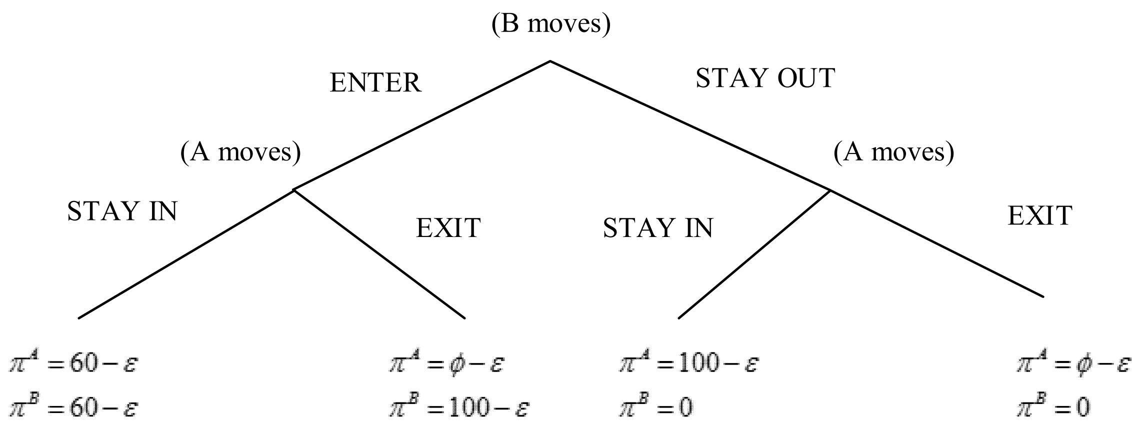
EconS 424 – Strategy and Game Theory
Midterm #2 – Answer Key

Exercise 1

Consider the entry-exit two-stage game represented in the figure below in which firm A is the incumbent firm that faces potential entrant firm B. In stage I, firm B decides whether to enter into A's market or whether to stay out. The cost of entry is denoted by ε. In stage II, the established firm, firm A, decides whether to stay in the market or exit.



The game tree reveals that firm A can recover some of its sunk entry cost by selling its capital for the price φ , where 0 ≤φ ≤ε . Solve the two problems:

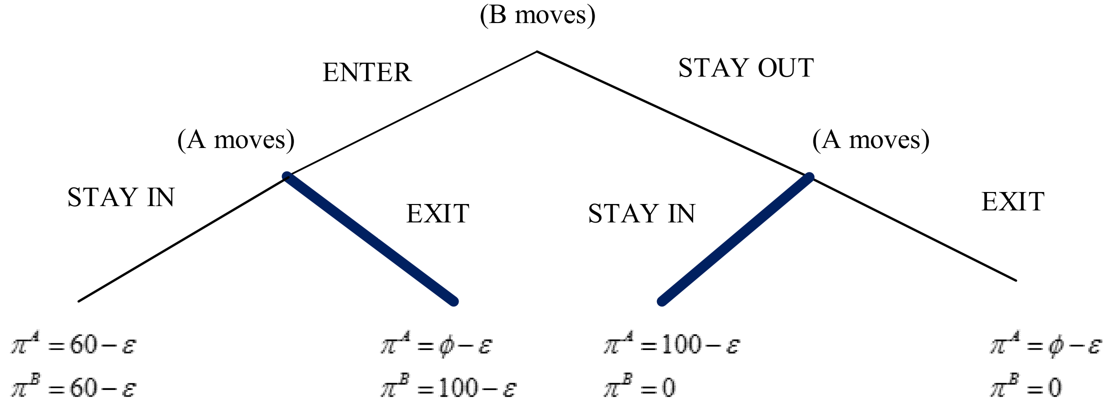
a) Compute the subgame-perfect equilibrium strategies of firms B and A assuming that ε < 60. Prove your answer.

Answer the above question assuming that $60 < \varphi \leq \varepsilon < 100$

Answer: Upon observing that firm B enters (in the LHS of the tree), firm A compares its profits from staying in, $60-\varepsilon$, against its profits from exiting, $\phi-\varepsilon$. Hence, firm A responds with exiting as long as $60-\varepsilon < \phi-\varepsilon \Rightarrow 60 < \phi$, which holds by assumption.

