

In the Name of God

Sharif University of Technology

Graduate School of Management and Economics

Microeconomics (for MBA students)

44111 (1393-94 1st term) - Group 2

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Game Theory

Game: A situation in which intelligent decisions are necessarily interdependent. A situation where utility (payoff) of an individual (player) depends upon her own action, but also upon the actions of other agents.

How does a rational individual choose?

My optimal action depends upon what my opponent does but his optimal action may in turn depend upon what I do

Player: A rational agent who participates in a game and tries to maximise her payoff.

Strategy (action): An action which a player can choose from a set of possible actions in every conceivable situation.

Strategy profile: a vector of strategies including one strategy for each player.

Payoff: The utility (payoff) that a player receives depending on the strategy profile chosen. Players need not be concerned only with money could be altruistic, or could be concerned not to violate a norm.

Each player seeks to maximize his expected payoff (in short, is **rational**).

Furthermore he knows that every other player is also rational, and knows that every other player knows that every player is rational and so on (**rationality is common knowledge**).

Pure Strategy: a strategy which consists of one single chosen by player for sure.

Mixed Strategy: a strategy which consists of a subset of the strategy set chosen by a player each with a probability.

Strategic form of a simultaneous move game:

		Column Player	
		C	D
Row Player	A	u_{AC}^r, u_{AC}^c	
	B		

u_{AC}^r is utility (payoff) of the row player in case the strategy profile AC were played.

Example: Suppose the demand in a duopoly market is

$$Q = 100 - 10P$$

Two firms operating in this market producing homogenous goods and with a constant marginal cost of zero. Firms should simultaneously decide to produce either 25 (collusion strategy) or 35 (deviation strategy). It is easy to follow that each firm's profit can be calculated by:

$$\pi_i(q_i, q_j) = \left(10 - \frac{q_i + q_j}{10}\right) q_i$$

Then we can calculate the payoff of each player for each strategy profile.

e.g.

$$\pi_A(25,35) = \left(10 - \frac{25 + 35}{10}\right) 25 = 100$$

		Firm B	
		C $q_B = 25$	D $q_B = 35$
		125 , 125	100 , 140
Firm A	C $q_A = 25$	125 , 125	100 , 140
	D $q_A = 35$	140 , 100	105 , 105

Now let's consider firm A's decision:

If firm B chooses strategy **C**, what is **the best response** by firm A?

A Can choose **C** and end up with profit of 125 or chooses **D** and make a profit of 140. Then the best response to **C** played by B is **D**.

If we continue to use the notation of best responses:

		Firm B	
		C	D
		$q_B = 25$	$q_B = 35$
Firm A	C	125 , 125	100 , <u>140</u>
	D	<u>140</u> , 100	<u>105</u> , <u>105</u>

Then firm A would be better off by choosing D regardless of the action of firm B. It is called a strictly dominant strategy.

Dominant strategy: where a player would be strictly better off by choosing one strategy independent of those chosen by others.

Prisoners' Dilemma

		Prisoner 2	
		Confess	Do not Confess
Prisoner 1	Confess	-6 , -6	0 , -9
	Do not Confess	-9 , 0	-1 , -1

Both prisoners will confess despite the fact that both will be better off if keep their mouths shut.

Iterated strict dominance

- 1) Eliminate strictly dominated strategies for all players. Left with reduced game, with each player having fewer strategies.
- 2) In the reduced game, eliminate strictly dominated strategies for all players ... further reduced game.
- 3) Repeat, till no further elimination is possible.

		Column		
		L	M	R
Row	U	1 , 0	1 , 2	0 , 1
	D	0 , 3	0 , 1	2 , 0

Return to the example of duopoly and consider the following possible strategies for two firms:

Price for each strategy profile		Firm B		
		$q_B = 25$	$q_B = 35$	$q_B = 45$
Firm A	$q_A = 15$	6	5	4
	$q_A = 25$	5	4	3
	$q_A = 35$	4	3	2

And the game:

		Firm B			
		$q_B = 25$	$q_B = 35$	$q_B = 45$	
Firm A		$q_A = 15$	90 , 150	75 , 175	60 , 180
		$q_A = 25$	125 , 125	100 , 140	75 , 135
		$q_A = 35$	140 , 100	105 , 105	70 , 90

As it can easily be checked the game does not have a strictly dominant strategy for each of the players. Now try the iteration of the strictly dominated strategies.

First we can see that $q_A = 15$ is strictly dominated by $q_A = 25$, then it never is going to be chosen by player A. The same is true for $q_B = 25$.

