Coalitional Game Theory

Game Theory

MohammadAmin Fazli

TOC

- Coalitional Games
- Fair Division and Shapley Value
- Stable Division and the Core Concept
- ϵ-Core, Least core & Nucleolus

- Reading:
  - Chapter 12 of the MAS book
  - Ferguson's Notes on Coalitional Game Theory

Coalitional Games

- Our focus is on what groups of agents, rather than individual agents, can achieve.
- Given a set of agents, a coalitional game defines how well each group (or *coalition*) of agents can do for itself.
- We are not concerned with:
  - How the agents make individual choices within a coalition;
  - How they coordinate;
  - ...instead, we take the payoffs to a coalition as given.
- Transferable utility assumption:
  - payoffs may be redistributed among a coalition’s members.
  - satisfied whenever payoffs are dispensed in a universal *currency*.
  - each coalition can be assigned a single value as its payoff.

Coalitional Games

- A coalitional game with transferable utility is a pair (*N, v*), where
  - *N* is a finite set of players, indexed by *i*; and
  - $v: 2^N \rightarrow R$ associates with each coalition $S \subseteq N$ a real-valued payoff *v*(*S*) that the coalition's members can distribute among themselves. We assume that *v*(*∅*) = 0.
- Questions:
  - Which coalition will form?
  - How should that coalition divide its payoff among its members?
  - The answer to (1) is often "the grand coalition" (all agents in *N*) though this can depend on having made the right choice about (2)

Example

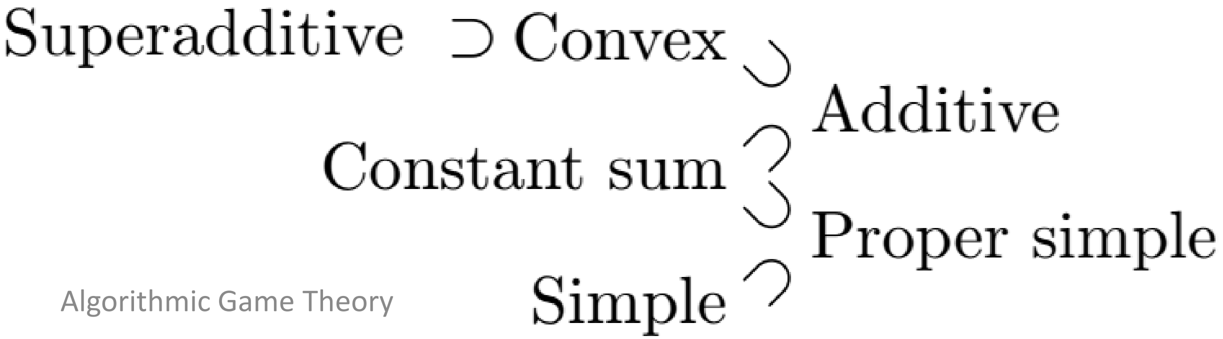
- A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively.
- They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties.
- A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

Superadditive Games

- A game $G = (N, v)$ is superadditive if for all $S; T \subset N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$.
- Superadditivity is justified when coalitions can always work without interfering with one another
  - The value of two coalitions will be no less than the sum of their individual values.
  - Implies that the grand coalition has the highest payoff
  - How should the coalition divide its payoff?
    - in order to be fair
    - in order to be stable

Other Types of Coalitional Games

- **Additive game:** A game G = (N,v) is additive (or inessential) if for all $S,T \subset N$, if $S \cap T = \emptyset$, then $v(S \cup T) = v(S) + v(T)$.
- **Constant-sum game:** A game G = (N,v) is constant sum if for all $S \subset N$, $v(S) + v(N \setminus S) = v(N)$.
- **Convex game:** *A game* G = (N,v) *is* convex *if for all* $S,T \subset N$, $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.
- **Simple game:** *A game* G = (N,v) *is* simple *if for all* $S \subset N$, $v(S) \in \{0,1\}$.



