xi$, Eq. (2.25) by η, Eq. (2.27) by ρ', Eq. (2.29) by ϕ', add the resulting equations, and substitute Eq. (2.36) into the sum to get

$$\begin{aligned}(1+\xi)(W_\xi)' + \eta(W_\eta)' + \rho'(W_{\rho'})' + \phi'(W_{\psi'})' - \rho' W_\rho + (1+\xi)\mathbf{f}\cdot\mathbf{a}_1 \\ + \eta\mathbf{f}\cdot\mathbf{a}_2 + \rho'\mathbf{g}\cdot\mathbf{a}_2 - (1+\rho)\phi'\mathbf{g}\cdot\mathbf{a}_1 = 0.\end{aligned} \tag{3.14}$$

If $\Theta = $ const., if W is independent of S, and if there exists a function $U(\mathbf{r}, \mathbf{p})$ such that

$$\begin{aligned}[U(\mathbf{r},\mathbf{p})]' &\equiv \frac{\partial U}{\partial \mathbf{r}}\cdot\mathbf{r}' + \frac{\partial U}{\partial \mathbf{p}}\cdot\mathbf{p}' \\ &\equiv \left(\frac{\partial U}{\partial x}\cos\phi + \frac{\partial U}{\partial y}\sin\phi\right)(1+\xi) + \left(-\frac{\partial U}{\partial x}\sin\phi + \frac{\partial U}{\partial y}\cos\phi\right)\eta \\ &\quad + \frac{\partial U}{\partial \rho}\rho' + \frac{\partial U}{\partial \phi}\phi' \\ &= (1+\xi)\mathbf{f}\cdot\mathbf{a}_1 + \eta\mathbf{f}\cdot\mathbf{a}_2 + \rho'\mathbf{g}\cdot\mathbf{a}_2 - (1+\rho)\phi'\mathbf{g}\cdot\mathbf{a}_1,\end{aligned} \tag{3.15}$$

then Eq. (3.14) can be integrated to yield

$$(1+\xi)W_\xi + \eta W_\eta + \rho' W_{\rho'} + \phi' W_{\psi'} - W + U = b\,(\text{const.}). \tag{3.16}$$

Condition (3.15) is certainly satisfied if $\mathbf{f}$ is hydrostatic (including the trivial case $\mathbf{f} = \mathbf{0}$) and if $\mathbf{g} = \mathbf{0}$, for in this case the right side of Eq. (3.15) vanishes so that we can take $U = 0$.

If the material is constrained so that there is no shear, i.e. if

$$\beta = 0 \tag{3.17}$$

no matter what the resultants are, then T_1 and T_2 reduce to N and Q. In this case Q is not prescribed as a constitutive function of the remaining strain variables; Q plays the role of a Lagrange multiplier in the equations in which it appears. We must accordingly modify those equations containing $\hat{T}_2(\mathbf{w}, S)$. Let

$$\bar{\mathbf{w}} = (\epsilon, \rho, \rho', \psi'). \tag{3.18}$$

We now replace Eqs. (3.4) and (3.5) by

$$\hat{T}_1(\bar{\mathbf{w}}, S) \equiv \hat{N}(\bar{\mathbf{w}}, S) = -\Lambda \cos(\phi - \alpha), Q(S) = \Lambda \sin(\phi - \alpha). \tag{3.19}$$

Similar minor adjustments are made in the integrals corresponding to Eqs. (3.9)–(3.11). To deal with Eq. (3.13), we rewrite Eq. (3.12) as

$$N^2 + Q^2 - 2qM = a \tag{3.20}$$

and use Eq. (2.28) with $\mathbf{p} \times \mathbf{g} = \mathbf{0}$ to get

$$(M')^2 + (1+\epsilon)^2[N^2 - 2qM - a] = 0. \tag{3.21}$$

The insertion of the constitutive relations into this equation yields

$$\left[\frac{d}{dS}\hat{M}(\bar{\mathbf{w}}, S)\right]^2 = (1+\epsilon)^2[a + 2q\hat{M}(\bar{\mathbf{w}}, S) - \hat{N}^2(\bar{\mathbf{w}}, S)]. \tag{3.22}$$

This we regard as an integral because the elimination of Q from Eqs. (2.23), (2.28) yields a differential equation of higher order. To get the constrained analog of Eq. (3.14), we multiply Eq. (2.22) by $1+\epsilon$, Eq. (2.27) by ρ', Eq. (2.28) by $\phi' = \theta'$, add the resulting equations, and substitute the reduced form of Eq. (2.36) into the sum to get

$$\begin{aligned}(1+\epsilon)(W_\epsilon)' + \rho'(W_{\rho'})' + \phi'(W_{\psi'})' - \rho' W_\rho + (1+\epsilon)\mathbf{f}\cdot\mathbf{a}_1 + \rho'\mathbf{g}\cdot\mathbf{a}_2 \\ - (1+\rho)\phi'\mathbf{g}\cdot\mathbf{a}_1 = 0.\end{aligned} \tag{3.23}$$

If $\Theta' = \text{const.}$, if W depends only on $\bar{\mathbf{w}}$, and if there exists a function $U(\mathbf{r}, \mathbf{p})$ (with $\mathbf{r}$ and $\mathbf{p}$ subject to Eq. (3.17)) such that

$$\frac{\partial U}{\partial \mathbf{r}}\cdot\mathbf{r}' + \frac{\partial U}{\partial \mathbf{p}}\cdot\mathbf{p}' = (1+\epsilon)\mathbf{f}\cdot\mathbf{a}_1 + \rho'\mathbf{g}\cdot\mathbf{a}_2 - (1+\rho)\phi'\mathbf{g}\cdot\mathbf{a}_1, \tag{3.24}$$

then Eq. (3.24) can be integrated to yield

$$(1+\epsilon)W_\epsilon + \rho' W_{\rho'} + \phi' W_{\psi'} - W + U = b. \tag{3.25}$$

We remark that a theory with the same mathematical structure as that induced by the constraint is a consequence of the more general constraint that

$$\beta = \hat{\beta}(\bar{\mathbf{w}}, S), \tag{3.26}$$

where $\hat{\beta}$ is some function of the indicated arguments. The method for incorporating this constraint follows the lines indicated in Ref. [15, Section 30]. This equivalence is clear in the hyperelastic case, for then the governing equations for conservative loads are the Euler equations for a variational problem based on a new strain-energy function depending on $\bar{\mathbf{w}}$. This strain-energy function is obtained by replacing β with $\hat{\beta}(\bar{\mathbf{w}}, S)$ in W.

A number of results in this section generalize those of Ref. [16]. Others are classical.

4 FLEXURE, EXTENSION, AND SHEAR UNDER TERMINAL LOADS

Our ultimate goal in this section is to examine the buckling problem for bodies with originally straight axes under compressive end loads. We begin by studying a somewhat more general problem and then reducing it to this buckling problem by introducing a sequence of specializations. This process makes clear the mathematical structure available at each intermediate step. We do not pause to explore the full implications of these intermediate problems, however.

To obtain mathematically simple theory, we assume that the constitutive functions $\hat{T}_1, \hat{T}_2, \hat{M}$ are independent of ρ and ρ'. For hyperelastic bodies, we correspondingly assume that W can be written as a sum of a function depending on ξ, η, ψ', S and a function depending on ρ, ρ', S. For the purposes of this section, there is no loss of generality in assuming that W itself depends only on ξ, η, ψ', S. When these special constitutive equations are substituted into the equilibrium equations, the latter decouple, with Eqs. (2.24), (2.25), (2.29) forming a determinate system for the unknown geometric variables ξ, η, ψ.

We assume that $\mathbf{f} = \mathbf{0}$, $\mathbf{p} \times \mathbf{g} = \mathbf{0}$. Then Eqs. (3.4)–(3.7) hold. The assumption that the symmetric part of matrix (2.35) is positive-definite implies that the symmetric part of

$$\begin{pmatrix} \dfrac{\partial \hat{T}_1}{\partial \xi} & \dfrac{\partial \hat{T}_2}{\partial \xi} \\ \dfrac{\partial \hat{T}_1}{\partial \eta} & \dfrac{\partial \hat{T}_2}{\partial \eta} \end{pmatrix} \tag{4.1}$$

is positive-definite. This condition, when coupled with the growth conditions described above Eqs. (2.36), permits us to invoke a global inverse function theorem (cf. Ref. [17]) ensuring that Eqs. (3.4), (3.5) can be solved uniquely for ξ, η as functions of the other variables in the equations. We denote this solution by

$$\xi = \hat{\xi}(\psi', -\Lambda \cos(\phi - \alpha), \Lambda \sin(\phi - \alpha), S), \tag{4.2}$$

$$\eta = \hat{\eta}(\psi', -\Lambda \cos(\phi - \alpha), \Lambda \sin(\phi - \alpha), S). \tag{4.3}$$

When Eqs. (4.2), (4.3) are substituted into the constitutive Eqs. (2.33), and these are then substituted into Eq. (2.29), we obtain

$$\frac{d}{dS}\hat{M}(\hat{\xi}, \hat{\eta}, \psi', S) + (1+\hat{\xi})\Lambda \sin(\phi-\alpha) + \hat{\eta}\Lambda \cos(\phi-\alpha) = 0. \tag{4.4}$$

The arguments of $\hat{\xi}$ and $\hat{\eta}$ are given in Eqs. (4.2), (4.3). This equation, in general, is a quasi-linear, non-autonomous second-order ordinary differential equation for ψ. Once ψ is found, ξ and η are determined from Eqs. (4.2), (4.3) and x and y are obtained by integrating Eq. (2.11). This equation is the natural generalization of Eq. (1.4) to account for coupled nonlinear response to flexure, extension, and shear. There has been no study of the global properties of the solutions of this equation. Indeed, the author knows of no theory capable of providing a detailed qualitative picture of the global behavior of Eq. (1.4) when B is not constant.†

When $\hat{T}_1$, $\hat{T}_2$, $\hat{M}$ are independent of S and when $\Theta' = K$ (const.), Eq. (4.4) reduces to an autonomous equation for ϕ or for

$$\gamma = \phi - \alpha, \tag{4.5}$$

which can be studied by phase plane methods. This study is simplified for hyperelastic materials because the equation of the phase plane trajectories can be obtained explicitly from the integral (3.16). In fact, by replacing W_ξ, W_η, ξ, η of Eq. (3.16) with the right sides of Eqs. (3.4), (3.5), (4.2), (4.3), we obtain the phase plane equation

$$\gamma' W_{\psi'} - W - \Lambda(1+\hat{\xi}) \cos\gamma + \Lambda\hat{\eta} \sin\gamma = b, \tag{4.6}$$

where the arguments of W and $W_{\psi'}$ are $\hat{\xi}$, $\hat{\eta}$, $\gamma' - K$, and the arguments of $\hat{\xi}$ and $\hat{\eta}$ are $\gamma' - K, -\Lambda \cos\gamma$, $\Lambda \sin\gamma$. Equation (4.6) is the generalization of Eq. (3.1). We note that Eq. (4.6) has period 2π in γ, just as Eq. (3.1) has in θ.

In the rest of this section we restrict our attention to bodies with straight reference axes: $\Theta = 0$ so that $\gamma = \psi - \alpha$. We assume that W is even in its argument ψ'. It follows that Eqs. (4.2), (4.3), and therefore Eq. (4.6), are even in γ'. To study the dependence of Eq. (4.6) on γ' we set

$$2\omega = (\gamma')^2. \tag{4.7}$$

†There is a wealth of detailed qualitative studies of boundary-value problems for nonlinear second-order ordinary differential equations. Consult the bibliography of [18]. Of more than passing note is the work of Refs. [19, 20]. But their assumptions, though quite general, preclude the sinusoidal nonlinearity in Eqs. (1.4) and (4.4).