Then the game is reduced to

		Firm B		
		$q_B = 25$	$q_B = 35$	$q_B = 45$
Firm A	$q_A = 15$	90, 150	75, 175	60, 180
	$q_A = 25$	125, 125	100, 140	75, 135
	$q_A = 35$	140, 100	105, 105	70, 90

Now it can be seen that $q_A = 25$ is strictly dominated by $q_A = 35$ and $q_B = 35$ is a dominant strategy for player B.

Weakly Dominance:

The money sharing game:

		B	
		Share	Grab
		M/2 , M/2	0 , M
A	Share	M/2 , M/2	0 , M
	Grab	M , 0	0 , 0

Then player A would at least as good as by choosing **Grab** regardless of the action of player B. It is called a weakly dominant strategy.

In other words, for player A; **Share** is weakly dominated by **Grab**.

Nash Equilibrium: A strategy profile is a Nash Equilibrium if every player's strategy in that profile is a best response to other players strategies.

		Prisoner 2	
		Confess	Do not Confess
Prisoner 1	Confess	-6 , -6	0 , -9
	Do not Confess	-9 , 0	-1 , -1

If prisoner 1 chooses to **Confess** the BR by 2 is to **Confess**.

If prisoner 1 chooses **Not to Confess** the BR by 2 is **Not to Confess**.

Because the game is symmetric it is the same for prisoner 1's BR.

Then the only Nash Eq. Of the game is (**Confess , Confess**).

Battle of Sexes: Two players are to choose simultaneously whether to go to the cinema or theatre. They have different preferences, but they both would prefer to be together rather than go on their own.

		Wife	
		Cinema	Theatre
Husband	Cinema	2 , 1	0 , 0
	Theatre	0 , 0	1 , 2

If husband chooses to go to **Cinema**, the BR by wife is to **Cinema**.

If husband chooses **Theatre**, the BR by wife is **Theatre**.

Because the game is symmetric it is the same for husband's BR.

Two Nash Eq. for this game: (**Cinema , Cinema**) & (**Theatre , Theatre**).

Note: Nash equilibrium is always a profile of strategies, one for each player.

Note: A strictly dominated strategy cannot be part of any Nash equilibrium since it is not the best response to any of other players' strategies.

Hawk-Dove game: Also known as chicken game.

		Driver 2	
		Swerve	Straight
		Swerve	0 , 0
Driver 1	Swerve	0 , 0	<u>-1</u> , <u>2</u>
	Straight	<u>2</u> , <u>-1</u>	-5 , -5

NE: (Sw , St) and (St , Sw).

Game is symmetric but the pure NE is asymmetric.

The money-sharing game:

		B	
		Share	Grab
		M/2 , M/2	0 , <u>M</u>
A	Share	<u>M</u> , 0	0 , 0
	Grab	0 , 0	0 , 0

Monetary payoffs in Shafted or Goldenballs (Played on TV show in UK, Netherlands, Australia, USA).

The best response to **S** is **G**. The best response to **G** is either **G** or **S**. The game has 3 Nash Eq. (Note: **S** is a weakly dominated Strategy)

In US and UK about 50% choose to share. In Netherlands, 38% share. Why?

Cournot Oligopoly:

Two firms, 1 and 2, producing a homogeneous good with inverse demand function of $P = 10 - \frac{1}{10}Q$.

They choose quantity $q_i \geq 0$ simultaneously. For simplicity suppose marginal costs are zero.

Total quantity $Q = q_1 + q_2$ is placed on the market and determines the price

Firm 1's Profit is:

$$\pi_1 = q_1 \cdot P = q_1(10 - 0.1q_1 - 0.1q_2) = 10q_1 - 0.1q_1^2 - 0.1q_1q_2$$

Suppose firm 2 fixes his production level at \hat{q}_2 ; then the best response by firm 1 should satisfy the first order condition:

$$\frac{\partial \pi_1}{\partial q_1} = 0 \quad \Rightarrow \quad 10 - 0.2q_1 - 0.1\hat{q}_2 = 0 \quad \text{or} \quad q_1 = 50 - 0.5\hat{q}_2$$

