Solutions to Exercise set 3 – MPC (SC42125)

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1 An MPC stability result

$$\begin{aligned}
\boldsymbol{u}_N^0 &= \big(u_N^0(0;x), u_N^0(1;x), \ldots, u_N^0(N-1;x)\big) \\
\boldsymbol{x}_N^0 &= \big(x_N^0(0;x), x_N^0(1;x), \ldots, x_N^0(N;x)\big) \\
x^+ &= f(x, u_N^0(0;x)) = x_N^0(1;x) \\
V_N(x, \boldsymbol{u}_N) &= \sum_{k=0}^{N-1} \ell(x(k), u(k))
\end{aligned}$$

1. Assume that the constrained optimization problem is feasible at each time step k. Let $V_N^0(x) = V_N(x, \boldsymbol{u}_N^0(x))$ be the optimal value of the cost function. In view of the assumption on ℓ, $V_N^0(x)$ is a positive definite function and $V_N^0(x) \geq \alpha(\|x\|)$. We want to show that $V_N^0(x)$ is a Lyapunov function for the closed loop system. Then, we have:

$$\begin{aligned}
V_N^0(x^+) = V_N(x^+, \boldsymbol{u}_N^0(x^+)) &= \\
&= \sum_{k=0}^{N-1} \ell(x_N^0(k;x^+), u_N^0(k,x^+)) \leq \\
&\leq \sum_{k=0}^{N-1} \ell(x_N^0(k;x^+), \tilde{u}(k)) = V_N(x^+, \tilde{u})
\end{aligned}$$

where

$$\tilde{\boldsymbol{u}} = \big(u_N^0(1;x), u_N^0(2;x), \cdots, u_N^0(N-1;x), 0\big).$$

By exploiting the final constraint, we have that $x_N^0(N,x) = 0$ and hence

$$V_N^0(x^+) \leq V_N^0(x) - \ell(x, u_N^0(0,x))$$

So V^0 is a Lyapunov function and the origin is globally asymptotically stable. We assumed that the optimization problem is feasible at each step. Indeed, for this condition to be ensured it is enough to choose the initial state $x(0)$ that is controllable to the origin in N steps. In fact, if at one time step the optimization problem is feasible with optimal input $\boldsymbol{u}_N^0$, the optimization problem at the next step admits at least one feasible input, i.e. $\tilde{\boldsymbol{u}} = \big(u_N^0(1;x), u_N^0(2;x), \cdots, u_N^0(N-1;x), 0\big)$.

2. Under the same controllability assumption of the previous point, exponential stability is guaranteed if, for some $c_1 > 0$, $a > 0$,

$$\ell(x,u) \geq c_1 \|x\|^a.$$

2 A simple example

See MATLAB code.

3 Computing the projection of $\mathbb{Z}$ onto $\mathcal{X}_N$

See MATLAB code.

4 Terminal constraint and region of attraction

See MATLAB code.

5 Trajectory tracking for unicycle-like vehicles

1. By using geometrical arguments, the expressions for $v_\mathrm{r}(t)$ and $\omega_\mathrm{r}(t)$ are given by:

$$\begin{cases} v_\mathrm{r}(t) = \sqrt{\dot{x}_r^2(t) + \dot{y}_r^2(t)} \\ \theta_\mathrm{r}(t) = \mathrm{atan2}(\dot{y}_\mathrm{r}(t), \dot{x}_\mathrm{r}(t)) \\ \omega_\mathrm{r}(t) = \dfrac{\dot{x}_\mathrm{r}(t)\ddot{y}_\mathrm{r}(t) - \dot{y}_\mathrm{r}(t)\ddot{x}_\mathrm{r}(t)}{\dot{x}_\mathrm{r}^2(t) + \dot{y}_\mathrm{r}^2(t)}. \end{cases}$$

2. By computing the time derivative of the given expression, we obtain

$$\dot{e}(t) = \begin{bmatrix} \cos(e_3(t)) & 0 \\ \sin(e_3(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_\mathrm{r}(t) \\ \omega_\mathrm{r}(t) \end{bmatrix} + \begin{bmatrix} -1 & e_2 \\ 0 & -e_1 \\ 0 & -1 \end{bmatrix} u(t).$$

3. By substituting the expression for $u(t)$, we get

$$\dot{e}(t) = \begin{bmatrix} \omega_{\mathrm{r}}(t)e_2(t) \\ -\omega_{\mathrm{r}}(t)e_1(t) + v_{\mathrm{r}}(t)\sin(e_3(t)) \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & e_2 \\ 0 & -e_1 \\ 0 & -1 \end{bmatrix} u_{\mathrm{b}}(t).$$

Note that the point $[e(t),\, u_{\mathrm{b}}(t)]^\top = [\mathbf{0}_3,\, \mathbf{0}_2]^\top$ is an equilibrium of the dynamical system above. Therefore, by linearizing around the reference trajectory we obtain:

$$\dot{e}(t) = \begin{bmatrix} 0 & \omega_{\mathrm{r}}(t) & 0 \\ -\omega_{\mathrm{r}}(t) & 0 & v_{\mathrm{r}}(t) \\ 0 & 0 & 0 \end{bmatrix} e(t) + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} u_{\mathrm{b}}(t),$$

which is a time-varying linear system. The Euler approximation gives the discrete-time counterpart, i.e.,

$$e(k+1) = A(k)e(k) + Bu_{\mathrm{b}}(k),$$

with

$$A(k) = \begin{bmatrix} 1 & \omega_{\mathrm{r}}(t)T & 0 \\ -\omega_{\mathrm{r}}(t)T & 1 & v_{\mathrm{r}}(t)T \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} -T & 0 \\ 0 & 0 \\ 0 & -T \end{bmatrix}.$$

4. See MATLAB code.

6 A vehicle on an elliptic road

See MATLAB code.