ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #8 – *ANSWER KEY*

Exercise 5-Chapter 28-Watson (*Signaling between a judge and a defendant*)

a. This game has a unique PBE. Find and report it.

After E^1, the judge chooses $\bar{y}$ such that:

$$\underset{\bar{y}}{Max} -(\bar{y}-1)^2$$

Taking FOCs with respect to $\bar{y}$, we obtain:

$$-2(\bar{y}-1)=0$$
$$\rightarrow \bar{y}=1$$

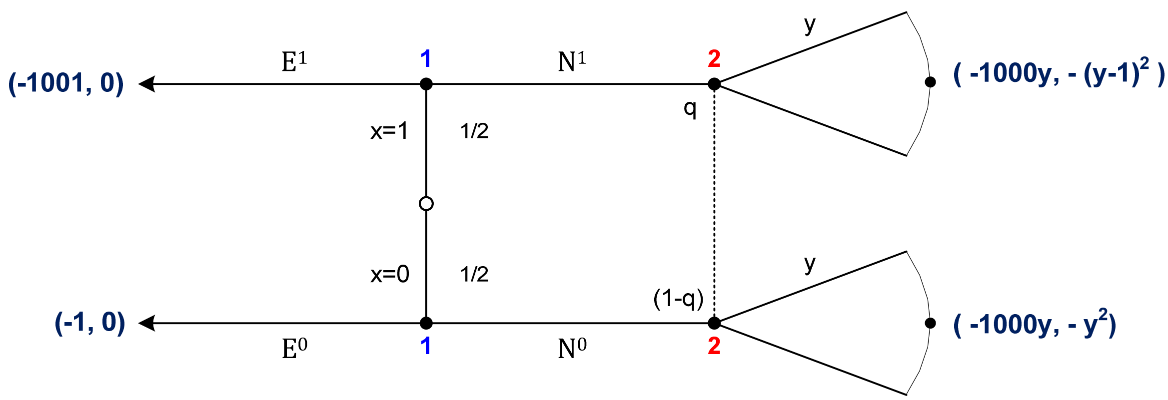
Similarly, after E^0, the judge chooses $\underline{y}$ such that:

$$\underset{\underline{y}}{Max} -\underline{y}^2$$

Taking FOCs with respect to $\underline{y}$, we obtain:

$$-2\underline{y} \leq 0 \quad \rightarrow \quad \underline{y}=0$$

Hence, the game becomes:



- ***Let us first check for the existence of a separating PBE where E^0 and N^1:***

1. *Belief*: q=1 since N only comes from x=1
2. *Judge (second mover)*: After observing N, the judge selects y assigning full probability to being in the open node of his information set (see figure below)

Hence,

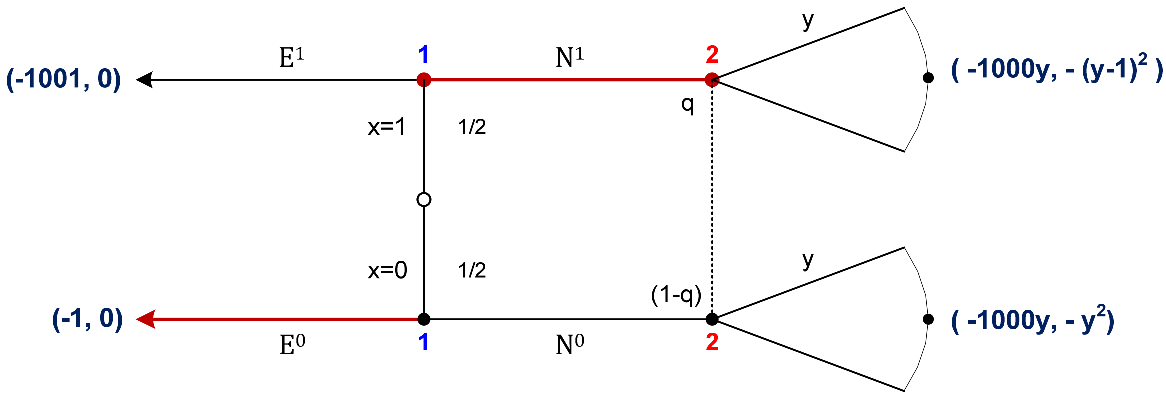
$$\underset{y}{Max} -(y-1)^2$$

Taking FOCs with respect to y, we obtain:

$$-2(y-1)=0, \text{ which implies } y=1$$

3. *Defendant (first mover)*:
   - If $x=1$, the defendant compares:
     - -1001 if he chooses E^1
     - -1000 if he chooses N^1
     - So, N^1 is better.

- If $x = 0$, the defendant compares:
  -1 if he chooses E^0
  -1000 if he chooses N^0
  So, E^0 is better.



Hence, this separating PBE can be supported.

- ***Let us now check the separating $N^0 E^1$***

1. *Beliefs*: $q = 0$ since N only comes from $x = 0$
2. *Judge*: After observing N, the judge assigns full probability to lower node of his information set. Then, he selects y such that:

$$\underset{y}{Max} -y^2$$

Taking FOCs with respect to y, we obtain:

$$-2y \leq 0, \text{ which implies } y = 0$$

3. *Defendant*:
   - If $x = 1$, the defendant compares:
     -1001 if he chooses E^1
     0 if he chooses N^1
     So, N^1 is better $\leftarrow$ Deviation from the prescribed separating $N^0 E^1$.
   - If $x = 0$, the defendant compares:
     -1 if he chooses E^0
     0 if he chooses N^0
     So, N^0 is better. Hence, the separating $N^0 E^1$ cannot be supported as PBE.

- ***Let us now check if a pooling PBE where $N^0 N^1$ can be sustained***

  1. *Beliefs*:

$$q = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times 1} = \frac{1}{2}$$

  2. *Judge*: After observing N, given his beliefs q=1/2, he must choose y in order to maximize his expected utility:

$$\underset{y}{Max} \ \frac{1}{2}[-(y-1)^2] + \frac{1}{2}[-y^2]$$

Taking FOCs with respect to y, we obtain:

$$-\frac{1}{2}\times 2(y-1)-\frac{1}{2}\times 2y=0, \text{ which implies } y=\frac{1}{2}$$

3. *Defendant*:
   - If $x=1$, the defendant compares:
     -1001 if he chooses E^1
     $-1000\times\frac{1}{2}=-500$ if he chooses N^1
     So, N^1 is better
   - If $x=0$, the defendant compares:
     -1 if he chooses E^0
     $-1000\times\frac{1}{2}=-500$ if he chooses N^0
     So, N^0 is better → Deviation from the prescribed pooling. Hence, the pooling N^0N^1 cannot be sustained.

