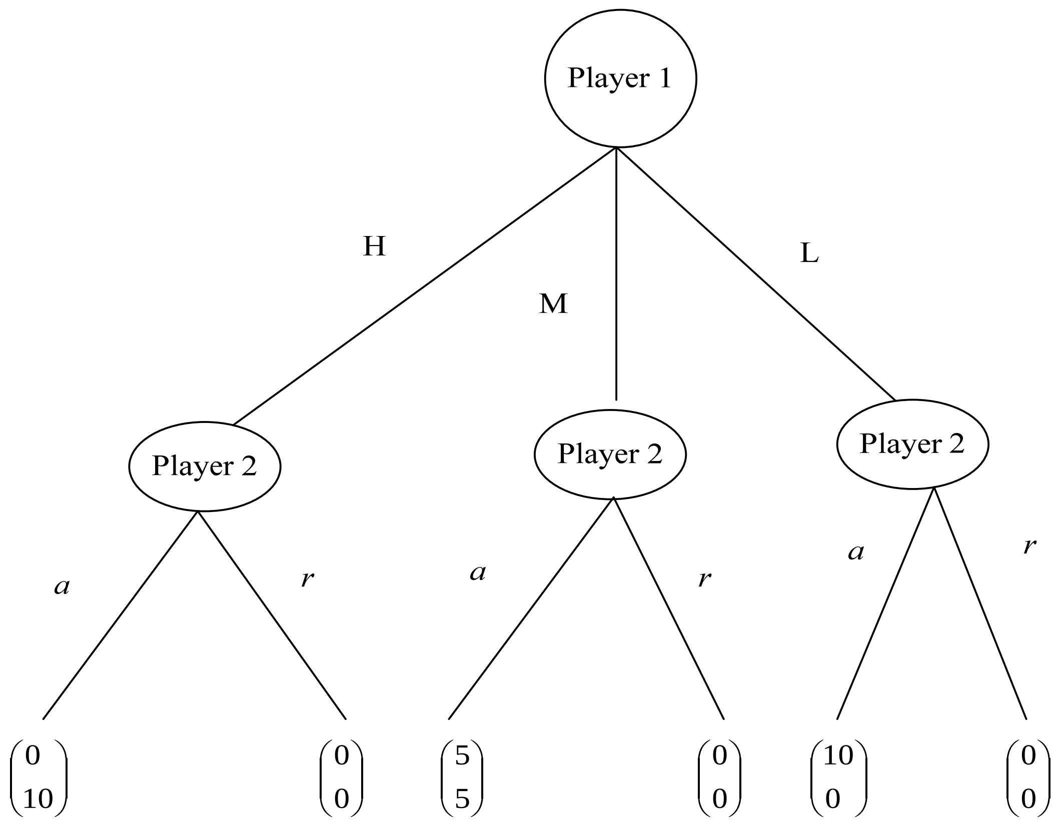
EconS 424
Strategy and Game Theory
Homework #1 – Answer Key

Exercise 1–*From extensive form to normal form representation*
Consider the following extensive form game



a) Which are the strategies for player 1?
   Three strategies S_1={H,M,L}

b) What are the strategies for player 2?
   Now the second mover can condition his move to the first player's action, since he is able to observe it (unlike in the prisoners' dilemma game). Hence, S_2={*aaa*, *aar*, *arr*, *rrr*, *rra*, *raa*, *ara*, *rar*} where each of them represents a complete plan of action that specifies what to do in the event that player 1 chooses H, in the event that player 1 chooses M and in the case that he chooses L, respectively.

c) Take your results from *a)* and *b)* and construct a matrix representing its normal form game representation.
   If you take the about three strategies for player 1, and the above eight strategies for player 2, we have the following normal form game.