The evenness of W in γ' permits us to introduce new functions V, $\tilde{\xi}$, $\tilde{\eta}$ by

$$W(\xi, \eta, \gamma') = V(\xi, \eta, \omega), \tag{4.8}$$

$$\hat{\xi}(\gamma', -\Lambda \cos\gamma, \Lambda \sin\gamma) = \tilde{\xi}(\omega, -\Lambda \cos\gamma, \Lambda \sin\gamma), \tag{4.9}$$

$$\hat{\eta}(\gamma', -\Lambda \cos\gamma, \Lambda \sin\gamma) = \tilde{\eta}(\omega, -\Lambda \cos\gamma, \Lambda \sin\gamma). \tag{4.10}$$

We wish to compute the derivative of the left side of Eq. (4.6) with respect to ω. We obtain some preliminary results. By applying the chain rule to Eq. (4.8), we find

$$\frac{\partial W}{\partial \psi'} = \gamma' \frac{\partial V}{\partial \omega}, \quad \gamma' \frac{\partial W}{\partial \psi'} = 2\omega \frac{\partial V}{\partial \omega}, \quad \frac{\partial^2 W}{\partial (\psi')^2} = \frac{\partial V}{\partial \omega} + 2\omega \frac{\partial^2 V}{\partial \omega^2}. \tag{4.11}$$

The substitution of Eqs. (4.8)–(4.10) into Eqs. (3.4), (3.5) yields

$$\frac{\partial V}{\partial \xi}(\tilde{\xi}(\omega, \ldots), \tilde{\eta}(\omega, \ldots), \omega) = -\Lambda \cos\gamma, \tag{4.12}$$

$$\frac{\partial V}{\partial \eta}(\tilde{\xi}(\omega, \ldots), \tilde{\eta}(\omega, \ldots), \omega) = \Lambda \sin\gamma, \tag{4.13}$$

from which we can readily compute $\partial\tilde{\xi}/\partial\omega$, $\partial\tilde{\eta}/\partial\omega$. The substitution of Eqs. (4.8) and (4.11) into the left side of Eq. (4.6) and the differentiation of the resulting expression with respect to ω gives the ω-derivative of the left side of Eq. (4.6) as

$$\frac{\partial V}{\partial \omega} + 2\omega \frac{\partial^2 V}{\partial \omega^2} + \left(2\omega \frac{\partial^2 V}{\partial \omega \partial \xi} - \frac{\partial V}{\partial \xi}\right)\frac{\partial \tilde{\xi}}{\partial \omega} + \left(2\omega \frac{\partial^2 V}{\partial \omega \partial \eta} - \frac{\partial V}{\partial \eta}\right)\frac{\partial \tilde{\eta}}{\partial \omega}$$
$$- \Lambda \frac{\partial \tilde{\xi}}{\partial \omega} \cos\gamma + \Lambda \frac{\partial \tilde{\eta}}{\partial \omega} \sin\gamma. \tag{4.14}$$

The substitution of Eqs. (4.12), (4.13) and the explicit formulas for $\partial\tilde{\xi}/\partial\omega$, $\partial\tilde{\eta}/\partial\omega$ into expression (4.14) and the re-introduction of W reduces expression (4.14) to

$$\det\begin{pmatrix} W_{\xi\xi} & W_{\xi\eta} & W_{\xi\psi'} \\ W_{\eta\xi} & W_{\eta\eta} & W_{\eta\psi'} \\ W_{\psi'\xi} & W_{\psi'\eta} & W_{\psi'\psi'} \end{pmatrix} \div \det\begin{pmatrix} W_{\xi\xi} & W_{\xi\eta} \\ W_{\eta\xi} & W_{\eta\eta} \end{pmatrix}, \tag{4.15}$$

which is positive by the positive-definiteness of the Hessian matrix of W. We can therefore invoke the (global) implicit function theorem to conclude that Eq. (4.6) has the form

$$(\gamma')^2 = F(\gamma, \Lambda, b). \tag{4.16}$$

We now further assume that W is also even in its argument η. It then follows that our specialization of Eqs. (3.4), (3.5), namely,

$$W_\xi(\xi, \eta, \gamma') = -\Lambda \cos\gamma, \tag{4.17}$$

$$W_\eta(\xi, \eta, \gamma') = \Lambda \sin\gamma, \tag{4.18}$$

are invariant under the replacement of γ and η by $-\gamma$ and $-\eta$. The uniqueness of the solutions $\hat{\xi}, \hat{\eta}$ of Eqs. (4.17), (4.18) then implies that

$$\hat{\xi}(\gamma', -\Lambda\cos\gamma, \Lambda\sin\gamma) = \hat{\xi}(\gamma', -\Lambda\cos\gamma, -\Lambda\sin\gamma), \tag{4.19}$$

$$\hat{\eta}(\gamma', -\Lambda\cos\gamma, \Lambda\sin\gamma) = -\hat{\eta}(\gamma', -\Lambda\cos\gamma, -\Lambda\sin\gamma). \tag{4.20}$$

Thus the replacement of γ by $-\gamma$ in Eq. (4.6) leaves this integral unchanged. Since Eq. (4.16) is just another version of Eq. (4.6), we have

$$F(\gamma, \Lambda, b) = F(-\gamma, \Lambda, b). \tag{4.21}$$

Thus the phase plane diagram corresponding to Eq. (4.6) or Eq. (4.16) is symmetric about both the γ and γ' axes and has period 2π in γ.

To determine further properties of the phase plane diagram, we examine the singular points. Since Eq. (4.16) is just the integral of Eq. (4.4) with $\hat{M} = W_{\psi'}$, these singular points are the pairs $(\gamma, 0)$ where γ is a zero of the algebraic equation obtained from Eq. (4.4) by setting $\gamma' = 0$ there. We assume that

$$W_{\psi'}(\xi, \eta, 0) \equiv 0. \tag{4.22}$$

The singular points are then solutions γ of

$$\tan\gamma = -\frac{\hat{\eta}(0, -\Lambda\cos\gamma, \Lambda\sin\gamma)}{1+\hat{\xi}(0, -\Lambda\cos\gamma, \Lambda\sin\gamma)} \equiv h(\gamma, \Lambda). \tag{4.23}$$

Since a (classical) solution to the governing equations must satisfy $0 < 1+\xi < \infty$, $|\eta| < \infty$, the right side $h(\gamma, \Lambda)$ of Eq. (4.23) is a bounded function of γ that vanishes only where $\hat{\eta}(0, -\Lambda\cos\gamma, \Lambda\sin\gamma)$ vanishes. Equations (4.19) and (4.20) imply that $h(\gamma, \Lambda)$ is an odd function of γ with period 2π in γ that vanishes at integral multiples of π. We further assume that

$$W_\eta(\xi, 0, 0) \equiv 0. \tag{4.24}$$

Since $W_{\eta\eta} > 0$, for any fixed ξ, $W_\eta(\xi, \eta, 0)$ is an increasing function of η that vanishes at $\eta = 0$. From Eq. (4.18) we then conclude that $\hat{\eta}(0, -\Lambda\cos\gamma, \Lambda\sin\gamma)$ has the same sign as $\sin\gamma$. (Recall that $\Lambda \geqslant 0$.) Moreover, Eq. (4.23) has solutions $\gamma = n\pi$, n an integer.

It follows from Eq. (2.10) that $h(\gamma, \Lambda)$ is just the value of $-\tan\beta$.

Since $\theta-\alpha=\gamma+\beta$,

$$\tan(\theta-\alpha)=\frac{\tan\gamma+\tan\beta}{1-\tan\gamma\tan\beta}, \tag{4.25}$$

so that Eq. (4.23) implies that

$$\theta-\alpha=n\pi, \quad n \text{ an integer}, \tag{4.26}$$

at any singular point, just as for the classical elastica problem. The boundedness of $h(\gamma,\Lambda)$ (cf. the second of conditions (2.8)) ensures that Eq. (4.26) holds under no circumstances other than the satisfaction of Eq. (4.23). Since θ and γ are constants for these singular solutions, β is also constant. Thus these solutions represent bodies with straight axes and with constant shear. The solutions $\gamma=n\pi$ of Eq. (4.23) correspond to zero shear. Whether there are singular solutions with nonzero shear depends on W and the size of Λ.

In Fig. 3 we plot $\tan\gamma$ and a typical form of $h(\gamma,\Lambda)$ embodying the properties we have just determined for it. The values of γ, indicated by large dots, at which these curves intersect are the solutions of Eq. (4.23). Note that solutions which are not integral multiples of π occur only in intervals of the form $((2n+\frac{1}{2})\pi,\ (2n+\frac{3}{2})\pi)$, n an integer.

Using representations of $\partial\hat{\xi}/\partial\gamma$ and $\partial\hat{\eta}/\partial\gamma$ for $\gamma'=0$ determined from Eqs. (4.17), (4.18), we obtain

$$\frac{\partial h}{\partial\gamma}(\gamma,\Lambda) = -\frac{\Lambda[(1+\hat{\xi})(W_{\xi\xi}\cos\gamma-W_{\xi\eta}\sin\gamma)-\hat{\eta}(W_{\eta\eta}\sin\gamma-W_{\xi\eta}\cos\gamma)]}{(1+\hat{\xi})^2(W_{\xi\xi}W_{\eta\eta}-W_{\xi\eta}^2)}, \tag{4.27}$$

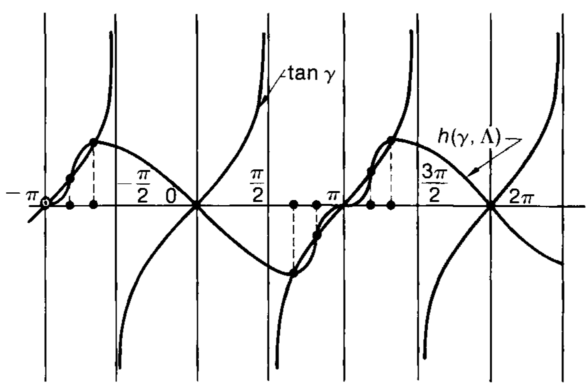

Fig. 3 $\Lambda>\Lambda^*,\ h_\gamma(\pi,\Lambda)<1$.

where the arguments of the derivatives of W are $\hat{\xi}, \hat{\eta}, 0$ and the arguments of $\hat{\xi}, \hat{\eta}$ are $0, -\Lambda\cos\gamma, \Lambda\sin\gamma$. From Eq. (4.27) we readily observe that $h_\gamma(0, \Lambda) < 0$ and $h_\gamma(\pi, \Lambda) > 0$ for $\Lambda > 0$, in agreement with previously obtained results.

Since $h(\gamma, 0) = 0$, there are singular points only at $\gamma = n\pi$ for $\Lambda = 0$. As Λ increases, $h(\gamma, \Lambda)$, which is a continuous function of Λ, may get large enough to intersect $\tan\gamma$ at other values of γ. This is shown in Fig. 3. As Λ increases further, the number and placement of these singular points change. We can obtain sufficient conditions ensuring that the only singular points are $\gamma = n\pi$. For example, the requirement that $h_\gamma(\gamma, \Lambda) < 1$ for all γ serves this purpose. Since $h_\gamma(\gamma, 0) = 0$ and since h_γ depends continuously on Λ, there is a positive value of Λ, call it Λ^*, such that the only singular points are at $\gamma = n\pi$ when $\Lambda < \Lambda^*$.

