14.12 Game Theory – Final

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Please answer only three of the following four questions. Each question is 35 points. Good luck!

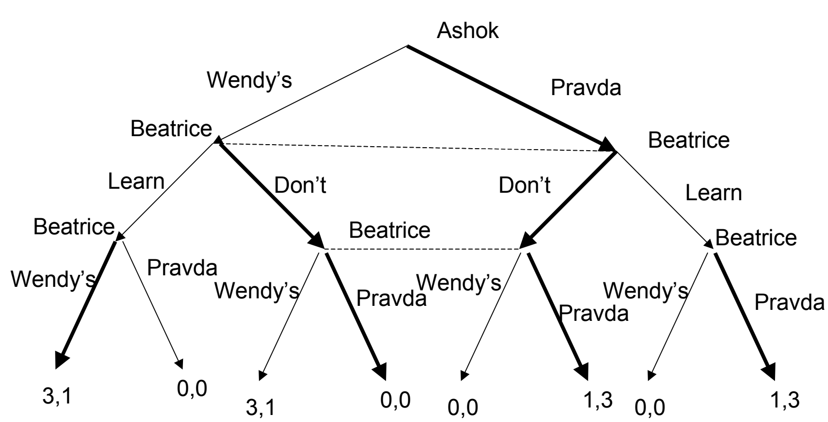
1. Ashok and Beatrice would like to go on a date. They have two options: a quick dinner at Wendy's, or dancing at Pravda. Ashok first chooses where to go, and knowing where Ashok went Beatrice also decide where to go. Ashok prefers Wendy's, and Beatrice prefers Pravda. A player gets 3 out his/her preferred date, 1 out of his/her unpreferred date, and 0 if they end up at different places. All these are common knowledge.

   (a) Find a subgame-perfect Nash equilibrium. Find also a non-subgame-perfect Nash equilibrium with a different outcome.

   ANSWER: SPE: Beatrice goes wherever Ashok goes, and Ashok goes Wendy's. The outcome is both go to Wendy's. Non-subgame-perfect Nash Equilibrium: Beatrice goes to Pravda at any history, so Ashok goes to Pravda. The outcome is each goes to Pravda. This is not subgame-perfect because it is not a Nash equilibrium in the subgame after Ashok goes to Wendy's.

   (b) Modify the game a little bit: Beatrice does not automatically know where Ashok went, but she can learn without any cost. (That is, now, without knowing where Ashok went, Beatrice first chooses between Learn and Not-Learn; if she chooses Learn, then she knows where Ashok went and then decides where to go; otherwise she chooses where to go without learning where Ashok went. The payoffs depend only on where each player goes —as before.) Find a subgame-perfect equilibrium of this new game in which the outcome is the same as the outcome of the non-subgame-perfect equilibrium in part (a). (That is, for each player, he/she goes to the same place in these two equilibria.)

   ANSWER: The following is a SPE, whose outcome is that each goes to Pravda.



2. We have two partners who simultaneously invest in a project, where the level of investment can be any non-negative real number. If partner i invests x_i and the other partner j invests x_j, then the payoff of partners i will be

$$\theta_i x_i x_j - x_i^3.$$

Here, θ_i is privately known by partner i, and the other partner believes that θ_i is uniformly distributed on $[0,1]$. All these are common knowledge. Find a symmetric Bayesian Nash equilibrium in which the investment of partner i is in the form of $x_i = a + b\sqrt{\theta_i}$.

ANSWER: We construct a Bayesian Nash equilibrium (x_1^*, x_2^*), which will be in the form of $x_i^*(\theta_i) = a + b\sqrt{\theta_i}$. The expected payoff of i from investment x_i is

$$U(x_i;\theta_i) = E\left[\theta_i x_i x_j^* - x_i^3\right] = \theta_i x_i E\left[x_j^*\right] - x_i^3.$$

Of course, $x_i^*(\theta_i)$ satisfies the first order condition

$$0 = \partial U(x_i;\theta_i)/\partial x_i|_{x_i^*(\theta_i)} = \theta_i E\left[x_j^*\right] - 3\left(x_i^*(\theta_i)\right)^2,$$

i.e.,

$$x_i^*(\theta_i) = \sqrt{\theta_i E\left[x_j^*\right]/3} = \sqrt{\frac{E\left[x_j^*\right]}{3}}\sqrt{\theta_i}.$$

That is, $a = 0$, and the equilibrium is in the form of $x_i^*(\theta_i) = b\sqrt{\theta_i}$ where

$$b = \sqrt{\frac{E\left[x_j^*\right]}{3}}.$$

But $x_j^* = b\sqrt{\theta_j}$, hence

$$E\left[x_j^*\right] = E\left[b\sqrt{\theta_j}\right] = bE\left[\sqrt{\theta_j}\right] = 2b/3.$$

