Normal Form Games

Game Theory

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- Games in Normal Form
- Analyzing Games
- Some other Solution Concepts for NFGs
- Reading:
  - Chapter 3 of the MAS book
  - Christos Papadimitriou lecture on Nash theorem

Self Interested Agents

- What does it mean to say that an agent is self-interested?
  - Not that they want to harm others or only care about themselves
  - Only that the agent has its own description of states of the world that it likes, and acts based on this description

- Each such agent has a utility function
  - Is a mapping from states of the world to real numbers.
  - Quantifies degree of preference across alternatives
  - Explains the impact of uncertainty
  - Decision-theoretic rationality: act to maximize expected utility

Utility Maximization

- Example:
  - Consider an agent Alice, who has three options: going to the club (c), going to a movie (m), or watching a video at home (h). If she is on her own, Alice has a utility of 100 for c, 50 for m, and 50 for h.
  - Bob is at the club 60% of the time, spending the rest of his time at the movie theater. He reduces Alice's utility by 90 at the club and by 40 at the movie theater.
  - Carol can be found at the club 25% of the time, and the movie theater 75% of the time. He increases Alice's utility for either activity by a factor of 1.5 .
  - What should Alice do?

	$B = c$	$B = m$
$C = c$	15	150
$C = m$	10	100

$A = c$

	$B = c$	$B = m$
$C = c$	50	10
$C = m$	75	15

$A = m$

Why Utility?

- It might seem obvious that preferences can be described by utility functions. But:
  - Why is a single-dimensional function enough?
  - Why should an agent’s response to uncertainty be captured purely by an expected value?
- von Neumann & Morgenstern,1944: A single dimensional function is enough for preferences with some properties

Von Neumann & Morgenstern's Theorem

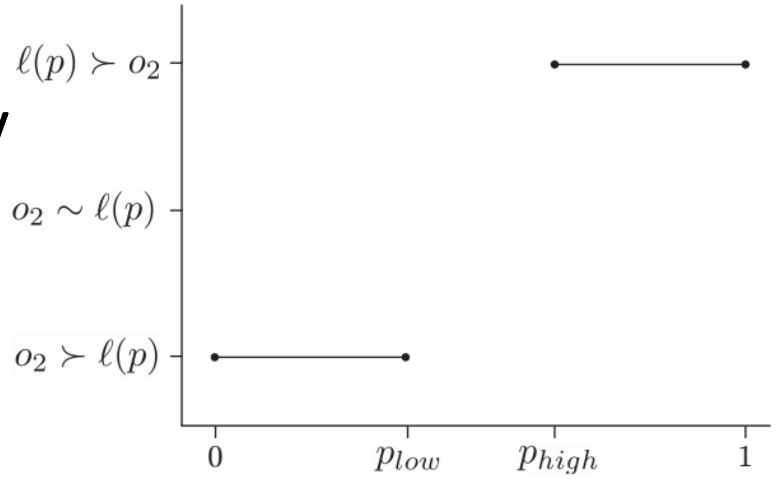
- Let O denote a finite set of outcomes. For any pair $o_1, o_2 \in O$,
  - $o_1 \succcurlyeq o_2$ denotes the proposition that the agent weakly prefers o_1 to o_2.
  - $o_1 \sim o_2$ denotes the proposition that the agent is indifferent between o_1 to o_2.
  - $o_1 \succ o_2$ denotes the proposition that the agent strictly prefers o_1 to o_2.
- A lottery is a probability distribution over the outcomes: $[p_1: o_1, p_2: o_2, \cdots, p_k: o_k]$
- Axioms:
  - Completeness: $\forall o_1, o_2: o_1 \succ o_2 \text{ or } o_1 \sim o_2 \text{or } o_1 \prec o_2$.
  - Transitivity: $\text{If } o_1 \succcurlyeq o_2 \text{ and } o_2 \succcurlyeq o_3 \text{ then } o_1 \succcurlyeq o_3$.

