# Exercise set 5 – MPC (SC42125)

Mattia Bianchi Barbara Franci Sergio Grammatico

## 1 Unconstrained tracking problem

Let us consider the following discrete-time, unconstrained system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k), \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$. Assume that a set of $n_c$ controlled (hence measured) variables has an assigned setpoint $r \in \mathbb{R}^{n_c}$ to be tracked, i.e., $Hy = r$, for some $H \in \mathbb{R}^{n_c \times p}$.

1. Show that the following condition is sufficient for the feasibility of the target problem for any $r$

$$\operatorname{rank} \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} = n + n_c. \tag{1}$$

2. Show that (1) implies $n_c \leq m$ and $n_c \leq p$, i.e., the number of controlled variables without offset is less than or equal to the number of manipulated variables and the number of measurements.

3. Show that (1) implies the rows of $H$ are independent. Determine if also the rows of $C$ are linearly independent, otherwise provide a counterexample.

4. Determine how to choose $H$ if one has installed redundant sensors, i.e., several rows of $C$ are identical.

## 2 Offset-free MPC for tracking

Let us consider the following LTI system

$$
\begin{cases} x^+ = Ax + Bu \\ y = Cx + d + v, \end{cases} \tag{2}
$$

where $d = [0.1 \quad 1]^\top$ is a constant disturbance, , $v$ a Gaussian measurement noise with $\mathbb{E}[v(t)] = 0$ and $\mathrm{Cov}\,[v(t)] = 10^{-4} I_2$. The initial condition is $x(0) = [0.01 \; -100 \; 0.01]^\top$ and

$$
A = \begin{bmatrix} 0.2682 & -0.0034 & -0.0073 \\ 9.6985 & 0.3277 & -25.4364 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -0.0054 & 0.1655 \\ 1.2964 & 97.8998 \\ 0 & -6.6368 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

1. Consider the following optimal target selection problem

$$
\begin{cases} \underset{x_{\text{ref}},\, u_{\text{ref}}}{\operatorname{argmin}} & \|u_{\text{ref}}\|^2 \\ \text{s.t.} & \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_{\text{ref}} \\ u_{\text{ref}} \end{bmatrix} = \begin{bmatrix} 0 \\ y_{\text{ref}} - \hat{d} \end{bmatrix} \end{cases}
$$

   Explain why the optimization problem is feasible for any $\hat{d} \in \mathbb{R}^2$ and $y_{\text{ref}} \in \mathbb{R}^2$. Then, add the input constraint $\|u\| \leq 2$ and argue about the problem feasibility for any couple $(\hat{d}, y_{\text{ref}})$.

2. Assume that both the state $x$ and the controlled output $y$ in 2 are measured. Design an offset-free MPC controller to solve the tracking problem with constant reference trajectory

$$
y_{\text{ref}} = \begin{bmatrix} 0.8780 \\ 0.6590 \end{bmatrix}
$$

   and simulate the closed-loop system on MATLAB.

   *Hint: Remember that the setpoint for the MPC problem depends on the disturbance estimation.*

## 3 Output offset-free MPC: a simple example

Consider the following LTI system

$$\begin{cases} x^+ = Ax + Bu + B_\mathrm{d}d \\ y = Cx + C_\mathrm{d}d, \end{cases} \tag{3}$$

where $d = [0.1 \quad 1]^\top$ is a constant disturbance, initial state $x(0) = [0.01 \; -100 \; 0.01]^\top$ and

$$A = \begin{bmatrix} 0.2682 & -0.0034 & -0.0073 \\ 9.6985 & 0.3277 & -25.4364 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -0.0054 & 0.1655 \\ 1.2964 & 97.8998 \\ 0 & -6.6368 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C_d = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}, \; B_d = \begin{bmatrix} 1 & 0.5 \\ 0.4 & 0 \\ 0 & 1 \end{bmatrix}.$$

The control objective is to track a reference trajectory

$$y_\mathrm{ref} = \begin{bmatrix} 0.8780 \\ 0.6590 \end{bmatrix}.$$

1. Check if the theoretical conditions that allow for an offset-free implementation are verified.

2. Assume the state $x$ and the disturbance $d$ are measured and design an MPC controller to solve the tracking problem, without offset-free formulation. Simulate the closed-loop system on MATLAB. Determine if it is possible to tune the MPC controller (for example, using a high gain) to track the reference $y_\mathrm{ref}$ without offset.

3. Assume the state $x$ and the disturbance $d$ are measured and design an offset-free MPC controller to solve the tracking problem. Simulate the closed loop system on MATLAB.

4. Assume the output $y$ is measured, but the state $x$ and the disturbance $d$ are unknown and to be estimated. Design an offset-free output MPC controller to solve the tracking problem. Simulate the closed loop system on MATLAB.

