





## An Extendable Python Implementation of Robust Optimisation Monte Carlo

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### Abstract

Performing inference in statistical models with an intractable likelihood is challenging, therefore, most likelihood-free inference (LFI) methods encounter accuracy and efficiency limitations. In this paper, we present the implementation of the LFI method Robust Optimisation Monte Carlo (ROMC) in the Python package **ELFI**. ROMC is a novel and efficient (highly-parallelisable) LFI framework that provides accurate weighted samples from the posterior. Our implementation can be used in two ways. First, a scientist may use it as an out-of-the-box LFI algorithm; we provide an easy-to-use API harmonised with the principles of **ELFI**, enabling effortless comparisons with the rest of the methods included in the package. Additionally, we have carefully split ROMC into isolated components for supporting extensibility. A researcher may experiment with novel method(s) for solving part(s) of ROMC without reimplementing everything from scratch. In both scenarios, the ROMC parts can run in a fully-parallelised manner, exploiting all CPU cores. We also provide helpful functionalities for (i) inspecting the inference process and (ii) evaluating the obtained samples. Finally, we test the robustness of our implementation on some typical LFI examples.

*Keywords:* Bayesian inference, implicit models, likelihood-free, Python, **ELFI**.

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## 1. Introduction

Simulator-based models are particularly captivating due to the modeling freedom they provide. In essence, any data generating mechanism that can be written as a finite set of algorithmic steps can be programmed as a simulator-based model. Hence, these models are often used to model physical phenomena in the natural sciences such as, e.g., genetics, epidemiology or neuroscience Gutmann and Corander (2016); Lintusaari, Gutmann, Dutta, Kaski, and Corander (2017); Franks (2020); Cranmer, Brehmer, and Louppe (2020). In simulator-based models, it is feasible to generate samples using the simulator but is infeasible to evaluate

the likelihood function. The intractability of the likelihood makes the so-called likelihood-free inference (LFI), i.e., the approximation of the posterior distribution without using the likelihood function, particularly challenging.

Optimization Monte Carlo (OMC), proposed by Meeds and Welling (2015), is a novel LFI approach. The central idea is to convert the stochastic data-generating mechanism into a set of deterministic optimization processes. Afterwards, Forneron and Ng (2016) described a similar method under the name ‘reverse sampler’. In their work, Ikonov and Gutmann (2019), identified some critical limitations of OMC and proposed Robust OMC (ROMC) as an improved version of OMC with appropriate modifications.

In this paper, we present the implementation of ROMC at the Python package **ELFI** (Engine for likelihood-free inference) Lintusaari, Vuollekoski, Kangasrääsiö, Skytén, Järvenpää, Marttinen, Gutmann, Vehtari, Corander, and Kaski (2018). The implementation has been designed to ensure extensibility. ROMC is an LFI framework; it defines a sequence of algorithmic steps for approximating the posterior without enforcing a specific algorithm for solving each step. Therefore, a researcher may use ROMC as the backbone algorithm and develop novel methods to solve each separate step.<sup>1</sup> We have designed our software for facilitating such experimentation.

To the best of our knowledge, this is the first implementation of the ROMC inference method to a generic LFI framework. We organize the evaluation in three steps. First, for securing that our implementation is accurate, we test it against an artificial example with a tractable likelihood. The artificial example also serves as a step-by-step guide for showcasing how to use the various functionalities of our implementation. Second, we use the second-order moving average (MA2) example from the **ELFI** package, using as ground truth the samples obtained with Rejection ABC Lintusaari *et al.* (2017) using a very high number of samples. Finally, we present the execution times of ROMC, measuring the speed-up achieved by the parallel version of the implementation.

## 2. Background

We first give a short introduction to simulator-based models, we then focus on OMC and its robust version, ROMC, and we, finally, introduce **ELFI**, the Python package used for the implementation.

### 2.1. Simulator-based models and likelihood-free inference

An implicit or simulator-based model is a parameterized stochastic data generating mechanism, where we can sample data points but we cannot evaluate the likelihood. Formally, a simulator-based model is a parameterized family of probability density functions  $\{p(\mathbf{y}|\boldsymbol{\theta})\}_{\boldsymbol{\theta}}$  whose closed-form is either unknown or computationally intractable. In these cases, we can only access the simulator  $m_r(\boldsymbol{\theta})$ , i.e., the black-box mechanism (computer code) that generates samples  $\mathbf{y}$  in a stochastic manner from a set of parameters  $\boldsymbol{\theta}$ . We denote the process of obtaining samples from the simulator with  $m_r(\boldsymbol{\theta}) \rightarrow \mathbf{y}$ . As shown by Meeds and Welling (2015), it is feasible to isolate the randomness of the simulator by introducing a set of nuisance

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<sup>1</sup>For being a ready-to-use algorithm, Ikonov and Gutmann (2019) proposed a default method for each step, but this choice is by no means restrictive.

random variables denoted by  $\mathbf{u} \sim p(\mathbf{u})$ . Therefore, for a specific tuple  $(\boldsymbol{\theta}, \mathbf{u})$  the simulator becomes a deterministic mapping  $g$ , such that  $\mathbf{y} = g(\boldsymbol{\theta}, \mathbf{u})$ . In terms of computer code, the randomness of a random process is governed by the global seed. Although each package may handle the randomness in slightly different ways<sup>2</sup>, in all cases, setting the initial seed to a specific integer converts the simulation to a deterministic piece of code.

As stated at the Introduction, the modeling freedom of simulator-based models comes at the price of difficulties in inferring the parameters of interest. Denoting the observed data as  $\mathbf{y}_0$ , the main difficulty lies at the intractability of the likelihood function  $L(\boldsymbol{\theta}) = p(\mathbf{y}_0|\boldsymbol{\theta})$ . To better see the sources of the intractability, and to address them, we go back to the basic characterization of the likelihood as the (rescaled) probability of generating data  $\mathbf{y}$  that is similar to the observed data  $\mathbf{y}_0$ , using parameters  $\boldsymbol{\theta}$ . Formally, the likelihood  $L(\boldsymbol{\theta})$  is:

$$L(\boldsymbol{\theta}) = \lim_{\epsilon \rightarrow 0} c_\epsilon \int_{\mathbf{y} \in B_{d,\epsilon}(\mathbf{y}_0)} p(\mathbf{y}|\boldsymbol{\theta}) d\mathbf{y} = \lim_{\epsilon \rightarrow 0} c_\epsilon \Pr(g(\boldsymbol{\theta}, \mathbf{u}) \in B_{d,\epsilon}(\mathbf{y}_0) \mid \boldsymbol{\theta}) \quad (1)$$

where  $c_\epsilon$  is a proportionality factor that depends on  $\epsilon$  and  $B_{d,\epsilon}(\mathbf{y}_0)$  is an  $\epsilon$  region around  $\mathbf{y}_0$  that is defined through a distance function  $d$ , i.e.,  $B_{d,\epsilon}(\mathbf{y}_0) := \{\mathbf{y} : d(\mathbf{y}, \mathbf{y}_0) \leq \epsilon\}$ . Equation 1 highlights the main source of intractability; computing  $\Pr(g(\boldsymbol{\theta}, \mathbf{u}) \in B_{d,\epsilon}(\mathbf{y}_0) \mid \boldsymbol{\theta})$  as the fraction of samples that lie inside the  $\epsilon$  region around  $\mathbf{y}_0$  is computationally infeasible in the limit where  $\epsilon \rightarrow 0$ . Hence, the constraint is relaxed to  $\epsilon > 0$ , which leads to the approximate likelihood:

$$L_{d,\epsilon}(\boldsymbol{\theta}) = \Pr(\mathbf{y} \in B_{d,\epsilon}(\mathbf{y}_0) \mid \boldsymbol{\theta}), \quad \text{where } \epsilon > 0. \quad (2)$$

and, in turn, to the approximate posterior:

$$p_{d,\epsilon}(\boldsymbol{\theta}|\mathbf{y}_0) \propto L_{d,\epsilon}(\boldsymbol{\theta})p(\boldsymbol{\theta}) \quad (3)$$

The approximation in Equations 2, 3 is by no means the only strategy to deal with the intractability of the likelihood function in Equation 1. Other strategies include modeling the (stochastic) relationship between  $\boldsymbol{\theta}$  and  $\mathbf{y}$ , and its reverse, or framing likelihood-free inference as a ratio estimation problem, see for example [Blum and Francois \(2010\)](#); [Wood \(2017\)](#); [Papamakarios and Murray \(2016\)](#); [Papamakarios, Sterratt, and Murray \(2020\)](#); [Chen and Gutmann \(2019\)](#); [Thomas, Dutta, Corander, Kaski, and Gutmann \(2020\)](#); [Hermans, Begy, and Louppe \(2020\)](#). However, both OMC and robust OMC, which we introduce next, are based on the approximation in Equation 2.

