

Stat 155 Midterm Spring 2014

Name:

SID:

This exam has 5 problems and a total of 75 points. Attempt all questions and show your working - solutions without explanation will **not** receive full credit. One double sided sheets of notes are permitted. Answer questions in the space provided. If additional space is needed use the additional space on the final pages.

Q 1 Find the value and optimal strategies for both players in the following payoff matrices for zero-sum games.

$$(a)[5 \text{ points}] \quad \begin{pmatrix} 2 & 4 \\ 7 & 3 \end{pmatrix} \quad (b)[10 \text{ points}] \quad \begin{pmatrix} 2 & 3 & 8 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 8 \end{pmatrix}$$

(c) [10 points] Player 1 and 2 each choose a number from $\{1, 2, 3\}$. If they choose the same number then player 1 receives \$1 from player 2 while if player 2 chooses a larger number then player 1 pays \$1 to player 2. If player 2 chooses a smaller number then nothing is exchanged. Write down the payoff matrix and calculate the value of the game.

Solutions

(a) This 2 by 2 matrix has no saddle points so we can use the formula. The optimal strategies are

$$x = \left(\frac{3-7}{3-7+2-4}, \frac{2-4}{3-7+2-4} \right) = \left(\frac{2}{3}, \frac{1}{3} \right), \quad y = \left(\frac{3-4}{3-4+2-7}, \frac{2-7}{3-4+2-7} \right) = \left(\frac{1}{6}, \frac{5}{6} \right).$$

The value is

$$v = \frac{2 \cdot 3 - 4 \cdot 7}{3 - 7 + 2 - 4} = \frac{11}{3}.$$

(b) The second row is dominated by the average of rows 1 and 3 since

$$\frac{1}{2}(2, 3, 8, 1) + \frac{1}{2}(3, 2, 1, 8) = \left(\frac{5}{2}, \frac{5}{2}, \frac{9}{2}, \frac{9}{2} \right) \geq (1, 2, 3, 4).$$

This reduces the problem to

$$\begin{pmatrix} 2 & 3 & 8 & 1 \\ 3 & 2 & 1 & 8 \end{pmatrix}$$

Now note the matrix is symmetric between rows 1 and 2. If we switch rows 1 and 2 we are left with

$$\begin{pmatrix} 3 & 2 & 1 & 8 \\ 2 & 3 & 8 & 1 \end{pmatrix}.$$

If we now switch columns 1 and 2 and switch columns 3 and 4 we are back to

$$\begin{pmatrix} 2 & 3 & 8 & 1 \\ 3 & 2 & 1 & 8 \end{pmatrix}$$

Thus $x_1 = x_2 = \frac{1}{2}$. While $y_1 = y_2$ and $y_3 = y_4$. When player 1 plays $(\frac{1}{2}, \frac{1}{2})$ then the payoffs for player 2 are $(\frac{5}{2}, \frac{5}{2}, \frac{9}{2}, \frac{9}{2})$. This player 2 uses only strategies 1 and 2 and so $y_1 = y_2 = \frac{1}{2}$. In the original game the optimal strategies are

$$x = (\frac{1}{2}, 0, \frac{1}{2}), \quad , y = (\frac{1}{2}, \frac{1}{2}, 0, 0)$$

and the value is $x^T A y = \frac{5}{2}$.

(c) The payoff matrix is

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

which is upper triangular. Thus we can solve the system of equations

$$\begin{aligned} y_1 - y_2 - y_3 &= V \\ y_2 - y_3 &= V \\ y_3 &= V \\ y_1 + y_2 + y_3 &= V \end{aligned}$$

So we have $y_3 = V$, $y_2 = V + y_3 = 2V$, $y_1 = V + y_2 + y_3 = 4V$. Thus $1 = 4V + 2V + V = 7V$ and so $V = \frac{1}{7}$.

Q 2

- (a) [5 points] Find the Nim-Sum of 15, 17, 19 and 31.
- (b) [5 points] In the game of Nim if the piles are 6, 10, 17 and 23 find all winning moves.
- (c) [5 points] In a subtraction game players may remove either 1, 3 or 5 chips each turn. Find the P positions and determine which player wins if the game starts with a pile of 100 chips.

Solution

- (a) In binary the numbers are 1111, 10001, 10011, 11111. The nim sum is $10010 = 18$.
- (b) In binary they are 110, 1010, 10001, 10111 and so the nim sum is $01010 = 10$. The only number with a 1 in its fourth binary digit is 10. Thus the only winning move is to remove all 10 chips from the pile with 10.
- (c) By backwards induction we can evaluate the first few states

0	1	2	3	4	5	6	7	8	9
<i>P</i>	<i>N</i>	<i>P</i>	<i>N</i>	<i>P</i>	<i>N</i>	<i>P</i>	<i>N</i>	<i>P</i>	<i>N</i>

A natural guess is that the P states are exactly the even sites. We will check this:

- The terminal state 0 is in P since 0 is even.
- From every P state you can only move to an N state which is true since all of the values 1, 3 and 5 are odd.
- From every N state one can move to a P state. This holds since from state $x \in N$, since x is odd one can remove one chip and move to an even number of chips.