Substituting this in the previous equation we obtain

$$b^2 = \frac{E\left[x_j^*\right]}{3} = \frac{2b/3}{3} = \frac{2b}{9},$$

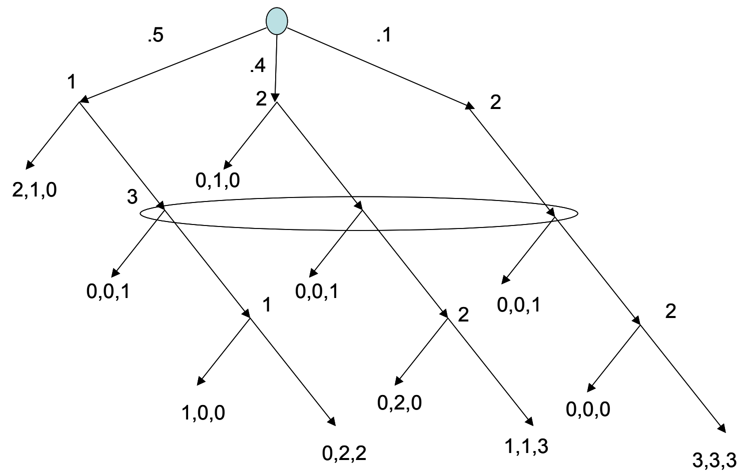
i.e.,

$$b = 2/9.$$

In summary,

$$x_i^*(\theta_i) = \frac{2}{9}\sqrt{\theta_i}.$$

3. Find a perfect Bayesian Nash equilibrium in the following game.



ANSWER: There is a unique perfect Bayesian Nash equilibrium in this game. Clearly, 1 must exit at the beginning and 2 has to go in on the right branch as he does not have any choice. The behavior at the nodes in the bottom layer is given by sequential rationality as in the figure below. Write α for the probability that 2 goes in in the center branch, β for the probability that 3 goes right, and μ for the probability 3 assigns to the center branch. In equilibrium, 3 must mix (i.e., $\beta \in (0,1)$). [Because if 3 goes left, then 2 must exit at the center branch, hence 3 must assign probability 1 to the node at the right (i.e., $\mu = 0$), and hence she should play right —a contradiction. Similarly, if 3 plays right, then 2 must go in at the center branch. Given his prior beliefs (.4 and .1), $\mu = 4/5$, hence 3 must play left — a contradiction again.] In order 3 to mix, she must be indifferent, i.e.,

$$1 = 0\mu + 3(1-\mu),$$

hence

$$\mu = 2/3.$$

By the Bayes' rule, we must have

$$\mu = \frac{.4\alpha}{.4\alpha + .1} = 2/3,$$

i.e.,

$$\alpha = 1/2.$$

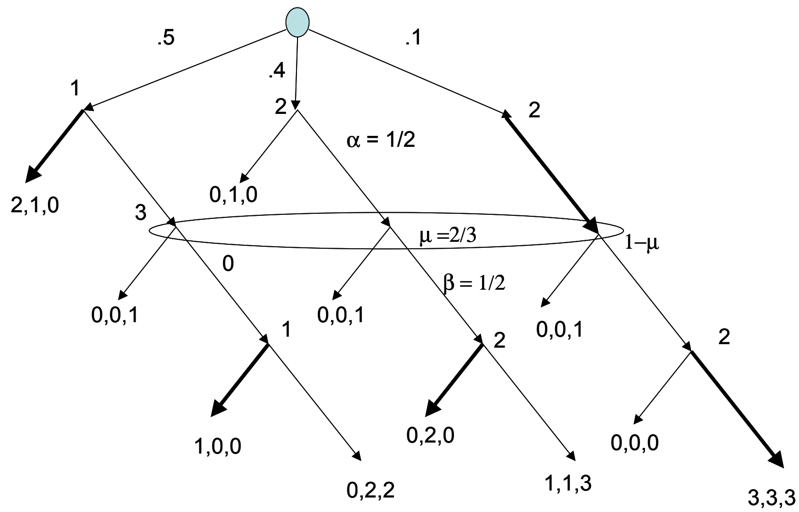
That is player 2 must mix on the center branch, and hence she must be indifferent, i.e.,

$$1 = 2\beta.$$

That is,

$$\beta = 1/2.$$

The equilibrium is depicted in the following figure.



4. We have a Judge and a Plaintiff. The Plaintiff has been injured. Severity of the injury, denoted by v, is the Plaintiff's private information. The Judge does not know v and believes that v is uniformly distributed on $\{0, 1, 2, \ldots, 99\}$ (so that the probability that $v = i$ is $1/100$ for any $i \in \{0, 1, \ldots, 99\}$). The Plaintiff can verifiably reveal v to the Judge without any cost, in which case the Judge will know v. The order of the events is as follows. First, the Plaintiff decides whether to reveal v or not. Then, the Judge rewards a compensation R. The payoff of the Plaintiff is $R - v$, and the payoff of the Judge is $-(v - R)^2$. Everything described so far is common knowledge. Find a perfect Bayesian Nash equilibrium.

ANSWER: We consider a symmetric Bayesian Nash equilibrium (s^*, R^*), where $s^*(v) \in \{v, NR\}$ determines whether the Plaintiff of type v reveals v or does Not Reveal, and R^* determines the reward, which is a function from $\{NR, 0, 1, \ldots, 99\}$. Given the Judge's preferences, if the Plaintiff reveals her type v, the Judge will choose the reward as

$$R^*(v) = v$$

and

$$R^*(NR) = E[v|NR].$$

In equilibrium, the Plaintiff gives her best response to R^* at each v. Hence, she must reveal her type whenever $v > R^*(NR)$, and she must not reveal her type whenever $v < R^*(NR)$. Suppose that $R^*(NR) > 0$. Then, $s^*(0) = NR$, and hence NR is reached with positive probability. Thus,

$$R^*(NR) = E[v|s^*(v) = NR] \le E[v|v \le R^*(NR)] \le R^*(NR)/2,$$

which could be true only when $R^*(NR) = 0$, a contradiction. Therefore, we must have

$$R^*(NR) = 0,$$

and thus

$$s^*(v) = v$$

at each $v > 0$. There are two equilibria (more or less equivalent).

- $s^*(v) = v$ for all v; $R^*(v) = v$; $R^*(NR) = 0$, and the Judge puts probability 1 to $v = 0$ whenever the Plaintiff does not reveal her type.
- $s^*(0) = NR$; $s^*(v) = v$ for all $v > 0$; $R^*(v) = v$; $R^*(NR) = 0$, and the Judge puts probability 1 to $v = 0$ whenever the Plaintiff does not reveal her type.