# Solutions to Exercise set 2 – MPC (SC42125)

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## 1 LQR and DARE

See the MATLAB code file.

## 2 MPC formulation

We preliminary define $\boldsymbol{u}_3 = [u(2)\ u(1)\ u(0)]^\top$.

1. The constraint $|u(k)| \leq 1$ can be rewritten as:

$$\forall k \in \{0,1,2\} : \begin{cases} u(k) \leq 1 \\ -u(k) \leq 1, \end{cases}$$

which leads to the following block matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \boldsymbol{u}_3 \leq \mathbf{1}_6,$$

where $\mathbf{1}_N$ is a vector of dimension $N$ of all elements equal to one. The same can be done with $|u(k) - u(k-1)| \leq \frac{1}{5}$, i.e.,

$$\forall k \in \{1,2\} : \begin{cases} u(k) - u(k-1) \leq \dfrac{1}{5} \\ -u(k) + u(k-1) \leq \dfrac{1}{5}, \end{cases}$$

and in vector form reads as:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \boldsymbol{u}_3 \leq \frac{1}{5} \cdot \mathbf{1}_4.$$

Finally, the constraint $|u(k)| \leq 1$ can be rewritten as:

$$\forall k \in \{1,2,3\} : \begin{cases} \ \ x(k) \leq 3 \\ -x(k) \leq 3, \end{cases}$$

where the state is propagated by means of the formula to obtain:

$$\begin{aligned} x(1) &= ax_0 + bu(0) \\ x(2) &= a^2x_0 + abu(0) + bu(1) \\ x(3) &= a^3x_0 + a^2bu(0) + abu(1) + bu(2). \end{aligned}$$

By substituting into the constraints, we obtain in vector form

$$\begin{bmatrix} 0 & 0 & b \\ 0 & b & ab \\ b & ab & a^2b \\ 0 & 0 & -b \\ 0 & -b & -ab \\ -b & -ab & -a^2b \end{bmatrix} \boldsymbol{u}_3 \leq 3 \cdot \mathbf{1}_6 + a \begin{bmatrix} -1 \\ -a \\ -a^2 \\ 1 \\ a \\ a^2 \end{bmatrix} x_0.$$

The quadratic cost function can be written as $V_3(x_0, \boldsymbol{u}_3) = \sum_{k=0}^{2} x^2(k) + u^2(k)$.

2. In this case, we have $V_3(x_0, \boldsymbol{u}_3) = \sum_{k=0}^{2} |x(k)| + |u(k)|$.

3. The first two constraints remain precisely the same. The only one that changes is the third. By adopting the same procedure above, we propagate the state as

$$\begin{aligned} x(1) &= Ax_0 + Bu(0) \\ x(2) &= A^2x_0 + ABu(0) + bu(1) \\ x(3) &= A^3x_0 + A^2Bu(0) + ABu(1) + Bu(2). \end{aligned}$$

By substituting into the constraints, we obtain in vector form

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -2 \\ -1 & -1 & -2 \\ -1 & -2 & -3 \end{bmatrix} \boldsymbol{u}_3 \leq 3 \cdot \mathbf{1}_{12} + \begin{bmatrix} 0 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -2 \\ -1 & -2 \\ -2 & -3 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} x_0.$$

The quadratic cost function can be written as $V_3(x_0, \boldsymbol{u}_3) = \sum_{k=0}^{2} \|x(k)\|^2 + u^2(k)$.

# 3 Method of Lagrange Multipliers

- The necessary and sufficient conditions for a global minimizer are that the partial derivatives of $L$ with respect to $x$ and $\lambda$ vanish. Hence:

$$\frac{\partial L(x,\lambda)}{\partial x} = Hx + h - D^\top \lambda = 0 \tag{1}$$
$$\frac{\partial L(x,\lambda)}{\partial \lambda} = -(Dx - d) \tag{2}$$

- By 1 and 2

$$Hx^\star = D^\top \lambda^\star \qquad -Dx^\star = -d, \tag{3}$$

hence

$$V^\star = \frac{1}{2} x^{\star T} H x^\star = \frac{1}{2} x^{\star\top}(D^\top \lambda^\star) = \frac{1}{2}(Dx^\star)^\top \lambda^\star = \frac{1}{2} d^\top \lambda^\star. \tag{4}$$

# 4 Steady-state Riccati equation

See MATLAB code.

## 5 Rate-of-change penalty

1. $\boldsymbol{x}_{N+1} = Tx_0 + S\boldsymbol{u}_N$,

$$T = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},\ S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & A^{N-3} & \cdots & B \end{bmatrix}.$$

$V_N(\boldsymbol{u_N}) = \frac{1}{2}\boldsymbol{u}_N^\top H\boldsymbol{u}_N + h^\top \boldsymbol{u}_N + c$,

$$\begin{aligned} H &= \left(\bar{R} + S^\top \bar{Q} S + 2S^\top \bar{M}\right), \\ h &= \left(S^\top \bar{Q} T x_0 + \bar{M}^\top T x_0\right), \\ c &= x_0^\top T^\top \bar{Q} T x_0, \end{aligned}$$

where

$$\bar{Q} = \begin{bmatrix} I_N \otimes Q & 0 \\ 0 & P \end{bmatrix},\ \bar{R} = I_N \otimes R,\ \text{and}\ \bar{M} = \begin{bmatrix} I_N \otimes M \\ 0 \end{bmatrix}$$

2. $\Delta u(k) = u(k) - u(k-1)$,

$$V_N(x(0), \boldsymbol{u}_N) = \frac{1}{2}\sum_{k+0}^{N-1}\left\{x(k)^\top Q x(k) + u(k)^\top R u(k) + \Delta u(k)^\top L \Delta u(k)\right\} + \frac{1}{2}x(N)^\top P x(N).$$

The augmented state $\tilde{x}(k) = [x(k) \quad u(k-1)]^\top$ follows the dynamics $\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u$, starting from $\tilde{x}(0) = [x_0 \ \ u(k-1)]^\top$, where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B \\ I \end{bmatrix}.$$

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & L \end{bmatrix},\ \tilde{R} = R + L,\ \tilde{M} = -\begin{bmatrix} 0 \\ L \end{bmatrix},\ \text{and}\ \tilde{P} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}.$$

The cost function becomes:

$$V_N(\tilde{x}(0), \boldsymbol{u}_N) = \frac{1}{2}\sum_{k=0}^{N-1}\left(\tilde{x}(k)^\top \tilde{Q}\tilde{x}(k) + u(k)^\top \tilde{R} u(k) + 2\tilde{x}(k)^\top \tilde{M} u(k)\right) + \frac{1}{2}\tilde{x}(N)^\top \tilde{P}\tilde{x}(N).$$

## 6 An LQR problem: finite horizon and instability

See MATLAB code.

## 7 Destabilization with state constraints

See MATLAB code.

## 8 Computing the maximal output admissible set

See MATLAB code.