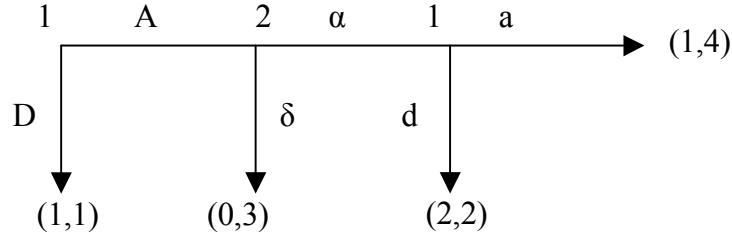


## 14.12 Game Theory – Final

**Instructions.** This is an open book exam; you can use any written material. You have 2 hours 50 minutes. Each question is 20 points. Good luck!

1. Consider the following extensive form game.



- (a) Find the normal form representation of this game.

$1 \setminus 2$	$\alpha$	$\delta$
Aa	1,4	0,3
Ad	2,2	0,3
Da	1,1	1,1
Dd	1,1	1,1

- (b) Find all rationalizable pure strategies.

$1 \setminus 2$	$\alpha$	$\delta$
Ad	2,2	0,3
Da	1,1	1,1
Dd	1,1	1,1

- (c) Find all pure strategy Nash equilibria.

$1 \setminus 2$	$\alpha$	$\delta$
Ad	2,2	0,3
Da	1,1	<b>1,1</b>
Dd	1,1	<b>1,1</b>

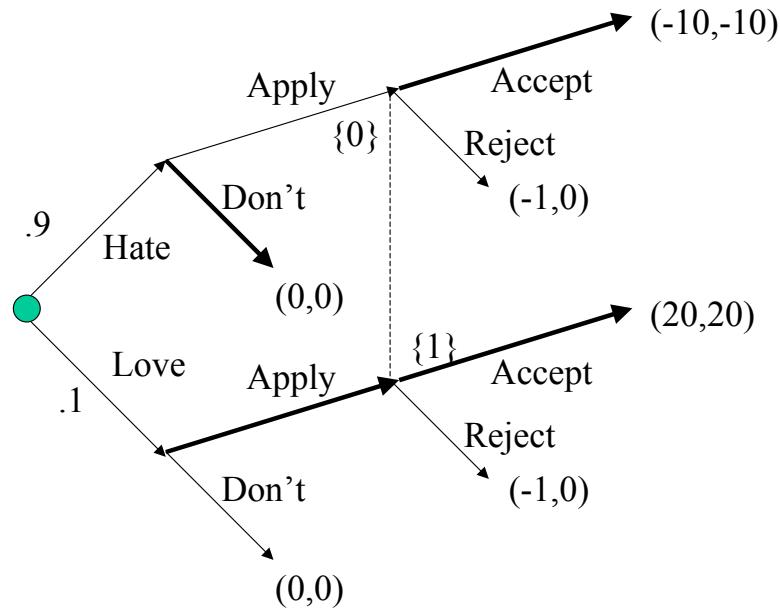
- (d) Which strategies are consistent with all of the following assumptions?

- (i) 1 is rational.
- (ii) 2 is sequentially rational.
- (iii) at the node she moves, 2 knows (i).
- (iv) 1 knows (ii) and (iii).

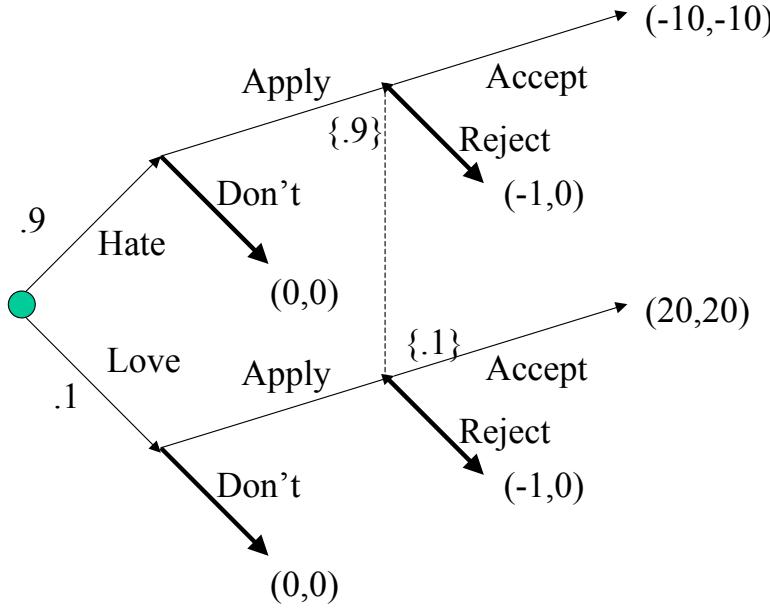
**ANSWER:** By (i) 1 does not play Aa. Hence, by (iii), at the node she moves, 2 knows that 1 does not play Aa, hence he knows that 1 plays Ad. Then, by (ii), 2 must play  $\delta$ . Therefore, by (i) and (iv), 1 must play Ad or Aa. The answer is 1 plays A, given chance 2 would play  $\delta$ .

