Solutions to Exercise set 1 – MPC (SC42125)

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1 Inverted pendulum: modeling and control

See the MATLAB code file.

2 State-space representation

We recall that a state-space realization is in general *not* unique. As an example, we adopt the classical controllable canonical form.

a) $G(s) = \dfrac{1}{(2s+1)(3s+1)} = \dfrac{1}{6s^2+5s+1} = \dfrac{1}{6}\dfrac{1}{s^2+\frac{5}{6}s+\frac{1}{6}}$

The SISO system has 2 poles, thus we look for a realization of order 2: $x = [x_1\ x_2]^\top$ and the vectors/matrix are

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} \dfrac{1}{6} & 0 \end{bmatrix},\ D = 0.$$

b) $G(s) = \dfrac{2s+1}{3s+1} = \dfrac{\frac{2}{3}s+\frac{1}{3}}{s+\frac{1}{3}} = \dfrac{2}{3} + \dfrac{\frac{1}{9}}{s+\frac{1}{3}}$

Here, the number of poles and zeros coincides (both equal to 1, hence the order of the system is 1). Therefore, we expect a nonzero value for the D matrix. In this case, we have:

$$A = -\frac{1}{3},\ B = 1,\ C = \frac{1}{9},\ D = \frac{2}{3}.$$

c) $y(k) = y(k-1) + 2u(k-1)$

In view of the properties of the Z-transform, the difference equation can be equivalently rewritten as

$$Y(z) - z^{-1}Y(z) = 2z^{-1}U(z) \implies G(z) = \frac{2z^{-1}}{1-z^{-1}} = \frac{2}{z-1}.$$

Since we have only one pole, the order of the system is 1 with the following quadruplet (A, B, C, D) for the state-space model:

$$A = 1,\ B = 1,\ C = 2,\ D = 0.$$

d) $y(k) = a_1 y(k-1) + a_2 y(k-2) + b_1 u(k-1) + b_2 u(k-2)$

By using the properties of the Z-transform, we have

$$(1 - a_1 z^{-1} - a_2 z^{-2})Y(z) = (b_1 z^{-1} + b_2 z^{-2})U(z) \implies G(z) = \frac{b_1 z + b_2}{z^2 - a_1 z - a_2}.$$

In this case, we have two poles, hence the order of the system is 2, and the vectors/matrix are

$$A = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = [b_2 \quad b_1],\ D = 0.$$

1. In general, the poles of a transfer function corresponds to the a subset of the eigenvalues of the matrix A, i.e., the roots of the characteristic polynomial $(\lambda I - A) = 0$.

2. The matrix D is zero when the degree of the numerator of G is less than the degree of the denominator (strictly proper transfer function).

3. For the transfer functions in the complex domain s, the static gain μ (if exists finite) is determined by evaluating $G(0)$ (or, equivalently, $\mu = -CA^{-1}B + D$). Thus, we have

   a) $\mu = G(0) = 1$ *b)* $\mu = G(0) = 1.$

   For the transfer functions in the domain z, the static gain μ (if exists finite) is determined by evaluating $G(1)$ (or, equivalently, $\mu =$

$-C(I-A)^{-1}B+D)$. Thus, we have

$$c)\ \mu = G(1) = +\infty \qquad d)\ \mu = G(1) = -\frac{b_1+b_2}{a_1+a_2-1}.$$

4. The order of the system is independent on the number of inputs and outputs. It corresponds to the dimensions of the square matrix $A \in \mathbb{R}^{n\times n}$ and, consequently, on the number of eigenvalues only.

5. Since there are no pole-zero cancellations, all the realizations are minimal. The case *d*) is non-minimal for all those values of the parameters a_1, a_2, b_1 and b_2 such that there is no cancellation between the unique zero and one of the two poles.

