14.12 Game Theory
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Solution to Homework 3

Problem One

a) This game has two subgames. One subgame is the game as a whole and the other is the subgame after both players say yes. In this last subgame we have 3 Nash equilibria (NE) which are (I, I), (C, C) and a mixed strategy. The mixed strategy NE is computed as follows:

Let's say player 1 plays I with probability μ and player 2 plays I with probability λ. Then the mixed equilibrium should satisfy the following equations:

Player 2 is indifferent between I and $C \Leftrightarrow \mu * 100 + (1-\mu) * 0 = \mu * 60 + (1-\mu) * 60 \Leftrightarrow u = 0.6$

Player 1 is indifferent between I and $C \Leftrightarrow \lambda * 100 + (1-\lambda) * 0 = \lambda * 60 + (1-\lambda) * 60 \Leftrightarrow \lambda = 0.6$

Therefore the solution is $\mu = \lambda = 0.6$.

This implies that the mixed Nash equilibrium is $(0.6 * I + 0.4C, 0.6I + 0.4 * C)$.

To compute the subgame-perfect equilibrium (SPE) we have to check if the strategy profile is a NE in every subgame. Thus, to obtain the SPE we have to assume that outcome of the smaller subgame will be one of those NE.

When both players choose I in the smaller subgame, the reduced form of the game is:

	Y	N
Y	$100, 100$	$80, 80$
N	$80, 80$	$80, 80$

There are 2 types of NE in this case (Y, Y) and (N, N). Therefore the SPE are (YI, YI) and (NI, NI)

When both players choose C in the smaller subgame. The reduced form of the game is

	Y	N
Y	$60,60$	$80,80$
N	$80,80$	$80,80$

There are 3 pure NE in this case $(Y,N),(NY)$ and (N,N) and 2 mixed NE $(N,pY+(1-p)N)$ and $(pY+(1-p)N,N)$, $0 \leq p \leq 1$.

Therefore the SPE are $(YC,NC),(NC,YC),(NC,NC),(NC,(pY+(1-p)N)C)$ and $((pY+(1-p)N)C,NC)$.

When both players choose to play the mixed NE in the smaller subgame the reduced form is the same as in the former case and therefore the NE are also the same. Thus, SPE of the game are $(YC,NC).(NC,YC),(NC,NC),(NC,(pY+(1-p)N)C)$ and $((pY+(1-p)N)C,NC)$.

Problem Two

Denote V_t^i the continuation value to player i when the game goes to the period t. Clearly, in the last period, $t = 3n$, player 2 will offer $(0,1)$ thus $V_{3n}^1 = 0$ and $V_{3n}^2 = 1$.

In period $t = 3n-3k+3,\ (k = 1,2,\ldots n-1),$ player 2 makes an offer. He will offer $(\delta V_{3n-3k+4}^1, 1-\delta V_{3n-3k+4}^1)$ that will be accepted by player 1.

In period $t = 3n-3k+2$, player 1 is the one to make an offer and will offer $(1-\delta V_{3n-3k+3}^2, \delta V_{3n-3k+3}^2)$.

In period $t = 3n-3k+1$, player 1 will offer $(1-\delta V_{3n-3k+2}^2, \delta V_{3n-3k+2}^2)$.

In period $t = 3n-3k$, player 2 will offer $(\delta V_{3n-3k+1}^1, 1-\delta V_{3n-3k+1}^1)$.

Therefore:

$$V_{3n-3k}^2 = 1-\delta V_{3n-3k+1}^1 = 1-\delta(1-\delta V_{3n-3k+2}^2) \Leftrightarrow V_{3n-3k}^2 = 1-\delta+\delta^2 V_{3n-3k+2}^2$$

$$\Leftrightarrow V_{3n-3k}^2 = 1-\delta+\delta^2(\delta * V_{3n-3k+3}^2) \Leftrightarrow V_{3n-3k}^2 = 1-\delta+\delta^3 * V_{3n-3k+3}^2$$

$$V_{3n-3k}^2 = 1-\delta+\delta^3(1-\delta+\delta^3 V_{3n-3k+6}^2) \Leftrightarrow V_{3n-3k}^2 = 1-\delta+\delta^3-\delta^4+\delta^6 V_{3n-3k+6}^2$$

$$V_{3n-3k}^2 = 1-\delta+\delta^3-\delta^4+\delta^6-\delta^7+\ldots\delta^{3k-3}-\delta^{3k-2}+\delta^{3k}V_{3n}^2$$

