Solution to Practice Problems: Game Theory

1. **Modeling a situation as a game**

   For each of the following situations, come up with a game (in normal form – the matrix representation we have been using) to describe the situation. For each one, decide if the game is zero-sum or not. Also, if the game appears to be equivalent to some game we've already seen in lecture, indicate which one.

   There are many possible solutions to these. The important thing is NOT to get the exact same numbers as I have. But you should try to get the same relative ordering between the payoffs.

   a. Two gentlemen are engaged in a duel. They each have two choices: shoot straight (to try to kill the other) or duck (to try to avoid being killed). If they both shoot straight, they both die; if they both duck, they live, but both lose some "face". If one ducks and one shoots straight, they both live, but the one who shoots earns "honor" and "respect" from his peers, while the one who ducks loses "face" and is branded a coward. Shooting and living gives a higher reward than ducking and living, regardless of what the other person does.

	Shoot	Duck
Shoot	-10, -10	-1, +1
Duck	+1, -1	-1, -1

This is essentially a variant of the game of Chicken, which I showed in class.

b. A river flows through two countries, A and B. It starts in country A, and then flows through B to the ocean. Both countries can either dam the river (and get electricity) or fish the river. If either country dams the river, it hurts the fish population in the river (either by decreasing the water flow downstream, or preventing fish from swimming upstream). So if one dams and the other fishes, it's bad for the one who fishes. If they both fish, it's good for both, but not as good as if they both dam, since the electric power is worth more than the fish.

	A Dams	A Fishes
B Dams	+2, +2	0, +2
B Fishes	+2, 0	+1, +1

c. Minority game: Three agents each have two possible actions. Whichever agent ends up in the minority (choosing a different action from the other two) wins. (This one is tricky because there are three players – you'll need to modify the matrix representation that we've been using to solve this.)

C: 1

	A: 1	A: 2
B: 1	A:0, B:0, C:0	A:1, B:0, C:0
B: 2	A:0, B:1, C:0	A:0, B:0, C:1

C: 2

	A: 1	A: 2
B: 1	A:0, B:0, C:1	A:0, B:1, C:0
B:2	A:1, B:0, C:0	A:0, B:0, C:0

2. Finding solution concepts

For each game below, find

- all *dominant strategies* for either player;
- all *pareto optimal* outcomes;
- and all *pure-strategy Nash equilibrium* outcomes.

2, 4	0, 1
2, 5	3, 3

Row 2 is a dominant strategy for the row player (player 2) . In column 1, playing row 2 gives a higher payoff for player 2 (5 > 4). And in column 2, playing row 2 gives a higher payoff for player 2 (3 > 1). Since it's a higher payoff either way, row 2 is a dominant strategy.

Player 1, the column player, doesn't have a dominant strategy in this game. Column 1 is better for player 1 in the top row (2 > 0), but column 2 is better for player 1in the bottom row (3 > 2).

The top-left square can't be pareto optimal, since the bottom-left square leaves everyone at least as well off, and player 2 better off. Likewise, the top-right square can't be pareto optimal, since every other square makes both players better off (this is sometimes called "pareto pessimal"). The two bottom squares are pareto optima: For each one of these squares, there is no other square where both players can get at least as good of a payoff, and one player gets a better payoff.

To find a Nash equilibrium, this game can be solved by *removing strategies that are dominated by other strategies.* First, since row 2 is a dominant strategy, that makes row 1 a *dominated* strategy. That basically means that the row player will never play row 1, so we can basically eliminate it from the game. If we just consider the bottom row, Player 1 (the column player) has a dominant strategy: pick column 2 (3 > 2). So that must be the Nash equilibrium. It's the only Nash equilibrium in this game.

5, 0	3, 2
3, 2	1, 4

The row player again has a dominant strategy, and again it is row 2: 2 > 0 in column 1, and 4 > 2 in column 2.

This time, the column player also has a dominant strategy: column 1. In row 1, 5 > 3. In row 2, 3 > 1.