		Player 2							
Player 1		*aaa*	*aar*	*arr*	*rrr*	*rra*	*raa*	*ara*	*rar*
	H	0,10	0,10	0,10	0,0	0,0	0,0	0,10	0,0
	M	5,5	5,5	0,0	0,0	0,0	5,5	0,0	5,5
	L	10,0	0,0	0,0	0,0	10,0	10,0	10,0	0,0

Exercise 2 – *Iterated Deletion of Strictly Dominated Strategies (IDSDS)*
Consider the following normal form game

		Player 2		
Player 1		*C1*	*C2*	*C3*
	R1	4,3	2,4	0,2
	R2	5,5	5,-1	-4,-2

Is there some strictly dominated strategy for player 2, i.e. a strategy she would never use whatever the strategy finally chosen by player 1?
Yes, strategy C3 is strictly dominated for player 2 by strategy C1, since player 2 would obtain higher payoffs choosing C3 than selecting C1 regardless of the actual strategy finally chosen by player 1.

If you apply iterative deletion of strictly dominated strategies (IDSDS), what is the surviving strategy pair? Explain the steps you use in IDSDS, and why you use them.
Once C3 is deleted because of being strictly dominated for player 2, then the resulting matrix is

		Player 2	
Player 1		*C1*	*C2*
	R1	4,3	2,4
	R2	5,5	5,-1

As one can observe, player 1 will never select strategy R1 in this reduced game, because of being strictly dominated by strategy R2. Once this strategy is deleted, we have

		Player 2	
Player 1		*C1*	*C2*
	R2	5,5	5,-1

And clearly, now player 2's strategy C2 is strictly dominated by C1 in this reduced game. Hence, this game is dominance solvable, and the only strategy pair surviving IDSDS is {R2,C1}.

Exercise 3 – *Two people in a partnership.*
Guided Exercise from Chapter 6 in Watson, pages 62-64
Problem: Suppose that two people decide to form a partnership firm. The revenue of the firm depends on the amount of effort expended on the job by each person and is given by:
$r(e_1, e_2) = a_1 e_1 + a_2 e_2,$

Where e_1 is the effort level of person 1 and e_2 is the effort level of person 2. The numbers a_1 and a_2 are positive constants. The contract that was signed by the partners stipulates that person 1 receives a fraction t (between 0 and 1) of the firm's revenue and person 2 receives a 1-t fraction. That is, person 1 receives the amount $t\,r(e_1, e_2)$, and person 2 receives (1-t) $r(e_1, e_2)$. Each person dislikes effort, which is measured by a personal cost of e_1^2 for person 1 and e_2^2 for person 2. Person i's utility in this endeavor is the amount of revenue that this person receives, minus the effort cost e_i^2. The effort levels (assumed nonnegative) are chosen by the people simultaneously and independently.

a. Define the normal form of this game (by describing the strategy spaces and payoff functions).
b. Using dominance, compute the strategies that the players rationally select (as a function of t, a_1, and a_2).
c. Suppose that you could set t before the players interact. How would you set t to maximize the revenue of the firm?

Solution:

a.

The game has two players. Each player selects an effort level, which is greater than or equal to zero. Thus, $S_i = [0, \infty)$ for i=1,2. As described, each player's payoff is the amount of revenue he receives, minus his effort cost. Thus, the payoff functions are
$u_1(e_1, e_2) = t[a_1 e_1 + a_2 e_2] - e_1^2$ and $u_2(e_1, e_2) = (1-t)[a_1 e_1 + a_2 e_2] - e_2^2$.

b.

In this game, each player has a strategy that dominates all others. To see this, observe how player 1's payoff changes as e_1 is varied. Taking the derivative of u_1 with respect to e_1, we get $ta_1 - 2e_1$. Setting this equal to zero and solving for e_1 reveals that player 1 maximizes his payoff by selecting $e_1^* = \frac{ta_1}{2}$. Similar analysis for player 2 yields $e_2^* = \frac{(1-t)a_2}{2}$.

