

Enumerating Disjoint Partial Models without Blocking Clauses

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Abstract

A basic algorithm for enumerating disjoint propositional models (disjoint AllSAT) is based on adding blocking clauses incrementally, ruling out previously found models. On the one hand, blocking clauses have the potential to reduce the number of generated models exponentially, as they can handle partial models. On the other hand, they need exponential space and slow down unit propagation. We propose a new approach that allows for enumerating disjoint partial models with no need for blocking clauses by integrating: Conflict-Driven Clause-Learning (CDCL), Chronological Backtracking (CB), and methods for shrinking models (Implicant Shrinking). Experiments clearly show the benefits of our novel approach.

1 Introduction

All-Solution Satisfiability Problem (AllSAT) is an extension of SAT that requires finding all possible solutions of a propositional formula. AllSAT has been heavily applied in the field of hardware and software verification. For instance, AllSAT can be used to automatically generate test suites for programs (Khurshid et al. 2004) and for bounded and unbounded model checking (Jin, Han, and Somenzi 2005). Recently AllSAT has found applications in artificial intelligence. For example, (Spallitta et al. 2022) exploits AllSMT (a variant of AllSAT dealing with first-order logic theories) to perform probabilistic inference in hybrid domains. AllSAT has also been applied to data mining to deal with the frequent itemset mining problem (Dlala et al. 2016).

Exploring the complete search space efficiently is a major concern in AllSAT. A naive approach would try to obtain all satisfying assignments one by one, gradually adding new constraints to prune the search and avoid being stuck in the same regions. Nevertheless, this approach is impractical for real-world issues with numerous variables and constraints. For a formula F with n variables, there are 2^n possible total assignments. If we were to generate all of these assignments explicitly, it would require exponential space complexity. To mitigate the issue, we can use *partial* models to obtain compact representations of a set of solutions. If a partial model does not explicitly assign the truth value of a variable, then it means that its truth value does not impact the satisfiability

of that assignment, thus two assignments are represented by the partial one. Extending the concept to a larger amount of unassigned variables, we have that in problems with n variables a partial assignment with m variables covers 2^{n-m} total assignments in one shot.

Although minimizing assignments is essential to enumerate large propositional instances in a feasible amount of time, naive implementations may lead to the coverage of the same total assignments by two or more partial assignments. Whereas this may not be problematic for certain applications (e.g. predicate abstraction (Lahiri, Bryant, and Cook 2003)), it can result in an incorrect final solution for other contexts. For this reason, literature distinguishes between enumeration with repetitions (AllSAT) and enumeration without repetitions (*disjoint AllSAT*). In this paper, we will focus on the latter.

Whereas disjoint AllSAT is strongly related to propositional model counting (#SAT), the necessity to produce all assignments for AllSAT led the two problems to develop different algorithms. Propositional enumeration algorithms can be grouped into three main categories: BDD-based solvers, blocking solvers, and non-blocking solvers.

BDD-based AllSAT solvers (Huang and Darwiche 2005) compile a Boolean formula F into a Binary Decision Diagram (BDD), using the latter to return all assignments. Having BDDs makes counting trivial since we only need to check all paths starting from the root of the BDD, which can be done in polynomial time. Despite this, the conversion of a formula into a BDD for AllSAT could result in an exponential blow-up in space and is heavily dependent on the order of variables chosen to generate it. Another intrinsic issue with BDD-based AllSAT solvers is the impossibility to provide implementations using any-time algorithms: no lower bound in the number of models of the problems can be provided unless the entire formula is compiled into a BDD.

Blocking AllSAT solvers (McMillan 2002; Jin, Han, and Somenzi 2005; Yu et al. 2014) rely on Conflict Driven Clause-Learning (CDCL) and non-chronological backtracking (NCB) to return the set of all satisfying assignments. They work by repeatedly adding blocking clauses to the formula after each model is found, which rules out the previous set of satisfying assignments until all possible satisfying assignments have been found. These blocking clauses ensure that the solver does not return the same satisfying

assignment multiple times and that the search space is efficiently scanned (Morgado and Marques-Silva 2005). Although blocking solvers are straightforward to implement and can be adapted to retrieve partial assignments, they become inefficient when the input formula F has a high number of models, as an exponential number of blocking clauses might be added to make sure the entire search space is visited. As the number of blocking clauses increases, unit propagation becomes more difficult, resulting in degraded performance.

Non-blocking AllSAT solvers (Grumberg, Schuster, and Yadgar 2004; Li, Hsiao, and Sheng 2004) overcome this issue by not introducing blocking clauses and by implementing *chronological backtracking* (CB) (Nadel and Ryvchin 2018): after a conflict arises, they backtrack on the search tree by updating the most recently instantiated variable. Chronological backtracking guarantees not to cover the same model of a formula multiple times without the typical CPU-time blow-up caused by blocking clauses. The major drawback of this approach is that there is no method known in the literature which would allow it to be combined with partial assignments. Moreover, regions of the search space with no solution cannot be escaped easily if CDCL is not integrated.

In this context, (Möhle and Biere 2019b) proposes a new formal calculus of a disjunctive model counting algorithm combining the best features of chronological backtracking and CDCL, but without providing an implementation or experimental results. Our goal was to apply these ideas in the context of AllSAT solving, and achieve better overall performance compared to existing approaches, particularly when instances with high numbers of solutions are considered.

Contributions In this work, we propose a novel AllSAT procedure to perform disjoint partial enumeration of propositional formulae by combining the best of current AllSAT state-of-the-art literature: (i) CDCL, to escape search branches where no satisfiable assignments can be found; (ii) chronological backtracking, to ensure no blocking clauses are introduced; (iii) efficient implicant shrinking, to reduce in size partial assignments, by exploiting the 2-literal watching scheme. We have implemented the aforementioned ideas in TABULARALLSAT and compared its performance against other publicly available state-of-the-art AllSAT tools using a variety of benchmarks, including both crafted and SATLIB instances. Our experimental results show that TABULARALLSAT outperforms all other solvers on most benchmarks, demonstrating the benefits of our approach.

Summary The paper is structured as follows. Section 2 presents the notation and the theoretical foundations for the rest of the work. In Section 3 we present the novel framework to perform propositional enumeration. We focus on combining CDCL and chronological backtracking in the search algorithm for AllSAT (Section 3.1), chronological implicant shrinking algorithms (Section 3.2), the novel notion of virtual backtrack reason for good learning (Section 3.3), and the impact of decision heuristics in chronological backtracking (Section 3.4). Section 4 is devoted to an extensive experimental evaluation. Lastly, we conclude

in Section 5, also highlighting future directions.