3 Predicted output computation

1. We shall propagate the state solution for $N = 25$, i.e.,

$$y(k+N) = Cx(k+N) = A^N x(k) + \underbrace{\left[A^{N-1}B \;\; A^{N-2}B \;\; \cdots \;\; B\right]}_{\mathcal{C}_N} \boldsymbol{u}_N.$$

In view of the structure of the matrix A, we note that, for a generic $p > 0$, we have

$$A^p = \begin{bmatrix} 1 & p\,a \\ 0 & 1 \end{bmatrix}.$$

Then, the only one nonzero component of $\boldsymbol{u}_N$ is the fourth, which "selects" the column of $\mathcal{C}_N$ given by A^4B. Thus, we obtain

$$\begin{aligned} y(k+25) &= C\,A^{25}\,x(k) + C\,A^4\,B\,u(k+20) \\ &= [1 \quad 0]\begin{bmatrix} 1 & 25a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1 \quad 0]\begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= 1 + 25a + 8a = 1 + 33a. \end{aligned}$$

4 Lyapunov function for exponential stability

By equation (1), we get $\|x(k)\|^\sigma \geq \frac{1}{a_2}V(x(k))$, which substituted in (2) gives us $V(f(x(k))) - V(x(k)) \leq -\frac{a_3}{a_2}V(x(k))$, with $\frac{a_3}{a_2} \in (0,1)$. Hence, we have:

$$V(f(x(k))) \leq \left(1 - \frac{a_3}{a_2}\right) V(x(k)) = \gamma V(x(k)). \tag{1}$$

Therefore, by exploiting the relation above, at the generic step $k > 0$ and for any initial condition $x(0) \in \mathbb{R}^n$, we have:

$$V(x(k)) \leq \gamma^k V(x(0)).$$

From (1), we obtain the following inequalities that prove the statement:

$$\begin{aligned}\|x(k)\| &\leq \left(\frac{1}{a_1} V(x(k))\right)^{\frac{1}{\sigma}} \leq \left(\frac{1}{a_1}\gamma^k V(x(0))\right)^{\frac{1}{\sigma}} \\ &\leq \left(\frac{a_2}{a_1}\gamma^k \|x(0)\|^\sigma\right)^{\frac{1}{\sigma}} = \left(\frac{a_2}{a_1}\right)^{\frac{1}{\sigma}} \|x(0)\| \gamma^{\frac{k}{\sigma}}.\end{aligned}$$

5 Exponential stability implies existence of a Lyapunov function

To alleviate the notation, let us omit some dependences on k. Let

$$V(x) \coloneqq \sum_{k=0}^{K-1} \phi^\top(k; x)\, \phi(k; x),$$

with index K to be designed. Note that $\phi(k; f(x)) = \phi(k; \phi(1; x)) = \phi(k+1; x)$. Hence, we have:

$$\begin{aligned}V(f(x)) - V(x) &= \sum_{k=0}^{K-1} \phi^\top(k+1; x)\phi(k+1; x) - \sum_{k=0}^{K-1} \phi^\top(k; x)\phi(k; x) \\ &= \sum_{h=1}^{K} \phi^\top(h; x)\phi(h; x) - \sum_{k=0}^{K-1} \phi^\top(k; x)\phi(k; x) \\ &= \phi^\top(K; x)\phi(K; x) - \|x\|^2 \leq -\left(1 - a^2 e^{-2\lambda K}\right) \|x\|^2,\end{aligned}$$

where the last inequality follows by (3). In this case, K can be chosen large enough so that $\left(1 - a^2 e^{-2\lambda K}\right) > 0$. This proves (4).

Since f is continuously differentiable, then it is also Lipschitz with constant L over the bounded domain $\mathcal{D}$, i.e., $\|f(x) - f(y)\| \leq L\|x - y\|$ for all x, $y \in \mathcal{D}$. This implies that

$$\|\phi(k+1; x) - \phi(k+1; y)\| = \|f(\phi(k; x)) - f(\phi(k; y))\| \leq L\|\phi(k; x) - \phi(k; y)\|,$$

and by induction

$$\|\phi(k;x) - \phi(k;y)\| \leq L^k \|x - y\|.$$

Then, by evaluating the relation in (5), we obtain:

$$\begin{aligned}
|V(x) - V(y)| &= \left| \sum_{k=0}^{K-1} \Big(\phi^\top(k;x)\phi(k;x) - \phi^\top(k;y)\phi(k;y) \Big) \right| \\
&= \left| \sum_{k=0}^{K-1} \Big(\phi^\top(k;x) \left(\phi(k;x) - \phi(k;y)\right) + \phi^\top(k;y) \left(\phi(k;x) - \phi(k;y)\right) \Big) \right| \\
&\leq \sum_{k=0}^{K-1} \|\phi(k;x)\| \|\phi(k;x) - \phi(k;y)\| + \|\phi(k;y)\| \|\phi(k;x) - \phi(k;y)\| \\
&\leq \sum_{k=0}^{K-1} \left(\|\phi(k;x)\| + \|\phi(k;y)\| \right) L^k \|x - y\| \\
&\leq \left(\sum_{k=0}^{K-1} a e^{-\lambda k} L^k \right) \left(\|x\| + \|y\| \right) \|x - y\| = c_4 \left(\|x\| + \|y\| \right) \|x - y\|.
\end{aligned}$$