- ***Let us now check if a pooling PBE where E^0E^1 can be sustained.***

1. *Beliefs*: $q\in[0,1]$ since N is only observed off-the-equilibrium
2. *Judge*: From his beliefs, he chooses y in order to maximize his expected utility:

$$\underset{y}{Max}\ \ q[-(y-1)^2]+(1-q)[-y^2]$$

Taking FOCs with respect to y, we obtain:

$$-q\times 2(y-1)-(1-q)\times 2y=0, \text{ which implies } y=q$$

3. *Defendant*:
   - If $x=1$, the defendant compares:
     -1001 if he chooses E^1
     $-1000q$ if he chooses N^1
     So, N^1 is better for any $q<1$ → Deviation from the prescribed pooling.
   - If $x=0$, the defendant compares:
     -1 if he chooses E^0
     $-1000q$ if he chooses N^0
     So, E^0 is better for any $q>1/1000$ → the pooling E^0E^1 cannot be supported as PBE either.

b. Explain why the result of part (a) is interesting from an economic standpoint?

The only equilibrium that we can support in this game is the separating equilibrium in which the innocent defendant provides evidence of his innocence, whereas the guilty defendant does not provide such evidence. This is something desirable, since the judge can perfectly infer the true innocence of a defendant by simply observing whether he/she presented evidences.

c. When $x\in[0,K]$ with each value equally likely, compute the PBE.

We are going to test the equilibrium where all types of x={0,…,K-1} present evidence (E), but the last type x=K presents no evidence (N).

1) <u>Beliefs</u>

   After observing the evidence presented by the defendant, the judge can perfectly observe his type 0,1,2,…,K-1. In these cases we don't need to specify beliefs. When no evidence(N) is presented, the judge's beliefs are:

$$\mu(t_j|N) = 0 \;\; \forall \;\; j = \{0, \dots, K-1\}$$

$$\mu(t_K|N) = 1$$

Which implies that after receiving no evidence, the judge assigns full probability to the K-type, and therefore no probability to any of the 0,1,2,…,K-1 types.

2) Judge's Best Response:

Given N:

$$\max_y -(y-K)^2 \quad \rightarrow \quad y^N = K$$

Given E(where there is no information set and the judge knows what type has played E):

$$\max_y [-(y-x)^2]$$

where x is the specific type of the defendant that presented evidence (a type that is observed by the judge thanks to the presentation of evidence). Taking FOCs with respect to y, we obtain

$$-2y + 2x = 0$$

$$\rightarrow y^E = x$$

3) Defendant's Best Response:

For types 0,…,K-1 : if he provides evidence, E, then they get y^E from the judge, providing:

$$-1000y^E - 1$$

which must exceed his payoff from not presenting evidence: $-1000y^N = -1000K$

(in this case the judge interprets that the defendant is a K-type and chooses a sentence $y^N = K$)

- Note, type x=0 prefers the payoff he obtains by presenting evidence, $-1000y^E - 1 = -1$, than his payoff from not presenting evidence, $-1000K$ (since K > 2 given that there are more than two types of defendants).
- Similarly for type x=1, where $-1000y^E - 1 = -1001 > -1000 * K$; and for all other types x=2,3,…
- The defendant who obtains the lowest equilibrium payoff from providing evidence is x=K-1, who obtains $-1000y^E - 1 = -1000(K-1) - 1$. Let us check if his equilibrium payoff from providing evidence is larger than from deviating, that is:

$$-1000K + 1000 - 1 > \; -1000K$$

$$-1000K + 999 > \; -1000K$$

$$999 > 0$$

This obviously holds, so the defendant behaves as prescribed when his type is x=0,…,K-1

For type K: if he doesn't provide evidence, N, (as initially prescribed) then he gets a sentence $y^N = K$ from the judge, providing a payoff of:

$$-1000K$$

This must exceed his alternative payoff from providing evidence (E):

$$-1000K - 1$$

The condition reduces to:

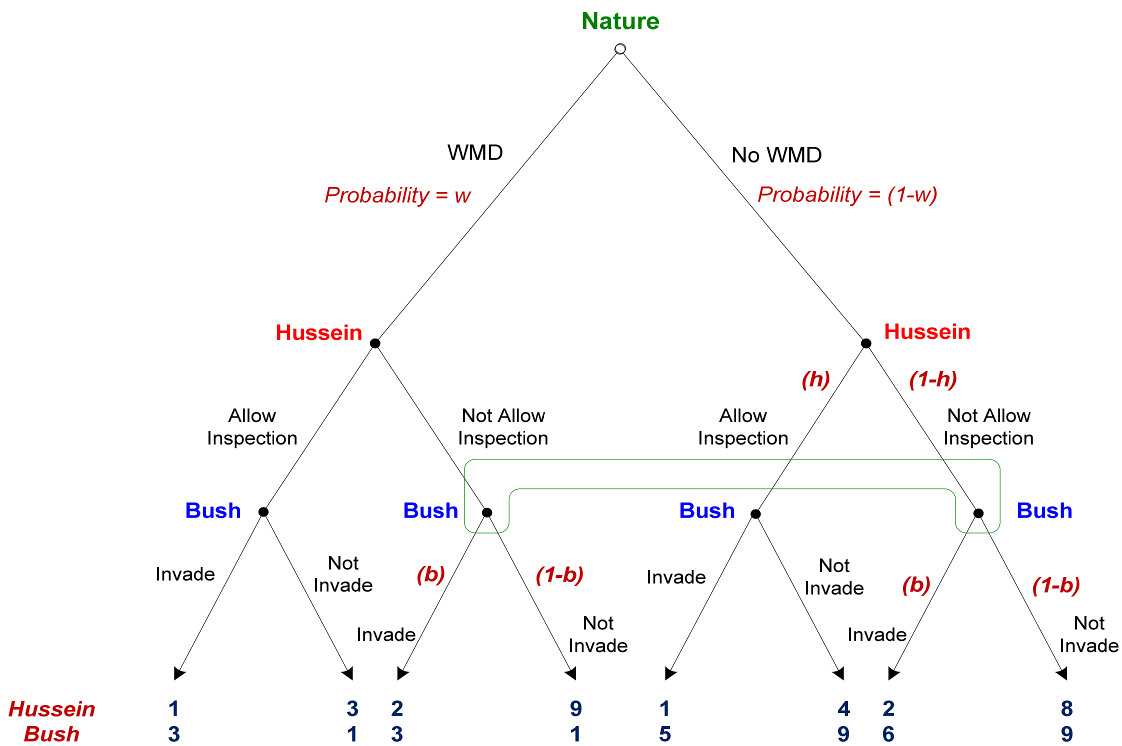
$$-1000K > -1000K - 1$$

$$0 > -1$$

This as well holds, showing the initially stated strategy, where types x=0,1,2,…,K-1 present evidence but type x=K does not, can be sustained as a PBE.

Exercise 4-Chapter 11-Harrington

The extensive form of the WMD game:



1. Nature moves first determining a presence of WMD:

- with probability w Hussein has WMD
- with probability $(1 - w)$ he does not, where $0 < w < 1/3$

2. After observing his own type, Hussein's strategies are the following:

- when he <u>has</u> WMD then he does not allow inspections with probability 1.
- when he <u>does not have</u> WMD then he can choose either to allow inspection with probability h or do not allow – with probability $(1 - h)$.