To determine the type of singular point, we expand Eq. (4.6) (or equivalently Eq. (4.16)) in powers of $(\gamma')^2$ and of γ. The derivative of the left side of Eq. (4.6) with respect to $(\gamma')^2$ is ratio (4.15), and by Eq. (4.22), the value of ratio (4.15) at $\gamma' = 0$ is just $W_{\psi'\psi'}(\hat{\xi}, \hat{\eta}, 0) > 0$ with $\hat{\xi}, \hat{\eta}$ having arguments $0, -\Lambda\cos\gamma, -\Lambda\sin\gamma$. The second derivative with respect to γ at a singular point is

$$\frac{\Lambda[1 + \hat{\xi}(0, -\Lambda\cos\gamma, \Lambda\sin\gamma)]}{\cos\gamma}[1 - h_\gamma(\gamma, \Lambda)\cos^2\gamma], \qquad (4.28)$$

when γ is a singular point. The second mixed partial derivative of the left side of Eq. (4.6) with respect to γ and γ' at $\gamma' = 0$ is 0. Thus, the singular point is a center when value (4.28) > 0 and a saddle when value (4.28) < 0. In particular, the points $(\gamma, \gamma') = (2n\pi, 0)$ are centers. The points $(\gamma, \gamma') = ((2n+1)\pi, 0)$ are saddles when $h_\gamma((2n+1)\pi, \Lambda) < 1$ and are centers when $h_\gamma((2n+1)\pi, \Lambda) > 1$. Since $\cos\gamma < 0$ in the interval $((2n+\frac{1}{2})\pi, (2n+\frac{3}{2})\pi)$, singular points in this interval are saddles if $h_\gamma < \sec^2\gamma$ (i.e. if the slope of $h(\gamma, \Lambda)$ is less than the slope of $\tan\gamma$) and are centers if $h_\gamma > \sec^2\gamma$. Thus, in the γ-interval $((2n+\frac{1}{2})\pi, (2n+\frac{3}{2})\pi)$ there may be a number of singular points distributed symmetrically about $\gamma = (2n+1)\pi$, the points most remote from $\gamma = (2n+1)\pi$ being saddles. This array of singular points behaves like a saddle "in the large." We do not catalog the possible types of phase plane diagrams that correspond to the various dispositions of these singular points, but merely observe that the presence of singular points at non-integral multiples of π gives rise to closed orbits not enclosing $\gamma = 2n\pi$. We indicate three of the simplest possibilities in Fig. 4. In Fig. 4a there are no singular points at non-integral multiples of π and the phase plane diagram has the same charac-

(a)

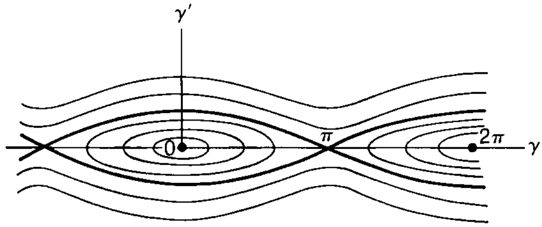

Fig. 4a $\Lambda < \Lambda^*$.

(b)

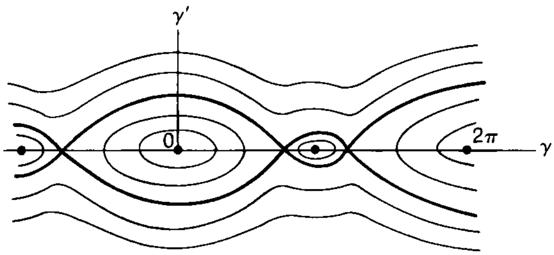

Fig. 4b $\Lambda > \Lambda^*, h_\gamma(\pi, \Lambda) > 1$.

(c)

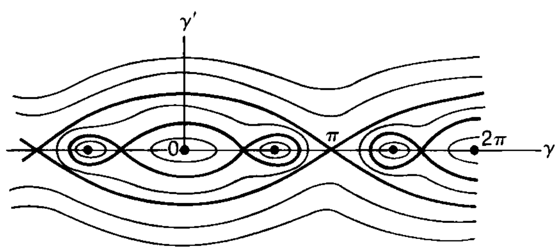

Fig. 4c $\Lambda > \Lambda^*, h_\gamma(\pi, \Lambda) < 1$.

ter as that for the classical elastica, Fig. 2. In Figs. 4a and b this pattern is disrupted by the presence of centers at values of γ other than $2n\pi$.

To analyze the properties of solutions, we strengthen Eq. (4.24) by requiring

$$W_\eta(\xi, 0, \psi') \equiv 0. \tag{4.29}$$

Then the positivity of $W_{\eta\eta}$ and Eq. (4.18) imply that $\hat{\eta}(\gamma', -\Lambda\cos\gamma, -\Lambda\sin\gamma)$ has the same sign as γ.

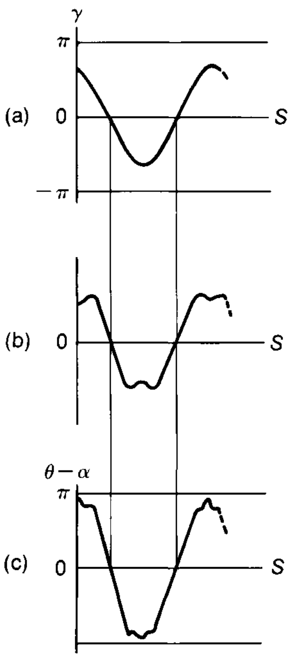

Fig. 5.

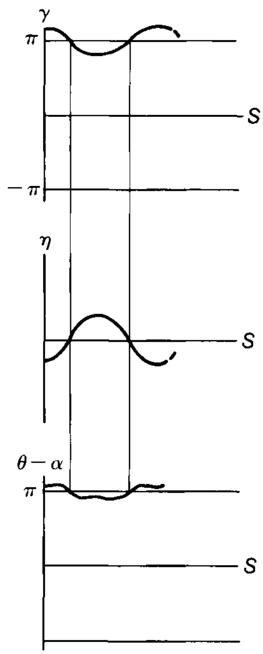

Fig. 6.

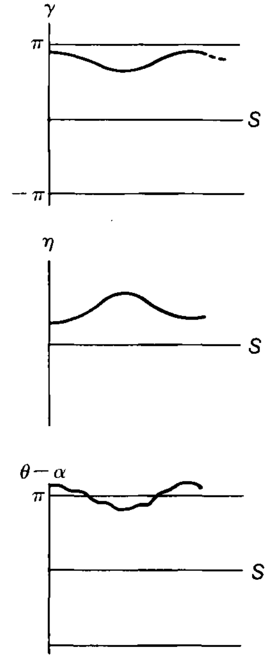

Fig. 7.

Let us now study the properties of solutions corresponding to the closed orbits about $(\gamma, \gamma') = (0, 0)$ in Figs. 4a, b, c. In Fig. 5a we sketch γ as a function of S for such orbits. Since $|\gamma| < \pi$, η has the same sign as γ. This is shown in Fig. 5b. Thus $\beta = \arctan(\eta/1+\xi)$ and therefore $\theta - \alpha = \beta + \gamma$ each have the same sign as γ. $\theta - \alpha$ is shown in Fig. 5c. Thus such solutions have substantially the same behavior as the corresponding solutions to the elastica. They correspond to buckled modes.

In Figs. 6 and 7 we perform a similar analysis on the closed orbits about $(\gamma, \gamma') = (\pi, 0)$ of Fig. 4b and about the center between $\gamma = \pi/2$ and $\gamma = \pi$ of Fig. 4c. In Fig. 8, we sketch the form of the axis of such a solution in the x, y-plane when $\alpha = 0$. We assume the ends are hinged, i.e. $M(S_1) = M(S_2) = 0$. By Eq. (4.22) and the positivity of $W_{\psi'\psi'}$, this

Fig. 8.

means that $\gamma'(S_1) = \gamma'(S_2) = 0$. Here the axis is not straight, though the body is in tension. This is a direct manifestation of the complexity of solutions that can be obtained for large shear. There are no analogous solutions for the classical elastica problem. We shall come upon a similar phenomenon in Section 7.

In summary, for sufficiently small $|\gamma|$, certainly for $|\gamma| < \pi/2$, the solutions have the same form of axis as for the classical elastica. For large Λ, there might occur solutions of the type of Fig. 8. These solutions indicate that simple characterizations about the properties and multiplicity of solutions break down for "large" solutions. We must point out that these considerations apply only to *possible* forms of solutions: The parametrization of a particular orbit in the phase plane must be adjusted so that the solutions can satisfy given boundary conditions. In the next section we indicate how to deal with more specific details of solutions such as the preservation of properties inherited from the bifurcation from a uniformly compressed state.

Studies of bifurcation for more special bodies without shear are given by Refs. [12, 14, 21]. Much of the analysis of this section is new.

5 FLEXURE AND EXTENSION UNDER HYDROSTATIC PRESSURE

In this section we assume that:

(i) The body is homogeneous with a straight or circular reference axis: Thus $\Theta' = K$ (const.).
(ii) The distributed load is hydrostatic.
(iii) The constraint (3.17) holds so that $Q = T_2$ plays the role of a multiplier.
(iv) The constitutive functions $\hat{N}$ and $\hat{M}$ depend only on ϵ and $\zeta \equiv \psi'$. In the hyperelastic case, the strain-energy function $W(\bar{\mathbf{w}})$ can be written as the sum of a function depending only on ϵ, ζ and a function depending only on ρ, ρ'. Since the resulting equations uncouple in this case, there is no loss of generality in denoting this first function by $W(\epsilon, \zeta)$. Note that the independence of S is consistent with (i).

Our governing equations are obtained by eliminating Q from Eqs. (2.22), (2.23), (2.28), with $\mathbf{f}$ given by Eq. (3.8) and $\mathbf{p} \times \mathbf{g} = \mathbf{0}$. We multiply Eq. (2.22) by $(1+\epsilon)$, multiply Eq. (2.28) by $\theta' = \phi' = K + \zeta$, add the resulting equations, and insert the constitutive relations to obtain

$$(1+\epsilon)[\hat{N}(\epsilon,\zeta)]'+(K+\zeta)[\hat{M}(\epsilon,\zeta)]'=0. \tag{5.1}$$

In the hyperelastic case

$$\hat{N}=W_{\epsilon},\quad \hat{M}=W_{\zeta} \tag{5.2}$$

and Eq. (5.1) has the integral

$$\Omega(\epsilon,\zeta)\ \equiv\ (1+\epsilon)W_{\epsilon}(\epsilon,\zeta)+(K+\zeta)W_{\zeta}(\epsilon,\zeta)-W(\epsilon,\zeta)=b. \tag{5.3}$$

From Eq. (3.22) we also have

$$\left[\frac{d}{dS}\hat{M}(\epsilon,\zeta)\right]^2=(1+\epsilon)^2[a+2q\hat{M}(\epsilon,\zeta)-\hat{N}^2(\epsilon,\zeta)]. \tag{5.4}$$

The relations (5.1) (or (5.3)) and (5.4) are completely equivalent to the equilibrium equations and constitutive equations under the assumptions listed above. The full set of governing equations is obtained by supplementing these with the geometric relations obtained from Eqs. (2.6) or (2.11) and (2.32):

$$x'=(1+\epsilon)\cos\theta,\quad y'=(1+\epsilon)\sin\theta,\quad \theta'=K+\zeta. \tag{5.5}$$

Our restrictions on matrix (2.35) reduce to the requirement that the symmetric part of

$$\begin{pmatrix}\hat{N}_{\epsilon} & \hat{M}_{\epsilon}\\ \hat{N}_{\zeta} & \hat{M}_{\zeta}\end{pmatrix} \tag{5.6}$$

be positive-definite.

We wish to solve Eq. (5.1) (or Eq. (5.3)) for ϵ as a function of ζ and substitute the resulting representation into Eq. (5.4) to reduce Eq. (5.4) to a first-order equation in ζ depending on some parameters. This equation will be amenable to a phase plane analysis. We accordingly begin by studying the family of curves in the ϵ, ζ-plane defined by Eq. (5.1). For this purpose we rewrite Eq. (5.1) as

$$[(1+\epsilon)\hat{N}_{\epsilon}+(K+\zeta)\hat{M}_{\epsilon}]\epsilon'+[(1+\epsilon)\hat{N}_{\zeta}+(K+\zeta)\hat{M}_{\zeta}]\zeta'=0. \tag{5.7}$$

We state our preliminary results in a series of lemmas.