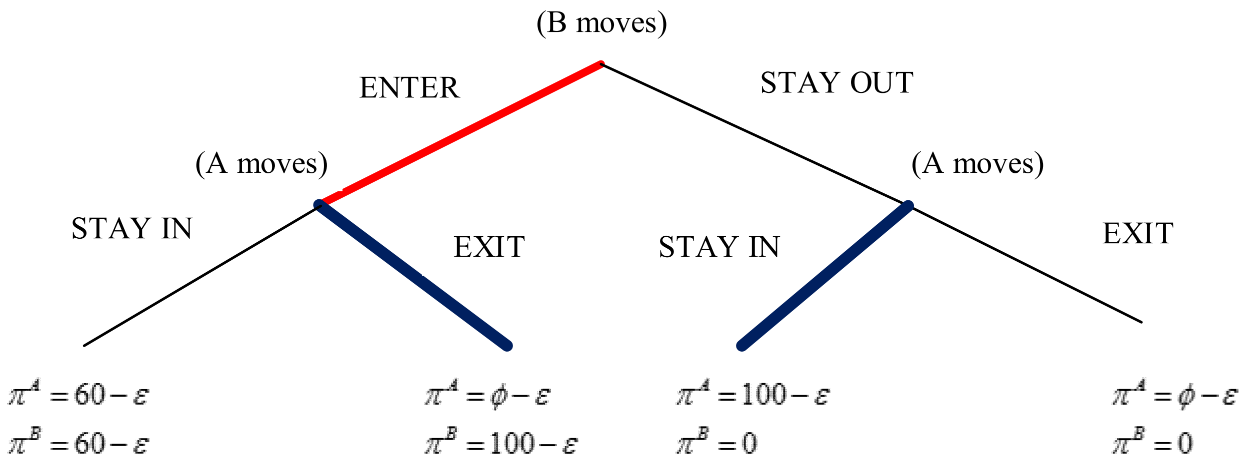
Upon observing that firm B stays out (in the RHS of the tree), firm A compares $100-\varepsilon$ against $\phi-\varepsilon$, and thus responds by staying in since $100-\varepsilon > \phi-\varepsilon \Rightarrow 100 > \phi$, which is true by assumption.

We can summarize these responses of firm A in the following figure:



Anticipating these optimal responses, firm B (first mover) compares its profit from entering (which is responded with exit by firm A), $100 - \varepsilon$, against its profit from staying out (which is responded by firm A with staying in), 0. Hence, firm B enters if $100 - \varepsilon > 0 \Rightarrow 100 > \varepsilon$, which is true by assumption.

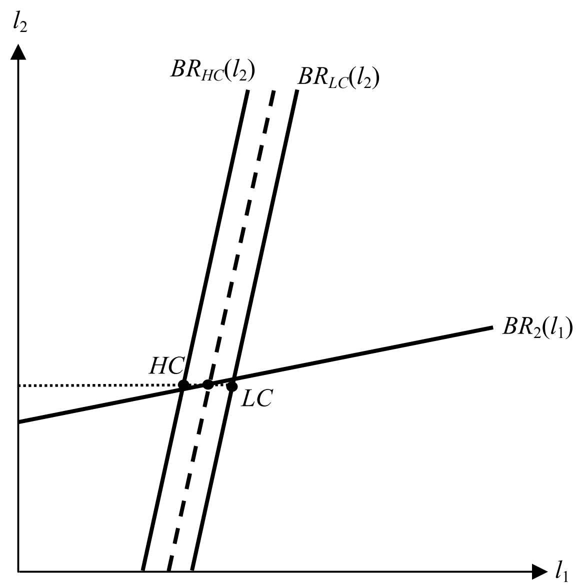
Hence, the SPNE of this game is {Enter, (Exit if firm B enters, Stay in if firm B stays out)}. In summary, the next figure shades the branches of this SPNE:



Exercise 2:

a. The best-response function is $l_{LC} = 3.5 + l_2 / 4$ for the low-cost type of player 1, $l_{HC} = 2.5 + l_2 / 4$ for the high-cost type, and $l_2 = 3 + \bar{l}_1 / 4$ for player 2, where $\bar{l}_1$ is the average for player 1. Solving these equations yields $l_{LC}^* = 4.5$, $l_{HC}^* = 3.5$, and $l_2^* = 4$.

b. Player 2 best responds to the average best response across the two types of player 1, given by the dashed line between the two best responses, and resulting in a choice of landscaping level given by the dotted horizontal line. The two types of player 1 best respond to the equilibrium landscaping effort of player 2, resulting in the outcome labeled *HC* if player 1 is the high-cost type and *LC* if player 1 is the low-cost type.



c. The low-cost type of player 1 earns 20.25 in the Bayesian-Nash equilibrium and 20.55 in the full-information game, so would prefer to signal its type if it could. Similar calculations show that the high-cost player would like to hide its type.