Q 3 [10 points] Recall in the game of chomp in each move a player removes a square and any squares to the right or above it. The player who removes the final square loses. In the following configuration which player has a winning strategy?

Solution

There are two main ways to solve this problem. One is to use backwards induction to classify all the possible states.

The quicker way is to use the argument we used in class for rectangles. Let x be the initial state and let y be the state after removing a single block leaving a 2 by 3 rectangle. If y is in P then x must be in N . If $y \in N$ then there must be some state $z \in P$ from which you can move to from state y . But one can easily check that every state z that you can move to from state y can also be moved to from state x since these moves will also remove the rightmost block. Thus there is a P state which can be reached from x so $x \in N$. Hence we have that the configuration is an N state.

Q 4 [10 points] A company conducts market research and finds that its potential customers value a new product according to a distribution with density

$$f(x) = \begin{cases} \frac{2}{3} & 0 \leq x < 1 \\ \frac{1}{6} & 1 \leq x \leq 3. \end{cases}$$

If the product costs $\frac{1}{3}$ to produce find the price that maximizes the profit for the company.

Solution

The CDF for the distribution is

$$F(x) = \begin{cases} \frac{2}{3}x & 0 \leq x < 1 \\ \frac{2}{3} + \frac{1}{6}(x - 1) & 1 \leq x \leq 3. \end{cases}$$

The expected profit per customer if the price is p is

$$(p - \frac{1}{3})(1 - F(p)) = \begin{cases} (p - \frac{1}{3})(1 - \frac{2}{3}p) & 0 \leq x < 1 \\ (p - \frac{1}{3})(\frac{1}{3} - \frac{1}{6}(p - 1)) & 1 \leq x \leq 3. \end{cases}$$

We should maximize this on separately on the two intervals $[0, 1]$ and $[1, 3]$. On $[0, 1]$,

$$\frac{d}{dp}(p - \frac{1}{3})(1 - \frac{2}{3}p) = (1 - \frac{2}{3}p) - \frac{2}{3}(p - \frac{1}{3}) = \frac{11}{9} - \frac{4}{3}p$$

and so is maximized at $p = \frac{11}{12}$ with an expected profit of $(\frac{11}{12} - \frac{1}{3})(1 - \frac{2}{3}\frac{11}{12}) = \frac{49}{216}$. On $[1, 3]$

$$\frac{d}{dp}(p - \frac{1}{3})(\frac{1}{3} - \frac{1}{6}(p - 1)) = (\frac{1}{3} - \frac{1}{6}(p - 1)) - \frac{1}{6}(p - \frac{1}{3}) = \frac{5}{9} - \frac{1}{3}p$$

and so is maximized at $p = \frac{5}{3}$ with an expected profit of $(\frac{5}{3} - \frac{1}{3})(\frac{1}{3} - \frac{1}{6}(\frac{5}{3} - 1)) = \frac{8}{27}$. Since $\frac{8}{27} > \frac{49}{216}$ the optimal price is $p = \frac{5}{3}$ and the profit per potential customer is $\frac{8}{27}$.

Q 5

Consider a two person auction where the agents values are independent $U(0,1)$ random variables and the item goes to the highest bidder who pays the average of the two bids.

(i) [10 points] Find a such that $\beta(v) = av$ is a symmetric Bayes-Nash strategy.

(ii) [5 points] Using this or otherwise calculate the expected revenue from the auction.

Solution

(i) Let $\beta(v) = av$. If player 2 bids according to β , player 1 has value v and player 2 bids b then the expected utility is

$$\begin{aligned}\mathbb{E}(v - \frac{b + \beta(V_2)}{2})I(b > \beta(V_2)) &= \mathbb{E}(v - \frac{b + aV_2}{2})I(b > aV_2) \\ &= \int_0^{b/a} (v - \frac{b + av_2}{2})dv_2 \\ &= \frac{b(4v - 3b)}{4a}\end{aligned}$$

Differentiating with respect to b we get

$$\frac{d}{db} \frac{b(4v - 3b)}{4a} = \frac{2v - 3b}{2a}$$

and so solving this equal to 0 we get $b = \frac{2}{3}v$. Hence $a = \frac{2}{3}$ and the Bayes-Nash Equilibrium is $\beta(v) = \frac{2}{3}v$.

(ii) The revenue is the average of the bids which is $\frac{2}{3} \frac{V_1 + V_2}{2}$. So the expected revenue is

$$\mathbb{E}R = \mathbb{E} \frac{2}{3} \frac{V_1 + V_2}{2} = \frac{1}{3} \mathbb{E}[V_1 + V_2] = \frac{1}{3} (\frac{1}{2} + \frac{1}{2}) = \frac{1}{3}.$$

Alternatively we could use revenue equivalence to note that the expected revenue is the same as in the second price auction. The expected revenue in the second price auction is

$$\mathbb{E} \min[V_1, V_2].$$

Now $\mathbb{P}[\min[V_1, V_2] > s] = (1 - s)^2$ and so $\min[V_1, V_2]$ has density $\frac{d}{ds} 1 - (1 - s)^2 = 2(1 - s)$. Then

$$\mathbb{E}R = \mathbb{E} \min[V_1, V_2] = \int_0^1 s(2(1 - s))ds = \frac{1}{3}.$$