2 Background

2.1 Notation

We assume F is a propositional formula defined on the set of Boolean variables $V = \{v_1, \dots, v_n\}$, with cardinality $|V|$. A *literal* ℓ is a variable v or its negation $\neg v$. We implicitly remove double negations: if ℓ is $\neg v$, by $\neg \ell$ we mean v rather than $\neg \neg v$. A *clause* is the disjunction of one or more literals $\bigvee_{\ell \in c} \ell$. A *cube* is the conjunction of one or more literals $\bigwedge_{\ell \in c} \ell$.

The function $M : V \mapsto \{\top, \perp\}$ mapping variables in F to their truth value is known as *assignment*. An assignment can be represented by either a set of literals $\{\ell_1, \dots, \ell_n\}$ or a cube conjoining all literals in the assignment $\ell_1 \wedge \dots \wedge \ell_n$. We distinguish between *total assignments* η or *partial assignments* μ depending on whether all variables are mapped to a truth value or not, respectively.

A *trail* is an ordered sequence of literals $I = \ell_1, \dots, \ell_n$ with no duplicate variables. The empty trail is represented by ε . Two trails can be conjoined one after the other $I = KL$, assuming $K \cap L = \emptyset$. We use superscripts to mark literals in a trail I : ℓ^d indicates a literal assigned during the decision phase, whereas ℓ^* refers to literals whose truth value is negated due to chronological backtracking after finding a model (we will refer to this action as *flipping*). Trails can be seen as ordered total/partial assignments; for the sake of simplicity, we will refer to *total* (resp. *partial*) *trails* to define a trail assigning all (resp. a subset of) variables in F a truth value.

Definition 1 The decision level function $\delta(V) \mapsto \mathbb{N} \cup \infty$ returns the decision level of variable V . If the variable is unassigned, we return ∞ . We extend this concept to literals ($\delta(\ell) = \delta(V(\ell))$) and clauses ($\delta(C) = \{\max(\delta(\ell)) \mid \ell \in C\}$).

Definition 2 The decision literal function $\sigma(dl) \mapsto \ell \cup \varepsilon$ returns the decision literal of level dl . If we have not decided on a literal at level dl yet, we return ε .

Definition 3 The reason function $\rho(\ell)$ returns the reason that forced literal ℓ to be assigned a truth value. We distinguish among:

- **DECISION**, if the literal assignment is due to the decision procedure;
- **UNIT**, if the literal is unit propagated at decision level 0, thus it is an initial literal;
- **PROPAGATED(c)**, if the literal is unit propagated at a decision level higher than 0 due to clause c ;

2.2 The 2-watched literal scheme

The *2-watched literal scheme* is an indexing technique that efficiently checks if the literals currently assigned do not cause a conflict, first introduced in the SAT solver Chaff (Moskewicz et al. 2001). For every clause, two literals are tracked. If at least one of the two literals is set to \top , then the clause is satisfied. If one of the two literals is set to \perp , then we scan the clause searching for a new literal ℓ' that can be

paired with the remaining one, being sure ℓ' is not mapped to \perp . If we reach the end of the clause, then we know the current assignment falsifies the formula. The 2-watched literal scheme is implemented through watch lists.

Definition 4 The watch list function $\omega(\ell)$ returns the set of clauses $\{c_1, \dots, c_n\}$ currently watched by literal ℓ .

2.3 CDCL and non-chronological backtracking

Conflict Driven Clause Learning (CDCL) is the most popular SAT solving technique (Marques-Silva and Sakallah 1999). It is an extension of the older Davis-Putnam-Logemann-Loveland (DPLL) algorithm (Davis, Logemann, and Loveland 1962), improving the latter by dynamically learning new clauses during the search process and using them to drive backtracking.

Every time the current trail falsifies a formula F , the SAT solver generates a conflict clause c starting from the falsified clause, by repeatedly resolving against the clauses which caused unit propagation of falsified literals. This clause is then learned by the solver and added to F . Depending on c , we backtrack to flip the value of one literal, potentially jumping more than one decision level (thus we talk about *non-chronological backtracking*). CDCL and non-chronological backtracking allow for escaping regions of the search space where no satisfying assignments are admitted, which benefits both standard SAT and AIIISAT solving. The idea behind conflict clauses has been extended in AIIISAT to learn clauses from partial satisfying assignments (known in the literature as *good learning* or *blocking clauses*) to ensure no total assignment is covered twice.

2.4 Chronological backtracking

The idea of exploiting chronological backtracking (CB) was first presented in GRASP (Marques-Silva and Sakallah 1999) in the context of clause learning, and then revamped for both SAT and AIIISAT in (Nadel and Ryvchin 2018; Möhle and Biere 2019a). The intuition is that non-chronological backtracking after conflict analysis can lead to redundant work, due to some assignments that could be repeated later on during the search. Instead, independent of the generated conflict clause c we chronologically backtrack and flip the last decision literal in the trail. Consequently, we explore the search space in a systematic and efficient manner, ensuring no assignment is covered twice during execution. Chronological backtracking combined with CDCL has been shown to be effective in SAT solving when dealing with satisfiable instances. In AIIISAT solving, it guarantees blocking clauses are no more needed to ensure the termination of the search.

3 Enumerating disjoint partial models without blocking clauses

In this section, we propose a novel approach that allows for enumerating disjoint partial models with no need for blocking clauses, by integrating: Conflict-Driven Clause-Learning (CDCL), to escape search branches where no satisfiable assignments can be found; Chronological Backtracking (CB),

to ensure no blocking clauses are introduced; and methods for shrinking models (Implicant Shrinking), to reduce in size partial assignments, by exploiting the 2-watched literal schema.

To this extent, (Möhle and Biere 2019b) discusses a formal calculus to combine CDCL and CB for propositional model counting, strongly related to the task we want to achieve. We take the calculus presented in that paper as the theoretical foundation on top of which we build our algorithms.

3.1 AIIISAT search algorithm by integrating CDCL and CB

The work in (Möhle and Biere 2019b) exclusively describes the calculus and a formal proof of correctness for a model counting framework on top of CDCL and CB, with neither any algorithm nor any reference in modern state-of-the-art solvers. To this extent, we start by presenting an AIIISAT procedure for the search algorithm combining the two techniques, which is reported in Algorithm 1.

The goal of this algorithm is to find a total trail I that satisfies F . At each decision level, it iteratively decides one of the unassigned variables in F and assigns a truth value (lines 32-35); it then performs unit propagation (line 4) until either conflict is reached (lines 5-18), or no other variable can be unit propagated (lines 19-30) or DECIDE has to be called again.

