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Flexural instabilities of elastic rods

P. PODIO-GUIDUGLI*

Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA 15213, U.S.A.

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ABSTRACT

In the framework of Antman's theory of elastic rods, it is shown that the behavior of a rod subject to end couples is ruled by a non-linear ordinary differential equation whose solutions describe the instability phenomena observed in severe bending tests of tubes, such as ovalization and necking of the cross section.

SOMMARIO

Nell'ambito della teoria di Antman per le travi elastiche, si mostra che il comportamento di una trave soggetta a coppie concentrate agli estremi è regolato da un'equazione differenziale ordinaria non lineare, le cui soluzioni descrivono fenomeni di instabilità osservati nelle prove di grande flessione di tubi, quali l'ovalizzazione e la strizione della sezione trasversale.

1. Introduction

In a number of papers (see *e.g.* [1], [2], [3]) Antman has formulated a fairly general theory of elastic rods. In this theory rods are modeled as one-dimensional continuous bodies with structure, whose behavior in statics is governed by a quasi-linear system of ordinary differential equations. Although this system is in general too complicated to solve in closed form, Antman has pushed the qualitative analysis of solutions remarkably far; from the point of view of the engineer, his description of shear and necking instabilities in tension is perhaps the most striking achievement (see [4], [5], [6]).

In this paper I use Antman's framework to study the problem of an initially straight rod under the action of terminal couples, and show that one can except necking instabilities also in flexure.

The rod I consider is capable only of planar deformations. These may consist of any suitable combination of axial and (in-plane) transverse stretching, shearing and

* Present address: Instituto di Scienza delle Costruzioni, Facoltà di Ingegneria, Università di Pisa, Italy.

bending; for simplicity, out-of-plane stretching is ignored. In Section 2, I formulate the *pure flexure problem* for elastic rods of this kind. In Section 3, I show that constitutive restrictions quite close to those already exploited in [4] are sufficient to delimit substantially the solution class of the flexure problem, which is reduced to a much simpler one. Then, in Section 4, by applying the so-called "shooting method" to this simpler boundary-value problem,* I obtain conditions for bifurcation of solutions at certain critical values of the terminal couples, and determine the local behavior of the bifurcated solutions. Finally, in Section 5, I discuss how these results may be used to interpret, at least qualitatively, the instabilities observed in pure bending tests of tubes.

According to Timoshenko (*cf.* [8], Art. 84), Bantlin first produced, in 1910, experimental evidence that a curved tube, when bent by end couples tending to increase the tube's initial curvature, had lower stiffness than a straight tube of identical cross section. One year later, v. Kármán gave a rationale for this fact, arguing that stiffness was reduced by the flattening of the cross section into an oval shape due to the compressive radial components of the normal stresses acting on an element of tube.

Such an ovalization, which has rather limited extent even for tubes, is obviously hard to detect when the rod is not hollow. As a matter of fact, I know of no experiments on purely flexural instabilities of solid rods.

The ovalization of an initially straight cylindrical tube subject to terminal couples is generally known as the *Brazier effect.* In 1927, Brazier [9] used Southwell's energy method combined with a clever adaptation of St. Venant's solution of the flexure problem to show that uniform ovalization should occur in bending and, therefore, the curve of terminal couple *vs.* curvature should have decreasing slope and a maximum at, say, (κ_c, μ_c), with μ_c to be interpreted as the collapse load for the tube. Brazier also performed some experiments with long, thin celluloid tubes, where however collapse occurred for μ slightly less than μ_c, with the median portion of the tube exhibiting the necked shape of Fig. 3b, Section 5.

Brazier's work indicates that there exist two competing mechanisms of flexural buckling: his analysis points to what nowadays one would call snap-buckling, or rather, a discontinuous phase transition; his experiments to the occurrence of a continuous transition (an observable bifurcation) from a stable branch along which both curvature and ovalization are independent of the axial coordinate to another stable branch along which a localized necking takes place.

My present analysis allows for both possibilities and could be pushed further to decide under which circumstances the one or the other mechanism comes first. As to this last issue, Donnell contends that snap-buckling is in order for tubes with $R/T<80$, where R is the outer radius and T is the wall thickness.† It may be difficult to confirm this view experimentally, at least for the following two reasons:

(a) The mountings usually do not permit ovalization at the ends; one is driven to

* The shooting method is discussed by J. B. Keller in [7].

† *Cf.* [10], p. 405. Notice that μ_c does not depend on R/T according to Brazier's formula.

use long tubes, and forget about end effects, but with long tubes instability phenomena are much more sensitive to imperfections.

(b) The presence of plastic effects may reduce the significance of a purely elastic analysis, a rule of thumb being that elastic buckling takes place only when R/T is of the same order as the ratio of Young modulus to yield stress [11].

2. Formulation of the flexure problem

For easier reference, I shall use notations as close to those of [4], [5] and [6] as possible. Also, I shall but list balance equations and constitutive assumptions; greater details and motivations are to be found in the papers just quoted and in [12].