## 2.2. Optimization Monte Carlo (OMC)

Our description of OMC [Meeds and Welling \(2015\)](#) follows [Ikonov and Gutmann \(2019\)](#).

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<sup>2</sup>For example, at **Numpy** [Harris, Millman, van der Walt, Gommers, Virtanen, Cournapeau, Wieser, Taylor, Berg, Smith, Kern, Picus, Hoyer, van Kerkwijk, Brett, Haldane, del Río, Wiebe, Peterson, Gérard-Marchant, Sheppard, Reddy, Weckesser, Abbasi, Gohlke, and Oliphant \(2020\)](#), the pseudo-random number generation is based on a global state, whereas, in **JAX** [Bradbury, Frostig, Hawkins, Johnson, Leary, Maclaurin, Necula, Paszke, VanderPlas, Wanderman-Milne, and Zhang \(2018\)](#) random functions consume a key that is passed as parameter.

We define the indicator function (boxcar kernel) that equals one only if  $\mathbf{x}$  lies in  $B_{d,\epsilon}(\mathbf{y})$ :

$$\mathbb{1}_{B_{d,\epsilon}(\mathbf{y})}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in B_{d,\epsilon}(\mathbf{y}) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

We, then, rewrite the approximate likelihood function  $L_{d,\epsilon}(\boldsymbol{\theta})$  of Equation 2 as:

$$L_{d,\epsilon}(\boldsymbol{\theta}) = \Pr(\mathbf{y} \in B_{d,\epsilon}(\mathbf{y}_0) | \boldsymbol{\theta}) = \int_{\mathbf{u}} \mathbb{1}_{B_{d,\epsilon}(\mathbf{y}_0)}(g(\boldsymbol{\theta}, \mathbf{u})) d\mathbf{u} \quad (5)$$

which can be approximated using samples from the simulator:

$$L_{d,\epsilon}(\boldsymbol{\theta}) \approx \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{B_{d,\epsilon}(\mathbf{y}_0)}(g(\boldsymbol{\theta}, \mathbf{u}_i)) \quad \text{where } \mathbf{u}_i \sim p(\mathbf{u}). \quad (6)$$

In Equation 6, for each  $\mathbf{u}_i$ , there is a region  $C_\epsilon^i$  in the parameter space  $\boldsymbol{\theta}$  where the indicator function returns one, i.e.,  $C_\epsilon^i = \{\boldsymbol{\theta} : g(\boldsymbol{\theta}, \mathbf{u}_i) \in B_{d,\epsilon}(\mathbf{y}_0)\}$ . Therefore, we can rewrite the approximate likelihood and posterior as:

$$L_{d,\epsilon}(\boldsymbol{\theta}) \approx \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{C_\epsilon^i}(\boldsymbol{\theta}) \quad (7)$$

$$p_{d,\epsilon}(\boldsymbol{\theta} | \mathbf{y}_0) \propto p(\boldsymbol{\theta}) \sum_i^N \mathbb{1}_{C_\epsilon^i}(\boldsymbol{\theta}). \quad (8)$$

As argued by [Ikonov and Gutmann \(2019\)](#), these derivations provide a unique perspective for likelihood-free inference by shifting the focus onto the geometry of the acceptance regions  $C_\epsilon^i$ . Indeed, the task of approximating the likelihood and the posterior becomes a task of characterizing the sets  $C_\epsilon^i$ . OMC by [Meeds and Welling \(2015\)](#) assumes that the distance  $d$  is the Euclidean distance  $\|\cdot\|_2$  between summary statistics  $\Phi$  of the observed and generated data, and that the  $C_\epsilon^i$  can be well approximated by infinitesimally small ellipses. These assumptions lead to an approximation of the posterior in terms of weighted samples  $\boldsymbol{\theta}_i^*$  that achieve the smallest distance between observed and simulated data for each realization  $\mathbf{u}_i \sim p(\mathbf{u})$ , i.e.,

$$\boldsymbol{\theta}_i^* = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \|\Phi(\mathbf{y}_0) - \Phi(g(\boldsymbol{\theta}, \mathbf{u}_i))\|_2, \quad \mathbf{u}_i \sim p(\mathbf{u}). \quad (9)$$

The weighting for each  $\boldsymbol{\theta}_i^*$  is proportional to the prior density at  $\boldsymbol{\theta}_i^*$  and inversely proportional to the determinant of the Jacobian matrix of the summary statistics at  $\boldsymbol{\theta}_i^*$ . For further details on OMC we refer the reader to ([Meeds and Welling 2015](#); [Ikonov and Gutmann 2019](#)).

### 2.3. Robust optimization Monte Carlo (ROMC)

[Ikonov and Gutmann \(2019\)](#) showed that considering infinitesimally small ellipses can lead to highly overconfident posteriors. We refer the reader to their paper for the technical details and conditions for this issue to occur. Intuitively, it happens because the weights in OMC are only computed from information at  $\boldsymbol{\theta}_i^*$ , and using only local information can be misleading.

For example, if the curvature of  $\|\Phi(\mathbf{y}_0) - \Phi(g(\boldsymbol{\theta}, \mathbf{u}_i))\|_2$  at  $\boldsymbol{\theta}_i^*$  is nearly flat, it may wrongly indicate that  $C_\epsilon^i$  is much larger than it actually is. In our software package we implement the robust generalization of OMC by [Ikonov and Gutmann \(2019\)](#) that resolves this issue.

ROMC approximates the acceptance regions  $C_\epsilon^i$  with  $D$ -dimensional bounding boxes  $\hat{C}_\epsilon^i$ . A uniform distribution,  $q_i(\boldsymbol{\theta})$ , is defined on top of each bounding box and serves as a proposal distribution for generating posterior samples  $\boldsymbol{\theta}_{ij} \sim q_i$ . The samples get an (importance) weight  $w_{ij}$  that compensate for using the proposal distributions  $q_i(\boldsymbol{\theta})$  instead of the prior  $p(\boldsymbol{\theta})$ :

$$w_{ij} = \mathbb{1}_{C_\epsilon^i}(\boldsymbol{\theta}_{ij}) \frac{p(\boldsymbol{\theta}_{ij})}{q(\boldsymbol{\theta}_{ij})}. \quad (10)$$

Given the weighted samples, any expectation  $\mathbb{E}_{p(\boldsymbol{\theta}|\mathbf{y}_0)}[h(\boldsymbol{\theta})]$  of some function  $h(\boldsymbol{\theta})$ , can be approximated as

$$\mathbb{E}_{p(\boldsymbol{\theta}|\mathbf{y}_0)}[h(\boldsymbol{\theta})] \approx \frac{\sum_{ij} w_{ij} h(\boldsymbol{\theta}_{ij})}{\sum_{ij} w_{ij}} \quad (11)$$

The approximation of the acceptance regions contains two compulsory and one optional step: (i) solving the optimization problems as in OMC, (ii) constructing bounding boxes around  $C_\epsilon^i$  and, optionally, (iii) refining the approximation via a surrogate model of the distance.

*(i) Solving the deterministic optimization problems*

For each set of nuisance variables  $\mathbf{u}_i, i = \{1, 2, \dots, n_1\}$ , we search for a point  $\boldsymbol{\theta}_i^*$  such that  $d(g(\boldsymbol{\theta}_i^*, \mathbf{u}_i), \mathbf{y}_0) \leq \epsilon$ . In principle,  $d(\cdot)$  can refer to any valid distance function. For the rest of the paper we consider  $d(\cdot)$  as the squared Euclidean distance, as in [Ikonov and Gutmann \(2019\)](#). For simplicity, we use  $d_i(\boldsymbol{\theta})$  to refer to  $d(g(\boldsymbol{\theta}, \mathbf{u}_i), \mathbf{y}_0)$ . We search for  $\boldsymbol{\theta}_i^*$  solving:

$$\boldsymbol{\theta}_i^* = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} d_i(\boldsymbol{\theta}) \quad (12)$$

and we accept the solution only if it satisfies the constraint  $d_i(\boldsymbol{\theta}_i^*) \leq \epsilon$ . If  $d_i(\boldsymbol{\theta})$  is differentiable, Equation 12 can be solved using any gradient-based optimizer. The gradients  $\nabla_{\boldsymbol{\theta}} d_i(\boldsymbol{\theta})$  can be either provided in closed form or approximated by finite differences. If  $d_i$  is not differentiable, Bayesian Optimization ([Shahriari, Swersky, Wang, Adams, and Freitas 2016](#)) can be used instead. In this scenario, apart from obtaining an optimal  $\boldsymbol{\theta}_i^*$ , we can also automatically build a surrogate model  $\hat{d}_i(\boldsymbol{\theta})$  of the distance function  $d_i(\boldsymbol{\theta})$ . The surrogate model  $\hat{d}_i$  can then substitute the actual distance function in downstream steps of the algorithms, with possible computational gains especially in cases where evaluating the actual distance  $d_i(\boldsymbol{\theta})$  is expensive.