On conflict (s.t. UNITPROPAGATION returns one clause c in F which is falsified by the current trail I), we first compute the maximum assignment level of all literals in the conflicting clause c and backtrack to that decision level (lines 6-7). This is necessary to support out-of-order assignments, the core insight in chronological backtracking into CDCL as described in (Nadel and Ryvchin 2018). If the solver reaches decision level 0 at this point, it means there are no more variables to flip and the whole search space has been visited, and we can terminate the algorithm (lines 8-9). Otherwise, we perform conflict analysis up to the last decision Unique Implication Point (last UIP), retrieving the conflict clause c' (line 10), as proposed in (Möhle and Biere 2019b).

If the conflict clause c' is a single unit, then uip can be unit-propagated at decision level 0: then we backtrack to level 0, unit propagates $\neg uip$, and restart our search (lines 11-14). Otherwise, we chronologically backtrack, push the flipped UIP into the trail, and set c' as its assignment reason for the flipping (lines 15-18).

Now suppose every variable is assigned a truth value (line 19); then the current total trail I satisfies F . If so, the procedure IMPLICANT-SHRINKING checks if, for some decision level dl' , we can backtrack up to $dl' < dl$ and obtain a partial trail still satisfying the formula (lines 20-22). (We discuss the details of chronological implicant shrinking in Section 3.2.) We can produce the partial assignment from the shrunk trail I (line 23). Then we check if all variables in I are assigned at decision level 0. If this is the case, then this means that we found the last assignment to cover F , so that we can end the search (lines 24-25). Otherwise, we perform chronological backtracking, flipping the truth value of

Algorithm 1: CHRONOLOGIC-CDCL(F, V)

```

1:  $I \leftarrow \varepsilon$  // Current trail
2:  $dl \leftarrow 0$  // Current decision level
3: while true do
4:  $I, c \leftarrow \text{UNITPROPAGATION}()$ 
5: if  $c \neq \varepsilon$  then
6:   if  $\delta(c) < dl$  then
7:      $I \leftarrow \text{BACKTRACK}(\delta(c))$ 
8:   if  $dl = 0$  then
9:     return
10:   $\langle uip, c' \rangle \leftarrow \text{CONFLICTANALYSIS}()$ 
11:  if  $|c'| = 1$  then
12:     $I \leftarrow \text{BACKTRACK}(0)$ 
13:     $I.\text{push}(\neg uip)$ 
14:     $\rho(\neg uip) \leftarrow \text{UNIT}$ 
15:  else
16:     $I \leftarrow \text{BACKTRACK}(dl - 1)$ 
17:     $I.\text{push}(\neg uip)$ 
18:     $\rho(\neg uip) \leftarrow \text{PROPAGATED}(c')$ 
19:  else if  $|I| = |V|$  then
20:     $dl' \leftarrow \text{IMPLICANT-SHRINKING}(I)$  //
    See Section 3.2
21:    if  $dl' < dl$  then
22:       $I \leftarrow \text{BACKTRACK}(dl')$ 
23:    produce  $I$ 
24:    if  $dl = 0$  then
25:      return
26:    else
27:       $\ell_{flip} \leftarrow \neg(\sigma(dl))$ 
28:       $I \leftarrow \text{BACKTRACK}(dl - 1)$ 
29:       $I.\text{push}(\ell_{flip})$ 
30:       $\rho(\ell_{flip}) = \text{BACKTRUE}$  //
      See Section 3.3
31:    else
32:       $\ell \leftarrow \text{DECIDE}()$ 
33:       $\rho(\ell) \leftarrow \text{DECISION}$ 
34:       $I.\text{push}(\ell)$ 
35:       $dl \leftarrow dl + 1$ 
36: return

```

the currently highest decision variables and searching for a new total trail I satisfying F (lines 27-30).

The need for implicant shrinking comes from the fact that, in (Möhle and Biere 2019b) it is implicitly assumed that they can determine if a partial trail satisfies the formula right after being generated, whereas modern SAT solvers cannot check this fact efficiently, so they detect satisfaction only when trails are total. Therefore, the partial trail satisfying the formula is computed a posteriori from the total one by implicant shrinking.

Algorithm 1 could be adapted to produce total assignments too, by modifying the implicant shrinking function in line 20 and setting dl' to the current decision level dl . Lines 27-30 would be skipped, and the resulting algorithm would work similarly to the DPLL algorithm.

The calculus discussed in (Möhle and Biere 2019b) assumes the last UIP is the termination criteria for the con-

Algorithm 2: IMPLICANT-SHRINKING(I)

```

1:  $b \leftarrow 0$  // Implicant shrinking backtrack level
2:  $I' \leftarrow I$  // Clone current trail keeping  $\delta$  and  $\rho$ 
3: while  $I' \neq \varepsilon$  do
4:   $\ell \leftarrow I'.\text{top}()$ 
5:   $I'.\text{pop}()$ 
6:  if  $\rho(\ell) \neq \text{DECISION}$  then
7:     $b \leftarrow \max(b, \delta(\ell))$ 
8:  else if  $\delta(\ell) > b$  then
9:     $b \leftarrow \text{SIMPLIFY-LITERAL}(\ell, b, I')$ 
10: else if  $\delta(\ell) = 0$  or  $(\delta(\ell) = b \text{ and } \rho(\ell) = \text{DECISION})$  then
11:   break
12: return  $b$ 

```

flict analysis. The possibility to lift this requirement and use other criteria s.t. first UIP had to be proved. We provide the following counter-example to show that the first UIP does not guarantee mutual exclusivity between returned assignments.

Example 1 Let F be the propositional formula

$$F = \overbrace{(x_1 \vee \neg x_2)}^{c_1} \wedge \overbrace{(x_1 \vee \neg x_3)}^{c_2} \wedge \overbrace{(\neg x_1 \vee \neg x_2)}^{c_3}$$

For the sake of simplicity, we assume CHRONOLOGIC-CDCL to return total truth assignments. If the initial variable ordering is x_3, x_2, x_1 (all set to false) then the first two total and the third partial trails generated by Algorithm 1 are the following:

$$\begin{aligned} I_1 &= \neg x_3^d \neg x_2^d \neg x_1^d \\ I_2 &= \neg x_3^d \neg x_2^d x_1^* \\ I_3 &= \neg x_3^d x_2^* \end{aligned}$$

Notice how I_3 leads to a falsifying assignment: x_2 forces x_1 due to c_1 and $\neg x_1$ due to c_3 at the same time. A conflict arises and we adopt the first UIP algorithm to stop conflict analysis. We identify x_2 as the first unique implication point (UIP) and construct the conflict clause $\neg x_2$. Since this is a unit clause, we force its negation x_2 as an initial unit. We can now set x_3 and x_1 to \perp and obtain a satisfying assignment. The resulting total trail $I = \neg x_3 \neg x_2 \neg x_1$ is covered twice during the search process. \diamond

We also emphasize that the incorporation of restarts in the search algorithm (or any method that implicitly exploits restarts, such as rephasing) is not feasible, as reported in (Möhle and Biere 2019b).

