Exercise set 3 – MPC (SC42125)

Mattia Bianchi Barbara Franci Sergio Grammatico

1 An MPC stability result

Consider the nonlinear dynamical system

$$x^+ = f(x, u),$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, with the origin as equilibrium point, i.e., $f(0, 0) = 0$. Then consider the following optimal control problem with terminal constraint:

$$\begin{cases} \min\limits_{\boldsymbol{u}_N} V_N(x_0, \boldsymbol{u}_N) \\ \text{s.t. } x(N) = 0, \end{cases}$$

with cost function $V_N(x_0, \boldsymbol{u}_N) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$, $x(0) = x_0$, where ℓ is a continuous, positive definite function such that $\ell(x, u) \geq \alpha(\|x\|)$, for some class $\mathcal{K}$ function α.

If $\boldsymbol{u}_N^0 = \big(u_N^0(0; x), u_N^0(1; x), \ldots, u_N^0(N-1; x)\big)$ is the optimal control sequence, the closed-loop system is

$$x^+ = f(x, u_N^0(0, x)).$$

1. Prove that the origin is (locally) asymptotically stable for the closed-loop system. In particular, state the assumptions required so that the control problem is *feasible* for some set of initial conditions.

2. Derive some additional assumptions on the function $\ell(x, u)$ so that the origin is *exponentially* stable for the closed-loop system. Discuss how does the controllability assumption change for this case.

2 A simple example

Consider the discrete-time linear system

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),$$

and the cost function

$$V_N(x_0, \boldsymbol{u}_N) = \frac{1}{2} \sum_{k=0}^{N-1} x(k)^\top Q x(k) + u(k)^\top R u(k),$$

with weight matrices

$$Q = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}, \ R = 0,$$

which essentially means that the use of control input is not penalized, hence can be chosen freely to maximize the performance.

1. Check the controllability of the system.

   By using MATLAB, simulate the unconstrained MPC regulator for some initial condition $[x_1(0) \ x_2(0)]^\top$ with $x_1(0) \neq 0$ and $N = 2$.

   Determine if the closed-loop system is stable and derive an explicit expression for the closed-loop.

2. For $N = 2$, impose the terminal constraint $x(N) = 0$, for the set of initial conditions in the same form as in point 1.

   Determine if the conditions derived for Exercise 1 are satisfied and if the closed-loop is stable.

3. By considering $N = 3$, repeat point 1.

   Determine if the system is stable now. Discuss advantages and disadvantages of choosing a longer horizon and of imposing a terminal constraint to ensure stability.

3 Computing the projection of $\mathbb{Z}$ onto $\mathcal{X}_N$

A polytope $\mathbb{Z}$ is usually defined as

$$\mathbb{Z} \coloneqq \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Gx + Hu + \psi \leq 0\},$$

where $G \in \mathbb{R}^{s \times n}$, $H \in \mathbb{R}^{s \times m}$ and $\psi \in \mathbb{R}^s$. We are interested in the projection of $\mathbb{Z}$ onto $\mathbb{R}^n$, i.e.,

$$\mathcal{X}_N \coloneqq \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in \mathbb{Z}\}.$$

Consider the discrete-time linear system

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

Algorithm 1: Solution algorithm for $m = 1$ (1 input)

Input: $G \in \mathbb{R}^{s\times n}$, $H \in \mathbb{R}^{s\times 1}$ and $\psi \in \mathbb{R}^{s\times 1}$

Single iteration:

1. Identify the following subset of $\mathcal{S} \coloneqq \{1, 2, \ldots, s\}$
   - $I^0 \coloneqq \{i \in \mathcal{S} \mid H(i) = 0\}$, with cardinality $s^0 \coloneqq |I^0|$
   - $I^+ \coloneqq \{i \in \mathcal{S} \mid H(i) > 0\}$, with cardinality $s^+ \coloneqq |I^+|$
   - $I^- \coloneqq \{i \in \mathcal{S} \mid H(i) < 0\}$, with cardinality $s^- \coloneqq |I^-|$
2. Let $C \coloneqq [G\ \psi]$ and define the matrix $D \in \mathbb{R}^{r\times(n+1)}$, $r = s^0 + s^+s^-$, as follows

$$D(i,:) \coloneqq \begin{cases} C(i,:), & \forall i \in I^0 \\ H(i)C(j,:) - H(j)C(i,:), & \forall i \in I^+, \forall j \in I^- \end{cases}$$