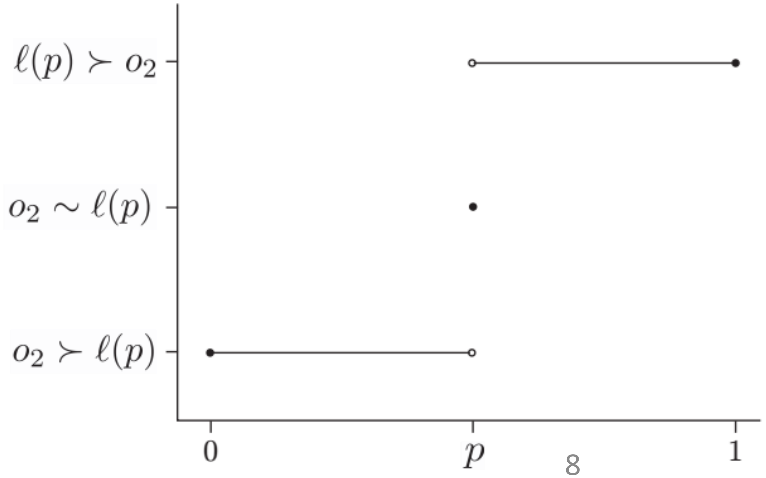
Von Neumann & Morgenstern's Theorem

- Axioms:
  - Substitutability: If $o_1 \sim o_2$ then for all sequences of one or more outcomes $o_3, o_4, \cdots, o_k$ and sets of probabilities $p, p_3, p_4, \cdots, p_k$ for which $p + \sum_{i=3}^{k} p_i = 1$, $[p: o_1, p_3: o_3, \cdots, p_k: o_k] \sim [p: o_2, p_3: o_3, \cdots, p_k: o_k]$
  - Decomposability: If $\forall o_i \in O, P_{l_1}(o_i) = P_{l_2}(o_i)$ then $l_1 \sim l_2$. $P_l(o_i)$ is the probability that outcome o_i is selected by lottery l
  - Monotonicity: If $o_1 \succ o_2$ and $p > q$ then $[p: o_1, 1-p: o_2] \succ [q: o_1, 1-q: o_2]$

Von Neumann & Morgenstern's Theorem

- Lemma: If a preference relation $\succcurlyeq$ satisfies the axioms completeness, transitivity, decomposability and monotonicity, and if $o_1 \succ o_2$ and $o_2 \succ o_3$, then there exists some probability p such that for all $p' < p$, $o_2 \succ [p': o_1, (1-p'): o_3]$, and for all $p'' > p$, $[p'': o_1, (1-p''): o_3] \succ o_2$.
  - Proof: see the blackboard
- Axiom:
  - Continuity: If $o_1 \succ o_2$ and $o_2 \succ o_3$, then $\exists p \in [0,1]$ such that $o_2 \sim [p: o_1, (1-p): o_3]$





Von Neumann & Morgenstern's Theorem

- Theorem: If a preference relation $\succcurlyeq$ satisfies the axioms completeness, transitivity, substitutability, decomposability, monotonicity and continuity, then there exist a function $u: \mathcal{L} \rightarrow [0,1]$ with the properties that
  - $u(o_1) \geq u(o_2)$ iff $o_1 \succcurlyeq o_2$ and
  - $u([p_1: o_1, \cdots, p_k: o_k]) = \sum_{i=1}^{k} p_i u(o_i)$
- Proof: see the blackboard.

Defining Games

- Players: who are the decision makers?
  - People? Governments? Companies? Somebody employed by a Company?...
- Actions: what can the players do?
  - Enter a bid in an auction? Decide whether to end a strike? Decide when to sell a stock? Decide how to vote?
- Payoffs: what motivates players?
  - Do they care about some profit? Do they care about other players?...

Defining Games

- Normal Form (Matrix Form, Strategic Form) List what payoffs get as a function of their actions
  - It is *as if* players moved simultaneously
  - But strategies encode many things...

- Extensive Form Includes timing of moves (later in course)
  - Players move sequentially, represented as a tree
    - Chess: white player moves, then black player can see white's move and react...
  - Keeps track of what each player knows when he or she makes each decision
    - Poker: bet sequentially – what can a given player see when they bet?