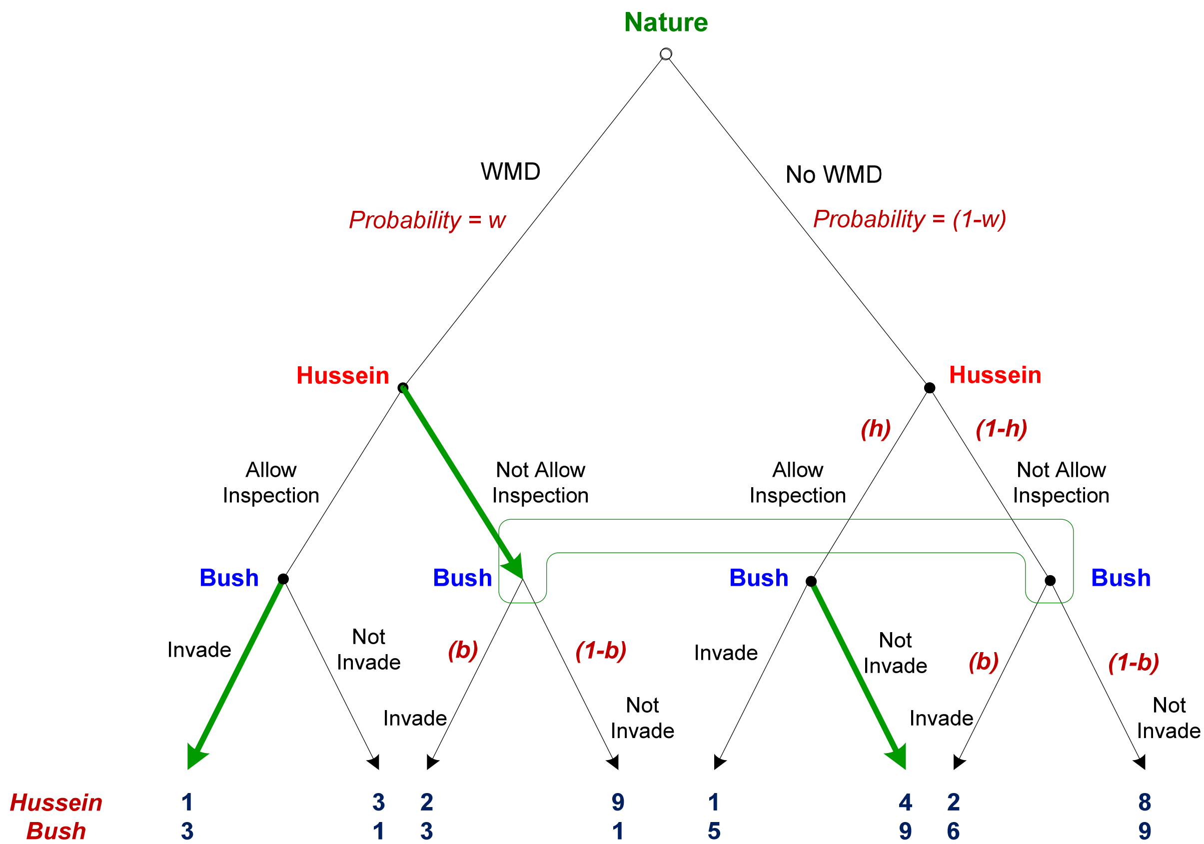
3. Assumptions:

- If Hussein has WMD, then Bush found out it and then Bush wants to invade;
- If Hussein does not have WMD, then Bush does not found it and he prefers not to invade.

After observing Hussein's decision about inspection, Bush strategies are:

- if Hussein allows inspections and WMD are found, then invade with probability 1.
- if Hussein allows inspections and WMD are not found, then do not invade with probability 1.
- if Hussein does not allow inspections, then Bush can guess that with some probability Hussein has WMD, so that Bush invade with probability b.

See graph 1 below:



Steps:

- **Step 1: Bush's beliefs**

  If Hussein does not allow inspections, then the probability of Hussein's having WMD is given by Bayes's rule:

$$P(WMD|Not\ Allow) = \frac{P(WMD) \times P(Not\ Allow, WMD)}{P(Not\ Allow)} = \frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}$$

  where Sadam has WMD with probability w, and in that event, he does not allow inspections with probability 1; and while with probability $(1-w)$, Sadam has WMD and, in that event, he does not allow inspections with probability $(1-h)$.

- **Step 2: Bush's optimal strategy given his beliefs**

  Its optimality is clear when there are inspections, whether WMD are found or not.

  - When inspections <u>are not allowed</u>, Bush is content to randomize (that is, $0 < b < 1$) if and only if:

$$E^{Bush}[INV|WMD\ or\ No\ WMD] = E^{Bush}[No\ INV|WMD\ or\ No\ WMD\ and\ NA]$$

$$3\left[\frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}\right] + 6\left[\frac{w \times (1-h)}{w \times 1 + (1-w) \times (1-h)}\right]$$
$$= \left[\frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}\right] + 9\left[\frac{w \times (1-h)}{w \times 1 + (1-w) \times (1-h)}\right]$$

The left-hand expression is the expected payoff from invading, and the right-hand expression is the expected payoff from not invading. Solving this equation for h yields:

$$h = \frac{3-5w}{3(1-w)}$$

Note that:

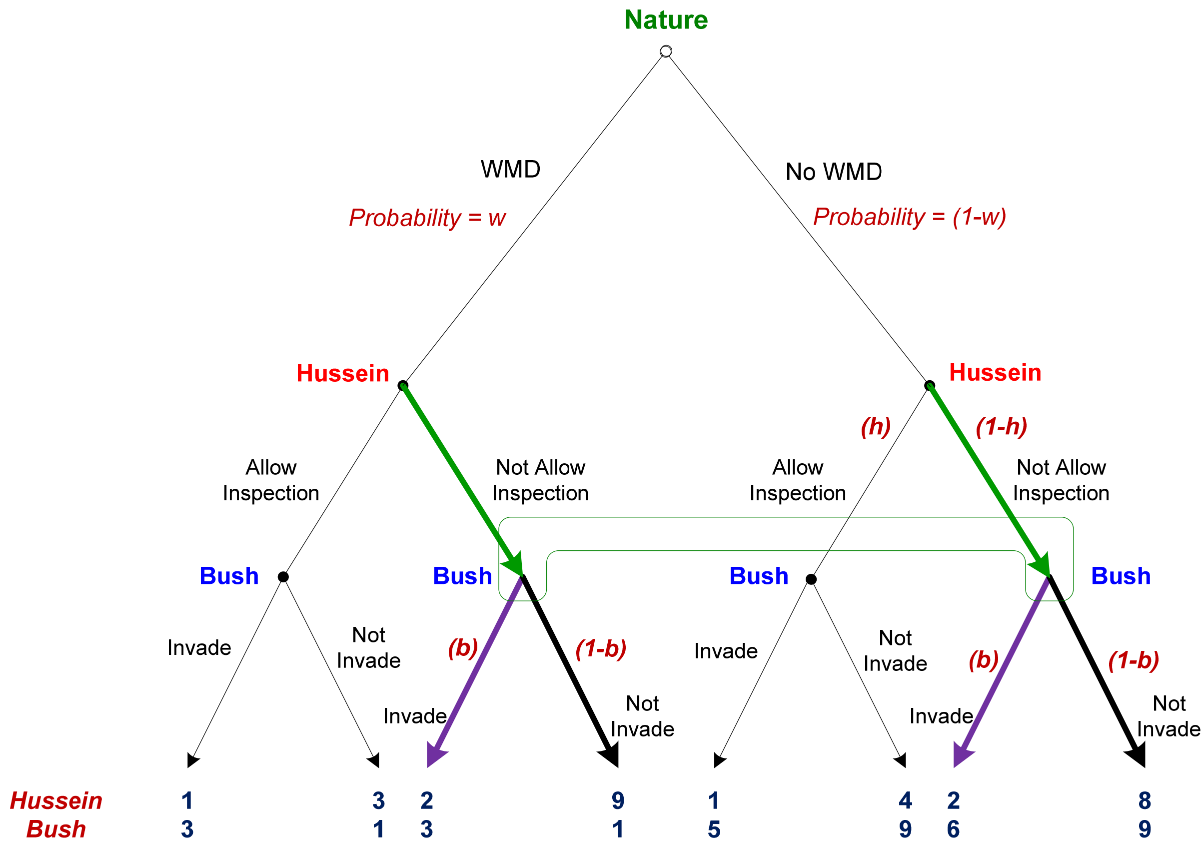
$$0 < \frac{3-5w}{3(1-w)} < 1 \quad when \ \ 0 < w < \frac{3}{5}$$