*(ii) Estimating the acceptance regions*

Each acceptance region  $C_\epsilon^i$  is approximated by a bounding box  $\hat{C}_\epsilon^i$ . The acceptance regions  $C_\epsilon^i$  can contain any number of disjoint subsets in the  $D$ -dimensional space and any of these subsets can take any arbitrary shape. We should make three important remarks. First, since the bounding boxes are built around  $\boldsymbol{\theta}_i^*$ , we focus only on the connected subset of  $C_\epsilon^i$  that contains  $\boldsymbol{\theta}_i^*$ , which we denote as  $C_{\epsilon, \boldsymbol{\theta}_i^*}^i$ . Second, we want to ensure that the bounding box  $\hat{C}_\epsilon^i$  is big enough to contain on its interior all the volume of  $C_{\epsilon, \boldsymbol{\theta}_i^*}^i$ . Third, we want  $\hat{C}_\epsilon^i$  to

be as tight as possible to  $C_{\epsilon, \theta_i^*}^i$  to ensure high acceptance rate on the importance sampling step that follows. Therefore, the bounding boxes are built in two steps. First, we compute their axes  $\mathbf{v}_m$ , for  $m = \{1, \dots, D\}$  based on the (estimated) curvature of the distance at  $\theta_i^*$ , and, second, we apply a line-search method along each axis to determine the size of the box. We refer the reader to Algorithm 2 for the details. After the bounding boxes construction, a uniform distribution  $q_i$  is defined on each bounding box, and is used as the proposal region for importance sampling.

(iii) *Refining the estimate via a local surrogate model (optional)*

For computing the weight  $w_{ij}$  in Equation 10, we need to check whether the samples  $\theta_{ij}$  drawn from the bounding boxes are inside the acceptance region  $C_\epsilon^i$ . This can be considered to be a safety-mechanism that corrects for any inaccuracies in the construction of  $\hat{C}_\epsilon^i$  above. However, this check involves evaluating the distance function  $d_i(\theta_{ij})$ , which can be expensive if the model is complex. Ikonov and Gutmann (2019) proposed fitting a surrogate model  $\tilde{d}_i(\theta)$  of the distance function  $d_i(\theta)$ , on data points that lie inside  $\hat{C}_\epsilon^i$ . In principle, any regression model can be used as surrogate model. Ikonov and Gutmann (2019) used a simple quadratic model because it has ellipsoidal isocontours, which facilitates replacing the bounding box approximation of  $C_\epsilon^i$  with a tighter-fitting ellipsoidal approximation.

The training data for the quadratic model is obtained by sampling  $\theta_{ij} \sim q_i$  and accessing the distances  $d_i(\theta_{ij})$ . The generation of the training data adds an extra computational cost, but leads to a significant speed-up when evaluating the weights  $w_{ij}$ . Moreover, the extra cost is largely eliminated if Bayesian Optimization with a Gaussian process (GP) surrogate model  $\hat{d}_i(\theta)$  was used to obtain  $\theta_i^*$  in the first step. In this case, we can use  $\hat{d}_i(\theta)$  instead of  $d_i(\theta)$  to generate the training data. This essentially replaces the global GP model with a simpler local quadratic model which is typically more robust.

## 2.4. Engine for likelihood-free inference (ELFI)

**Engine for Likelihood-Free Inference (ELFI)**<sup>3</sup> Lintusaari *et al.* (2018) is a Python package for LFI. We selected to implement ROMC in **ELFI** since it provides convenient modules for all the fundamental components of a probabilistic model (e.g. prior, simulator, summaries etc.). Furthermore, **ELFI** already supports some recently proposed likelihood-free inference methods. **ELFI** handles the probabilistic model as a Directed Acyclic Graph (DAG). This functionality is based on the package **NetworkX** Hagberg, Schult, and Swart (2008), which supports general-purpose graphs. In most cases, the structure of a likelihood-free model follows the pattern of Figure 1; some edges connect the prior distributions to the simulator, the simulator is connected to the summary statistics that, in turn, lead to the output node. Samples can be obtained from all nodes through sequential (ancestral) sampling. **ELFI** automatically considers as parameters of interest, i.e., those we try to infer a posterior distribution, the ones included in the `elfi.Prior` class.

All inference methods of **ELFI** are implemented following two conventions. First, their constructor follows the signature `elfi.<Class name>(<output node>, *arg)`, where `<output node>` is the output node of the simulator-based model and `*arg` are the parameters of the method. Second, they provide a method `elfi.<Class name>.sample(*args)` for drawing samples from the approximate posterior.

<sup>3</sup>Extended documentation can be found <https://elfi.readthedocs.io>

```

# Define the simulator, the summary and the observed data
def simulator(t1, t2, batch_size=1, random_state=None):
    # Implementation comes here. Return 'batch_size'
    # simulations wrapped to a NumPy array.
def summary(data, argument=0):
    # Implementation comes here...
y = # Observed data, as one element of a batch.

# Specify the ELFI graph
t1 = elfi.Prior('uniform', -2, 4)
t2 = elfi.Prior('normal', t1, 5) # depends on t1
SIM = elfi.Simulator(simulator, t1, t2, observed=y)
S1 = elfi.Summary(summary, SIM)
S2 = elfi.Summary(summary, SIM, 2)
d = elfi.Distance('euclidean', S1, S2)

# Run the rejection sampler
rej = elfi.Rejection(d, batch_size=10000)
result = rej.sample(1000, threshold=0.1)

```

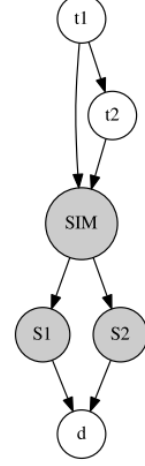


Figure 1: Baseline example for creating an **ELFI** model. Image taken from [Lintusaari et al. \(2018\)](#)

### 3. Implementation

This section is split in two parts. We first express ROMC as an algorithm and then we present the general implementation principles we follow.

#### 3.1. Algorithmic view of ROMC

For designing an extendable implementation, we firstly define ROMC as a sequence of algorithmic steps, which is the driver for the implementation that follows. At a high level, ROMC can be split into the training and the inference part. In broad view, the training part covers all steps for estimating the proposal regions and the inference part calculates the weighted samples. Algorithm 1 defines ROMC formally; Steps 2-11 (before the horizontal line) form the training part and steps 13-18 the inference part.

##### *Training part*

At the training (fitting) part, the goal is the estimation of the proposal regions  $\hat{C}_\epsilon^i$ . The tasks are; (a) sampling the nuisance variables  $\mathbf{u}_i \sim p(\mathbf{u})$  for obtaining the deterministic functions  $d_i(\boldsymbol{\theta})$ , (b) solving the optimisation problems  $\min_{\boldsymbol{\theta}} d_i(\boldsymbol{\theta})$  for obtaining  $\boldsymbol{\theta}_i^*$ ,  $d_i^*$ , and (c) estimate the proposal distribution  $q_i$ .

If  $d_i(\boldsymbol{\theta})$  is differentiable, using a gradient-based method is advised for obtaining  $\boldsymbol{\theta}_i^*$  faster. In this case, the gradients  $\nabla_{\boldsymbol{\theta}} d_i$  gradients are approximated automatically with finite-differences. This approximation requires two evaluations of  $d_i$  for *each* parameter  $\theta_m, m \in \{1, \dots, D\}$ <sup>4</sup>, which works in low-dimensional problems. If  $d_i(\boldsymbol{\theta})$  is not differentiable, Bayesian Optimisation can be used as an alternative solution. In this scenario, the training part becomes slower due to fitting of the surrogate model and the blind optimisation steps.