The current configuration of a rod is described by a pair of vector fields

$$\boldsymbol{r} = \boldsymbol{r}(S), \qquad \boldsymbol{p} = \boldsymbol{p}(S) \tag{2.1}$$

subject to the condition

$$\boldsymbol{r}'(S) \cdot \boldsymbol{p}(S) \times \boldsymbol{e} > 0, \tag{2.2}$$

where $S \in [0, 1]$ is the arc length coordinate in some reference configuration, a prime denotes differentiation with respect to S, and $\boldsymbol{e}$ is a fixed unit vector. $\boldsymbol{r}(S)$ is interpreted as the position vector of a point of the axis; $\boldsymbol{p}(S)$ and $\boldsymbol{e}$ together determine the plane, and characterize the deformation, of the cross section of the rod.

In the reference configuration the rod has straight axis and undeformed cross section throughout, *i.e.*, it occupies the interval $[0, 1]$ of the X_1-axis of a given orthonormal Cartesian reference $\{0; \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$ in such a way that $S = X_1/L$, $\boldsymbol{E}_3 = \boldsymbol{e}$ and the pair (1) has the form

$$\boldsymbol{R} = \boldsymbol{R}(S) = S\boldsymbol{E}_1, \qquad \boldsymbol{P} = \boldsymbol{P}(S) \equiv \boldsymbol{E}_2. \tag{2.3}$$

In the situations I envisage here the curve $\boldsymbol{r}(S)$ lies in X_1, X_2-plane. I let θ represent the inclination of $\boldsymbol{r}'$ to $\boldsymbol{E}_1$ and φ the inclination of $\boldsymbol{p}$ to $\boldsymbol{E}_2$, and introduce the unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$; $\boldsymbol{a}_1, \boldsymbol{a}_2$ (see Fig. 1):

$$\begin{aligned} \boldsymbol{e}_1(S) &:= |\boldsymbol{r}'(S)|^{-1}\boldsymbol{r}'(S) = \cos\theta(S)\boldsymbol{E}_1 + \sin\theta(S)\boldsymbol{E}_2, \\ \boldsymbol{e}_2(S) &= -\sin\theta(S)\boldsymbol{E}_1 + \cos\theta(S)\boldsymbol{E}_2; \end{aligned} \tag{2.4}$$

$$\begin{aligned} \boldsymbol{a}_1(S) &= \cos\varphi(S)\boldsymbol{E}_1 + \sin\varphi(S)\boldsymbol{E}_2, \\ \boldsymbol{a}_2(S) &:= |\boldsymbol{p}(S)|^{-1}\boldsymbol{p}(S) = -\sin\varphi(S)\boldsymbol{E}_1 + \cos\varphi(S)\boldsymbol{E}_2; \end{aligned} \tag{2.5}$$

$$\theta = \beta + \varphi, \quad \text{with} \quad \beta \in (-\pi/2, \pi/2). \tag{2.6}$$

Furthermore, I set

$$\boldsymbol{r}'(S) =: (1 + \epsilon(S))\boldsymbol{e}_1(S), \qquad \boldsymbol{p}(S) =: (1 + \rho(S))\boldsymbol{a}_2(S); \tag{2.7}$$

in view of (2), the stretches $(1+\epsilon)$ and $(1+\rho)$ defined by (7) obey the inequalities

$$1 + \epsilon > 0, \qquad 1 + \rho > 0. \tag{2.8}$$

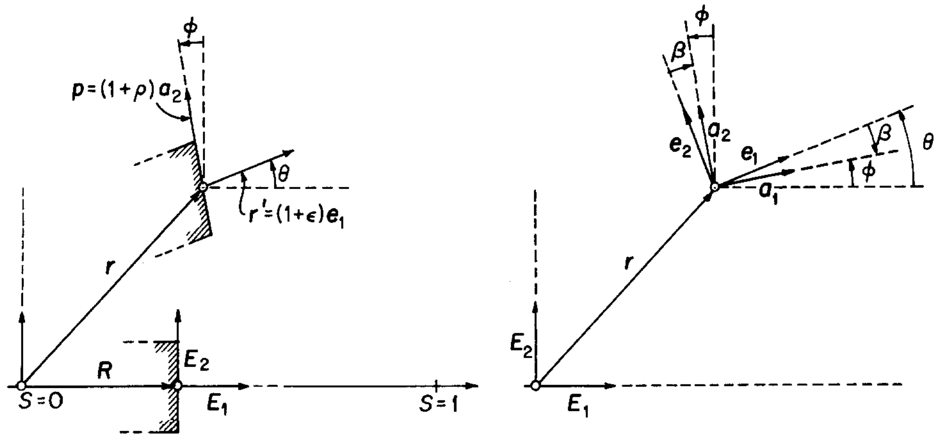

Figure 1

Finally, the current curvature κ of the axis is given by

$$\kappa(S)=(1+\epsilon(S))^{-1}\theta'(S). \tag{2.9}$$

It follows from (3) and the above definitions that θ, φ, ϵ, ρ and κ vanish identically in the reference configuration.