Exercise #3

a. α measures the degree of publicness of the good.

b. $U^1 = \log(y_1^1 + \alpha y_1^2) + x_2^1$, where y1i is the purchase of good 1 by individual *i*. Using the budget constraint (and assuming both goods have unit price) obtains

$$U^1 = \log(y_1^1 + \alpha y_1^2) + M - y_1^1$$

Taking first order conditions with respect to $y_1{}^1$, we obtain

$$\frac{1}{y_1^1 + \alpha y_1^2} - 1 = 0.$$

The game is symmetric. So the solution is $y_1^1 = y_1^2 = y_1 = \dfrac{1}{1+\alpha}$ Hence the consumption level in equilibrium is

$$x_1^1 = x_1^2 = x_1 = [1+\alpha]\, y_1 = 1$$

c. The level of social welfare is

$$W = \log(y_1^1 + \alpha y_1^2) + M - y_1^1 + \log(y_1^2 + \alpha y_1^2) + M - y_1^2$$

Applying symmetry obtains

$$W = 2\log([1+\alpha]\, y_1) + 2[M - y_1]$$

So

$$\frac{\partial W}{\partial y_1} = \frac{2}{y_1} - 2 = 0$$

Hence $y_1 = 1$ and $x_1 = 1+\alpha$. The two outcomes are the same if $\alpha = 0$.

d. Utility now becomes

$$U^1 = \log(y_1^1 + \alpha y_1^2) + M - y_1^1 + \alpha(M - y_1^2).$$

The Nash equilibrium remains at $y_1^1 = y_1^2 = y_1 = \dfrac{1}{1+\alpha}$ with symmetry the level of welfare is

$$W = 2\log((1+\alpha)\, y_1) + 2(1+\alpha)(M - y_1), \text{ so}$$

$y_1 = \dfrac{1}{1+\alpha}$. The two outcomes are identical for all α.

e. In part b there is one private good and one public good when $\alpha \neq 0$. So free riding takes place when $\alpha \neq 0$. With $\alpha = 0$, there are two private goods, so the outcome is efficient. In part *d* both goods have an identical degree of publicness so the consumption externalities are balanced. It is not possible to free-ride on both goods, so efficiency results.

Exercise #4

a. The expected payoffs of players. 1 and 2 are $\dfrac{\phi x_1}{\phi x_1 + x_2} R - x_1$ and $\dfrac{x_2}{\phi x_1 + x_2} R - x_2$ respectively. Taking derivatives with respect to own spending and simplifying, we obtain the best-response functions

$$x_1 = \frac{\sqrt{\phi R x_2}}{\phi} - \frac{x_2}{\phi},$$

$$x_2 = \sqrt{\phi R x_1} - \phi x_1$$

Each best-response function is a concave parabola with zeros at $x_2 = \{0; \phi R\}$ and $x_1 = \left\{0; \dfrac{R}{\phi}\right\}$ respectively for player 1 and 2. The best-response function of player 1 has a peak at $x_2 = \phi \dfrac{R}{4}$, and the best-response function of player 2 has a peak at $x_1 = \frac{R}{4\phi}$. The best-response functions are illustrated in figure.

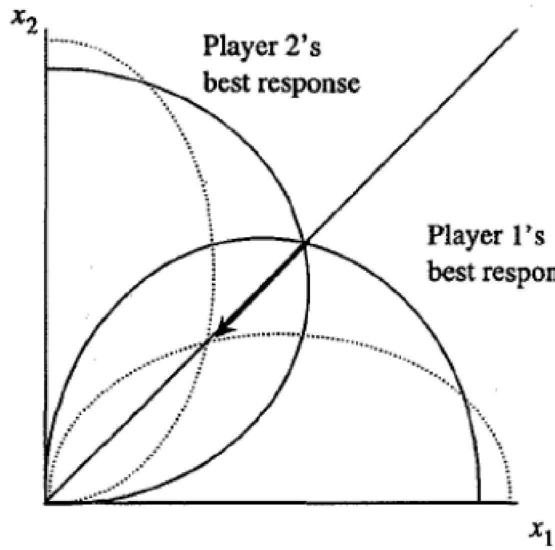


Figure. Equilibrium

b. Note that firm l's best response implies $\phi x_1 = \sqrt{\phi R x_2} - x_2$, and firm 2's best response implies $\phi x_1 = \sqrt{\phi R x_1} - x_2$. So we must have $x_1=x_2$ in equilibrium. Solving for the symmetric equilibrium, we obtain