Output: Extract $P \coloneqq D(:, 1 : \text{end} - 1)$, $\gamma \coloneqq -D(:, \text{end})$.

subject to state constraints $\mathbb{X} \coloneqq \left\{x \in \mathbb{R}^2 \mid x_1 \leq 2\right\}$ and input constraints $\mathbb{U} \coloneqq \{u \in \mathbb{R} \mid |u| \leq 1\}$. Let us consider the MPC problem with $N = 2$, such that $\boldsymbol{u}_2 = (u(0), u(1))^\top$.

1. Find matrices and vector G, H and ψ such that

$$\mathbb{Z} \coloneqq \left\{(x, \boldsymbol{u}_2) \mid x, \phi(1; x, \boldsymbol{u}_2), \phi(2; x, \boldsymbol{u}_2) \in \mathbb{X},\ \boldsymbol{u}_2 \in \mathbb{U}^2\right\}.$$

2. Implement Algorithms 1 and 2 (which follows the typical MATLAB notation to select elements/rows/columns and to append matrices/vectors) to compute $\mathcal{X}_N$ and, for the numerical example above, verify that the set $\mathcal{X}_2 \coloneqq \left\{x \in \mathbb{R}^2 \mid \exists \boldsymbol{u}_2 \in \mathbb{R}^2 \text{ s.t. } (x, \boldsymbol{u}_2) \in \mathbb{Z}\right\} = \left\{x \in \mathbb{R}^2 \mid Px + \gamma \leq 0\right\}$ is given by

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},\ \gamma = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}.$$

Algorithm 2: Solution algorithm for $m > 1$

Input: $G \in \mathbb{R}^{s\times n}$, $H \in \mathbb{R}^{s\times m}$ and $\psi \in \mathbb{R}^{s\times 1}$

Initialization: Set $j \coloneqq m$, $G_j \coloneqq [G\ H(:, 1 : m-1)]$, $H_j \coloneqq H(:, m)$ and $\psi_j = \psi$.

Iteration:

1. Compute P_j and γ_j by means of Algorithm 1 with G_j, H_j and ψ_j
2. **if** $j = 0$

   Output: Set $P = P_0$, $\gamma = \gamma_j$ and `exit from Algorithm 2`

   else Set $j \coloneqq j - 1$, update

$$G_j = P_{j+1}(:, 1 : \text{end} - 1),\ H_j = P_{j+1}(:, \text{end}),\ \psi_j = \gamma_{j+1},$$

   and go to 1.

4 Terminal constraint and region of attraction

Consider the discrete-time linear system

$$x(k+1) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k),$$

subject to state constraints $\mathbb{X} \coloneqq \left\{x \in \mathbb{R}^2 \mid x_1 \leq 5\right\}$ and input constraints $\mathbb{U} \coloneqq \left\{u \in \mathbb{R}^2 \mid -\mathbf{1}_2 \leq u \leq \mathbf{1}_2\right\}$. Consider the following cost function with terminal penalty on the final state, $V_\mathrm{f}(x(N))$,

$$V(x(0), \boldsymbol{u}_N) = \frac{1}{2}\sum_{k=0}^{N-1} \left\{x(k)^\top Q x(k) + u(k)^\top R u(k)\right\} + V_\mathrm{f}(x(N)),$$

where $x(0) = x_0 = [0.5,\ 0.5]^\top$, $Q = \alpha I_2$ and $R = I_2$.

1. Set $N = 3$ and implement the *unconstrained* MPC with no terminal cost ($V_\mathrm{f}(x(N)) = 0$) for some values of α.

   Select a value of α for which the resultant closed-loop system is unstable.

2. Now implement the *constrained* MPC with no terminal cost for the value of α obtained in the previous point.

   Determine if the resulting closed-loop system is stable or unstable.

