Introduction to Probabilities

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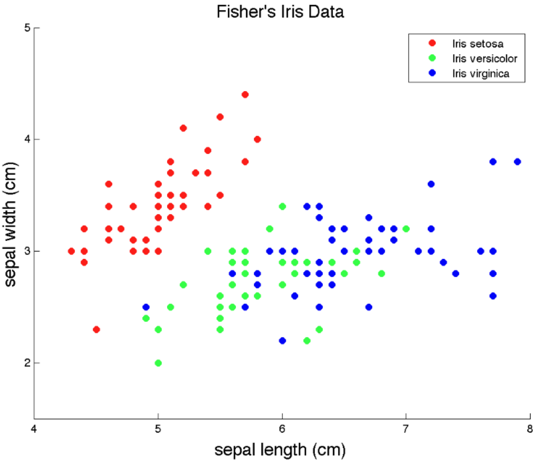
Helping Material

- **Mathematics of Machine Learning**
  https://mml-book.github.io/

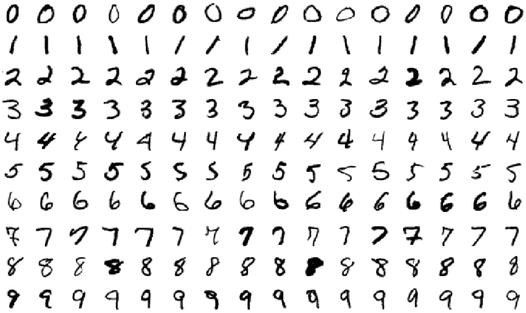
Variability

- Variability is part of nature.
- Data for 3 species of iris, from Ronald Fisher (1936).



Variability

- Our handwriting is unique.
- Variability leads to uncertainty: e.g., distinguishing between 1 vs 7 or 4 vs 9.

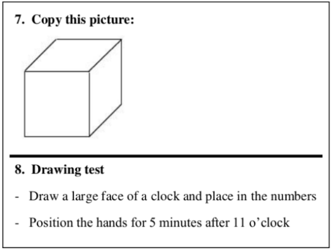


Variability

- Variability leads to uncertainty.
- Reading handwritten text in a foreign language.

Example: Screening and Diagnostic Tests

- Early warning test for Alzheimer's disease (Scharre, 2010, 2014).
- Detects "mild cognitive impairment".
- Takes 10–15 minutes.
- Freely available.
- Assume a 70-year-old man tests positive.
- Should he be concerned?



7. Copy this picture:

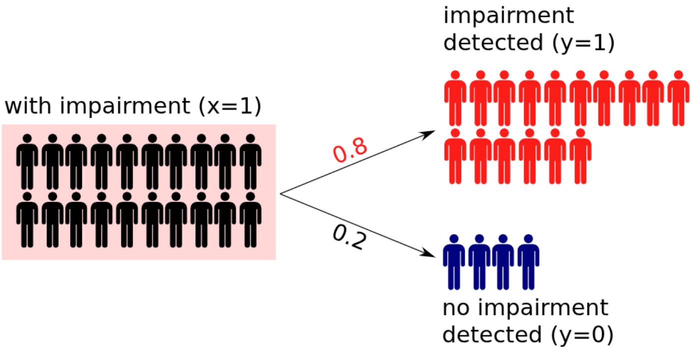
8. Drawing test

- Draw a large face of a clock and place in the numbers
- Position the hands for 5 minutes after 11 o'clock

(Example from sagetest.osu.edu)

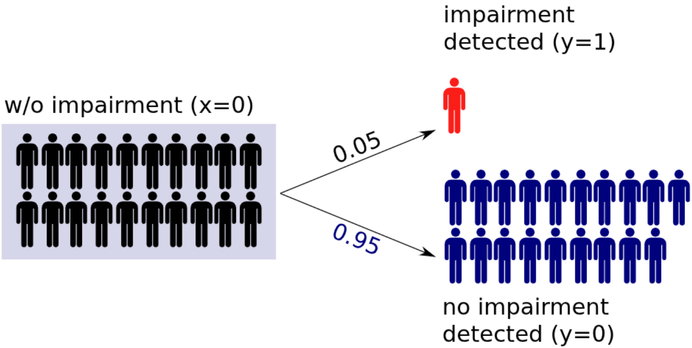
Accuracy of the Test

- Sensitivity of 0.8 and specificity of 0.95 (Scharre, 2010).
- 80% correct for people with impairment.



Accuracy of the Test

- Sensitivity of 0.8 and specificity of 0.95 (Scharre, 2010).
- 95% correct for people without impairment.



