**University of Illinois** **Spring 2013**

# ECE 586BH: Problem Set 4: Problems and Solutions
# Extensive form games: Normal form representation, behavioral strategies, sequential rationality

**Due:** Thursday, March 28 at beginning of class

**Reading:** Fudendberg and Triole, Sections 3.1-3.4, & 8.3

Section III of Osborne and Rubenstein, available online, covers this material too.

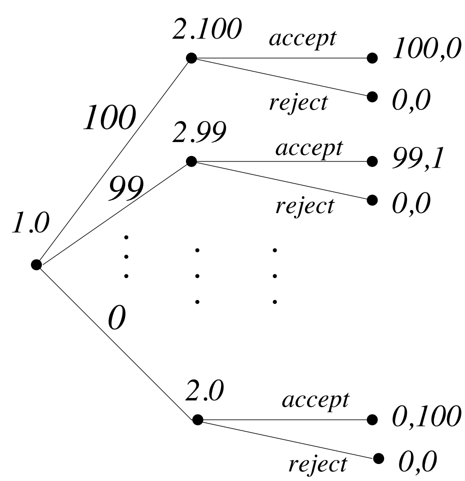
Lectures also relied on Myerson's book.

1. **[Ultimatum game]**

   Consider the following two stage game, about how two players split a pile of 100 gold coins. The action (strategy) set of player 1 is given by $S_1 = \{0, \ldots, 100\}$, with choice $i$ meaning that player 1 proposes to keep $i$ of the gold coins. Player 2 learns the choice of player one, and then takes one of two actions in response: A (accept) or R (reject). If player two plays accept, the payoff vector is $(i, 100 - i)$. If player two plays reject, the payoff vector is $(0, 0)$.

   (a) Describe the extensive form version of the game using a tree labeled as in class.

   **Solution:**




   (b) Describe the normal form of the game. It suffices to specify the strategy spaces and payoff functions. (Hint: Player 2 has $2^{101}$ pure strategies.)

   **Solution:** The strategy spaces are $S_1 = \{0, 1, \ldots, 100\}$ and $S_2 = \{0, 1\}^{100}$. The interpretation of $d = (d(i) : 0 \leq i \leq 100) \in S_2$ is $d(i) = 1$ if player 2 accepts when player 1 plays $i$, and $d(i) = 0$ if player 2 rejects when player 1 plays $i$. The payoff functions are given by $u_1(i, d) = i \cdot d(i)$ and $u_2(i, d) = (100 - i)d(i)$.

   (c) Identify a Nash equilibria of the normal form game with payoff vector $(50, 50)$.

   **Solution:** The pair $(50, d_{50})$ is such an NE, where $d_{50}(i) = 1$ for $i \leq 50$ and $d_{50}(i) = 0$ for $i > 50$. That is, when player 2 uses the strategy $d_{50}$, she accepts if player 1 plays $i \leq 50$ and she rejects otherwise.

   (d) Identify the subgame perfect equilibria of the extensive form game. (Hint: There are two of them.)

**Solution:** In the subgame beginning at node $2.i$, if $i \leq 99$, player 2 is strictly better off to accept (payoff $100 - i$) than to reject, so in a subgame perfect equilibrium, player 2 must accept whenever $i \leq 99$. The only degree of freedom left for player 2 is the value of $d(100)$, which could either be zero or one.

Let $d_j$ be the strategy for player 2 such that she accepts if and only if $i$, the action of player 1, satisfies $i \leq j$. Then $(99, d_{99})$ and $(100, d_{100})$ are subgame perfect equilibria.

(e) Identify the trembling hand perfect pure strategy equilibria of the normal form version of the game.

