Fall 2007 Final Solutions

1. (25%)
   a. (T,L) is the unique pure strategy NE.

   b. Grim trigger strategies that would support (B,R) as the SPNE of the infinitely repeated game are as follows:
      i. For player 1, play B in the first period. For all other periods, as long as no player has deviated (not played (B,R)) in any previous period, play B. If any deviation by any player has occurred, play T forever.
      ii. For player 2, play R in the first period. For all other periods, as long as no player has deviated (not played (B,R)) in any previous period, play R. If any deviation by any player has occurred, play L forever.

   c. Equilibrium path payoffs are $4(1+\delta+\delta^2+\ldots) = 4/(1-\delta)$. Player 2 does NOT have a unilateral profitable deviation. Player 1, however, can deviate to T and get a payoff of 20, but that pulls the trigger. So player 1's deviation payoff is $20 + 3(\delta + \delta^2 + \ldots) = 20 + 3\delta/(1-\delta)$. Our trigger strategies form a SPNE where collusion is sustained if.

$$4/(1-\delta) >= 20 + 3\delta/(1-\delta) \qquad (*)$$
$$4 >= 20 - 20\delta + 3\delta$$
$$17\delta >= 16$$
$$\delta^* >= 16/17$$

Note this is very close to 1. This is because the one period deviation payoffs to player 1 is relatively high compared to the equilibrium path payoff (20 versus 4). As we increase the deviation payoff, the critical discount factor also would get closer to 1.

2. (25%)
   a. Player 1 chooses from (A,B,C) and player 2 chooses from (L,M,R).
   b. Consider the normal form of the game:

Player 2

Player 1		L	M	R
	A	1,4	2,0	9,2
	B	4,2	9,0	6,4
	C	7,0	8,0	9,0

So there are two pure strategy NE at (C,L) and (C,R).

   c. There is only one subgame so SPNE = NE.
   d. Note that in both of the SPNE, player 2's information set is ON-the equilibrium path so the beliefs that are consistent with each just put probability 1 on the node following player 1's choice of C. Thus:

PBE = { (C,L), p, q, 1-p-q | p = 0, q= 0,
(C,R), p, q, 1-p-q | p = 0, q= 0 }

3. (25%)
    a. See Osborne.
    b. So we start with the sender's strategy of $\sigma_S = (Q,Q)$. This implies that $p=0.25$ because the sender's equilibrium strategy tells us nothing about his type if the receiver sees Quiche. Thus, the receiver compares
$$E_R[\text{Duel}|Q] = 2(0.25) + 4(0.75) = 3.5$$
with
$$E_R[\text{Not Duel}|Q] = 3(0.25) + 1(0.75) = 1.5.$$
So a receiver will duel with the sender if he sees him having Quiche for breakfast. Now we need the strategy and belief of the receiver following the Beer signal. A strong sender, if he plays Q, gets a payoff of 3. A weak sender, if he plays Q, gets a payoff of 4, his highest possible payoff. So a weaker sender never wants to deviate and a strong sender doesn't want to deviate as long as the receiver doesn't duel if he sees the beer signal. So we require that q satisfies:
$$E_R[\text{Not Duel}|B] \geq E_R[\text{Duel}|B]$$
$$1*q + 3*(1-q) \geq 4*q+2*(1-q)$$
$$3-2q \geq 2+2q$$
$$4q \leq 1$$
$$q \leq 0.25$$
So our complete PBE is as follows :
PBE $=\{$(Quiche,Quiche),(Duel,Not Duel), (p,1-p), (q,1-q) $|$ $p = 0.25, q \leq 0.25\}$.

    c. In a separating equilibrium on (Beer, Quiche), we start with $\sigma_s = (B,Q)$. This immediately tells us the receiver's beliefs: $p = 0$, $q = 1$. Only strong senders have beer for breakfast (according to the strategy) so the only belief that is consistent with this is $q = 1$. Only weak senders have quiche for breakfast so $p = 0$, ie, $1-p = 1$. Since the receiver knows he's either at the southwest or northeast corner of the game, he will play σ_R = (Duel, Duel). The strong sender is getting his highest payoff (4) from having beer for breakfast and the weak sender is as well. Thus neither have an incentive to deviate and we get a PBE at :
PBE = $\{$(Beer,Quiche),(Duel, Duel), (p,1-p), (q,1-q) $|$ $p = 0, q = 1\}$.