The latter condition was assumed. See graph 2 below.

  - When Sadam <u>has WMD</u>, it is clearly optimal for him to not allow inspections. When he does not have WMD, it is optimal to randomize if and only if:

$$2b + 8(1-b) = 4$$

where he earns a payoff of 4 by allowing inspections – in which case there is no invasion-- and gets an expected payoff of $2b + 8(1-b) = 4$ from not allowing inspections (where there is an invasion with probability b). Solving this equation, we can get $b = 2/3$.



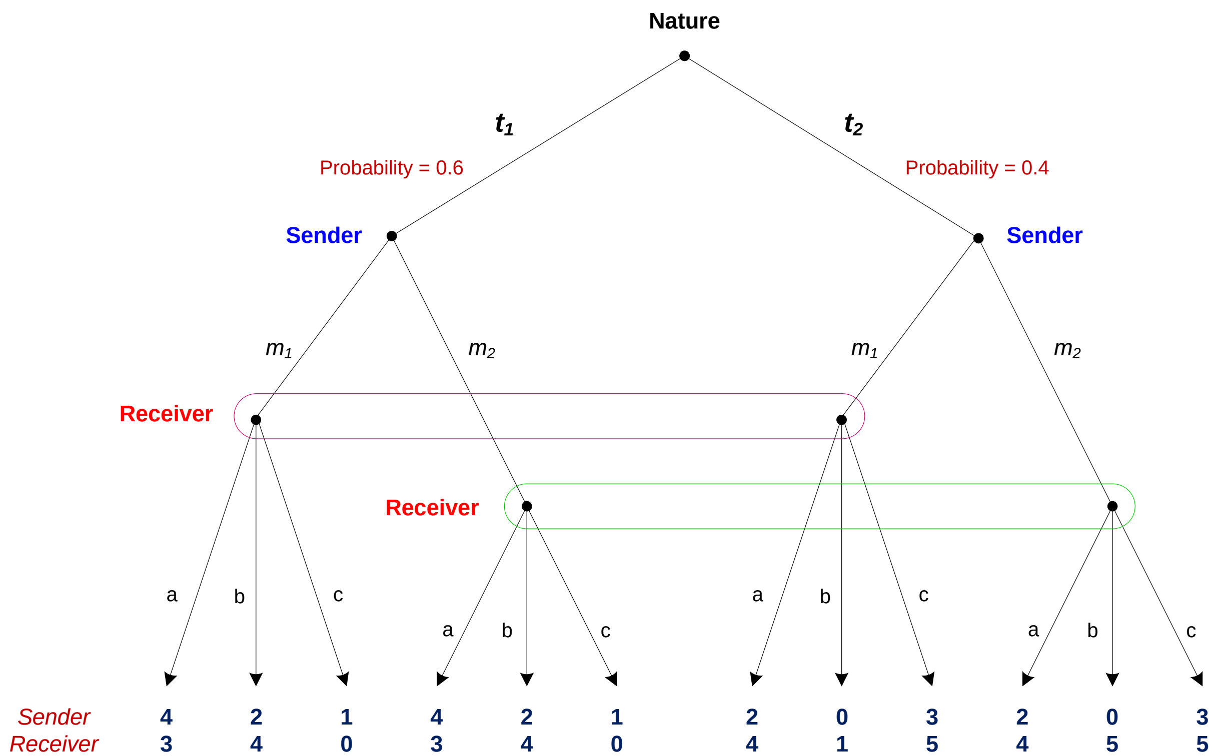
$$E^{Bush}[INV|WMD\ or\ No\ WMD] = P(NA|WMD) \times U(INV) + P(NA|No\ WMD) \times U(INV)$$

$$= 3 \times P(NA|WMD) + 6 \times P(NA|No\ WMD)$$

$$E^{Bush}[NINV|WMD\ or\ No\ WMD\ and\ NA] = 1 \times P(NA|WMD) + 9 \times P(NA|No\ WMD)$$

<u>Exercise 5-Chapter 12-Harrington</u>

Consider the cheap talk game:



a. Find a separating PBNE.

With a separating equilibrium, the sender chooses distinct messages, so let us presume that the sender chooses m_1 when his type is t_1 and chooses m_2 when his type is t_2. (We could instead have supposed that the sender's strategy is to choose m_2 when his type is t_1, and m_1 when his type is t_2.)

<u>Receiver's beliefs</u>

After observing message m_1,

$$\mu(t_1|m_1) = 1$$

$$\mu(t_2|m_1) = 0$$

And after observing message m_2,

$$\mu(t_1|m_2) = 0$$

$$\mu(t_2|m_2) = 1$$

Receiver's optimal response

- After observing m_1, the receiver believes that such a message can only originate from a t_1-type of sender. Hence, his optimal response is b given that it yields a payoff of 4 (higher than what he gets from a, 3, and c, 0.)
- After observing message m_2, the receiver believes that such a message can only originate from a t_2-type of sender. Hence, his optimal response is either b or c, since both yield a payoff of 5, rather than a, which only provides a payoff of 4. For simplicity, we choose c.