Defining Games-The Normal Form

- Finite, *n*-person normal form game: ⟨*N, A, u*⟩:
  - Players: $N = \{1, \ldots, n\}$ is a finite set of *n* , indexed by *I*
  - Action set for player i A_i
    - $a = (a_1, a_2, \cdots, a_n) \in A = A_1 \times A_2 \times \cdots \times A_n$ is an action profile
  - Utility function or Payoff function for player i: $A \to R$
    - $u = (u_1, u_2, \cdots, u_n)$ is a profile of utility functions

Normal Form Games- The Standard Matrix Representation

- Writing a 2-player game as a matrix:
  - "row" player is player 1, "column" player is player 2
  - rows correspond to actions $a_1 \in A_1$, columns correspond to actions $a_2 \in A_2$
  - cells listing utility or payoff values for each player: the row player first, then the column
- Here's the TCP Backoff Game written as a matrix

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

A Large Example

- Players: *N* = *{*1*, . . . ,* 10*,* 000*,* 000*}*

- Action set for player *i* A_i = *{Revolt, Not}*

- Utility function for player *i*:
  - $u_i(a) = 1$ if $\#\{j: a_j = Revolt\} \geq 2{,}000{,}000$
  - $u_i(a) = -1$ if $\#\{j: a_j = Revolt\} < 2{,}000{,}000$ and $a_i = Revolt$
  - $u_i(a) = 0$ if $\#\{j: a_j = Revolt\} < 2{,}000{,}000$ and $a_i = Not$

Prisoner's Dilemma

- Prisoner's dilemma is the following game with $c > a > d > b$.

	C	D
C	a, a	b, c
D	c, b	d, d

Common-Payoff Games

- A common-payoff game is a game in which for all action profiles $a \in A_1 \times A_2 \times \cdots \times A_n$ and any pair of agents i,j, it is the case that $u_i(a) = u_j(a)$
- Example: Coordination Game-Modeling Cooperation

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Constant-sum Games

- A two-player normal-form game is constant-sum if there exists a constant c such that for each strategy profile $a \in A_1 \times A_2$ it is the case that $u_1(a) + u_2(a) = c$

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Matching Pennies game

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

Rock, Paper, Scissors game.

Strategies in Normal Form Games

- Pure Strategy: To select a single action and play it. i.e. the set of pure strategies for player i is $S_i = A_i$.
- Mixed Strategy: Let (N,A,u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then the set of mixed strategies for player i is $S_i = \Pi(A_i)$.
- Strategy Profile: $S_1 \times S_2 \times \cdots \times S_n$

Mixed Strategies

- By $s_i(a_i)$ we denote the probability that an action a_i will be played under mixed strategy s_i.
- The support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$
- Expected Utility of a Mixed Strategy: Given a normal-form game (N,A,u), the expected utility u_i for player i of the mixed-strategy profile $s = (s_1, s_2, \cdots, s_n)$ is defined as

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j)$$

Best Response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $a_{-i} = \langle a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \rangle$
  - Now $a = (a_i, a_{-i})$

- Best response: Player i's best response to the strategy profile s_{-i} is a mixed (pure) strategy $s_i^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$.

Nash Equilibrium

- Really, no agent knows what the others will do?
- What can we say about which actions will occur?
- Nash equilibrium: A strategy profile $s = (s_1, s_2, \cdots, s_n)$ is a Nash equilibrium if, for all agents i, s_i is a best response to s_{-i}.
- Strict Nash: A strategy profile $s = (s_1, s_2, \cdots, s_n)$ is a strict Nash equilibrium if, for all agents i and for all strategies $s'_i \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.
- Weak Nash: A strategy profile $s = (s_1, s_2, \cdots, s_n)$ is a weak Nash equilibrium if, for all agents i and for all strategies $s'_i \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

Keynes Beauty Contest Game

- Each player names an integer between 1 and 100.
- The player who names the integer closest to two thirds of the *average* integer wins a prize, the other players get nothing.
- Ties are broken uniformly at random.