Then the BR function for firm 1 is: $q_1 = 50 - 0.5\hat{q}_2$

And similarly for firm 2: $q_2 = 50 - 0.5\hat{q}_1$

To find the NE we have to set $q_2 = \hat{q}_2(q_1)$

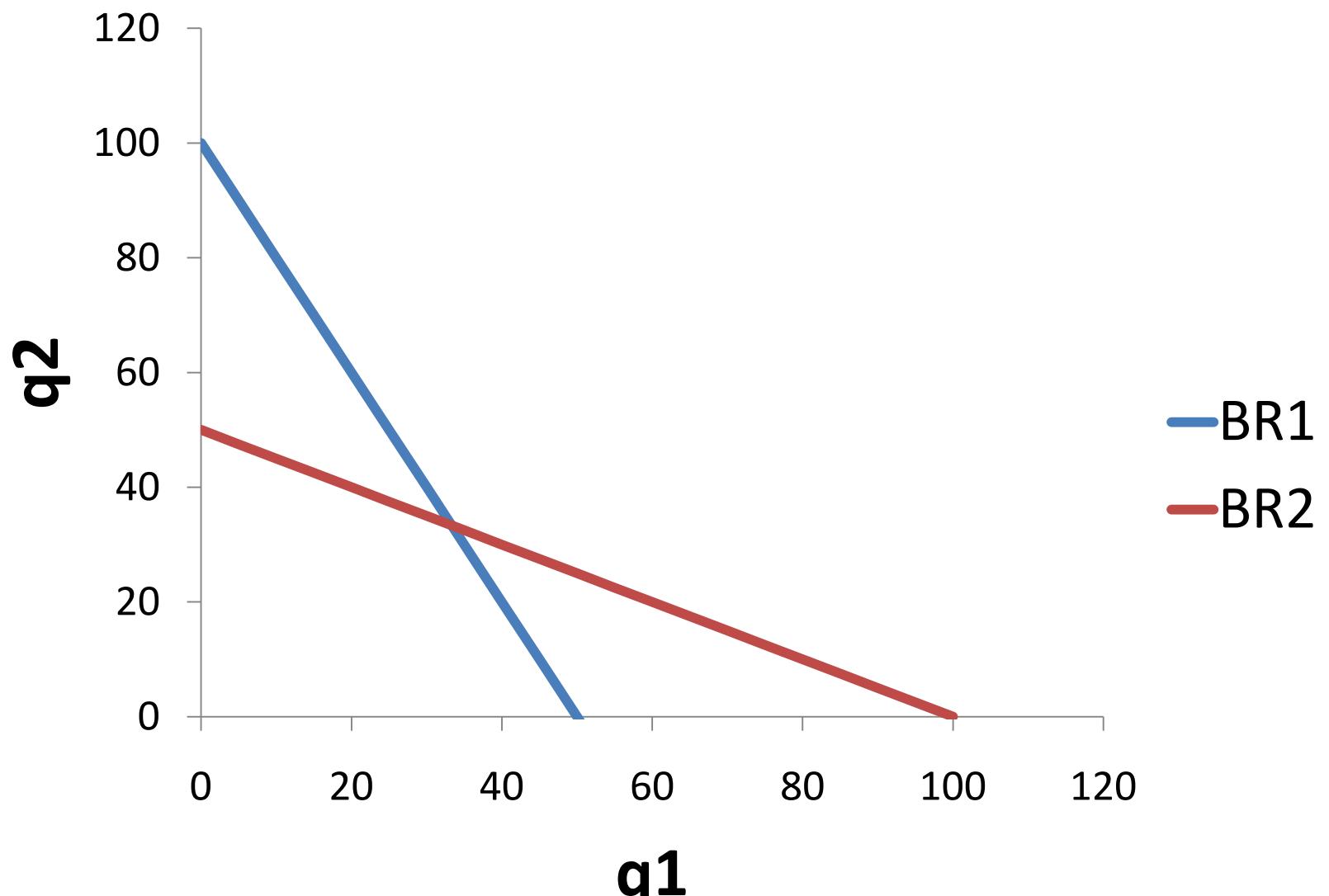
$$q_2 = 50 - 0.5(50 - 0.5q_2)$$

And the only NE is:

$$q_1^C = q_2^C = \frac{100}{3}$$

Remember the monopoly quantity is: $q^M = 50$.

Easy to calculate that: $P^M = 5$, $\pi^M = 250$ and $P^C = \frac{10}{3} = 3.33$, $\pi_1^C = \pi_2^C = 111.1$



Cournot Oligopoly with N firms:

It can be shown that for our example with N firms we have:

$$q_i^C = \frac{100}{N + 1}$$

$$P^C = \frac{10}{N + 1}$$

If the market was a perfectly competitive market we had

$$q_i^C \rightarrow 0 \quad \text{and} \quad P^C = 0$$

Which is the same as when in Cournot setting $N \rightarrow \infty$

Bertrand Oligopoly:

Same context, but firms choose prices. Prices can be continuously varied, i.e. p_i is any real number. Firms have the same marginal cost of mc .