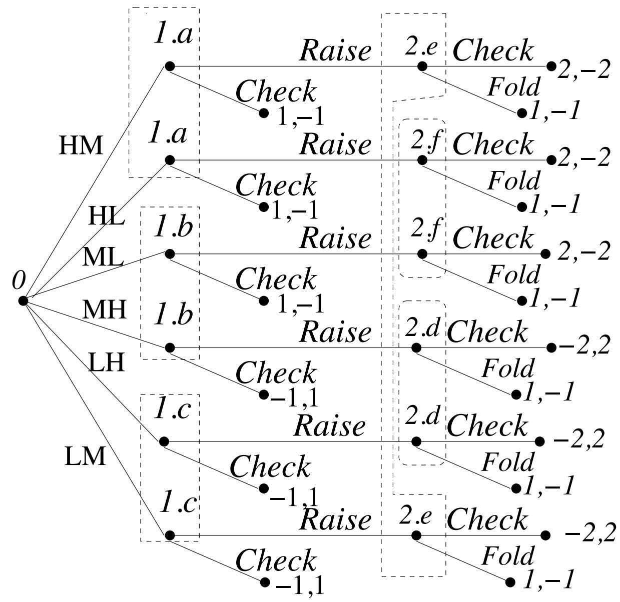
**Solution:** Let $(i^*, d^*)$ be a trembling hand perfect equilibrium. By definition it means there is sequence of fully mixed strategies $(p^{(n)}, q^{(n)})$ converging to $(i^*, d^*)$ such that $d^*$ is a best response to $p^{(n)}$ and $i^*$ is a best respond to $q^{(n)}$. It follows that $d^*(i) = 1$ for $i \leq 99$. Also, $i^*$ must be a best response to $d^*$. So $(i^*, d^*)$ must be one of the two subgame perfect equilibria found in part (d).

In fact, both $(99, d_{99})$ and $(100, d_{100})$ are trembling hand perfect equilibria. To see this, note that both $d_{99}$ and $d_{100}$ are weakly dominant strategies for player 2, and hence they are both best responses for any choice of $p^{(n)}$. In the other direction, $i = 99$ is a best response to any $q^{(n)}$ which puts probability sufficiently close to one on the pure strategy $d_{99}$, and $i = 100$ is a best response to any $q^{(n)}$ which puts probability sufficiently close to one on the pure strategy $d_{100}$.

2. **[Half Kuhn poker]**
Players 1 and 2 engage in the following game. There is a deck of three cards, $L, M$, and $H$, such that $L$ is the low card, $M$ is the medium card, and $H$ is the high card. After both players inspect the deck, the deck is shuffled and one card is dealt to each player, face down. The third card is put aside. Each player looks at his card. Player 1 can raise or check. If player 1 raises, player 2 can either check or fold. If player 2 checks, the players show their cards, and the player with the larger card wins two dollars from the player with the smaller card. If player 2 folds, player 2 pays one dollar to player 1. If player 1 checks, the players show their cards, and the player with the larger card wins one dollar from the player with the smaller card.

(a) Draw a tree diagram modeling this as an extensive form game, and label it using the conventions from class (as in Myerson's book). For each node of the tree, indicate wether it is controlled by chance (use label 0), player 1, or player 2, and indicated an information state for each node controlled by a player. Group together nodes with the same player and information state using dashed lines.

**Solution:**




The node labeled 0 is the chance node. Branch $HM$ from the zero node indicates player 1 is dealt $H$ and player 2 is dealt $M$. Other branches from node zero are labeled similarly. Information states $a, b$, or $c$ indicate that player 1 was dealt $H$, $M$, or $L$, respectively. Information state $d, e$, or $f$ indicates that player 1 raised and player 2 was dealt $H$, $M$, or $L$, respectively.

(b) Derive the normal form of this game. (Use of a computer or spreadsheet is recommended, even if you don't completely automate the calculation.)