LEMMA 1. *The direction field in the ϵ, ζ-plane defined by Eq. (5.7) has no singularities for $\epsilon>-1$.*

Proof. If this lemma were false, there would be a point (ϵ,ζ), $\epsilon>-1$, such that the coefficients of ϵ' and ζ' in Eq. (5.7) would vanish simultaneously, thus implying that

$$(1+\epsilon)[(1+\epsilon)\hat{N}_{\epsilon}+(K+\zeta)\hat{M}_{\epsilon}]+(K+\zeta)[(1+\epsilon)\hat{N}_{\zeta}+(K+\zeta)\hat{M}_{\zeta}] \tag{5.8}$$

vanishes. But this is impossible by the positive-definiteness of the symmetric part of matrix (5.6) and by the requirement that $1+\epsilon$ be positive.

Thus there is exactly one trajectory of Eq. (5.7) passing through a given point (ϵ, ζ) with $1+\epsilon > 0$; if $(1+\epsilon)\hat{N}_\epsilon(\epsilon,\zeta) + (K+\zeta)\hat{M}_\epsilon(\epsilon,\zeta) = 0$, then the curve has a tangent parallel to the line $\zeta = 0$.

LEMMA 2a. *Let*

$$\hat{M}(\epsilon, 0) \equiv 0. \tag{5.9}$$

Then no trajectory of Eq. (5.7) intersects the semi-axis $\zeta = 0, -1 < \epsilon < \infty$ more than once (without leaving the half-plane $\epsilon > -1$). At any such intersection $d\epsilon/d\zeta$ *is finite.*

Proof. The last statement of the lemma is obtained by evaluating Eq. (5.7) at $\zeta = 0$:

$$\frac{d\epsilon}{d\zeta} = -\frac{(1+\epsilon)\hat{N}_\zeta(\epsilon,0) + K\hat{M}_\zeta(\epsilon,0)}{(1+\epsilon)\hat{N}_\epsilon(\epsilon,0)}. \tag{5.10}$$

The positive-definiteness of the symmetric part of matrix (5.6) shows that matrix (5.10) is finite for $-1 < \epsilon < \infty$.

Now suppose that a trajectory were to intersect the semi-axis $\zeta = 0$, $-1 < \epsilon < \infty$ twice. Let C denote the portion of such a trajectory joining the intersection points. Then each trajectory entering the region bounded by C and the ϵ-axis would have to pierce the ϵ-axis transversely. Since such trajectories could not cross C (by Lemma 1), there would have to be a singularity in the region bounded by C and the ϵ-axis, and this is impossible by Lemma 1.

In the hyperelastic case we can obtain the same result by an analytic proof.

LEMMA 2b. *Let Eqs. (5.2) and (5.9) hold. Then the conclusion of Lemma 2a is valid. (Note that the trajectories of Eq. (5.7) are the curves of the family (5.3).)*

Proof. Curves that intersect the semi-axis $\zeta = 0, -1 < \epsilon < \infty$ have a parameter b of the form

$$b = (1+\epsilon_0)W_\epsilon(\epsilon_0, 0) - W(\epsilon_0, 0) \equiv \Omega(\epsilon_0, 0). \tag{5.11}$$

Since the derivative of the right side of Eq. (5.11) with respect to ϵ_0 is

$$\Omega_\epsilon(\epsilon_0, 0) = (1+\epsilon_0)W_{\epsilon\epsilon}(\epsilon_0, 0) > 0, \tag{5.12}$$

the (global) inverse function theorem ensures that there is a one-to-one correspondence between the parameter b of the form (5.11) and the point of intersection $(\epsilon_0, 0)$, $\epsilon_0 > -1$. The boundedness of $d\epsilon/d\zeta$ at $\zeta = 0$ is a consequence of the (local) implicit function theorem that guarantees the solution of Eq. (5.3) for ϵ in terms of ζ when $\Omega_\epsilon(\epsilon_0, 0) > 0$.

To avoid minor inconvenience on the boundary $\epsilon = -1$ of the half-plane, we assume that

$$\frac{(1+\epsilon)\hat{N}_\zeta + (K+\zeta)\hat{M}_\zeta}{(1+\epsilon)\hat{N}_\epsilon + (K+\zeta)\hat{M}_\epsilon} \to 0 \quad \text{as } \epsilon \to -1, \tag{5.13}$$

uniformly in ζ for ζ in any bounded interval. This means that $\epsilon = -1$ is a trajectory of Eq. (5.7) when Eq. (5.7) is defined on the closed half-plane $\epsilon \geqslant -1$. Thus no trajectory of Eq. (5.7) can leave the half-plane $\epsilon > -1$ at any finite value of ζ. The requirement (5.13) is a growth (or coercivity) condition of the type described in the paragraph preceding Eq. (2.36). We also assume in the sequel that the hypotheses of Lemma 2a are satisfied.

Let b denote the parameter for the family of trajectories of Eq. (5.7). (In the hyperelastic case, b is given by Eq. (5.11) and has a one-to-one correspondence with the intersection points of trajectories with the half-line $\zeta = 0$, $\epsilon > -1$.) Let E be an open subset of the half-plane $-1 < \epsilon < \infty$, $-\infty < \zeta < \infty$ having the property that all the trajectories passing through E have single-valued representations

$$\epsilon = \hat{\epsilon}(\zeta, b) \tag{5.14}$$

on E with $[1+\hat{\epsilon}(\zeta, b)]\hat{N}_\epsilon(\hat{\epsilon}(\zeta, b), \zeta) + (K+\zeta)\hat{M}_\epsilon(\hat{\epsilon}(\zeta, b), \zeta)$ uniformly bounded away from zero on E. We suppress the explicit dependence of this representation upon E. Lemmas 2a, 2b imply that there is certainly such a set E containing the half-line $\zeta = 0$, $\epsilon > -1$. These sets E can be described by taking account of the places where $(1+\epsilon)\hat{N}_\epsilon + (K+\zeta)\hat{M}_\epsilon$ vanishes. (Cf. Ref. [13].) For any particular material, there is no difficulty in constructing such sets.

Now a given boundary-value problem may have a multiplicity of solutions. For all solutions whose values ϵ, ζ lie in E, this multiplicity cannot be due to the possibility of more than one axial strain field ϵ corresponding to a single flexural strain field ζ, because Eq. (5.14) ensures that ϵ is uniquely determined by ζ.

To describe the actual shape of the deformed body, we introduce the curvature

$$k(S) = \frac{K+\zeta(S)}{1+\epsilon(S)}. \tag{5.15}$$

LEMMA 3. *Let ϵ, ζ represent a solution of the governing equations. If $k'(S_0) = 0$, then $\zeta'(S_0) = 0$. If $\zeta'(S_0) = 0$ and if $(1+\epsilon)\hat{N}_\epsilon + (K+\zeta)\hat{M}_\epsilon \neq 0$ at S_0, then $k'(S_0) = 0$.*

Proof. Since

$$k' = \frac{\zeta'}{1+\epsilon} - \frac{(K+\zeta)\epsilon'}{(1+\epsilon)^2}, \tag{5.16}$$

k' has zeros only where

$$(1+\epsilon)\zeta' = (K+\zeta)\epsilon'. \tag{5.17}$$

The substitution of Eq. (5.7) into Eq. (5.17) yields

$$\begin{aligned}\{(1+\epsilon)[(1+\epsilon)\hat{N}_\epsilon + (K+\zeta)\hat{M}_\epsilon] \\ + (K+\zeta)[(1+\epsilon)\hat{N}_\zeta + (K+\zeta)\hat{M}_\zeta]\}\zeta' = 0.\end{aligned} \tag{5.18}$$

The coefficient of ζ' in Eq. (5.18) is just value (5.8), which cannot vanish. Thus $\zeta'(S_0) = 0$ when $k'(S_0) = 0$. Conversely, when $\zeta'(S_0) = 0$, Eq. (5.7) and our hypothesis on $(1+\epsilon)\hat{N}_\epsilon + (K+\zeta)\hat{M}_\epsilon$ implies that $\epsilon'(S_0) = 0$, so $k'(S_0) = 0$ by Eq. (5.16).

The substitution of Eq. (5.14) into Eq. (5.4) and some straightforward manipulation involving Eq. (5.7) reduces Eq. (5.4) to the first-order equation

$$(\zeta')^2 = H(\zeta, a, b, q) \tag{5.19}$$

on E, where

$$H(\zeta, a, b, q) \equiv (a + 2q\hat{M} - \hat{N}^2)\left\{\frac{[1+\hat{\epsilon}(\zeta, b)]\hat{N}_\epsilon + (K+\zeta)\hat{M}_\epsilon}{\hat{N}_\epsilon\hat{M}_\zeta - \hat{N}_\zeta\hat{M}_\epsilon}\right\}^2, \tag{5.20}$$

and the arguments of $\hat{N}$, $\hat{M}$ and their derivatives are $\hat{\epsilon}(\zeta, b), \zeta$. Note that the denominator of the right side of Eq. (5.20) is the square of the determinant of matrix (5.6), which is strictly positive.

We now specialize to the problem of an initially circular ring. Without loss of generality we take its initial radius to be 1 so $K = 1$. The following periodicity conditions apply:

$$\epsilon(S + 2\pi) = \epsilon(S), \tag{5.21}$$

$$\phi(S + 2\pi) = \phi(S) + 2\pi, \tag{5.22}$$

$$x(S+2\pi) = x(S), \quad y(S+2\pi) = y(S). \tag{5.23}$$

Since $\phi' = 1 + \zeta$, condition (5.22) is equivalent to the two conditions

$$\zeta(S + 2\pi) = \zeta(S), \tag{5.24}$$

$$\int_0^{2\pi} \zeta(S)\, dS = 0. \tag{5.25}$$

Thus our boundary-value problem consists of the integral of Eq. (5.7) depending on the parameter b, Eq. (5.4) depending on parameters a and q, Eqs. (5.5), (5.21), (5.24), (5.25), and (5.23). Since q is given, we must determine the parameters a, b in such a way that the unknown functions satisfy the governing equations and side conditions. In our work we do not have to concern ourselves with finding the actual (multiple-valued) dependence of a, b on q. We find it convenient, however, to denote the values of a solution ζ by $\hat{\zeta}(S, a, b, q)$, where a, b, q are the values of the parameters corresponding to this solution.

LEMMA 4. *Let E be a connected set containing a segment of the half-line $\zeta = 0$, $\epsilon > -1$. Let $\hat{\zeta}$ be any solution of the boundary value problem with corresponding parameters (a, b, q) that satisfies*

$$(\hat{\epsilon}(\hat{\zeta}(S, a, b, q), b), \quad \zeta(S, a, b, q)) \in E \tag{5.26}$$

for all S and that gives rise to a simple closed deformed axis. Then either $\zeta(S, a, b, q) \equiv 0$ or else $\hat{\zeta}(S, a, b, q)$ defines a non-constant function with the following properties:

(i) *There are numbers $\zeta_1(a, b, q) < 0$, $\quad \zeta_2(a, b, q) > 0$ such that*

$$\zeta_1(a, b, q) \leqslant \hat{\zeta}(S, a, b, q) \leqslant \zeta_2(a, b, q). \tag{5.27}$$

(ii) *$\hat{\zeta}$ is symmetric about its points of tangency to the lines $\zeta = \zeta_1(a, b, q)$ and $\zeta = \zeta_2(a, b, q)$.*

(iii) *$\hat{\zeta}'$ vanishes only on these lines.*

(iv) *$\hat{\zeta}$ is monotone between successive contact points with these lines.*

Proof. If $\hat{\zeta}$ is a constant solution, then Eq. (5.25) implies that $\hat{\zeta} = 0$. Otherwise Eq. (5.26) shows that Eqs. (5.4), (5.7) of our boundary-value problems are equivalent to Eq. (5.19). Let $\zeta_1(a, b, q)$ and $\zeta_2(a, b, q)$ respectively denote the largest nonpositive zero and the smallest nonnegative zero of

$$a + 2q\hat{M}(\hat{\epsilon}(\zeta, b), \zeta) - \hat{N}^2(\hat{\epsilon}(\zeta, b), \zeta) \tag{5.28}$$

satisfying $(\epsilon(\zeta, b), \zeta) \in E$, if these numbers exist. Now Eq. (5.22) implies that $\zeta_1(a, b, q)$ is the largest nonpositive zero and $\zeta_2(a, b, q)$ is the smallest nonnegative zero of $H(\zeta, a, b, q)$. If these zeros exist, are simple, and do not vanish, a standard phase plane analysis shows that every solution of Eq. (5.19) that vanishes somewhere has the properties (i)–(iv). (The phase plane diagram is just a smooth oval symmetric about the ζ-axis.) Condition (5.25) requires a solution of the boundary-value problem to vanish some place, so the lemma is proved in this case. If these zeros exist and are simple, but at least one of them vanishes, Eq. (5.25) would prevent the existence of a nonzero solution $\hat{\zeta}$.

