14.12 Game Theory
Fall 2005

Answers to Midterm 2, Fall 2005

Answer to Problem 1

This game has two subgames. One subgame is the game as a whole and the other is the subgame after players 1 and 2 choose to play R and r, respectively. In this last subgame we have 2 pure Nash equilibria (NE) which are (A, a) and (B, b).

To compute the subgame-perfect equilibrium (SPE) we have to check if the strategy profile is a NE in every subgame. Thus, to obtain the SPE, we have to assume that outcome of the smaller subgame will be one of those NE.

When players 1 and 2 play (A, a) in the smaller subgame, the reduced form of the game is:

	l	r
L	$3,1$	$2,0$
R	$2,0$	$3,3$

There are two NE in this case (L, l) and (R, r). Therefore, the SPE are (LA, la) and (RA, ra).

When they play (B,b) in the smaller subgame, the reduced form of the game is:

	l	r
L	$3,1$	$2,0$
R	$2,0$	$1,1$

In this case there is just one NE (L, l).Therefore, the SPE is (LB, lb).

Therefore, this game has three SPE: (LA, la), (RA, ra) and (LB, lb).

Answer to Problem 2

1. (a) This is not SPE because in the punishment mode each player wants to deviate and play L instead of M increasing his payoff from 3 to 7 in the current period. This has no impact on the future payoffs because in the following period they will automatically go to the collusion mode no matter what they played in the punishment mode.

   (b) This is SPE. If player i sticks to his equilibrium strategy, then he gets 6 forever. The present value of this is $6/(1-\delta)$. If player deviates, and chooses M he gets 8 in the current period and the game switches to the punishment mode for just one period where he gets 2. Thereafter, the game switches back to the collusion mode and the player i will get 6 forever. Thus his total discounted payoff will be $8+\delta 2++\delta^2 \frac{6}{1-\delta}$. Therefore, he does not have an incentive to deviate if:

$$6/(1-\delta) \geq 8 + \delta 2 + +\delta^2 \frac{6}{1-\delta} \Leftrightarrow 6 + 6\delta \geq 8 + \delta 2 \Leftrightarrow \delta \geq 1/2$$

In the Punishment mode, if player i sticks to his equilibrium strategy, he gets 2 in the current period and gets 6 thereafter. If player i deviates and plays H or M he gets 0 in the current period instead of 2 and 6 thereafter, since the game switches back to the collusion mode no matter what they choose in the punishment mode. Thus, player i doesn't have any incentive to deviate. Therefore, since $\delta = 0.99 > 1/2$ the strategy profile is SPE.

Answer to Problem 3

First compute total profits from collusion and from deviation when all other firms are colluding:

If each firm produces $\frac{1}{2n}$ units, $P = 1 - n\left(\frac{1}{2n}\right) = \frac{1}{2}$.

Therefore, profit to each firm $= \left(\frac{1}{2n}\right)\left(\frac{1}{2} - 0\right) = \frac{1}{4n}$.

If $n-1$ firms produce $\frac{1}{2n}$ units, the profit-maximising level of output for the n^{th} firm is given by

$$\arg\max_q \left\{1 - (n-1)\left(\frac{1}{2n}\right) - q - 0\right\} q$$

$$= \frac{n+1}{4n}$$

Therefore, profit for the firm from the most profitable deviation $= \left(\frac{n+1}{4n}\right)^2$.

Now compute total profits from the 'punishment-phase' and from deviation in this phase when all other firms are producing x.

If each firm produces x units, $P = 1 - nx$.

Profit to each firm $= (1 - nx)\, x$.

If $n-1$ firms produce x units, profit maximising level of output for the n^{th} firm is given by

$$\arg\max_q \{1 - (n-1)\, x - q\}\, q$$

$$= \frac{1 - (n-1)\, x}{2}$$

Therefore, profit for the firm from the most profitable deviation $= \left[\frac{1-(n-1)x}{2}\right]^2$.

Now, we are able to write down the conditions under which a single period deviation is profitable in the collusion and punishment modes.

In the collusion mode, gain from deviation

$$= \left(\frac{n+1}{4n}\right)^2 - \frac{1}{4n}$$

and loss from punishment in the following period

$$= \frac{1}{4n} - (1 - nx)\, x$$

Therefore, for this kind of deviation to be unprofitable we need

$$\left(\frac{n+1}{4n}\right)^2 - \frac{1}{4n} \leq \delta \left[\frac{1}{4n} - (1 - nx)\, x\right]$$

In the punishment mode, gain from deviation

$$= \left[\frac{1 - (n-1)\, x}{2}\right]^2 - (1 - nx)\, x$$

and loss from punishment in the following period

$$= \frac{1}{4n} - (1 - nx)\, x$$

Therefore, for this kind of deviation to be unprofitable, we need

$$\left[\frac{1 - (n-1)\, x}{2}\right]^2 - (1 - nx)\, x \leq \delta \left[\frac{1}{4n} - (1 - nx)\, x\right]$$

The given strategy profile is a subgame perfect equilibrium if and only if the two inequalities above are satisfied.

