

ECE 368BH: Problem Set 1: Problems and Solutions

Analysis of static games

Due: Thursday, January 31 at beginning of class

Reading: Menache and Ozdaglar, Part I

1. [Guessing 2/3 of the average]

Consider the following game for n players. Each of the players selects a number from the set $\{1, \dots, 100\}$, and a cash prize is split evenly among the players whose numbers are closest to two-thirds the average of the n numbers chosen.

- (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ'_i that always does at least as well as μ_i , and is strictly better than μ_i for some vector of strategies of the other players.)

Solution: Any choice of number in the interval $\{68, \dots, 100\}$ is weakly dominated, because replacing a choice in that interval by the choice 67 (here 67 is $(2/3)100$ rounded to the nearest integer) would not cause a winning player to lose, while, for some choices of the other players, it could cause a losing player to win. Thus, after one step of elimination, we assume all players select numbers in the interval $\{1, \dots, 67\}$. After two steps of elimination we assume players select numbers in the set $\{1, \dots, 45\}$. After three steps, $\{1, \dots, 30\}$, and so on. At each step the set of remaining strategies has the form $\{1, \dots, k\}$, and as long as $k \geq 2$ the set shrinks at the next step. So the procedure terminates when all players choose the number one.

- (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium.

Solution: A bimatrix game with $A_1 = A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives such an example. Playing 2 is weakly dominated for each player, and eliminating those choices leads to the Nash equilibrium $(1, 1)$. However, $(2, 2)$ is also a Nash equilibrium.

- (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^* = 1$.)

Solution: Consider a Nash equilibrium of mixed strategies. Let k^* be the largest integer such that there exists at least one player i choosing k^* with strictly positive probability. To complete the proof, we show that $k^* = 1$, meaning all players always choose the number one. For the sake of argument by contradiction, suppose $k^* \geq 2$. Let player i denote a player that plays k^* with positive probability. For any choice of strategies of other players, player i has a pure strategy with a strictly positive probability of winning. Since k^* must be a best response for player i , it must therefore also have a strictly positive probability of winning. It is impossible for player i to win if no other chosen numbers are equal to k^* . (Indeed, if player i were the only one to choose k^* , the second highest chosen number would be strictly closer to $2/3$ of the average than k^* .) Thus, at

least one of the other players must have a strictly positive probability of choosing k^* . But this means that player i^* could strictly increase her payoff by selecting $k^* - 1$ instead of k^* (indeed, such change would never change her from winning to losing, and in case she wins, she would win strictly more with positive probability) which contradicts the requirement that k^* be a best response for player i . This completes the argument by contradiction.

2. [A game for allocation proportional to bid]

Suppose an amount C of a divisible resource such as communication bandwidth is to be allocated to n buyers. Each buyer i submits a positive bid, b_i , and the vector of all bids is denoted by $\mathbf{b} = (b_1, \dots, b_n)$. In return, the buyer pays the amount b_i and receives an amount $x_i = \frac{Cb_i}{B}$ of the resource, where $B = b_1 + \dots + b_n$. The payoff for buyer i is $\pi_i(\mathbf{b}) = U_i(x_i) - b_i$, where U_i is a concave, continuously differentiable function on $(0, \infty)$ with $\lim_{x \rightarrow 0} U'_i(x) = +\infty$ and $U'_i(x) > 0$ for $0 < x_i \leq C$.

- (a) Characterize the value(s) of $\mathbf{x} = (x_1, \dots, x_n)$ such that the social welfare, $\sum_i U_i(x_i)$, is maximized, subject to the constraints $x_i \geq 0$ for $1 \leq i \leq n$ and $\sum_i x_i \leq C$. In addition, show that if the functions U_i are strictly concave then the allocation maximizing the social welfare is unique.

