Exercise set 4 – MPC (SC42125)

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1 Infinite horizon cost and constraints

Consider the discrete-time system

$$x(k+1) = \begin{bmatrix} 0.3 & -0.9 \\ -0.4 & -2.1 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k),$$

subject to the constraints

$$|x_1 + x_2| \leq 1, \quad |x_1 - x_2| \leq 1.$$

1. By considering the proportional feedback law $u(k) = Kx(k)$, $K = [0.4 \quad 1.8]$, show *by hand* that the following set is a valid terminal constraint set:

$$\mathbb{X}_{\mathrm{f}} := \left\{ x \in \mathbb{R}^2 \mid |x_1 + x_2| \leq 1, \ |x_1 - x_2| \leq 1 \right\}.$$

   Hint: Consider what happens on the vertices of $\mathbb{X}_{\mathrm{f}}$.

2. Describe a procedure for determining the largest terminal constraint set for a given, stabilizing feedback gain K.

2 Infinite horizon cost to go as terminal penalty

Consider the discrete-time system

$$x(k+1) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k),$$

subject to the following constraints:

$$x \in \mathbb{X} := \left\{ x \in \mathbb{R}^2 \mid |x_1| \leq 5 \right\}, \ u \in \mathbb{U} := \left\{ u \in \mathbb{R}^2 \mid -\mathbf{1}_2 \leq u \leq \mathbf{1}_2 \right\}.$$

Algorithm 1:

Initialization: Set $k \coloneqq 0$

Iteration:

For all $i = 1, 2, \ldots, s$

$$x_i^\star \coloneqq \begin{cases} \underset{x}{\operatorname{argmax}} & f_i(A_K^{k+1}x) \\ \quad \text{s.t.} & f_j(A_K^t x) \leq 0, \quad \forall j \in \{1, 2, \ldots, s\},\ \forall t \in \{0, 1, \ldots, k\} \end{cases}$$

End

If $f_i(A_K^{k+1}x_i^\star) \leq 0 \ \ \forall i \in \{1, 2, \ldots, s\} \longrightarrow$ **then** set $k^\star = k$ and define

$$\mathbb{X}_f \coloneqq \left\{x \in \mathbb{R}^n \mid f_i(A_K^t x) \leq 0,\ \forall i \in \{1, 2, \ldots, s\},\ \forall t \in \{0, 1, \ldots, k^\star\}\right\}.$$

Else set $k \coloneqq k + 1$ and continue.

The cost function of the optimal control problem is

$$V_N(x_0, \boldsymbol{u}_N) = \frac{1}{2}\sum_{k=0}^{N-1}\left(\|x(k)\|_Q^2 + \|u(k)\|^2\right) + V_{\mathrm{f}}(x),$$

where $Q = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right]$, $V_{\mathrm{f}}(x) = \frac{1}{2}x^\top(N)Px(N)$ and P is the solution of the steady-state Riccati equation. Set $N = 3$ and $\alpha = 10^{-5}$.

1. Compute the infinite horizon optimal cost and control law for the unconstrained system.

2. By using the control law for the unconstrained system, let us consider the following set:

   $$\mathcal{X} \coloneqq \left\{x \in \mathbb{R}^2 \mid x \in \mathbb{X},\ Kx \in \mathbb{U}\right\} = \left\{x \in \mathbb{R}^2 \mid f_i(x) \leq 0,\ i = 1, 2, \ldots, s\right\}.$$

   Find the explicit expression for $f_i(\cdot)$, for all $i = 1, 2, \ldots, s$.

   Apply Algorithm 1 to find $\mathbb{X}_f$, the maximal constraint admissible set for the system $x(k+1) = (A + BK)x(k) = A_K x(k)$ with constraint $x \in \mathcal{X}$. Numerically check that

Algorithm 2: Solution algorithm for $m = 1$ (1 input)

Input: $G \in \mathbb{R}^{s\times n}$, $H \in \mathbb{R}^{s\times 1}$ and $\psi \in \mathbb{R}^{s\times 1}$

Single iteration:

