Econ 383 (Applied Game Theory) Midterm Exam

Instructor: Songzi Du

Monday October 28, 2013, 2:30 – 4:20 PM

Write your name, SFU ID number, and tutorial section number **both on the exam booklets and on the questionnaire**. But note that only the exam booklets are graded.

- Name:

- SFU ID number:

- Tutorial section number:

General instructions

1. This is a closed-book exam: no books, notes, computer, cellphone, internet, or other aids. While it is not necessary, a simple, non-graphing calculator.

2. If you use decimals in calculation, keep two decimal places.

3. Write clearly. Illegible answers will receive no credit.

4. Show your work! Partial credits are given. Answers without proper explanation/calculation will be penalized.

5. A request for regrade can be accepted **only if** the exam is written with a pen.

6. Stay in your seat if less than 20 minutes remain in the exam. You may leave early if you finish 20 minutes before the exam is over.

7. Hand in both the exam booklets and this questionnaire.

1. (10 points) Four people see a hundred-dollar bill on the street. Each person i $(1 \leq i \leq 4)$ has two potential actions (i.e., strategies): claim the money as his/her own, or not make such claim. If multiple people claim the money as their own, police will be called and nobody gets anything, i.e., each person gets a payoff of 0. On the other hand, if only one person claims the money as his/her own, that person gets a payoff of 100. Finally, if nobody claim the money, each person also gets a payoff of 0.

(i) Is this a symmetric game? Explain.

Yes, because the *payoff* to a *strategy* only depends on the other strategies, and not on who uses them. In other words, analyzing one player's behavior is exactly the same as analyzing every other player's behavior.

(ii) Is there a dominant strategy in this game? Explain.

Yes, C (i.e. "claim") is a dominant strategy. If a player chooses N (i.e. "not claim"), then her payoff is always zero, regardless of the other players' strategies. When all other players choose N, the player choosing C gets positive payoff; and when played against other strategies, C does equally well as N. Therefore, C weakly dominates N for every player.

Remark 1. Notice the difference between a *strategy* and a *strategy profile*. A **strategy** is a plan of action for <u>one particular player</u>; whereas a **strategy profile** is a collection of strategies of <u>all players</u>. Hence, it is incorrect to answer, for example, (C, C, C, C), because this is a strategy profile, not a strategy.

(iii) What are the (pure-strategy) Nash equilibria in this game, and how many of them exist? Explain.

There are four classes of pure NEs in this game. Basically a NE requires that *at least one* player chooses C.

- First and foremost, everyone playing their dominant strategy is a NE: (C, C, C, C).
- There are 4 NEs in which exactly three players choose C.
- There are 6 NEs in which exactly two players choose C.
- There are 4 NEs in which exactly one player choose C.

Thus, there are 15 pure NEs in total.

An alternative way to think about the problem is the following. As long as a strategy profile has at least one player choosing C, it is a NE. Since there are four players and each has two strategies, there are a total of $2^4 = 16$ possible strategy profiles. There is only one among the 16 strategy profiles where nobody plays C, i.e. (N, N, N, N). Therefore, there are $16 - 1 = 15$ NEs.

2. (10 points) Consider an all-pay auction of a single object and with n bidders. Suppose that each bidder i has an independent private value v_i (which is his private information) for the object, where v_i is distributed according to the probability distribution function $F(x) = x^2$: that is, for any $0 \le x \le 1$, the probability that v_i is less than or equal to x is $F(x) = x^2$.

Derive the differential equation that the Nash equilibrium bidding strategy $s(v_i)$ must satisfy. You do not have to solve the differential equation.