$$x_1 = x_2 = x^* = \frac{R\phi}{[1+\phi]^2}.$$

Since both firms bid the same, firm 1 is more likely to win the rent in equilibrium, Since

$$p_1 = \frac{\phi x^*}{\phi x^* + x^*} > \frac{x^*}{\phi x^* + x^*} = p_2.$$

c. From part b we have that the symmetric equilibrium spending is proportional to $\frac{\phi}{(1+\phi)^2}$, which is decreasing in ϕ for all $\phi \geq 1$. So we expect more spending on rent-seeking activities when players are identical. Therefore it is when contestants are in a symmetric position that competition is the most severe. If one of the contestants has a comparative advantage in the contest, there will be less spending on rent-seeking activities.

Exercise #6
Harrington Ch. 14 Exercise 1

A.

ANSWER: This game has two Nash equilibria, (a, w) and (b, x). Thus, consider a strategy for player 1 in which, in period 1, he chooses c; in a future period, chooses c if the outcome was (c, y) in all past periods; and, in a future period, chooses b if the outcome was not (c, y) in some past period. For player 2, in period 1, she chooses y; in a future period, chooses y if the outcome was (c, y) in all past periods; and, in a future period, chooses x if the outcome was not (c, y) in some past period. The equilibrium condition is

$$\frac{7}{1-\delta} \geq 9 + \delta\left(\frac{4}{1-\delta}\right) \Rightarrow \delta \geq \frac{2}{5}.$$

B.

ANSWER: Suppose it is either period 1 or it is some future period in which the outcome was either (c, y) or (d, z) in the previous period. The prescribed action of c for player 1 is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7 + 7\delta \geq 9 \Rightarrow \delta \geq \frac{2}{7}.$$

Now consider a history in which in the previous period the outcome was neither (c, y) nor (d, z). The prescribed action of d for player 1 is optimal if

$$0 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 4 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7\delta \geq 4 \Rightarrow \delta \geq \frac{4}{7}.$$

The conditions are the same for player 2. It is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{7} \text{ and } \delta \geq \frac{4}{7}, \text{ or } \delta \geq \frac{4}{7}.$$

C.

ANSWER: Consider period 1 or a future period in which the previous period's outcome was either (c, y), (d, w), or (a, z). Player 1's prescribed action of c is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times (-4) + \delta^2 \times 7 + \cdots$$
$$\Rightarrow 7 + 7\delta \geq 9 - 4\delta \Rightarrow \delta \geq \frac{2}{11}.$$

Next, consider a history in which, in the previous period, player 1 did not choose c and player 2 chose y. Player 1's prescribed action of d is optimal if

$$-4 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 2 + \delta \times 4 + \delta^2 \times 4 + \cdots$$
$$\Rightarrow -4 + 7\frac{\delta}{1-\delta} \geq 2 + 4\frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{2}{3}.$$

Next, consider a history in which, in the previous period, player 1 chose c and player 2 did not choose y. Player 1's prescribed action of a is optimal since any other action lowers the current period payoff and reduces the future payoff stream from

$$\delta \times 7 + \delta^2 \times 7 + \cdots$$

to

$$\delta \times 4 + \delta^2 \times 4 + \cdots.$$

Finally, for any other history, player 1 is to choose b. Note that b maximizes his current payoff given player 2 is to choose x. Furthermore, the future payoff is the same since, come next period, the previous period's history will involve player 2 choosing x and thus the outcome will be (b, x). This applies as well to all ensuing periods. The analysis is analogous for player 2. This strategy profile is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{11} \text{ and } \delta \geq \frac{2}{3} \Rightarrow \delta \geq \frac{2}{3}.$$

Harrington Ch. 15 Exercise 1

A.

