Stat 155 Homework # 2 Due February 10

Problems:

Q 1 Consider a 2 person first price auction with reserve price r (in such an auction the highest bidder pays his bid and wins the item if it is greater than r while if neither bidder bids more than r then no one gets it). Suppose that the agents have independent $U[0,1]$ values.

(a) Calculate the expected revenue if both bid according to the strategy $\beta(v) = v/2$.

(b) Explain why this not a Bayes-Nash equilibrium?

(c) Harder bonus question: Find a Bayes-Nash equilibrium for this auction.

Solution

(a) If the values are V_1, V_2 then payment is

$$f(V_1, V_2) = \begin{cases} \frac{1}{2}\max\{V_1, V_2\} & \text{if } \frac{1}{2}\max\{V_1, V_2\} > r \\ 0 & \text{otherwise.} \end{cases}$$

As we saw in class the density of $Y = \max\{V_1, V_2\}$ is $2y$ and hence the expected revenue is

$$\mathbb{E}R = \mathbb{E}\frac{1}{2}YI(Y/2 > r) = \int_{2r}^{1} \frac{y}{2} 2y dy = \frac{1}{3} - (2r)^3/3.$$

also so the revenue is $\frac{1}{3} - \frac{8}{3}r^3$. This is not a Bayes-Nash equilibrium because if a agent has a value between r and $2r$ then they have no chance of winning the bid and gaining any utility. If instead they increase their bid to r then and the other agent has a value less than r then they improve their utility by purchasing the item for r. For part (c) I can discuss this in office hours.

Q 2

Consider an auction, with reserve price X chosen from distribution G where the item is allocated to a uniformly chosen random bidder among those who bid above the reserve price. The winner pays X the reserve. Show that this auction is truthful, that is that the optimal strategy is to bid your value.

If there are two agents with independent $U[0,1]$ values and X is also $U[0,1]$ find the expected revenue.

Solution

Let X be the random reserve price. Then if $V_1 > X$ then any bid greater than X yields the same expected utility, while bidding less than X yields 0 utility so no change in bidding improves the expected utility. If $V_1 < X$ then any bid greater X will have a negative expected utility while any bid less than X will have 0 utility. Again bidding the agents value is an optimal bid. Thus bidding the agents value is optimal.

The 6 possible orderings of V_1, V_2 and X are all equally likely since they are IID. Thus there is a one third probability that the reserve is the smallest of the three and conditional on this event it is the smallest of 3 order statistics and so has density $3(1-x)^2$. Similarly there is a one third probability that the reserve is the middle of the three and conditional on this event it is the second of 3 order statistics and so has density $6x(1-x)$. When the reserve is the largest then no revenue is paid. Combining we get that the expected revenue is

$$\mathbb{E}R = \frac{1}{3}\int_0^1 3x(1-x)^2 dx + \frac{1}{3}\int_0^1 6x^2(1-x)dx = \frac{1}{4}.$$

Q 3

In an auction with $n \geq 3$ agents, two identical items are for sale. The two highest bidders get one item each and both pay the third highest bid. Show that this auction is truthful and find its expected revenue if the values are IID $U[0,1]$.

Solution

This is similar to a second price auction. Let b be the bid of the second highest bid amongst the other bidders. Then if the agent has utility $V_1 > b$ then any bid greater than b yields utility $V_1 - b$ and any bid less than b yields 0. Thus a non-truthful bid does not increase her utility. If the agents value is smaller than b then a bid less than b yields 0 utility while a bid above b yields negative utility. SO in each case, changing the bid does not increase the utility and thus truthful bidding is the optimal strategy.

The third highest bid is the third highest order statistic and so has density $n\binom{n-1}{n-3}x^{n-3}(1-x)^2 = \frac{n(n-1)(n-2)}{2}x^{n-3}(1-x)^2$. Hence the expected revenue is

$$\mathbb{E}R = 2\int_0^1 x\frac{n(n-1)(n-2)}{2}x^{n-3}(1-x)^2 dx = n(n-1)(n-2)\int_0^1 x^n - 2x^{n-1} + x^{n-2} dx = \frac{2(n-2)}{n+1}$$