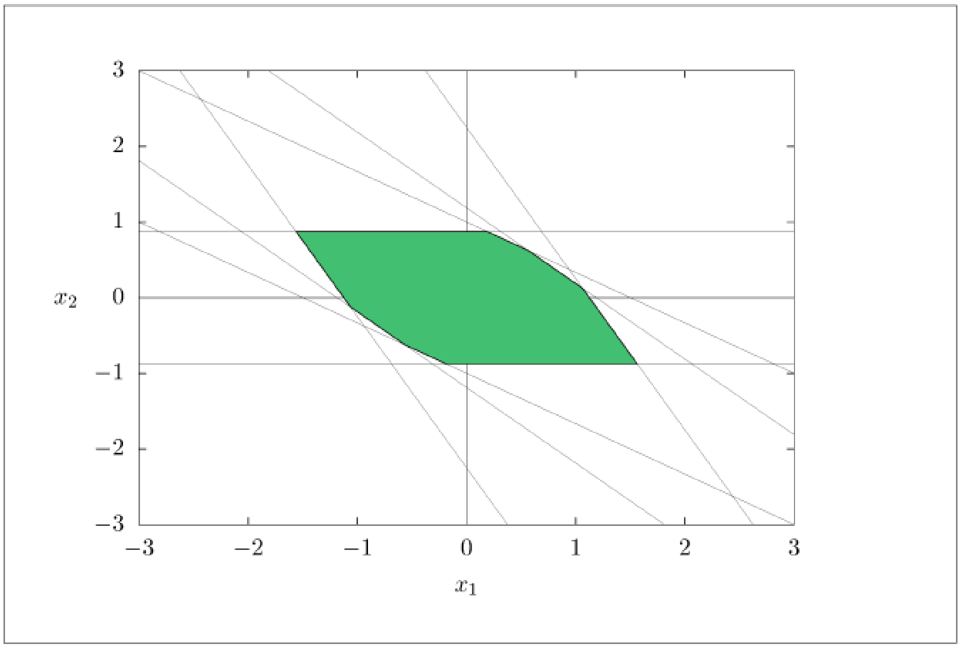

Figure 1: Region of attraction for the constrained MPC controller.

3. Finally, implement the *constrained* MPC with terminal equality constraint $x(N) = 0$, for the same value of α as in the previous points. Find the region of attraction for the constrained MPC controller using the projection algorithm from Exercise 3. The result should resemble that in Figure 1.

5 Trajectory tracking for unicycle-like vehicles

Figure 2 depicts a planar mobile robot, commonly called *unicycle*, whose dynamics are typically described by the following nonlinear differential equations:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\,\cos(\theta) \\ v\,\sin(\theta) \\ \omega \end{bmatrix}, \tag{1}$$

where $q = [x,\, y,\, \theta]^\top$ is the state of the vehicle, composed by the x-coordinate, the y-coordinate and the orientation θ, respectively, while $u = [v,\, \omega]^\top$ is the control input, i.e., linear and angular velocity, respectively.

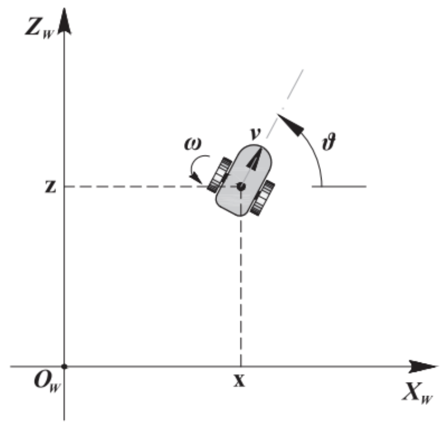

Figure 2: Planar unicycle.

The trajectory tracking problem requires to control the state of the robot $q(t)$ to a reference, given trajectory $q_\mathrm{r}(t) = [x_\mathrm{r}(t),\, y_\mathrm{r}(t),\, \theta_\mathrm{r}(t)]^\top$.

1. Given some trajectory $q_\mathrm{r}(t) = [x_\mathrm{r}(t),\, y_\mathrm{r}(t),\, \theta_\mathrm{r}(t)]^\top$, compute $v_\mathrm{r}(t)$ and $\omega_\mathrm{r}(t)$, i.e., the control inputs required to keep a perfect trajectory tracking.