In the current configuration the rod is subject to a resultant force $\boldsymbol{n}$ and to director forces $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$;* the resultant moment $\boldsymbol{m}$ is defined to be

$$\boldsymbol{m}=\boldsymbol{p}\times\boldsymbol{\mu}. \tag{2.10}$$

In the absence of body forces and couples, the balance of resultant force, resultant moment and director forces is expressed by

$$\boldsymbol{n}'=\boldsymbol{0}, \qquad (\boldsymbol{r}\times\boldsymbol{n}+\boldsymbol{m})'=\boldsymbol{0}, \qquad \boldsymbol{\mu}'-\boldsymbol{\nu}=\boldsymbol{0}, \tag{2.11}$$

respectively.† I assume that

$$\boldsymbol{n}=N\boldsymbol{e}_1+Q\boldsymbol{e}_2; \tag{2.12}$$

$$\boldsymbol{\mu}=K\boldsymbol{a}_1+\Sigma\boldsymbol{a}_2, \qquad \boldsymbol{\nu}=P\boldsymbol{a}_2. \tag{2.13}$$

It then follows from $(7)_2$, (10) and $(13)_1$ that

$$\boldsymbol{m}=-(1+\rho)K\boldsymbol{e}, \tag{2.14}$$

and for convenience I set

$$M:=-(1+\rho)K. \tag{2.15}$$

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* In the sense of a principle of virtual work these forces would correspond to the "displacements" $\boldsymbol{r}'$, $\boldsymbol{p}'$ and $\boldsymbol{p}$, respectively.

† The mechanical meaning of statements of balance of director forces is usually left somewhat obscure by experts in theories of oriented continuous media. However, equations such as (11) are carefully motivated by Antman in [1] and [6]. The approach exploited in [13] is used in [12] to show that this equation, together with $(11)_2$, amounts to stating a suitably generalized notion of balance of angular momentum (see also [14]).

One interprets $N(S)$, $Q(S)$ and $M(S)$, respectively, as the axial force, shear force and bending moment acting on the cross section at S (see Fig. 2).

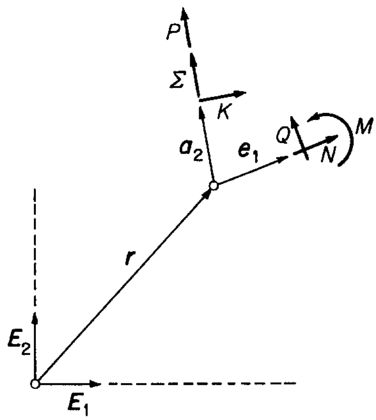

Figure 2

It is easily seen that the balance equations (11) can be given the scalar forms:

$$N' - \theta' Q = 0, \qquad Q' + \theta' N = 0; \tag{2.16}$$

$$M' + (1+\epsilon) Q = 0; \tag{2.17}$$

$$K' - \varphi' \Sigma = 0, \quad \Sigma' + \varphi' K - P = 0. \tag{2.18}$$

Eqs. (15), (17) and $(18)_1$ yield

$$(1+\epsilon) Q - \rho' K - \varphi'(1+\rho)\Sigma = 0. \tag{2.19}$$

Following Antman (see *e.g.* [5]) I regard (19) as identically satisfied in any configuration of the rod as a consequence of appropriate constitutive choices. Thus, I drop the balance equation $(18)_1$, use (15) to dispense with K, and rewrite $(18)_2$ and (19) as respectively,

$$\Sigma' - \varphi'(1+\rho)^{-1} M - P = 0, \tag{2.20}$$

$$(1+\epsilon) Q + \rho'(1+\rho)^{-1} M - \varphi'(1+\rho)\Sigma = 0. \tag{2.21}$$

The rod is comprised of a homogeneous elastic material; *i.e.*, any dynamic variable of the list (N, Q, M, P, Σ) is given as a function of the list of strain variables $(\epsilon, \beta, \kappa, \rho, \rho')$:

$$N = N(S) = \mathcal{N}(\epsilon(S), \beta(S), \kappa(S), \rho(S), \rho'(S)), \quad \text{etc.}$$

Beside some smoothness and growth conditions which I leave tacit, the constitutive functions are to obey certain *a priori* restrictions that will be laid down explicitly later on.

For the class of rods just described, the *pure flexure problem* is to find a pair of smooth vector fields $\boldsymbol{r}(S)$ and $\boldsymbol{p}(S)$ over $[0, 1]$, which are consistent with the geometrical relations (4)–(9); the balance equations (16), (17) and (20); the constitutive equations (22), subject to the requirement (21); the boundary conditions

$$N(0)=N(1)=0, \qquad Q(0)=Q(1)=0 \tag{2.23}$$

and

$$M(0)=M(1)=\mu \qquad \Sigma(0)=\Sigma(1)=0. \tag{2.24}$$

Remark. Other flexure problems are obtained either by replacing $(24)_1$ by

$$-\theta(0)=\theta(1)=\lambda, \tag{2.25}$$

or $(24)_2$ by

$$\rho(0)=\rho(1)=0, \tag{2.26}$$

or $(24)_{1,2}$ by (25) and (26). Each of the above four combinations of boundary conditions amounts to postulating a specific operation of the loading device, the current nomenclature being that a loading device is *soft* (*hard*) if end "forces" ("displacements") are prescribed. Accordingly, the pure flexure problem corresponds to a fully soft loading device.