Keynes Beauty Contest Game

- Suppose a player believes the average play will be *X* (including his or her own integer)
- That player's optimal strategy is to say the closest integer to $\frac{2}{3}X$.
- *X* has to be less than 100, so the optimal strategy of any player has to be no more than 67.
- If *X* is no more than 67, then the optimal strategy of any player has to be no more than $\frac{2}{3}67$.
- If *X* is no more than $\frac{2}{3}67$, then the optimal strategy of any player has to be no more than $\left(\frac{2}{3}\right)^2 67$.
- Iterating, the unique Nash equilibrium of this game is for every player to announce 1!

Nash Equilibrium

- Each player's action maximizes his or her payoff given the actions of the others.

- Nobody has an incentive to *deviate* from their action if an equilibrium profile is played.

- Someone has an incentive to *deviate* from a profile of actions that do *not* form an equilibrium.

Pareto Optimality

- Sometimes, one outcome *o* is at least as good for every agent as another outcome *o′*, and there is some agent who strictly prefers *o* to *o′*
  - in this case, it seems reasonable to say that *o* is better than *o*
- Pareto domination: Strategy profile s Pareto dominates profile s′ if for all $i \in N$, $u_i(s) \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$.
- Pareto optimality: Strategy profile s is Pareto optimal, or strictly Pareto efficient, if there does not exist another strategy profile $s' \in S$ that Pareto dominates s.
- Can a game have more than one Pareto-optimal outcome?
- Does every game have at least one Pareto-optimal outcome?

Pareto Optimality

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

the paradox of Prisoner's dilemma: the Nash equilibrium is the only non-pareto optimal outcome

Finding Nash Equilibria

	LW	WL
LW	$\boxed{2, 1}$	$0, 0$
WL	$0, 0$	$\boxed{1, 2}$

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

$$U_{\text{wife}}(\text{LW}) = U_{\text{wife}}(\text{WL})$$
$$2 * p + 0 * (1 - p) = 0 * p + 1 * (1 - p)$$
$$p = \frac{1}{3}$$

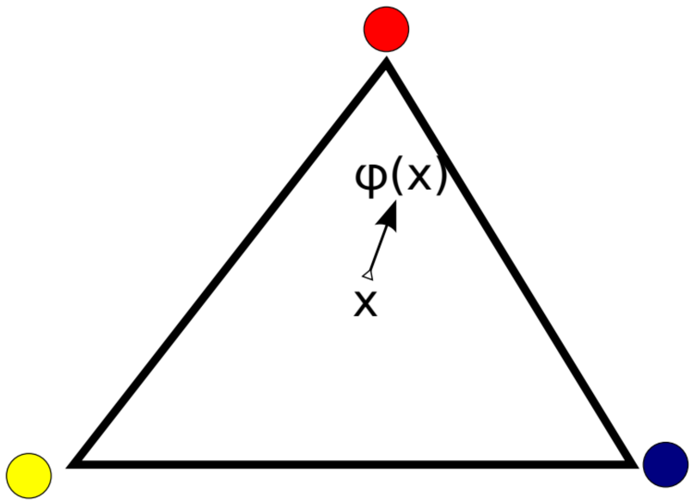
Nash’s Theorem

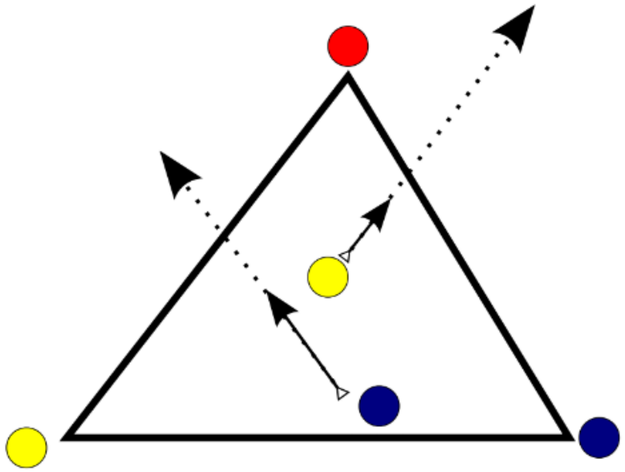
- Nash’s theorem: Every game with a finite number of players and action profiles has at least one Nash equilibrium.
  - Proof: see the blackboard
  - The idea is to use Brouwer’s fixed point theorem
  - $\phi(x_1, x_2, \dots, x_n) = (z_1, z_2, \dots, z_n)$:

$$z_i = \arg\max_{z_i'} \left[u_i \left(z_i'; x_{-i} \right) - ||z_i' - x_i||^2 \right]$$