where $V_{3n}^2 = 1$

$$V_{3n-3k}^2 = 1-\delta+(1-\delta)\delta^3+(1-\delta)\delta^6+\ldots+(1-\delta)\delta^{3k-3}+\delta^{3k}$$

$$V_{3n-3k}^2 = (1-\delta)(1+\delta^3+\delta^6+\ldots\delta^{3k-3})+\delta^{3k}$$

$$V_{3n-3k}^2 = \frac{(1-\delta)(1-\delta^{3k})}{1-\delta^3}+\delta^{3k} \Leftrightarrow V_{3n-3k}^2 = \frac{(1-\delta)(1-\delta^{3k})}{(1-\delta)(1+\delta+\delta^2)}+\delta^{3k}$$

$$\Leftrightarrow V^2_{3n-3k} = \frac{(1-\delta^{3k})}{(1+\delta+\delta^2)} + \delta^{3k} \Leftrightarrow V^2_{3n-3k} = \frac{1+\delta^{3k+1}+\delta^{3k+2}}{(1+\delta+\delta^2)}$$

This gives the continuation value of player 2 in the periods $t = 3n-3k$, $k = 0,1\ldots n-1$.

In period $t = 3n-3k-1$ player 1 proposal is given by:

$$(V^1_{3n-3k-1} = 1 - V^2_{3n-3k-1}, V^2_{3n-3k-1} = \delta V^2_{3n-3k})$$
$$= (\frac{1-\delta+\delta^2-\delta^{3k+2}-\delta^{3k+3})}{(1+\delta+\delta^2)}, \frac{(\delta+\delta^{3k+2}+\delta^{3k+3})}{(1+\delta+\delta^2)})$$

In period t=3n-3k-2 player 1 proposal is given by:

$$(V^1_{3n-3k-2} = 1 - V^2_{3n-3k-2}, V^2_{3n-3k-2} = \delta V^2_{3n-3k-1})$$
$$= (\frac{(1+\delta-\delta^{3k+3}-\delta^{3k+4})}{(1+\delta+\delta^2)}, \frac{(\delta^2+\delta^{3k+3}+\delta^{3k+4})}{(1+\delta+\delta^2)})$$

Therefore, in period 1, $k = n-1$, player 1 proposes:

$$(V^1_1 = \frac{(1+\delta-\delta^{3(n-1)+3}-\delta^{3(n-1)+4})}{(1+\delta+\delta^2)} = \frac{(1-\delta)(1+\delta-\delta^{3n}-\delta^{3n+1})}{(1-\delta)(1+\delta+\delta^2)} =$$
$$\frac{(1-\delta)(1+\delta)(1-\delta^{3n})}{(1-\delta^3)} = \frac{(1-\delta^2)(1-\delta^{3n})}{(1-\delta^3)},\ V^2_1 = 1 - V^1_1 = 1 - \frac{(1-\delta^2)(1-\delta^{3n})}{(1-\delta^3)})$$

This offer will be accepted by player 2 and thus the bargaining process ends in the first period.

b) In the infinite each subgame is similar to subgames starting three periods before it and three periods after it, so there are three types of subgames to consider. Thus we have that $V^2_{3n} = V^2_{3n+3}$ and equation (1) becomes:

$$V^2_{3n} = 1 - \delta + \delta^3 * V^2_{3n+3}$$
$$V^2_{3n} = 1 - \delta + \delta^3 * V^2_{3n} \Leftrightarrow V^2_{3n} = \frac{1-\delta}{1-\delta^3}$$

This is what player 2 proposes to keep to himself. He offers $V^2_{3n} = 1 - V^2_{3n} = \frac{\delta-\delta^3}{1-\delta^3}$ to player 1.

In period 3n-1 player 1 offers $V^2_{3n-1} = \delta V^2_{3n} = \frac{\delta-\delta^2}{1-\delta^3}$ to player 2 and keeps $V^1_{3n-1} = 1 - V^2_{3n} = \frac{1-\delta^3-(\delta-\delta^2)}{1-\delta^3}$ to himself.

In period 3n-2 player 1 offers $V^2_{3n-2} = \delta^2 V^2_{3n} = \frac{\delta^2-\delta^3}{1-\delta^3}$ to player 2 and keeps to $V^1_{3n-2} = 1 - V^2_{3n} = \frac{1+\delta^2}{1-\delta^3}$ to himself.

Therefore the equilibrium strategies are:

Player 1

period 3n-2 offers $(\frac{1+\delta^2}{1-\delta^3}, \frac{\delta^2-\delta^3}{1-\delta^3})$

period 3n-1 offers $(\frac{1+\delta^2}{1-\delta^3}, \frac{\delta-\delta^2}{1-\delta^3})$

period 3n accepts any offer greater or equal to $\frac{\delta-\delta^3}{1-\delta^3}$.

