ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #3 – ANSWER KEY

<u>Exercise #1:</u> Harrington Chapter 7 – Exercise 12

12. Saddam Hussein is deciding where to hide his weapons of mass destruction (WMD), while the United Nations is deciding where to look for them. The payoff to Saddam from successfully hiding WMD is 5 and from having them found is 2. For the UN, the payoff to finding WMD is 9 and from not finding them is 4. Saddam can hide them in facility *X*, *Y*, or *Z*. The UN inspection team has to decide which facilities to check. Because the inspectors are limited in terms of time and personnel, they cannot check all facilities.
    a. Suppose the UN has two pure strategies: It can either inspect facilities *X* and *Y* (both of which are geographically close to each other) or inspect facility *Z*. Find a Nash equilibrium in mixed strategies.

Part A

		UN	
		a	**b = (1-a)**
Saddam		X&Y	Z
x	X	2, $\underline{9}$	$\underline{5}$, 4
y	Y	2, $\underline{9}$	$\underline{5}$, 4
z = (1-x-y)	Z	$\underline{5}$, 4	2, $\underline{9}$

ANSWER: Let x denote the probability that Saddam Hussein hides WMD in facility X, y is the probability for facility Y, and $z = 1 - x - y$ is the probability for facility Z. For the United Nations, a is the probability that it inspects X and Y, and $b = 1 - a$ is the probability that it inspects Z. If we conjecture that both players randomize over all of their pure strategies, the expected payoffs from Saddam's three pure strategies are

Expected payoff from X: $a \times 2 + (1 - a) \times 5 = 5 - 3a$.

Expected payoff from Y: $a \times 2 + (1 - a) \times 5 = 5 - 3a$.

Expected payoff from Z: $a \times 5 + (1 - a) \times 2 = 2 + 3a$.

a and b must equate these three expected payoffs:

$$5 - 3a = 2 + 3a \Rightarrow a = \frac{1}{2}.$$

Turning to the UN, it is optimal for it to randomize over its two pure strategies (in particular, assigning probability $\frac{1}{2}$ to each) if and only if x and y equate the UN's two expected payoffs:

$$x \times 9 + y \times 9 + (1 - x - y) \times 4 = x \times 4 + y \times 4 + (1 - x - y) \times 9 \Rightarrow x + y = \frac{1}{2}$$

where the left-hand expression is the expected payoff from inspecting facilities X and Y. To be part of a Nash equilibrium, it is only necessary that Saddam hide WMD at either facility X or Y with probability $\frac{1}{2}$. There is then an infinite number of Nash equilibria, which differ only in terms of how this probability $\frac{1}{2}$ is allocated between facility X and facility Y. For example, one Nash equilibrium has Saddam hide the WMD at X with probability $\frac{1}{6}$, Y with probability $\frac{1}{3}$, and Z with probability $\frac{1}{2}$, while the UN inspects X and Y with probability $\frac{1}{2}$ and Z with probability $\frac{1}{2}$. Another Nash equilibrium is for Saddam to hide the WMD at X with probability $\frac{3}{8}$, Y with probability $\frac{1}{8}$, and Z with probability $\frac{1}{2}$, while the UN inspects X and Y with probability $\frac{1}{2}$ and Z with probability $\frac{1}{2}$.

Part B

b. Suppose the UN can inspect any two facilities, so that it has three pure strategies. The UN can inspect X and Y, X and Z, or Y and Z. Find a Nash equilibrium in mixed strategies.

		UN		
		a	b	c = (1-a-b)
Saddam		X&Y	X&Z	Y&Z
x	X	2, 9	2, 9	5, 4
y	Y	2, 9	5, 4	2, 9
z = (1-x-y)	Z	5, 4	2, 9	2, 9

ANSWER: Let x denote the probability that Saddam hides WMD in facility X, y the probability for facility Y, and $z = 1 - x - y$ the probability for facility Z. For the UN, a is the probability that it inspects X and Y, b is the probability that it inspects X and Z, and $c = 1 - a - b$ is the probability that it inspects Y and Z. Conjecture that both players randomize over all of their pure strategies. The expected payoffs from Saddam's three pure strategies are

Expected payoff from X: $a \times 2 + b \times 2 + (1 - a - b) \times 5 = 5 - 3a - 3b$.

Expected payoff from Y: $a \times 2 + b \times 5 + (1 - a - b) \times 2 = 2 + 3b$.