3.2 Chronological implicant shrinking

Effectively shrinking a total trail I when chronological backtracking is enabled is not as trivial as it may seem. Suppose some literals at level dl are required to satisfy the formula. Given a literal ℓ assigned at a lower decision level $\delta(\ell) < dl$, we have no guarantee that this assignment is needed to satisfy F (e.g. we could have done a bad initial

choice by deciding a variable with no impact on the satisfiability of F). If we had a blocking AllSAT solver, we could simply learn a blocking clause ignoring ℓ and further shrink the current trail with no drawbacks. This behavior cannot be replicated easily using chronological backtracking. Removing ℓ would imply backtracking up to its level, which could drop literals (such as the ones at level dl) essential for the satisfiability of the formula.

Now let us assume we have a total trail I satisfying the formula F where the maximum decision level is dl , and that there exists a decision level $b < dl$ s.t. the shrunken trail I' to this decision level (consisting of exactly the literals in I assigned up to decision level b) still satisfies the formula. Then all literals assigned at level dl' , with $b < dl' \leq dl$ are assigned by DECIDE: UNIT-PROPAGATION cannot force the truth value of the unassigned variables, and Algorithm 1 must call DECIDE to set their truth value and obtain a total trail. Being assigned by DECIDE is then necessary for a literal to be dropped from the trail. The opposite is not necessarily true: not every variable assigned by DECIDE can be dropped from the trail ensuring the shrunk trail still satisfies the formula. For instance, if we have the formula $F = x_1 \vee x_2$ and obtain the trail $I = x_1^d x_2^d$, then removing all variables assigned by DECIDE would drop all literals from I , and the empty trail does not satisfy F .

Finally, the following can be observed if a literal ℓ is not assigned by DECIDE: in that case ℓ is unit propagated because of either: (i) UNIT-PROPAGATION; (ii) conflict analysis; (iii) chronological backtracking after a model is found. In all three cases, ℓ cannot be dropped from I if we want to perform disjoint AllSAT. If ℓ is assigned because of reason (i), then ℓ is the only literal in I that satisfies a clause in F . If ℓ is assigned because of reason (ii), then ℓ is the only literal in I that satisfies an active conflict clause. If ℓ is assigned because of reason (iii), then ℓ cannot be dropped to ensure no total assignments would be covered twice by two different partial assignments.

On top of these observations, we propose the main idea behind chronological implicant shrinking, which is described in Algorithm 2.

The idea is to pick literals from the current trail starting from the latest assigned literals (line 3) and determine the lowest decision level b to backtrack and shrink the implicant. First, we check if ℓ was not assigned by DECIDE (line 6). If this is the case, we set b to be at least as high as the decision level of ℓ ($\delta(\ell)$), ensuring that it will not be dropped by implicant shrinking (line 7), since ℓ has a role in performing disjoint AllSAT.

If this is not the case, we compare its decision level $\delta(\ell)$ to b (line 8). If $\delta(\ell) > b$, then we actively check if it is necessary for I to satisfy F (line 9) and set b accordingly. Two versions of literal simplification are presented in Section 3.2 and Section 3.2.

If ℓ is either an initial literal (i.e. assigned at decision level 0) or both $\rho(\ell) = \text{DECISION}$ and $\delta(\ell) = b$ hold, all literals in the trail assigned before ℓ would have a decision level lower or equal than b . This means that we can exit the loop early (lines 10-11), since scanning further the trail would be unnecessary. Finally, if none of the above conditions holds,

Algorithm 3: SIMPLIFY-LITERAL-2WATCHES-LITERALS(ℓ, b, I')

```

1: for  $c \in \omega(\ell)$  do
2:   if  $\exists \ell_c \in c$  s.t.  $\ell_c \neq \ell$  and  $\ell_c \in I'$  then
3:     Watch  $c$  by  $\ell_c$  instead of  $\ell$ 
4:   else
5:      $b \leftarrow \max(b, \delta(\ell))$ 
6: return  $b$ 

```

we can assume that b is already greater than $\delta(\ell)$, and we can move on the next literal in the trail.

Variable lifting using 2-watched literals In (Déharbe et al. 2013) the authors propose an algorithm to shorten total assignments and obtain a prime implicant by using watch lists. We adopted the ideas from this work and adapted them to be integrated into CB-based AllSAT solving, which we present in Algorithm 3.

For each literal ℓ we check its watch list $\omega(\ell)$ (line 1). For each clause c in $\omega(\ell)$ we are interested in finding a literal ℓ_c such that: (i) ℓ_c is literal in the current trail that satisfies c and it has not already been checked by IMPLICANT-SHRINKING; (ii) ℓ_c is not ℓ itself (line 2). If it exists, we update the watch lists, so that now ℓ_c watches c instead of ℓ , then we move on to the next clause (line 3). If no replacement for ℓ is available, then ℓ is the only remaining literal that guarantees c is satisfied, and we cannot reduce it. We update b accordingly, ensuring ℓ would not be minimized by setting b to a value higher or equal than $\delta(\ell)$ (line 5). We stress the fact that once IMPLICANT-SHRINKING terminates, all watch lists should be restored to their value before the procedure was called, otherwise some of the admissible models of F would not be found by the search algorithm.

Example 2 Let F be the following propositional formula:

$$F = \overbrace{(x_1 \vee x_2 \vee x_3)}^{c_1}$$

F is satisfied by 7 different total assignments:

$$\begin{aligned} & \left\{ \begin{array}{ccc} x_1, & x_2, & x_3 \end{array} \right\}, \\ & \left\{ \neg x_1, & x_2, & x_3 \right\}, \\ & \left\{ x_1, & \neg x_2, & x_3 \right\}, \\ & \left\{ \neg x_1, & \neg x_2, & x_3 \right\}, \\ & \left\{ x_1, & x_2, & \neg x_3 \right\}, \\ & \left\{ \neg x_1, & x_2, & \neg x_3 \right\}, \\ & \left\{ x_1, & \neg x_2, & \neg x_3 \right\} \end{aligned}$$

When initialized, our solver has the following watch lists:

$$\begin{aligned} \omega(x_1) &= \{c_1\} \\ \omega(x_2) &= \{c_1\} \\ \omega(x_3) &= \emptyset \end{aligned}$$

Algorithm 1 can produce the total trail $I_1 = x_3^d x_2^d x_1^d$. SIMPLIFY-LITERAL-2WATCHES-LITERALS starts by minimizing the value of x_1 . The watch list associated with x_1 contains c_1 , hence we need to substitute x_1 with a new literal in clause c_1 . A suitable substitute exists, namely x_3 . We

update the watch lists accordingly to Algorithm 3. The watch lists are now:

$$\begin{aligned}\omega(x_1) &= \emptyset \\ \omega(x_2) &= \{c_1\} \\ \omega(x_3) &= \{c_1\}\end{aligned}$$

Next, SIMPLIFY-LITERAL-2WATCHES-LITERALS eliminates x_2 from the current trail: x_1 was already cut off, x_2 and x_3 are the current indexes for c_1 and x_3 is assigned to \top . Since no other variables are available in c_1 , we must force x_3 to be part of the partial assignment, and we set b to 1 to prevent its minimization. This yields the partial trail $I_1 = x_3$.