2. This question is about a game between a possible applicant (henceforth student) to a Ph.D. program in Economics and the Admission Committee. Ex-ante, Admission Committee believes that with probability .9 the student hates economics and with probability .1 he loves economics. After Nature decides whether student loves or hates economics with the above probabilities and reveals it to the student, the student decides whether or not to apply to the Ph.D. program. If the student does not apply, both the student and the committee get 0. If student applies, then the committee is to decide whether to accept or reject the student. If the committee rejects, then committee gets 0, and student gets -1. If the committee accepts the student, the payoffs depend on whether the student loves or hates economics. If the student loves economics, he will be successful and the payoffs will be 20 for each player. If he hates economics, the payoffs for both the committee and the student will be -10. Find a separating equilibrium and a pooling equilibrium of this game.

ANSWER: A separating equilibrium:



A pooling equilibrium:



3. Consider a bargaining problem where two risk-neutral players are trying to divide a dollar they own, which they cannot use until they reach an agreement. The players do not discount the future, but at the end of each rejection of an offer the bargaining breaks down with probability  $1 - \delta \in (0, 1)$  and each player gets 0.

- (a) Consider the following bargaining procedure. Player 1 makes an offer  $(x, 1 - x)$ , where  $x$  is player 1's share. Then, player 2 decides whether or not to accept the offer. If she accepts, they implement the offer, yielding division  $(x, 1 - x)$ . If she rejects the offer, then with probability  $1 - \delta$ , the bargaining breaks down and each gets 0; with probability  $\delta$ , player 1 makes another offer, which will be accepted or rejected by player 2 as above. (If player 2 rejects the offer, bargaining will break down with probability  $1 - \delta$  again.) If the offer is rejected and the bargaining did not break down, now player 2 makes a counter offer, and player 1 accepts or rejects this counter offer as above. If the offer is rejected, this time the game will end, and each will get 0. Find the subgame-perfect Nash equilibrium of this game. Compute the expected payoff of each player at the beginning of the game in this equilibrium.

ANSWER: On the last day, 1 accepts any offer, so 2 offers  $(0, 1)$ . Hence, on the previous day, 2 accepts an offer iff she gets at least  $\delta$ . Hence, 1 offers  $(1 - \delta, \delta)$  — accepted. Thus, in the first day, 2 accepts an offer iff she gets at least  $\delta^2$ . Hence, 1 offers  $(1 - \delta^2, \delta^2)$  — accepted. The expected payoffs are  $(1 - \delta^2, \delta^2)$ .

- (b) Compute the subgame-perfect equilibrium of the game in which the procedure in part (a) is repeated 2 times. (The probability of bargaining breakdown after each rejection is  $1 - \delta$ , except for the end of the game.)

ANSWER: The last period as above. Let's look at the first period. On the last day of the first period, 1 accepts an offer iff he gets at least  $\delta(1 - \delta^2)$ , so 2 offers

$(\delta(1 - \delta^2), 1 - \delta(1 - \delta^2))$ . Hence, on the previous day, 2 accepts an offer iff she gets at least  $\delta(1 - \delta(1 - \delta^2))$ . Hence, 1 offers

$$(1 - \delta(1 - \delta(1 - \delta^2)), \delta(1 - \delta(1 - \delta^2))) = (1 - \delta + \delta^2(1 - \delta^2), \delta(1 - \delta(1 - \delta^2)))$$

— accepted. Thus, in the first day, 2 accepts an offer iff she gets at least  $\delta^2(1 - \delta(1 - \delta^2))$ . Hence, 1 offers

$$(1 - \delta^2(1 - \delta(1 - \delta^2)), \delta(1 - \delta(1 - \delta^2))) = (1 - \delta^2 + \delta^3(1 - \delta^2), \delta^2(1 - \delta(1 - \delta^2)))$$

— accepted.