Sender's optimal messages

- If his type is t_1, by sending m_1 he obtains a payoff of 2 (since m_1 is responded with b), but a lower payoff of 1 if he deviates towards message m_2 (since such message is responded with c). Hence, the sender doesn't want to deviate from m_1. [Note that if m_2 were responded with b, then the sender would be indifferent between m_1 and m_2 (both would yield a payoff of 2). Strictly speaking, he wouldn't have incentives to deviate from message m_1].
- If his type is t_2, he obtains a payoff of 3 by sending message m_2 (which is responded with c) and a payoff of 0 if he deviates to message m_1 (which is responded with b). Hence, he doesn't have incentives to deviate from m_2. [Similarly as above, note that if message m_2 were responded with b the sender would obtain the same payoff sending m_2 and m_1, 0. Nonetheless, t_2-sender wouldn't have incentives to deviate from his initially prescribed message of m_2].

Hence, the initially prescribed separating strategy profile can be supported as a PBE.
Also note that there is another separating equilibrium in which m_1 and m_2 are exchanged.

b. Find a pooling PBNE.

With a pooling PBNE, the sender chooses the same message regardless of his type. Let this message be m_1.

Receiver's beliefs

After observing message m_1 (in problem),

$$\mu(t_1|m_1) = \frac{0.6 * 1}{0.6 * 1 + 0.4 * 1} = 0.6$$

$$\mu(t_2|m_1) = \frac{0.4 * 1}{0.6 * 1 + 0.4 * 1} = 0.4$$

After receiving message m_2 (off-the-equilibrium),

$$\mu(t_1|m_1) = \frac{0.6 * 0}{0.6 * 0 + 0.4 * 0} = \frac{0}{0}$$

and hence beliefs must be arbitrarily specified, i.e. $\mu \in [0,1]$.

<u>Receiver's optimal response</u>

- After receiving a message m_1, the receiver's expected utility from responding with actions a, b, and c are

$$\text{Action } a\text{:} \quad 0.6 \times 3 + 0.4 \times 4 = 3.4$$
$$\text{Action } b\text{:} \quad 0.6 \times 4 + 0.4 \times 1 = 2.8$$
$$\text{Action } c\text{:} \quad 0.6 \times 0 + 0.4 \times 5 = 2.0$$

Hence, the receiver's optimal strategy is to choose action a in response to message m_1.

- After receiving message m_2 (off-the-equilibrium), the receiver's EUs from each of his three possible responses are

$$EU_{Receiver}(a|m_2) = \mu * 3 + (1-\mu) * 4 = 4 - \mu$$
$$EU_{Receiver}(b|m_2) = \mu * 4 + (1-\mu) * 5 = 5 - \mu$$
$$EU_{Receiver}(c|m_2) = \mu * 0 + (1-\mu) * 5 = 5 - 5\mu$$

  Where it's clear that the EU from responding with b, $5 - \mu$, is the highest EU payoff the responder can obtain given that $\mu \in [0,1]$.

<u>Sender's optimal message</u>

- If his type is t_1, the sender obtains a payoff of 4 from sending m_1 (since it is responded with a), but a payoff of only 2 when deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.
- If his type is t_2 , the sender obtains a payoff of 2 by sending m_1 (since it is responded with a), but a payoff of only 0 by deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.

Therefore, the initially prescribed pooling strategy profile where both types of sender select $m_1$can be sustained as a PBE of the game.

There are other babbling equilibria that differ in terms of the message sent by the sender and the receiver's beliefs in response to a message that the sender never sends (according to his strategy). For any babbling equilibrium, it must be the case that the receiver ends up choosing action a.

c. Suppose the probability that the sender is type t_1 is p and the probability that the sender is type t_2 is $(\mathbf{1}-\boldsymbol{p})$. Find the values for p such that there is a pooling PBNE in which the receiver chooses action b.

For any pooling equilibrium, the sender's strategy has him choose the same message—let it be m_1 – for any type and, in response to observing that message; the receiver's beliefs are her prior beliefs.

<u>Receiver's beliefs</u>

After observing message m_1,

$$\mu(t_1|m_1) = \frac{p * 1}{p * 1 + (1-p) * 1} = p$$

$$\mu(t_2|m_1) = \frac{(1-p) * 1}{p * 1 + (1-p) * 1} = 1 - p$$

After receiving message m_2 (off-the-equilibrium),

$$\mu(t_1|m_1) = \frac{p * 0}{p * 0 + (1-p) * 0} = \frac{0}{0}$$

And hence beliefs must be arbitrarily specified, i.e. $\mu \in [0,1]$.

<u>Receiver's optimal response</u>

- After receiving a message m_1 (in equilibrium), the receiver's expected utility from responding with actions a, b, and c are

$$\text{Action } a\text{:}\quad p \times 3 + (1-p) * 4 = 4 - p$$

$$\text{Action } b\text{:}\quad p \times 4 + (1-p) * 1 = 1 + 3p$$

$$\text{Action } c\text{:}\quad p \times 0 + (1-p) * 5 = 5 - 5p$$

For it to be optimal to choose action b, it must be the case that

$$1 + 3p \geq 4 - p \quad \rightarrow \quad p \geq \frac{3}{4} \quad \textit{and} \quad 1 + 3p \geq 5 - 5p \quad \rightarrow \quad p \geq \frac{1}{2}$$

Thus if $p < 3/4$, then the receiver does not choose action b at a pooling equilibrium, as she would prefer action a. If $p \geq \frac{3}{4}$, then it is the receiver's optimal strategy to choose b in response to message m_1.

- After receiving message m_2 (off-the-equilibrium), the receiver's EUs from each of his three possible responses are

$$EU_{Receiver}(a|m_2) = \mu * 3 + (1-\mu) * 4 = 4 - \mu$$

$$EU_{Receiver}(b|m_2) = \mu * 4 + (1-\mu) * 5 = 5 - \mu$$

$$EU_{Receiver}(c|m_2) = \mu * 0 + (1-\mu) * 5 = 5 - 5\mu$$

Where it's clear that the EU from responding with b, $5-\mu$, is the highest EU payoff the responder can obtain given that $\mu \in [0,1]$.