Consider bidder i with value v_i (obviously $v_i \in [0,1]$). Because of *identical distribution* of all bidder values, for any $j \ne i$, the probability that i's value is greater than any particular j's value is

$$\Pr(v_j \le v_i) = v_i^2.$$

Moreover, since bidder values are *independently distributed*, the probability that i's value is the largest among all bidders is

$$\Pr\left(\max_{j \ne i} v_j \le v_i\right) = (v_i^2)^{n-1}.$$

In a symmetric NE where each bidder follows a strictly increasing bidding strategy $s(v_i)$, bidder i's problem is

$$\max_{v_i' \in [0,1]} (v_i')^{2(n-1)} v_i - s(v_i'), \tag{1}$$

where v_i' is the "type" that bidder i can potentially imitate when submitting her bid.

The FOC associated with (1) is

$$2(n-1)(v_i')^{2n-3} v_i - s'(v_i')\Big|_{v_i' = v_i} = 0. \tag{2}$$

This is the differential equation from which we can solve for the equilibrium strategy $s(v_i)$.

[The following is not required by the question, but just to show how $s(v_i)$ looks like.] From (2), we get

$$s'(v_i) = 2(n-1)(v_i)^{2n-2}.$$

Integrating on both sides with respect to v_i, we get[1]

$$s(v_i) = 2(n-1)\frac{(v_i)^{2n-1}}{2n-1}.$$

Remark 2. Recall that when bidder values are *uniformly distributed*, i.e. $\Pr(v_j \leq v_i) = v_i$, the equilibrium bidding strategy is

$$s_U(v_i) = \frac{n-1}{n}(v_i)^n.$$

Figure 1 compares the two bidding strategies for the case of two bidders ($n = 2$). Notice that it is much more likely to draw a high value under quadratic distribution ($F(x) = x^2$) than under a uniform distribution ($G(x) = x$). Hence, we see that for values below some threshold—in this case, $\bar{v}$, at the intersection of the two curves—bidder i shades her bid more under quadratic distribution than under uniform distribution. The reason being that she is not confident enough that her bid will be the winning bid, and so shading helps her reduce the cost of participating in the auction. On the other hand, if her value is above the threshold $\bar{v}$, she shades less under quadratic distribution than under uniform distribution. The reason being that her value is in a region where the probability of winning is high, but the competition is also high, and thus bidder i wants to bid higher (relative to the case under uniform distribution) to secure the win.

3. (10 points) (i) Explain why bidding one's private value (truthful bidding) is a dominant strategy in the second price auction of a single object. To get full credit, your explanation must cover all cases (what happens when one bids above one's value, and what happens when one bids below one's value).

Consider a bidder i with private value v_i. Let $\bar{b}$ be the highest competing bid from the other bidders.

Under truthful bidding, $b_i = v_i$, i's payoff is $v_i - \bar{b} > 0$ if he wins ($b_i > \bar{b}$), and 0 otherwise ($b_i \leq \bar{b}$).

If i overbids, $b_i' > v_i$, his payoff is the same as that under truthful bidding except when $\bar{b} \in [v_i, b_i')$, in which case his payoff is $v_i - \bar{b} < 0$. Hence, truthful bidding weakly dominates over-bidding.

If i underbids, $b_i'' < v_i$, his payoff is the same as that under truthful bidding except when

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[1]Recall that $\int x^n dx = \frac{x^{n+1}}{n+1}$, and $\int_0^x s'(t)dt = s(x)$.
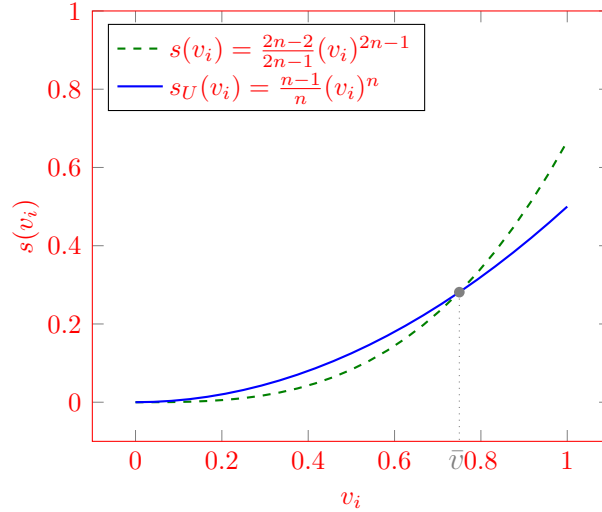
Figure 1: Comparison of bidding strategies under different value distributions $(n = 2)$