<sup>4</sup> $D$  is the dimensionality of  $\boldsymbol{\theta}$ , i.e.,  $\boldsymbol{\theta} \in \mathbb{R}^D$



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**Algorithm 1** ROMC. Requires the prior  $p(\boldsymbol{\theta})$ , the simulator  $M_r(\boldsymbol{\theta})$ , number of optimisation problems  $n_1$ , number of samples per region  $n_2$ , acceptance limit  $\epsilon$

---

```

1: procedure ROMC
2:   for  $i \leftarrow 1$  to  $n_1$  do
3:      $\mathbf{u}_i \sim p(\mathbf{u})$  ▷ Draw nuisance variables
4:     Convert  $M_r(\boldsymbol{\theta})$  to  $g(\boldsymbol{\theta}, \mathbf{u} = \mathbf{u}_i)$  ▷ Define deterministic simulator
5:      $d_i(\boldsymbol{\theta}) = d(g(\boldsymbol{\theta}, \mathbf{u} = \mathbf{u}_i), \mathbf{y}_0)$  ▷ Define distance function
6:      $\boldsymbol{\theta}_i^* = \operatorname{argmin}_{\boldsymbol{\theta}} d_i, d_i^* = d_i(\boldsymbol{\theta}_i^*)$  ▷ Solve optimisation problem
7:     if  $d_i^* > \epsilon$  then
8:       Go to 2 ▷ Filter solution
9:     end if
10:    Estimate  $\hat{C}_\epsilon^i$  and define  $q_i$  ▷ Estimate proposal area
11:    (Optional) Fit  $\tilde{d}_i$  on  $\hat{C}_\epsilon^i$  ▷ Fit surrogate model
12:  

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13:  for  $j \leftarrow 1$  to  $n_2$  do
14:     $\boldsymbol{\theta}_{ij} \sim q_i$ , compute  $w_{ij}$  as in Algorithm 3 ▷ Sample
15:  end for
16: end for
17:  $\mathbb{E}_{p(\boldsymbol{\theta}|\mathbf{y}_0)}[h(\boldsymbol{\theta})]$  as in eq. (11) ▷ Estimate an expectation
18:  $p_{d,\epsilon}(\boldsymbol{\theta})$  as in eq. (8) ▷ Evaluate the unnormalised posterior
19: end procedure

```

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After obtaining the optimal points  $\boldsymbol{\theta}^*$ , we estimate the proposal regions. Algorithm 2 describes the line search approach for finding the region's boundaries. An important step for the proposal region estimation is deciding the axes of the bounding box; the directions  $\mathbf{v}_m, m = \{1, \dots, D\}$  we follow for reaching the boundaries. We approximate them as the direction of the highest curvature of  $d_i$  at  $\boldsymbol{\theta}_i^*$ . This estimation is given by the eigenvalues of the Hessian matrix  $\mathbf{H}_i$  of  $d_i$ <sup>5</sup> at  $\boldsymbol{\theta}_i$ . The Hessian matrix is approximated numerically. In case where the distance function is the Euclidean, the Hessian matrix can be also computed as  $\mathbf{H}_i = \mathbf{J}_i^T \mathbf{J}_i$ , where  $\mathbf{J}_i$  is the Jacobian matrix of the summary statistics  $\Phi(g(\boldsymbol{\theta}, \mathbf{u} = \mathbf{u}_i))$  at  $\boldsymbol{\theta}_i^*$ . The approximation through the Jacobian matrix has the computational advantage of using only first-order derivatives.

### Inference Part

Performing the inference includes one or more of the following three tasks; (a) sampling from the posterior  $\boldsymbol{\theta}_i \sim p_{d,\epsilon}(\boldsymbol{\theta}|\mathbf{y}_0)$ , (b) computing an expectation  $\mathbb{E}_{\boldsymbol{\theta}|\mathbf{y}_0}[h(\boldsymbol{\theta})]$  and (c) evaluating the unnormalised posterior  $p_{d,\epsilon}(\boldsymbol{\theta}|\mathbf{y}_0)$ . Sampling is performed by getting  $n_2$  samples from each proposal distribution  $q_i$ . For each sample  $\boldsymbol{\theta}_{ij}$ , the distance function<sup>6</sup> is evaluated for checking if it lies inside the acceptance region. Algorithm 3 defines the steps for computing a weighted sample. Computing the expectation is easy after weighted samples are obtained using the equation 11, so we do not discuss it separately. Evaluating the unnormalised posterior requires the deterministic functions  $g_i$  and the prior distribution  $p(\boldsymbol{\theta})$ <sup>7</sup>. The evaluation

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<sup>5</sup>Either the real distance  $d_i$  or the Gaussian Process approximation  $\hat{d}_i$

<sup>6</sup>As before, if a surrogate model  $\hat{d}$  is available, it can be utilised as the distance function.

<sup>7</sup>There is no need for solving the optimisation problems and building the proposal regions.



requires iterating over all  $g_i$  and evaluating the distance from the observed data.

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**Algorithm 2** Approximation  $C_\epsilon^i$  with a bounding box  $\hat{C}_\epsilon^i$ ; Requires: a model of distance  $d_i(\boldsymbol{\theta})$ , an optimal point  $\boldsymbol{\theta}_i^*$ , a number of refinements  $K$ , a step size  $\eta\_start$ , maximum iterations  $M$  and a curvature matrix  $\mathbf{H}_i$  ( $\mathbf{J}_i^T \mathbf{J}_i$  or GP Hessian)

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```

1: Compute eigenvectors  $\mathbf{v}_m$  of  $\mathbf{H}_i$  ( $d = 1, \dots, D$ )
2: for  $m \leftarrow 1$  to  $D$  do
3:    $\tilde{\boldsymbol{\theta}} \leftarrow \boldsymbol{\theta}_i^*$ 
4:    $k \leftarrow 0$ 
5:    $\eta \leftarrow \eta\_start$  ▷ Initialize  $\eta$ 
6:   repeat
7:      $j \leftarrow 0$ 
8:     repeat
9:        $\tilde{\boldsymbol{\theta}} \leftarrow \tilde{\boldsymbol{\theta}} + \eta \mathbf{v}_m$  ▷ Large step size  $\eta$ .
10:       $j \leftarrow j + 1$ 
11:      until  $d(g(\tilde{\boldsymbol{\theta}}, \mathbf{u} = \mathbf{u}_i), \mathbf{y}_0) > \epsilon$  or  $j \geq M$  ▷ Check distance or maximum iterations
12:       $\tilde{\boldsymbol{\theta}} \leftarrow \tilde{\boldsymbol{\theta}} - \eta \mathbf{v}_m$ 
13:       $\eta \leftarrow \eta/2$  ▷ More accurate region boundary
14:       $k \leftarrow k + 1$ 
15:    until  $k = K$ 
16:    if  $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_i^*$  then ▷ Check if no step has been done
17:       $\tilde{\boldsymbol{\theta}} \leftarrow \tilde{\boldsymbol{\theta}} + \frac{\eta\_start}{2^K} \mathbf{v}_m$  ▷ Then, make the minimum step
18:    end if
19:    Set  $\tilde{\boldsymbol{\theta}}$  as the positive end point along  $\mathbf{v}_m$ 
20:    Run steps 3 - 18 for  $\mathbf{v}_m = -\mathbf{v}_m$  and set  $\tilde{\boldsymbol{\theta}}$  as the negative end point along  $\mathbf{v}_m$ 
21:  end for
22: Fit a rectangular box around the region end points and define  $q_i$  as uniform distribution

```

---



---

**Algorithm 3** Sampling. Requires a function of distance  $d_i$ , the prior distribution  $p(\boldsymbol{\theta})$ , the proposal distribution  $q_i$

---