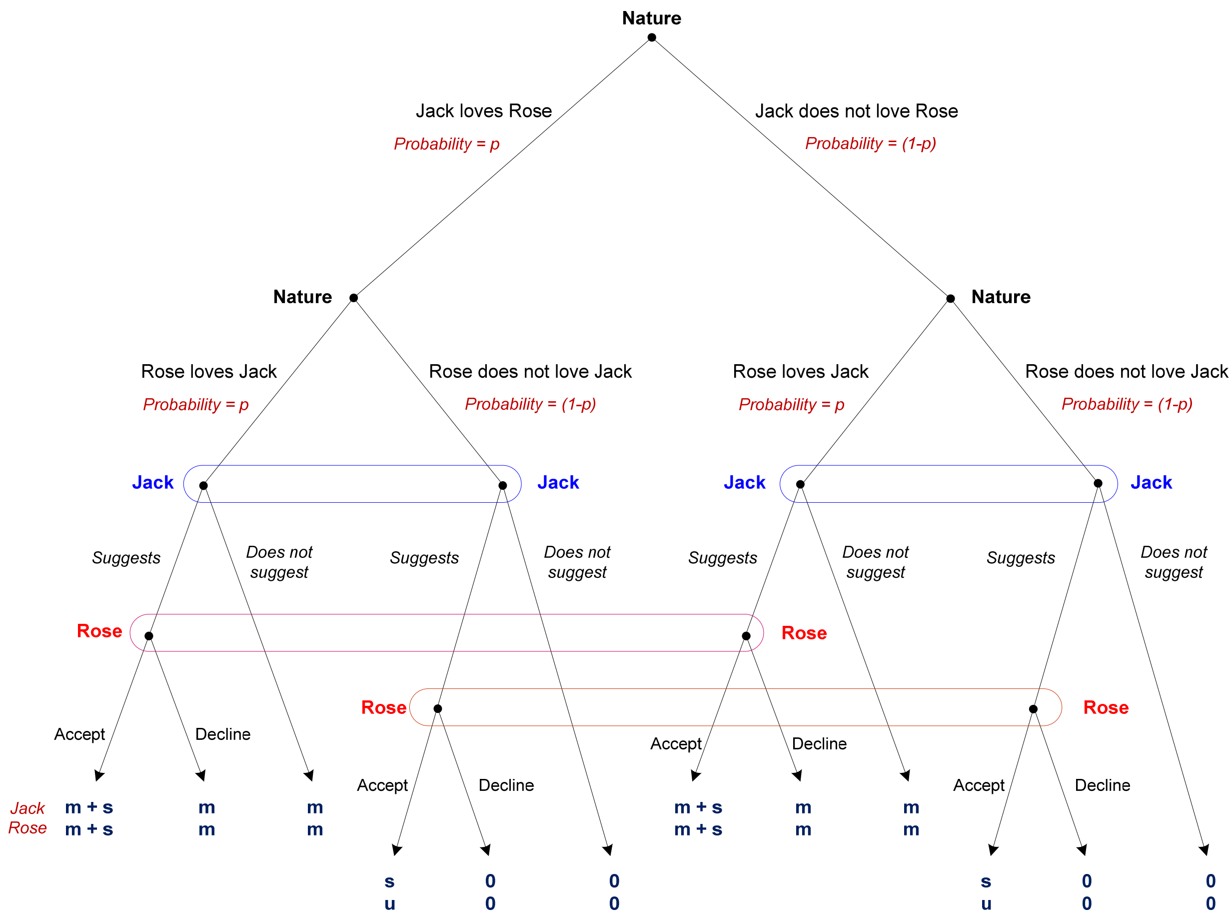
Sender's optimal message

- If his type is t_1, the sender obtains a payoff of 2 from sending m_1 (since it is responded with b), but a payoff of only 2 when deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.
- If his type is t_2 , the sender obtains a payoff of 0 by sending m_1 (since it is responded with b), but a payoff of only 0 by deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.

Therefore, the initially prescribed pooling strategy profile where both types of sender select $m_1$can be sustained as a PBE of the game when $p \geq \frac{3}{4}$.

Exercise 7-Chapter 12-Harrington

Consider the Courtship Game with Cheap Talk



Show that there is no PBNE in which premarital sex occurs.

Consider a strategy profile in which Rose accepts Jack's proposal when it is made. To begin, it is clear that she would never accept having sex with someone she doesn't love. Doing so results in a payoff of $u < 0$ (as she knows she isn't going to marry Jack), while not having sex results in a payoff of zero. Thus, if there is an equilibrium with premarital sex, it would only involve Rose having sex with Jack if she loves him. Suppose Rose does act in that manner, accepting if she loves Jack but declining if she does not.

Jack. *What is an optimal response for Jack?* Regardless whether or not he loves Rose, his payoff is higher by having sex. Thus, he'll ask for sex; he has nothing to lose. If Rose doesn't love him, then she'll decline and his payoff is zero. If she does love him, then his payoff is higher by s. More specifically, if he loves Rose, then his expected payoff from asking for sex is:

$$EU_{Jack}(sex|loves\ Rose) = p \times (m + s) + (1 - p) \times 0 = p(m + s)$$

and from not asking is $EU_{Jack}(not\ sex|loves\ Rose) = p * m + (1 - p) * 0 = p * m$. Thus, Jack asks for sex.

Rose. Is Rose's strategy of accepting if she loves Jack optimal given Jack asks regardless whether he loves her? Her payoff from accepting his proposition is:

$$EU_{Rose}(accept|loves\ Jack) = p \times (m + s) + (1 - p) \times u$$

Since both Jack types ask, Rose doesn't learn anything about whether he wants to marry her from the fact that he wants to have sex with her. Recall that we are evaluating this in the case when she loves Jack. If she doesn't have sex, then her expected payoff is:

$$EU_{Rose}(reject|loves\ Jack) = p \times m + (1 - p) \times 0 = pm$$

Thus, it is indeed optimal for Rose to accept if and only if:

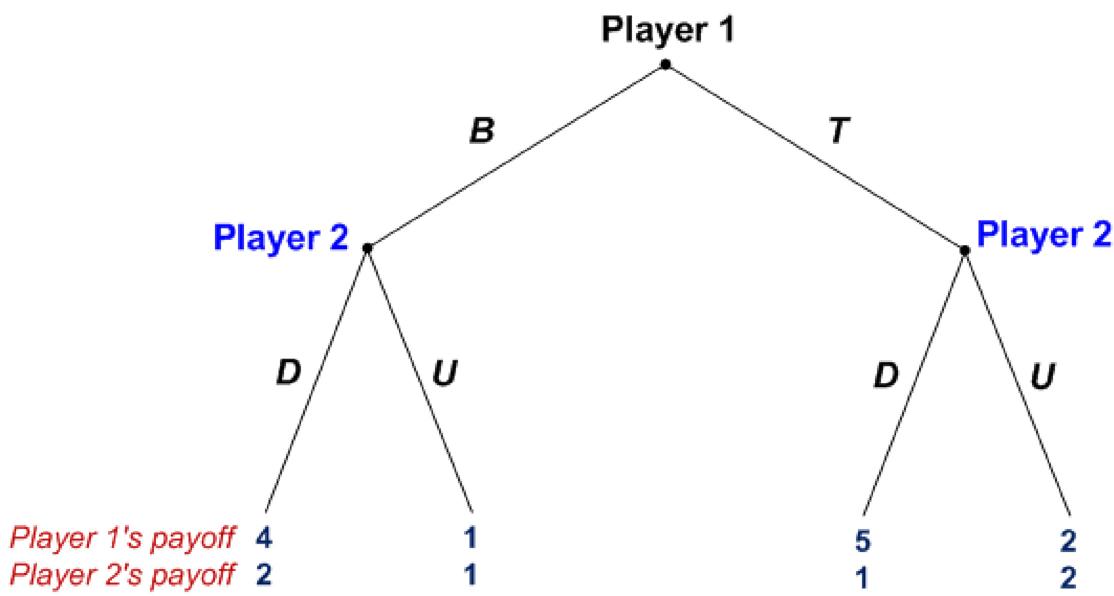
$$p \times (m + s) + (1 - p) \times u > p * m$$

which is equivalent to $ps + (1 - p)u > 0$.