$\bar{b} \in (b_i'', v_i]$, in which case his payoff is 0. If i had bid truthfully, his payoff would have been positive. Therefore, truthful bidding weakly dominates under-bidding.

Since we have exhausted all possible strategies, we conclude that truthful bidding is a weakly dominant strategy in the second price auction of a single object.

(ii) Give an example (i.e., values and bids) that shows that bidding one's private value is not a dominant strategy in the third price auction of a single object (that is, the highest bidder wins the object and pays the third highest bid; everyone else does not pay).

Consider three bidders. Suppose bidders 1 and 2 submit bids b_1, b_2, respectively, and bidder 3's value is such that $b_1 < v_3 < b_2$. If bidder 3 bids truthfully, her payoff is 0, because bidder 2 will win the object. However, if bidder 3 overbids, so that $b_3 > b_2 > v_3$, then she would win the auction and get a positive payoff $v_3 - b_1 > 0$. This shows that truthful bidding is not a dominant strategy: there exists a situation where playing truthful bidding is not a best response.

Remark 3. Note that simply because there exist other alternative best responses does not mean that a strategy is not weakly dominant. For instance, suppose $b_1 < b_2 < v_3$, so that bidder 3 bidding any value strictly greater than b_2 is a best response. But this does NOT mean that truthful bidding is not a (weakly) dominant strategy.[2]

4. (10 points) Consider the second price auction of a single object and with a reserve

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[2]Multiple best responses, however, does mean that there exists no *strictly* dominant strategy. But this is not what the question is asking for.

price R. There are two bidders who have independent private values v_i, which are either 1, 2, or 3. For each bidder i, the probabilities of $v_i = 1$, $v_i = 2$ and $v_i = 3$ are each $1/3$. If there is a tie at a bid of $x \geq R$ for the highest bid, the winner is selected at random from among the highest bidders and the price is x.

(i) Calculate seller's revenue when $R = 3$ and everyone plays the dominant strategy.

First, observe that there are 9 possible combinations of bidder values: there are two bidders, and each has three possible values (so $3^2 = 9$). Each of these combinations occur with probability $1/3$. When $R = 3$, the object is sold only if the highest bid is at least 3. Since bidders follow their dominant strategy, i.e. to bid truthfully, we know that seller's revenue is positive only if *at least one bidder has value $v_i = 3$*. There are five combinations of bidder values that satisfy this requirement: $\{(1,3), (2,3), (3,3), (3,2), (3,1)\}$. In each of these five cases, the selling price is 3. Therefore, the expected revenue for the seller is

$$\frac{5}{9}(3) = \frac{15}{9} = \frac{5}{3}.$$

(ii) Calculate seller's revenue when $R = 2$ and everyone plays the dominant strategy.

When $R = 2$, the object is sold only if *at least one bidder has value $v_i \geq 2$*. There are eight combinations of bidder values that satisfy this requirement (basically every combination except $(1,1)$). The selling price is 3 when bidder values are $(3,3)$, and 2 otherwise. Hence, the expected revenue for the seller is

$$\frac{1}{9}(3) + \frac{7}{9}(2) = \frac{17}{9}.$$

**For Question 5 and 6 assume the prices can only be integers.**

5. (10 points) Suppose we have four buyers (x, y, z and w) and four sellers (a, b, c and d). The valuations of the buyers for the goods of the sellers are as follows.