To treat the remaining possibilities, we invoke the Four Vertex Theorem, which asserts that the curvature of every smooth simple closed curve must have at least two maxima and two minima. By Lemma 3 and the hypotheses, $\hat{\zeta}$ must have at least four extrema. Now if $\zeta_1(a, b, q)$ and $\zeta_2(a, b, q)$ both exist, but one of these, say $\zeta_1(a, b, q)$ is not simple, then the phase plane diagram would have a singular point at $\zeta_1(a, b, q)$ and no solution could reach the singular point in a finite S-interval. It would therefore be impossible for such a solution to have the necessary four extrema. If at most one zero of H exists, the corresponding solution would likewise fail to have four extrema.

It should be noted that the conditions (5.5) are needed for the Four Vertex Theorem.

If the hypotheses of Lemma 4 are violated, the boundary-value problem may have a solution (with parameters a, b, q), but this solution need not have the properties (i)–(iv). We illustrate this possibility in Fig. 9. In Fig. 9a we show a trajectory of Eq. (5.7) with parameter b. On this trajectory $(1+\epsilon)\hat{N}_\epsilon + (1+\zeta)\hat{M}_\epsilon$ vanishes twice, and the values of ζ at which it vanishes lie between 0 and $\zeta_2(a, b, q)$. By allowing H to be regarded as a multiple-valued function of ζ induced by a multiple-valued representation of the solutions of Eq. (5.7) for ϵ as a function of ζ, we can study Eq. (5.4) by this generalized interpretation of Eq. (5.19). In Fig. 9b we sketch a typical H. (Note that H has zeros at the zeros of $(1+\epsilon)\hat{N}_\epsilon +$ $(1+\zeta)\hat{M}_\epsilon$.) The phase plane diagram of Eq. (5.19) is sketched in Fig. 9c. A solution may traverse the three loops, switching loops at the tangents. If the parameters are adjusted appropriately, one or several such solutions of Eq. (5.19) may be solutions of the boundary-value problem. It is clear that such solutions need not enjoy the properties described in Lemma 4. Indeed, the multiple-valuedness of ϵ as a function of ζ as illustrated in Fig. 9a provides a mathematical mechanism by which the qualitative properties of solutions can break down. The phase plane diagram (Fig. 9c) would be considerably more complicated if $(1+\epsilon)\hat{N}_\epsilon +$ $(1+\zeta)\hat{M}_\epsilon$ had more zeros.

LEMMA 5. *Let the hypotheses of Lemma 4 hold and let the solution $\hat{\zeta}$ have period $2\pi/n$ in S (n an integer). Then the corresponding curvature*

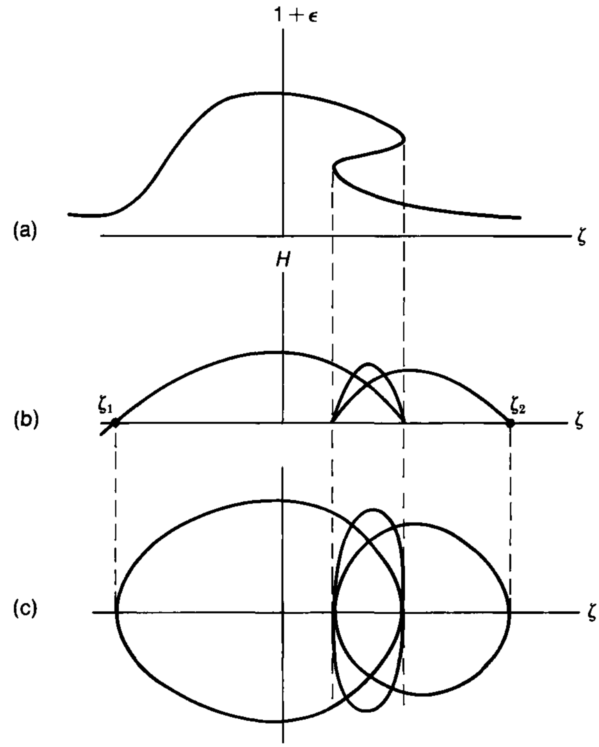

Fig. 9 (From Antman, S. S. "The Shape of Buckled Nonlinearly Elastic Rings," *ZAMP*, Vol. 21, 1970).

k with values $\hat{k}(S, a, b, q)$ also has period $2\pi/n$ in S and has extrema at exactly the same values of S at which $\hat{\zeta}$ has extrema. Moreover, if $k_1(a, b, q)$ and $k_2(a, b, q)$ respectively denote the smaller and larger of the quantities

$$\frac{1+\zeta_1(a, b, q)}{1+\hat{\epsilon}(\zeta_1(a, b, q), b)}, \quad \frac{1+\zeta_2(a, b, q)}{1+\hat{\epsilon}(\zeta_2(a, b, q), b)}, \tag{5.29}$$

then

(i)

$$k_1(a, b, q) \leq \hat{k}(S, a, b, q) \leq k_2(a, b, q), \tag{5.30}$$

(ii) *$\hat{k}$ is symmetric about its points of tangency to the lines $k = k_1(a, b, q)$ and $k = k_2(a, b, q)$.*
(iii) *$\hat{k}'$ vanishes only on these lines.*
(iv) *$\hat{k}$ is monotone between successive contact points with these lines.*

Proof. Since $\hat{\zeta}$ has period $2\pi/n$ and since ϵ is given by Eq. (5.14), ϵ also has period $2\pi/n$. Equation (5.15) then implies that $\hat{k}$ has period $2\pi/n$ in S.

Lemma 3 ensures that $\hat{k}'$ and $\hat{\zeta}'$ have extrema at the same values of S.

Let S_0 be a zero of $\hat{\zeta}'$. Property (ii) of Lemma 4 implies that for arbitrary σ

$$\hat{\zeta}(S_0-\sigma) = \hat{\zeta}(S_0+\sigma),$$

(where we have suppressed the dependence on a, b, q) so that

$$\epsilon(S_0-\sigma) = \hat{\epsilon}(\hat{\zeta}(S_0-\sigma), b) = \hat{\epsilon}(\hat{\zeta}(S_0+\sigma), b) = \epsilon(S_0+\sigma).$$

Equation (5.15) then implies that

$$\hat{k}(S_0-\sigma) = \hat{k}(S_0+\sigma),$$

which is property (ii).

Let s be the arc length parameter of the deformed configuration of the ring. s is given as a function of S by

$$s = \hat{s}(S) = \int_0^S [1+\epsilon(\bar{S})]\, d\bar{S}. \tag{5.31}$$

The total length of the deformed ring is

$$l = \hat{s}(2\pi). \tag{5.32}$$

LEMMA 6. *Let the hypotheses of Lemma 4 hold and let the solution $\hat{\zeta}$ have least period $2\pi/n$ in S. Then $\hat{k}$ has least period l/n in s.*

Proof. Equations (5.31), (5.32) and the periodicity of ϵ imply that

$$\begin{aligned}\hat{s}(S+2\pi/n) &= S+2\pi/n+\int_0^{2\pi/n} \epsilon(\bar{S})\, d\bar{S} + \int_{2\pi/n}^{2\pi/n+S} \epsilon(\bar{S})\, d\bar{S} \\ &= S + l/n + \int_0^S \epsilon(\bar{S})\, d\bar{S} = \hat{s}(S) + l/n. \end{aligned} \tag{5.33}$$

We define

$$\kappa(s, a, b, q) = \hat{k}(S, a, b, q). \tag{5.34}$$

Since $\hat{\zeta}$ has period $2\pi/n$ in S, $\hat{k}$ also has period $2\pi/n$ in S by Lemma 5. Thus

$$\kappa(\hat{s}(S+2\pi/n), a, b, q) = \kappa(\hat{s}(S), a, b, q)$$

and Eq. (5.33) then implies

$$\kappa(s+l/n, a, b, q) = \kappa(s, a, b, q).$$

LEMMA 7. *Let the hypotheses of Lemma 4 hold. Let S_0 be a zero of $\hat{\zeta}'$. Then $\kappa(s, a, b, q)$ is symmetric about $s_0 = \hat{s}(S_0)$ and therefore satisfies properties (i)–(iv) of Lemma 5.*

Proof. We suppress the dependence on a, b, q. By Lemma 5, $\hat{k}(S_0 - \sigma) = \hat{k}(S_0 + \sigma)$ for all real σ, whence $\kappa(\hat{s}(S_0 - \sigma)) = \kappa(\hat{s}(S_0 + \sigma))$. Now

$$s(S_0 + \sigma) = s_0 + \sigma + \int_{S_0}^{S_0+\sigma} \epsilon(\bar{S})\, d\bar{S}, \quad \hat{s}(S_0 - \sigma) = s_0 - \sigma - \int_{S_0-\sigma}^{S_0} \epsilon(\bar{S})\, d\bar{S}.$$

Setting

$$\tau = \sigma + \int_{S_0}^{S_0+\sigma} \epsilon(\bar{S})\, d\bar{S}$$

and using the symmetry of ϵ about S_0, we have

$$s(S_0 - \sigma) = s_0 - \tau, \quad \hat{s}(S_0 + \sigma) = s_0 + \tau,$$

so that

$$\kappa(s_0 - \tau) = \kappa(s_0 + \tau)$$

for all real τ.

We denote by $\zeta_{(n)}$ any solution of the boundary-value problem having least period $2\pi/n$, where n is a positive integer. By Lemma 4, if $(\hat{\epsilon}(\zeta_{(n)}(S), b), \zeta_{(n)}(S)) \in E$ for all S, then $\zeta_{(n)}$ has exactly $2n$ extrema (and $2n$ zeros) in any half-open interval of length 2π.