If this condition does not hold, then Rose would prefer to decline even if she loves Jack. In that case, there is no PBE in which premarital sex occurs.

Bonus Exercise #1 (Based on work by Kyle Bagwell, and developed as an exercise by Eric Maskin)

Consider the following extensive form game.



a) Find a subgame perfect Nash equilibrium (SPNE) of this game.
   - Is it unique? Are there any other Nash equilibria?

ANSWER: The extensive form game is given in the following figure.

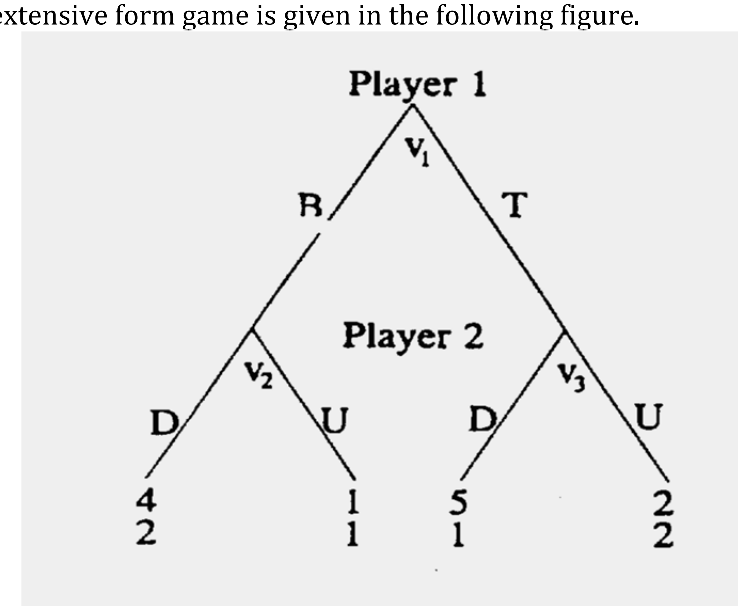


Figure 9.C.7(a)

Set of pure strategies: The set of pure strategies for player 1 is $S_1 = \langle B, T \rangle$. For player 2 his set of pure strategies is $S_2 = \langle DD, DU, UD, UU \rangle$ where the first component of every pair means playing a given action (either D or U) after observing that player 1 chose B (i.e., at node V_2), while the second component of the pair describes the action that player 2 selects after observing that player 1 chose T, i.e., at node V_3.

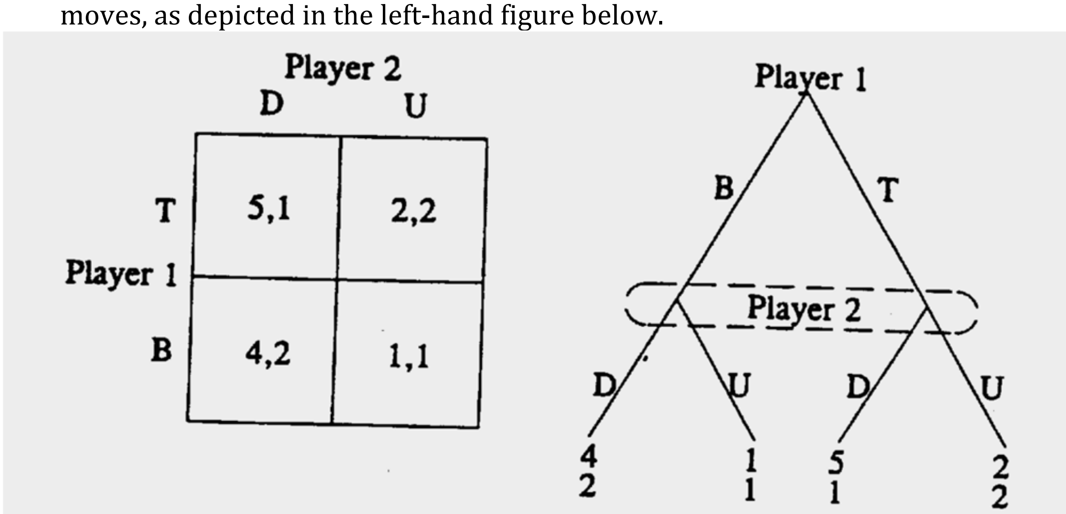
SPNE: By backward induction it's easy to see that the unique SPNE is (B,DU).
NE: There are two more classes of NE:

1. Player 1 plays T, and player 2 plays UU with probability p and DU with probability 1-p, where $p \geq \frac{2}{3}$; and
2. Player 1 plays B, and player 2 plays DU with probability p and DD with probability 1-p, where $p \geq \frac{1}{3}$.

b) Now suppose that player 2 cannot observe player 1's move. Draw the new extensive form. What is the set of Nash equilibria?

ANSWER: When player 2 cannot observe player 1's action before choosing D or U, the extensive form game must include a information set when player 2 is called on to move, as depicted in the right-hand figure below.

- Regarding the normal form representation of the game, notice that the game became a simultaneous-move game in which players are unable to observe each others' moves, as depicted in the left-hand figure below.