Expected payoff from Z: $a \times 5 + b \times 2 + (1 - a - b) \times 2 = 2 + 3a$.

a and b must equate these three expected payoffs:

$$5 - 3a - 3b = 2 + 3b.$$

$$5 - 3a - 3b = 2 + 3a.$$

$$2 + 3a = 2 + 3b.$$

The last condition implies $a = b$; let the common value be denoted d. Then the first (and second) condition become

$$5 - 3d - 3d = 2 + 3d \Rightarrow d = \frac{1}{3}.$$

Therefore, Nash equilibrium has the UN uniformly randomize over its three pure strategies, assigning $\frac{1}{3}$ to each of them. The UN's expected payoffs are

Expected payoff from X and Y: $x \times 9 + y \times 9 + (1 - x - y) \times 4 = 4 + 5x + 5y$.

Expected payoff from X and Z: $x \times 9 + y \times 4 + (1 - x - y) \times 9 = 9 - 5y$.

Expected payoff from Y and Z: $x \times 4 + y \times 9 + (1 - x - y) \times 9 = 9 - 5x$.

For the UN to be content to randomize over its three pure strategies, it must be the case that

$$4 + 5x + 5y = 9 - 5y.$$

$$4 + 5x + 5y = 9 - 5x.$$

$$9 - 5x = 9 - 5y.$$

The last condition implies $x = y$. Letting this common value be denoted w, then the first condition becomes

$$4 + 5w + 5w = 9 - 5w \Rightarrow w = \frac{1}{3}.$$

Therefore, Nash equilibrium has Saddam uniformly randomize over its three pure strategies; assigning $\frac{1}{3}$ to each of them.

Exercise 2 – Mixed strategy Nash equilibrium

a) The normal form representation of the game for *n*=2 players is given below.

		Player 2	
Player 1		*X*	*Y*
	X	3,3	4,3
	Y	3,4	2,2

There are three pure strategy Nash equilibria in this game, (X,X), (X,Y) and (Y,X).

b) When introducing *n*=3 players, the normal form representation of the game is:

- First, if Player 3 chooses X,

		Player 2	
Player 1		*X*	*Y*
	X	0,0,0	3,3,3
	Y	3,3,3	2,2,4

- And if Player 3 chooses Y,

		Player 2	
Player 1		*X*	*Y*
	X	3,3,3	4,2,2
	Y	2,4,2	1,1,1

Hence, the pure strategy Nash equilibria of the game with n=3 players are (X,Y,X), (Y,X,X) and (X,X,Y).

c) If every player is choosing X with probability p and Y with probability 1-p, the expected utility that player 1 obtains by playing X is:

- $EU_1(X)=p^2 0+p(1-p)3+(1-p)p3+(1-p)^2 4=p(1-p)6+4(1-p)^2$

And player 1's utility from playing Y is:

- $EU_1(Y)=p^2 3+p(1-p)2+(1-p)p2+(1-p)^2 1=3p^2+4(1-p)p+(1-p)^2$

Player 1 is indifferent between choosing strategy X and Y for values of p such that $EU_1(X)=EU_1(Y)$. That is,

- $p(1-p)6+4(1-p)^2=3p^2+4(1-p)p+(1-p)^2$, and simplifying, $2p^2+4p-3=0$

- Solving for p, we find that either p= -1-$\sqrt{2.5}$ <0 (which cannot be a solution to our problem, since $p \in [0,1]$, or p= -1+$\sqrt{2.5}$ =0.58, which is the solution to our problem.

Hence, every player in this game randomizes between X and Y (using mixed strategies) assigning probability p=0.58 to strategy X, and 1-p=0.42 to strategy Y.

Exercise 3 – Security (Max-min) strategies

a)

