

# Stat 155 Homework # 10 Due April 29 (Tuesday)

## Problems:

**Q 1** Karlin-Peres Chapter 7 Q 7.1

If the voting in an election with 3 candidates was

$$35\% : A > B > C \quad (1)$$

$$35\% : B > C > A \quad (2)$$

$$30\% : C > A > B \quad (3)$$

Then the winning candidate in a runoff election is candidate  $A$  since once candidate  $C$  is removed  $A$  gets 65% of the vote. But head to head between  $A$  and  $C$  candidate  $C$  would get 65% of the vote and beat  $A$ .

**Q 2** For each of the following voting systems on  $n > 1$  voters and  $q > 2$  alternatives, find rankings such that at least one voter will want to manipulate

- Almost dictator: the winner is the most preferred candidate  $A$  of voter 1 **unless** all other voters have some  $B$  (different than  $A$ ) as the most preferred candidate.
- Almost majority: Assume  $n$  is odd and fix two candidates  $A, B$ . The winner is the majority winner between  $A, B$  unless all voters have candidate  $C$  as their most preferred candidate.

Solution:

Almost dictator: If voter 1 prefers candidate  $A$  and  $n - 2$  of the other voters prefer  $B$  and the final voter has a ranking of  $C > B > A$  then the final voter can manipulate by switching to  $B > C > A$  in which case the winner will switch from  $A$  to  $B$ .

Almost majority: If All but one voter ranks  $C$  first and has a relative ranking of  $C > B > A$  but one voter ranks  $A > C > B$ . Then the winner will be candidate  $B$ . But the final voter can manipulate and vote  $C$  first in which case  $C$  will win the election which they prefer to  $B$ .

**Q 3** Four competitors compete in a 2 round knockout singing competition. In one semi-final A and B compete and in the other semi-final C competes against D. The winners progress to the final and the winner in the final becomes the champion. There are 3 judges and in each competition the judges vote and the singer with the majority of the votes wins.

The judges each have their own preferences of the singers as to who they want to win. Find a set of preferences where strategic manipulation is possible for one of the judges.

Solution:

Suppose the 3 judges  $X, Y, Z$  have preferences:

$$\text{For } X : A > B > C > D \quad (4)$$

$$\text{For } Y : B > C > A > D \quad (5)$$

$$\text{For } Z : C > D > B > A \quad (6)$$

Under these preferences  $B$  will win the first semi-final and  $C$  will win the second. Then in the final voters  $X$  and  $Y$  will choose  $B$  who will win. If judge  $Z$  changed his preference to  $C > D > A > B$  then  $A$  would win the first semi-final and in the final between  $A$  and  $C$ , singer  $C$  would win 2 to 1. Since  $C$  is the first preference for  $Z$  then this is an opportunity for manipulations by  $Z$ .

**Q 4** In an election there are 3 candidates,  $A, B$  and  $C$  and 3 voters whose preferences are:

$$\text{Voter 1 : } A > B > C$$

$$\text{Voter 2 : } B > C > A$$

$$\text{Voter 3 : } C > A > B$$

The election is by plurality (most votes wins) except if there is a tie in which case the choice of Voter 1 wins the election. We can consider voting for each of the 3 candidates as a pure strategy. Find the pure Nash equilibria in this election.

Solution:

If all voters vote for one candidate then that is a Nash equilibrium since no voter can change the result by changing their vote. In a 3-way tie, at least one voter will get their last preference and so can change the result by changing their vote, thus it is not Nash. If one candidate gets 2 votes then this is Nash if neither of those voters place that candidate last and if the other voter did not vote for the top candidate of either of them and if player 1 can't change to their winning voter. Thus apart from unanimous votes the Nash equilibria are:

- Voters 1 and 3 vote for  $A$  while voter 2 votes for  $B$ .
- Voters 2 and 3 vote for  $C$  while voter 1 votes for  $A$ .

This gives a total of 5 Nash equilibria.