Variability Implies Uncertainty

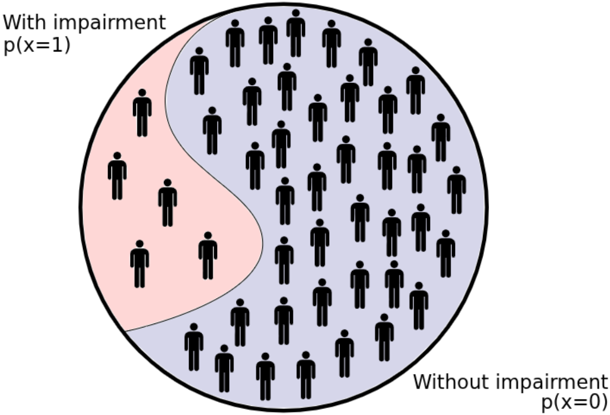
- People of the same group do not have the same test results
  - Test outcome is subject to variability.
  - The data are noisy.
- Variability leads to uncertainty.
  - Positive test $\equiv$ true positive?
  - Positive test $\equiv$ false positive?
- What can we safely conclude from a positive test result?
- How should we analyze such ambiguous data?

Probabilistic Approach

- $P(y \mid x)$: model of the test specified in terms of (conditional) probabilities.
- $x \in \{0, 1\}$: quantity of interest (cognitive impairment or not).
- $y \in \{0, 1\}$: test outcome (negative or positive).
- The test outcomes y can be described with probabilities:
  - Sensitivity $= 0.8$
    - $\Rightarrow P(y = 1 \mid x = 1) = 0.8$
    - $\Rightarrow P(y = 0 \mid x = 1) = 0.2$
  - Specificity $= 0.95$
    - $\Rightarrow P(y = 0 \mid x = 0) = 0.95$
    - $\Rightarrow P(y = 1 \mid x = 0) = 0.05$

Prior Information

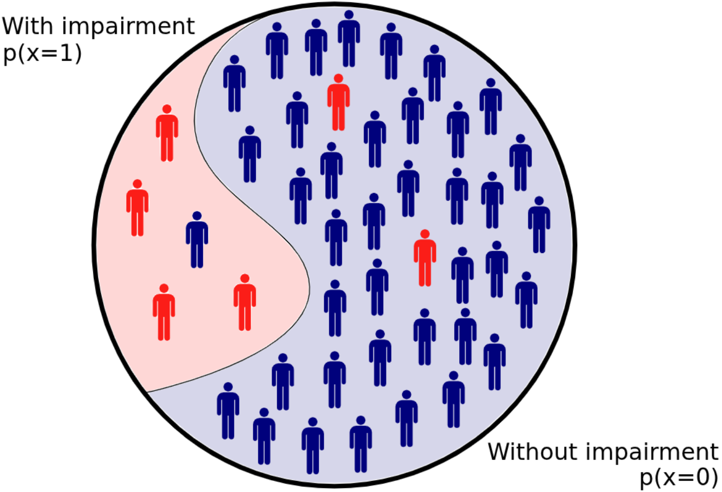
- Among people like the patient, $P(x = 1) = \frac{5}{45} \approx 11\%$ have a cognitive impairment (plausible range: $3\% - 22\%$, Geda, 2014).



Probabilistic Model

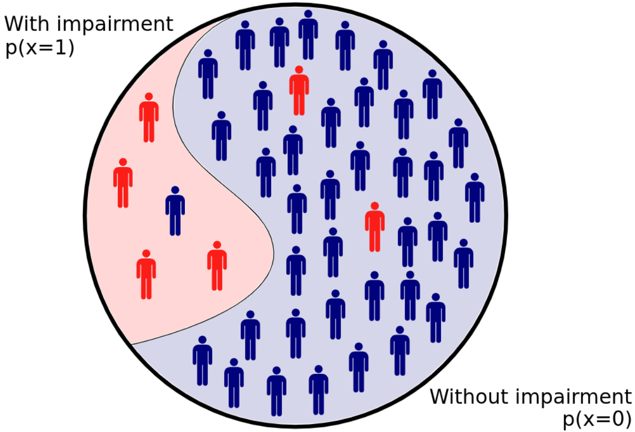
- Reality:
  - Properties/characteristics of the group of people like the patient
  - Properties/characteristics of the test
- Probabilistic model:
  - $P(x = 1)$: probability of cognitive impairment
  - $P(y = 1|x = 1)$ or $P(y = 0|x = 1)$: probability of positive or negative test given cognitive impairment
  - $P(y = 1|x = 0)$ or $P(y = 0|x = 0)$: probability of positive or negative test given no cognitive impairment
  - Fully specified by three numbers.
- A probabilistic model is an abstraction of reality that uses probability theory to quantify the chance of uncertain events.

