Exercise set 2 – MPC (SC42125)

Mattia Bianchi   Barbara Franci   Sergio Grammatico

1 LQR and DARE

Given the following model of a spring,

$$\ddot{z}(t) + \dot{z}(t) + z(t) = u(t),$$

1. Write the system in a continuous-time state-space form. Discretize the model, with sampling time $T = 1$ s and assuming zero-order-hold for the input $u(t)$. Compare the discrete-time model obtained with the one generated via MATLAB by using the function `c2d`.

2. Consider the discrete-time LTI model computed in the previous step and the problem to find the input trajectory $u_d(k)$ that minimizes the cost function

$$V(x_d, u_d) = \sum_{k=0}^{\infty} x_d(k)^\top Q x_d(k) + 2u_d^2(k).$$

   with $Q = \mathrm{diag}(1, 2)$. Check if the system is controllable. If so, compute the solution to the optimization problem by using LQR (use the MATLAB function `dare`). Simulate the discrete-time system with the optimal control law.

2 MPC formulation

Consider the following discrete-time system, for $k \geq 0$:

$$x(k+1) = ax(k) + bu(k)\,,$$

with parameters a, $b > 0$ and assigned $x_0 = x(0) \in (-3, 3)$, subject to the following constraints:

- $|u(k)| \leq 1, \quad \forall k \geq 0;$
- $|u(k) - u(k-1)| \leq \frac{1}{5}, \quad \forall k \geq 1;$
- $|x(k)| \leq 3, \quad \forall k \geq 0.$

1. Formulate the MPC control problem: given the horizon $N = 3$, determine the matrices that define the constraints and define a quadratic objective function, involving both the state and the control input, without terminal cost (i.e., with stage cost only).

2. Define a 1-norm type objective function (with stage cost only).

For the discrete-time system

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k),$$

for a given $x_0 = x(0) \in (-3, 3) \times (-3, 3)$:

3. With the same constraints introduced above, formulate the MPC control problem: given the horizon $N = 3$, determine the constraint matrices and define a quadratic objective function, involving both the state and the control input, without terminal cost (i.e., with stage cost only).

3 Method of Lagrange multipliers

Consider the quadratic objective function $V(x) = \frac{1}{2}x^\top Hx + h^\top x$ and optimization problem

$$\begin{cases} \min\limits_{x \in \mathbb{R}^n} & V(x) \\ \text{s.t.} & Dx = d \end{cases} \tag{1}$$

in which $H \succ 0$, $d \in \mathbb{R}^m$, $m < n$, i.e., there are fewer constraints than decision variables. In the method of Lagrange multipliers, we augment the objective function with the constraints to form the Lagrangian function, L,

$$L(x, \lambda) = \tfrac{1}{2}x^\top Hx + h^\top x - \lambda^\top (Dx - d) \tag{2}$$

in which $\lambda \in \mathbb{R}^m$ is the vector of so-called Lagrange multipliers.

1. Show that the (KKT) necessary and sufficient condition for optimality are equivalent to the matrix equation

$$\begin{bmatrix} H & -D^\top \\ -D & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = - \begin{bmatrix} h \\ d \end{bmatrix} \tag{3}$$

2. We note one other important feature of the Lagrange multipliers, their relation with the optimal cost of the purely quadratic case. Show that, for $h = 0$, the cost function can be expressed in terms of $\lambda^\star$ by

$$V^\star = \tfrac{1}{2} d^\top \lambda^\star. \tag{4}$$

4 Steady-state Riccati equation

Let us consider the system

$$x(k+1) = Ax(k) + Bu(k), \tag{5}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and consider the cost function

$$V_N(x(0), \boldsymbol{u}_N) = \frac{1}{2} \sum_{k=0}^{N-1} x(k)^\top Q x(k) + u(k)^\top R u(k). \tag{6}$$

Choose weights $Q \succcurlyeq 0$, $R \succ 0$ of appropriate size.

1. Iterate the DARE by hand with MATLAB until the matrix Π stops changing. Hold this numerical result. Now call the MATLAB function `dare`. Determine whether or not the two numerical solutions coincide. Determine where are the eigenvalues of $A + BK$ placed in the complex plane.