This game is constant-sum (all 4 squares add up to 5). As is usual in constant-sum games, every square is a pareto optimum. There's no square where both players are better off or at least as well off with one being better off, since then it couldn't be constant-sum.

The intersection of the dominant strategies gives the only Nash equilibrium.

6, 6	4, 0
0, 4	5, 5

This game is a variant of the "Stag Hunt" coordination game. Neither player has a dominant strategy: the best row for the row player depends on what the column player does, and the best column for the column player depends on what the row player does.

There is only one pareto optimum: the upper-left. In every other square, both players would benefit by moving to the upper-left, so no other square can be a pareto optimum.

There's two pure-strategy Nash equilibria (as well as mixed-strategy ones, but we won't do those here). In the upper-left, neither player has an incentive to switch if the other player sticks to their strategy, and the same for the lower-right.

6, 6	0, 5
5, 0	4, 4

This game looks a lot like a coordination game, but it's not. The difference is that the payoffs in the lower-left and upper-right give both agents an incentive to move away from the lower-right.

As a result, each player has a dominant strategy: player 1's strategy is to play column 1. Player 2's strategy is to play row 1.

There is only one pareto optimum, the top-left.

There is only one Nash equilibrium, the top-left (the intersection of the dominant strategies).

3. Finding mixed-strategy Nash equilibria for zero sum games

Below are two zero-sum games. For each one, find a mixed-strategy Nash equilibrium that is not also a pure-strategy Nash equilibrium.

6, -6	-1, 1
-5, 5	4, -4

Let c1 = the probability that the column player (player 1) plays column 1.

Let r1 = the probability that the row player plays row 1.

For an equilibrium, the column player needs to choose c1 so that the row player's two rows give equal payoff. So:

-6 c1 + 1 (1-c1) (the payoff to player 2 for playing row 1)

=

5 c1 + -4 (1-c1) (the payoff to player 2 for playing row 2)

- ⇨ 5 = 8c1
- ⇨ c1 = 5/8

By the same line of reasoning, for an equilibrium, the row player needs to choose r1 so that the column player's two columns give equal payoff. So:

6 r1 + -5 (1-r1) (the payoff to player 1 for playing column 1)

=

-1 r1 + 4 (1-r1) (the payoff to player 1 for playing column 2)

=> 16r1 = 9

=> r1 = 9/16

So the mixed-strategy Nash equilibrium is:

player 1 chooses column 1 with probability 5/8, and column 2 with probability 3/8

player 2 chooses row 1 with probability 9/16, and row 2 with probability 7/16

The value of the game for player 1 is:

6 r1 + -5 (1-r1) = 6*9/16 – 5*7/16 = 1.1875

The value of the game for player 2 is -1.1875.

1, -1	3, -3
2, -2	-6, 6

Let c1 = the probability that the column player (player 1) plays column 1.

Let r1 = the probability that the row player plays row 1.

For an equilibrium, the column player needs to choose c1 so that the row player's two rows give equal payoff. So:

-1 c1 -3 (1-c1) (the payoff to player 2 for playing row 1)

=

-2 c1 + 6 (1-c1) (the payoff to player 2 for playing row 2)

- ⇨ 10c1 = 9
- ⇨ c1 = 9/10

By the same line of reasoning, for an equilibrium, the row player needs to choose r1 so that the column player's two columns give equal payoff. So:

1 r1 + 2 (1-r1) (the payoff to player 1 for playing column 1)

=

3 r1 + -6 (1-r1) (the payoff to player 1 for playing column 2)

- ⇨ 10r1 = 8
- ⇨ r1 = 8/10 = 4/5

So the mixed-strategy Nash equilibrium is:

player 1 chooses column 1 with probability 9/10, and column 2 with probability 1/10

player 2 chooses row 1 with probability 4/5, and row 2 with probability 1/5

The value of the game for player 1 is:

1 r1 + 2 (1-r1) = 1* 4/5 + 2*1/5 = 6/5 = 1.2.

The value of the game for player 2 is -1.2.