14.12 Game Theory-Midterm I
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Instructions: This is an open book exam, you can use any written material. You have 1 hour and 20 minutes. Each question is 35 points where the bonus question 3(c) accounts for the extra 5 points. Good luck!

1. Consider the following two player game where Player 1 chooses one of the three rows and Player 2 chooses one of the three columns:

	C_1	C_2	C_3
R_1	2,-1	4,2	2,0
R_2	3,3	0,0	1,1
R_3	1,2	2,8	5,1

   (a) What are the strategies that survive IESDS?

   Solution: *Strategy C_3 is strictly dominated by mixture $\frac{1}{2}C_1 + \frac{1}{2}C_2$; once C_3 is out, R_3 is strictly dominated by R_1. The answer is R_1, R_2, C_1, C_2.*

   (b) At each step of the elimination what were your rationality and knowledge assumptions?

   Solution: *At the first step we assume that player 2 is rational; at the second step we assume that player 1 is rational and knows that player 2 is rational.*

   (c) Find all Nash equilibria, including the mixed one.

   Solution: *By inspection, (R_1, C_2) and (R_2, C_1) are pure strategy Nash equilibria. To find the mixed one, assume that player 1 plays $\alpha R_1 + (1-\alpha)R_2$. Playing C_1 gives player 2 the payoff of $-\alpha + 3(1-\alpha) = 3 - 4\alpha$, while playing C_2 gives him 2α. He will be willing to mix if $\alpha = \frac{1}{2}$. Likewise, assume that player 2 plays $\beta C_1 + (1-\beta)C_2$. Playing R_1 gives player 1 the payoff of $2\beta + 4(1-\beta) = 4 - 2\beta$, while playing R_2 gives her 3β. She will be willing to randomize if $\beta = \frac{4}{5}$. So the mixed Nash equilibrium is $(\frac{1}{2}R_1 + \frac{1}{2}R_2, \frac{4}{5}C_1 + \frac{1}{5}C_2)$.*

2. Consider the following extensive form game with perfect information:

   (a) Find out the backwards induction outcome.

   Solution:

   i. *in the last stage of the game, player 1 chooses to play a.*

   ii. *in the previous stage, player 2 chooses to play L.*

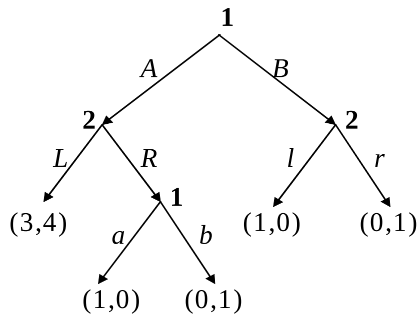

Figure 1:

iii. *if we'd been in right hand side part of the game (following a move to the right B in the first stage), player 2 would have chosen r.*

iv. *in the first stage of the game player 1 choses A.*

So the backward induction outcome of this game is "player 1 plays A, player 2 responds by playing L; if player 1 had played B in the first place, player 2 would have responded by playing r; if A and R had been played, player 1 would have responded by playing a".

(b) At each step of the backwards induction, what were your sequential rationality and knowledge assumptions?

Solution:

i. *we assume player 1 is sequentially rational and knows the history of the game (he knows he's on that particular node) and the payoffs.*

ii. *we just need to assume that player 2 is sequentially rational, knows the previous history of the game, and knows the payoffs. Then no matter what he expects player 1 to play subsequently, he would still play L. If on top of that, he knows that the other player is sequentially rational, all the better.*

iii. *we assume that player 2 is rational, knows the history of the game, and the payoffs.*

iv. *we need to assume that player 1 knows that player 2 is sequentially rational (so that he can expect player 2 not to go Right if he moves to A), and knows the rules of the game. We do not need to assume any higher order knowledge of rationality (namely we don't need to assume that player 1 knows that player 2 knows that he is sequentially rational).*

(c) Determine the strategies of each player and write out the corresponding normal form game.

Solution: *player 1 has 3 strategies: Aa, Ab and B. Most of you answered that he has 4 (Aa, Ab, Ba, Bb). It does not change your understanding of the game or any result, but you had some useless information: if player 1 moves to B, there is no way he will be asked to decide between a or b.*
Player 2 has 4 strategies: Ll, Lr, Rr and Rr.
The corresponding normal form game is:

	Ll	Lr	Rl	Rr
Aa	$(3,4)^*$	$(3,4)^*$	$(1,0)$	$(1,0)$
Ab	$(3,4)^*$	$(3,4)^*$	$(0,1)$	$(0,1)$
B	$(1,0)$	$(0,1)$	$(1,0)$	$(0,1)$

(d) Find all four pure strategy Nash equilibria.
Solution: *by inspection, we can see that the 4 pure strategy NE of this game are the one indicated with an asterix on the table. Note that only (Aa,Lr) is also a subgame perfect NE. The other equilibria aren't since they rely on the threats/promises to make moves that are irrational in case we go out of equilibrium. For instance, (Aa,Ll) is not subgame perfect, because it would imply that player 2 would play l as a response to B, which would be irrational.*

3. (Cournot quantity competition with prior technology choice) There are two competing firms $N = \{1, 2\}$ and two periods. Firm i has to invest $f(c_i) = (1 - c_i)^2$ in period 1, in order to adopt a technology where its marginal cost of production in period 2 will be $c_i \in [0, 1]$. Note that a technology with a lower marginal cost requires a higher initial investment. In period 2, the firms engage in Cournot quantity competition given their previously determined technologies.