Note that, although each player's payoff depends on the strategy of the other player, a player's optimal strategy does not depend on the other's strategy. The set of undominated strategies is therefore

$$UD_1 = \{\frac{ta_1}{2}\} \text{ and } UD_2 = \{\frac{(1-t)a_2}{2}\}.$$

c.

Because they depend on a_1, a_2, and t, let us write the optimal strategies e_1^* and e_2^* as functions of these parameters. The revenue of the firm is then given by

$$a_1 e_1^*(a_1, a_2, t) + a_2 e_2^*(a_1, a_2, t)$$

Plugging in the values e_1^* and e_2^* from part (b), the revenue is

$$a_1 \frac{ta_1}{2} + a_2 \frac{(1-t)a_2}{2} = t\frac{a_1^2}{2} + (1-t)\frac{a_2^2}{2}.$$

Note that the objective function is linear in t. Thus, maximization occurs at a "corner," where either t=0 or t=1. If a_1>a_2, then it is best to set t=1; otherwise, it is best to set t=0.

Incidentally, one can also consider the problem of maximizing the firm's revenue minus the partners' effort costs. Then the problem is to maximize

$$a_1 \frac{ta_1}{2} + a_2 \frac{(1-t)a_2}{2} - (\frac{ta_1}{2})^2 - (\frac{(1-t)a_2}{2})^2$$

And, using calculus, the solution is to set $t = \dfrac{a_1^2}{a_1^2 + a_2^2}$.

Exercise 4 – *Pure strategies that are only strictly dominated by a mixed strategy*
Consider the following normal form game

		Player 2	
Player 1		*Left*	*Right*
	Up	4,1	0,2
	Middle	0,0	4,1
	Down	1,3	1,2

Is there some strictly dominated strategy for player 1 involving only the use of pure strategies?
No, there is no strategy for player 1 such that the payoff received by player 1 is always higher, regardless of the strategy chosen by player 2. Neither U nor M dominates the other, and D does not dominate these strategies. In addition, neither U nor M dominates D. For instance, although U is better than D when player 2 selects L, D performs better than U when player 2 selects R.

Is there some strictly dominated strategy for player 1 when mixed strategies are allowed? [*Hint: you may assign probabilities to two of her strategies, similarly as we did in class*].
There is a mixed strategy for player 1 that dominates D. Consider 1's mixed strategy of selecting U with probability 1/2, M with probability 1/2 and D with probability 0. We represent this mixed strategy as (1/2, 1/2, 0). If player 2 selects L, then this mixed strategy yields an expected payoff to player 1 of: EU(mixed)=1/2*4+1/2*0+0*1=2, and player 1 does worse by playing D (a payoff of only 1).
The same is true when player 2 selects R: EU(mixed)= 1/2*0+1/2*4+0*1=2, and player 1 does worse by playing D (a payoff of only 1).
Therefore, strategy D is strictly dominated by the mixed strategy (1/2, 1/2, 0).

Delete the strictly dominated strategies for player 1 that you found in the previous question. Then, represent the remaining (undeleted) strategies.
Once strategy D is deleted because of being strictly dominated by the above mixed strategy, we have a (reduced) normal form game given by