6 Comparison functions

• *a*) By computing $\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x) = \frac{1}{1+x^2} > 0$, hence $\alpha(x)$ is strictly increasing. However, by evaluating $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2} < \infty$, we see that $\alpha(x)$ belongs to class $\mathcal{K}$ only.

b) In this case, $\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x) = c\, x^{c-1} > 0$ for any $c > 0$. Moreover, $\lim_{x\to\infty} x^c = \infty$, hence $\alpha(x)$ belongs to class $\mathcal{K}_\infty$.

c) The function can be written as

$$\alpha(x) = \begin{cases} x^2 & x \in [0,1] \\ x & x \in (1, \infty), \end{cases}$$

and it is a continuous, strictly increasing function. Moreover, $\lim_{x\to\infty} \alpha(x) = \infty$ and hence it belongs to the class $\mathcal{K}_\infty$.

d) We see that the function is strictly increasing in x and strictly decreasing in y for any $k > 0$, since

$$\frac{\partial}{\partial x}\beta(x,y) = \frac{1}{(kxy+1)^2} > 0$$
$$\frac{\partial}{\partial y}\beta(x,y) = \frac{-kx^2}{(kxy+1)^2} < 0.$$

Moreover, $\beta(x,y) \to 0$ as $y \to \infty$. Therefore, it belongs to the class $\mathcal{KL}$.

e) With the same arguments above, case d), the function belongs to the class $\mathcal{KL}$ for any $c > 0$.

• Since α is strictly increasing on the interval $[0,a)$, if $x \geq y$, we have $x + y \leq 2x$. Hence,

$$\alpha(x+y) \leq \alpha(2x) \leq \alpha(2x) + \alpha(2y).$$

The same argument holds in the case $y \geq x$ and hence the inequality is always satisfied.

7 Positive semidefinite Q penalty and its square root

- Q can be written as $T^\top \Lambda T$, with T an orthogonal matrix and Λ the diagonal matrix containing the eigenvalues of Q. The latter is true because a change of coordinates does not change the eigenvalues of a matrix. Then $Q^{1/2} = T^\top \Lambda^{1/2} T$ (where $\Lambda^{1/2}$ is the diagonal matrix containing the square root of eigenvalues of Q), as

$$Q^{1/2}Q^{1/2} = T^\top \Lambda^{1/2} T T^\top \Lambda^{1/2} T = T^\top \Lambda T. \tag{2}$$

  It is easy to check that $Q^{1/2}$ is symmetric; moreover the eigenvalues of $Q^{1/2}$ are the diagonal elements of $\Lambda^{1/2}$, that are all ≥ 0; hence $Q^{1/2}$ is positive semidefinite.

- According to Hautus test (after Prof. Malo Hautus, TU Eindhoven), a pair (A, C) is detectable if and only if

$$\operatorname{rank}\begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n \qquad \forall |\lambda| \geq 1. \tag{3}$$

  Also,

$$\operatorname{rank}\begin{bmatrix} \lambda I - A \\ Q^{1/2} \end{bmatrix} = \operatorname{rank}\left(\begin{bmatrix} I & 0 \\ 0 & T^\top \end{bmatrix}\begin{bmatrix} \lambda I - A \\ Q^{1/2} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}\right) = \operatorname{rank}\begin{bmatrix} \lambda I - A \\ \Lambda^{1/2} \end{bmatrix},$$

since the multiplication by a nonsingular matrix does not change the rank. Clearly,

$$\operatorname{rank}\begin{bmatrix}\lambda I - A\\ Q^{1/2}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\lambda I - A\\ \Lambda^{1/2}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\lambda I - A\\ \Lambda\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\lambda I - A\\ Q\end{bmatrix},$$

as multiplying the row of a matrix by a nonzero real does not change the rank either. We conclude that, for all $|\lambda| \geq 1$,

$$\operatorname{rank}\begin{bmatrix}\lambda I - A\\ Q^{1/2}\end{bmatrix} = n \iff \operatorname{rank}\begin{bmatrix}\lambda I - A\\ Q\end{bmatrix} = n. \tag{4}$$

8 An MPC problem

See MATLAB code.