Table 1: Parameters

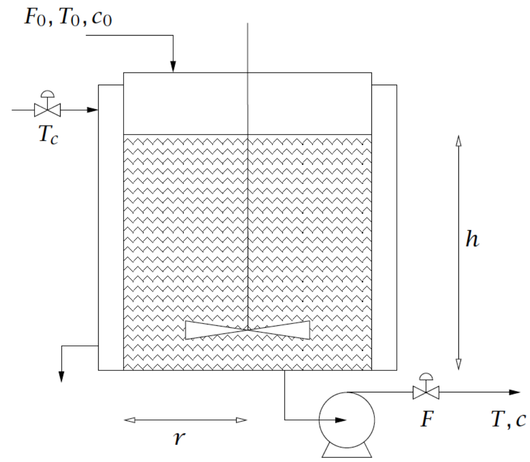


Figure 1: Schematic view of the well-stirred reactor (from "Disturbance Models for Offset-Free Model-Predictive Control“, AlChE journal, 2003, 49.2: 426-437, by G. Pannocchia and J. B. Rawlings).

| Parameter | Nominal value | |
|---|---|---|
| $F_0$ | 0.1 | [m$^3$/min] |
| $T_0$ | 350 | [K] |
| $c_0$ | 1 | [kmol/m$^3$] |
| $r$ | 0.219 | [m] |
| $k_0$ | $7.2 \cdot 10^{10}$ | [min$^{-1}$] |
| $E/R$ | 8750 | [K] |
| $U$ | 54.94 | [kJ/min·m$^2$·K] |
| $\rho$ | 1000 | [kg/m$^3$] |
| $C_p$ | 0.239 | [kJ/kg·K] |
| $\Delta H$ | $-5 \cdot 10^4$ | [kJ/kmol] |

# 4 Offset-free linear MPC of a stirred-tank reactor

The mass and energy balances that characterize a reaction $\mathcal{A} \to \mathcal{B}$ occurring in the well-stirred chemical reactor in Fig. 1 is described by the following nonlinear state space model:

$$\begin{cases} \dot{c}(t) = \dfrac{F_0(c_0 - c(t))}{\pi r^2 h(t)} - k_0\, c(t)\, e^{-\frac{E}{RT}} \\ \dot{T}(t) = \dfrac{F_0(T_0 - T(t))}{\pi r^2 h(t)} - \dfrac{\Delta H}{\rho C_p} k_0 c(t)\, e^{-\frac{E}{RT}} + \dfrac{2U}{r\rho C_p}(T_c - T(t)) \\ \dot{h}(t) = \dfrac{F_0 - F}{\pi r^2}, \end{cases} \tag{4}$$

where $h$ is the level of the tank, $c$ the molar concentration of species $\mathcal{A}$ and $T$ represents the temperature of the reactor. The input control are $T_c$ and $F$, the coolant liquid temperature and the outlet flow-rate, respectively. Moreover, the inlet flow-rate $F_0$ acts as an unmeasured disturbance. With the nominal values summarized in Table 1, the open-loop steady-state operating conditions are:

$$c_{\text{ref}} = 0.878\,[\text{kmol/m}^3],\ T_{\text{ref}} = 324.5\,[\text{K}],\ h_{\text{ref}} = 0.659\,[\text{m}]$$

$$T_{c_{\text{ref}}} = 300\,[\text{K}],\ F_{\text{ref}} = 0.1\,[\text{m}^3/\text{min}].$$

1. By introducing the following measurable state variables, control inputs and disturbance

$$x = \begin{bmatrix} c - c_{\text{ref}} \\ T - T_{\text{ref}} \\ h - h_{\text{ref}} \end{bmatrix},\ u \begin{bmatrix} T_c - T_{c_{\text{ref}}} \\ F - F_{\text{ref}} \end{bmatrix},\ d = F_0 - F_{0_{\text{ref}}},$$

linearize the nonlinear system in (4) around the steady-state operating conditions and discretize by using a sampling time of 1 [min] to get:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_{\text{d}} d \\ y(k) = Cx(k), \end{cases}$$

with the following numerical values

$$A = \begin{bmatrix} 0.2682 & -0.0034 & -0.0073 \\ 9.6985 & 0.3277 & -25.4364 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} -0.0054 & 0.1655 \\ 1.2964 & 97.8998 \\ 0 & -6.6368 \end{bmatrix}$$

$$B_{\text{d}} = \begin{bmatrix} -0.1174 \\ 69.7264 \\ 6.6368 \end{bmatrix},\ C = I_3.$$

2. Solve the following optimization problem to compute the infinite horizon optimal cost and control law for the unconstrained system

$$\mathbb{P}_{\infty}^{\text{uc}}(x) \,:\, \min_{\boldsymbol{u}} \sum_{k=0}^{\infty} \left( \|y(k)\|_Q^2 + \|u(k)\|_R^2 \right),$$

with

$$Q = \begin{bmatrix} 1.2977 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2.3027 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 \\ 0 & 100 \end{bmatrix}.$$

3. Model the disturbance $d$ with two integrating output disturbances on the two state variables $c$ and $h$. Simulate the response of the closed-loop system after a 10% increase in the inlet flow-rate $F_0$ at time $t = 10$ [min]. Use the nonlinear differential equations in (4) for the plant model. Determine if there are any steady offset on the outputs.

4. Choose a disturbance model with three integrating modes. Assess that, according with the theory, you can choose three integrating output disturbances for this plant.

5. Choose a disturbance model with three integrating modes and choose two integrating output disturbances on the two state variables. Choose one integrating input disturbance on the outlet flow-rate $F$. Argue on the detectability of the augmented system. Successively, repeat the simulation at item 3) and determine if there are any steady offset on the outputs. Compare and contrast the closed-loop performance for the design with two integrating disturbances and the design with three integrating disturbances.