Brouwer’s Theorem

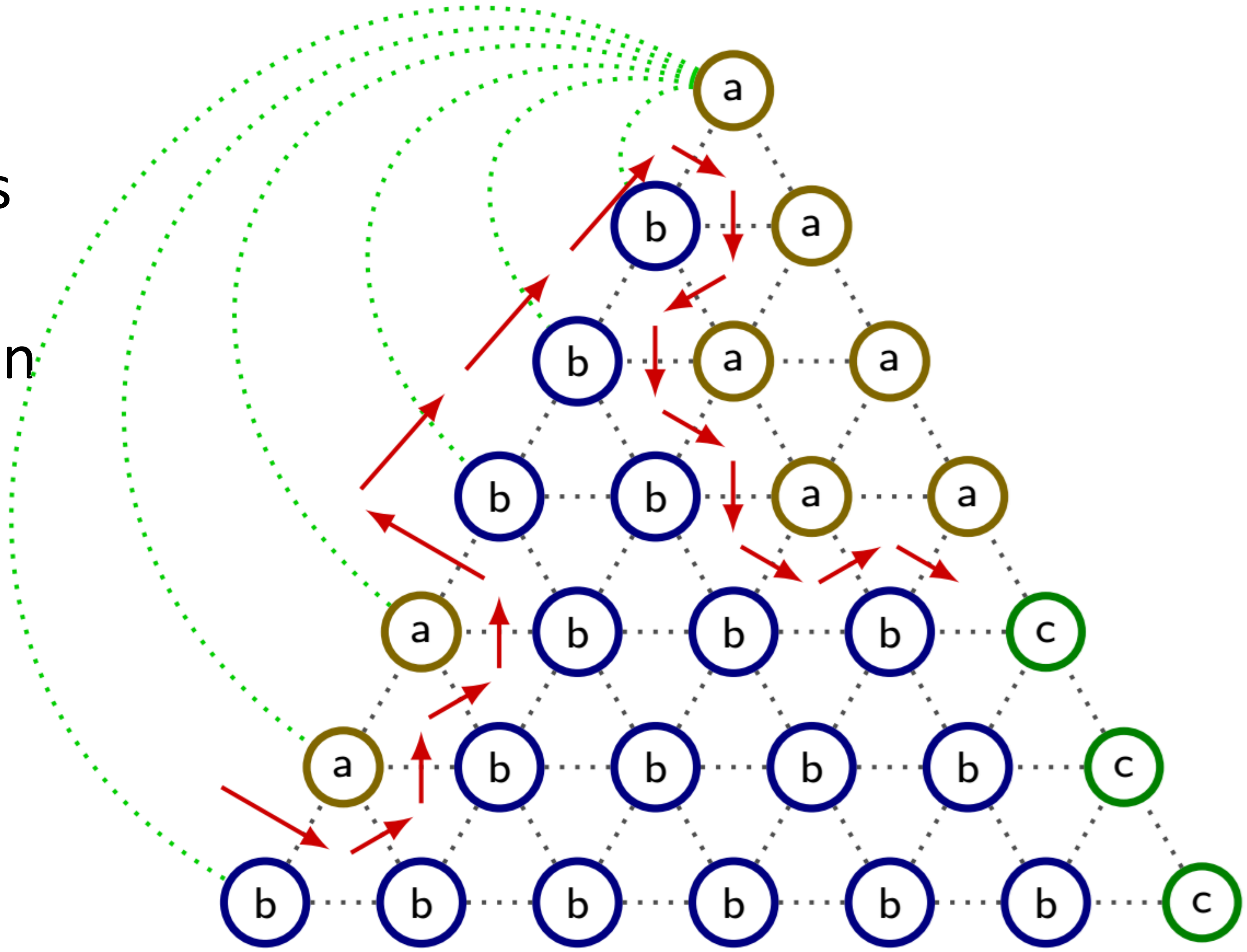
- Brouwer’s theorem: Every continuous function from a closed compact convex (c.c.c.) set to itself has a fixed point.
  - Proof: see the blackboard
  - The idea is to use Sperner’s theorem