ANSWER: Suppose the occupant of cubicle 101 enters in even periods and the occupant of cubicle 102 in odd periods. Consider the following strategy for the occupant of 101. If in all past periods the outcome was either (c, x) or, (a, z), then he chooses a if it is the first period of his life and chooses c if it is the second period. For any other history, he chooses b. For the occupant of 102, if in all past periods the outcome was either (c, x) or (a, z), then he chooses x if it is the first period of his life and chooses z if it is the second period. For any other history, he chooses y. Since the game and strategy are symmetric, it is sufficient to show the optimality of the strategy for one player, and we'll take the occupant of cubicle 101. Consider a history in which all past outcomes were either (c, x) or (a, z). If this is the first period of his life, his payoff from choosing a is $5 + 10$ or 15, as he expects the other player to choose z this period and expects an outcome of (c, x) next period. Alternatively, he can choose b and get a payoff of $8 + 3$ or 11; note that this deviation induces an outcome of (b, y) in his last period. Or he can choose c and get a payoff of $0 + 3$ or 3. Thus, it is optimal to choose a. Now consider the second period of his life. His strategy has him choose c, which results in a payoff of 10, given the other player chooses x. This is clearly optimal, as choosing a or b delivers a lower payoff of 8 or 4, respectively. Next, consider a history in which, in some past period, the outcome was neither (c, x) nor (a, z). If it is the first period of his life, he knows that the occupant of cubicle 102 will choose y in the current and next period. Thus, choosing b yields a payoff of $3 + 3$ or 6, and that is higher than choosing either a or c. In his second period, the same calculus applies.

B.

ANSWER: This is a trick question, as there is no such subgame perfect Nash equilibrium. Consider a strategy profile that has, say, the occupant of cubicle 102 choosing x in the second period of her life. Doing so yields a payoff of 8, while instead playing z results in a higher payoff of 10.

Watson Ch. 26 Exercise 1

As the problem hints, because A and B have equal probabilities it is sufficient to average the payoffs for each outcome to form the normal-form matrix of this Bayesian game. After constructing the normal form, the survivors of IDSDS shows that only Z and V survive as rationalizable strategies.

(a) The Bayesian normal form is:

1 \ 2	V	W
X	3, 0	2, 1
Y	3, 0	2, 1
Z	5, 1	3, 0

(Z, V) is the only rationalizable strategy profile.

Here, player 1 can now condition his response to what state the game is in, so he has nine different strategy profiles essentially. The payoffs for player 1 in the normal form game will be the average of the payoffs player 1 receives from playing any one of the 9 strategies. Player 2 still faces the same situation as in part A, so player 2 still has only two strategies and the payoffs in the normal form game are the averages of the payoffs given player 1's strategies.

It is then possible using IDSDS so find the rationalizable strategies:

1. Player 1 obtains a strictly higher expected payoff choosing Z than select X or Y. Hence, we can claim that Z is a strictly dominant strategy for player 1. Alternatively, X and Y are both strictly dominated by Z, which allows us to delete them from the matrix (using IDSDS).
2. The above deletion leaves only the last row of the matrix. Therefore, player 2 (anticipating that player 1 chooses Z given that Z is player 1's strictly dominant strategy), selects V given that it provides him with an expected payoff of 1 rather than 0.
3. Hence, the only strategy pair surviving IDSDS is (Z,V).

(b) The Bayesian normal form is:

1 \ 2	V	W
X^AX^B	3, 0	2, 1
X^AY^B	6, 0	4, 1
X^AZ^B	5.5, .5	3.5, .5
Y^AX^B	0, 0	0, 1
Y^AY^B	3, 0	2, 1
Y^AZ^B	2.5, .5	1.5, .5
Z^AX^B	2.5, .5	1.5, .5
Z^AY^B	5.5, .5	3.5, .5
Z^AZ^B	5, 1	3, 0

X^AY^B is a dominant strategy for player 1. Thus, the rationalizable set is (X^AY^B, W).

c) False.

Here, may simply compare player 1's payoff from part A of 5, and the payoff from part B of 4. 5 > 4, so more information has not benefitted player 1.

Watson Ch. 26 Exercise 5

$$\text{(a) } u_1(p_1, p_2) = 42p_1 + p_1 p_2 - 2p_1^2 - 220 - 10p_2$$

To construct the payoff function for firm 1 we want to find their profit function. Therefore you simply multiply the demand function, q1, by their prices, p1, and then subtract the costs of 10 per unit, 10*q1.