THEOREM 1. *Let E be as in Lemma 4. Let m and n be two unequal integers. If there exists a family of solutions $\{\zeta(S, c)\}$ in the space C^1 of continuously differentiable functions depending continuously on the parameter $c \in [0, 1]$ with $(\hat{\epsilon}(\zeta(S, c), b),\ \zeta(S, c)) \in E$ for all S and if $\zeta(S, 0) = \zeta_{(m)}(S), \zeta(S, 1) = \zeta_{(n)}(S)$, then this family must contain the solution $\zeta(S) \equiv 0$ and the solutions can change least period only at the zero solution. Indeed, the only solutions which, with their first derivatives, are uniformly near a given nonzero solution "in E" are solutions with the same least period as the given solution.*

Proof. Since solutions are continuously differentiable, we employ the C^1 norm to measure their size and accordingly define

$$\|\zeta_{(m)} - \zeta_{(n)}\| \equiv \max_S |\zeta_{(m)}(S) - \zeta_{(n)}(S)| + \max_S |\zeta'_{(m)}(S) - \zeta'_{(n)}(S)|. \qquad (5.35)$$

We show that if $\|\zeta_{(m)} - \zeta_{(n)}\|$ is arbitrarily small, then $\zeta_{(m)}$ and $\zeta_{(n)}$ must each be arbitrarily small. Assume without loss of generality that

$$\zeta_{(m)}(\Sigma) \equiv \max_S \zeta_{(m)}(S) \geqslant \max_S \zeta_{(n)}(S).$$

Then

$$\zeta_{(m)}(\Sigma) - \max_S \zeta_{(n)}(S) \leq \zeta_{(m)}(\Sigma) - \zeta_{(n)}(\Sigma) \leq \max_S |\zeta_{(m)}(S) - \zeta_{(n)}(S)|. \tag{5.36}$$

Hence

$$|\max_S \zeta_{(m)}(S) - \max_S \zeta_{(n)}(S)| \leq \max_S |\zeta_{(m)}(S) - \zeta_{(n)}(S)|, \tag{5.37}$$

and similarly

$$|\min_S \zeta_{(m)}(S) - \min_S \zeta_{(n)}(S)| \leq \max_S |\zeta_{(m)}(S) - \zeta_{(n)}(S)|. \tag{5.38}$$

Also

$$\begin{aligned}\max_S |\zeta'_{(m)}(S) - \zeta'_{(n)}(S)| &\geq \frac{1}{2\pi}\int_0^{2\pi} |\zeta'_{(m)}(S) - \zeta'_{(n)}(S)|\, dS \\ &\geq \frac{1}{2\pi}\left|\int_0^{2\pi} |\zeta'_{(m)}(S)|\, dS - \int_0^{2\pi} |\zeta'_{(n)}(S)|\, dS\right|. \end{aligned}\tag{5.39}$$

From Lemma 4, it follows that

$$\int_0^{2\pi} |\zeta'_{(m)}(S)|\, dS = 2m[\max_S \zeta_{(m)}(S) - \min_S \zeta_{(m)}(S)]. \tag{5.40}$$

Combining Eqs. (5.35), (5.37)–(5.40), we have

$$\begin{aligned}\|\zeta_{(m)} - \zeta_{(n)}\| &\geq \tfrac{1}{2}|\max_S \zeta_{(m)}(S) - \max_S \zeta_{(n)}(S)| + \tfrac{1}{2}|\min_S \zeta_{(m)}(S) - \min_S \zeta_{(n)}(S)| \\ &\quad + \frac{1}{\pi}|m[\max_S \zeta_{(m)}(S) - \min_S \zeta_{(m)}(S)] - n[\max_S \zeta_{(n)}(S) - \min_S \zeta_{(n)}(S)]|. \end{aligned}\tag{5.41}$$

If $\|\zeta_{(m)} - \zeta_{(n)}\|$ is to equal the arbitrarily small positive number δ, then each term on the right side of Eq. (5.41) must be less than δ. Hence

$$\begin{aligned}\pi\delta &\geq |m[\max_S \zeta_{(m)}(S) - \min_S \zeta_{(m)}(S)] - n[\max_S \zeta_{(n)}(S) - \min_S \zeta_{(n)}(S)]| \\ &= |(m-n)[\max_S \zeta_{(m)}(S) - \min_S \zeta_{(m)}(S)] + n[\max_S \zeta_{(m)}(S) - \max_S \zeta_{(n)}(S)] \\ &\quad - n[\min_S \zeta_{(m)}(S) - \min_S \zeta_{(n)}(S)]| \\ &\geq |m-n||\max_S \zeta_{(m)}(S) - \min_S \zeta_{(m)}(S)| - 4n\delta \\ &\geq |m-n| \max_S |\zeta_{(m)}(S)| - 4n\delta. \end{aligned}$$

The last inequality in this series is a consequence of Eq. (5.25). Thus

$$\max_S |\zeta_{(m)}(S)| \leq (4n+\pi)\delta/|m-n|$$

and we get an analogous inequality for $\zeta_{(n)}$.

To obtain a similar estimate for the derivatives we first observe that as a consequence of Eq. (5.13) there is a solution $\zeta(S) \equiv 0$ for all q greater than some negative number. This may be seen by examining the original differential equations. (Cf. Ref. [13].) In this case, a is a well-defined function of the constant axial extension (which need not be a single-valued function of q). We can now use the differential Eq. (5.19) to show that ζ' is uniformly small when ζ is. For this purpose we note that H is continuous in ζ, uniformly for all q in a bounded interval. That b can always be chosen as a continuous functional of ζ is a consequence of the properties of Eq. (5.7). (The integral (5.3) makes this especially evident.) To show that a depends continuously on ζ requires a bifurcation analysis; this is supplied in Ref. [13]. Thus ζ' must be small when ζ is.

It should be noted that this result depends critically upon Lemma 4 and is not a property of periodic functions in general. Indeed, by perturbing a function with period P in neighborhoods of the points $\{2nP\}$, one can obtain a function with least period $2P$.

From this theorem we conclude that every branch of solutions, except the trivial branch $\zeta = 0$, lying in E is characterized by its least period (or by its number of zeros). If in E a secondary bifurcation can occur from a given branch away from the trivial solution $\zeta = 0$, the solution on the original and secondary branches must have the same least period. This is a sort of Sturm–Liouville theorem for a nonlinear system of order greater than 2. The set E provides an estimate of the region of (ϵ, ζ)-space in which are preserved the qualitative properties that solutions inherit from their linearizations at the trivial solutions.

THEOREM 2. *Every simple configuration of the ring for which the hypotheses of Lemma 4 hold must have at least two axes of symmetry.*

Proof. Lemma 6 and the Four Vertex Theorem imply that κ must have least period l/n in S with $n \geqslant 2$. That each such simple configuration of the ring must have at least two axes of symmetry is a consequence of this result and the symmetry of κ about its tangents to the lines $\kappa = k_1(a, b, q)$, $\kappa = k_2(a, b, q)$ ensured by Lemma 7.

If the configuration of the ring is not simple, the Four Vertex Theorem does not apply and there may be deformed configurations with just two curvature extrema. This possibility is discussed and illustrated in Ref. [13].

The results of this section are a generalization of some of the material of Ref. [13].† This paper contains some further related results as well as a

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bifurcation analysis. A global existence and multiplicity theory for the hyperelastic problem is given in Ref. [10]. Results for the classical ring elastica appear in Ref. [22].

6 FLEXURE, EXTENSION, AND SHEAR UNDER HYDROSTATIC PRESSURE

In this section we indicate how to generalize the treatment of Section 5 to account for shear, but do not carry out a full analysis. We find that there are significant complications engendered by the introduction of shear. We assume that:

(i) The body is homogeneous with a straight or circular reference axis. ($\Theta' = K$.)
(ii) The distributed load is hydrostatic.
(iii) The constitutive functions $\hat{T}_1, \hat{T}_2, \hat{M}$ depend only on $\xi, \eta, \zeta \equiv \psi'$. In the hyperelastic case W is taken to depend only on these arguments.

The equilibrium equations are Eqs. (2.24), (2.25), (2.29) with $\mathbf{f}$ given by Eq. (3.8) and $\mathbf{p} \times \mathbf{g} = \mathbf{0}$. The integral (3.13) has the form

$$[\hat{T}_1(\xi, \eta, \zeta)]^2 + [\hat{T}_2(\xi, \eta, \zeta)]^2 - 2q\hat{M}(\xi, \eta, \zeta) = a. \tag{6.1}$$

It is occasionally convenient to use the differential form of Eq. (6.1):

$$\left(T_1 \frac{\partial \hat{T}_1}{\partial \xi} + T_2 \frac{\partial T_2}{\partial \xi} - q \frac{\partial \hat{M}}{\partial \xi}\right)\xi' + \left(\hat{T}_1 \frac{\partial \hat{T}_1}{\partial \eta} + T_2 \frac{\partial \hat{T}_2}{\partial \eta} - q \frac{\partial M}{\partial \eta}\right)\eta'$$
$$+ \left(T_1 \frac{\partial \hat{T}_1}{\partial \zeta} + \hat{T}_2 \frac{\partial \hat{T}_2}{\partial \zeta} - q \frac{\partial \hat{M}}{\partial \zeta}\right)\zeta' = 0. \tag{6.2}$$

Multiplying Eq. (2.24) by $1 + \xi$, Eq. (2.25) by η, Eq. (2.29) by $\phi' = K + \zeta$, adding the resulting equations, and using the constitutive relations, we obtain

$$\left[(1+\xi) \frac{\partial \hat{T}_1}{\partial \xi} + \eta \frac{\partial \hat{T}_2}{\partial \xi} + (K+\zeta) \frac{\partial \hat{M}}{\partial \xi}\right]\xi'$$
$$+ \left[(1+\xi) \frac{\partial \hat{T}_1}{\partial \eta} + \eta \frac{\partial \hat{T}_2}{\partial \eta} + (K+\zeta) \frac{\partial \hat{M}}{\partial \eta}\right]\eta'$$
$$+ \left[(1+\xi) \frac{\partial \hat{T}_1}{\partial \zeta} + \eta \frac{\partial \hat{T}_2}{\partial \zeta} + (K+\zeta) \frac{\partial \hat{M}}{\partial \zeta}\right]\zeta' = 0. \tag{6.3}$$

In the hyperelastic case this has the integral

$$(1+\xi)W_\xi+\eta W_\eta+(K+\zeta)W_\zeta-W=b, \tag{6.4}$$

which is a special case of Eq. (3.16). When the constitutive relations are substituted into Eq. (2.29), it becomes

$$\hat{M}'+(1+\xi)\hat{T}_2-\eta\hat{T}_1=0. \tag{6.5}$$

The three equations (6.2), (6.3), (6.5) are equivalent to the original set of equilibrium equations. In the hyperelastic case, Eqs. (6.1), (6.4), (6.5) are equivalent to the equilibrium equations.

Now Eqs. (6.2) and (6.3) define a family of curves in the half-space $-1<\xi<\infty$, $-\infty<\eta<\infty$, $-\infty<\zeta<\infty$. To imitate the procedure of Section 5, we should try to solve Eqs. (6.2) and (6.3) for ξ, η as functions of ζ and insert the resulting expressions into Eq. (6.5). We observe, however, that Eq. (6.5) is linear in ζ' whereas Eq. (5.4), the corresponding equation for shearless deformation, is quadratic in ζ'. At first sight Eq. (6.5) therefore seems inadequate for a phase plane analysis. This difficulty is illusory: The requisite multiple-valuedness of Eq. (6.5) in ζ' in fact arises from the squared terms in Eq. (6.1). Indeed, we can make Eq. (6.5) resemble Eq. (5.4) by substituting Eq. (6.1) into the identity

$$[(1+\xi)\hat{T}_2-\eta\hat{T}_1]^2=[(1+\xi)^2+\eta^2][\hat{T}_1{}^2+\hat{T}_2{}^2]-[(1+\xi)\hat{T}_1+\eta\hat{T}_2]^2 \tag{6.6}$$

and then inserting Eq. (6.6) into Eq. (6.5):

$$(M')^2=[(1+\xi)^2+\eta^2](a+2q\hat{M})-[(1+\xi)\hat{T}_1+\eta\hat{T}_2]^2. \tag{6.7}$$

(The identity (6.6) is just a two-dimensional consequence of the vector identity $|\mathbf{u}\times\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2$.)

We must now justify these formal operations. Our restrictions on matrix (2.35) reduce to the requirement that the symmetric part of the matrix

$$\begin{pmatrix} \dfrac{\partial\hat{T}_1}{\partial\xi} & \dfrac{\partial\hat{T}_2}{\partial\xi} & \dfrac{\partial\hat{M}}{\partial\xi} \\ \dfrac{\partial\hat{T}_1}{\partial\eta} & \dfrac{\partial\hat{T}_2}{\partial\eta} & \dfrac{\partial\hat{M}}{\partial\eta} \\ \dfrac{\partial\hat{T}_1}{\partial\zeta} & \dfrac{\partial\hat{T}_2}{\partial\zeta} & \dfrac{\partial\hat{M}}{\partial\zeta} \end{pmatrix} \tag{6.8}$$

be positive-definite. We also assume that $\hat{T}_1$ and $\hat{M}$ are even functions of η and that $\hat{T}_2$ is an odd function of η. In the hyperelastic case, these requirements reduce to the evenness of W in η. It follows that if (ξ, η, ζ) is a solution of the equations, then $(\zeta, -\eta, \zeta)$ is also a solution. We further require

$$\hat{M}(\xi, \eta, 0) \equiv 0, \quad \frac{\partial \hat{T}_1}{\partial \zeta}(\xi, \eta, 0) \equiv 0. \tag{6.9}$$

(The second assumption of Eq. (6.9) is a consequence of the first for hyperelastic bodies.) Then the equations admit the trivial solution

$$\xi = \text{const.}, \quad \eta = 0, \quad \zeta = 0. \tag{6.10}$$

For buckling problems, we wish to determine the behavior of solutions in a region of the ξ, η, ζ-space containing the circular solutions (6.10).