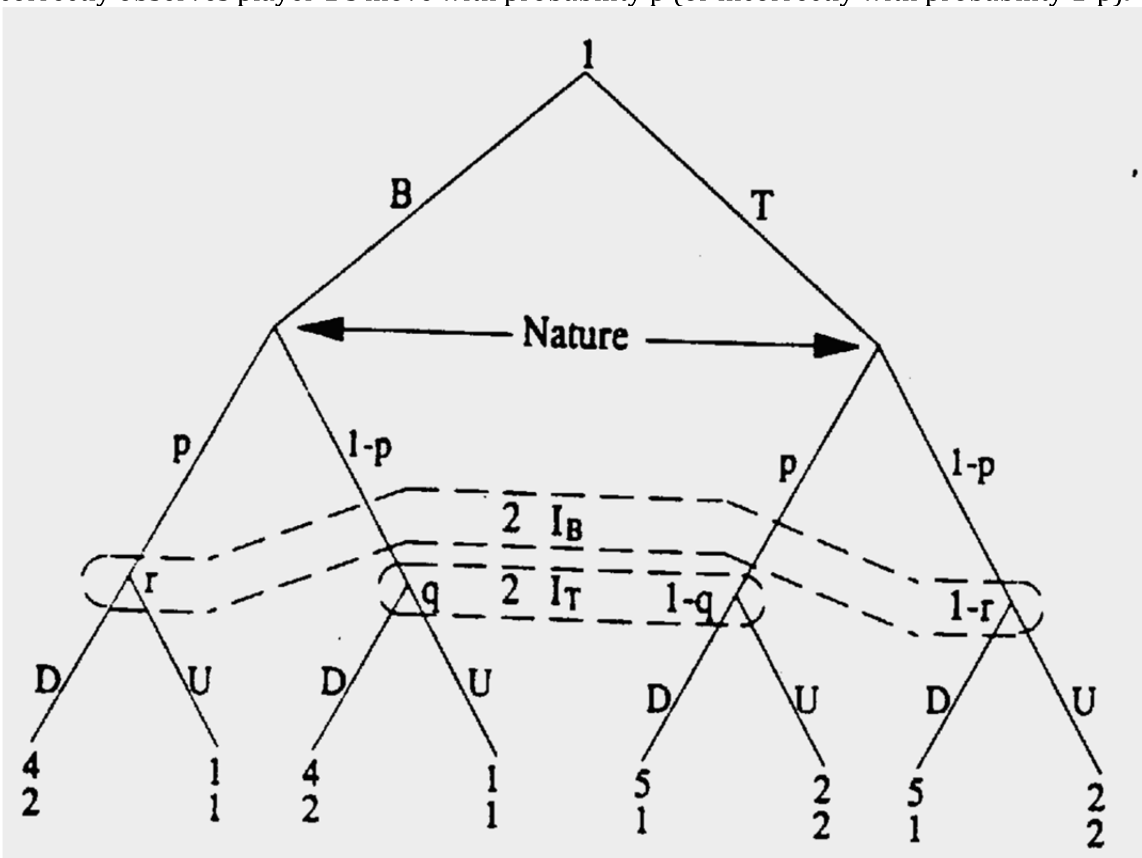


We can now identify players' best responses (you can do the standard underlining we did in class). Since playing T is strictly dominant strategy for player 1, we have a unique NE: (T,U).

Now suppose that player 2 observes player 1's move correctly with probability $p \in (0,1)$ and incorrectly with probability $1-p$ (e.g., if player 1 plays *T*, player 2 observes *T* with probability *p* and observes *B* with probability 1 – *p*). Suppose that player 2's propensity to observe player 1's move incorrectly (i.e., given by the value of *p*) is common knowledge to the two players.

c) What is the extensive form now? Draw it.

ANSWER: The modified game is depicted in the figure below. Importantly, note that nature acts in this game *after* player 1 chooses B or T, by introducing the possibility that player 2 correctly observes player 1's move with probability p (or incorrectly with probability 1-p).



Notation used in the figure:

- I_k denotes player 2's information set after she observes $k \in \langle B, T \rangle$,
- r is the belief she assigns to the event that player 1 played B after she finds herself in information set I_B. That is, after player 2 observes B, r is the belief that player 1 indeed played B while 1-r is the probability that, despite observing B, the action that player 1 selected was actually T.
- Similarly, q is the belief she assigns to the event that, despite observing action B, player 1 actually chose T. In contrast, 1-q denotes the probability that player 2 observes T originating from player 1 and, indeed, this was the action that player 1 selected.

d) Show that there is a unique Perfect Bayesian equilibrium (PBE). What is it?

ANSWER: Let $s \in [0,1]$ denote the probability that player 1 plays B. We can hence have three possible situations in a PBE:

1. First, player 1 plays B, i.e., s=1;
2. Second, player 1 plays T, i.e., s=0; and
3. Third, player 1 randomizes between B and T, i.e., $s \in (0,1)$.

Let us next separately analyze each of these three strategy profiles:

1. Player 1 playing B, i.e., s=1, cannot be part of a PBE. Indeed, if this were the case we must have that player 2's beliefs become q=r=1, which implies that player 2 will always respond playing D. But given that P2 always plays D, player 1 will prefer to deviate from B and play T.
2. Second, player 1 playing T, i.e., s=0, is part of a PBE. Let us show it by following the standard procedure examined in class. In this strategy profile, player 2's beliefs become q=r=0, which implies that player 2 will always respond playing U. Given that P2 always plays U, player 1 will prefer to play T (as prescribed). Thus, player 1 playing T and player 2 responding with U in each of her information sets is a PBE.
3. To consider the possibility of a PBE in which player 1 randomizes between B and T, i.e., $s \in (0,1)$, we first note that this will induce player 2 to a update his posterior beliefs q and r using Bayes' rule. In particular, in such an equilibrium we must have:

$$r = \frac{s \cdot p}{(1-s)(1-p)+s \cdot p}, \text{ and } q = \frac{s(1-p)}{s(1-p)+p(1-s)}$$

Simple algebra shows that $s \overset{>}{\underset{<}{=}} p$ if and only if $s \overset{>}{\underset{<}{=}} \frac{1}{2}$, and that $s \overset{>}{\underset{<}{=}} (1-p)$ if and only if $r \overset{>}{\underset{<}{=}} \frac{1}{2}$. This observation allows us to concentrate on 4 cases as follows:

- **<u>s > p and s > (1-p)</u>**: In this case we must have that beliefs satisfy $q > \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will always respond with D, which in turn implies that player 1's best response is to play T in pure strategies, i.e., s=0, thus violating the initially prescribed strategy profile that predicted player 1 randomizes between B and T. Therefore, we cannot support a PBE in case (a).
- **<u>s < p and s < (1-p):</u>** In this case we must have that beliefs satisfy $q < \frac{1}{2}$ and $r < \frac{1}{2}$. This implies that player 2 will always respond with U, which in turn implies that player 1's best response is to choose T in pure strategies, i.e., s=0, thus violating the initially prescribed strategy profile that predicted player 1 randomizes between B and T. Therefore, we cannot support a PBE in case (b) either.
- **<u>(1 – p) < s < p:</u>** (which implies that $p > \frac{1}{2}$). In this case we must have beliefs must satisfy $q < \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will respond with U in information set I_T but respond with D in information set I_B. Player 1's best response will now depend on p. In particular, player 1 selecting B gives him an expected payoff of $4p+1(1-p)$, and playing T gives him an expected payoff of $2p+5(1-p)$. If $p \neq \frac{2}{3}$ then player 1 will have a unique best response (i.e., he plays pure strategies, either selecting B or T), which rules out such PBE. However, if $p = \frac{2}{3}$ then we have that player 1 is indifferent between selecting B and T, and a mixed-strategy PBE can be sustained as follows:
    - Player 1 plays B with probability $s \in (\frac{1}{3}, \frac{2}{3})$, and

  - Player 2 responds with U in information set I_T and with D in information I_B.

- **p < s < (1-p):** (which implies $p < \frac{1}{2}$) This case is symmetric to case (c) above. If $p \neq \frac{1}{3}$ then player 1 will have a unique best response (i.e., he plays pure strategies, either selecting B or T), which rules out such PBE. However, if $p = \frac{1}{3}$ then we have a mixed-strategy PBE as follows:
  - Player 1 plays B with probability $s \in (\frac{1}{3}, \frac{2}{3})$, and
  - Player 2 will respond with D in information set I_T and with U in information set I_B.

To summarize, there exists a unique pure strategy PBE as described earlier, and if p is randomly drawn from the interval (0,1) then the pure strategy NE identified in exercise (b), i.e., (T,U), is the unique PBE with probability 1. However, if $p = \frac{1}{3}$ or $p = \frac{2}{3}$ then in addition there exists a mixed-strategy PBE as described in cases (c) and (d).