Buyer	value for a's good	value for b's good	value for c's good	value for d's good
x	5	5	0	0
y	3	3	0	0
z	1	1	0	0
w	4	4	0	0

Run the bipartite-graph auction to find the market-clearing prices. For each step of the

auction, write down the current prices and the preferred-seller graph. Find a matching that is implied by the market-clearing prices. What is this auction?

The auction will run for four rounds, as illustrated below. In Round 1, a constricted set is $\{x, y, z, w\}$. In Rounds 2 and 3, the constricted set is $\{x, y, w\}$. Observe that this is a third price auction for two identical items: buyers with the two highest values win the objects, and they pay the third highest price.

Round 1

Round 2

Round 3

Round 4
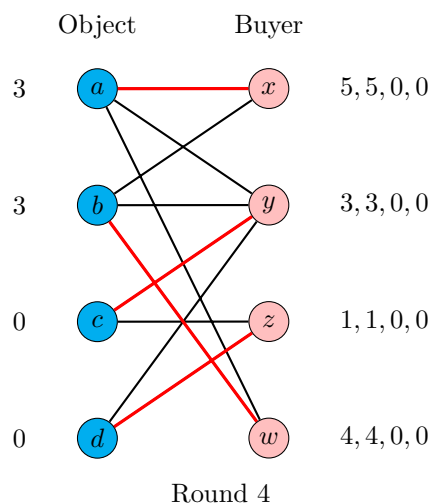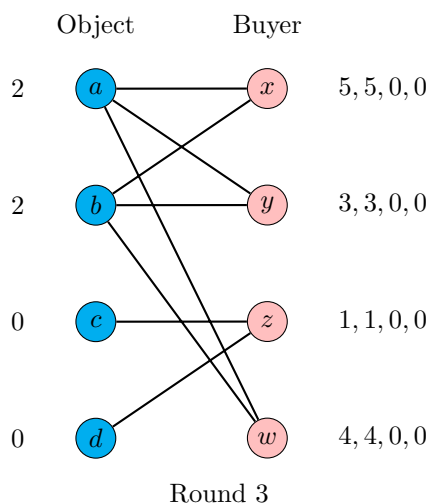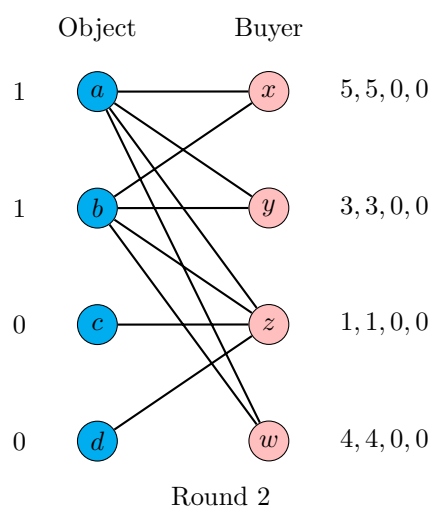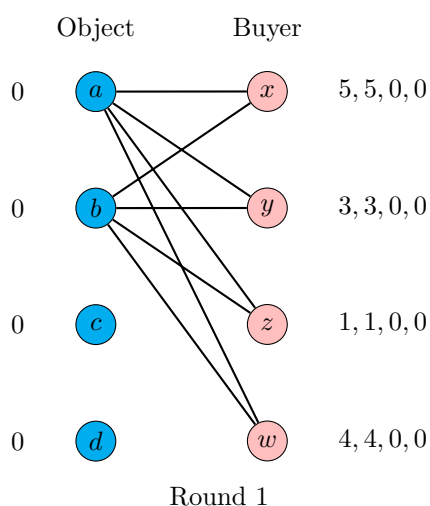
6. (10 points) Suppose we have three buyers (x, y and z) and three sellers (a, b and c). The valuations of the buyers for the goods of the sellers are as follows.

Buyer	value for a's good	value for b's good	value for c's good
x	5	3	3
y	4	3	0
z	4	2	1

(i) There are six possible matchings from buyers to sellers. Calculate the total value (i.e., social welfare) for each of them.