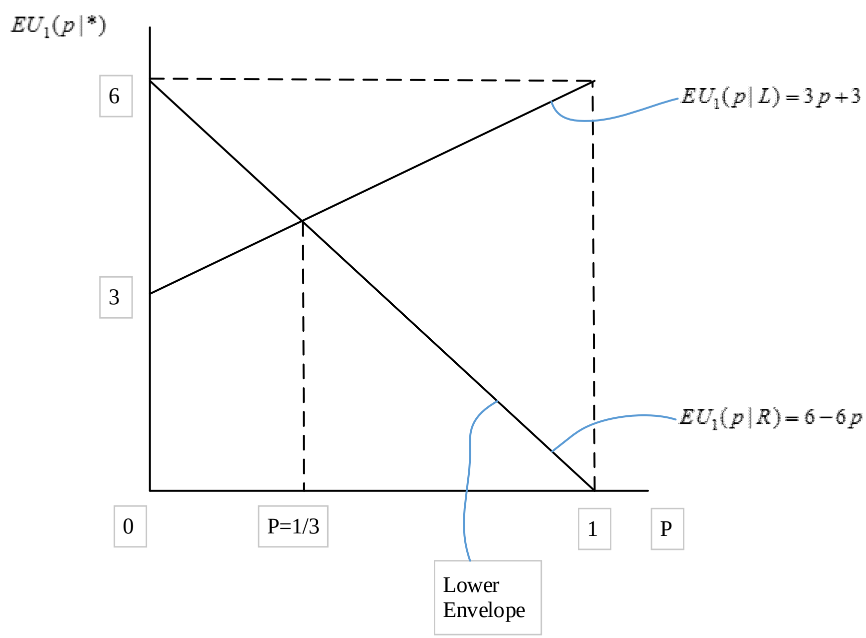
		q	1-q
	P1/P2	L	R
p	T	6,0	0,6
1-p	B	3,2	6,0

Player 1:

When player 2 chooses left, $EU_1(p \mid L) = 6p + 3(1-p) = 3p + 3$

When player 2 chooses right, $EU_1(p \mid R) = 0p + 6(1-p) = 6 - 6p$

These two expected utilities cross at: $3p + 3 = 6 - 6p \Rightarrow p = 1/3$, as the following figure depicts.

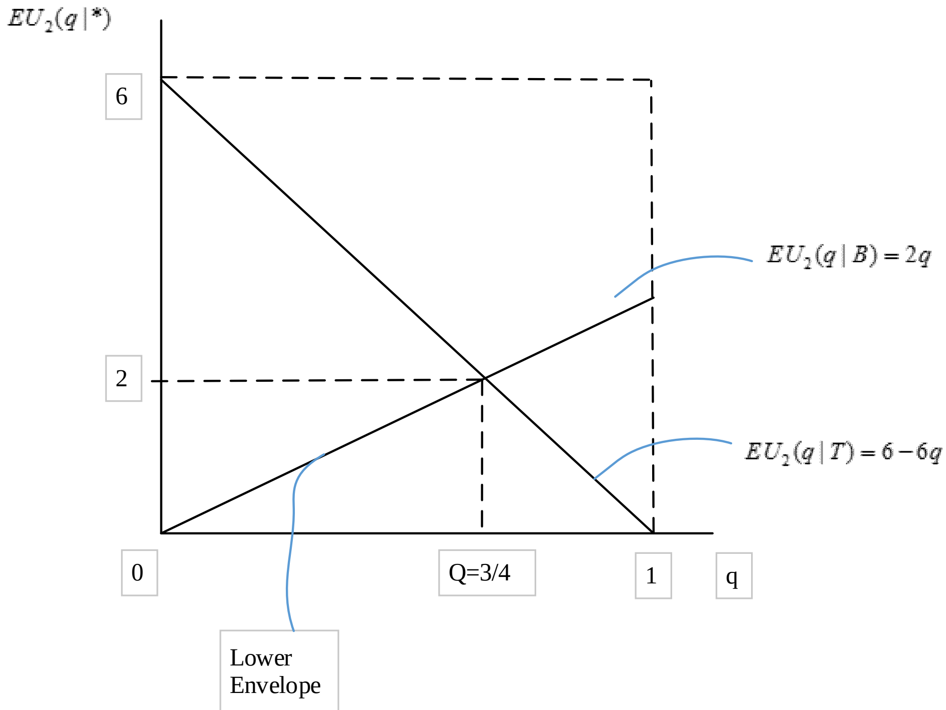


Player 2:

When player 1 chooses top, $EU_2(p \mid T) = 0q + 6(1-q) = 6-6q$

When player 1 chooses bottom, $EU_2(p \mid B) = 2q + 0(1-q) = 2q$

As the following figure depicts, these two expected utilities cross at: $6-6q = 2q \Rightarrow q = 3/4$



Hence, we can summarize the MAXMIN strategy profile as:

$$Player1\{\frac{1}{3}top, \frac{2}{3}bottom\}$$
$$Player2\{\frac{3}{4}left, \frac{1}{4}right\}$$

b) In order to find EU_1, we can use either $EU_1(p \mid L)$ or $EU_1(p \mid R)$ since at p=1/3 they cross each other.

