1 may Tissue curvature and apicobasal mechanical tension imbalance instruct cancer morphogenesis

nature SUPPLEMENTARY INFORMATION

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Mechanical equations

this is independent on ϑ! Hence, we are ruling out instabilities that are localized along the axis.

Nota guse:

$$P(\mathrm{div} N + b) + G F^{-1}(P\,\mathrm{div} M + c) = 0 \qquad P = I - n\otimes n$$

$$\mathrm{div}\left(F^{-1}(P\,\mathrm{div} M + c) - N\cdot G - n\cdot b\right) = 0$$

If ref = curr. then P = proj. $\perp$ normal $F = P$

$$\begin{cases} P(\mathrm{div} N + b) + G(\mathrm{div} M + c) = 0 \\ \mathrm{div}\,\mathrm{div} M + \mathrm{div}\, c - N\cdot G - n\cdot b = 0 \end{cases} \iff$$

check:

Componentwise expression:

$$P\,\mathrm{div} N = P\, N_{,\alpha} a^\alpha = P\left(N^{\alpha\beta} a_\alpha \otimes a_\beta\right)_{,\gamma} a^\gamma$$

$$= P N^{\alpha\beta} a_{\alpha,\gamma}\delta_\beta^\gamma + N^{\alpha\beta} a_\alpha a_{\beta,\gamma} a^\gamma + N^{\alpha\beta}{}_{,\gamma} a_\alpha \delta_\beta^\gamma$$

$$= N^{\alpha\beta}(a_{\alpha,\gamma}\cdot a^\delta) a_\delta \delta_\beta^\gamma + N^{\alpha\beta} a_\alpha \Gamma^\gamma_{\beta\gamma} + N^{\alpha\beta}{}_{,\beta} a_\alpha$$

$$= N^{\alpha\beta}\Gamma^\delta_{\alpha\beta} a_\delta + N^{\alpha\beta}\Gamma^\gamma_{\beta\gamma} a_\alpha + N^{\alpha\beta}{}_{,\beta} a^\alpha$$

$$= N^{\alpha\beta}\big|_\beta a_\alpha = \nabla_\beta N^{\alpha\beta} a_\alpha$$

covariant derivative

Constitutive equations:

$$N = K\,\frac{a - a_0}{a_0} P \iff$$

$$M = (k\,\mathrm{tr} G + \mathfrak{J}) P \qquad G_0 = k\, m_0 P$$

Balance equations

$$\mathrm{div}_s\left(K\frac{a-a_0}{a_0}P\right) = \nabla K\left(\frac{a-a_0}{a_0}\right) + K\frac{a-a_0}{a_0}\underset{\substack{\shortparallel \\ 0}}{\mathrm{div} P}$$

$$\begin{cases} K\nabla\frac{a-a_0}{a_0} + G\,\nabla\left(K(\mathrm{tr} G - \mathfrak{J})\right) = 0 \\ \Delta(k\,\mathrm{tr} G - \mathfrak{J}) - K(\mathrm{tr} G)\frac{a-a_0}{a_0} = 0 \end{cases} \iff$$

Remark on the Laplace Beltrami operator

Remark on the Laplace Beltrami

Prop: $\mathrm{div}\,\nabla\varphi = \frac{1}{\sqrt{a}}\left(\varphi_{,\alpha} a^{\alpha\beta}\sqrt{a}\right)_{,\beta}$ where

$a_1 \times a_2 \cdot n = \sqrt{a}$ = area element

Basic identity: $\frac{(\sqrt{a})_{,\alpha}}{\sqrt{a}} = a_{\gamma,\alpha}\cdot a^\gamma$.

Proof: $a_{1,\alpha}\cdot a^1 + a_{2,\alpha}\cdot a^2 = a_{1,\alpha}\cdot\frac{a_2\times n}{\sqrt{a}} + a_{2,\alpha}\cdot\frac{n\times a_1}{\sqrt{a}} = \frac{1}{\sqrt{a}}(n$

$$\nabla\varphi = \varphi_{,\alpha}\otimes a^\alpha$$

$$\mathrm{div}\,\nabla\varphi = (\varphi_{,\alpha} a^\alpha)_{,\beta}\cdot a^\beta = (\varphi_{,\alpha} a^\alpha\cdot a^\beta)_{,\beta} - \varphi_{,\alpha}\underbrace{a^\alpha\cdot a^\beta_{,\beta}}_{(a^\alpha\cdot a^\beta)(a_\beta\cdot a^\gamma_{,\gamma})}$$

$$= (\varphi_{,\alpha} a^{\alpha\beta})_{,\beta} - \varphi_{,\alpha} a^{\alpha\beta}\, a_\beta\cdot a^\gamma_{,\gamma}$$

$$= \cdots + \varphi_{,\alpha} a^{\alpha\beta}\, a_{\gamma,\beta} a^\gamma$$

$$a_{\beta,\gamma}\cdot a^\gamma = a_{\gamma,\beta}\cdot a^\gamma$$

$$= \frac{\sqrt{a}}{\sqrt{a}}(\varphi_{,\alpha} a^{\alpha\beta})_{,\beta} + \frac{\sqrt{a}_{,\beta}}{\sqrt{a}}\varphi_{,\alpha} a^{\alpha\beta}$$

$$= \frac{1}{\sqrt{a}}\left(\varphi_{,\alpha} a^{\alpha\beta}\sqrt{a}\right)_{,\beta} \quad \text{(Laplace-Beltrami operator)}$$