If We Tested the Whole Population



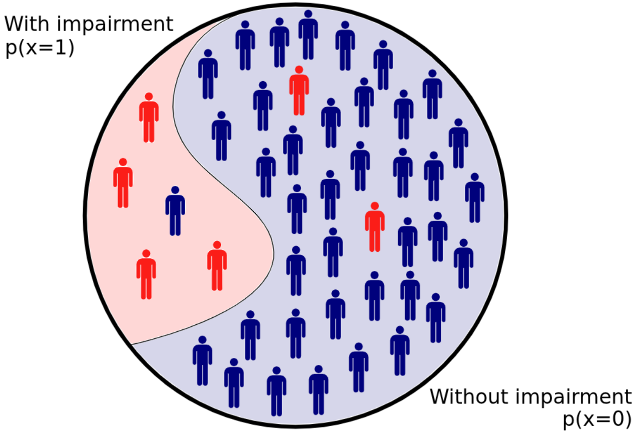
If We Tested the Whole Population

- Fraction of people who are impaired and test positive:
  $P(x = 1, y = 1) = P(y = 1|x = 1)P(x = 1) = 0.8 \cdot \frac{5}{45} = \frac{4}{45} \approx 9\%$



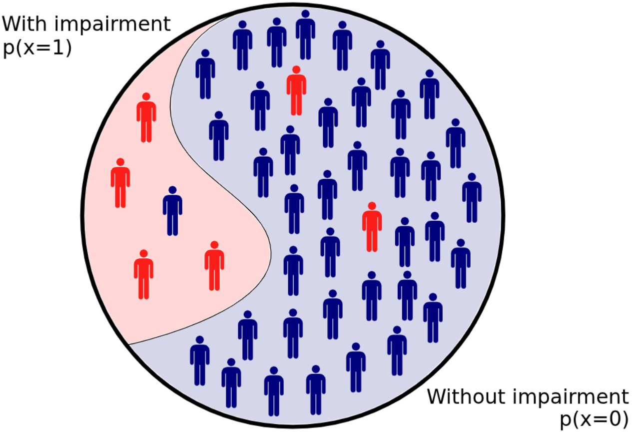
If We Tested the Whole Population

- Fraction of people who are not impaired and test positive:
  $P(x = 0, y = 1) = P(y = 1|x = 0)P(x = 0) = 0.05 \cdot \frac{40}{45} = \frac{2}{45} \approx 4\%$



If We Tested the Whole Population

- Fraction of people where the test is positive:
  $P(y = 1) = P(x = 1, y = 1) + P(x = 0, y = 1) = \frac{4}{45} + \frac{2}{45} = \frac{6}{45} \approx 13\%$



Putting Everything Together

- Among those with a positive test, fraction with impairment:

$$P(x = 1|y = 1) = \frac{P(y = 1|x = 1)P(x = 1)}{P(y = 1)} = \frac{4}{6} = \frac{2}{3}$$

- Fraction without impairment:

$$P(x = 0|y = 1) = \frac{P(y = 1|x = 0)P(x = 0)}{P(y = 1)} = \frac{2}{6} = \frac{1}{3}$$

- Equations are examples of Bayes' rule.
- Positive test increased probability of cognitive impairment from 11% (prior belief) to 67%, or from 6% to 51%.
- 51% (coin flip)

Probabilistic Modeling and Reasoning

- Bayesian analysis is probabilistic modeling and reasoning using Bayes' rule.
- **Probabilistic modeling:**
  - Model the world (events) using probabilities and random variables.
  - Example random variables:
    - y: test outcome
    - x: cognitive impairment
- **Probabilistic reasoning (inference):**
  - Compute probabilities of events given other events.
  - Infer probabilities of unobserved events from observed data.
  - Use Bayes' rule to update beliefs based on evidence.

Probabilistic Modeling and Reasoning

- In our example:
  - Unobserved/uncertain event: cognitive impairment $x = 1$
  - Observed event (evidence): test result $y = 1$
  - **Prior**: probability before seeing evidence, e.g., $P(x = 1)$
  - **Posterior**: updated probability after evidence, e.g., $P(x = 1|y = 1)$
- Key idea: The posterior quantifies what we believe about x after seeing the test result y.

Key Rules of Probability

- **Product rule:**
  - $P(x=1, y=1) = P(y=1|x=1)P(x=1)$
  - $P(x=1, y=1) = P(x=1|y=1)P(y=1)$
- **Sum rule:**
  - $P(y=1) = P(x=1, y=1) + P(x=0, y=1)$
- **Bayes' rule (conditioning) as consequence of product rule:**
  - $P(x=1|y=1) = \frac{P(x=1,y=1)}{P(y=1)} = \frac{P(y=1|x=1)P(x=1)}{P(y=1)}$
- **Denominator from sum rule and product rule:**
  - $P(y=1) = P(y=1|x=1)P(x=1) + P(y=1|x=0)P(x=0)$

Key Rules of Probability

- The rules generalize to multivariate random variables
  - $\mathbf{x} = (x_1, x_2, \ldots)$: vector of random variables
  - $\mathbf{y} = (y_1, y_2, \ldots)$: vector of random variables
- The rules generalize to continuous random variables
- **Product rule:**
  - $P(\mathbf{x}, \mathbf{y}) = P(\mathbf{y}|\mathbf{x})P(\mathbf{x})$
  - $P(\mathbf{x}, \mathbf{y}) = P(\mathbf{x}|\mathbf{y})P(\mathbf{y})$
- **Sum rule:**
  - $P(\mathbf{y}) = \sum_{\mathbf{x}} P(\mathbf{x}, \mathbf{y})$ (discrete case)
  - $P(\mathbf{y}) = \int P(\mathbf{x}, \mathbf{y}) d\mathbf{x}$ (continuous case)