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Game Theory

Problem Set 11

1. A seller owns an object that a buyer wants to buy. The value of the object to the seller is c. The value of the object to the buyer is private information. The buyer's valuation v is a random variable distributed over the interval $[0, V]$ according to the (continuous) c.d.f. F. Assume that $[1 - F(v)] / f(v)$ is a decreasing function of v. The von Neumann-Morgenstern utility of a type v from getting a unit at price p is $v - p$ and the utility of no purchase in 0.

(i) Suppose the seller is constrained to charge just one price. Show that the profit maximizing price satisfies $p = c + [1 - F(p)] / f(p)$.

(ii) Suppose that the seller can commit to a menu of offers $[q(v), p(v)]$, where $q(v)$ is the probability with which a consumer who chooses offer v will get a unit, and $p(v)$ is the price she will pay in the event that she gets a unit. Prove that the menu that maximizes the seller's profit consists of a single price, which is the one found in (i), and that any buyer can get the good at this price with probability 1.

2. Consider the following auction environment. A seller has a single object for sale and can commit to any selling mechanism (the seller's valuation of the object is zero). There are two potential bidders, indexed by $i = 1, 2$. The valuation of the object of bidder $i = 1, 2$ is denoted by v_i and is distributed uniformly over the unit interval. Valuations are independent between the two bidders. Bidder 1 knows her own valuation v_1. However, bidder 2 does *not* know v_2.

The bidders' payoffs are as follows. Suppose bidder $i = 1, 2$ has type v_i and pays the amount t_i to the seller. Her payoff is equal to $v_i - t_i$ if she gets the object, and equal to $-t_i$ otherwise.

(i) Construct the optimal direct mechanism for the seller (i.e., find the incentive compatible, individually rational mechanism that maximizes the seller's expected revenues). Compute the seller's revenues.

(ii) Can you find a simple *indirect* mechanism that gives to the seller the same expected revenues as the optimal direct mechanism?

3. A seller has a unit for sale. Its quality is either high (H) or low (L). The quality is known to the seller but not to the buyer, whose prior probability that the quality is high is 1/2. Their valuations of the unit are as follows.

	Quality H	Quality L
Buyer	V	2
Seller	7	0

where $V > 7$. Thus, the utility to the buyer of getting the unit at price p is $2 - p$ if it is of the low quality, and $V - p$ if it is of the high quality. Similarly, the utility to the seller is p and $p - 7$, respectively.

(i) Find the ex-post efficient outcomes.

(ii) Identify the range of V (above 7) for which there is, and the range of V for which there is no incentive compatible, individual rational mechanism that will achieve the ex-post efficient outcome.

(iii) Describe the best outcome (in the maximizing of the sum of expected utilities) that can be achieved for each V (above 7) and the mechanism that achieves it.

HINT: A mechanism for this Bayesian bargaining problem consists of a pair of functions $q : \{L, H\} \to [0, 1]$ and $t : \{L, H\} \to \mathbb{R}$, where $q(i)$ is the probability that the object will be sold to the buyer and $t(i)$ is the expected net payment from the buyer to the seller if $i = L, H$ is the type reported by the seller to a mediator.

4. A seller owns an object that a buyer wants to buy. The quality of the object is a random variable v, with support $[0, 1]$ and distribution function $F(v) = v^{\alpha}$, where $\alpha > 0$. The seller knows the quality of the object but the buyer does not. When the quality of the object is v, the value of the object is v to the seller and zv to the buyer, where $z > 1$. Thus, if the object of quality v is traded at price p, the seller gets $p - v$ and the buyer gets $zv - p$. Both players have utility equal to zero if there is no trade.

Consider the function $G : (0, \infty) \times (1, \infty) \to [0, 1]$ defined as follows. For each pair (α, z) construct the incentive-compatible individually rational mechanism that maximizes the (ex-ante) probability of trade. Denote this probability by $G(\alpha, z)$. Derive the function G.

(N.B. If the probability of trade is $q(v)$ when the quality is v, then the (ex-ante) probability of trade is equal to $\int_0^1 q(v)\, dF(v)$.)