# Explaining Reverse Chain Rule

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## 1 The Equation

Here is a more consise form of the equation for reverse chain rule.

$$\int f \circ g dx = \int \frac{f \circ g}{g'} dg$$

## 2 Using My Method

Instead of defining a $u$, we do the same process of finding the "inner" function, but set that as the expression you are integrating by, rather than first defining it as $u$.

$$\int x \sin x^2 dx \to \int x \sin x^2 dx^2$$

This is essentially the same as.

$$\int x \sin x^2 dx \to \int x \sin u du$$

Then you divide the expression being integrated by the derivative of the inside function.

$$\int x \sin x^2 dx^2 \to \int \frac{x \sin x^2}{2x} dx^2$$

Which is essentially the same as.

$$\int x \sin u du \to \int \frac{x \sin u}{2x} du$$

Next you simplify.

$$\int \frac{x \sin x^2}{2x} dx^2 \to \frac{1}{2} \int \sin x^2 dx^2$$

And finish integrating, keeping in mind that you are integrating by the "inner" function, not $x$.

$$\frac{1}{2} \int \sin x^2 dx^2 \to \frac{1}{2}(-\cos x^2)$$

In summary, notice how little we had to actually write to integrate.

$$\int x \sin x^2 dx$$

$$\int \frac{x \sin x^2}{2x} dx^2$$

$$\int \frac{1}{2} \sin x^2 dx^2$$

$$-\frac{1}{2} \cos x^2$$

# 3 Understanding My Method

Perhaps you might agree that this method saves time on some problems. But you might wonder if this method comes at the cost of understanding, making someone more confused. I would argue that this method has the potential to do the opposite. Let me show how this function is extracted from the chain rule.

$$(f(g(x)))' = f'(g(x))g'(x)$$

Now we are going to take the antiderivative of $f$, the rule is still the same, just easier for us to work with since now the we have an $f$ on the right side rather than an $f'$.

$$(\int f(g(x))dg(x))' = f(g(x))g'(x)$$

Notice that we are integrating by $g(x)$, not by $x$, this is since we only took the antiderivative of $f$, NOT $g$. We are basically composing the functions $\int f(u)du$ (which is the antiderivative of $f$) and $g(x)$ to get $\int f(g(x))dg(x)$.

Now we are going to replace $f$ with $f/g'(x)$, so that after we simplify, we can get $f(g(x))$ on one side of the equation.

$$(\int \frac{f(g(x))}{g'(x)} dg(x))' = \frac{f(g(x))}{g'(x)} g'(x)$$

$$(\int \frac{f(g(x))}{g'(x)} dg(x))' = f(g(x))$$

Now take the antiderivative on both sides. This cancels out with the derivative on the left side, which leaves us with:

$$\int \frac{f(g(x))}{g'(x)} dg(x) = \int f(g(x))dx$$

That is the reverse chain rule.

Now lets prove that this is truely the reverse of the chain rule by taking the derivative of both sides.

$$(\int \frac{f(g(x))}{g'(x)} dg(x))' = f(g(x))$$

If this is really the reverse of the chain rule you would expect applying the chain rule to the left side will result in $f(g(x))$. Of course the "inner" function on the left side is $g(x)$ and the "outer" function is $\int \frac{f(u)}{g'(x)} du$. (The derivative of the outer function should be $\frac{f(u)}{g'(x)}$)

$$\frac{f(g(x))}{g'(x)} g'(x) = f(g(x))$$

Which simplifies.

$$f(g(x)) = f(g(x))$$

I am not adding $+C$ because screw you.