Fair Division

- Perhaps the most straightforward answer to the question of how payoffs should be divided is that the division should be fair.
- Lioyd Shapley's idea: members should receive payments or shares proportional to their marginal contributions.
- But this is not easy:
  - Suppose $v(N) = 1$ but $v(S) = 0$ if $N \neq S$.
  - Then $v(N) - v(N \setminus \{i\}) = 1$ for every i: everybody's marginal contribution is 1, everybody is essential to generating any value.
  - We can not pay everyone their marginal contribution

Fair Division

- Feasible Payoffs: Given a coalitional game (N,v), the feasible payoff set is defined as $\{x \in R^N | \sum_{i \in N} x_i \leq v(N)\}$
  - Budget Balanced
- Pre-Imputation: Given a coalitional game (N,v), the pre-imputation set, denoted P, is defined as $\{x \in R^N | \sum_{i \in N} x_i = v(N)\}$
  - Efficient
- Imputation: Given a coalitional game (N,v), the imputation set, I, is defined as $\{x \in P \mid \forall i \in N, x_i \geq v(i)\}$.
  - Individually rational

Fair Division

- Symmetry:
  - *i* and *j* are interchangeable relative to *v* if they always contribute the same amount to every coalition of the other agents, i.e. for all *S* that contains neither *i* nor *j*, $v(S \cup \{i\}) = v(S \cup \{j\})$
  - For any v, if i and j are interchangeable then $\psi_i(N,v) = \psi_j(N,v)$
- Dummy Players:
  - *i* is a dummy player if the amount that *i* contributes to any coalition is 0, i.e. for all *S*: $v(S \cup \{i\}) = v(S)$.
  - For any v, if i is a dummy player then $\psi_i(N,v) = v(\{i\})=0$.
- Additivity:
  - For any two v_1 and v_2, we have for any player i that $\psi_i(N,v_1+v_2) = \psi_i(N,v_1)+\psi_i(N,v_2)$, where the game (N,v_1+v_2) is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for every coalition S.

The Shapley Value

- Theorem: Given a coalitional game (N,v), there is a unique pre-imputation *φ(N,v)* that satisfies the Symmetry, Dummy player, Additivity axioms.
  - Proof: see the blackboard.
- Given a coalitional game (*N, v*), the Shapley Value divides payoffs among players according to:

$$\phi_i(N,v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)!\Big[v(S \cup \{i\}) - v(S)\Big]$$

  for each player i.

Stable Division

- The Shapley value defined a fair way of dividing the grand coalition's payment among its members. However, this analysis ignored questions of stability.
- Would the agents be willing to form the grand coalition given the way it will divide payments, or would some of them prefer to form smaller coalitions? Unfortunately, sometimes smaller coalitions can be more attractive for subsets of the agents, even if they lead to lower value overall.

Ex. Stable Division

- Our parliament example:
  - A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.
- Shapley values: (50, 16.67, 16.67, 16.67) [see the blackboard)
- While *A* can't obtain more than 50 on its own, *A* and *B* have incentive to defect and divide the $100 million between them (e.g., (75,25))

The Core

- Under what payment divisions would the agents want to form the grand coalition?
- They would want to do so if and only if the payment profile is drawn from a set called the core.
- Core: A payoff vector *x* is in the core of a coalitional game (*N, v*) iff

$$\forall_{S \subseteq N}, \sum_{i \in S} x_i \geq v(S)$$

- Analogous to Nash equilibrium, except that it allows deviations by groups of agents.

The Core

- Core is not always non-empty:
  - Consider again the voting game.
  - The set of minimal coalitions that meet the required 51 votes is *{A, B}*, *{A, C}*, *{A, D}* and *{B, C, D}*.
  - If the sum of the payoffs to parties *B*, *C*, and *D* is less than $100 million, then this set of agents has incentive to deviate.
  - If *B*, *C*, and *D* get the entire payoff of $100 million, then *A* will receive $0 and will have incentive to form a coalition with whichever of *B*, *C*, and *D* obtained the smallest payoff.
  - Thus, the core is empty for this game.

The Core

- The core is not unique:
  - Consider changing the example so that an 80% majority is required
  - The minimal winning coalitions are now *{A, B, C}* and *{A, B, D}*.
  - Any complete distribution of the $100 million among *A* and *B* now belongs to the core
    - all winning coalitions need the support of these two parties.