3. Partial characterization of the solution class

In this section, my purpose is to show that the solution class of the pure flexure problem can be substantially delimited by introducing suitable restrictions on the elastic response.

I begin by assuming that

(i) $\mathcal{N}\equiv 0$ when $\epsilon=0, \quad \dfrac{\partial\mathcal{N}}{\partial\epsilon}>0$

(ii) $\mathcal{Q}\equiv 0$ when $\beta=0, \quad \mathcal{N}+\dfrac{\partial\mathcal{Q}}{\partial\beta}>0.$

These conditions guarantee, among other things, that axial and shear forces vanish identically in the reference configuration.*

* It is proven in [12] that the inequalities in (i) and (ii) follow from the constitutive assumptions accepted in [5] and [6]; moreover, under the same circumstances,

$$\left(\frac{\partial\mathcal{N}}{\partial\epsilon}>0\right)\Leftrightarrow\left(\mathcal{N}+\frac{\partial\mathcal{Q}}{\partial\beta}>0\right),$$

so that either one of the inequalities in (i) and (ii) could then be omitted.

Assumption (ii), in contrast with Antman's (3.3) in [4], does not require $\mathcal{Q}$ to be an increasing function of β, a constitutive hypothesis regarded as generally inappropriate by Antman himself (see *e.g.* [5], Sect. 8). Moreover, it is obvious from the developments below that Antman's semi-uniqueness theorem for rods in tension holds even if one replaces his assumption (3.3) with (ii).

With a view toward analyzing the consequences of assumptions (i) and (ii), I first notice that Eqs. (2.16) yield

$$(N^2(S)+Q^2(S))' \equiv 0.$$

Thus, by the use of boundary conditions $(2.23)_{1,2}$ and $(2.23)_{3,4}$, respectively, I obtain

$$N(S) \equiv 0 \quad \text{and} \quad Q(S) \equiv 0. \tag{3.1}$$

Now, $(1)_1$ and (i) imply that

$$\epsilon(S) \equiv 0, \tag{3.2}$$

whereas $(1)_{1,2}$ and (ii) yield

$$\beta(S) \equiv 0. \tag{3.3}$$

This proves the first portion of the following proposition, the assertions of the second portion being equally easy consequences of (2.6), (3) and (4), (2.9), (2), respectively.

PROPOSITION 1. *The solutions of any flexure problem must correspond to current configurations where both the axial force and the shear force vanish identically, as well as the axial elongation and the shear. Moreover,*

$$\theta(S) \equiv \varphi(S) \quad \textit{and} \quad \kappa(S) \equiv \theta'(S) \equiv \varphi'(S). \tag{3.4}$$

I now derive further results which are valid only for the pure flexure problem. It follows from $(1)_2$, (2.17) and the boundary conditions $(2.24)_{3,4}$ that

$$M(S) \equiv \mu, \tag{3.5}$$

i.e., in the current configuration the bending moment has the constant value of the terminal couples. In view of (5), one is led to consider the algebraic equation

$$\mathcal{M}(0, 0, \kappa, \rho, \rho') = \mu. \tag{3.6}$$

By analogy with (i) and (ii), I make a further physically reasonable assumption, namely,

$$\text{(iii)}\ \mathcal{M} = 0 \quad \text{when} \quad \kappa = 0, \qquad \frac{\partial \mathcal{M}}{\partial \kappa} > 0.^*$$

For definiteness, I also choose μ to be positive. It then follows from (5) and (iii) that Eq. (7) is uniquely solvable for κ:

$$\kappa = \mathcal{K}(\rho, \rho'; \mu); \tag{3.7}$$

moreover,

$$\kappa(S) > 0 \qquad \forall S \in [0, 1]. \tag{3.8}$$

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* Condition (iii) is equivalent to Antman's (3.4) in [4]. As was the case for conditions (i) and (ii), the inequality in (iii) also follows from the constitutive assumptions accepted in [5] and [6].

Equations $(1)_2$, (5) and (2.21), in view also of (4) and (8), determine the director force Σ:

$$\Sigma = \frac{\mu}{\hat{\mathscr{K}}(\rho, \rho'; \mu)} \frac{\rho'}{(1+\rho)^2}. \tag{3.9}$$

The last formula suggests an expedient change of strain variable:

$$u := \frac{1}{1+\rho}.^* \tag{3.10}$$

I define

$$\bar{\mathscr{K}}(u, u'; \mu) := \hat{\mathscr{K}}(\rho, \rho'; \mu), \tag{3.11}$$

$$\bar{\mathscr{R}}(u, u'; \mu) := \mathscr{R}(0, 0, \hat{\mathscr{K}}(\rho, \rho'; \mu), \rho, \rho'), \tag{3.12}$$

whenever the pair (u, ρ) obeys (10). Accordingly, I rewrite (9) as

$$\Sigma = -\frac{\mu u'}{\bar{\mathscr{K}}(u, u'; \mu)}; \tag{3.13}$$

I then turn to (2.20), the only balance equation which is not trivially satisfied in the pure flexure problem, and to the boundary conditions $(2.24)_{3,4}$, and conclude with