Chronological backtracking now restores the watched literal indexing to its value before implicant shrinking (in this case the initial state of watch lists) and flips x_3 into $\neg x_3$. DECIDE will then assign \top to both x_2 and x_1 . The new trail $I_2 = \neg x_3^d x_2^d x_1^d$ satisfies F . Algorithm 3 drops x_1 since c_1 is watched by x_2 and thus we would still satisfy F without it. x_2 , on the other hand, is required in I_2 : x_3 is now assigned to \perp and thus cannot substitute x_2 . We obtain the second partial trail $I_2 = \neg x_3 x_2^d$. Last, we chronologically backtrack and set x_2 to \top . Being x_3 and x_2 both \perp , UNIT-PROPAGATION forces x_1 to be \top at level 0. We obtain the last trail satisfying F , $I_3 = \neg x_3 \neg x_2 x_1$.

The final solution is then:

$$\left\{ \begin{array}{l} \{ \quad \quad \quad x_3 \} , \\ \{ \quad \quad \quad \neg x_3 \} , \\ \{ x_1, \neg x_2, \neg x_3 \} \end{array} \right\},$$

Variable lifting using blocking literals The idea from (Déharbe et al. 2013) has been natively suggested from SAT solving, where restoring watch lists to their value after implicant shrinking was not necessary. However, in AllSAT, updating the 2-watched literal schema after every model is found to be computationally expensive and may cause a bottleneck for formulas with many solutions. In light of this, we propose a modified version of Algorithm 3 that leverages blocking literals (Chu, Harwood, and Stuckey 2009; Sörensson and Eén 2009) to determine the lowest decision level b to backtrack. We will refer to this version of implicant shrinking as SIMPLIFY-LITERAL-BLOCKING-LITERALS. The main idea behind (Chu, Harwood, and Stuckey 2009) is to update the data structure of watch lists so that we cache the other literal ℓ_ω paired to c in $\omega(\ell)$ (from now on we will refer to ℓ_ω as *blocking literal*). Moreover, they propose *inline binary clauses*, so that a binary clause in F is not stored in the clause database but they are inlined into the watch lists.

Suppose that the current trail I satisfies F , which implies that for each clause c in F , at least one of its two blocking literals is surely in I . We also assume that Algorithm 2 is performing implicant shrinking and encounters a decision literal ℓ with a decision level $dl > b$. Rather than scanning the entire clause watched by ℓ to find a replacement, we project the clauses onto their watched literals, thus we check the status of blocking literals to state if a clause is satisfied or not by the current trail. If the other blocking literal is not in the trail, then ℓ is the only literal of the pair satisfying c ,

and it cannot be dropped. Otherwise, if both ℓ and the other blocking literal are in I , we can drop one of them with no impact on the satisfiability of c . The backtracking level b is then set to a value that is at least as high to not discard both literals.

It is worth noting that this variant of implicant shrinking is conservative when it comes to dropping literals from the trail. We do not consider the possibility of another literal ℓ' in c that is not a blocking literal, is in the current trail I , and has a lower decision level than the two literals watching c . In such a case, we could set b to $\delta(\ell')$, resulting in a more compact partial assignment. Nonetheless, not scanning the clause can significantly improve performance, making our approach a viable alternative when covering many solutions for AllSAT. We plan to extend this variant to incorporate the more precise concepts introduced in (Nadel 2022) as future work.

3.3 Implicit solution reasons

Incorporating chronological backtracking into the AllSAT algorithm makes blocking clauses unnecessary: the search space is scanned systematically, with no assignment being repeated twice, and adding some relevant clauses to not replicate an assignment is no more needed. Instead, upon discovering a model, we backtrack chronologically to the most recently assigned decision variable ℓ and flip its truth value, as if there is a reason clause c - containing the negated decision literals of I - that forces the flip. These reason clauses c are typically irrelevant to SAT solving and are not stored in the system. On the other hand, when CDCL is combined with chronological backtracking, these clauses are required for conflict analysis, as demonstrated by the forthcoming example.

Example 3 Let F be the same formula from Example 1. We assume the first trail generated by Algorithm 1 is $I_1 = \neg x_3^d \neg x_2^d \neg x_1^d$. Algorithm 2 can reduce x_1 since $\neg x_2$ is enough to satisfy both c_1 and c_3 . Consequently, we obtain the assignment $\mu_1 = \neg x_3 \wedge \neg x_2$, then flip $\neg x_2$ to x_2 . The new trail $I_2 = \neg x_3^d x_2^d$ forces x_1 to be true due to the c_1 ; c_3 would then be not satisfiable anymore and generate a conflict. The last UIP is x_3 : that means the reason clause c' forcing x_2 to be flipped must be handled by the solver to compute the conflict clause. \diamond

A straightforward approach would be storing these clauses in memory with no update to the literal watching indexing; this approach allows for c to be called exclusively by the CDCL procedure without affecting variable propagation. We remark that if F admits a large number of models, storing these clauses will negatively affect performances: either we have to frequently call flushing procedures to remove no more active backtrack reason clauses, or we risk going out of memory to store them.

To overcome the issue, we introduce the notion of *virtual backtrack reason clauses*. When a literal ℓ is flipped after a satisfying assignment is found, its reason clause contains the negation of decision literals assigned at a level lower than $\delta(\ell)$ and ℓ itself. Consequently, we introduce an additional value, BACKTRUE, to the possible answers of the

reason function ρ . This value is used to tag literals flipped after a (possibly partial) assignment is found. When the conflict analysis algorithm encounters a literal ℓ having $\rho(\ell) = \text{BACKTRUE}$, the resolvent can be easily reconstructed by collecting all the decision literals with a lower level than ℓ and negating them. This way we do not need to physically store these clauses for conflict analysis, allowing us to save time and memory for clause flushing.