$EU_1(p \mid L) = 3 + 3p = 4$

$EU_2(q \mid T) = 6 - 6q = 1.5$

c) Notice that this game is a strictly competitive game. This allows us to claim that the strategy profiles that can be sustained as Security (or Max-min) strategies from part (a) of the exercise must also be Nash equilibria in this part of the exercise, yielding exactly the same expected payoff for each player, i.e., $4 for player 1 and $1.5 for player 2. As a practice, we next show that, indeed, the only Nash equilibrium of the game coincides with the Security strategy we found in part (a).

		q	1-q
	P1/P2	L	R
p	T	<u>6</u>,0	0,<u>6</u>
1-p	B	3,<u>2</u>	<u>6</u>,0

$BR_1(L) = T$ and $BR_2(T) = R$

$BR_1(R) = B$ and $BR_2(B) = L$

There are hence no pure strategy NE

<u>Let us now find MSNE:</u>

The value of q that makes player 1 indifferent between T and B is:

$$EU_1(T) = EU_1(B)$$

$$6q + 0(1-q) = 3q + 6(1-q)$$
$$q = 2/3$$

And the value of p that makes player 2 indifferent between L and R is:

$$EU_2(L) = EU_2(R)$$

$$0p + 2(1-p) = 6p + 0(1-p)$$
$$p = 1/4$$

Hence, the MSNE can be summarized as:

- Player 1 plays {1/4 Top, 3/4 Bottom}
- Player 2 plays {2/3 Left, 1/3 Right}

d) Let us now find the EU_1 from the previous MSNE:

$$EU_1 = 6pq + 0p(1-q) + 3q(1-p) + 6(1-q)(1-p)$$
$$= 6*1/4*2/3 + 3*2/3*3/4 + 6*1/3*3/4 = 1 + 1.5 + 1.5 = 4$$

And similarly for player 2:

$$EU_2 = 0pq + 6p(1-q) + 2q(1-p) + 0(1-q)(1-p)$$
$$= 6*1/4*1/3 + 2*2/3*3/4 = .5 + 1 = 1.5$$

e)

- Player 1's expected utility from playing the MSNE of the game, 4; coincides with that from playing maxmin strategy, 4.
- Player 2's expected utility from playing the MSNE of the game, 1.5, coincides with the expected utility from playing the maxmin strategy, 1.5.

Exercise 4 – Cournot competition with N firms

Consider three firms competing *a la* Cournot, in a market with inverse demand function $P(Q) = 1 - Q$, and production costs normalized to zero.

a. Find the psNE of the game when firms simultaneously and independently choose quantities. Determine the equilibrium profit level for each firm.
b. Consider now that two (out of three) firms merge, and thus choose their output decision in order to maximize their joint profits. Find the psNE in this game for the merged firms and the unmerged firms. Identify the equilibrium profits for each firm, and compare them with your results pre-merger in part (a)
c. Consider now that all three firms merge. Find their profit maximizing output and profits, comparing them with your results in (a) and (b).
d. Repeat parts (a)-(b), but considering that firms compete in a Bertrand model of price competition with differentiated products, with demand function $q_i = a - bp_i + dp_j + dp_k$, for any firm *i* and its rival *k≠i*. Show that in this case, the merger between firm 1 and 2 would yield profits above those these firms obtain before the merger.

Answer:

a. The profits for firm *i* are

$$\pi_i = (1-Q)q_i = \left(1 - q_i - q_j - q_k\right)q_i$$

Taking first order conditions with respect to q_i, we obtain:

$$1 - 2q_i - q_j - q_k = 0 \quad \Rightarrow \quad q_i\left(q_j, q_k\right) = \frac{1 - q_j - q_k}{2}$$

And in a symmetric Nash equilibrium in which all firms are producing the same output, i. e., $q_i = q_j = q_k = q$, we find $q_i = \frac{1}{4}$ for every firm i.

Hence, equilibrium prices are $p = 1 - Q = 1 - 3\frac{1}{4} = \frac{1}{4}$

And, therefore, profits for every firm are (recall that there are no production costs)

$$\pi_i = pq_i = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

b. There are only two firms in the market now: the merge of firms 1 and 2, and the (unmerged) firm 3. The profits for either of these *two* firms are:

$$\pi_i = (1 - Q)q_i = \left(1 - q_i - q_j\right)q_i$$

And taking FOCs with respect to q_i, we obtain:

$$1 - 2q_i - q_j = 0 \quad \Longrightarrow \quad q_i\left(q_j\right) = \frac{1 - q_j}{2}$$

and in a symmetric Nash equilibrium in which all firms are producing the same output, i. e., $q_i = q_j = q$, we find $q_i = \frac{1}{3}$ for every firm i.