If prices are unequal, all consumers go to lower price firm.

If equal, market is shared.

- 1) There is a Nash equilibrium where $p_1 = p_2 = mc$: None of the firms has incentive to deviate from this strategy.
- 2) There is no other Nash equilibrium in pure strategies.

Note: With just two firms we had the same outcome as a perfectly competitive market in Bertrand competition.

Why should we believe that players will play a Nash equilibrium?

- a) Self enforcing agreement: If the players reach an agreement about how to play the game, then a necessary condition for the agreement to hold up is that the strategies constitute a Nash equilibrium. Otherwise, at least one player will have an incentive to deviate.
- b) Any prediction about the outcome of a non-cooperative game is self-defeating if it specifies an outcome that is not a Nash equilibrium.
- c) Result of process of adaptive learning: Suppose players play a game repeatedly. The idea is that a reasonable learning process, if it converges, should converge to a Nash equilibrium.

2nd Price Sealed-Bid Auction:

Suppose there are n bidders competing for one unit of an indivisible good.

Each player (i) has a valuation (v_i) for the good which is independently drawn from a distribution.

Players place their bids (b_i) simultaneously.

The winner is the player with the highest bid and pays the second highest bid. (for simplicity assume $b_i \neq b_j \therefore \forall i \neq j$)

Then the payoff of each player can be written as:

$$\pi_i = \begin{cases} v_i - \max_{j \neq i} b_j & b_i = \max_j b_j \\ 0 & b_i \neq \max_j b_j \end{cases}$$

It is a weakly dominant strategy for each player to bid his true valuation.

In order to make an easier representation suppose we are considering the i^{th} player and we define $\bar{b}_{-i} = \max_{j \neq i} b_j$ which the highest bid for all other players.

The payoff function can be rewritten as (this time including the possibility of having a tie):

$$\pi_i = \begin{cases} v_i - \bar{b}_{-i} & b_i > \bar{b}_{-i} \\ 1/m(v_i - \bar{b}_{-i}) & b_i = \bar{b}_{-i} \\ 0 & b_i < \bar{b}_{-i} \end{cases}$$

For a m -way tie ($m-1$ other players bid exactly b_i which is the highest bid) each highest bidder has $1/m$ chance of obtaining the object.

First, we show that for player i bidding $b_i < v_i$ is weakly dominated by bidding $\hat{b}_i = v_i$:

Table below shows the payoff for player i , fixing the actions of other players:

	$\bar{b}_{-i} < b_i < v_i$	$\bar{b}_{-i} = b_i < v_i$	$b_i < \bar{b}_{-i} < v_i$	$b_i < \bar{b}_{-i} = v_i$	$b_i < v_i < \bar{b}_{-i}$
$b_i < v_i$	$v_i - \bar{b}_{-i}$	$\frac{1}{m}(v_i - \bar{b}_{-i})$	0	0	0
$\hat{b}_i = v_i$	$v_i - \bar{b}_{-i}$	$v_i - \bar{b}_{-i}$	$v_i - \bar{b}_{-i}$	0	0

Then it is clear bidding his true valuation is weakly dominant action to bidding anything less than that.

Second, we show that for player i bidding $b_i > v_i$ is weakly dominated by bidding $\hat{b}_i = v_i$:

Table below shows the payoff for player i , fixing the actions of other players:

	$\bar{b}_{-i} < v_i < b_i$	$\bar{b}_{-i} = v_i < b_i$	$v_i < \bar{b}_{-i} < b_i$	$v_i < \bar{b}_{-i} = b_i$	$v_i < b_i < \bar{b}_{-i}$
$b_i > v_i$	$v_i - \bar{b}_{-i}$	$v_i - \bar{b}_{-i} = 0$	$v_i - \bar{b}_{-i} < 0$	0	0
$\hat{b}_i = v_i$	$v_i - \bar{b}_{-i}$	$\frac{1}{m}(v_i - \bar{b}_{-i}) = 0$	0	0	0

Then it is clear bidding his true valuation is weakly dominant action to bidding anything more than that.

From this two, we can conclude that bidding his true valuation is a weakly dominant action for every player regardless of actions by other players.

The important issue here is that this mechanism is a truth revealing mechanism. Players would reveal their true valuation for the object when this mechanism is used.

It can be shown that for example in the 1st sealed-bid auctions players have incentive to bid different from their true valuation.