The Core

- Balanced weights: A set of nonnegative weights (over 2^N), λ, is balanced if

$$\forall_{i \in N}, \sum_{S: i \in S} \lambda(S) = 1$$

- Theorem (Bondereva–Shapley): A coalitional game (N,v) has a nonempty core if and only if for all balanced sets of weights λ,

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S) \,.$$

  - Proof: See the blackboard.

Some Facts

- Theorem: Every constant-sum game that is not additive has an empty core.
- Theorem: In a simple game the core is empty iff there is no veto player. If there are veto players, the core consists of all payoff vectors in which the nonveto players get zero.
  - We say that a player i is a veto player if $v(N \setminus \{i\}) = 0$.
- Theorem: Every convex game has a nonempty core.
- Theorem: In every convex game, the Shapley value is in the core.

ϵ-Core & Least Core

- $\boldsymbol{\epsilon}$-Core: A payoff vector x is in the ϱ-core of a coalitional game (N,v) if and only if

$$\forall_{S \subset N}, \sum_{i \in S} x_i \geq v(S) - \epsilon$$

- Least Core: A payoff vector x is in the least core of a coalitional game (N,v) if and only if x is the solution to the following linear program.

$$\begin{aligned} &\text{minimize} \quad \epsilon \\ &\text{subject to} \quad \sum_{i \in S} x_i \geq v(S) - \epsilon \qquad \forall S \subset N \end{aligned}$$

Nucleolus

- Nucleolus: A payoff vector x is in the nucleolus of a coalitional game (N,v) if it is he solution to the series of optimization programs $O_1, O_2, \ldots, O_{|N|}$, where these programs are defined as follows, where ϵ_{i-1} is the optimal objective value to program O_{i-1} and S_{i-1} is the set of coalitions for which in the optimal solution to O_{i-1}, the constraints are realized as equalities.

$$(O_1) \quad \begin{cases} \text{minimize} & \epsilon \\ \text{subject to} & \sum_{i \in S} x_i \geq v(S) - \epsilon \quad \forall S \subset N \end{cases}$$

$$(O_i) \quad \begin{cases} \text{minimize} & \epsilon \\ \text{subject to} & \sum_{i \in S} x_i = v(S) - \epsilon_0 \qquad \forall S \in \mathcal{S}_1 \\ & \vdots \\ & \sum_{i \in S} x_i = v(S) - \epsilon_{i-1} \quad \forall S \in \mathcal{S}_{i-1} \setminus \mathcal{S}_{i-2} \\ & \sum_{i \in S} x_i \geq v(S) - \epsilon \qquad \forall S \in 2^N \setminus \mathcal{S}_{i-1} \end{cases}$$

Nucleolus

- Theorem: For any coalitional game (N,v), the nucleolus of the game always exists and is unique.
  - Proof Sketch: See the blackboard.

Ex. Weighted Majority Games

- Weighted majority game: A weighted majority game is defined majority game by weights $w(i)$ assigned to each player $i \in N$. Let W be $\sum_{i \in N} w(i)$. The value of a coalition is 1 if $\sum_{i \in S} w(i) \geq \frac{W}{2}$ and 0 otherwise.
- Theorem: Computing the Shapley value in weighted majority games is #P-complete.

Ex. Weighted Graph Games

- Weighted graph game: Let(V,W) denote an undirected weighted graph, where V is the set of vertices and $W \in R^{V \times V}$ is the set of edge weights; denote the weight of the edge between the vertices i and j as w(i,j). This graph defines a weighted graph game (WGG), where the coalitional game is constructed game as follows:
  - N = V
  - $v(S) = \sum_{i,j \in S} w(i,j)$
- Proposition: If all the weights are nonnegative then the game is convex.
- Proposition: If all the weights are nonnegative then membership of a payoff vector in the core can be tested in polynomial time.

Ex. Weighted Graph Games

- Theorem: The Shapley value of the coalitional game (N,v) induced by a weighted graph game (V,W) is
$$\phi_i(N, v) = \frac{1}{2} \sum_{j \neq i} w(i, j)$$
    - Proof: See the blackboard.
- Theorem: The core of a weighted graph game is nonempty if and only if there is no negative cut in the weighted graph.
    - Proof: See the blackboard.
- Theorem: Testing the non-emptiness of the core of a general WGG is NP-complete.