**Solution:** Table 1 shows the payoff matrix for player 1. The game is zero sum, so the payoffs of player 2 are opposite the payoffs of player 1. Strategy RRC for player 1, for example, means "raise if card is H or M, check if card is L." Similarly, strategy CFF for player 2 means "check if card is high, fold if card is M or L." To calculate the table accurately it helps to work out the payoffs to player 1 for each of the six possible ways cards are assigned. See code attached.

Table 1: Payoff matrix for player 1 in the normal form equivalent of Half Kuhn Poker.

| | CCC | CCF | CFC | CFF | FCC | FCF | FFC | FFF |
|---|---|---|---|---|---|---|---|---|
| RRR | 0 | -1/3 | 1/3 | 0 | 1 | 2/3 | 4/3 | 1 |
| RRC | 1/3 | 0 | 1/6 | -1/6 | 5/6 | 1/2 | 2/3 | 1/3 |
| RCR | 0 | -1/6 | 1/3 | 1/6 | 1/2 | 1/3 | 5/6 | 2/3 |
| RCC | 1/3 | 1/6 | 1/6 | 0 | 1/3 | 1/6 | 1/6 | 0 |
| CRR | -1/3 | -1/2 | 1/6 | 0 | 2/3 | 1/2 | 7/6 | 1 |
| CRC | 0 | -1/6 | 0 | -1/6 | 1/2 | 1/3 | 1/2 | 1/3 |
| CCR | -1/3 | -1/3 | 1/6 | 1/6 | 1/6 | 1/6 | 2/3 | 2/3 |
| CCC | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

```
AHM=[2 2 1 1 2 2 1 1  %player one payoffs for cards HM
     2 2 1 1 2 2 1 1
     2 2 1 1 2 2 1 1
     2 2 1 1 2 2 1 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1]

AHL=[2 1 2 1 2 1 2 1  %player one payoffs for cards HL
     2 1 2 1 2 1 2 1
     2 1 2 1 2 1 2 1
     2 1 2 1 2 1 2 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1]

AMH=[-2 -2 -2 -2  1  1  1  1   %player one payoffs for cards MH
     -2 -2 -2 -2  1  1  1  1
     -1 -1 -1 -1 -1 -1 -1 -1
     -1 -1 -1 -1 -1 -1 -1 -1
     -2 -2 -2 -2  1  1  1  1
     -2 -2 -2 -2  1  1  1  1
     -1 -1 -1 -1 -1 -1 -1 -1
     -1 -1 -1 -1 -1 -1 -1 -1]

AML=[2 1 2 1 2 1 2 1  %player one payoffs for cards ML
     2 1 2 1 2 1 2 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1
     2 1 2 1 2 1 2 1
     2 1 2 1 2 1 2 1
     1 1 1 1 1 1 1 1
     1 1 1 1 1 1 1 1]

ALH=[-2 -2 -2 -2  1  1  1  1   %player one payoffs for cards LH
     -1 -1 -1 -1 -1 -1 -1 -1
     -2 -2 -2 -2  1  1  1  1
     -1 -1 -1 -1 -1 -1 -1 -1
     -2 -2 -2 -2  1  1  1  1
     -1 -1 -1 -1 -1 -1 -1 -1
     -2 -2 -2 -2  1  1  1  1
     -1 -1 -1 -1 -1 -1 -1 -1]

ALM=[-2 -2  1  1 -2 -2  1  1    %player one payoffs for cards LM
     -1 -1 -1 -1 -1 -1 -1 -1
     -2 -2  1  1 -2 -2  1  1
     -1 -1 -1 -1 -1 -1 -1 -1
     -2 -2  1  1 -2 -2  1  1
     -1 -1 -1 -1 -1 -1 -1 -1
     -2 -2  1  1 -2 -2  1  1
     -1 -1 -1 -1 -1 -1 -1 -1]

 A=(AHM+AHL+AMH+AML+ALH+ALM)/6;
```

