Collegio Carlo Alberto

Game Theory

Solutions to Problem Set 5

# 1 Bertrand duopoly with incomplete information

The game is defined by

$$
\begin{aligned}
I &= \{1,\ 2\}\,,\ \text{set of players}\\
A_i &= [0,\infty)\\
T_i &= \{b_L, b_H\}\,,\ \text{with } p(b_L)=\theta\\
u_i(b_i,p_i,p_j) &= \begin{cases} (a-p_i-b_ip_j)p_i & ,\ \text{if } p_i \leqslant a-b_ip_j \\ 0 & ,\ \text{otherwise}\end{cases}.
\end{aligned}
$$

In addition, the strategy sets $S_i$ ($\forall i = 1,2$) are the sets of all functions $s_i : T_i \to A_i$, i.e. mappings from types into actions. The symmetric pure-strategy BNE are the BNE is which $s_i(b_i = b_L) = s_j(b_j = b_L)$ and $s_i(b_i = b_H) = s_j(b_j = b_H)$.

Now, in order to find the symmetric pure strategy BNE, first suppose that $p_i \leqslant a - b_ip_j$, $\forall i = 1,2$. Since we are focusing on symmetric equilibria, we have that:

$$
\begin{aligned}
p_i^*(b_i) &= \arg\max_{p_i}\left\{\theta(a-p_i-b_ip_j^L)p_i + (1-\theta)(a-p_i-b_ip_j^H)p_i\right\}\\
FOC:&\ \ \theta(a-p_i-b_ip_j^L) - \theta p_i + (1-\theta)(a-p_i-b_ip_j^H) - (1-\theta)p_i = 0\\
&a - \theta b_ip_j^L - 2p_i - (1-\theta)b_ip_j^H = 0\\
p_i^*(b_i) &= \frac{a-\theta b_ip_j^L-(1-\theta)b_ip_j^H}{2}
\end{aligned}
$$

Now, imposing symmetry and denoting $p_L^*$ and $p_H^*$ the equilibrium prices of the low types and high types respectively, we have that:

$$
\begin{aligned}
p_L^* &= \frac{a-\theta b_Lp_L^*-(1-\theta)b_Lp_H^*}{2}\\
p_L^*(2+\theta b_L) &= a-(1-\theta)b_Lp_H^*\\
p_L^* &= \frac{a-(1-\theta)b_Lp_H^*}{(2+\theta b_L)}
\end{aligned}
$$

Similarly, for the high type:

$$p_H^* = \frac{a - \theta b_H p_L^*}{(2 + (1-\theta) b_H)}$$

Combining the expressions for $p_L^*$ and $p_H^*$, we get:

$$\begin{aligned} p_L^* &= \frac{a}{2} \frac{(2 + (1-\theta)(b_H - b_L))}{(2 + (1-\theta) b_H + \theta b_L)} && (1) \\ p_H^* &= \frac{a}{2} \frac{(2 - \theta(b_H - b_L))}{(2 + (1-\theta) b_H + \theta b_L)} && (2) \end{aligned}$$

If the equilibrium prices and quantities are all positive (i.e. if we in fact have an interior solution), $p_L^*$ and $p_H^*$ as defined above constitute a symmetric BNE. Note that $p_L^*>0$ always hold, so the condition that have to hold for both prices to be positive is:

$$2 - \theta(b_H - b_L) > 0.$$

The quantities are positive if:

$$\begin{aligned} q(p_H^*, p_H^*) &= a - p_H^* - b_H p_H^* > 0 \\ q(p_H^*, p_L^*) &= a - p_H^* - b_H p_L^* > 0 \\ q(p_L^*, p_H^*) &= a - p_L^* - b_L p_H^* > 0 \\ q(p_L^*, p_L^*) &= a - p_L^* - b_L p_L^* > 0 \end{aligned}$$

Remember that $b_H > b_L$, and notice that $p_L^* > p_H^*$ .This implies that if the inequality $q(p_H^*, p_L^*) > 0$ holds, the other three inequalities will also hold. Hence, in order to check that the equilibrium quantities are positive, it is enough to check that the following holds:

$$\begin{aligned} q(p_H^*, p_L^*) &= a - p_H^* - b_H p_L^* > 0 \\ p_H^* &< a - b_H p_L^* \\ \frac{a}{2} \frac{(2 - \theta(b_H - b_L))}{(2 + (1-\theta) b_H + \theta b_L)} &< a - b_H p_L^* \\ a\,(2 - \theta(b_H - b_L)) &< 2a(2 + b_H - \theta(b_H - b_L)) - b_H a\,(2 + (1-\theta)(b_H - b_L)) \\ (2 - \theta(b_H - b_L)) &< 2(2 + b_H - \theta(b_H - b_L)) - b_H\,(2 + b_H - b_L - \theta(b_H - b_L)) \\ &= (2 - b_H)(2 + b_H - \theta(b_H - b_L)) + b_L b_H \\ &= (2 - b_H)(2 - \theta(b_H - b_L)) + b_H(2 - b_H + b_L) \\ 0 &< (1 - b_H)(2 - \theta(b_H - b_L)) + b_H(2 - b_H + b_L) \\ 0 &< (1 - b_H)(2 + b_H - \theta(b_H - b_L)) + b_H(1 + b_L) \\ &\equiv (1 - b_H)A + B \\ if\ b_H &< 1 \text{ then } q(p_H^*, p_L^*) > 0. \end{aligned}$$

Now, note that the expression A is always positive, and so is B. Hence, if we assume that $b_H < 1$ (which is sufficient but not necessary), we have that $q(p_H^*, p_L^*) > 0$, and expressions (1) and (2) describe a symmetric pure strategy BNE.

