Problem 1 *(Existence of Pure Strategy Nash Equilibrium)*

Note that a function $u_i(s)$ is *upper semi-continuous* at s, if, for any sequence s^n converging to s, $\limsup_{n\to\infty} u_i(s^n) \le u_i(s)$. Note also that a function u_i has a *continuous maximum* if $u_i^*(s_{-i}) \equiv \max_{s_i} u_i(s_i, s_{-i})$ is continuous in s_{-i}.

Let S_i be a nonempty, convex, and compact subset of a finite-dimensional Euclidean space, for all i. Assume, for all i, that u_i is quasiconcave in s_i, is upper semi-continuous in s, and has a continuous maximum. Show that there exists a pure strategy Nash equilibrium.

Problem 2 *(Free-riding)*

Consider the following game-theoretic model of the equilibrium determination of the level of effort team members put into a team project, with team size of 2. In this game, the two team members simultaneously choose the level of effort, e_1, e_2, to spend on the project. They each get utility from progress on the project (which is a function of the sum of the efforts) and disutility from the effort they personally expend. Player 1, values the achievement on the project more. Specifically, assume that e_1 and e_2 are chosen from the set of nonnegative real numbers and that

$$u_1(e_1, e_2) = k\log(e_1 + e_2) - e_1,$$
$$u_2(e_1, e_2) = \log(e_1 + e_2) - e_2,$$

where $k > 1$.

(a) Find best response functions for each of the players.

(b) Find the pure strategy Nash equilibria of the game. How does the distribution of effort in equilibrium reflect the difference in the player's preferences?

(c) Try to write down a modification of the model above in which the outcome is more fair.

Problem 3 *(Correlated Equilibrium)*

Consider the finite game

	L	R
U	10, 10	2, 14
D	12, 4	0, 0

(a) Let Δ be a probability distribution on $S_1 \times S_2$. For example, $\Delta(U, L)$ is the probability (U,L) is played. Write out the full set of constraints that Δ must satisfy in order to be a correlated equilibrium.

(b) Find a correlated equilibrium in which the sum of the players' payoff is greater than the sum of the payoffs in any Nash equilibrium of G.

(c) What is the largest possible value for the sum of the players' payoff in a correlated equilibrium of G.

Problem 4 *(Bertrand Competition with Mixed Strategy Nash Equilibrium)*

Consider a market with 2 firms which produce the same good. Firms 1 and 2 simultaneously choose price p_1 and p_2 from the set of nonnegative real numbers. Assume that there are no costs of production. If $p_i < p_j$ then firm i gets demand $D(p_i)$ and firm j gets demand 0. The demand function takes value of $D(p) = \frac{1}{\sqrt{1+p}}$. If the firms charge equal prices, assume that demand is split evenly. While it only has one pure strategy Nash equilibrium, the model above actually has an infinite number of mixed strategy Nash equilibria. Show by direct construction that the model has a symmetric mixed strategy Nash equilibrium where the players choose strategies from a distribution with full support on $[10, \infty)$. What is interesting about the set up of this model that lets the firm put positive weight on price range that is arbitrarily large?

Problem 5 *(Hotelling Competition)*

Each of n candidates chooses a position to take on the real line in the interval [0,1]. There is a continuum of citizens, whose favorite positions are uniformly distributed between [0,1]. A candidate attracts votes of citizens whose favorite positions are closer to his position than to the position of any other candidate; if k candidates choose the same position, then each receives the fraction $1/k$ of the votes that the position attracts. The payoff of each candidate is his vote share.

(a) Find all pure strategy Nash equilibria when $n = 2$.

(b) Show that there does not exist a pure strategy Nash equilibrium when $n = 3$. Find a mixed strategy Nash equilibrium.

Problem 6 *(Pricing-Congestion Game)* **[Bonus]**

Consider the following pricing-congestion game as presented in Lecture 5. In a two link network, let the total flow be $d = 1$, reservation utility be $R = 1$. Assume the latency functions are given by

$$l_1(x) = 0, \quad l_2(x) = \begin{cases} 0, & \text{if } 0 \le x \le 1/2, \\ \frac{x - 1/2}{\epsilon}, & \text{if} x \ge 1/2, \end{cases} \tag{1}$$

for some $\epsilon > 0$. Construct a mixed strategy Nash Equilibrium when link 2 has a capacity constraint, i.e. $\epsilon \to 0$, $l_2(x) = \infty$ when $x > \frac{1}{2}$.

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