- (c) Find the subgame-perfect equilibrium of the game in which this procedure is repeated until they reach an agreement. Note that player 1 makes two offers, then 2 makes one offer, then 1 makes two offers, and so on. You need to show that the proposed strategy profile is in fact a subgame-perfect equilibrium. (The probability of bargaining breakdown after each rejection is  $1 - \delta$ .)

[Hint: One way is to compute the SPE for the game in which the procedure is repeated  $n$  times and let  $n \rightarrow \infty$ . A somewhat easier way is to consider an alternating offer bargaining procedure with some effective discount rates — different for a different player.]

ANSWER: If you compare the calculations above with the calculations with the alternating offer case with asymmetric discount rates, you should realize that the first offer player 1 makes and the offer player 2 makes are identical to the offers players 1 and 2 make, respectively, if the discount rates were  $\delta_1 = \delta$  and  $\delta_2 = \delta^2$ . Intuitively, in his second offer player 1 makes player 2 indifferent between accepting 1's second offer and making an offer next day, and in his first offer he makes her indifferent between accepting the offer and waiting for the second offer. Therefore, 2 is indifferent between accepting 1's first offer and waiting two days two make an offer, as in the the alternating offer case when her discount rate is  $\delta^2$ . Now conjecture that the subgame-perfect equilibrium would be as in the alternating offer game with above discount rates. That is,

- in his first offer, player 1 offers

$$\begin{aligned} \left( \frac{1 - \delta_2}{1 - \delta_1 \delta_2}, 1 - \frac{1 - \delta_2}{1 - \delta_1 \delta_2} \right) &\equiv \left( \frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2(1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \equiv \left( \frac{1 - \delta^2}{1 - \delta^3}, \frac{\delta^2(1 - \delta)}{1 - \delta^3} \right) \\ &\equiv \left( \frac{1 + \delta}{1 + \delta + \delta^2}, \frac{\delta^2}{1 + \delta + \delta^2} \right); \end{aligned}$$

- in his secon offer, he will offer

$$\left( 1 - \frac{\delta(1 - \delta_1)}{1 - \delta_1 \delta_2}, \frac{\delta(1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \equiv \left( \frac{1 + \delta^2}{1 + \delta + \delta^2}, \frac{\delta}{1 + \delta + \delta^2} \right);$$

- player 2 will offer

$$\left( 1 - \frac{1 - \delta_1}{1 - \delta_1 \delta_2}, \frac{1 - \delta_1}{1 - \delta_1 \delta_2} \right) \equiv \left( \frac{\delta_1(1 - \delta_2)}{1 - \delta_1 \delta_2}, \frac{1 - \delta_1}{1 - \delta_1 \delta_2} \right) \equiv \left( \frac{\delta + \delta^2}{1 + \delta + \delta^2}, \frac{1}{1 + \delta + \delta^2} \right).$$

Player 1's first offer and player 2's offer are by formula for alternating offer, player 1's second offer is calculated by backward induction using the player 2's offer in the next period. Using single deviation property, you need to check that this is an equilibrium.

4. We have an employer and a worker, who will work as a salesman. The worker may be a good salesman or a bad one. In expectation, if he is a good salesman, he will make \$200,000 worth of sales, and if he is bad, he will make only \$100,000. The employer gets 10% of the sales as profit. The employer offers a wage  $w$ . Then, the worker accepts or rejects the offer. If he accepts, he will be hired at wage  $w$ . If he rejects the offer, he will not be hired. In that case, the employer will get 0, the worker will get his outside option, which will pay \$15,000 if he is good, \$8,000 if he is bad. Assume that all players are risk-neutral.

- (a) Assume that the worker's type is common knowledge, and compute the subgame-perfect equilibrium.

ANSWER: A worker will accept a wage iff it is at least as high as his outside option, and the employer will offer the outside option — as he still makes profit. That is, 15,000 for the good worker 8,000 for the bad.

- (b) Assume that the worker knows his type, but the employer does not. Employer believes that the worker is good with probability 1/4. Find the perfect Bayesian Nash equilibrium.