```

1:  $\boldsymbol{\theta}_{ij} \sim q_i$ 
2: if  $d_i(\boldsymbol{\theta}_{ij}) > \epsilon$  then
3:   Go to 2 ▷ Reject sample
4: else
5:    $w_{ij} = \frac{p(\boldsymbol{\theta}_{ij})}{q(\boldsymbol{\theta}_{ij})}$  ▷ Compute weight
6:   Store  $(w_{ij}, \boldsymbol{\theta}_{ij})$  ▷ Store weighted sample
7: end if

```

---

### 3.2. General implementation principles

The overview of our implementation is illustrated in Figure 2. Following Python's naming principles, the methods starting with an underscore (green rectangles) represent internal (private) functions, whereas the rest (blue rectangles) are the methods exposed as the public API. In Figure 2, it can be easily observed that the implementation follows Algorithm 1. The

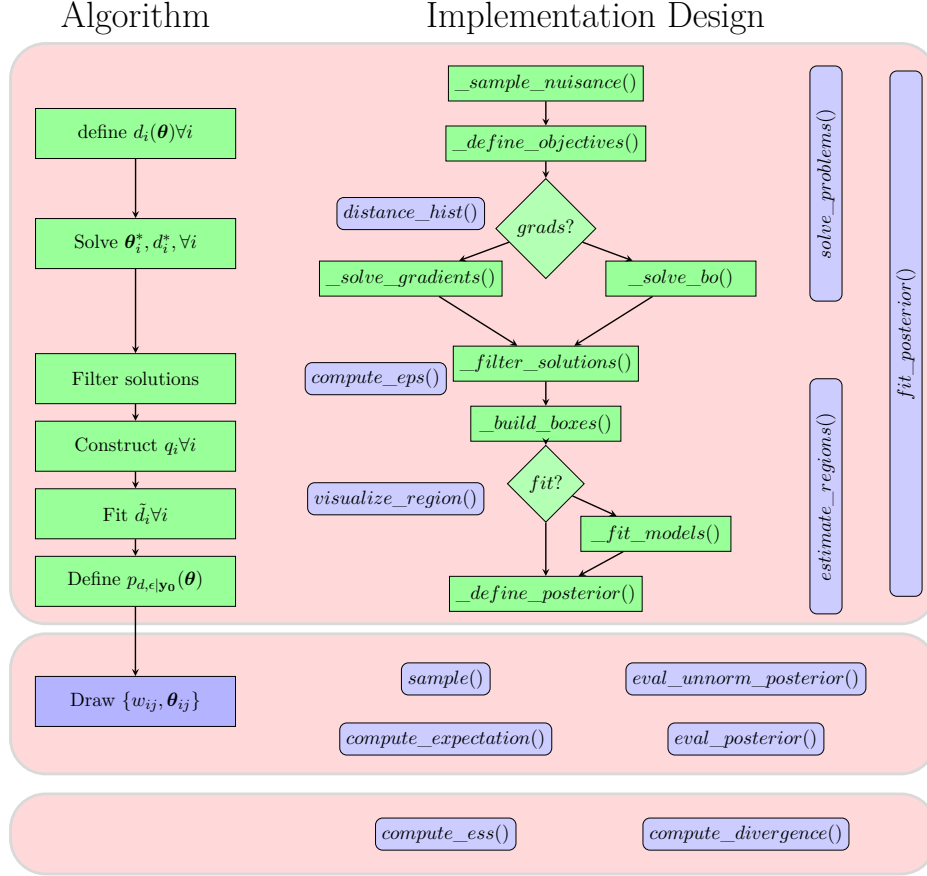


Figure 2: Overview of the ROMC implementation. On the left side, we depict ROMC as a sequence of algorithmic steps. On the right side, we present the functions that form our implementation; the green rectangles (starting with underscore) are the internal functionalities and the blue rectangles the publicly exposed API. This side-by-side illustration highlights that our implementation follows strictly the algorithmic view of ROMC.

training part includes all the steps until the computation of the proposal regions, i.e., sampling the nuisance variables, defining the optimisation problems, solving them, constructing the regions and fitting local surrogate models. The inference part comprises of evaluating the unnormalised posterior (and the normalised when is possible), sampling and computing an expectation. We also provide some utilities for inspecting the training process, such as plotting the histogram of the final distances or visualising the constructed bounding boxes. Finally, in the evaluation part, we provide two methods for evaluating the inference; (a) computing the Effective Sample Size (ESS) of the samples and (b) measuring the divergence between the approximate posterior the ground-truth, if the latter is available.<sup>8</sup>

<sup>8</sup>Normally, the ground-truth posterior is not available; However, this functionality is useful in cases where the posterior can be computed numerically or with an alternative method, i.e., ABC Rejection Sampling), and we would like to measure the discrepancy between the two approximations.

*Parallelising ROMC method*

As discussed, ROMC has the significant advantage of being fully parallelisable. We exploit this fact by implementing a parallel version of the major fitting components; (a) solving the optimisation problems, (b) constructing bounding box regions. We parallelise these processes using the built-in Python package **multiprocessing**. The specific package enables concurrency, using subprocesses instead of threads, for side-stepping the Global Interpreter (GIL). For activating the parallel version of the algorithm, the user simply has to pass the argument `parallelize=True` when instantiating ROMC.

```
elfi.ROMC(<output_node>, parallelize = True)
```

*Simple one-dimensional example*

For illustrating the functionalities we will use the following running example introduced by [Ikononov and Gutmann \(2019\)](#),

$$p(\theta) = \mathcal{U}(\theta; -2.5, 2.5) \quad (13)$$

$$p(y|\theta) = \begin{cases} \theta^4 + u & \text{if } \theta \in [-0.5, 0.5] \\ |\theta| - c + u & \text{otherwise} \end{cases} \quad (14)$$

$$u \sim \mathcal{N}(0, 1) \quad (15)$$

The prior is a uniform distribution in the range  $[-2.5, 2.5]$  and the likelihood is defined at 14. The constant  $c$  is  $0.5 - 0.5^4$  ensures the continuity of the pdf. There is only one observation  $y_0 = 0$ . The inference in this particular example can be performed quite easily without using a likelihood-free inference approach. We can exploit this fact for validating the accuracy of our implementation.

In the following code snippet, we code the model at **ELFI**:

```
import elfi
import scipy.stats as ss
import numpy as np

def simulator(t1, batch_size = 1, random_state = None):
    c = 0.5 - 0.5**4
    if t1 < -0.5:
        y = ss.norm(loc = -t1-c, scale = 1).rvs(random_state = random_state)
    elif t1 <= 0.5:
        y = ss.norm(loc = t1**4, scale = 1).rvs(random_state = random_state)
    else:
        y = ss.norm(loc = t1-c, scale = 1).rvs(random_state = random_state)
    return y

# observation
y = 0
```

```
# Elfi graph
t1 = elfi.Prior('uniform', -2.5, 5)
sim = elfi.Simulator(simulator, t1, observed = y)
d = elfi.Distance('euclidean', sim)

# Initialise the ROMC inference method
bounds = [(-2.5, 2.5)] # bounds of the prior
parallelize = True # activate parallel execution
romc = elfi.ROMC(d, bounds = bounds, parallelize = parallelize)
```

## 4. Implemented functionalities

In this section, we present the implementation, dividing it into three parts; at 4.1 we present the fitting part, at 4.2 the inference part and at 4.3 the evaluation part. Finally, at 4.4 we describe how a user may extend ROMC with its custom modules.

### 4.1. Training part

```
romc.solve_problems(n1, use_bo = False, optimizer_args = None)
```

This method (a) draws the nuisance variables, (b) defines the optimisation problems and (c) solves them using either a gradient-based optimiser or the Bayesian optimisation (BO) scheme. The three tasks are completed sequentially, as shown in Figure 2. The definition of the optimisation problems is performed by drawing  $n1$  integer numbers from a discrete uniform distribution  $u_i \sim \mathcal{U}\{1, 2^{32} - 1\}$ . Each integer  $u_i$  is the seed used in **ELFI**'s random simulator. The user may select the Bayesian Optimisation scheme by setting the argument `use_bo = True` and pass custom arguments to the optimizer through `optimize_args`.

```
romc.distance_hist(**kwargs)
```

This function helps the user decide which threshold  $\epsilon$  to use, by plotting a histogram of the distances at the optimal point  $d_i(\theta_i^*) : \{i = 1, 2, \dots, n_1\}$  or  $\hat{d}_i^*$  in case `use_bo=True`. The function accepts all keyword arguments and forwards them to the underlying `pyplot.hist()` function of the package **matplotlib**. In this way the user may customise some properties of the histogram, such as the number of bins or the range of values.

```
romc.estimate_regions(eps_filter, use_surrogate=None, fit_models=False)
```

This method estimates the proposal region around the optimal points, following Algorithm 2. By default, the distance model that will be used follows the decision from the previous step; if a gradient-based optimizer has been used, then the real distance function  $d$  will be chosen. If BO, then then the surrogate model  $\hat{d}$ . In case, the user wants to enforce using the original distance function  $d$  after BO, they may set `use_surrogate=False`. Finally, the option `fit_models` defines whether to fit local surrogate models  $\tilde{d}$  after estimating the proposal regions.

```
romc.fit_posterior(args*) # training in a single call
romc.visualize_region(i)  # acceptance region
romc.compute_eps(quantile) # estimates eps
```

---

The function `fit_posterior` is a syntactic sugar for applying `solve_problems` and `estimate_regions` into a single step. The function `visualize_region` can be used for plotting the bounding box around the optimal point, when the parameter space is up to 2D. The argument `i` is the index of the corresponding optimisation problem. Finally, `compute_eps` returns the appropriate distance value  $d_{i=\kappa}^*$  where  $\kappa = \lfloor \text{quantile} \cdot n \rfloor$  from the collection  $\{d_i^*\} \forall i = \{1, \dots, n\}$  where  $n$  is the number of accepted solutions. It can be used to automate the selection of the threshold  $\epsilon$ , e.g., `eps=romc.compute_eps(quantile=0.9)`.

### Example

In the following snippet, we put together the routines described above to perform the training part at our running example.