Player 2

In period 3n-2 accepts any offer greater or equal to $\frac{\delta^2-\delta^3}{1-\delta^3}$

In period 3n-1 accepts any offer greater or equal to $\frac{\delta-\delta^3}{1-\delta^3}$

In period 3n offers $(\frac{\delta-\delta^3}{1-\delta^3}, \frac{1-\delta}{1-\delta^3})$

For n=1,2.....

In the first period player 1 will offer $\frac{\delta^2-\delta^3}{1-\delta^3}$ to player 2 and the offer will be accepted.

c) In the infinite period game, denote by V1 and V2 the expected payoffs to players in the beginning of each period before the uncertainty about who is making the offer is realized. Clearly the game is stationary, so these continuation payoffs are the same in the beginning of each period.

Now with probability 1/2 player 1 will be selected and he will offer $(1-\delta V2, \delta V2)$ that will be accepted. If Player 2 gets to make an offer, which happens also with probability 1/2, Player 1 will be offered $\delta V1$ and will accept.

Overall, the payoff to player 1 is $V1 = 1/2(1-\delta V2) + 1/2\delta V1$.

Similar considerations for players 2 result in $V2 = 1/2\delta V2 + 1/2(1-\delta V1)$

This gives a system of two equations with two unknowns, which is easily solved. The answer is $V1 = 1/2, V2 = 1/2$, and the offers are calculated accordingly (for instance, if player 1 gets to make an offer, he will offer $(1-\delta/2, \delta/2)$.That constructed strategies are indeed an SPE follows almost immediately from construction.

In the finite period game we can solve it backwards. Denote V_t^i, the payoffs to player i in the beginning of each period t, before the uncertainty about who is making the offer is realized.

In the last period, period 3n, if player 1 is selected he will offer $(1,0)$ and this will be accepted. If player 2 is chosen he will offer (0,1). Thus, before the uncertainty about who is going to make the offer in the last period is solved, the expected payoff of player 1 is $V_{3n}^1 = 1/2 * 0 + 1/2 * 1 = 1/2$ (Doing the same for player 2 we obtain $V_{3n}^2 = 1/2$).

In the previous period, period $3n-1$, if player 1 is selected to make an offer, the minimal amount that he can offer that will be accepted by player 2 is $(1-\delta * 1/2, \delta * 1/2)$. If player 2 is to make an offer he will offer $(\delta * 1/2, 1-\delta * 1/2)$. Thus, the expected payoff of player 1 in the period 3n-1, before the uncertainty is realized, is $V_{3n-1}^1 = (1/2) * (\delta/2) + (1/2)(1-\delta/2) = 1/2$. (we obtain the same for player 2). Since the expected payoff in the beginning of period 3n-1 is the same as in the last period. The offers in period 3n-2 will be the same as in period 3n-1 and the same for all previous periods. Therefore, at each period t the player that is selected to make an offer will offer $\delta/2$ to the other player and will keep $(1-\delta/2)$ for himself. In the first period the player who is selected to offer will offer $\delta/2$ to the other player which will accepted.

Problem Three

We'll assume $C > 0$. Suppose k firms have entered the market. Let's solve for the quantities the firms will choose in a subgame-perfect equilibrium. (Note that a firm will never produce a quantity greater than 1 because producing a quantity of 0, for instance, is always better.) If $k = 0$, we have nothing to solve for. If $k = 1$, the firm's problem is $\max\limits_{0 \leqslant q \leqslant 1} q(1-q) - C$. The first order condition is $1 - 2q = 0$, so the optimal quantity here is $1/2$. If $k \geqslant 2$, then we get into a situation like the Cournot game considered in class. The best response function for firm i can be obtained by solving $\max\limits_{0 \leqslant q_i \leqslant 1} q_i(1 - q_i - \sum\limits_{j \neq i} q_j) - C$. The best response function is $q_i^*(\sum\limits_{j \neq i} q_j) = (1 - \sum\limits_{j \neq i} q_j)/2$ if $\sum\limits_{j \neq i} q_j < 1$ and $q_i^*(\sum\limits_{j \neq i} q_j) = 0$ if $\sum\limits_{j \neq i} q_j \geqslant 1$. If you solve all the best response functions simultaneously (it's easy to do this by just solving all the first order conditions for the best response functions simultaneously), you get that each firm produces a quantity of $1/(k+1)$. The market quantity will thus be $k/(k+1)$, the price will be $1/(k+1)$, and the firms who enter will each receive a profit of $1/(k+1)^2 - C$. Before moving on, note that this expression for profits also holds for the case of $k = 1$.