ANSWER: Again a worker will accept an offer iff his wage at least as high as his outside option. Hence if  $w \geq 15,000$  the offer will be accepted by both types, yielding

$$U(w) = (1/4)(.1)200,000 + (3/4)(.1)100,000 - w = 12,500 - w < 0$$

as the profit for the employer. If  $8,000 \leq w < 15,000$ , then only the bad worker will accept the offer, yielding

$$U(w) = (3/4)[(.1)100,000 - w] = (3/4)[10,000 - w]$$

as profit. If  $w < 8,000$ , no worker will accept the offer, and the employer will get 0. In that case, the employer will offer  $w = 8,000$ , hiring the bad worker at his outside option.

- (c) Under the information structure in part (b), now consider the case that the employer offers a share  $s$  in the sales rather than the fixed wage  $w$ . Compute the perfect Bayesian Nash equilibrium.

ANSWER: Again a worker will accept the share  $s$  iff his income is at least as high as his outside option. That is, a bad worker will accept  $s$  iff

$$100,000s \geq 8,000$$

i.e.,

$$s \geq s_B = \frac{8,000}{100,000} = 8\%.$$

A good worker will accept  $s$  iff

$$s \geq s_G = \frac{15,000}{200,000} = 7.5\%.$$

In that case, if  $s < s_G$  no one will accept the offer, and the employer will get 0; if  $s_G \leq s < s_B$ , the good worker will accept the offer and the employer will get

$$(1/4)(10\% - s) 200,000 = 50,000(10\% - s),$$

and if  $s \geq s_B$ , each type will accept the offer and the employer will get

$$(10\% - s) [(1/4) 200,000 + (3/4) 100,000] = 125,000(10\% - s).$$

Since  $125,000(10\% - s_B) = 2\%125,000 = 2,500$  is larger than  $50,000(10\% - s_G) = 2.5\%50,000 = 1,250$ , he will offer  $s = s_B$ , hiring both types.

5. As in question 4, We have an employer and a worker, who will work as a salesman. Now the market might be good or bad. In expectation, if the market is good, the worker will make \$200,000 worth of sales, and if the market is bad, he will make only \$100,000 worth of sales. The employer gets 10% of the sales as profit. The employer offers a wage  $w$ . Then, the worker accepts or rejects the offer. If he accepts, he will be hired at wage  $w$ . If he rejects the offer, he will not be hired. In that case, the employer will get 0, the worker will get his outside option, which will pay \$12,000. Assume that all players are risk-neutral.

- (a) Assume that whether the market is good or bad is common knowledge, and compute the subgame-perfect equilibrium.

ANSWER: A worker will accept a wage iff it is at least as high as his outside option 12,000. If the market is good, the employer will offer the outside option  $w = 12,000$ , and make  $20,000 - 12,000 = 8,000$  profit. If the market is bad, the return 10,000 is lower than the worker's outside option, and the worker will not be hired.

- (b) Assume that the employer knows whether the market is good or bad, but the worker does not. The worker believes that the market is good with probability 1/4. Find the perfect Bayesian Nash equilibrium.

ANSWER: As in part (a). [We will have a separating equilibrium.]

- (c) Under the information structure in part (b), now consider the case that the employer offers a share  $s$  in the sales rather than the fixed wage  $w$ . Compute a perfect Bayesian Nash equilibrium.

ANSWER: Note that, since the return is 10% independent of whether the market is good or bad, the employer will make positive profit iff  $s < 10\%$ . Hence, except

for  $s = 10\%$ , we must have a pooling equilibrium. Hence, at any  $s$ , the worker's income is

$$[(1/4) 200,000 + (3/4) 100,000] s = 125,000s.$$

This will be at least as high as his outside option iff

$$s \geq s^* = \frac{12,000}{125,000} = 9.6\% < 10\%.$$

Hence an equilibrium: the worker will accept an offer  $s$  iff  $s \geq s^*$ , and the employer will offer  $s^*$ . The worker's beliefs at any offer  $s$  is that the market is good with probability  $1/4$ . [Note that this is an inefficient equilibrium. When the market is bad, the gains from trade is less than the outside option.]

There are other inefficient equilibria where there is no trade (i.e., worker is never hired). In any such equilibrium, worker take any high offer as a sign that the market is bad, and does not accept an offer  $s$  unless  $s \geq 12,000/100,000 = 12\%$ , and the employer offers less than 12%. When the market is good, in any such pure strategy equilibrium, he must in fact be offering less than  $s^*$ . (why?) For instance, employer offers  $s = 0$  independent of the market, and the worker accept  $s$  iff  $s > 12\%$ .