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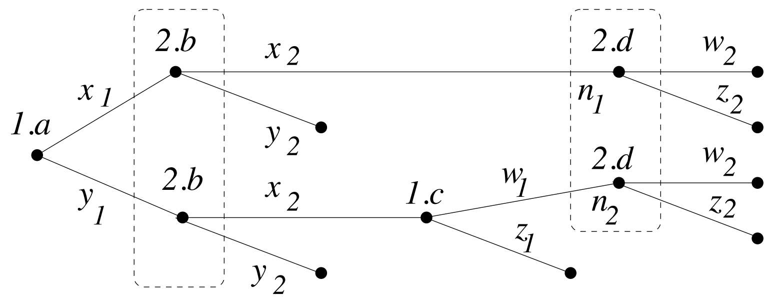
(c) Identify a pair of saddle point strategies and the value of the game to player 1. (May need to do this numerically. The program saddlepoint.m posted to the webpage could be useful.)

**Solution:** Numerical computation identified the following maxmin strategy for player 1: $(0,0,1/3,2/3,0,0,0,0)$ and the following minmax strategy for player 2: $(0,1/3,0,2/3,0,0,0,0)$. Under these strategies, player 1 always raises on H, always checks on M, and, if his card is L, he raises with probability 1/3, while player two always checks on H, folds on L, and, if her card is M, she checks with probability 1/3. The value of the game is 1/18 for player 1. Intuitively, it makes sense for player 1 not to randomize if his card is M. In that case he knows that player 2 will have either H and check or L and fold. But if player 1 has L, he knows that player 2 could have M, in which case player 2 might not check if player 1 raises, even though player 2 has a higher card than player 1.

3. **[Finding a behavioral representation]**
Consider the extensive form game shown, and consider the mixed strategy profile $\tau$ for the




normal representation of the game, given by $\tau = (\tau_1, \tau_2)$ where

$$\begin{aligned} \tau_1 &= .5[x_1 w_1] + .5\alpha[y_1 w_1] + .5(1-\alpha)[y_1 z_1] \\ \tau_2 &= \beta[x_2 w_2] + (1-\beta)[y_2 z_2] \end{aligned}$$

for some constants $\alpha, \beta \in (0, 1)$.

(a) Does this game satisfy the perfect recall condition? Briefly explain.

**Solution:** Yes. Each player must take an action for two information states. The second information state for player 1 has only one node, so there is nothing to check for player 1. The second information state for player 2 is $d$. Both nodes in that state come after a branch labeled $x_2$ out of a node with information state $b$. That is, the prior knowledge of player 2 about her prior move is the same for both nodes of information state $d$.

(b) Find the behavioral representation $\sigma$ of $\tau$. (Hint: You should specify distributions player 1 uses in states $a$ and $c$, and the distribution player 2 uses in states $b$ and $d$.)

**Solution:** For the behavioral strategy of player 1, note that information state $c$ can be reached only if player 1's choice of strategy is $[y_1 w_1]$ or $[y_1 z_1]$. For the behavioral strategy of player 2, note that information state $d$ can be reached only if player 2 plays $x_2$ in information state $b$, which, under $\tau_2$, means player 2 will play $w_2$ in state $d$. We thus get:

$$\begin{aligned} \sigma_1 &= (.5[x_1] + .5[y_1], \alpha[w_1] + (1-\alpha)[z_1]) \\ \sigma_2 &= (\beta[x_2] + (1-\beta)[y_2], [w_2]) \end{aligned}$$

As expected, given that the game satisfies the perfect recall condition, $\sigma_1$ does not depend on $\tau_2$ and $\sigma_2$ does not depend on $\tau_1$.

(c) Note that $n_1$ and $n_2$ are the two nodes labeled with information state $d$. Find the conditional probability node $n_1$ is reached given information state $d$ is reached. Does it depend on $\tau_1$ (i.e. on $\alpha$)? Does $\sigma_2$ depend on $\tau_1$ (i.e. on $\alpha$)?