Now that we know what will happen in the second stage of a subgame-perfect equilibrium, let's look at the first stage where firms must decide whether to enter. We can see from above that profits fall as the number of firms who enter the market rises. There are a couple "corner" cases to consider, which are when it is not profitable for any firm to enter and when there are nonnegative profits to be obtained even if every firm enters the market. But if we are not at one of these corner cases, then the number of firms in the market will be such that profits are positive for the entrants but would be nonpositive if another firm entered (or it could also happen that profits are exactly zero for the entrants but would be negative if another firm entered). Profits would be zero if $1/(k+1)^2 - C = 0$, or $k = 1/\sqrt{C} - 1$. They're positive if $k < 1/\sqrt{C} - 1$ and negative if $k > 1/\sqrt{C} - 1$. If $C > 1/4$, then profits to entering will always be negative (since in this case we have $1 > 1/\sqrt{C} - 1$), and no firm should enter. If $C = 1/4$, then there can be either zero firms or one firm in the market. If $C < (\frac{1}{n+1})^2$, then there will be positive profits even if every firm enters the market (since in this case we have $n < 1/\sqrt{C} - 1$), so every firm should indeed enter. If $C = (\frac{1}{n+1})^2$, then there are positive profits with $n - 1$ firms in the market but there would be 0 profits if the last firm entered, so in this case there should be either n or $n - 1$ firms in the market. For $(\frac{1}{n+1})^2 < C < 1/4$, then the number of firms in the market will be the largest integer that is less than or equal to $1/\sqrt{C} - 1$ (and if $1/\sqrt{C} - 1$ is an integer, then there could also be $1/\sqrt{C} - 2$ firms who enter the market).

So in the end, the answer depends on C. If $C > 1/4$, then the unique subgame-perfect equilibrium is for each firm to not enter and for entrants to produce $1/(k+1)$, where k is the number of entrants. If $C < (\frac{1}{n+1})^2$, then the unique subgame-perfect equilibrium is for all firms to enter and for an entrant to produce $1/(k+1)$, where k is the number of entrants. If $(\frac{1}{n+1})^2 \leqslant C \leqslant 1/4$, then subgame-perfect equilibria occur when the number of firms who enter the market is the largest integer that is less than or equal to $1/\sqrt{C} - 1$ and an entrant produces $1/(k+1)$, where k is the number of entrants (note that an actual subgame-perfect equilibrium specifies *which* firms enter and which don't). For the case where $(\frac{1}{n+1})^2 \leqslant C \leqslant 1/4$ and $1/\sqrt{C} - 1$ is an integer, then there are also subgame-perfect equilibria where $1/\sqrt{C} - 2$ firms enter the market and entrants produce $1/(k+1)$, where k is the number of entrants.

Problem Four

Here we have a finite (albeit long) perfect information game, and so the set of subgame-perfect equilibria correspond with the strategy profiles computed via backward induction. Let's find the backward induction outcome.

Let's look at the last round of the game. (Note that there are actually many different potential last rounds, one for each possible history of play from the first 999 years, but it turns out that the actions the firms choose in the last round in the backward induction outcome will not depend on what has been done up to that point). Goliath moves last and can observe what the startup has done in this round. If the startup produces the browser, then updating would add 1 to Goliath's final total sum of annual profits and not updating would add 2, so Goliath should update in this case. Similarly, if the startup produces the search engine, then Goliath should not update because not updating adds 4 to Goliath's final total sum of annual profits and updating only adds 3. Knowing that Goliath will not update, the startup should produce the browser because doing so will give the startup a profit of 2, which is higher than the profit of 1 that the startup gets from producing the search engine.

Rolling back the tree to year 999, we see that, no matter what happens this year, Goliath will earn a profit of 4 next year. So things are very similar in this year to what they are in year 1000. If the year 999 startup produces the browser, then the final sum of Goliath's annual profits will be whatever it has earned in years 1-998 plus 4 plus 1 if it updates and will be whatever it has earned in years 1-998 plus 4 plus 2 if it doesn't update; so in this case, Goliath shouldn't update. Similar reasoning will show that Goliath should also update if the year 999 startup produces the search engine.

Knowing that Goliath won’t update, the year 999 startup should produce the browser because doing so will result in a payoff of 2, whereas producing the search engine will result in a payoff of 1.

This same reasoning will hold for any period, so the unique subgame perfect Nash equilibrium is for Goliath to not update in each possible case in which it would be called upon to act and for each startup to produce the browser in each possible case in which it would be called upon to act.