3.4 Decision variable ordering

One problem affecting chronological backtracking is variable ordering during DECIDE. As shown in (Möhle and Biere 2019b), different orders can lead to a different number of partial trails retrieved. There is little research on this topic when chronological backtracking is enabled. Our experiments have shown that the *Variable State Aware Decaying Sum* (VSADS) heuristic (Huang and Darwiche 2005) has overall good performances in this context. The idea is to set the priority of a variable according to two weighted factors: (i) the count of variable occurrences in the formula, as in the Dynamic Largest Combined Sum (DLCS) heuristics; and (ii) an "activity score," which increases when the variable appears in conflict clauses and decreases otherwise, as in the Variable State Independent Decaying Sum (VSIDS) heuristic. It has the merit of making sure variables that never appear in clauses have the lowest priority, drastically reducing the number of partial assignments we can get through our minimization techniques. Nevertheless, when the algorithm proposed in 3.2 is used for implicant shrinking, VSADS initial scores can still lead to bad variable ordering choices for trivial problems, as shown in the following example.

Example 4 Let $F = \overbrace{x_1 \vee x_2 \vee x_3 \vee x_4}^{c_1}$. The VSADS heuristic sets the same initial score for every variable since they appear once in the problem.

If we start by prioritizing x_1 and x_2 during the decision phase and obtain the trail $I_1 = x_1^d x_2^d x_3^d x_4^d$, then x_3 and x_4 can be easily removed from the trail given the absence of watch lists for them. We can then drop x_2 since SIMPLIFY-LITERAL-BLOCKING-LITERALS allows for that and obtain a partial trail with one single literal $I' = x_1$. Now suppose x_3 and x_4 are first chosen before x_1 and x_2 , obtaining the trail $I_2 = x_4^d x_3^d x_2^d x_1^d$. We could be able to drop x_1 , but all the other literals could not be removed since x_2 is the other blocking literal watching c_1 , forcing b to be set to 3. \diamond

To avoid this situation, we enforce the decision algorithm to privilege variables watching at least one clause when they share the same VSADS score. This condition helps in making good initial variable choices, leading to shorter assignments on average.

4 Experimental evaluation

To validate both the efficiency and effectiveness of our techniques, we implemented all the ideas discussed in Section 3 in TABULARALLSAT. The code of the algorithm and all benchmarks are publicly accessible here: TO-UPLOAD.

TABULARALLSAT is built on top TABULASAT, a new simple SAT solver focusing on minimality and ease of understanding. Besides chronological backtracking, it does not have any preprocessing, restarts and rephasing are disabled, and watching data structures are similar to MiniSAT.

Experiments are performed on an Intel Xeon Gold 6238R @ 2.20GHz 28 Core machine with 128 GB of RAM, running Ubuntu Linux 20.04. Timeout has been set to 1200 seconds.

4.1 Benchmarks

We opted for benchmarks having two characteristics: (i) each problem admits a high number of total assignments; (ii) the problem structure allows for some minimization of assignments, to test the efficiency of the chronological implicant shrinking algorithms. To assess the effectiveness of chronological implicant shrinking, we provide a set of crafted datasets.

Binary clauses contain problems defined by binary clauses in the form:

$$(x_1 \vee x_n) \wedge (x_2 \vee x_{n-1}) \wedge \dots \wedge (x_{n/2-1} \vee x_{n/2})$$

where n represents the number of variables. Although these instances are structurally straightforward, finding all solutions poses a significant challenge: if a problem has n variables, then retrieving all possible assignments requires returning $3^{n/2}$ assignments within a feasible timeframe.

Rnd3sat contains 410 random 3-SAT problems of varying sizes, with the number of variables n ranging from 10 to 50, and 10 problems for each size. We choose not to use the instances uploaded to SATLIB (Hoos and Stützle 2000), which were intended for testing the performance of SAT solving. In SAT instances, the ratio of clauses to variables needed to achieve maximum hardness is about 4.26, but in AllSAT, it should be set to approximately 1.5 (Bayardo Jr and Schrag 1997). To address this, we created a new set of random 3-SAT problems that conform to this ratio.

We also tested our algorithms over SATLIB benchmarks. We opted for two benchmarks, both proposed in (Singer, Gent, and Smaill 2000).

CBS contains random-3-SAT problems with the backbone size set to 10. The backbone of a formula is a set of variables whose truth value is always either true or false in each satisfying assignment. The backbone size is the number of backbone variables for a specific problem. We choose a set of problems with small backbone sizes and a low number of clauses since they appear to be the ones admitting more models.

BMS instances are structurally similar to CBS, with the only difference being the backbone minimization procedure applied on top of the random 3-SAT problems being more aggressive to obtain a backbone-minimal sub-instance, i.e. for each clause c in F there exists a literal ℓ such that ℓ is no more in the backbone of F if c is removed.

4.2 Comparing implicant shrinking techniques

We start by evaluating the performance between the two implicant shrinking algorithms from Section 3.2. We compare