**Solution:** $p(n_1|\tau) = .5\beta$, $p(n_2|\tau) = .5\alpha\beta$ and $p(d|\tau) = p(n_1|\tau) + p(n_2|\tau)$. Therefore by Bayes formula,

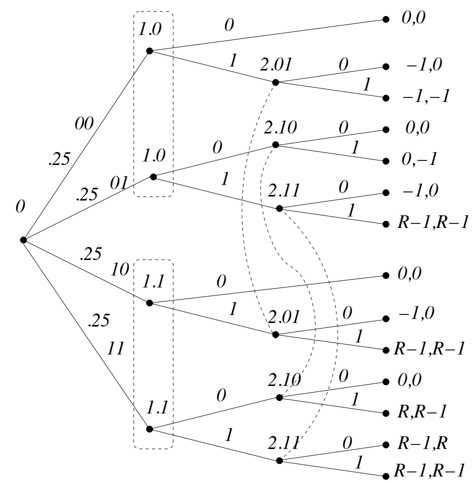
$$p(n_1|d,\tau) = \frac{p(n_1|\tau)}{p(n_1|\tau) + p(n_2|\tau)} = \frac{.5\beta}{.5\beta + .5\alpha\beta} = \frac{1}{1+\alpha}$$

Yes, $p(n_1|d,\tau)$ does depend on $\tau_1$. No, as noted above, $\sigma_2$ doesn't depend on $\alpha$. In particular, either way information state $d_2$ is reached, player 2 must have played $x_2$ at the earlier information state $b$, which determines the conditional distribution of the action of player 2 in state $d$ (which in this case, is the pure action $[w_2]$.)

4. **[Joint funding with incomplete information]**
Two players each observe a random bit. Let $X_i$ be the random bit observed by player $i$. Assume $X_1$ and $X_2$ are independent, Bernoulli(0.5) random variables. Each player $i$ determines a bit $D_i$. Let $R > 0$. The payoff of player $i$ is given by $u_i(X,D) = R\mathbf{1}_{\{X_1+X_2+D_1+D_2 \geq 3\}} - D_i$. That is, both players receive benefit $R$ if the sum of the random bits and the decision bits is greater than or equal to three, and player $i$ must pay $D_i$, even if the sum $X_1+X_2+D_1+D_2$ is not greater than or equal to three. Suppose neither player observes the random bit of the other player, and suppose player 2 observes the value of $D_1$ before making her decision.

(a) Sketch a decision tree giving the extensive form of this game. To be definite, suppose that the root node is a chance node with four branches, one for each of the four possible values of $(X_1, X_2)$. Also, it is clear that player 2 should always play 0 if she observes $X_2 = D_1 = 0$, and the payoff vector in that case is $(0,0)$. For simplicity, reduce the size of the tree by implicitly assuming player 2 aways plays 0 if she observes $X_2 = D_1 = 0$. This reduces the number of information states for player 2 from four to three.

**Solution:**




An information state of the form $1.i$ means the private bit of player 1 is $i$. An information state of the form $2.jk$ means player 2's private bit is $j$ and player 1 played $k$.

(b) Indicate the payoff matrix for the normal representation of the game. For simplicity, it might be easier to give four times the payoff matrix rather than the payoff matrix itself.

**Solution:** Strategies for player 1 have the form $d_0d_1$ and strategies for player 2 have the form $d_{01}d_{10}d_{11}$.