We examine Eqs. (6.2), (6.3) to see if they can be solved for ξ, η as functions of ζ. We observe that the coefficients of η' in each expression vanish at $\eta = 0$, $\zeta = 0$ in virtue of our assumptions. Consequently, the standard existence theories of ordinary differential equations do not apply. (This difficulty is due to the nature of the nonlinearity of Eq. (6.1) or Eq. (6.2), which was the source of the beneficial double-valuedness suggested by Eq. (6.7). We should expect such difficulty because we know that if Eqs. (6.2), (6.3) admit a solution for η in terms of ζ, then the negative of this solution is also a solution.) On the other hand, the determinant of the coefficients of ξ' and ζ' evaluated at $\eta = 0$, $\zeta = 0$, will vanish only in exceptional circumstances, so Eqs. (6.2), (6.3) can usually be solved for ξ and ζ as functions of η near $\eta = 0, \zeta = 0$.

To deal with these technical aggravations, we merely set

$$2\omega = \eta^2 \tag{6.11}$$

and define

$$\begin{aligned}
\hat{T}_1(\xi, \eta, \zeta) &= \tilde{T}_1(\xi, \omega, \zeta), \\
\hat{T}_2(\xi, \eta, \zeta) &= \eta \tilde{T}_2(\xi, \omega, \zeta), \\
\hat{M}(\xi, \eta, \zeta) &= \tilde{M}(\xi, \omega, \zeta),
\end{aligned} \tag{6.12}$$

these representations being justified by our symmetry assumptions. Note that the positivity of $\partial \hat{T}_2 / \partial \eta$ implies that $\tilde{T}_2(\xi, 0, \zeta) > 0$. The substitution of Eqs. (6.11), (6.12) into Eqs. (6.2), (6.3) reduces these equations to

$$\begin{aligned}
&\left[\tilde{T}_1 \frac{\partial \tilde{T}_1}{\partial \xi} + 2\omega \tilde{T}_2 \frac{\partial \tilde{T}_2}{\partial \xi} - q \frac{\partial \tilde{M}}{\partial \xi}\right]\xi' + \left[\tilde{T}_1 \frac{\partial \tilde{T}_1}{\partial \omega} + \tilde{T}_2\left(\tilde{T}_2 + 2\omega \frac{\partial \tilde{T}_2}{\partial \omega}\right) - q \frac{\partial \tilde{M}}{\partial \omega}\right]\omega' \\
&\qquad + \left[\tilde{T}_1 \frac{\partial \tilde{T}_1}{\partial \zeta} + 2\omega \tilde{T}_2 \frac{\partial \tilde{T}_2}{\partial \zeta} - q \frac{\partial \tilde{M}}{\partial \zeta}\right]\zeta' = 0,
\end{aligned} \tag{6.13}$$

$$\left[(1+\xi)\frac{\partial\tilde{T}_1}{\partial\xi}+2\omega\frac{\partial\tilde{T}_2}{\partial\xi}+(K+\zeta)\frac{\partial\tilde{M}}{\partial\xi}\right]\xi'$$
$$+\left[(1+\xi)\frac{\partial\tilde{T}_1}{\partial\omega}+\left(\tilde{T}_2+2\omega\frac{\partial\tilde{T}_2}{\partial\omega}\right)+(K+\zeta)\frac{\partial\tilde{M}}{\partial\omega}\right]\omega'$$
$$+\left[(1+\xi)\frac{\partial\tilde{T}_1}{\partial\zeta}+2\omega\frac{\partial T_2}{\partial\zeta}+(K+\zeta)\frac{\partial\tilde{M}}{\partial\zeta}\right]\zeta'=0. \qquad (6.14)$$

Using Eq. (6.8), we find that the determinant of coefficients of ξ' and ω' at $\eta=0$, $\zeta=0$ is

$$\frac{\partial\tilde{T}_1}{\partial\xi}\tilde{T}_2[\tilde{T}_1-(1+\xi)\tilde{T}_2]. \qquad (6.15)$$

It can vanish only when the term in brackets vanishes and this can occur only when T_1 (which equals N since $\eta=0$) is positive, i.e. when the ring is in tension. Note further that this bracketed term is just the limit of $-(1+\epsilon)\hat{Q}/\eta$ as $\eta\to 0$ and this equals $-\hat{Q}_\beta$. Thus Eqs. (6.13), (6.14) can be solved for ξ, ω in terms of ζ in the neighborhood of all but a few exceptional points on the half-line $-1<\xi<\infty$, $\omega=0$, $\zeta=0$. These solutions give rise to the two-parameter family of double-valued solutions of Eqs. (6.2), (6.3), which we denote by

$$\xi=\hat{\xi}(\zeta,a,b),\quad \eta=\pm\hat{\eta}(\zeta,a,b). \qquad (6.16)$$

These solutions can be substituted into Eq. (6.5). After some manipulation involving Eqs. (6.2), (6.3), Eq. (6.5) assumes the form

$$\zeta'=\pm h(\zeta,a,b,q), \qquad (6.17)$$

where

$$\Delta h=\det\begin{pmatrix}\hat{T}_1\frac{\partial\hat{T}_1}{\partial\xi}+\hat{T}_2\frac{\partial\hat{T}_2}{\partial\xi}-q\frac{\partial M}{\partial\xi} & \hat{T}_1\frac{\partial\hat{T}_1}{\partial\eta}+\hat{T}_2\frac{\partial\hat{T}_2}{\partial\eta}-q\frac{\partial\hat{M}}{\partial\eta}\\ (1+\hat{\xi})\frac{\partial\hat{T}_1}{\partial\xi}+\hat{\eta}\frac{\partial\hat{T}_2}{\partial\xi}+(K+\zeta)\frac{\partial\hat{M}}{\partial\xi} & (1+\hat{\xi})\frac{\partial\hat{T}_1}{\partial\eta}+\hat{\eta}\frac{\partial\hat{T}_2}{\partial\eta}+(K+\zeta)\frac{\partial\hat{M}}{\partial\eta}\end{pmatrix}, \qquad (6.18)$$

Δ is the determinant of matrix (6.8), and the arguments of the constitutive functions and their derivatives are $\hat{\xi}(\zeta,a,b),\hat{\eta}(\zeta,a,b),\zeta$. Equation (6.17) is obviously equivalent to

$$(\zeta')^2=H(\zeta,a,b,q)\equiv h^2(\zeta,a,b,q), \qquad (6.19)$$

wherever Eq. (6.16) is valid. This equation is readily treated by phase plane methods and the program of investigation for shearless deformation described in Section 5 can be contemplated for this more general theory. We conjecture that related results can be obtained.

Note that our formulation here differed in several respects from that of Section 5. The Eqs. (6.2), (6.3) produced the double-valued solution (6.16). If we regard the constraint of zero shear as a limit of small shear, then Eq. (6.16) shows that this limit is non-uniform, the two branches of solutions emanating from the semi-axis $-1 < \xi < \infty$, $\eta = 0$, $\zeta = 0$ in the ξ, η, ζ-space collapsing to a single branch in the ξ, ζ-plane with the properties of the solutions at $\eta = 0$, $\zeta = 0$ being lost.

The development in this section is new.

7 NECKING

We now treat the deformation of a body with a straight reference axis under the action of tensile terminal loads. Here we employ the full kinematic structure of the director theory. We take $S_1 = 0$, $S_2 = 1$. We assume that the end $S = 0$ is fixed and hinged at the origin and the end $S = 1$ is free to slide along the x-axis, hinged, and subject to a force of magnitude λ acting in the positive x-direction. Thus these boundary conditions are

$$x(0) = 0, \quad y(0) = 0, \quad M(0) = 0, \tag{7.1}$$

$$N(1)\cos\theta(1) - Q(1)\sin\theta(1) = \lambda, \quad y(1) = 0, \quad M(1) = 0. \tag{7.2}$$

We allow any suitable conditions on ρ or G. Note that in the notation of Eq. (3.3), $\lambda = -\Lambda$, $\alpha = \pi$. We assume that

$$\hat{M} \equiv 0 \quad \text{when } \psi' = 0, \qquad \hat{Q} \equiv 0 \quad \text{when } \beta = 0. \tag{7.3}$$

Further, we require that the symmetric part of matrix (2.35) be positive-definite.

The following result enables us to simplify the ensuing analysis.

THEOREM. *If $\lambda \geqslant 0$ and if Eqs. (7.1), (7.2) hold, then any solution of the governing equations that satisfies the additional restrictions*

$$-\pi < \theta < \pi, \tag{7.4}$$

$$\hat{Q}_\beta(\mathbf{w}, S) > 0, \tag{7.5}$$

must have a straight axis with zero shear, i.e.

$$\beta(S) \equiv 0, \quad \phi(S) \equiv 0, \quad \theta(S) \equiv 0, \quad y(S) \equiv 0. \tag{7.6}$$

Proof. Equations (3.4), (3.5), (3.7) are equivalent to

$$\hat{N}(\mathbf{w}(S), S) = \lambda\cos\theta(S), \tag{7.7}$$

$$\hat{Q}(\mathbf{w}(S), S) = -\lambda \sin\theta(S), \tag{7.8}$$

$$\hat{M}(\mathbf{w}(S), S) = \lambda y(S). \tag{7.9}$$

If $\lambda = 0$, inequality (7.5) and Eq. (7.8) imply that $\beta(S) \equiv 0$, the positivity of $\hat{M}_{\psi'}$ (from the positive-definiteness of matrix (2.35)) and Eq. (7.9) imply that $\psi'(S) \equiv 0$, and Eq. (2.4) then implies $\theta'(S) \equiv 0$. The relations (7.6) follow from Eqs. (2.6), (7.1), (7.2).

If $\lambda > 0$, we assume for contradiction that $y(S) \not\equiv 0$. Then Eqs. (7.1), (7.2) require y or $-y$ to have a positive interior maximum. We just treat the case in which y is maximized at $\Sigma \in (0, 1)$:

$$y(\Sigma) > 0, \quad y'(\Sigma) = 0, \quad y''(\Sigma) \leqslant 0. \tag{7.10}$$

Since $1+\epsilon > 0$ by Eq. (2.8), Eq. (2.6) and the second relation of Eq. (7.10) imply that $\sin\theta(\Sigma) = 0$. The bound (7.4) then implies $\theta(\Sigma) = 0$. Evaluating the S-derivative of Eq. (2.6) at Σ, we obtain

$$y''(\Sigma) = [1+\epsilon(\Sigma)]\theta'(\Sigma). \tag{7.11}$$

The third of inequalities (7.10) then yields

$$\theta'(\Sigma) \leqslant 0. \tag{7.12}$$

Evaluating the S-derivative of Eq. (7.8) at Σ, we have (in virtue of Eq. (7.3)) that

$$\hat{Q}_\beta(\mathbf{w}(\Sigma), \Sigma)\beta'(\Sigma) = -\lambda\theta'(\Sigma) = -\lambda[\beta'(\Sigma) + \psi'(\Sigma)], \tag{7.13}$$

whence

$$\theta' = \hat{Q}_\beta(\mathbf{w}(\Sigma), \Sigma)[\lambda + \hat{Q}_\beta(\mathbf{w}(\Sigma), \Sigma)]^{-1}\psi'(\Sigma). \tag{7.14}$$

But the positivity of $\hat{M}_{\psi'}$, Eq. (7.9), and the first relation of Eq. (7.3) imply that $\psi'(\Sigma) > 0$ so that $\theta'(\Sigma) > 0$ by inequalities (7.5), (7.4), in contradiction to inequality (7.12).