Sperner's Theorem

- Given a triangle whose vertices are colored a, b and c
- Proper coloring: every vertex on the edge colored (a,b), is colored with a or b.
- Sperner's Theorem: *Every proper coloring of a triangulation has a panchromatic triangle*
  - Proof: see the blackboard



Nash Equilibria and Symmetric Games

- A symmetric game is one where each utility function $u_i(\cdot)$ does not change under permutations of the strategies played: more specifically,

$$u_i(s_1, s_2, \ldots s_n) = u_{\Pi(i)}(s_{\Pi(1)}, s_{\Pi(2)}, \ldots s_{\Pi(n)}) \text{ for any permutation function } \Pi(i)$$

- Theorem: Every symmetric game has a symmetric Nash equilibrium.
  - Proof: See the blackboard
- Theorem: Finding the Nash equilibrium of a general two-player game reduces to finding the Nash equilibrium of a symmetric two-player game.
  - Proof: See the blackboard

Maxmin Strategy

- Maxmin is a strategy that maximizes i's worst-case payoff, in the situation where all the other players happen to play the strategies which cause the greatest harm to I (the security level)

- Maxmin: The maxmin strategy for player i is $argmax_{s_i} \min\limits_{s_{-i}} u_i(s_i, s_{-i})$, and the maxmin value for player i is $\max_{s_i} \min\limits_{s_{-i}} u_i(s_i, s_{-i})$.

Minmax Strategy

- In two-player games the minmax strategy for player i against player -i is a strategy that keeps the maximum payoff of -i at a minimum, and the minmax value of player -i is that minimum.
- Minmax in two-player games: In a two-player game, the minmax strategy for player i against player -i is $\mathrm{argmin}_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player -i's minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.
- Minmax, n-player: In an n-player game, the minmax strategy for player i against player $j \neq i$ is i's component of the mixed-strategy profile s_{-j} in the expression $\mathrm{argmin}_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$. As before, the minmax value for player j is $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$.

Minmax Theorem

- Minmax Theorem (Von Neumann): In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.
  - Proof: See the blackboard.

Minmax Regret

- It can make sense for agents to care about minimizing their worst-case *losses*, rather than maximizing their worst-case payoffs.
- Regret: An agent i's regret for playing an action a_i if the other agents adopt action profile a_{-i} is defined as

$$\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i})$$

- Max Regret: An agent i's maximum regret for playing an action a_i is defined as

$$\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i}) \right)$$

Minmax Regret

- Minmax regret: Minimax regret actions for agent i are defined as

$$argmin_{a_i \in A_i} \left(\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i}) \right] - u_i(a_i, a_{-i}) \right) \right)$$

- Example:
  - Player 1's maxmin strategy is to play B
  - If player 1 does not believe that player 2 is malicious, he might reason in another way
  - If player 2 were to play R then it would not matter very much how player 1 plays: loss = ϵ
  - If player 2 were to play L then player 1's action would be very significant: loss = 98
  - Thus player 1 might choose to play T in order to minimize his worst-case loss.

	L	R
T	$100, a$	$1-\epsilon, b$
B	$2, c$	$1, d$

Domination

- **Domination:** Let s_i and s_i' be two strategies of player i, and S_{-i} the set of all strategy profiles of the remaining players. Then
  - s_i strictly dominates s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
  - s_i weakly dominates s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and for at least one $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
  - s_i very weakly dominates s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

Domination

- **Dominant strategy:** A strategy is strictly (resp., weakly; very weakly) dominant for an agent if it strictly (weakly; very weakly) dominates any other strategy for that agent.

- **Dominated strategy:** A strategy s_i is strictly (weakly; very weakly) dominated for an agent i if some other strategy s_i' strictly (weakly; very weakly) dominates s_i

Another Forms of Equilibria

- Correlated Equilibrium
- Trembling-hand Perfect Equilibrium
- ϵ-Nash Equilibrium
- Stackelberg Equilibrium (Competition)
- Cournot Equilibrium (Competition)
- Bertrand Equilibrium (Competition)
- And ….