```
# Training (fitting) part
n1 = 500 # number of optimisation problems
seed = 21 # seed for solving the optimisation problems
use_bo = False # set to True for switching to Bayesian optimisation

# Training step-by-step
romc.solve_problems(n1 = n1, seed = seed, use_bo = use_bo)
romc.theta_hist(bins = 100) # plot hist to decide which eps to use

eps = .75 # threshold for the bounding box
romc.estimate_regions(eps = eps) # build the bounding boxes

romc.visualize_region(i = 1) # for inspecting visually the bounding box

# Equivalent one-line command
# romc.fit_posterior(n1 = n1, eps = eps, use_bo = use_bo, seed = seed)
```

## 4.2. Inference part

```
romc.sample(n2)
```

This is the basic inference utility of the ROMC implementation; we draw  $n_2$  samples for each bounding box region. This gives a total of  $k \times n_2$ , where  $k < n_1$  is the number of the optimal points remained after filtering<sup>9</sup>. The samples are drawn from a uniform distribution  $q_i$  defined over the corresponding bounding box and the weight  $w_i$  is computed as in Algorithm 3.

---

<sup>9</sup>From the  $n_1$  optimisation problems, only the ones with  $d_i(\theta^*) < \epsilon$  are maintained for building a bounding box

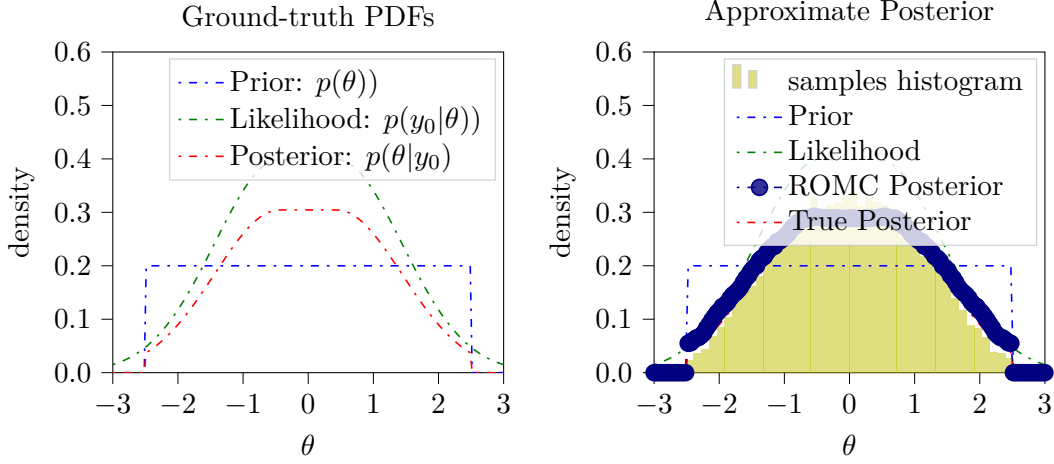


Figure 3: Histogram of distances and visualisation of a specific region.

```
romc.compute_expectation(h) # E[h(x)|theta]
romc.eval_unnorm_posterior(theta, eps_cutoff = False) # unnorm p(theta)
romc.eval_posterior(theta, eps_cutoff = False) # normalised p(theta)
```

The function `compute_expectation` computes the expectation  $E_{p(\theta|y_0)}[h(\theta)]$  using expression (11). The argument `h` can be any python Callable. The method `eval_unnorm_posterior` computes the unnormalised posterior approximation using the expression (3). The method `eval_posterior` evaluates the normalised posterior estimating the partition function  $Z = \int p_{d,\epsilon}(\theta|y_0)d\theta$ ; with Riemann's integral approximation. The approximation is computationally feasible only in a low-dimensional parametric space.

#### Example - Sampling and compute expectation

In the following code snippet, we use the inference utilities to (a) get weighted samples from the approximate posterior, (b) compute an expectation and (c) evaluate the approximate posterior. We also use some of **ELFI**'s built-in tools to get a summary of the obtained samples. For the utility `compute_expectation`, we demonstrate how to use it in order to compute the samples mean and variance. Finally, we evaluate `eval_posterior` at multiple points to plot the approximate posterior of Figure 3. We observe that the approximation is quite close to the ground-truth.

```
# Inference part
seed = 21
n2 = 50
romc.sample(n2 = n2, seed = seed)

# visualize region, adding the samples now
romc.visualize_region(i = 1)

# Visualise marginal (built-in ELFI tool)
```

```

weights = romc.result.weights
romc.result.plot_marginals(weights = weights, bins = 100, range = (-3,3))

# Summarize the samples (built-in ELFI tool)
romc.result.summary()
# Number of samples : 19300
# Sample means: theta: -0.0116

# compute expectation
exp_val = romc.compute_expectation(h = lambda x: np.squeeze(x))
print("Expected value : %.3f" % exp_val)
# Expected value: -0.012

exp_var = romc.compute_expectation(h = lambda x: np.squeeze(x)**2)
print("Expected variance: %.3f" % exp_var)
# Expected variance: 1.120

# eval unnorm posterior
romc.eval_unnorm_posterior(theta = 0)

# check eval posterior
romc.eval_posterior(theta = 0)

```

### 4.3. Evaluation part

```

romc.compute_divergence(gt_posterior, bounds, step, distance)
romc.compute_ess()

```

The utility `compute_divergence` estimate the divergence between the ROMC approximation and the ground truth posterior. Since the estimation is performed using the Riemann's approximation, the method can only work in low dimensional spaces. As mentioned at the beginning of this chapter, in a real-case scenario it is not expected the ground-truth posterior to be available. In cases the ground truth posterior is not available (as in real-scenarios), the user may select the approximation obtained with any other inference approach for comparing the two methods. The argument `step` defines the step used in the Riemann's approximation and the argument `distance` can be either "Jensen-Shannon" or "KL-divergence".

The method `compute_ess` computes the Effective Sample Size (ESS) using the following expression,

$$ESS = \frac{(\sum_i w_i)^2}{\sum_i w_i^2} \quad (16)$$

The ESS is a valuable measure of the **actual** sample size in cases of weighted samples. For example, there are cases where in a big population, a single sample with huge weight dominates. The ESS provides a nice metric in these cases.



```

# Evaluation part
res = romc.compute_divergence(wrapper, distance = "Jensen-Shannon")
print("Jensen-Shannon divergence: %.3f" % res)
# Jensen-Shannon divergence: 0.035

nof_samples = len(romc.result.weights)
ess = romc.compute_ess()
print("Nof Samples: %d, ESS: %.3f" % (nof_samples, ess))
# Nof Samples: 19300, ESS: 16196.214

```

#### 4.4. Extensibility

ROMC describes a sequence of steps for approximating the posterior distribution, without explicitly enforcing the methods that solve these steps. Even though for each step a specific algorithm is proposed by [Ikonov and Gutmann \(2019\)](#), in general ROMC is functional if a practitioner thinks of an alternative way of approaching a specific step. Considering this particularity, we designed the implementation to support extensibility.

We have specified four specific points where a user may intervene with their custom modules; (a) the gradient-based optimisation, (b) the Bayesian Optimisation, (c) the proposal region construction and (d) the surrogate model fitting. These are the four critical parts of the ROMC procedure, whereas the rest of the code is the backbone of the algorithm.

The four replaceable parts described above, are solved using the four methods of the `OptimisationProblem` class; (a) `solve_gradients(**kwargs)`, (b) `solve_bo(**kwargs)`, (c) `build_region(**kwargs)`, (d) `fit_local_surrogate(**kwargs)`. The user can create a custom class that inherits the basic `OptimisationProblem` and, then, overwrite any of the four functions with custom ones. Suppose a user wants to fit Neural Networks as local surrogate models  $\tilde{d}_i$ . The user should overwrite the `fit_local_surrogate(**kwargs)` function with one that internally trains a neural network. In the following snippet we illustrate how to do that, using the `neural_network.MLPRegressor` class of the **scikit-learn** package.

```

class CustomOptim(OptimisationProblem):
    def __init__(self, **kwargs):
        super(CustomOptim, self).__init__(**kwargs)

    def fit_local_surrogate(self, **kwargs):
        nof_samples = 500
        objective = self.objective

        # helper function
        def local_surrogate(theta, model_scikit):
            assert theta.ndim == 1
            theta = np.expand_dims(theta, 0)
            return float(model_scikit.predict(theta))

        # create local surrogate model as a function of theta
        def create_local_surrogate(model):

```

```

        return partial(local_surrogate, model_scikit = model)

    local_surrogates = []
    for i in range(len(self.regions)):
        # prepare dataset
        x = self.regions[i].sample(nof_samples)
        y = np.array([objective(ii) for ii in x])

        # train Neural Network
        mlp = MLPRegressor(hidden_layer_sizes = (10,10), solver = 'adam')
        model = Pipeline(['linear', mlp])
        model = model.fit(x, y)
        local_surrogates.append(create_local_surrogate(model))

    self.local_surrogates = local_surrogates
    self.state["local_surrogates"] = True

```

In the same way, the user may replace each of the other three functionalities. The only restriction that must be respected concerns the side-effects each method has at the `OptimisationProblem` class level. In the following four snippets we present the signature of each method and we name the class-level variables that must be set under the comment `# side-effects`.