The process is similar for firm 2, except that the marginal cost is c instead of 10. For example:

$$u_2(p_1, p_2) = q_2 * p_2 - q_2 * c$$
$$= 22 - 2p_2 + p_1 * (p_2 - c) = 22p_2 - 2p_2^2 + p_1 p_2 - 22c + 2p_2 c - p_1 c$$

B.

To find the best response functions, simply take the partial of each firm's profit function with respect to their choice variable, set the resulting equation equal to zero, and solve for their choice variable. Firm 1's best response function for example is:

$$\frac{\partial u_1}{\partial p_1} = 42 + p_2 - 4p_1 = 0$$
$$4p_1 = 42 + p_2$$
$$BR_1(p_2) = p_1 = \frac{42 + p_2}{4}$$

$$\textit{The best response for player 2 is similarly:}$$
$$BR_2(p_1) = \frac{22 + p_1 + 2c}{4}$$

C.

When C=10, the firms are identical, and it is possible to plug each firm's BR into the other firm's BR to solve the system of equations, which results in a symmetric equilibrium of:

$$p_1^* = 14, \qquad p_2^* = 14$$

D.

Calculating the price that player 1 will play requires calculated the expected value of player 2's marginal cost, which is .5*6 + .5*14 = 10. This happens to be exactly the same calculation as in part

C, so $p_1^* = 14$ still. We have to calculate player 2's prices given player 1's price of 14 at each marginal cost for player 2:

$$p^*_{2,c=14} = \frac{22 + 14 + 2 * 14}{4} = 16$$
$$p^*_{2,c=6} = \frac{22 + 14 + 2 * 6}{4} = 12$$

<u>Exercise #6</u>

A.

	Join		Not Join	
Join	<u>1</u>	<u>1</u>	-1	0
Not Join	0	-1	<u>0</u>	<u>0</u>

Player A's best response: If player B joins, then player A's best response is to join as well; while if player B does not join, then player A's best response is not join. That is, player A seeks to mimic player B's action.
A similar argument applies to player B: if player A joins, his best response is to join, while if player A does not join, player B's best response is not to join.
Hence, we found 2 psNE: { (J, J) , (NJ, NJ) }

msNE: Let us now find the msNE. Making player A indifferent between joining and not joining,

$$EU^A(J) = EU^A(NJ)$$

$q * 1 + (1 - q) * -1 = 0$, and solving for q, we obtain $q = \frac{1}{2}$

And since players' payoffs are symmetric, player A also mixes with probability p=1/2. Hence, the msNE of the game is

$$\left\{ \left(\frac{1}{2}J, \frac{1}{2}NJ\right), \left(\frac{1}{2}J, \frac{1}{2}NJ\right) \right\}$$

B. Now parameter t is distributed according to a uniform distribution, i.e., t ~ U[0,1]

	Join	Not Join
Join	$\underline{1}$ $\underline{1+t}$	-1 0
Not Join	0 -1	$\underline{0}$ $\underline{0}$

Let us start analyzing the informed player (player B). When player A joins (top row), player B's best response is to join as well, since 1+t>0, which holds for all values of t. In contrast, when player A does not join (bottom row), player B's best response is to not join since 0>-1.

Let us now move to the uninformed player A. His payoffs are unaffected by t, thus implying that his best response coincides with that of part (a) of the exercise, namely, mimicking his opponent's action.

At this point, we obtained two BNE, one in which both players join and another in which no player does. Which equilibrium emerges, however, depends on players' beliefs. In particular, player B chooses to join rather than not join when his expected utility from joining is larger, where

$$EU^B(J) = \mu * (1+t) + (1-\mu)(-1) = \mu + \mu t - 1 + \mu = \mu(2+t) - 1$$
$$EU^B(NJ) = 0$$

Hence, P_B will decide to join if and only if:

$$EU^B(J) \geq EU^B(NJ)$$
$$\leftrightarrow \mu(2+t) - 1 \geq 0 \leftrightarrow \mu \geq \frac{1}{2+t}$$

We can now summarize the two BNEs of this game.

<u>1st BNE</u>

If $\mu \geq \frac{1}{2+t}$ $then\ P_B\ Joins,\ then\ P_A\ will\ also\ Join$. More compactly, $\left\{J, J;\ \mu \geq \frac{1}{2+t}\right\}$

<u>2nd BNE</u>

If $\mu < \frac{1}{2+t}$ $then\ P_B\ NJ,\ then\ P_A\ NJ$. More compactly, $\left\{NJ, NJ;\ \mu < \frac{1}{2+t}\right\}$.