1. Identify the following subset of $\mathcal{S} \coloneqq \{1, 2, \ldots, s\}$
   - $I^0 \coloneqq \{i \in \mathcal{S} \mid H(i) = 0\}$, with cardinality $s^0 \coloneqq |I^0|$
   - $I^+ \coloneqq \{i \in \mathcal{S} \mid H(i) > 0\}$, with cardinality $s^+ \coloneqq |I^+|$
   - $I^- \coloneqq \{i \in \mathcal{S} \mid H(i) < 0\}$, with cardinality $s^- \coloneqq |I^-|$
2. Let $C \coloneqq [G\ \psi]$ and define the matrix $D \in \mathbb{R}^{r\times(n+1)}$, $r = s^0 + s^+ s^-$, as follows

$$D(i,:) \coloneqq \begin{cases} C(i,:), & \forall i \in I^0 \\ H(i)C(j,:) - H(j)C(i,:), & \forall i \in I^+, \forall j \in I^- \end{cases}$$

Output: Extract $P \coloneqq D(:, 1 : \text{end} - 1)$, $\gamma \coloneqq -D(:, \text{end})$.

$$\mathbb{X}_\text{f} = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} -1.23 & -0.9 \\ 1.35 & 0.9 \\ -0.225 & -1.65 \\ 0.225 & 1.65 \end{bmatrix} x \leq \mathbf{1}_4 \right\}.$$

3. Add a terminal constraint $x(N) \in \mathbb{X}_\text{f}$ and implement the constrained MPC with $x(0) = x_0 = [0.5\ 0.5]^\top$. Find matrices and vector G, H and ψ that described the linear representation of the following set:

$$\mathbb{Z} \coloneqq \left\{ (x, \boldsymbol{u}_3) \mid x, \phi(1; k, \boldsymbol{u}_3), \phi(2; k, \boldsymbol{u}_3), \phi(3; k, \boldsymbol{u}_3) \in \mathbb{X},\ \boldsymbol{u}_3 \in \mathbb{U}^6 \right\}.$$

   Apply Algorithm 2-3 to compute $\mathcal{X}_N$, the region of attraction for the MPC problem with $V_\text{f}(\cdot)$ as the terminal cost and $x(N) \in \mathbb{X}_\text{f}$ as the terminal constraint.

4. Estimate *by hand* the set of initial states ($\mathcal{X}_N^0$) for which the MPC control sequence with control horizon N is equal to the MPC control sequence for an infinite horizon.

   Hint: Recall that $x \in \mathcal{X}_N^0$ *if and only if* $x^0(N; x) \in \text{int}(\mathbb{X}_\text{f})$.

Algorithm 3: Solution algorithm for $m > 1$

Input: $G \in \mathbb{R}^{s\times n}$, $H \in \mathbb{R}^{s\times m}$ and $\psi \in \mathbb{R}^{s\times 1}$

Initialization: Set $j \coloneqq m$, $G_j \coloneqq [G\ H(:,1:m-1)]$, $H_j \coloneqq H(:,m)$ and $\psi_j = \psi$.

Iteration:

1. Compute P_j and γ_j by means of Algorithm 2 with G_j, H_j and ψ_j
2. **if** $j = 0$

   Output: Set $P = P_0$, $\gamma = \gamma_j$ and `exit from Algorithm 2`

   else Set $j \coloneqq j - 1$, update

   $$G_j = P_{j+1}(:,1:\text{end}-1),\ H_j = P_{j+1}(:,\text{end}),\ \psi_j = \gamma_{j+1},$$

   and go to 1.

3 Terminal penalty with and without terminal constraint

Consider the discrete-time system

$$x(k+1) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k),$$

subject to the constraints

$$x \in \mathbb{X} \coloneqq \left\{x \in \mathbb{R}^2 \mid |x_1| \le 15\right\},\ u \in \mathbb{U} \coloneqq \left\{u \in \mathbb{R}^2 \mid -5\cdot\mathbf{1} \le u \le 5\cdot\mathbf{1}_2\right\},$$

and with finite-horizon cost function

$$V_N(x_0, \boldsymbol{u}_N) = \frac{1}{2}\sum_{k=0}^{N-1}\left(\|x(k)\|_Q^2 + \|u(k)\|^2\right) + V_\mathrm{f}(x),$$

where $Q = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right]$, $V_\mathrm{f}(x) = \frac{1}{2}x^\top(N)Px(N)$ and P is the solution of the steady-state Riccati equation. Use $N = 3$ and $\alpha = 10^{-5}$.

1. As in Exercise 2, compute the infinite horizon control law for the unconstrained system and find regions $\mathbb{X}_\mathrm{f}$, $\mathcal{X}_N$ and $\mathcal{X}_N^0$.

2. Remove the terminal constraint and estimate *by hand* the new domain of attraction $\tilde{\mathcal{X}}_N$. Compare $\tilde{\mathcal{X}}_N$ with $\mathcal{X}_N$ and $\mathcal{X}_N^0$ computed at the previous point.

