14.12 Game Theory
Muhamet Yildiz
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Solution to Homework 5

1. **Answer to Problem 1**

   (a) The game can be written formally as follows:
   Actions space:$A_1 = \{X, Y, Z\}$, $A_2 = \{L, R\}$
   Types space: $T_1 = \{t_1\}$, $T_2 = \{\theta = -1, \theta = 1\}$
   Beliefs: $p(\theta = 1|t_1) = p(\theta = -1|t_1) = 1/2$ and $p(t_1|\theta_1) = p(t_1|\theta_2) = 1$
   Finally, the payoffs for the two players are given by:
   $u_1(X, L; \theta = -1, t_1) = 3$ and $u_2(X, L; \theta = -1, t_1) = -1$
   $u_1(Y, L; \theta = -1, t_1) = 2$ and $u_2(Y, L; \theta = -1, t_1) = -2$
   $u_1(Z, L; \theta = -1, t_1) = 0$ and $u_2(Z, L; \theta = -1, t_1) = 0$
   $u_1(X, R; \theta = -1, t_1) = 0$ and $u_2(X, R; \theta = -1, t_1) = 0$
   $u_1(Y, R; \theta = -1, t_1) = 2$ and $u_2(Y, R; \theta = -1, t_1) = -1$
   $u_1(Z, R; \theta = -1, t_1) = 3$ and $u_2(Z, R; \theta = -1, t_1) = 1$
   $u_1(X, L; \theta = 1, t_1) = 3$ and $u_2(X, L; \theta = 1, t_1) = 1$
   $u_1(Y, L; \theta = 1, t_1) = 2$ and $u_2(Y, L; \theta = 1, t_1) = 2$
   $u_1(Z, L; \theta = 1, t_1) = 0$ and $u_2(Z, L; \theta = 1, t_1) = 0$
   $u_1(X, R; \theta = 1, t_1) = 0$ and $u_2(X, R; \theta = 1, t_1) = 0$
   $u_1(Y, R; \theta = 1, t_1) = 2$ and $u_2(Y, R; \theta = 1, t_1) = 1$
   $u_1(Z, R; \theta = 1, t_1) = 3$ and $u_2(Z, R; \theta = 1, t_1) = -1$

   (b) Starting with player 2, for type $\theta = -1$ R strictly dominates L and thus $s_2(\theta = -1) = R$. For type $\theta = 1$, playing L strictly dominates R and thus $s_2(\theta = 1) = L$. Player 1 anticipates that with probability 1/2 player 2 is type $\theta = -1$ and will play R and that with probability 1/2 player 2 is type $\theta = 1$ and will play L. Therefore, his expected payoffs are given by:
   $E(U_1(X)) = 1/2 * U_1(X, R) + 1/2 * U_1(X, L) = 1/2 * 0 + 1/2 * 3 = 3/2$
   $E(U_1(y)) = 1/2 * U_1(Y, R) + 1/2 * U_1(Y, L) = 1/2 * 2 + 1/2 * 2 = 2$
   $E(U_1(Z)) = 1/2 * U_1(Z, R) + 1/2 * U_1(Z, L) = 1/2 * 3 + 1/2 * 0 = 3/2$
   Therefore, the optimal strategy for player 1 is to play Y.
   Thus, the Bayesian Nash equilibrium is given by ($s_1 = y$, $s_2(\theta = -1) = R$, $s_2(\theta = 1) = L$).

i. If it is common knowledge that $\theta = -1$, the payoff matrix is given by:

	L	R
X	$3, -1$	$0, 0$
Y	$2, -2$	$2, -1$
Z	$0, 0$	$3, 1$

By inspection we can verify that the Nash equilibrium is (Z,R).

ii. If it is common knowledge that $\theta = 1$, the payoff matrix is given by:

	L	R
X	$3, 1$	$0, 0$
Y	$2, 2$	$2, 1$
Z	$0, 0$	$3, -1$

By inspection we can verify that the Nash equilibrium is (X,L).

2. **Answer to Problem 2**

(a) The game can be written formally as

$$G = \{A_1, .., A_n; T_1, .., T_n; p_1, .., p_n; u_1, .., u_n\}$$

where $A_i = [0, \infty)$ denotes the action space of player i,
$T_i = \{1, 2\}$ is the type space of player i,
$p_i\left(t_{-i}|t_i\right) = 2^{1-n}$ for each $t_{-i} \in T_1 \times T_2..T_{i-1} \times T_{i+1} \times ..T_n$, describes the beliefs of player i,
and $u_i(x_1, x_2..x_n; t_i) = \theta_i x_i - x_i \sum_{j=1}^{n} x_j$ is the payoff of player i.

(b) Denote by $x_j\left(\theta_j\right)$ the strategy adopted by each player $j \neq i$. Then the expected payoff to player i from transmitting x_i units of data equals

$$E\left[\theta_i x_i - x_i \sum_{j=1}^{n} x_j\left(\theta_j\right)\right]$$

$$= \theta_i x_i - x_i \sum_{j=1}^{n} E x_j\left(\theta_j\right)$$

Then the marginal gain from transmitting an additional unit of data equals

$$\frac{d}{dx_i}\left[\theta_i x_i - x_i \sum_{j=1}^{n} E x_j\left(\theta_j\right)\right]$$

$$= \theta_i - 2x_i - \sum_{j \neq i} E x_j\left(\theta_j\right) \tag{1}$$

This expression is larger for $\theta_i = 2$ for any value of x_i. Therefore, in equilibrium, the higher θ type should choose a (weakly) higher value of x_i. Therefore, we conjecture that the following is a symmetric BNE. $x_j(1) = 0$, $x_j(2) = y > 0$, for $j = 1..n$.

For this to be an equilibrium, the marginal gain from transmitting data should be zero at $x_i = y$ for $\theta_i = 2$ when $x_j(1) = 0$, $x_j(2) = y$ for each $j \neq i$; i.e.

$$2 - 2y - \sum_{j \neq i} \left[\frac{1}{2} 0 + \frac{1}{2} y \right] = 0$$

$$\implies y = \frac{4}{n+3}$$

We also require that the marginal gain from transmitting data should be smaller than or equal to zero at $x_i = 0$ for $\theta_i = 1$ when $x_j(1) = 0$, $x_j(2) = y$ for each $j \neq i$; i.e.

$$1 - 2.0 - \sum_{j \neq i} \left[\frac{1}{2} 0 + \frac{1}{2} y \right] \leq 0$$

$$\implies 0 + (n-1) \frac{y}{2} \geq 1$$

Substituting with $y = \frac{4}{n+3}$, we obtain

$$\begin{aligned} &\iff \quad 2 \frac{(n-1)}{(n+3)} \geq 1 \\ &\iff \quad n \geq 5 \end{aligned}$$

Therefore, if $n \geq 5$ we have a symmetric BNE in which each student sends no data if $\theta = 1$ and $\frac{4}{n+3}$ units of data if $\theta = 2$.