Answer to Problem 4

1. (a) For the one-period game, suppose player 1 is selected to propose. Player 2 will receive zero if she rejects an offer. Therefore, she accepts any offer that provides her something greater than or equal to zero. Therefore, player 1 will offer $(1, 0)$.

   Similarly, if player 2 is selected to propose, then player 1 accepts any offer that provides him something greater than or equal to zero. Therefore, player 2 will offer $(0, 1)$.

   Thus, for each player, the value of being selected as the proposer is 1; and the value of not being selected as the proposer is 0. Therefore, before the selection is made, expected payoff to player 1 is $p\,(1) + (1-p)\,0 = p$; and the expected payoff to player 2 is $(1-p)\,1 + p\,(0) = 1 - p$.

   (b) For the two-period game, suppose player 1 is selected to propose at $t = 1$. If player 2 rejects an offer she will have an expected payoff of $(1-p)$ from the 1-period game to come, as computed in part (a), discounted by the factor δ. Therefore, at $t = 1$, player 2 will accept any offer that provides her something greater than or equal to $\delta(1-p)$. Therefore, player 1 will offer $(1-\delta(1-p), \delta(1-p))$.

Similarly, if player 2 is selected to propose at $t=1$, then player 1 will accept any offer that provides him something greater than or equal to δp. Therefore, player 2 will offer $(1-\delta p, \delta p)$.

Therefore, player 1 receives a payoff of $1-\delta(1-p)$ if selected to propose at $t=1$; and δp if not. Therefore, before a selection is made at $t=1$, the expected payoff to player 1 from the game is $p[1-\delta(1-p)]+(1-p)\delta p$ which simplifies to p.

Similarly, player 2 receives a payoff of $1-\delta p$ if selected to propose at $t=1$; and $\delta(1-p)$ if not. Therefore, before a selection is made at $t=1$, the expected payoff to player 2 from the game is $(1-p)(1-\delta p)+p[\delta(1-p)]=(1-p)$.

(c) Given the answers to parts (a) and (b), the following strategy profile seems to be a reasonable candidate for subgame perfect equilibrium: If selected to propose, player 1 offers $(1-\delta(1-p), \delta(1-p))$; when required to accept/decline, player 1 accepts an offer if and only if it leaves him δp dollars or more. If selected to propose, player 2 offers $(1-\delta p, \delta p)$; when required to accept/decline, player 2 accepts an offer if and only if it leaves her $\delta(1-p)$ dollars or more. For this strategy profile, whoever is selected to propose, the offer made is immediately accepted. As in part (b), we can compute that the expected payoff in any period, before the proposer is selected, equals p for player 1 and $(1-p)$ for player 2.

We use the single-deviation principle to verify if this strategy profile constitutes a subgame perfect equilibrium. If selected to propose, player 1 can deviate by offering a smaller or a larger share to player 2. Obviously, a larger share will be accepted, leaving player 1 with a smaller payoff. Therefore such a deviation is not profitable. A smaller share will be rejected, according to player 2's strategy; thus, the game will continue into the next period, giving player 1 an expected discounted payoff of δp, assuming both players play according to their original strategies from then on. Now $\delta p < (1-\delta)+\delta p = 1-\delta(1-p)$, and this latter expression is player 1's payoff if he does not deviate. Therefore, this type of deviation will also be unprofitable. Furthermore, since δp is the payoff that player 1 receives in expectation if he rejects an offer, deviating to a strategy, where he rejects a share $x \geq \delta p$ or accepts a share $y < \delta p$, will be unprofitable. In the same manner, we can demonstrate that there is no profitable single-period deviation for player 2 (simply relace p with $q = 1-p$ in the arguments above). Thus, we have established, using the single-deviation principle, that this strategy profile constitutes a subgame

(d) From the analysis in part (c), we know that after investments have been made, the expected payoffs to the two players from the continuation game equal p and δp respectively. Therefore, before investments

have been made, the payoff to player 1 equals $p - x = \frac{x}{x+y} - x$, and the payoff to player 2 equals $(1-p) - x = \frac{y}{x+y} - y$.
For a subgame perfect equilibrium, we need to find investment levels x and y such that neither player has an incentive to deviate. Given y, the optimal level of investment for player 1 is given by

$$\max_x \left(\frac{x}{x+y}\right) - x$$

The first-order condition yields

$$\begin{aligned} &\frac{y}{(x+y)^2} - 1 = 0 \\ &\implies y = (x+y)^2 \qquad (1) \end{aligned}$$

Similarly, given x, the optimal level of investment for player 2 is given by

$$\max \left(\frac{y}{x+y}\right) - y$$

The first-order condition yields

$$\begin{aligned} &\frac{x}{(x+y)^2} - 1 = 0 \\ &\implies x = (x+y)^2 \qquad (2) \end{aligned}$$

To satisfy (1) and (2), we must have $x = y = \frac{1}{4}$. Therefore, in equilibrium, each person invests $\frac{1}{4}$, and in the continuation game, they use the strategies described in part (c).