Similar theorems can be obtained for different boundary-value problems. This theorem can be strengthened to treat compressive loads by utilizing the minimizing properties of eigenvalues for certain associated linear problems. (Cf. Refs. [21, 23].) A maximum principle argument was employed in Ref. [12] to obtain such a uniqueness theorem for bodies that could suffer only flexure and extension. The bound (7.4) excludes solutions that are just rigid rotations of the straight solution. For other boundary conditions, it would exclude loops. The requirement (7.5) ensures that the material can resist shearing. It prevents the appearance of non-straight solutions corresponding to shear or Lüder's band type instability. The local existence of such solutions can be demonstrated by a bifurcation analysis.

By virtue of this theorem and Eq. (7.3), we have that

$$Q(S) \equiv 0, \quad M(S) \equiv 0, \tag{7.15}$$

so that the nontrivial equilibrium Eqs. (2.22) or (3.4) and (2.27) reduce to

$$N' = 0 \quad \text{or} \quad N = \lambda, \tag{7.16}$$

$$G' - J = 0. \tag{7.17}$$

These equations are supplemented by the reduced form of Eq. (2.6):

$$x' = 1 + \epsilon. \tag{7.18}$$

We assume that the material is homogeneous so that our constitutive relations are

$$\begin{aligned} N(S) &= \hat{N}(\epsilon(S), \rho(S), \rho'(S)), \\ G(S) &= \hat{G}(\epsilon(S), \rho(S), \rho'(S)), \\ J(S) &= \hat{J}(\epsilon(S), \rho(S), \rho'(S)). \end{aligned} \tag{7.19}$$

Since $\hat{N}_\epsilon > 0$ (as a consequence of the positive-definiteness of the symmetric part of matrix (2.35)), we can substitute Eqs. (7.19) into the second of Eqs. (7.16) and solve the resulting algebraic equation uniquely for ϵ:

$$\epsilon = \hat{\epsilon}(\rho, \rho', \lambda). \tag{7.20}$$

The substitution of Eqs. (7.19) and (7.20) into Eq. (7.17) reduces the latter to a second-order autonomous equation for ρ,

$$\frac{d}{dS} \hat{G}(\hat{\epsilon}(\rho, \rho', \lambda), \rho, \rho') - \hat{J}(\hat{\epsilon}(\rho, \rho', \lambda), \rho, \rho') = 0, \tag{7.21}$$

which can be studied by phase plane techniques. We limit our attention however to the hyperelastic case for which this equation has the integral

$$\begin{aligned} \rho' W_{\rho'}(\hat{\epsilon}(\rho, \rho', \lambda), \rho, \rho') + \lambda[1 + \hat{\epsilon}(\rho, \rho', \lambda)] \\ - W(\hat{\epsilon}(\rho, \rho', \lambda), \rho, \rho') = b. \end{aligned} \tag{7.22}$$

This is nothing more than the specialization of Eq. (3.16) consistent with Eqs. (7.6), (7.15), (7.20). This is the equation for the trajectories in the (ρ, ρ')-phase plane. The phase plane diagram has singular points $(\rho, 0)$, where ρ is a solution of

$$\hat{J}(\hat{\epsilon}(\rho, 0, \lambda), \rho, 0) \equiv W_\rho(\hat{\epsilon}(\rho, 0, \lambda), \rho, 0) = 0. \tag{7.23}$$

These singular points represent solutions with constant thickness ρ.

We assume that W is even in ρ', so that we can set

$$2\omega = (\rho')^2, \quad W(\epsilon, \rho, \rho') = V(\epsilon, \rho, \omega). \tag{7.24}$$

We substitute Eq. (7.24) into Eq. (7.22) and take the derivative of the left side with respect to ω. Restoring the original variables, we find this derivative to be

$$\frac{W_{\epsilon\epsilon}W_{\rho'\rho'} - W_{\epsilon\rho'}^2}{W_{\epsilon\epsilon}}, \tag{7.25}$$

where the arguments of the derivatives of W are those indicated in Eq. (7.22). Since term (7.25) is strictly positive by virtue of our assumptions on matrix (2.35), Eq. (7.22) can be written in the alternative form

$$(\rho')^2 = H(\rho, \lambda, b), \tag{7.26}$$

which is especially easy to analyze. To determine the nature of the singular points, we expand Eq. (7.22) in powers of ρ, ρ' about the singular point. The coefficient of $(\rho')^2/2$ is value (7.25), the coefficient of $\rho^2/2$ is

$$\frac{W_{\epsilon\rho}^2 - W_{\epsilon\epsilon}W_{\rho\rho}}{W_{\epsilon\epsilon}}, \tag{7.27}$$

and the coefficients of ρ, ρ', $\rho\rho'$ vanish. Thus the singular point is a saddle when value (7.27) is negative and a center when value (7.27) is positive.

Let us assume that

$$W_{\rho\rho} > 0. \tag{7.28}$$

(When the model is given a two-dimensional interpretation, this is also a consequence of the strong ellipticity condition, provided Eq. (7.6) holds.) When value (7.27) is negative, the saddle is the only singular point because the Hessian matrix of second derivatives of W with respect to ϵ, ρ is positive-definite so the equations

$$W_\epsilon(\epsilon, \rho, 0) = \lambda, \quad W_\rho(\epsilon, \rho, 0) = 0 \tag{7.29}$$

have a unique solution by the global implicit function theorem. When the term (7.27) is positive we can reach no such conclusion.

The transition of the singular point from saddle to center accompanying the change of sign of term (7.27), corresponds to the onset of necking. (If the material is such that term (7.27) is always negative, then such a transition cannot occur.) This effect is most strikingly illustrated when

$$\hat{G}(\epsilon, \rho, 0) \equiv 0 \tag{7.30}$$

and the boundary conditions are

$$G(0) = G(1) = 0, \tag{7.31}$$

for in this case Eqs. (7.30), (7.31) and the positivity of $G_{\rho'}$ imply that

$$\rho'(0) = \rho'(1) = 0 \tag{7.32}$$

and the equations admit the trivial solution

$$\epsilon = \text{const.}, \quad \rho = \text{const.} \tag{7.33}$$

Under these conditions, the only solution of Eq. (7.22) when term (7.27) is negative is the singular solution $\rho = \text{const.}$ corresponding to the saddle point. On the other hand, when term (7.27) is positive, there can also be non-constant solutions which describe the necking phenomenon. These solutions correspond to portions of closed orbits about the center. Thus there are a variety of possible solutions manifesting a periodic structure. A number of these forms of solution have been observed. (Cf. the photographs of Ref. [24].) Proof of the local existence of such necking solutions by bifurcation theory and of the global existence by variational methods have been given in Ref. [11]. For the problem with conditions (7.32), a nontrivial solution does not appear until term (7.27) actually reaches a certain positive number.

To interpret the significance of term (7.27) we first solve the second equation of Eq. (7.29) for ρ as a function of ϵ:

$$\rho = \hat{\rho}(\epsilon). \tag{7.34}$$

This is possibly by inequality (7.28). Then

$$N(S) = \hat{N}(\epsilon, \hat{\rho}(\epsilon), 0) = W_\epsilon(\epsilon, \hat{\rho}(\epsilon), 0) \equiv \text{const.} \tag{7.35}$$

Here the constant ϵ represents the average extension. Then

$$\begin{aligned} \frac{d}{d\epsilon}\hat{N}(\epsilon, \hat{\rho}(\epsilon), 0) &= \hat{N}_\epsilon + \hat{N}_\rho \hat{\rho}_\epsilon = \hat{N}_\epsilon - \hat{N}_\rho \hat{J}_\epsilon / \hat{J}_\rho \\ &= (W_{\epsilon\epsilon} W_{\rho\rho} - W_{\epsilon\rho}^2) / W_{\rho\rho}. \end{aligned} \tag{7.36}$$

Thus the vanishing of term (7.27) implies the vanishing of $(d/d\epsilon)\hat{N}(\epsilon, \hat{\rho}(\epsilon), 0)$ (which is at a local maximum of $N(\epsilon, \rho(\epsilon), 0)$ as a function of ϵ) and this is the usual criterion for the onset of necking.

This material generalizes some of the results of Ref. [11], in which a full bifurcation analysis is given.

8 CONCLUSION

In the problems we have treated there are two recurrent themes. The first is the continual reliance of the analysis on the positive-definiteness of the symmetric part of matrix (2.35). This condition enabled us to apply a variety of implicit function theorems, which were instrumental in reducing the problems to tractable forms. This same condition plays a fundamental role in the existence theory and the bifurcation theory [6, 8–11, 13].

The second theme concerns the complications attending large shear. In Section 4 we found that the presence of shear gave rise to a variety of singular solutions not available in a theory without shear. The qualitative behavior of the family of solutions is altered by the presence of these solutions. In Section 6 we found that there are marked distinctions of treatment between problems with and without shear. In Section 7 we explicitly prohibited the difficulties due to large shear by considering only those solutions satisfying inequality (7.5). Otherwise, we would have had to contend with solutions representing the shear states associated with Lüder's bands. We remark that inequality (7.5) should not be expected to hold universally because Q is always taken normal to the deformed axis, whereas the resultant that actually "effects" the shear is T_2, which is along the deformed material section. (*See* Fig. 10.) Thus it is reasonable to expect $\partial \hat{T}_2/\partial\eta$ to be positive as our conditions on matrix (2.35) imply.

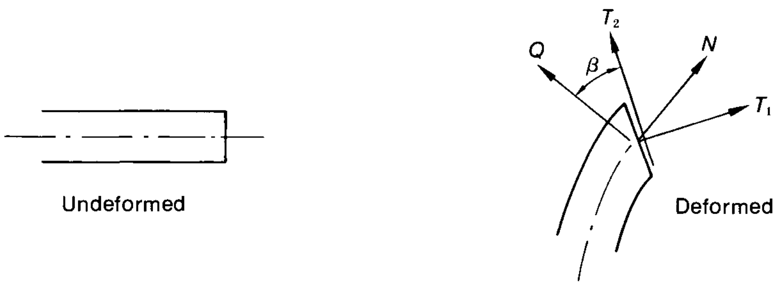

Fig. 10.

The problems we have treated were among the simplest possible for fully nonlinear one-dimensional theories of elasticity. They each led to an autonomous second-order ordinary differential equation (involving several parameters), although the original system is of high order. Problems not having this property would require a much deeper analysis. It may be possible to perform such an analysis by the classical method of shooting, a very pretty example of which is given by Ref. [25]. Most recent work, however, has used topological methods, which may be regarded as the natural generalizations of the topological concepts underlying the phase plane method. In addition to the works cited in Section 4, we just mention Ref. [26], which treats the axisymmetric deformation of a circular von Kármán plate. We note that non-autonomous problems appear naturally in the axisymmetric deformation of shells because the reference radius is generally not a constant function of S. In general, the ordinary differential equations of nonlinear elasticity furnish a rich and untapped source of significant research problems. The same holds *a fortiori* for the partial differential equations, but these are less accessible to analysis.

We have not treated bifurcation problems in this article. Bifurcation theory furnishes a rigorous description of the behavior of solutions near a trivial solution. It proves a fruitful way of determining effects due to the rich kinematic and constitutive structure of the equations described in Section 2 and does not require the independence of S that was so essential for our global analyses. The method of Poincaré is described in Ref. [27], that of Lyapunov and Schmidt in Ref. [28]. References [11–14] furnish bifurcation analyses of specific problems.

Acknowledgment. This work was supported by National Science Foundation Grant GP-32996X.

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