We begin by finding the payoff matrices for each of the four branches from the chance node. An asterisk is used to indicate that in addition to the payments shown, both players receive payoff $R$.

| | ↓ | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 00 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | (player 2 sees 00) |
| 00 | 01 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | ( ” ” ” ” ) |
| | 10 | -1,0 | -1,0 | -1,0 | -1,0 | -1,-1 | -1,-1 | -1,-1 | -1,-1 | ($1^{st}$ bit of player 2 relevant) |
| | 11 | -1,0 | -1,0 | -1,0 | -1,0 | -1,-1 | -1,-1 | -1,-1 | -1,-1 | ( ” ” ” ” ) |

| | ↓ | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 00 | 0,0 | 0,0 | 0,-1 | 0,-1 | 0,0 | 0,0 | 0,-1 | 0,-1 | ($2^{nd}$ bit of player 2 relevant) |
| 01 | 01 | 0,0 | 0,0 | 0,-1 | 0,-1 | 0,0 | 0,0 | 0,-1 | 0,-1 | ( ” ” ” ” ) |
| | 10 | -1,0 | -1,-1* | -1,0 | -1,-1* | -1,0 | -1,-1* | -1,0 | -1,-1* | ($3^{rd}$ bit of player 2 relevant) |
| | 11 | -1,0 | -1,-1* | -1,0 | -1,-1* | -1,0 | -1,-1* | -1,0 | -1,-1* | ( ” ” ” ” ) |

| | ↓ | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 00 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | (player 2 sees 00) |
| 10 | 01 | -1,0 | -1,0 | -1,0 | -1,0 | -1,-1 * | -1,-1 * | -1,-*1 | -1,-1* | ($1^{st}$ bit of player 2 relevant) |
| | 10 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | 0,0 | (player 2 sees 00) |
| | 11 | -1,0 | -1,0 | -1,0 | -1,0 | -1,-1* | -1,-1* | -1,-1* | -1,-1* | ($1^{st}$ bit of player 2 relevant) |

| | ↓ | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 00 | 0,0 | 0,0 | 0,-1* | 0,-1* | 0,0 | 0,0 | 0,-1* | 0,-1* | ($2^{nd}$ bit of player 2 relevant) |
| 11 | 01 | -1,0* | -1,-1* | -1,0* | -1,-1* | -1,0* | -1,-1* | -1,0* | -1,-1* | ($3^{rd}$ bit of player 2 relevant) |
| | 10 | 0,0 | 0,0 | 0,-1* | 0,-1* | 0,0 | 0,0 | 0,-1* | 0,-1* | ($2^{nd}$ bit of player 2 relevant) |
| | 11 | -1,0* | -1,-1* | -1,0* | -1,-1* | -1,0* | -1,-1* | -1,0* | -1,-1* | ($3^{rd}$ bit of player 2 relevant) |

Adding together the above matrices yields four times the payoff matrix.

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| 00 | 0,0 | 0,0 | R,R-2 | R,R-2 | 0,0 | 0,0 | R,R-2 | R,R-2 |
| 01 | R-2,R | R-2,R-1 | R-2,R-1 | R-2,R-2 | 2R-2,2R-1 | 2R-2,2R-2 | 2R-2,2R-2 | 2R-2,2R-3 |
| 10 | -2,0 | R-2,R-1 | R-2,R-1 | 2R-2,2R-2 | -2,-1 | R-2,R-2 | R-2,R-2 | 2R-2,2R-3 |
| 11 | R-4,R | 2R-4,2R-2 | R-4,R | 2R-4,2R-2 | 2R-4,2R-2 | 3R-4,3R-4 | 2R-4,2R-2 | 3R-4,3R-4 |

(c) Identify all the NE in mixed strategies in case $0 < R < 1$, and identify which of those is a (trembling hand) perfect equilibrium of the normal form game and which of those is a sequential equilibrium scenario. (Hint: Player 1 has a strictly dominant strategy.) (As described in Fudenberg and Tirole Section 8.3, Kreps and Wilson (1982) defined the notion of a pair $(\sigma, \mu)$ being a *sequential equilibrium*, where $\sigma$ is a behavioral strategy and $\mu$ is a belief vector. The definition requires two conditions on $(\sigma, \mu)$: (S) sequential rationality–meaning the decision for each player in each information state maximizes the expected payoff of the player, and (C) consistency, which means that $(\sigma, \mu)$ can be obtained as the limit of some sequence $(\sigma^n, \mu^n)$ such that all branches at each decision node have strictly positive probability under $\sigma^n$, and $\mu^n$ is computed from $\sigma^n$ by Bayes formula. A strategy profile $\sigma$ is a *sequential-equilibrium scenario* if there exists $\mu$ so that $(\sigma, \mu)$ is a sequential equilibrium.)