Solution: By the KKT conditions, $\frac{\partial L(\lambda, \mathbf{x})}{\partial x_i} = 0$, where L is the Lagrangian function defined by $L(\lambda, \mathbf{x}) = \sum_{i=1}^n U_i(x_i) + \lambda(C - \sum_{i=1}^n x_i)$, and where $\lambda \geq 0$ is the Lagrange multiplier. This yields $U'_i(x_i) = \lambda$ for $1 \leq i \leq n$. Since the functions u_i are strictly increasing, it is clear that the sum constraint is binding, i.e. $\sum_{i=1}^n x_i = C$, and this equation can be used to determine $\lambda > 0$. An interpretation is that the marginal valuation for additional capacity is the same for all buyers, and is equal to λ . If the functions U_i are strictly convex, then the social welfare is a strictly concave function of \mathbf{x} over a compact, convex set, and hence achieves its maximum at a unique vector \mathbf{x} . Alternatively, more explicitly, we could note that the functions U'_i are strictly increasing, so the x_i 's are uniquely determined by λ , and the sum $\sum_i x_i$ becomes a strictly decreasing function of λ . Thus λ is uniquely determined and hence \mathbf{x} is uniquely determined.

- (b) Find an explicit expression for the allocation $\mathbf{x} = (x_1, \dots, x_n)$ that maximizes the social welfare, in case $U_i(x_i) = w_i \ln(x_i)$, where for each i , w_i is a given positive weight.

Solution: The equations in part (a) become $\frac{w_i}{x_i} = \lambda$, or $x_i = \frac{w_i}{\lambda}$. To satisfy the capacity constraint requires $\lambda = \left(\sum_j w_j\right) / C$, yielding $x_i = \frac{Cw_i}{\sum_j w_j}$. That is, the social welfare is maximized by the allocation that is proportional to the given weights in the valuation functions.

- (c) Show that there exists a unique Nash equilibrium, and characterize it. (Hint: A necessary condition for a Nash equilibrium is $\frac{\partial \pi_i(\mathbf{b})}{\partial b_i} = 0$ for $1 \leq i \leq n$. Show that this condition is equivalent to KKT conditions for the solution of maximizing a strictly concave function, as in part (a). Define new valuation functions \tilde{U}_i by $\tilde{U}'_i(x_i) = U'(x_i)(1 - \frac{x_i}{C})$ for $0 < x_i \leq C$.)

Solution: Let $B_{-i} = B - b_i$. Observe that $x_i = \frac{Cb_i}{B} = C \left(1 - \frac{B_{-i}}{B}\right)$. Therefore, $\frac{\partial x_i}{\partial b_i} = \frac{CB_{-i}}{B^2} = \frac{C(1-x_i)}{B}$. Therefore, by the chain rule of calculus, $\frac{\partial \pi_i(\mathbf{b})}{\partial b_i} = U'(x_i) \frac{C(1-x_i)}{B} - 1$. Therefore, the equation $\frac{\partial \pi_i(\mathbf{b})}{\partial b_i} = 0$ is equivalent to $\tilde{U}'_i(x_i) = \frac{B}{C}$ for each i . Comparing this to the solution to part (a), it means that \mathbf{b} is a Nash equilibrium if and only if \mathbf{x} is a solution to the social welfare maximization problem for the alternative valuation

functions \tilde{U}_i and, and the sum of the bids is λC . Since the alternative valuation functions are strictly concave, the Nash equilibrium is unique.

3. [Nash saddle point]

Consider a two person zero sum game represented by a finite $m \times n$ matrix A . The first player selects a probability vector p and the second selects a probability vector q . The first player wishes to minimize pAq^T and the second player wishes to maximize pAq^T . Let V denote the value of the game, so $V = \min_p \max_q pAq^T = \max_q \min_p pAq^T$.

- (a) Consider the following statement S : If \bar{p} and \bar{q} are probability distributions (of the appropriate dimensions) such that $\bar{p}A\bar{q}^T = V$, then (\bar{p}, \bar{q}) is a Nash equilibrium point (in mixed strategies). Either prove that statement S is true, or give a counter example.

Solution: The statement is false. For example, let $m = n = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Let $\bar{p} = (1, 0)$ and $\bar{q} = (0, 1)$. Then $\bar{p}A\bar{q}^T = 0 = V$. But if the second player were to switch from using \bar{q} to using $\hat{q} = (1, 0)$, the payoff would increase from zero to $\bar{p}A\hat{q}^T = 1$. Therefore, (\bar{p}, \bar{q}) is not a Nash equilibrium.