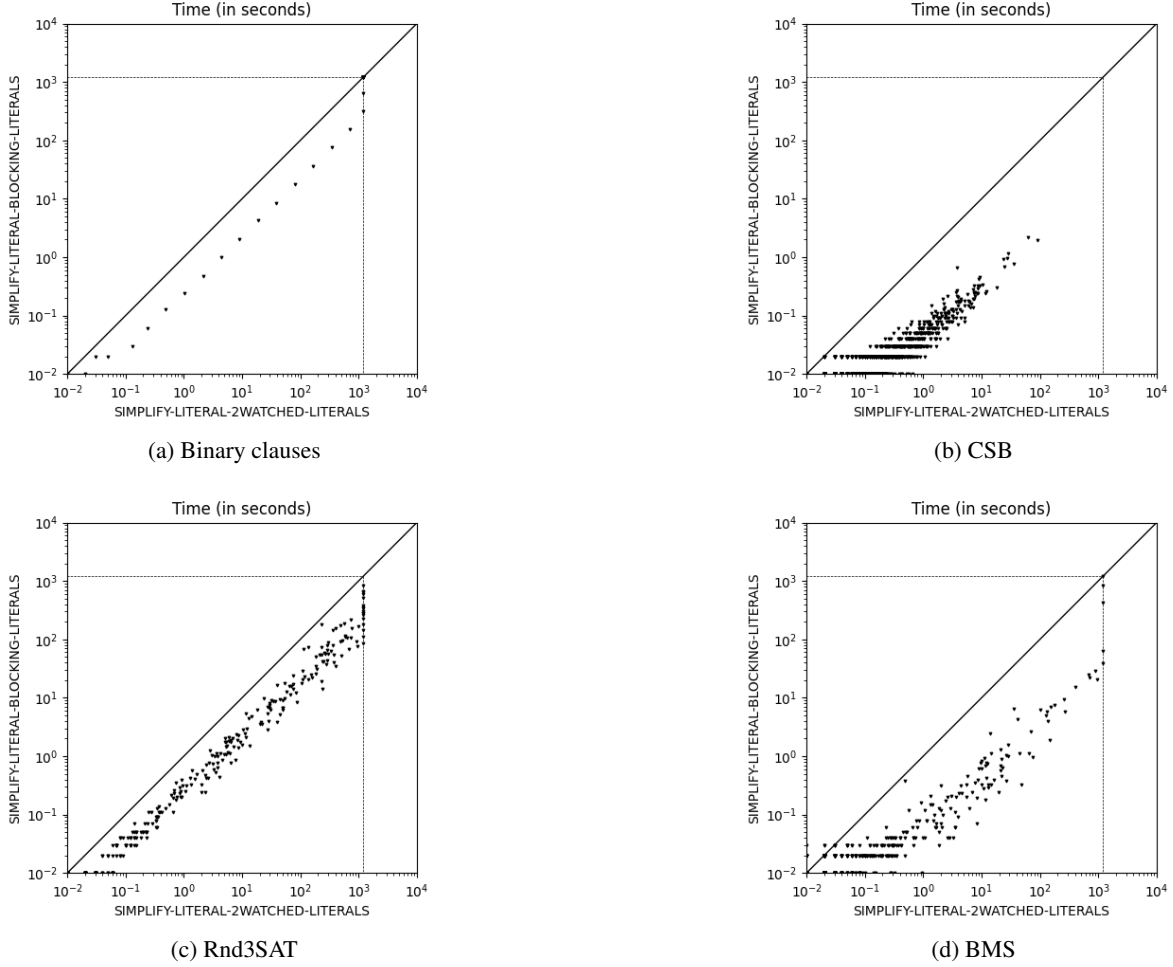


Figure 1: Scatter plot comparing CPU times of the two implicant shrinking algorithms. Both axes are log-scaled.

both CPU time and the number of disjoint partial assignments retrieved by the two algorithms and compare them using scatter plots. We checked the correctness of the enumeration by testing if the number of total trails covered by the set of partial solutions was the same as the model count reported by the #SAT solver Ganak (Sharma et al. 2019). Both implicant shrinking algorithms provided the same count as Ganak for all instances. Results are shown in Figures 1 and 2.

Results suggest that whereas SIMPLIFY-LITERAL-2WATCHED-LITERALS is slightly more effective than SIMPLIFY-LITERAL-BLOCKING-LITERALS in shrinking total assignments (this is more evident in the rnd3sat problems), in general, no one of the two versions clearly outperforms the other. When considering performances, however, SIMPLIFY-LITERAL-BLOCKING-LITERALS outperform the other variant. This outcome is expected: the computational cost of resetting each watch list $\omega(\ell)$ to its previous state upon finding a model significantly slows down the computation process the higher the number of total models satisfying F is.

All the experiments in the next subsection assume TABULARALLSAT relies on the SIMPLIFY-LITERAL-BLOCKING-LITERALS algorithm.

4.3 Baseline solvers

We compare TABULARALLSAT against publicly available state-of-the-art AllSAT solvers. We considered BC, NBC, and BDD (Toda and Soh 2016), respectively a blocking, a non-blocking, and a BDD-based disjoint AllSAT solver whose code is publicly available. BC also provides the option to obtain partial assignments. We will refer to this version of BC as BC_PARTIAL. Lastly, we considered MATHSAT5 (Cimatti et al. 2013), since it provides a publicly accessible interface to compute partial enumeration of propositional problems by exploiting blocking clauses.

4.4 Results

Table 1 reports the number of instances solved by each solver for each set of benchmarks before reaching timeout. The CPU times reported in Figure 3 excludes the time required for printing or storing each model and consider only

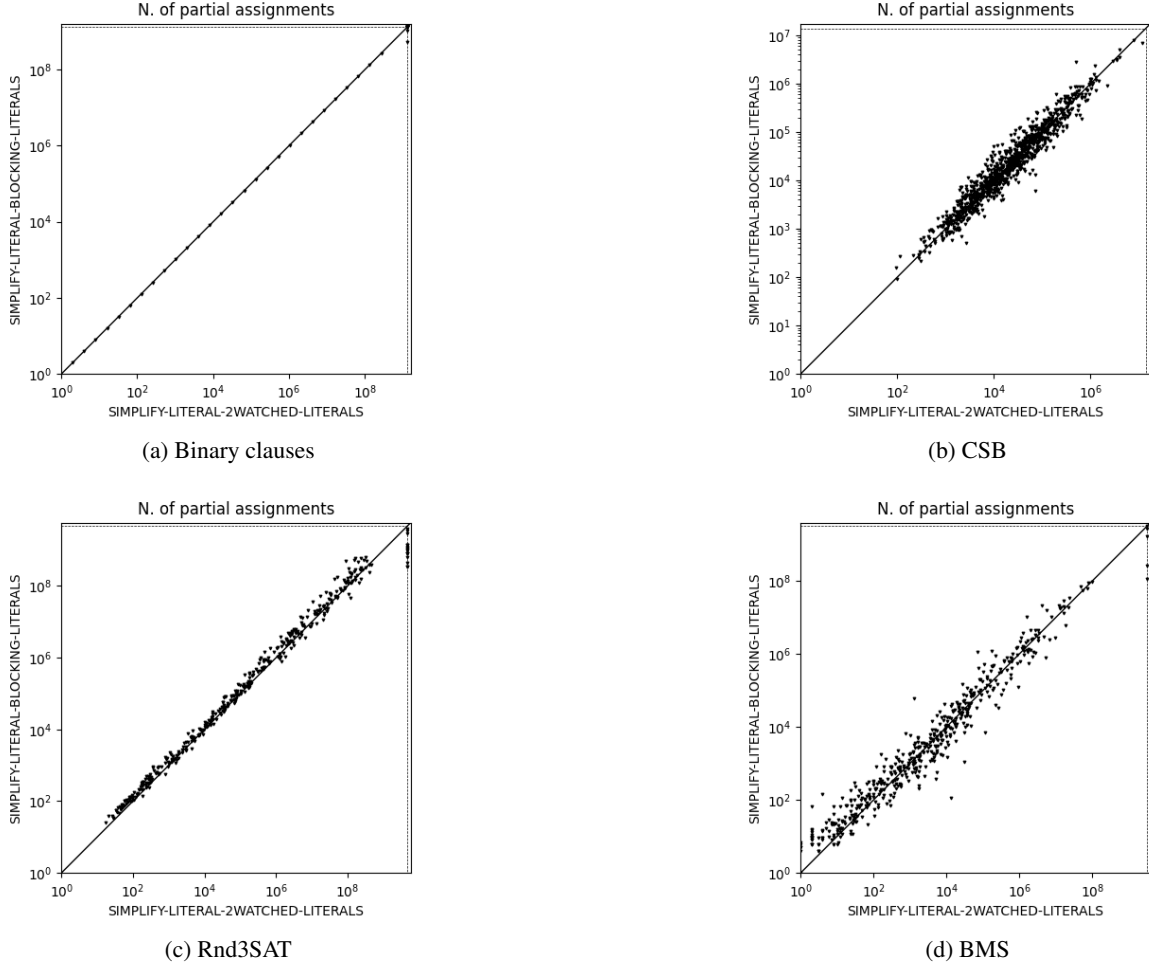


Figure 2: Scatter plot comparing number of partial models retrieved of the two implicant shrinking algorithms. Both axes are log-scaled. TODO: add blue points

the time taken to reach each assignment. We present our results using CDF plots. Based on the plots, we can draw the following conclusions:

1. In all benchmarks, TABULARALLSAT outperforms BC, BC_PARTIAL, NBC and MATHSAT5; it is also more efficient than BDD in three of the four sets of problems. The only benchmark set in which BDD outperforms TABULARALLSAT is the rnd3sat problems. Those instances are not structurally complex due to the low clause-to-variable ratio and can be compiled into BDDs with minimal inefficiencies. The higher the number of clauses in the problem instance, the more challenging the compilation of the propositional formula into a BDD is, as we can see in BMS and CSB. We remark the x-axis is log-scaled, thus the gaps between different solvers, in many cases, are by orders of magnitude. For instance, in Figure 3a, TabularAllSAT took 1 second to solve 20 instances, whereas NBC took more than 100 seconds.
2. If we consider only algorithms performing partial enumeration, we notice how the implicant shrinking proce-

cedure can negatively affect performances if not optimized (see BC_PARTIAL compared to its standard version BC in Figures 3b and 3d). To this extent, TABULARALLSAT proves its effectiveness in reducing total assignments without visible side effects in computational times, outperforming both BC_PARTIAL and MATHSAT5.

5 Conclusion

We presented an AllSAT procedure that combines CDCL, CB, and chronological implicant shrinking to perform partial disjoint enumeration, with the goal of getting the best from each technique. The experiments confirm the benefits of doing so, avoiding both performance degradations due to blocking clauses and computational bottlenecks generated by the solver being stuck in non-satisfiable branches for too long.

Nevertheless, there is still room for improvement. Future research directions include exploring novel decision heuristics that are suitable for chronological backtracking, as hinted in Section 3.4. Additionally, investigating the inte-

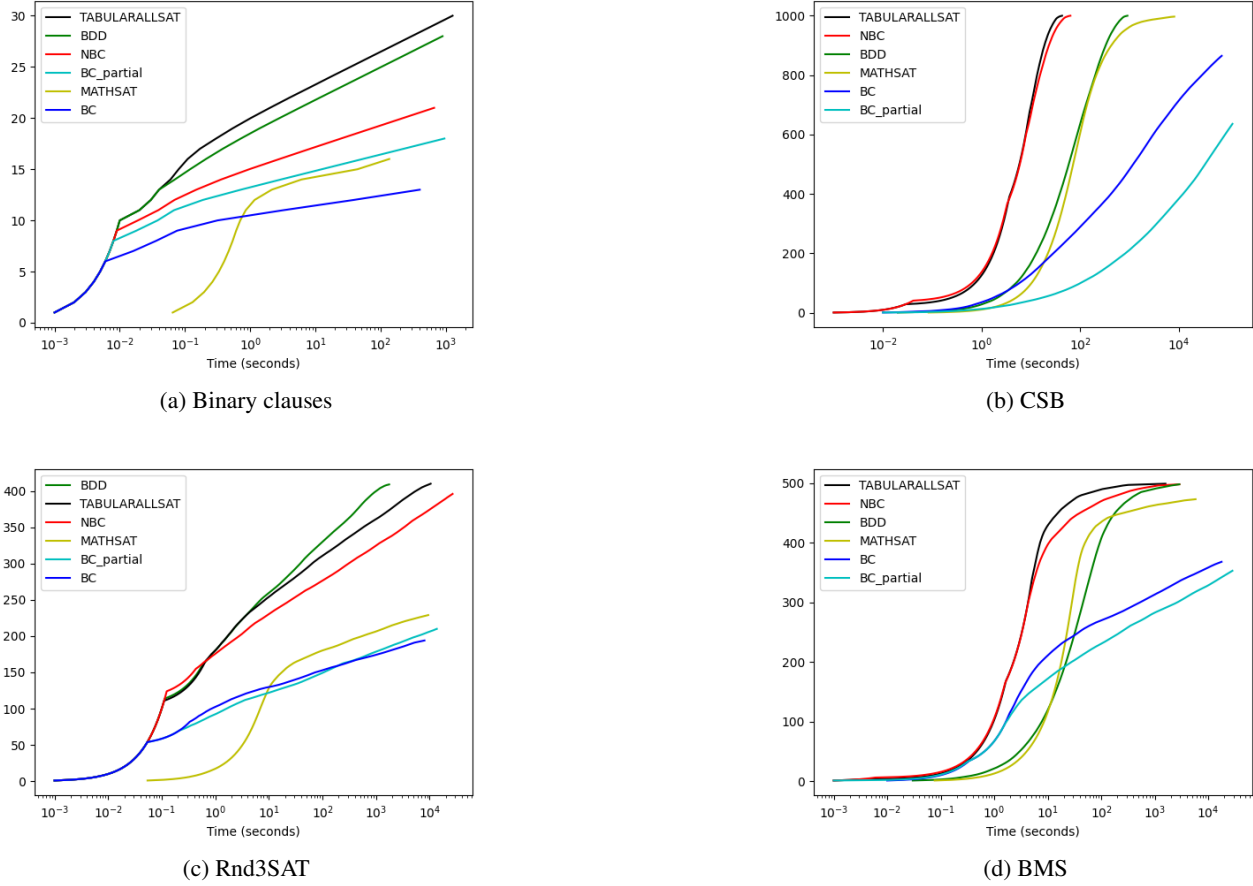


Figure 3: CDF plots comparing CPU times against the publicly available ALLSAT solvers. The x-axis is log-scaled.

gration of component caching with chronological backtracking could further advance the ideas presented in this work. Finally, extending TABULARALLSAT to handle projected enumeration could be an interesting direction.

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	TabularAllSAT	BDD	NBC	MathSAT	BC	BC_partial
binary clauses (50)	30	28	21	16	13	18
rnd3sat (410)	410	409	396	229	194	210
CSB (1000)	1000	1000	1000	997	865	636
BMS (500)	499	498	498	473	368	353
Total (1960)	1939	1935	1915	1715	1440	1217

Table 1: Table reporting the number of instances solved by each solver within the timeout time.

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