PROPOSITION 2. *Under the constitutive assumption* (i)–(iii), *the study of the solution class of the pure flexure problem is reduced to an analysis of the solutions of the quasi-linear ordinary differential equation of the second order*

$$\left(\frac{u'}{\bar{\mathscr{K}}(u, u'; \mu)}\right)' + \bar{\mathscr{K}}(u, u'; \mu)u + \frac{1}{\mu}\bar{\mathscr{R}}(u, u'; \mu) = 0, \tag{3.14}$$

with the boundary conditions

$$u'(0) = u'(1) = 0. \tag{3.15}$$

4. Bifurcation analysis

In this Section I determine circumstances under which the boundary-value problem (3.14), (3.15) admits a solution path $u = u_0(\mu)$ from which, at certain specified values of the parameter μ, other solution paths branch off. As was anticipated in the introduction, I find it convenient to use a suitably tailored version of the shooting method (see [7] and [4]). I also give a local description of the bifurcated branches.

I shall need further assumptions on the elastic response, the first ones being that

(iv) $\bar{\mathscr{K}}(\bar{\mathscr{R}})$ is an even function of u'.†

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* $(2.8)_2$ requires that $u > 0$.

† Conditions (iv) are implied by $(2.16)_c$ and $(2.16)_e$ of [6], respectively. Moreover, the first one of (iv), together with (3.13), agrees with $(i)_1$ and (ii) in [4], Sect. 4, and with one of $(2.16)_d$ in [6].

A consequence of (iv) is that it makes sense to restate problem (3.14), (3.15) as follows:

$$u''+f(u, u'; \mu)=0, \tag{4.1}$$

with f an even function of u' defined by

$$f(u, u'; \mu):=\frac{-\dfrac{\partial \bar{\mathcal{K}}}{\partial u}u'^2+\bar{\mathcal{K}}^2\left(\bar{\mathcal{K}}u+\dfrac{1}{\mu}\bar{\mathcal{R}}\right)}{\bar{\mathcal{K}}-\dfrac{\partial \bar{\mathcal{K}}}{\partial u'}u'}; \tag{4.2}$$

$$u'(0)=u'(1)=0. \tag{4.3}$$

I shall also assume that

(v) The algebraic equation

$$g(u, \mu):=\bar{\mathcal{K}}(u, 0; \mu)u+\frac{1}{\mu}\bar{\mathcal{R}}(u, 0, \bar{\mathcal{K}}(u, 0; \mu))=0 \tag{4.4}$$

has a unique solution

$$u=u_0(\mu) \tag{4.5}$$

which can be extended to pass through $u=1$, $\mu=0$; moreover, both $(-\partial g/\partial \mu)$ and $(\partial g/\partial u)$ are uniformly bounded away from zero.* It follows from (v) that, for any fixed μ, $u(S; \mu)\equiv u_0(\mu)$ is a solution of the boundary-value problem (1)–(3). The question is now to find values of μ, if any, at which other solutions originate.

I represent such putative bifurcated branches in the form

$$u(S; \mu, \alpha)=u_0(\mu)+\alpha v(S; \mu, \alpha). \tag{4.6}$$

For all values of the real parameter α in some neighborhood of zero, the smooth function v is subject to the initial condition

$$\alpha v(0; \mu, \alpha)=\alpha; \tag{4.7}$$

solves the differential equation

$$\alpha v''+f(u_0+\alpha v, \alpha v'; \mu)=0 \tag{4.8}$$

obtained by substituting (6) into (1) and (2); and obeys boundary conditions

$$\alpha v'(0; \mu, \alpha)=0, \tag{4.9}$$

$$\alpha v'(1; \mu, \alpha)=0 \tag{4.10}$$

following from (3).

To investigate the existence of a solution of type (6) to the *boundary-value problem* (8)–(10) is the usual point of departure of most perturbation methods.† On

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* It can be easily seen that (v) agrees with assumption (7.28) in [5]. Note that: $f(u, 0; \mu)=\bar{\mathcal{K}}(u, 0; \mu)g(u, \mu)$; $(\partial f/\partial \mu)(u, 0; \mu)=\bar{\mathcal{K}}(u, 0, \mu)(\partial g/\partial \mu)(u, \mu)$; $(\partial f/\partial u)(u, 0; \mu)=\bar{\mathcal{K}}(u, 0, \mu)(\partial g/\partial u)(u, \mu)$. Thus, in particular: $f(u, 0; \mu)=0$ whenever $g(u, \mu)=0$; $(\partial f/\partial \mu)(u, 0; \mu)<0$; $(\partial f/\partial u)(u, 0; \mu)>0$.