## 2 A war game

$$\begin{aligned} T_1 &= T_2 = \{w, s\}, \ p(w) = p(s) = \frac{1}{2} \\ A_1 &= A_2 = \{A, N\} \\ s_1 &: T_1 \to A_1 \text{ (pure strategies)} \\ s_2 &: T_2 \to A_2 \text{ (pure strategies)} \end{aligned}$$

Payoffs:

$t_1 = w, t_2 = w$

| | A | N |
|---|---|---|
| A | $-8, -8$ | $10, 0$ |
| N | $0, 10$ | $0, 0$ |

$t_1 = w, t_2 = s$

| | A | N |
|---|---|---|
| A | $-8, 4$ | $10, 0$ |
| N | $0, 10$ | $0, 0$ |

$t_1 = s, t_2 = w$

| | A | N |
|---|---|---|
| A | $4, -8$ | $10, 0$ |
| N | $0, 10$ | $0, 0$ |

$t_1 = s, t_2 = s$

| | A | N |
|---|---|---|
| A | $-6, -6$ | $10, 0$ |
| N | $0, 10$ | $0, 0$ |

The approach we take here to finding the pure strategy BNE is to look at all possible strategies (AA, AN, NA, NN) for one of the players, and check for each if it can be an equilibrium strategy.

First, suppose player 2 follows the following strategy:

$$s_2 = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right.$$

Then, we can derive the best-response of player 1:

$$\left. \begin{array}{l} Eu_1(A, w; s_2) = -8 \\ Eu_1(N, w; s_2) = 0 \\ Eu_1(A, s; s_2) = -1 \\ Eu_1(N, s; s_2) = 0 \end{array} \right\} \Rightarrow BR_1(s_2) = \left\{ \begin{array}{c} s \longrightarrow N \\ w \longrightarrow N \end{array} \right.$$

Similarly, if we suppose that $s_1 = \left\{ \begin{array}{c} s \longrightarrow N \\ w \longrightarrow N \end{array} \right.$, the best-response for player 2 is $s_2 = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right.$. Hence, by the symmetry of the game we have found two fixed-points. That is, we have the following two pure strategy BNE:

$$s_i = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right. , \; s_j = \left\{ \begin{array}{c} s \longrightarrow N \\ w \longrightarrow N \end{array} \right. , i \in \{1, 2\}$$

Now, suppose player 2 plays according to:

$$s_2 = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow N \end{array} \right.$$

Then, we can derive the best-response of player 1:

$$\left. \begin{array}{c} Eu_1(A, w; s_2) = 1 \\ Eu_1(N, w; s_2) = 0 \\ Eu_1(A, s; s_2) = 2 \\ Eu_1(N, s; s_2) = 0 \end{array} \right\} \Rightarrow \; BR_1(s_2) = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right.$$

But if $s_1 = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right.$, the best-response of player 2 is $BR_2(s_1) = \left\{ \begin{array}{c} s \longrightarrow N \\ w \longrightarrow N \end{array} \right.$. Hence, this is not a BNE.

Now, suppose player 2 plays according to:

$$s_2 = \left\{ \begin{array}{c} s \longrightarrow N \\ w \longrightarrow A \end{array} \right.$$

Then, we can derive the best-response of player 1:

$$\left. \begin{array}{c} Eu_1(A, w) = 1 \\ Eu_1(N, w) = 0 \\ Eu_1(A, s) = 7 \\ Eu_1(N, s) = 0 \end{array} \right\} \Rightarrow \; BR_1(s_2) = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right.$$

But if $s_1 = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right.$, the best-response of player 2 is $BR_2(s_1) = \left\{ \begin{array}{c} s \longrightarrow N \\ w \longrightarrow N \end{array} \right.$. Hence, this is not a BNE.

To summarize: the two BNE in pure strategies are:

$$s_i = \left\{ \begin{array}{c} s \longrightarrow A \\ w \longrightarrow A \end{array} \right. , \; s_j = \left\{ \begin{array}{c} s \longrightarrow N \\ w \longrightarrow N \end{array} \right. \text{,for } \; i, j \in \{1, 2\} \, , \; i \neq j.$$

## 3 Is Information always beneficial?

The game is described by:

$$\begin{aligned} T_1 &= \{a,b\}\,,\ \ p(a)=p(b)=\frac{1}{2} \\ A_1 &= \{U,D\}\,,\ \ A_2=\{L,C,R\} \end{aligned}$$

and the payoffs:

| | $t_1=a$ | | | | $t_1=b$ | | |
|---|---|---|---|---|---|---|---|
| | $L$ | $C$ | $R$ | | $L$ | $C$ | $R$ |
| $U$ | $4,-1$ | $3,0$ | $3,-3$ | $U$ | $4,-1$ | $3,-3$ | $3,0$ |
| $D$ | $5,4$ | $2,5$ | $2,0$ | $D$ | $5,4$ | $2,0$ | $2,5$ |

Finally, a pure strategy for player 1 is a function $s_1 : T_1 \to A_1$, whereas a pure strategy for player 2 is an $s_2 \in A_2$.

