Stat 155 Homework # 4 Solution

Problems:

Q 1 Ferguson Chapter II Section 2.6 Q 1

Solution The matrix

$$\begin{pmatrix} -1 & -3 \\ -2 & 2 \end{pmatrix}$$

has no saddle point. Thus applying the formulas for a two by two game, the optimal strategies are

$$x = (\frac{2-(-2)}{(-1-(-3)+(2-(-2))}, \frac{(-1-(-3)}{(-1-(-3)+(2-(-2))}) = (\frac{2}{3}, \frac{1}{3})$$

and

$$y = (\frac{2-(-3)}{(-1-(-2))+(2-(-3))}, \frac{(-1-(-2))}{(-1-(-2))+(2-(-3))} = (\frac{5}{6}, \frac{1}{6})$$

The value is

$$V = (-1)\cdot\frac{2}{3} - 2\cdot\frac{1}{3} = -\frac{4}{3}.$$

Q 2 Ferguson Chapter II Section 2.6 Q 2

Solution

When $t \leq 0$ then 0 is a saddle point and $v(t) = 0$. When $0 \leq t \leq 1$ then t itself is a saddle point and the value is t. For $t > 1$ the is no saddle point and so the 2 by 2 matrix formula gives

$$v(t) = \frac{-2t}{1-2-t} = \frac{2t}{t+1}.$$

Thus

$$v(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 1 \\ \frac{2t}{t+1} & 1 \leq t. \end{cases}$$

Q 3 Ferguson Chapter II Section 2.6 Q 4

Solution

(a)

$$\begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$$

The second column dominates the first column.

$$\begin{pmatrix} 4 & 1 & 0 \\ 3 & 2 & -1 \\ -1 & 4 & 3 \\ -2 & 1 & 2 \end{pmatrix}$$

The third row dominates the fourth row.

$$\begin{pmatrix} 4 & 1 & 0 \\ 3 & 2 & -1 \\ -1 & 4 & 3 \end{pmatrix}$$

The third column dominates the second column.

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \\ -1 & 4 \end{pmatrix}$$

We now solve this 3 by 2 matrix and let $y = (y_1, y_2)$ be the strategy for player 2. Then the value is given by

$$\max(4y_1 + (1 - y_1), 3y_1 + 2(1 - y_1), -y_1 + 4(1 - y_1))$$

This is minimized by taking $y_1 = \frac{1}{3}$ giving a value of $\frac{7}{3}$. In this case $Ay = (2, \frac{7}{3}, \frac{7}{3})$ and thus player 1 will use a combination of rows 2 and 3. So the resulting 2 by 2 matrix is

$$\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

The optimal strategy for player 1 in this 2 by 2 game is

$$x = (\frac{5}{6}, \frac{1}{6}).$$

In the original game thus the value is $\frac{7}{3}$ and the strategies are $x = (0, \frac{5}{6}, \frac{1}{6}, 0)$ and $y = (0, \frac{1}{3}, \frac{2}{3}, 0)$.

(b) Starting with this matrix.

$$\begin{pmatrix} 10 & 0 & 7 & 1 \\ 2 & 6 & 4 & 7 \\ 6 & 3 & 3 & 5 \end{pmatrix}$$

The last column is dominated by the second column.

$$\begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ 6 & 3 & 3 \end{pmatrix}$$

We have that

$$\frac{1}{2}(10, 0, 7) + \frac{1}{2}(2, 6, 4) = (6, 3, \frac{11}{2}) \geq (6, 3, 3)$$

and thus the combination of the first two rows dominates the third

$$\begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \end{pmatrix}$$

The function

$$\min(10x_1 + 2(1 - x_1), 6(1 - x_1), 7x_1 + 4(1 - x_1))$$

is maximized taking $x_1 = \frac{2}{7}$. This gives a value of $\frac{30}{7}$. With $x = (\frac{2}{7}, \frac{5}{7})$ the payoffs for the strategies of player 2 are

$$x^T A = (\frac{30}{7}, \frac{30}{7}, \frac{34}{7})$$

so player 2 uses a combination of columns 1 and 2. The resulting matrix is

$$\begin{pmatrix} 10 & 0 \\ 2 & 6 \end{pmatrix}$$

Solving for player 2's strategy in this 2 by 2 game we have

$$y = (\frac{3}{7}, \frac{4}{7})$$

Thus in the original game the value is $\frac{30}{7}$ and the strategies are $x = (\frac{2}{7}, \frac{5}{7}, 0)$ and $y = (\frac{3}{7}, \frac{4}{7}, 0, 0)$.

Q 4 Karlin-Peres Chapter 2 Exercise 2.7

Solution The payoff matrix is

$$\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

As this is a diagonal matrix the strategies are given by

$$x = y = (\frac{d_1^{-1}}{\sum_{i=1}^3 d_i^{-1}}, \frac{d_2^{-1}}{\sum_{i=1}^3 d_3^{-1}}, \frac{d_1^{-1}}{\sum_{i=1}^3 d_i^{-1}}) = (\frac{2}{11}, \frac{6}{11}, \frac{3}{11}).$$