† A formal perturbation scheme is used in [6] to study shear and necking instabilities in tension, and could be adapted – conceivably without loosing the essential feature of being convergent – to deal with the problem under study here.

the other hand, (7) may be seen as a normalization condition on the unknown function v, and is introduced with a view toward matching the strictly perturbational approach with the shooting method, where consideration of the *initial-value problem* (7)–(9) is in order. I find it expedient in the following to denote by v any solution of type (6) of either problem (8)–(10) or problem (7)–(9) (beside the full problem (7)–(10), as originally stipulated).

A necessary condition for bifurcation is to produce a pair $(\mu, \dot{v}(S;\mu))$, where the function $\dot{v}$, defined by

$$\dot{v}(S;\mu) := \frac{d(\alpha v)}{d\alpha}(S;\mu,0) = v(S;\mu,0), \tag{4.11}$$

solves the linear eigenvalue problem obtained by differentiating (7)–(10) about $\alpha = 0$, namely:

$$\dot{v}(0;\mu) = 1; \tag{4.12}$$

$$\dot{v}''(S;\mu) + \omega^2(\mu)\dot{v}(S;\mu) = 0; \tag{4.13}$$

$$\dot{v}'(0;\mu) = 0, \qquad \dot{v}'(1;\mu) = 0. \tag{4.14}$$

Here I have used (iv) and (v) to put

$$\omega^2(\mu) := \frac{\partial f}{\partial u}(u_0(\mu), 0; \mu). \tag{4.15}$$

It is easily seen that such a pair is $(\mu_n, \dot{v}(S;\mu_n))$, whenever the material response is such that any one of the algebraic equations

$$\omega^2(\mu) = n^2\pi^2, \qquad n = 1, 2, \ldots \tag{4.16}$$

has a real solution μ_n; and

$$\dot{v}(S;\mu_n) = \cos(\omega(\mu_n)S) = \cos(n\pi S). \tag{4.17}$$

In the present instance, the shooting method consists in the following steps:

(a) One observes that, essentially because of Theorem 1 of [7], the initial-value problem (7)–(9) has a unique solution $v(S;\mu,\alpha)$ for any (μ,α) in a proper neighborhood of $(\mu_n, 0)$.

(b) One substitutes this solution into the boundary condition (10) in order to obtain the "bifurcation equation":

$$v'(1;\mu,\alpha) = 0. \tag{4.18}$$

(c) One shows, by the use of the implicit function theorem, that equation (18), which in view of (11) and $(14)_2$ is solved by the pair $(\mu_n, 0)$, has solution (μ,α) with $\alpha \neq 0$ for $\mu \neq \mu_n$.

In order to carry out step (c), I begin be defining

$$\ddot{v}(S;\mu) := \frac{d^2(\alpha v)}{d\alpha^2}(S;\mu,0) = 2\frac{\partial v}{\partial \alpha}(S;\mu,0), \tag{4.19}$$

and noticing that

$$\frac{\partial v'}{\partial \alpha}(1;\mu_n,0)=\tfrac{1}{2}\ddot{v}'(1;\mu_n). \tag{4.20}$$

Then, I lay down the problem obtained by differentiating problem (7)–(10) twice at $\alpha=0$ and then setting $\mu=\mu_n$, namely:

$$\ddot{v}(0;\mu_n)=0; \tag{4.21}$$

$$\ddot{v}''(S;\mu_n)+\omega^2(\mu_n)\ddot{v}(S;\mu_n)=a(\mu_n)\dot{v}^2(S;\mu_n)+b(\mu_n)\dot{v}'^2(S;\mu_n); \tag{4.22}$$

$$\ddot{v}'(0;\mu_n)=0, \qquad \ddot{v}'(1;\mu_n)=0; \tag{4.23}$$

where

$$a(\mu_n):=-\frac{\partial^2 f}{\partial u^2}(u_0(\mu_n),0;\mu_n), \qquad b(\mu_n):=-\frac{\partial^2 f}{\partial u'^2}(u_0(\mu_n),0;\mu_n). \tag{4.24}$$

A routine calculation shows that the orthogonality condition

$$\int_0^1 (a\dot{v}^2+b\dot{v}'^2)\dot{v}\,ds=0 \tag{4.25}$$

holds whatever the coefficients a and b might be, *i.e.*, under no further restrictions on the material response. Thus, problem (21)–(23) has one and only one solution $\ddot{v}$. It then follows from (20) and $(23)_2$ that the bifurcation equation (18) cannot be solved directly for $\alpha(\mu)$.