2. Repeat point 1 by multiplying the matrix Q by some $\alpha > 1$. Determine where do the eigenvalues move, depending on α.

3. Repeat point 1 by multiplying the matrix R by some $\alpha > 1$. Determine do the eigenvalues move, depending on α.

5 Rate-of-change penalty

Given the linear system in (5) with initial state $x(0) = x_0$, let us consider the generalized LQR problem with the cross term between $x(k)$ and $u(k)$ in the finite-horizon cost function:

$$V_N(x(0), \boldsymbol{u}_N) = \frac{1}{2} \sum_{k=0}^{N-1} \Big\{ x(k)^\top Q x(k) + u(k)^\top R u(k) + 2x(k)^\top M u(k) \Big\} + \\ + \frac{1}{2} x(N)^\top P x(N).$$

1. Write the formula for the prediction matrices T and S such that

$$\boldsymbol{x}_{N+1} = T x_0 + S \boldsymbol{u}_N, \tag{7}$$

where $\boldsymbol{x}_{N+1} = (x(0), x(1), \ldots, x(N))$ and $\boldsymbol{u}_N = (u(0), u(1), \ldots, u(N-1))$ are column vectors, in $\mathbb{R}^{n(N+1)}$ and $\mathbb{R}^{mN}$, respectively.

Then, write the cost function V_N in the form

$$V_N(\boldsymbol{u}_N) = \frac{1}{2} \boldsymbol{u}_N^\top H \boldsymbol{u}_N + h^\top \boldsymbol{u}_N + \text{constant}\,,$$

where H is a square $mN \times mN$ matrix and h a column vector of dimension mN.

2. Now consider the following cost function:

$$V_N(x(0), \boldsymbol{u}_N) = \frac{1}{2} \sum_{k=0}^{N-1} \Big\{ x(k)^\top Q x(k) + u(k)^\top R u(k) + \Delta u(k)^\top L \Delta u(k) \Big\} + \\ + \frac{1}{2} x(N)^\top P x(N),$$

where with $\Delta u(k)$, we mean the rate of change of the control input, i.e., $\Delta u(k) = u(k) - u(k-1)$. Note that $\Delta u(0) = u(0) - u(-1)$, hence $u(-1)$ must be given. Show that, by augmenting the state with $u(k-1)$, i.e.,

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix},$$

this new optimal control problem reduces to a standard LQR with cross terms. Determine what are $\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{R}, \tilde{M}, \tilde{P}$ for the new problem.

6 An LQR problem: finite horizon and instability

Given the linear system in (5) with initial state $x(0) = x_0$, output $y(k) = Cx(k)$ and parameters

$$A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} -2/3 & 1 \end{bmatrix},\ x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

let us consider the finite horizon LQR problem with cost function

$$V(x(0), \boldsymbol{u}_N) = \frac{1}{2} \sum_{k=0}^{N-1} \left\{ x(k)^\top Q x(k) + u(k)^\top R u(k) \right\} + \frac{1}{2} x(N)^\top P x(N),$$

with

$$N = 5,\ R = 0.001,\ Q = P = \begin{bmatrix} 4/9 + 0.001 & -2/3 \\ -2/3 & 1 + 0.001 \end{bmatrix}.$$

By using MATLAB:

1. Compute the matrices T, S, H, h as in Exercise 5. Then solve the optimal control problem $\min_{\boldsymbol{u}_N} V_N(x_0, \boldsymbol{u}_N)$.

2. Iterate the backward Riccati equation to find the optimal gain $K(0)$. Compare the corresponding input with $u^\star(0)$ computed in previous point. Compute the eigenvalues of $A + BK(0)$ and determine whether or not the closed-loop system is stable.

3. Compute the optimal gain $K(0)$ for large values of N. Determine what happens to the eigenvalues of $A + BK(0)$ when N increases.

4. Check the controllability of the system. Compute the optimal gain for the the infinite-horizon LQR problem (without terminal constraint). Compare the result with the finite-horizon case, with $N = 25$.

5. Compute the finite-horizon optimal gains $K(0), K(1), \ldots, K(N-1)$, $N = 5$, with $P = \Pi$, the solution of the DARE. Explain the obtained result.