First note that if we had assumed that these were two separate games of complete information, we could have used repeated deletion of strictly dominated strategies to find unique pure strategy NE - (U, C) and (U, R) - for the games corresponding to "a" and "b" respectively. In both of these games, the equilibrium payoff for player 2 would have been 0 (and the payoff for player 1 would have been 3).

Now, let us return to the incomplete information game described above. Note first that now L strictly dominates both C and R for player 2. This implies that in any BNE, it must be the case that player 2 plays strategy L. Then, if player 2 plays L, the best-response for player 1 is strategy DD, i.e. to play D whe s/he is type a, and D when s/he is type b. Hence, we have found a unique BNE:

$$s = (s_1,\ s_2) = (DD, L).$$

Finally, note that the equilibrium payoff for player 2 in the incomplete information game is $4 > 0$. That is, if we believe that the solution concepts used here (NE and BNE) are appropriate as predictors of real-world outcomes, player 2 would actually benefit from having less information.

### 3.1 Alternative approach

In order to illustrate different ways of looking for BNE, this subsection presents an approach to question 3 that is not based on the fact that C and R are strictly dominated by L.

In this case, we need to consider the different possible strategies for player 2. The strategies for player 2 can take the following form:

- $L$
- $C$

- $R$
- $LC$ (i.e., a randomization between $L$ and $C$),
- $LR$
- $CR$
- $LCR$

We need to check each of these and see if we can have a BNE on each particular form. First, suppose

$$\begin{aligned} S_2 &= L \\ S_1 &= \begin{cases} D & if \ \mathrm{t}_1 = a \\ D & if \ \mathrm{t}_1 = b \end{cases} \\ &\Downarrow \\ Eu_2(L) &= \frac{1}{2}\{4+4\} = 4 \\ Eu_2(C) &= Eu_2(R) = 2.5 \end{aligned}$$

which implies that $((D, D), \ L)$ is a BNE. Now, suppose

$$\begin{aligned} S_2 &= C \\ S_1 &= \begin{cases} U & \text{if } t_1 = a \\ U & \text{if } t_1 = b \end{cases} \\ &\Downarrow \\ Eu_2(L) &= -1 \\ Eu_2(C) &= Eu_2(R) = -1.5 \end{aligned}$$

Hence, $BR_2(S_1) = L \neq C$, so this is not a BNE. Similarly, we can show that player 2 cannot play $S_2 = R$ in any BNE.

Suppose player 2 mixes between $\{C, R\}$ .Since for both $S_2 = C$ and $S_2 = R$, the best response of player 1 (both types) is $U$ , which means that this is te best-response also to a strategy in which player 2 mixes between $\{C, R\}$ . But if player 1 plays $U$, the best response of player 2 is $L$. Thus, a mix between $\{C, R\}$ cannot be played in a BNE.

Suppose player 2 mixes between $\{L, C\}$ with probabilities $\{\alpha, 1-\alpha\}$. For player 2 to mix it must be the case that her expected payoff of playing L and C are equal and greater than the expected utility of playing R. To compute the expected utility, we need to specify what player 2 believes that the two types of player 1 will do. Suppose player 2 believes that type $a$ mixes between $\{U, D\}$ with probability $\{p_1, 1-p_1\}$ and type $b$ mixes between $\{U, D\}$ with probability $\{p_2, 1-p_2\}$ .Then:

$$
\begin{aligned}
Eu_2(L) &= \frac{1}{2}\left[p_1(-1)+(1-p_1)4\right]+\frac{1}{2}\left[p_2(-1)+(1-p_2)4\right]=4-2.5(p_1+p_2) \\
Eu_2(C) &= \frac{1}{2}\left(0+5-5p_1-3p_2\right)=2.5-2.5p_1-1.5p_2 \\
Eu_2(R) &= \frac{1}{2}\left(-3p_1+5-5p_2\right)=2.5-2.5p_2-1.5p_1
\end{aligned}
$$

However, then for $Eu_2(L)=Eu_2(C)$ we must have that $p_2=1.5$, which is impossible. A similar analysis rules out a mix between $\{L,R\}$.

Finally, suppose player 2 is mixing among $\{L,C,R\}$ . As we have showed, for player 2 to mix between $\{L,C\}$ or $\{L,R\}$, it must be the case that some of the probabilities defined above exceed 1. Hence, it is impossible to induce player 2 to mix between these actions.

To summarize, the unique BNE is $((D,D),L)$, where:

$$
\begin{array}{l}
Eu_2(DD,L)=4 \\
u_1(DD,L\mid a)=5 \\
u_1(DD,L\mid b)=5.
\end{array}
$$