On the other hand, if one assumes that

$$\frac{\partial v'}{\partial \mu}(1;\mu_n,0)\neq 0, \tag{4.26}$$

equation (18) can be solved uniquely for $\mu(\alpha)$ when (μ,α) is near $(\mu_n,0)$; moreover, $\mu(\alpha)$ is smooth and $\mu(0)=\mu_n$. Accordingly, one can write

$$\mu(\alpha)\approx\mu_n+\dot{\mu}\alpha+\frac{1}{2!}\ddot{\mu}\alpha^2, \quad \text{with} \quad \dot{\mu}:=\frac{d\mu}{d\alpha}(0), \qquad \ddot{\mu}:=\frac{d^2\mu}{d\alpha^2}(0), \tag{4.27}$$

and obtain from (18), $(14)_2$, $(23)_2$ and (26) that, necessarily,

$$\dot{\mu}=0, \qquad \ddot{\mu}=-2\,\frac{\dfrac{\partial^2 v}{\partial\alpha^2}(1;\mu_n,0)}{\dfrac{\partial v'}{\partial\mu}(1;\mu_n,0)}. \tag{4.28}$$

Whenever $\ddot{\mu}\neq 0$, an expression for $\alpha(\mu)$ near $(\mu_n,0)$ can still be derived formally from (27):

$$\alpha_n^{\pm}(\mu)\approx\pm\left(\frac{2}{\ddot{\mu}}(\mu-\mu_n)\right)^{1/2}. \tag{4.29}$$

To determine conditions under which formula (29) makes sense, one might differentiate problem (7)–(10) three times at $\alpha = 0$; set $\mu = \mu_n$; obtain explicitly the unique solution $(\partial^2 v/\partial\alpha^2)\,(S;\,\mu_n,\,0)$ of the resulting initial-value problem; and then require that this function do not vanish at $S = 1$. To save labor, I define

$$\dddot{v}(S;\,\mu) := \frac{d^3(\alpha v)}{d\alpha^3}(S;\,\mu,\,0) = 3\,\frac{\partial^2 v}{\partial\alpha^2}(S;\,\mu,\,0), \tag{4.30}$$

and use (27), $(28)_1$, (11), (19) and (30) to write

$$\alpha v(S;\,\mu(\alpha),\,\alpha) \approx \dot{v}(S;\,\mu_n)\alpha + \frac{1}{2!}\,\ddot{v}(S;\,\mu_n)\alpha^2 + \frac{1}{3!}\,\dddot{v}(S;\,\mu_n)\alpha^3. \tag{4.31}$$

Successive differentiation of problem (8)–(10) yields a sequence of problems for the derivatives of αv and μ with respect to α at $\alpha = 0$. In particular, $\ddot{\mu}$ is determined by the orthogonality condition which guarantees the solvability of the third problem in that sequence. One finds that

$$\ddot{\mu} = -\frac{\int_0^1 (a\dot{v}\ddot{v} + b\dot{v}'\ddot{v}' + c\dot{v}^3 + d\dot{v}\dot{v}'^2)\dot{v}\,ds}{\int_0^1 e\dot{v}^2\,ds} \tag{4.32}$$

where the coefficients a and b are defined by (24) and

$$c(\mu_n) := \frac{1}{3}\frac{\partial^3 f}{\partial u^3}(u_0(\mu_n),\,0;\,\mu_n), \qquad d(\mu_n) := \frac{\partial^3 f}{\partial u\,\partial u'^2}(u_0(\mu_n),\,0;\,\mu_n),$$

$$e(\mu_n) := \frac{\partial^2 f}{\partial u\,\partial\mu}(u_0(\mu_n),\,0;\,\mu_n). \tag{4.33}$$

Now, observe that

$$\frac{\partial v'}{\partial\mu}(1;\,\mu_n,\,0) = \left(\frac{\partial\dot{v}}{\partial\mu}(S;\,\mu_n)\right)'(1) = (-1)^{n+1}\,\frac{1}{2}\,\frac{d\omega^2}{d\mu}(\mu_n); \tag{4.34}$$

therefore, if (26) is satisfied, the eigenvalue μ_n is simple.

My last assumption on the elastic response is the following one:

(vi) There exists an index n such that equation (16) has a real solution μ_n and, moreover, the material moduli $(d\omega^2/d\mu)(\mu_n)$, $c(\mu_n)$, $e(\mu_n)$ are all different from zero.*

When condition (vi) holds, formula (29) can be made meaningful; therefore, bifurcation does occur at point(s) $(u_0(\mu_n),\,\mu_n)$. I summarize and conclude the above bifurcation analysis by means of

Proposition 3. *Under the constitutive assumptions* (iv)–(vi), *the non-linear problem* (1)–(3) *has a solution path* $u_0(\mu)$ *from which, at* $\mu = \mu_n$, *two and only two other*

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*The hypotheses on the function $\omega^2(\mu)$ at $\mu = \mu_n$ are the same as in [4]; *cf.*, in particular, condition (4.23) of that paper. As to the *genericity* of the hypotheses on c and e, to the effect that they are compatible only with a *non-linear* material response, see the views expressed by Antman and Carbone on top of p. 136 of [6].

solutions $u_n^+(S;\mu)$ and $u_n^-(S;\mu)$ branch off. These are such that (cf. (6) *and* (29))

$$u_n^{\pm}(S;\mu) := u_0(\mu) + \alpha_n^{\pm}(\mu) v(S;\mu,\alpha_n^{\pm}(\mu)). \tag{4.35}$$

The bifurcated branches u_n^+ and u_n^- are distinct from each other and from u_0 for μ near and distinct from μ_n. If $\ddot{\mu}<0$ $(\ddot{\mu}>0)$, then, for μ near and greater (less) than μ_n, no solution which tends to $u_0(\mu_n)$ as μ tends to μ_n exists other than u_n.