   Hint: Refer to the optimal cost-to-go.

7 Destabilization with state constraints

Let us consider a state-feedback regulation problem for the discrete-time system described in Exercise 6 with the same numerical values, except for $x_0 = [3,\ 3]^\top$. Let us consider the cost function given in (6), where the control horizon is $N = 5$ and the weight matrices are

$$Q = I,\ \ R = 1.$$

1. Implement a receding-horizon control policy and plot the resulting variables (input, state, output) starting from the initial condition x_0.
2. Add the output constraint $|y(k)| \leq \frac{1}{2}$. Plot the resulting variables of the constrained regulator (input, state, output). Is this regulator stabilizing? Determine if the tuning parameters Q and R can be modified to affect the closed-loop stability.
3. Change the output constraint into $|y(k)| \leq 1 + \varepsilon$ and plot the closed-loop response for several values of $\varepsilon > 0$. Determine if any of these regulators is (de)stabilizing.
4. Set the output constraint back to $|y(k)| \leq \frac{1}{2}$ and add the terminal equality constraint $x(N) = 0$. Discuss what is the solution to the regulator problem in this case and what happens if the control horizon N increases or decreases.

8 Computing the maximal output admissible set

We say that the initial state of an autonomous linear system

$$\begin{cases} x(k+1) = Ax(k) \\ y(k) = Cx(k) \end{cases}$$

is *output admissible* with respect to a constraint set $\mathcal{Y} := \{y \in \mathbb{R}^p \mid f_i(y) \leq 0,\ \forall i = 1, 2, \ldots, s\}$ if the resulting output $y(k) \in \mathcal{Y}$, $k \in \mathbb{N}$. The set of all such initial conditions is the *maximal output admissible set*, which is formally defined as

$$\mathcal{O}_\infty := \left\{ x \in \mathbb{R}^n \mid CA^k x \in \mathcal{Y},\ \text{for all } k \in \mathbb{N} \right\}.$$

A possible way to iteratively compute an estimate of $\mathcal{O}_\infty$, i.e.,

$$\mathcal{O}_k := \left\{ x \in \mathbb{R}^n \mid CA^t x \in \mathcal{Y},\ \text{for all } t = 0, 1, \ldots, k \right\},$$

is summarized in the following algorithm.

Algorithm 1:

Initialization:

Set $k := 0$

Iteration:

For all $i = 1, 2, \ldots, s$

$$x_i^\star := \begin{cases} \underset{x}{\operatorname{argmax}} & f_i(CA^{k+1}x) \\ \text{s.t.} & f_j(CA^t x) \leq 0 \qquad \forall j \in \{1, 2, \ldots, s\},\ \forall t \in \{0, 1, \ldots, k\} \end{cases}$$

End

If $f_i(CA^{k+1}x_i^\star) \leq 0\ \forall i \in \{1, 2, \ldots, s\} \longrightarrow$ **then** set $k^\star = k$ and define

$$\mathcal{O}_\infty := \left\{x \in \mathbb{R}^n \mid f_i(CA^t x) \leq 0,\ \forall i \in \{1, 2, \ldots, s\},\ \forall t \in \{0, 1, \ldots, k^\star\}\right\}.$$

Else set $k := k + 1$ and continue.

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1. Implement Algorithm 1 and verify the code for the system

$$A = \begin{bmatrix} 0.9 & 1 \\ 0 & 0.09 \end{bmatrix},\ C = [1\ 1],$$

subject to the constraints $\mathcal{Y} := \{y \in \mathbb{R} \mid |y| \leq 1\}$ and that the maximal output admissible set is given by

$$\mathcal{O}_\infty := \left\{x \in \mathbb{R}^2 \mid Hx \leq h\right\},\ H = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 0.9 & 1.09 \\ -0.9 & -1.09 \\ 0.81 & 0.9981 \\ -0.81 & -0.9981 \end{bmatrix},\ h = \mathbf{1}_6.$$

2. Show that $k^* = 2$.

3. Discuss what happens to k^* when A_{22} increases and approaches 1. Then, conclude on the case $A_{22} \geq 1$.