3. Discuss what happens if the terminal cost function changes to $V_{\mathrm{f}}(x) = \frac{3}{2}x^\top Px$.

4 Unreachable setpoints in constrained versus unconstrained linear systems

Consider the discrete-time linear system

$$x(k+1) = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k),\ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{1}$$

and the cost function

$$\begin{aligned} V_N^{\mathrm{a}}(x_0, \boldsymbol{u}_N) &= \frac{1}{2}\sum_{k=0}^{N-1} \ell(x(k), u(k)) = \\ &= \frac{1}{2}\sum_{k=0}^{N-1} (x(k)-x^\star)^\top Q(x(k)-x^\star) + (u(k)-u^\star)^\top R(u(k)-u^\star), \end{aligned} \tag{2}$$

where $x^\star = [3,\ 3]^\top$, $u^\star = 1$ is a given set-point, with weight matrices

$$Q = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix},\ R = 2I.$$

1. Determine if $(x^\star,\ u^\star)$ is an equilibrium point for the system in (1) and compute the value of the cost function for the infinite horizon problem.

2. Compute the optimal steady state $(x_{\mathrm{s}}, u_{\mathrm{s}})$, i.e., the pair (x, u) that satisfies

$$(x_{\mathrm{s}}, u_{\mathrm{s}}) = \begin{cases} \underset{x\in\mathbb{R}^2, u\in\mathbb{R}}{\operatorname{argmin}} & \ell(x,u) \\ \text{s.t.} & (x,s) \text{ steady state for (1)} \end{cases}$$

   Then, consider the cost function

$$V_N^{\mathrm{b}}(x_0, \boldsymbol{u}_N) = \frac{1}{2}\sum_{k=0}^{N-1} (x(k)-x_{\mathrm{s}})^\top Q(x(k)-x_{\mathrm{s}}) + (u(k)-u_{\mathrm{s}})^\top R(u(k)-u_{\mathrm{s}}). \tag{3}$$

Set $x_0 = x(0)$, $N = 5$, and the terminal constraint $x(N) = x_{\mathrm{s}}$.

Solve two different MPC control problems for the cost functions (2) and (3), and compare the results.

Hint: You should get the same input sequence for the two control problems.

3. Consider the input constraint $|u(k)| < 2$. Compute the optimal steady state $(x_{\mathrm{c}}, u_{\mathrm{c}})$, i.e., the pair (x, u) that satisfies

$$(x_{\mathrm{c}}, u_{\mathrm{c}}) = \begin{cases} \underset{x \in \mathbb{R}^2,\, |u|<2}{\operatorname{argmin}} & \ell(x, u) \\ \text{s.t.} & (x, s) \text{ steady state for (1)} \end{cases}$$

Then, consider the cost function

$$V_N^{\mathrm{c}}(x_0, \boldsymbol{u}_N) = \frac{1}{2} \sum_{k=0}^{N-1} (x(k) - x_{\mathrm{c}})^\top Q (x(k) - x_{\mathrm{c}}) + (u(k) - u_{\mathrm{c}})^\top R (u(k) - u_{\mathrm{c}}). \tag{4}$$

Set $x_0 = x(0)$, $N = 5$, and the terminal constraint $x(N) = x_{\mathrm{c}}$.

Solve two different MPC control problems for the cost functions (2) and (4) and compare the results.

5 Terminal penalty and input constraints

Consider the discrete-time linear system

$$x(k+1) = \begin{bmatrix} 0.1 & 0 \\ 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(k), \; x(0) = \begin{bmatrix} 100 \\ 0.4 \end{bmatrix}$$

and the cost function

$$V_N(x_0, \boldsymbol{u}_N) = \frac{1}{2} \sum_{k=0}^{N-1} \Big\{ x(k)^\top Q x(k) + u(k)^\top R u(k) \Big\} + x(N)^\top P x(N)$$

with

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 0.1 \end{bmatrix}, \; R = 0.1\, I$$

Choose $N = 2$.

1. Determine if the system controllable. Choose P so that the unconstrained MPC is equivalent to the infinite horizon problem. Simulate the closed-loop system on MATLAB.

2. Consider the constraint $|u| < 0.6$. Simulate the closed-loop system with the same weight P as in point 1. Determine if the closed-loop system is stable.

3. Repeat point 2, but this time take $N = 10$. Determine if the closed-loop system is stable.