Finally, in view of (35), (29), (31) and $(17)_2$, I arrive at the following local representation of $u_n^{\pm}$:

$$u_n^{\pm}(S;\mu) \approx u_0(\mu) \pm \mathfrak{b}_{1n} \cos(n\pi S)\,|\mu-\mu_n|^{1/2} + (\mathfrak{a}_{2n} + \mathfrak{b}_{2n}\cos(n\pi S) + \\ + \mathfrak{c}_{2n}\cos(2n\pi S))\,|\mu-\mu_n|, \tag{4.36}$$

with constants $\mathfrak{b}_{1n}>0, \mathfrak{a}_{2n}, \mathfrak{b}_{2n}$ and $\mathfrak{c}_{2n}$ depending on various derivatives of f up to the third order evaluated at $(u_0(\mu_n), 0; \mu_n)$.

5. Interpretation – Flexural instabilities of tubes

Think of an initially straight cylindrical bar or tube being bent by two opposite end couples of mangitude μ. In view of $(2.7)_2$ and (3.10), the length of the material fiber of the cross section which currently lies along $\boldsymbol{p}(S)$ is

$$r(S;\mu) := \frac{R}{u(S;\mu)} \tag{5.1}$$

where R is the radius of the cross section. Moreover, the curvature of the axis is given by

$$\kappa(S;\mu) := \bar{\mathcal{K}}(u(S;\mu), u'(S;\mu);\mu). \tag{5.2}$$

By assumption (v), $\mu \mapsto (u_0(\mu)-1)$ is a monotonically increasing function of $\mathbb{R}^+$ into itself. Let $\bar{n}$ be the smallest index n for which the situation described in assumption (vi) takes place. Then, for μ in a right neighborhood of 0, there exists a solution of the pure flexure problem corresponding to *circular axis* and *uniform ovalization* of the cross section, described by

$$\kappa(S;\mu) \equiv \bar{\mathcal{K}}(u_0(\mu), 0;\mu), \qquad r(S;\mu) \equiv \frac{R}{u_0(\mu)}. \tag{5.3}$$

At $\mu=\mu_{\bar{n}}$ two other solutions branches off, corresponding to an axis of variable curvature and a *lobated profile* of ovalized cross sections. More particularly, if $\bar{n}=1$, one can envisage a lopsided profile with more pronounced ovalization at one side. If $\bar{n}=2$, either one of the profiles sketched in Fig. 3 becomes possible; the *necked profile* shown in Fig. 3b has been repeatedly observed, *e.g.* in buckling experiments performed on thick-walled rubber tubes (*cf.* [15]). For increasing values of $\bar{n}$, one obtains a sequence of *wrinkled profiles* with increasingly shorter axial wave length, which have also been observed for a variety of tubes (see [16] and the literature quoted therein).

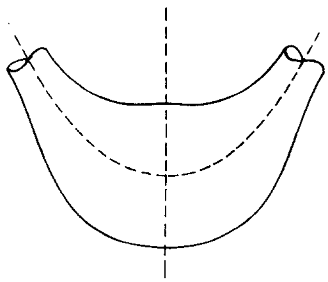

Figure 3a

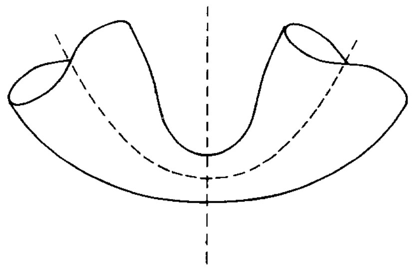

Figure 3b

The flexural instabilities which I have described occur when the material is such that the constitutive assumptions (i)–(vi) apply. Of these assumptions, (i)–(iv) directly reflect certain growth and symmetry properties of the material response for which one has, in a sense, an *a priori* feeling; on the other hand, (v) and (vi) are more technical in nature, as they are suggested by the procedure of attack to the problem. However, as was pointed out, all of these assumptions either coincide or agree substantially, to within differences dictated by the context or being a matter of taste, with the assumptions accepted in [1]–[6]. As customary in Antman's theory of rods, where a monotonicity condition on the elastic response rules out material instabilities, flexural instabilities are but special features of the solution class of a given boundary-value problem; moreover, their existence has been demonstrated without postulating any sort of plastic behavior.

Finally, I observe that ignoring out-of-plane stretching of the cross section does not seem to be a serious drawback, at least in the description of flexural instabilities. Indeed, when ovalization takes place, the ratio of principal stretches of material fibers in the cross sectional plane should stay reasonably constant; when axial wrinkling is in order and the tube wall is not exceptionally thin, it is observed that an axial wave "has practically zero amplitude over the entire circumference [of the cross section] except for a small region in the highly compressed part" (cf. [11]).

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