2. Compute the dynamics, $\dot{e}(t)$, of the tracking error $e(t) := [e_1(t),\, e_2(t),\, e_3(t)]^\top$, which can be written in the robot coordinate system as:
$$e = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} (q_\mathrm{r} - q),$$

3. Let the control input be $u = u_\mathrm{f} + u_\mathrm{b} = \left[\begin{smallmatrix} v_\mathrm{r} \\ \omega_\mathrm{f} \end{smallmatrix}\right] + u_\mathrm{b}$, where u_f is the *feedforward* component, and u_b is the *feedback* component.

   Linearize around the reference trajectory and discuss what kind of linear system is obtained.

   For a given sampling time T, compute a discretized version of such a linear system using Euler's approximation.

4. By using MATLAB, compute the reference input and generate a discrete-time (LTV) model, with $T = 0.001$, for the error e around

the following trajectory:

$$\begin{bmatrix} x_{\mathrm{r}}(t) \\ y_{\mathrm{r}}(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(3t) \end{bmatrix},$$

where $\theta_{\mathrm{r}}(t)$ is computed to obtain a compatible trajectory.

Check that the (discretized) reference input keeps the system on the reference trajectory. Then, implement an MPC controller for the LTV model with the following cost function:

$$V_N(e, \boldsymbol{u}_{\mathrm{b}}) = \frac{1}{2} \sum_{k=0}^{N-1} e(k)^\top Q e(k) + u_{\mathrm{b}}(k)^\top R u_{\mathrm{b}}(k),$$

with

$$Q = 100\,\mathrm{diag}([4\ 40\ 0.1]),\ R = \tfrac{1}{100} I_2,\ N = 10.$$

Use this result to implement a controller for the continuous-time dynamics in (1) (using also the sampled reference input) to solve the tracking problem. Then repeat adding the constraint $|\omega(k)| \leq 25$.

6 A vehicle on an elliptic road

Consider a vehicle on a road which forms an ellipse, described by the set $\left\{x \in \mathbb{R}^2 \mid x_1^2 + 4x_2^2 = 1\right\}$, as in Figure 3. The nonlinear dynamics of the vehicle are

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \sin(\vartheta(x) + u(k)) \\ \cos(\vartheta(x) + u(k))/2 \end{bmatrix},$$

with

$$\vartheta(x) = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$$

i.e., the vehicle moves on the ellipse for distance $u \in U = \mathbb{R}$ in one time step (and ϑ represents the angular distance traveled). For $u > 0$ the vehicle moves clockwise and for $u < 0$ it moves counterclockwise. Set the initial state $[0, 1/2]^\top$ and stage cost $\ell(x,u) = \|x - x_*\|^2 + u^2$ with $x_* = (0, -1/2)$ The vehicle is only allowed to move clockwise, hence we impose the constraint $u(k) > 0, \forall k$. Perform the following numerical simulations for this problem (implement your own NMPC solver or use the one in: `numerik.mathematik.uni-bayreuth.de/~lgruene/nmpc-book/matlab_nmpc/nmpc.m`).

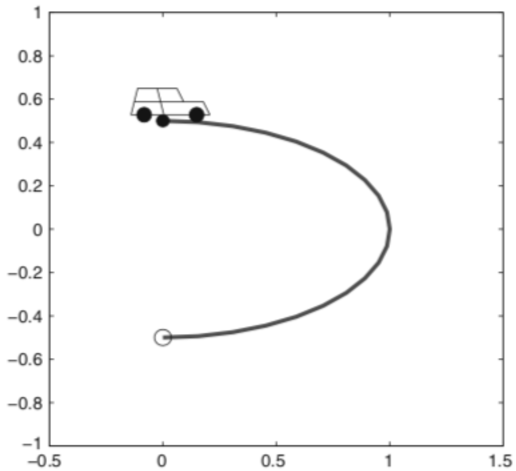

Figure 3: Elliptic road.

1. Implement the NMPC closed loop for $N = 8$ and confirm that the closed- loop trajectory does not converge toward x^*. Can you explain why? What is the stage cost along the ellipse? Repeat the experiment for $N = 11$.

2. Modify the NMPC problem by introducing the terminal constraint $x_0 = \{x^*\}$. Again considering the horizon length $N = 8$, verify that now the closed-loop trajectory converges to x^*.

3. Set the control horizon to $N = 11$ and consider the input constraint $u(k) \leq 0.1, \forall k$. Is the closed-loop trajectory converging to x^*? Repeat the experiment for $N = 20$.