**Solution:** For $0 < R < 1$, strategy 00 is a strictly dominant strategy of the normal form game, so that player 1 must use strategy 00 in any Nash equilibrium. Furthermore, any NE has the form $(00, q)$ where $q$ is a probability distribution over the strategies of player 2 with support set $\{000, 001, 010, 011, 100, 101\}$. If there is a strategy profile such that each strategy is weakly dominant (including the condition of being strictly better for some response of the other player) then the profile is a (trembling hand) perfect equilibrium and it is the only such equilibrium. Hence, $(00, 000)$ is the unique trembling hand perfect equilibrium. Since $(00, 000)$ is a (trembling hand) perfect equilibrium, it is also a sequential-equilibrium scenario of the extensive form game.

(In theory, not all sequential-equilibrium scenarios are trembling hand perfect, so we're not done yet.) There are no other sequential equilibria for the extensive form game. In particular, for each of the three information states at which player 2 must make a decision, for any belief distribution player 2 has over the two states of the information state, her unique best response is to always play 0. Similarly, player 1 is always better off playing 0 in either of his information states (it suffices to check this for the case player 2 always plays 0, but it is true no matter what strategy player 2 uses.)

(d) Find a pure strategy profile that is an NE for any $R \geq 1$, with strictly positive expected payoffs for $R > 1$. For which value(s) of $R > 1$, if any, is it (trembling hand) perfect? For which value(s) of $R > 1$, if any, is it a sequential-equilibrium scenario?

**Solution:** (01,100) is an NE for any $R \geq 1$ and it has strictly positive payoffs for both players if $R > 1$. For any value of $R > 1$, 01 is the unique best response of player 1 to 100, and 100 is the unique best response of player 2 to 01. Therefore, (01,100) is a trembling hand perfect equilibrium for any $R > 1$. It is therefore also a sequential-equilibrium scenario for any $R > 1$. (Interestingly, under strategy 01 for player 1, either $X_1 + D_1 = 0$ or $X_1 + D_1 = 2$, and player 2 can tell which of these two cases holds. If player 2 uses 100 in response to 01 by player 1, it means she plays 0 if see knows either $X_1 + D_1 = 0$ or $X_1 + X_2 + D_1 = 3$, and she plays 1 if she knows $X_1 + X_2 + D_1 = 2$.) )

(e) Find four pure strategy profiles different from the one in part (d) that are NE for any $R \geq 2$. For each of the two profiles, answer the following. For which value(s) of $R > 2$, if any, is the profile (trembling hand) perfect? For which value(s) of $R > 2$, if any, is the profile a sequential-equilibrium scenario?

**Solution:** For $R \geq 2$ four NEs appear in addition to the one found in part (d). They are listed below, along with a determination of whether they are trembling hand perfect and/or sequential-equilibrium scenarios, assuming $R > 2$.