Hence, equilibrium prices are $p = 1 - Q = 1 - 2\frac{1}{3} = \frac{1}{3}$

And, therefore, profits for every firm are

$$\pi_i = pq_i = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

This implies that

- ✓ Firms 1 and 2 obtain profits of $\frac{\frac{1}{9}}{2} = \frac{1}{18}$ after the merger, which are lower than the pre-merger profits of $\frac{1}{16}$
- ✓ Firm 3 obtains profits of $\frac{1}{9}$, which exceed its pre-merger profits of $\frac{1}{16}$

Intuition: the merged firms internalize part of price reduction that an increase in their aggregate production entails, i. e., they consider the profit loss that the increase in production by one of the firm participating in the merger entails on the other firm that joined the merger. As a consequence, the merged firms reduce their individual production relative to pre-merger levels (in the standard Cournot competition analyzed in part a). However, the unmerged Firm 3 does not take into these price effects, and must responds to a lower output level from both of its competitors by increasing its own production. Ultimately, the firms that merged obtain a lower profit than before the merger, while the merged firm earns a larger profit. This result is often referred as the "merger paradox".

c. If all firms merge, they form a cartel, acting as a monopolist. [Note that this is only true when they all merge, not when only two of them merge, as we examined in the previous section]. When they all merge their joint profits are

$$\pi_i = (1 - Q)Q = Q - Q^2$$

Taking first order conditions with respect to Q, we obtain

$$1 - 2Q = 0 \quad \Longrightarrow \quad Q = \frac{1}{2}$$

Which implies that equilibrium price is

$$p = 1 - Q = 1 - \frac{1}{2} = \frac{1}{2}$$

And equilibrium profits are

$$\pi = pQ = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Therefore, the individual profits of every firm participating in the merger are $\frac{\frac{1}{4}}{3} = \frac{1}{12}$, which are clearly higher than their profits pre-merger (when all firms compete as Cournot oligopolists) of $\frac{1}{16}$.

d. First, note that the inverse demand function is

$$q_i = a - bp_i + dp_j + dp_k$$

- **No merger** (Bertrand competition with product differentiation):

$$\max_{p_i} p_i q_i = p_i\left(a - bp_i + dp_j + dp_k\right)$$

Taking FOCs with respect to p_i, we obtain

$$a - 2bp_i + dp_j + dp_k = 0$$

And at a symmetric equilibrium where all firms set the same price, $p_i = p_j = p_k = p$ we have

$$a - 2bp + dp + dp = 0 \quad \Longrightarrow \quad p = \frac{a}{2(b-d)}$$

With associated equilibrium profits of

$$\pi = pq = \frac{a}{2(b-d)}\left(a - b\frac{a}{2(b-d)} + 2d\frac{a}{2(b-d)}\right)$$

And rearranging

$$\pi = \frac{ba^2}{4(b-d)^2}$$

- **Merger of two firms**: firms 1 and 2 merge, and choose p_1 and p_2 to maximize their joint profits, as follows

$$\max_{p_1,p_2} p_1(a - bp_1 + dp_2 + dp_3) + p_2(a - bp_2 + dp_1 + dp_3)$$

Taking FOCs with respect to p_1 and p_2 respectively, we obtain:

$$\begin{aligned} a - 2bp_1 + 2dp_2 + dp_3 &= 0 \\ a - 2bp_2 + 2dp_1 + dp_3 &= 0 \end{aligned} \qquad (1)$$

And the unmerged firm 3 chooses p_3 to maximize its own profits:

$$\max_{p_3} p_3(a - bp_3 + dp_1 + dp_2)$$

And taking FOCs with respect to p_3

$$a - 2bp_3 + dp_1 + dp_2 = 0 \qquad (2)$$

Solving (1) and (2), we obtain equilibrium prices of:

$$p_1 = p_2 = \frac{a(2b^2 - bd - d^2)}{2(b-d)(2b^2 - 2bd - d^2)}$$

And for firm 3:

$$p_3 = \frac{ab}{(2b^2 - 2bd - d^2)}$$

The joint profits post-merger for firms 1 and 2 are

$$\frac{(2b^2 - bd - d^2)a^2}{(b-d)2(2b^2 - 2bd - d^2)^2}$$

While the pre-merger profits for firms 1 and 2 were only

$$2\frac{ba^2}{4(b-d)^2}$$

Hence, the profits post-merger are higher than pre-merger.