Formally, in period 1, each firm i simultaneously chooses its marginal cost of production $c_i \in [0, 1]$. In period 2, firms learn each other's marginal costs and then each firm i simultaneously determines its quantity level $q_i \geq 0$. The net profit of firm i is:

$$\pi_i(q_i, q_j, c_i) = q_i[P(q_i + q_j) - c_i] - f(c_i) \qquad i \neq j$$

where market demand is given by $P(q_1 + q_2) = \max\{2 - (q_1 + q_2), 0\}$. (Note that the intercept of the demand is $a = 2$ not 1!) In parts (a) and (b), you are asked to compute the subgame perfect Nash equilibrium of this game.

(a) Given $c_1, c_2 \in [0, 1]$, what are the equilibrium quantity choices $q_1(c_1, c_2)$ and $q_2(c_1, c_2)$ in period 2?
Solution: Firm 1's profits are given by $\pi_1 = q_1[2 - (q_1 + q_2) - c_1] - f(c_1)$ (We ignore the possibility that $2 - (q_1 + q_2) < 0$. For it would then be possible for either firm to lower output and cut losses. Therefore it cannot happen in equilibirum).

Once c_1 and c_2 have been chosen, the optimal value of q_1 is given by $\frac{\partial \pi_1}{\partial q_1} = 0$
$\Longrightarrow q_1 = \frac{1}{2}(2 - q_2 - c_1)$
Similarly, the optimal value of q_2 is given by $\frac{\partial \pi_2}{\partial q_2} = 0 \Longrightarrow q_2 = \frac{1}{2}(2 - q_1 - c_2)$
The equilibrium quantity choices are given by the intersection of these two equations.
$\Longrightarrow q_1(c_1, c_2) = \frac{1}{3}(2 + c_2 - 2c_1)$, $q_2(c_1, c_2) = \frac{1}{3}(2 + c_1 - 2c_2)$

(b) Given that in period 2 the firms will set quantities according to (a), what are the equilibrium levels of marginal costs c_1 and c_2 in period 1? (Hint: First write down the equilibrium profit levels $\pi_1(q_1(c_1, c_2), q_2(c_1, c_2), c_1)$ and $\pi_2(q_1(c_1, c_2), q_2(c_1, c_2), c_2)$ as functions of c_1 and c_2 only.)

Solution: Substituting for quantities in the profit function of firm 1, using the expressions derived in part a, we obtain
$\pi_1(q_1(c_1, c_2), q_2(c_1, c_2), c_1) = \frac{1}{3}(2 + c_2 - 2c_1)[2 - \frac{1}{3}(2 + c_2 - 2c_1) - \frac{1}{3}(2 + c_1 - 2c_2) - c_1] - (1 - c_1)^2$
$= \frac{1}{9}(2 + c_2 - 2c_1)^2 - (1 - c_1)^2$
The optimal value for c_1 is given by $\frac{\partial \pi_1}{\partial c_1} = 0 \Longrightarrow 5c_1 = 5 - 2c_2$
Similarly, from firm 2's optimisation problem, we obtain $5c_2 = 5 - 2c_1$
Therefore, the equilibrium levels of marginal costs are given by the intersection of these two equations. $\Longrightarrow c_1 = c_2 = \frac{5}{7}$

(c) (Extra credit, take a shot at this only if you have extra time) Consider a modification of the above game where initially firm 1 determines its marginal cost c_1, then firm 2 observes c_1 and determines c_2, and finally firm 1 observes c_2 and the firms engage in simultaneous quantity competition as in above. What are the subgame perfect Nash equilibrium strategies and the equilibrium levels of marginal costs and quantities in this case?

Solution: To compute the subgame perfect Nash equilibrium strategies, we use backward induction. The final stage of the game, where the two firms simultaneously choose quantities is exactly as in the original setting considered in part a. Therefore, the optimal strategies for the final stage of the game are given by $q_1(c_1, c_2) = \frac{1}{3}(2 + c_2 - 2c_1)$, $q_2(c_1, c_2) = \frac{1}{3}(2 + c_1 - 2c_2)$
.

We plug in these expressions into the profit function of firm 2 to compute the optimal value of c_2 as a function of c_1.
Mechanically, this problem is exactly the same as that solved in part b. Hence we obtain $c_2(c_1) = 1 - \frac{2}{5}c_1$
Next, we substitute for quantities and the marginal cost of firm 2 in the profit function of firm 1 using the expressions above to obtain
$\pi_1(c_1) = \frac{1}{3}[2 + (1 - \frac{2}{5}c_1) - 2c_1][2 - \frac{1}{3}\{2 + (1 - \frac{2}{5}c_1) - 2c_1\} - \frac{1}{3}\{2 + c_1 - 2(1 - \frac{2}{5}c_1)\} - c_1] - (1 - c_1)^2$

$= \frac{1}{9}[2 + (1 - \frac{2}{5}c_1) - 2c_1]^2 - (1 - c_1)^2 = (1 - \frac{4}{5}c_i)^2 - (1 - c_i)^2$

The optimal value for c_1 is given by $\frac{\partial \pi_1}{\partial c_1} = 0 \Longrightarrow 2(1 - \frac{4}{5}c_1) + 2(1 - c_1) = 0 \Longrightarrow c_1 = \frac{5}{9}$

We can now write down the subgame perfect Nash equilibrium strategies:

Firm 1 chooses $c_1 = \frac{5}{9}$ and $q_1(c_1, c_2) = \frac{1}{3}(2 + c_2 - 2c_1)$

Firm 2 chooses $c_2(c_1) = 1 - \frac{2}{5}c_1$ and $q_2(c_1, c_2) = \frac{1}{3}(2 + c_1 - 2c_2)$.

Therefore, the equilibrium outcome is as follows:

$c_1 = \frac{5}{9}$, $c_2 = \frac{7}{9}$, $q_1 = \frac{5}{9}$, $q_2 = \frac{1}{3}$