Calculation of C_ϑ^ϑ.

$$X = (R + \delta r)\cos\vartheta\, e_x + (R + \delta r)\sin\vartheta\, e_y + z e_z$$

$$X = (R + \delta r)\, Q(\vartheta) e_x + z e_z \qquad Q(\vartheta) = \text{rotation about } e_z \text{ angle } \vartheta \text{ counterclockwise}$$

$$a_\vartheta = X_{,\vartheta} = e_z \times (R + \delta r) Q(\vartheta) e_x + \delta r' Q(\vartheta) e_x$$

$$a_{\theta,\theta} = e_z \times a_\theta + e_z \times \delta r' Q(\theta) e_x + \delta r'' Q(\theta) e_x$$

$$n = \frac{a_\theta \times a_z}{|a_\theta \times a_z|}$$

$$a^\theta = \frac{1}{|a_\theta|^2} a_\theta \Rightarrow \quad a^\theta\cdot a_\theta \qquad a^\theta\cdot a_z = 0$$

$$C_\theta^\theta = n_{,\theta}\cdot a^\theta = \left(\frac{a_\theta\times a_z}{|a_\theta\times a_z|}\right)_{,\theta}\cdot\frac{a_\theta}{|a_\theta|^2} = \frac{a_{\theta,\theta}\times a_z\cdot a_\theta}{|a_\theta\times a_z||a_\theta|^2}$$

$$a_{\vartheta,\vartheta}\times a_z\cdot a_\vartheta = |a_\vartheta|^2 + |\delta r' Q(\vartheta) e_x|^2 - \delta r''(R + \delta r)$$

$$= |a_\vartheta|^2 + |\delta r'|^2 - \delta r''(R + \delta r) \simeq |a_\vartheta|^2 - \delta r'' R$$

$$|a_\vartheta|^2 = |a_\vartheta\times a_z|^2 = (R + \delta r)^2 + (\delta r')^2 \simeq R^2 + 2R\delta r \Rightarrow \frac{1}{|a_\vartheta|} \simeq \frac{1}{R}\left(1 - \frac{\delta r}{R}\right)$$

$$\frac{|a_\vartheta|^2 - \delta r'' R}{|a_\vartheta|^2 |a_\vartheta\times a_z|} \simeq \frac{1}{|a_\vartheta\times a_z|} - \frac{\delta r''\cancel{R}}{R^2\cdot\cancel{R}} \simeq \boxed{\frac{1}{R}\left(1 - \frac{\delta r}{R} - \frac{\delta r''}{R}\right)}$$

$$\cdot a_1 \times a_2)_{,\alpha} - \overbrace{\frac{1}{\sqrt{a}} \; n_{,\alpha} \cdot a_1 \times a_2}^{=0}$$

We denote the tangential surface tension tensor t^{ij}, the tangential bending moment density tensor in the surface m^{ij} and its related tensor $\bar{m}^{ij} = -m^{ik}\epsilon_k{}^j$. We assume that no external force or torque density act on a surface element, and consider configurations of mechanical equilibrium with no flow. The tangential and normal force balance equations can then be written [3]

$$\nabla_i t^{ij} + C_i{}^j \nabla_k \bar{m}^{ki} = 0 \quad , \tag{21}$$

$$\nabla_i \nabla_j \bar{m}^{ji} - C_{ij} t^{ij} = 0 \quad . \tag{22}$$

We consider the following constitutive equations for the tension and bending moment density tensors

$$t_{ij} = K \frac{a - a_0}{a_0} g_{ij} \quad , \tag{23}$$

isotropic

$$\bar{m}_{ij} = (\kappa C_k{}^k + \zeta_c) g_{ij} \quad , \tag{24}$$

With the constitutive equations 23-24, the tangential and normal force balance equations can be rewritten

$$K\partial_i\left(\frac{a-a_0}{a_0}\right)+{C_i}^j\partial_j(\kappa {C_k}^k+\zeta_c)=0 \quad , \tag{25}$$

$$\Delta(\kappa {C_k}^k+\zeta_c)-K\frac{a-a_0}{a_0}{C_k}^k=0 \quad , \tag{26}$$

where we have introduced the Laplace-Beltrami operator, $\Delta=\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j f)$.