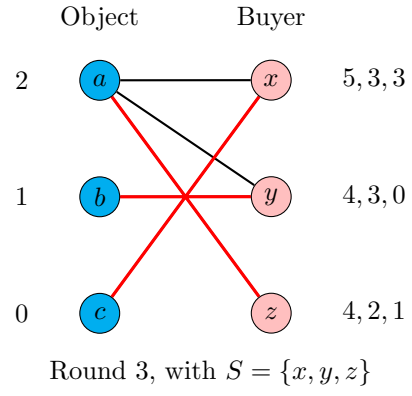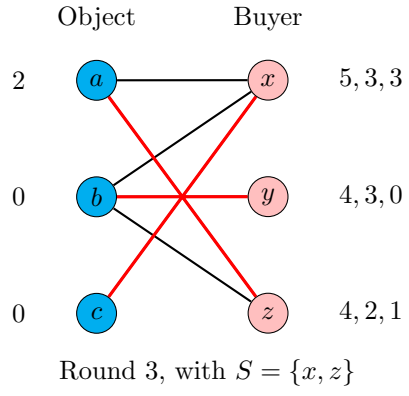| Matching	Total value
$x - a,\ y - b,\ z - c$	9
$x - a,\ y - c,\ z - b$	7
$x - b,\ y - a,\ z - c$	8
$x - b,\ y - c,\ z - a$	7
$x - c,\ y - a,\ z - b$	9
$x - c,\ y - b,\ z - a$	10

(ii) Run the bipartite-graph auction to find the market-clearing prices. For each step of the auction, write down the current prices and the preferred-seller graph. Find the matching that is implied by the market-clearing prices and check whether it maximizes the total value.

The first two rounds of the auction are illustrated as follows. In Round 1, set $\{x, y, z\}$ is constricted.

Round 1   Round 2
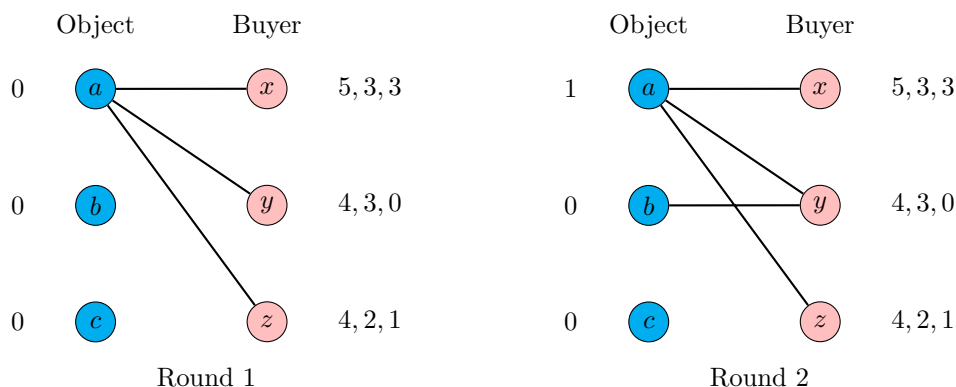
In Round 2, there are two constricted sets: $\{x, z\}$ and $\{x, y, z\}$, with corresponding neighbor sets $N(\{x, z\}) = \{a\}$ and $N(\{x, y, z\}) = \{a, b\}$. It turns out that picking different constricted sets in this case will result in different market-clearing prices:

Object		Buyer	
2	a	x	$5, 3, 3$
0	b	y	$4, 3, 0$
0	c	z	$4, 2, 1$

Round 3, with $S = \{x, z\}$

Object		Buyer	
2	a	x	$5, 3, 3$
1	b	y	$4, 3, 0$
0	c	z	$4, 2, 1$

Round 3, with $S = \{x, y, z\}$

Both set of prices nevertheless lead to the same matching, which is the one maximizing the total value.