Exercise 5 (Bonus Exercise) – Cournot mergers with efficiency gains

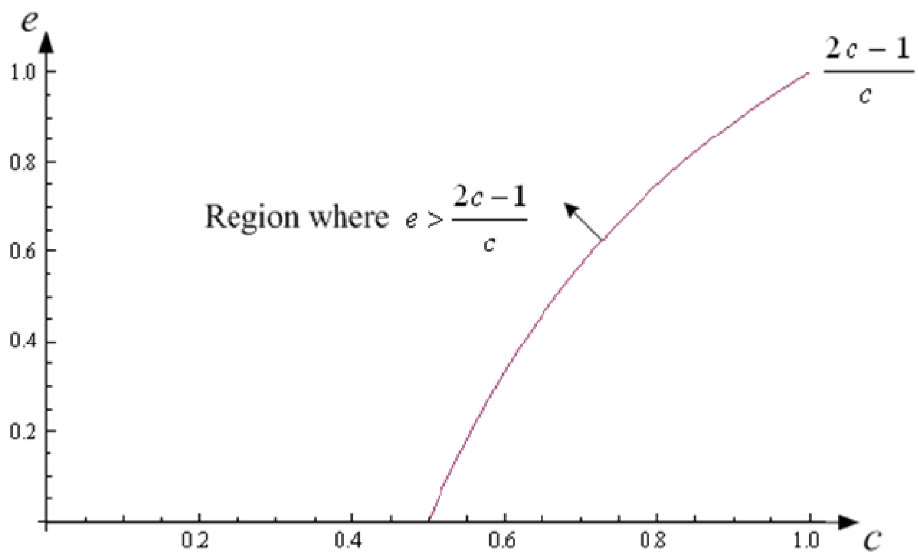
a. Each firm has a profit of $\pi_1 = (1 - c - Q)q_1$, for every firm $i = 1, 2, 3$. The FOCs are given by $1 - c - 2q_i - q_j - q_k = 0$, with *i, j, k*=1, 2, 3 and $i \neq j \neq k$.

- Solving for q at the symmetric equilibrium yields, $q_c = \frac{1-c}{4}$
- Hence, equilibrium prices are $p_c = \frac{1-3c}{4}$ and
- Equilibrium profits are $\pi_c = \frac{(1-c)^2}{16}$

b.

1. After the merger, two firms are left: firm 1, with cost ec, and firm 3, with cost c. It is easy to show that deriving the FOCs and solving them gives equilibrium outputs $q_1 = \frac{(1-c(2e-1))}{3}$ and $q_3 = \frac{(1-c(2-e))}{3}$.

   Note that the outsider firm can sell a positive output at equilibrium only if the merger does not give rise to strong cost saving: that is $q_3 \geq 0$ if $e \geq \frac{2c-1}{c}$ (if $c < \frac{1}{2}$, then the previous payoff becomes $\frac{2c-1}{c} < 0$, implying that $e \geq \frac{2c-1}{c}$ holds for all $e \geq 0$, ultimately that the outsider firm will always sell at the equilibrium.) The following figure illustrate the cutoff $e > \frac{2c-1}{c}$, where $c > \frac{1}{2}$, and the region of (e,c) combinations above this cutoff for which the region is sufficiently cost saving to induce the outside firm to produce positive output levels.



   - The equilibrium price is $p_m = \frac{(1+c(1+e))}{3}$ and equilibrium profits are given by $\pi_1 = \frac{(1-c(2e-1))^2}{9}$ and $\pi_3 = \frac{(1-c(2-e))^2}{9}$

2. Prices decrease after the merger only if there are sufficient efficiency gains: $p_m \leq p_c$ can be rewritten as $e \leq \frac{5c-1}{4c}$. Note that if $c < 1/5$, then $\frac{5c-1}{4c} < 0$, implying that $e \leq \frac{5c-1}{4c}$ cannot

hold for any $e \geq 0$. As a consequence, $p_m > p_c$, and prices will never fall no matter how strong efficiency gains, *e*, are.

3. To see if the merger is profitable, we have to study the inequality $\pi_1 \geq 2\pi_c$, which after some algebra can be seen to correspond to an inequality of the second order whose relevant solution is

$$e \leq \frac{4(1+c) - 3\sqrt{2}(1-c)}{8c}$$

In other words, the merger is profitable only if it gives rise to enough cost savings. The following figure depicts the cutoff of *e* where the region of costs is restricted to $c \in \left[\frac{1}{5}, 1\right]$