4. (25%)
    a. Consider the monopolist's problem :
$$\text{Max}(Q) \quad \{ Q(12-Q) \}$$
$$\text{FOC}(Q) : 12-2Q = 0 \rightarrow Q^m = 6$$
$$P^m = 12-6=6$$
$$\pi^m = 6*6 = 36$$
Therefore, $\frac{1}{2}\pi^m = 18$. Note that each firm is producing 3 units in the joint-monopoly outcome.
    b. With costs equal to zero, any prices strictly above zero leads to a profitable deviation by one of the firms to just below the other. The lower price firm steals the entire market and makes strictly positive profits. This undercutting can continue until $p_1 = p_2 = 0$ and firms split the market equally. At this price, both firms make zero profits. Unilaterally raising their price leads to

losing the market and still making zero profits. Lowering their price to below zero (if allowed) leads to negative profits. So at $p_1=p_2=0$, we have a NE of the Bertrand (price-setting) game.

c. If firms play Cournot, then firm 1 solves:

$$\text{Max}(q_1) \quad \{q_1(12-q_1-q_2)\}$$
$$\text{FOC}(q_1) : 12 - 2q_1 - q_2 = 0$$
$$q_1(q_2) = 6 - \tfrac{1}{2} q_2$$

Symmetrically,

$$q_2(q_1) = 6 - \tfrac{1}{2} q_1$$

Solve simultaneously:

$$q_1 = 6 - \tfrac{1}{2}(6 - \tfrac{1}{2} q_1)$$
$$\tfrac{3}{4} q_1 = 3$$
$$q^c_1 = 4$$
$$\rightarrow q^c_2(q_1) = 6 - \tfrac{1}{2}(4) = 4$$
$$\rightarrow Q = q_1 + q_2 = 8 \rightarrow P = 12-8 = 4$$
$$\rightarrow \pi^c_1 = \pi^c_2 = 4*4=16$$

d. For the infinitely repeated Bertrand game, firms split the monopoly profit each period along the equilibrium path. If one firm unilaterally deviates, they choose a price that is an infinitesimally small amount below the monopoly price and they earn the monopoly profit for one period. Then the punishment phase (Bertrand pricing resulting in zero profits) sets in. Thus:

$$\Pi^e_i = \tfrac{1}{2}\pi^m(1+\delta+\delta^2+\delta^3+\ldots) = 18(1+\delta+\delta^2+\delta^3+\ldots) = 18/(1-\delta)$$
$$\Pi^d_i = \pi^m + 0(\delta+\delta^2+\delta^3+\ldots) = 36$$

So we require:

$$\Pi^e_i \geq \Pi^d_i \rightarrow 18/(1-\delta) \geq 36 \rightarrow \delta \geq \tfrac{1}{2}$$

e. For the infinitely repeated Cournot game, firms split the monopoly profit each period along the equilibrium path. If one firm unilaterally deviates, they choose an optimal quantity off their best response function (noting that the other firm is producing half the monopoly quantity):

$$q_i^d = 6 - \tfrac{1}{2} q_j = 6 - \tfrac{1}{2}(\tfrac{1}{2} Q^m) = 6-\tfrac{1}{2}(3) = 4.5$$

In the one period that a firm successfully deviates the market price is then:

$$P = 12 - \tfrac{1}{2} Q^m - q_i^d = 12 - 3 - 4.5 = 4.5$$

Thus,

$$\pi^d_i = 4.5*4.5 = 81/4 = 20.25$$

Following the deviation, firms set the Cournot quantity and earn the Cournot profits each period, thus:

$$\Pi^d_i = 20.25 + 16(\delta+\delta^2+\delta^3+\ldots) = 20.25 + 16\delta/(1-\delta)$$

So we require:

$$\Pi^e_i \geq \Pi^d_i \rightarrow 18/(1-\delta) \geq 20.25 + 16\delta/(1-\delta)$$
$$18 \geq 20.25 - 20.25\delta + 16\delta$$
$$4.25\delta \geq 2.25$$
$$\delta \geq 2.25/4.25 \approx 0.529$$