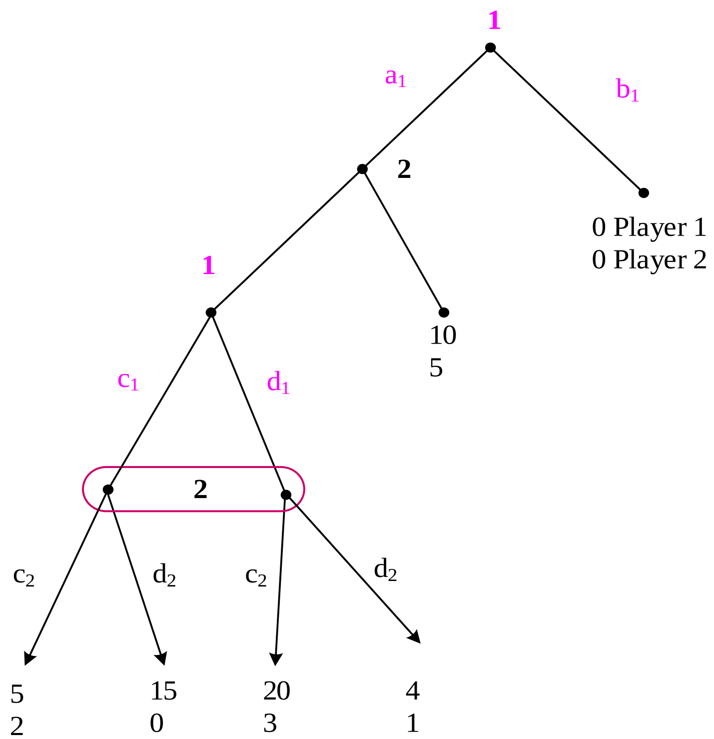
		Player 2	
Player 1		*Left*	*Right*
	Up	4,1	0,2
	Middle	0,0	4,1

Proceed with the IDSDS. What is the strategy pair surviving IDSDS?
In the above (reduced) normal form game, player 2 finds Left as being strictly dominated by strategy Right. Note that in the original game (before deleting any of player 1's strategies, we could not eliminate any of the player 2's strategies, since none of them were strictly dominated).
Hence, the remaining strategies are

Player 1		*Right*
	Up	0,2
	Middle	4,1

That makes strategy Up strictly dominated for player 1, in this iterative process of deletion of strictly dominated strategies. Hence, this game is dominance solvable, and the strategy pair surviving the IDSDS is {Middle, Right}.

Exercise 5: Exercise 8 from Chapter 2 in Harrington. FIGURE PR2.8



ANSWER: Player 1 has two information sets, the initial node and the information set associated with *a*1 and *a*2 having been played. Let *x*/*y* denote a strategy for player 1 that assigns action *x* to the initial node and action *y* to the other information set. Player 1's strategy then contains four elements: *a*1/*c*1, *a*1/*d*1, *b*1/*c*1, and *b*1/*d*1. Player 2 also has two information sets, the singleton associated with 1 having used *a*1 and the information set with two nodes—one when the path is *a*1 S *a*2 S *c*1 (read as "*a*1 is chosen then *a*2 is chosen then *c*1 is chosen") and one when the path is *a*1 S *a*2 S *d*1. If strategy *x*/*y* assigns action *x* to the first information set and action *y* to the second one, then player 2 has four strategies: *a*2/*c*2, *a*2/*d*2, *b*2/*c*2, and *b*2/*d*2. The payoff matrix associated with these strategies is shown in **FIGURE SOL2.8.1**.

	a2/c2	***a2/d2***	***b2/c2***	***b2/d2***
a1/c1	5,2	15,0	10,5	10,5
a1/d1	20,3	4,1	10,5	10,5
b1/c1	0,0	0,0	0,0	0,0
b1/d1	0,0	0,0	0,0	0,0

Exercise 6: Harrington, Chapter 3, Exercise 10.

(BONUS EXERCISE)

Harrington
Chapter 3

Student 2 Effort, x_2

Ex 10-a

Student 1 Effort, x_1

	0	1	2	3	4	5
0	10,8	10,7	8,8	8,7	8,6	8,5
1	9,8	9,7	9,6	7,7	7,6	7,5
2	8,8	8,7	8,6	8,5	6,6	6,5
3	~~7,8~~	~~7,7~~	~~7,6~~	~~7,5~~	~~7,4~~	~~5,5~~
4	~~6,8~~	~~6,7~~	~~6,6~~	~~6,5~~	~~6,4~~	~~6,3~~
5	~~5,8~~	~~5,7~~	~~5,6~~	~~5,5~~	~~5,4~~	~~5,3~~

Round 1 IDSDS:

$x_1 = \{3,4,5\}$ dominated by $x_1 = 0$

$x_2 = \{1,3,4,5\}$ dominated by $x_2 = 0$

Surviving Strategies:

x_2

x_1

	0	2
0	10,8	8,8
1	9,8	9,6
2	~~8,8~~	~~8,6~~

Round 2 IDSDS:

$x_1 = 2$ dominated by $x_1 = 1$

Exercise 11 from Chapter 3 in Harrington.