```

-----
def solve_gradients(self, **kwargs):
    # custom solution procedure
    x = ...
    y = ...
    jac = ...
    hess_inv = ...

    # side-effects
    self.state["attempted"] = True
    if success:
        self.result = RomcOpimisationResult(x, y, jac, hess_inv)
        self.state["solved"] = True
        return True
    else:
        return False
-----

def solve_bo(self, **kwargs):
    # custom procedure
    x = ...
    y = ...
    custom_surrogate = ...

    # side-effects
    self.state["attempted"] = True

```

```

    if success:
        self.result = RomcOpimisationResult(x, y)
        self.surrogate = custom_surrogate
        self.state["solved"] = True
        self.state["has_fit_surrogate"] = True
        return True
    else:
        return False
-----

def build_region(self, **kwargs):
    # custom build_region method
    bounding_box: List[NdimBoudningBox] = ...
    success = True/False # whether region built correctly

    # side-effects
    self.eps_region = eps_region
    if success:
        # construct region
        self.regions = bounding_box
        self.state["region"] = True
        return True
    else:
        return False
-----

def fit_local_surrogate(self, **kwargs):
    # custom local surrogates
    custom_surrogates = ...
    success = True/False # whether local surrogates fit correctly

    # side-effects
    if success:
        self.local_surrogate = local_surrogates
        self.state["local_surrogates"] = True
        return True
    else:
        return False
-----

```

The two classes that may be needed for creating the custom routines are

(a) `RomcOpimisationResult` and (b) `NdimBoundingBox`. We present their signatures below.

```

-----
class RomcOpimisationResult:
    def __init__(self, x_min, f_min, hess_appr):
        Parameters
        -----
        x_min: np.ndarray (D,) or float, the minimum point

```