2nd price sealed-bid auction is equivalent to an English auction when players valuations and bidding strategy are not affected by bidding behaviour of their rivals.

Sequential Games

When players move sequentially or repeat a simultaneous game.

The concept of equilibrium used here is sub-game perfect equilibrium; which consists of a NE for each sub-game.

We talk in class about the definition of sub-game and sub-game perfect equilibrium in class in more details.

Repeated Prisoners' Dilemma

Consider the following game:

A duopolistic market; firms have marginal cost of zero.

Firms can choose to either price high $P_H = 2$ or price low $P_L = 1$.

The demand is 4 units when the price is P_H and 6 units when the price is P_L .

Then the strategic form of the game is:

		Firm 2	
		P_H	P_L
		P_H	$4, 4$
Firm 1	P_H	$4, 4$	$0, 6$
	P_L	$6, 0$	$3, 3$

The game is a prisoners' dilemma and the Nash Eq. of the game is when both firms choose P_L , However by committing to P_H both can benefit.

Basic lesson of prisoner's dilemma:

In one-shot interaction, individual's have incentive to behave opportunistically

Leads to socially inefficient outcomes

What happens if interaction is repeated?

If you cooperate today, I will reward you tomorrow (by cooperating)

If you defect today, I will punish you tomorrow (by defecting)

This provides an incentive to cooperate today

Under what conditions does this argument work?

Finitely Repeated Prisoners' Dilemma

Suppose the duopoly game we introduced, is repeated T times.

Each time exactly with the same characteristics.

Another notion of equilibrium is needed for repeated games which is called sub-game perfect equilibrium.

A strategy profile for a repeated game is a SPE if it includes a NE for every stage.

In period T :

Both firms have incentive to defect and choose P_L ; since there is no prospect of further cooperation.

Then in period T they both choose P_L .

In period $T - 1$:

Firms know that (P_L, P_L) is the only NE for the next stage.

Then there would be no chance cooperation in the next stage.

It is again NE for both to play (P_L, P_L)

Then (P_L, P_L) is played in every stage of the game whatever the history.

The outcome is similar to one-shot game.

Infinitely Repeated Prisoners' Dilemma

Suppose the game is infinitely repeated

There is no final stage then the argument which made for finite case is useless.

In order to calculate the present value of future payoffs we define the discount factor $0 < \delta < 1$.

The today's value of \$1 of tomorrow's income is δ .

If the game was repeated twice and player i receives payoffs of π_i^1 and π_i^2 , the present value of this payoff can be written as:

$$\pi_i = \pi_i^1 + \delta\pi_i^2$$

Strategy for player i must specify action in period t for any history

Sub-game perfect equilibrium strategy profile must be a Nash equilibrium

Let's investigate whether the following strategy is a SPE:

Each players decides according to this rule in each period:

Play P_H in this stage if (P_H, P_H) is played in the previous stage or if it is the first period.

Play P_L for any other history.

Now if we can show that none of the firms have incentive to deviate from this strategy we have a SPE:

		Firm 2	
		P_H	P_L
		P_H	$4, 4$
Firm 1	P_H	$4, 4$	$0, 6$
	P_L	$6, 0$	$3, 3$

At any given period; fixing the action of firm 2 and check for firm 1:

Cooperation payoff:

$$\pi_1^C = 4 + 4\delta + 4\delta^2 + \dots$$

Deviation payoff:

$$\pi_1^D = 6 + 3\delta + 3\delta^2 + \dots$$

Note: Payoffs well defined since $0 < \delta < 1$

We have to check whether deviation can lead to higher payoffs?

$$\pi_1^D > \pi_1^C$$

$$6 + 3\delta + 3\delta^2 + \dots > 4 + 4\delta + 4\delta^2 + \dots$$

$$6 + 3(\delta + \delta^2 + \dots) > 4(1 + \delta + \delta^2 + \dots)$$

$$6 + 3\left(\frac{\delta}{1-\delta}\right) > 4\left(\frac{1}{1-\delta}\right)$$

$$6 > \left(\frac{4 - 3\delta}{1 - \delta}\right)$$

$$\delta < \frac{2}{3}$$

This results means if $\delta < \frac{2}{3}$, Then firms have incentive to deviate from cooperation.

Alternatively if firms are patient enough $\delta > \frac{2}{3}$, they will cooperate.

It is trivial that there is no need to check the deviation incentive after other histories.

Other alternatives also can be studied:

e.g. players punish the defected firm for only a given number of periods or tit for tat strategy.