i. **(00 010)** Player 1 never plays 1. Player 2 plays 1 if (and only if) $X_2 = 1$ and, as expected, player 1 plays 0. Player 2 believes by playing 1 in this case that she has probability 0.5 to meet the threshold. This NE is not a (trembling hand) perfect equilibrium because for player 2, 010 is weakly dominated by 111 (including 111 being strictly better than 010 for some plays of player 1) for $R > 2$. To see if this NE is a sequential-equilibrium scenario we seek a suitable belief vector $\mu$ so that $(\sigma, \mu)$ is a sequential equilibrium. There is a one dimensional family of $\mu$ so that $(\sigma, \mu)$ is consistent. Namely, for $0 \leq \theta \leq 1$, $\mu(0) = \mu(1) = \mu(10) = (0.5, 0.5)$ and $\mu(01) = \mu(11) = (\theta, 1-\theta)$. (About notation: Here for example, $\mu(01)$ represents the belief distribution for information state 10, which consists of two nodes as shown in the game tree, and $\mu(01) = (\theta, 1-\theta)$ means the top one of these two nodes shown in the diagram has conditional probability $\theta$ given information state 01 is reached. The fact $\mu(01) = \mu(11)$ comes from the fact that $X_2$ is independent of $(X_1, D_1)$.) Under this belief vector, in order for 0 to be a best response in state 01 requires $\theta \times 0 + (1-\theta) \times 0 \geq (\theta)(-1) + (1-\theta)(R-1)$ and for 0 to be a best response for state 11 requires $\theta \times 0 + (1-\theta)R \geq \theta(R-1) + (1-\theta)(R-1)$. This requires $1 \geq (1-\theta)R \geq R-1$ which requires $R \leq 2$. Therefore, the NE is not a sequential-equilibrium scenario for $R > 2$.

ii. **(10 011)** This is a NE and, for $R > 2$, the strategy of each player is the unique best response to the strategy of the other player. Hence, it is a (trembling hand) perfect equilibrium and, thus, also a sequential-equilibrium scenario. (Under this equilibrium, Player 1 acts to make $X_1 + D_1 = 1$ and player two plays 1 if and only if $X_2 = 1$.)

iii. **(11,101)** This is a (trembling hand) perfect equilibrium, and hence also a sequential-equilibrium, for any $R > 2$. To see this, define a sequence of fully mixed strategies $\sigma^{(n)}$ for $n \geq 1$ with $\epsilon = \frac{1}{2n}$, by $\sigma_1^{(n)} = \epsilon^2[00] + (1-\theta)\theta\epsilon[01] + \theta\epsilon[10] + (1-\epsilon-\epsilon^2)[11]$ and $\sigma_2^{(n)} = (1-\epsilon)[101] + \epsilon\widetilde{\sigma}_2$ where $0 < \theta < 1$ and $\widetilde{\sigma}_2$ is an arbitrary fully mixed strategy for player 2. Since [11] is the unique best response to [101], it follows that [11] is a best response to $\sigma_2^{(n)}$ for all sufficiently large $n$. There are two best responses for player 2 against [11], namely [101] and [111]. If

$$(1-\theta)(2R-2) + \theta(R-2) > (1-\theta)(2R-3) + \theta(2R-3)$$

or $\theta < \frac{1}{R}$, then for $n$ large, the response [101] is better than [111] against $\sigma_1^{(n)}$, and hence for $n$ large enough, $\sigma_i$ is a best response to $\sigma_{-i}^{(n)}$. By deleting a finite number of terms from the beginning of the sequence $\sigma^{(n)}$ we arrive at a sequence showing this NE is trembling-hand perfect.

Intuitively, the NEs (11,101) and (11,111) are very close. For both of these NE, player one always plays 1. The difference is what player 2 believes given player 1 plays 0, which is an event of probability zero. The parameter $\theta$ can be viewed as satisfying $\theta = P(X = 1|D_1 = 0)$.

iv. **(11,111)** By the analysis of the previous NE, we see that this NE is also a trembling hand perfect equilibrium and a sequential equilibrium scenario. The same sequence $\sigma^{(n)}$ can be used as before, but with $\theta$ in the range $\frac{1}{R} < \theta < 1$.

We remark that for $R > 2$, there are four NEs that are all trembling hand perfect and sequential equilibrium scenarios, namely (01, 100), (10,011), (11,101), and (11,111). For $R$ large, the payoffs of both players are approximately $2R$ for the first two of these equilibria, and $3R$ for the other two. (These are four times the payoffs for the original game.)