```

f_min: float, distance at the minimum point
hess_appr: np.ndarray (DxD), Hessian approximation at x_min
"""
-----
class NDimBoundingBox:
    def __init__(self, rotation, center, limits, eps_region):
        Parameters
        -----
        rotation: np.array (D,D), rotation matrix for the bounding box
        center: np.array (D,) center of the bounding box
        limits: np.ndarray, shape: (D,2), the limits of the bounding box
        eps_region: float, distance threshold
-----

```

## 5. Use-case illustration

In this section, we test the implementation using the second-order moving average (MA2) example, which is one of the standard models of **ELFI**. We perform the inference using three different versions of ROMC; (i) using a gradient-based optimiser, (ii) using the Bayesian Optimisation scheme and (iii) fitting a Neural Network as a surrogate model. The later illustrates how to extend the implementation, replacing part of ROMC with a user-defined component. Finally, we measure the execution speed-up achieved by the parallelised version of ROMC.

### *Model Definition*

MA2 is a probabilistic model for time series analysis. The observation at time  $t$  is given by,

$$y_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}, \quad t = 1, \dots, T \quad (17)$$

$$\theta_1, \theta_2 \in \mathbb{R}, \quad w_k \sim \mathcal{N}(0, 1), k \in \mathbb{Z} \quad (18)$$

The r.v.  $w_k \sim \mathcal{N}(0, 1)$  is white noise and the two parameters of interest,  $\theta_1, \theta_2$ , model the dependence from the previous observations. The number of sequential observations  $T$  is a constant and set to  $T = 100$ . For securing that the inference problem is identifiable, i.e., the likelihood has only one mode, we use the prior proposed by [Marin, Pudlo, Robert, and Ryder \(2012\)](#),

$$p(\boldsymbol{\theta}) = p(\theta_1)p(\theta_2|\theta_1) = \mathcal{U}(\theta_1; -2, 2)\mathcal{U}(\theta_2; \theta_1 - 1, \theta_1 + 1) \quad (19)$$

The observation vector  $\mathbf{y}_0 = (y_1, \dots, y_{100})$  is generated with  $\boldsymbol{\theta}^* = (0.6, 0.2)$ . The dimensionality of the output  $\mathbf{y}$  is high, therefore we use summary statistics. Considering that the output vector represents a time-series signal, we select the autocovariances with lag = 1 and lag = 2, as shown in equations (20) and (21). The final distance node is the squared Euclidean distance (23).

	$\mu_{\theta_1}$	$\sigma_{\theta_1}$	$\mu_{\theta_2}$	$\sigma_{\theta_2}$
Rejection ABC	0.516	0.142	0.07	0.172
ROMC (gradient-based)	0.501	0.142	0.033	0.169
ROMC (Bayesian optimisation)	0.494	0.16	0.086	0.167
ROMC (Neural Network)	0.491	0.138	0.04	0.172

Table 1: Comparison of the samples obtained from the estimated posterior with (a) Rejection sampling and (b) the different versions of ROMC. We observe that the obtained samples share similar statistics along all methods.

$$s_1(\mathbf{y}) = \frac{1}{T-1} \sum_{t=2}^T y_t y_{t-1} \quad (20)$$

$$s_2(\mathbf{y}) = \frac{1}{T-2} \sum_{t=3}^T y_t y_{t-2} \quad (21)$$

$$s(\mathbf{y}) = (s_1(\mathbf{y}), s_2(\mathbf{y})) \quad (22)$$

$$d = \|s(\mathbf{y}) - s(\mathbf{y}_0)\|_2^2 \quad (23)$$

### Inference

In order to show all the capabilities of the implementation, we perform the inference (i) using the gradient-based optimizer, (ii) using the Bayesian Optimisation scheme and (iii) fitting a Neural Network (NN) as a surrogate model. We use the Rejection ABC algorithm for evaluating all approaches. Replacing the typical quadratic surrogate model with a NN serves as an illustrator of the extensibility of our implementation. The replacement is done by coding a custom optimisation function with a surrogate model of our own preference, as shown in Chapter 4.4. For the definition of the NN, we use the **MLPRegressor** class of the **scikit-learn** package. Therefore, the NN substitutes the real distance function  $d_i$  inside the proposal region  $q_i \forall i$  at the inference phase, i.e., sampling, computing an expectation and evaluating the posterior. In our example we use a neural network of two hidden layers of 10 neurons each and we train it sampling 500 examples from each proposal region.

In Figure 4, we illustrate the acceptance region of the same deterministic simulator, in the gradient-based and the Bayesian optimisation case. The acceptance regions are quite similar even though the different optimisation schemes lead to different optimal points.

In Figure 5, we demonstrate the histograms of the marginal posteriors, for each approach; (a) Rejection ABC (first column), (b) ROMC with gradient-based optimisation (second column) (c) ROMC with Bayesian optimisation (third column) and (d) ROMC with the NN extension. We observe a significant agreement between the different approaches. At Table 1 we present the empirical mean  $\mu$  and standard deviation  $\sigma$  for each inference approach and finally, in Figure 6, we illustrate the unnormalised posterior for the three different variations of the ROMC method. The results show that all ROMC variations provide consistent results between them and in comparison with the standard Rejection ABC algorithm.

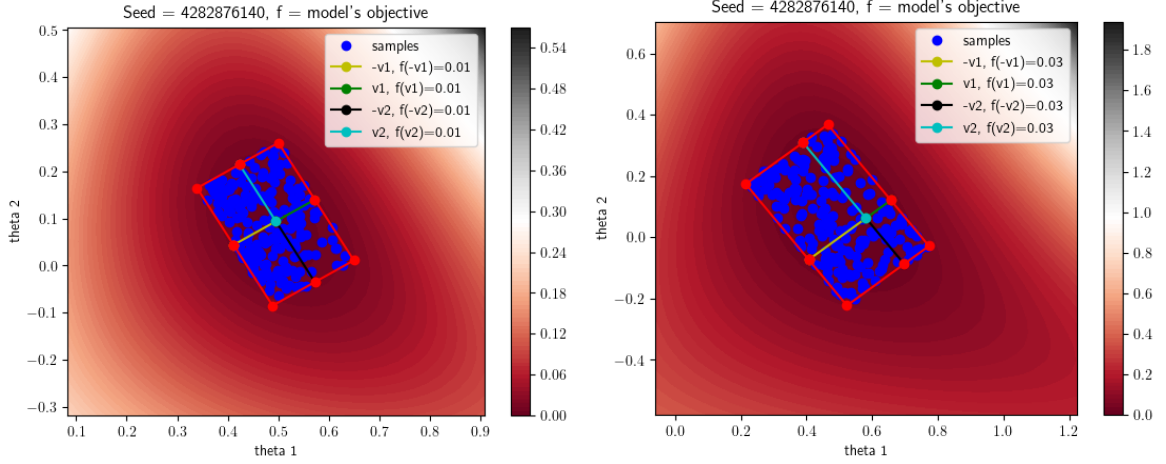


Figure 4: The acceptance region in a specific optimisation problem. In the left figure the region obtained with gradient-based optimiser and in the right one with Bayesian Optimisation.

### Parallelisation

As stated above, ROMC is a ridiculously parallelisable method. Therefore, it is straightforward to parallelise the fitting part, i.e., (i) solving the optimisation problems and (ii) estimating the proposal regions. Our implementation supports exploiting all the available CPU cores through the built-in Python package **multiprocess**<sup>10</sup>. In Figure 7 we observe the execution times for performing the inference on the MA2 model; the parallel version performs both tasks almost five times faster than the sequential. The result is reasonable given that the experiments have run in a single machine with the Intel® Core™ i7-8750H Processor, which has six separate cores.

## 6. Summary and discussion

In this paper, we presented the implementation details we followed for developing the LFI method ROMC at the **ELFI** package. We paid thorough attention to two specific use-case scenarios. Firstly, we illustrate how a user may take advantage of our ready-to-use API for solving its LFI problem. Secondly, we focus on the scenario where a researcher wants to intervene and alter parts of the method. Our implementation is designed to support this as well.

There are still open challenges for the left for future research. Two directions may enable ROMC to solve high-dimensional problems efficiently. The first one is enabling ROMC's execution into a cluster of computers. ROMC can be characterized as an *embarrassingly parallel* workload; each optimization problem is an entirely independent task. Therefore, supporting inference into a cluster of computers can radically speed up the inference. The second one refers to implementing the method in a framework that supports automatic differentiation. Automatic differentiation is necessary for efficiently solving optimisation problems, especially in high-dimensional parametric models.

<sup>10</sup><https://docs.python.org/3/library/multiprocessing.html>

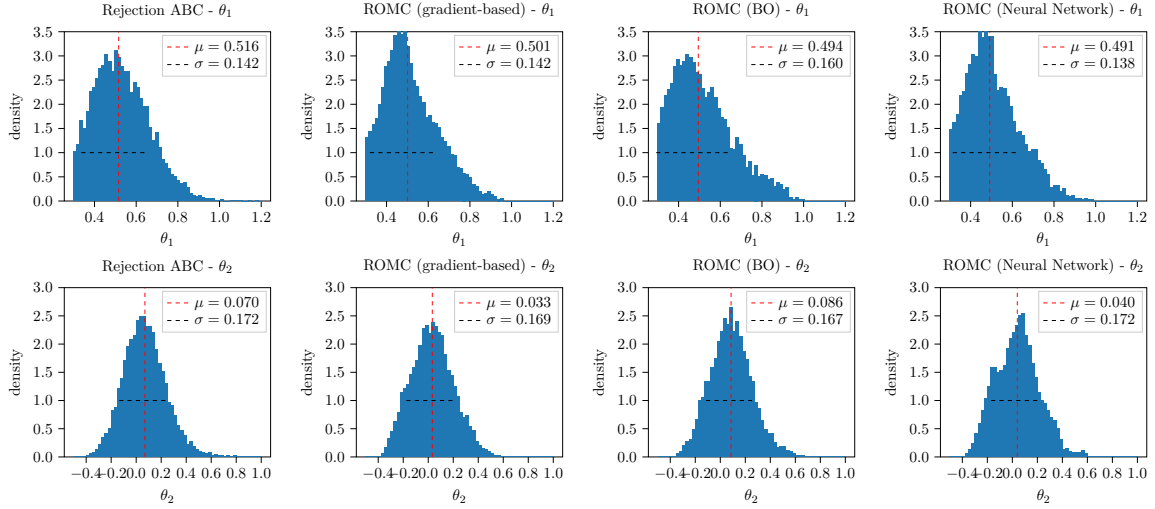


Figure 5: Histogram of the marginal posterior distributions using three different inference approaches; (a) in the first row, the samples are obtained using Rejection ABC sampling (b) in the second row, using ROMC with a gradient-based optimiser and (c) in the third row, using ROMC with Bayesian optimisation approach. The vertical (red) line represents the samples mean  $\mu$  and the horizontal (black) the standard deviation  $\sigma$ .

## Computational details

The results in this paper were obtained using Python 3.7.9 with the **ELFI** 0.8.3 package. The experiments have been executed in a single machine with an Intel® Core™ i7-8750H Processor and with Ubuntu 20.04 LTS operating system.

## Acknowledgments

HP was funded by European Research Council grant 742158 (SCARABEE, Scalable inference algorithms for Bayesian evolutionary epidemiology).



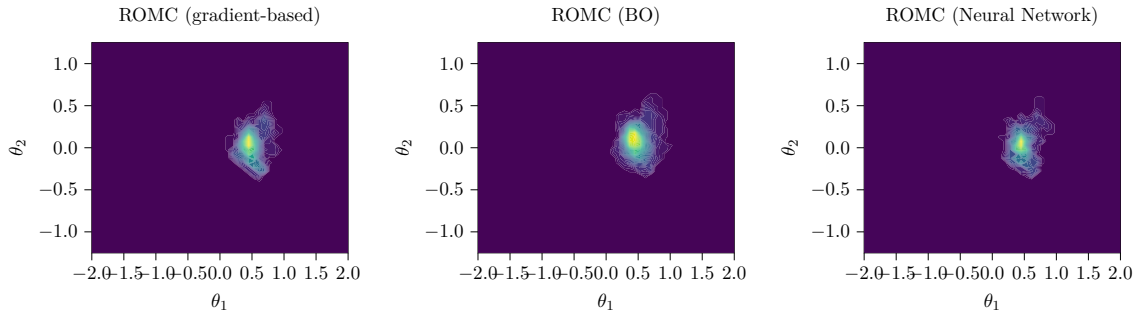


Figure 6: The unnormalised posterior distribution using the ROMC method with (a) a gradient-based optimisation (b) Bayesian Optimisation (c) gradient-based with a Neural Network as a surrogate model.

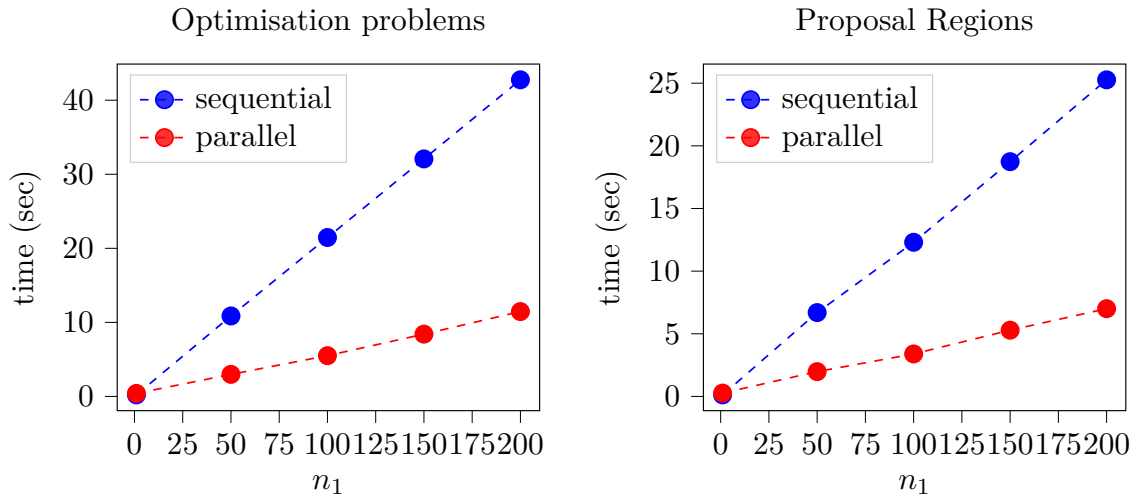


Figure 7: Comparison between parallel and sequential execution of ROMC. We observe that the parallel version runs almost 5 times faster.

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*Journal of Statistical Software*

published by the Foundation for Open Access Statistics

MMMMMM YYYY, Volume VV, Issue II

doi:10.18637/jss.v000.i00

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Submitted: yyyy-mm-dd

Accepted: yyyy-mm-dd

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