- (b) Consider the following statement T : A Nash equilibrium consisting of a pair of pure strategies exists if and only if $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Either prove that statement T is true, or give a counter example.

Solution: The statement is true. It is the same as a statement proved in class, but with p replaced by i and q replaced by j . The proof is repeated here.

(if part) Suppose (i^*, j^*) is a Nash equilibrium. By definition, this means $\min_i A_{i,j^*} = A_{i^*,j^*} = \max_j A_{i^*,j}$. It is easy to see that $\min_i \max_j A_{i,j}$ and $\max_j \min_i A_{i,j}$ are both contained in the length zero interval $[\min_i A_{i,j^*}, \max_j A_{i^*,j}]$, so they must be equal.

(only if part) Suppose $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Let i^* be a minmax optimal action for the first player and j^* be a maxmin optimal action for the second player. Then $\max_j A_{i^*,j} = \min_i \max_j A_{i,j} = \max_j \min_i A_{i,j} = \min_i A_{i,j^*}$. Therefore, $A_{i^*,j^*} \leq \max_j A_{i^*,j} = \min_i A_{i,j^*}$ and $A_{i^*,j^*} \geq \min_i A_{i,j^*} = \max_j A_{i^*,j}$, which by definition means (i^*, j^*) is a Nash equilibrium.

4. [Equilibria for a two player game]

Consider the two player game shown:

| | | | |
|---|--|-----|-----|
| | | L | R |
| T | | 6,6 | 2,8 |
| B | | 8,2 | 0,0 |

The first player selects T or B and the second player selects L or R.

- (a) Identify all pure strategy Nash equilibria (if any) and the payoff vector for each one.

Solution: There are two pure Nash equilibria: (T,R) with payoff vector (2,8), and (B,L) with payoff vector (8,2).

- (b) Identify all non-degenerate mixed strategy Nash equilibria and the payoff vector for each one.

Solution: It is easy to see there are no Nash equilibria in which exactly one of the players has a non degenerate mixed strategy. So we seek a pair of mixed strategies $(p, 1-p), (q, 1-q)$ so that either action is a best response for either player. That requires $6p + 2(1-p) = 8p$ and $6q + 2(1-q) = 8q$, or $(p, q) = (0.5, 0.5)$ which has payoff vector (4,4).

- (c) Identify the polytope of all correlated equilibria by giving the set of inequalities they satisfy, and find the correlated equilibria with largest sum of payoffs.

Solution: Each correlated equilibrium corresponds to a probability distribution (a, b, c, d) over the possible pairs of actions, $\{(T, L), (T, R), (B, L), (B, R)\}$. The conditions needed to be a correlated equilibrium, in addition to (a, b, c, d) being a probability distribution, are

$$\begin{aligned} (T \rightarrow B) \quad a(6 - 8) + b(2 - 0) &\geq 0 \\ (B \rightarrow T) \quad c(8 - 6) + d(0 - 2) &\geq 0 \\ (L \rightarrow R) \quad a(6 - 8) + c(2 - 0) &\geq 0 \\ (R \rightarrow L) \quad b(8 - 6) + d(0 - 2) &\geq 0, \end{aligned}$$

where, for example, the equation for $(T \rightarrow B)$ insures that the first player would not receive a higher expected payoff by using B whenever told to play T . The equations reduce to (a, b, c, d) is a probability vector such that $a \leq b, a \leq c, d \leq b$, and $d \leq c$.

The payoff vector for a given choice of (a, b, c, d) is $(6a + 2b + 8c, 6a + 8b + 2c)$. The sum of payoffs is $12a + 10(b + c)$. To maximize the sum of payoffs, we clearly should let $d = 0$. Then $b + c = 1 - a$ and the sum of payoffs is $10 + 2a$. The largest a can be is $\frac{1}{3}$, so the correlated equilibrium with the maximum sum of payoffs is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, and the payoff vector is $(5.333..., 5.3333...)$.