11. Groucho Marx once said, "I'll never join any club that would have me for a member." Well, Groucho is not interested in joining your investment club, but Julie is. Your club has 10 members, and the procedure for admitting a new member is simple: Each person receives a ballot that has two options: (1) admit Julie and (2) do not admit Julie. Each person can check one of those two options or abstain by not submitting a ballot. For Julie to be admitted, she must receive at least six votes in favor of admittance. Letting *m* be the number of ballots submitted with option 1 checked, assume that your payoff function is

{1 if m =6, 7, 8, 9, 10
{0 if m =0, 1, 2, 3, 4, 5

.

a. Prove that checking option 1 (admit Julie) is not a dominant strategy.

(Please note that if for the existence of a dominant strategy, there must be a corresponding strictly dominated strategy that the former dominates. Hence, if there is no dominant strategy then it must be that there is no strictly dominated strategy. Therefore, you can answer the question showing that there is at least one instance in which you as a player don't have any strictly dominated strategy. Hint: because you are indifferent among all possible strategies, i.e., option 1, option 2, or abstain).

ANSWER: If all the other players check option 2, then Julie is not admitted regardless whether you check option 1, check option 2, or abstain. Since all three strategies yield the highest payoff in that case, none of them is strictly dominated and thus there is no dominant strategy.

b. Prove that abstaining is a weakly dominated strategy.

(Here you need to identify all the cases (of other players' votes) in which you are indifferent between abstaining and voting for options 1 and 2, since your vote wouldn't change the outcome of the election. However, you must also show at least one case in which you strictly prefer to vote option 1. As a consequence, you can claim that option 1 weakly dominates abstaining, and thus abstention becomes a weakly dominated strategy.)

ANSWER: If at least six of the other members submit ballots with option 1 checked, then Julie is admitted regardless of what you do; your payoff is 1 with all strategies. If four or fewer of the other members vote in favor of option 1, then Julie is denied admittance regardless of what you do; your payoff is 0 with all strategies. This leaves only the case when five of the other members submit ballots in favor of admitting Julie. Abstaining results in a payoff of 0, as Julie ends up with only five supporting votes. Voting and checking option 1 results in her admittance and thus a payoff of 1. Hence, abstaining is weakly dominated by voting in favor of Julie.

c. Now suppose you're tired at the end of the day and so it is costly for you to attend the evening's meeting to vote. By not showing up, you abstain from the vote. This is reflected in your payoff function having the form

{1 if m _ 6, 7, 8, 9, 10 and you abstained
{.5 if m _ 6, 7, 8, 9, 10 and you voted
{0 if m _ 0, 1, 2, 3, 4, 5 and you abstained
{-.5 if m _ 0, 1, 2, 3, 4, 5 and you voted

Prove that abstaining is not a weakly dominated strategy.

(In this part of the exercise, you need to identify that there are cases of other players' votes) in which you strictly prefer to vote in favor of option 1. This implies that you are no longer indifferent between option 1, option 2, and abstention (you just showed these cases in question b), but rather strictly prefer option 1. This implies that, in the cases you identified, voting for option 1 strictly dominates abstention, and thus abstaining is a strictly (not weakly) dominated strategy.)

ANSWER: If the other members vote so that $m \geq 6$, then the payoff from showing up and voting is, while it's 1 from not showing up and thus abstaining. Since there are strategies for the other players whereby abstention is the unique optimal strategy, then abstention cannot be weakly dominated.