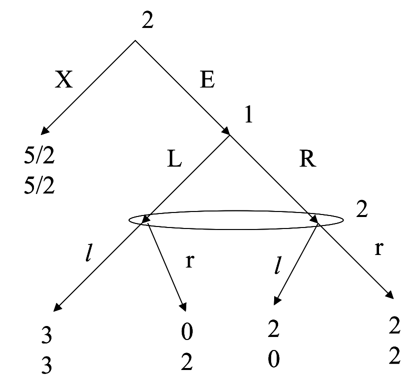
14.12 Game Theory – Midterm II

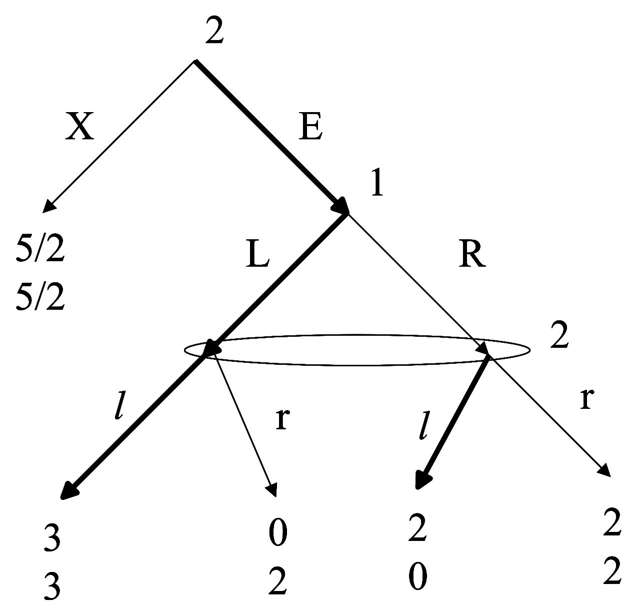
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

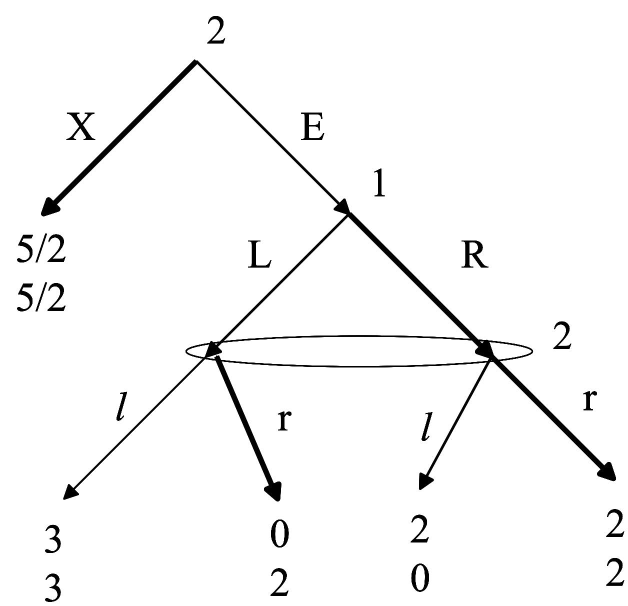
1. Consider the following game:



Compute all the pure-strategy subgame-perfect equilibria. Use a forward induction argument to eliminate one of these equilibria.

Answer: There are two pure strategy Nash equilibria in the proper subgame, yielding subgame-perfect equilibria:





For player 2, Er is strictly dominated by Xr, while El is not dominated. Hence, if player 1 keeps believing that 2 is rational whenever it is possible, then when he sees that 2 played E, he ought to believe that 2 plays strategy El — not the dominated strategy Ex. In that case, 1 would play L, and 2 would play E. Therefore, the equilibrium on the left is eliminated.

2. Below, there are pairs of stage games and strategy profiles. For each pair, check whether the strategy profile is a subgame-perfect equilibrium of the game in which the stage game is repeated infinitely many times. Each agent tries to maximize the discounted sum of his expected payoffs in the stage game, and the discount rate is $\delta = 0.99$.

   (a) **Stage Game:**

1\2	L	M	R
T	2,-1	0,0	-1,2
M	0,0	0,0	0,0
B	-1,2	0,0	2,-1

   Strategy profile: Until some player deviates, player 1 plays T and player 2 alternates between L and R. If anyone deviates, then each play M thereafter.

   Answer: It is subgame perfect. Since (M,M) is a Nash equilibrium of the stage game, we only need to check if any player wants to deviate when player 1 plays T and player 2 alternates between L and R. In this regime, the present value of player 1's payoffs is

$$V_{1L} = \frac{2}{1-\delta} - \frac{\delta}{1-\delta} = \frac{2-\delta}{1-\delta} > 0$$

   when 2 is to play L and

$$V_{1R} = \frac{2\delta}{1-\delta} - \frac{1}{1-\delta} = \frac{2\delta - 1}{1-\delta} = 98$$

   when 2 is to play R. When 2 plays L, 1 cannot gain by deviating. When 2 plays R, the best 1 gets by deviating is

$$2 + 0 < 98$$

   (when he plays B). The only possible profitable deviation for player 2 is to play R when he is supposed to play left. In that contingency, if he follows the strategy he gets $V_{1R} = 98$; if he deviates, he gets $2 + 0 < V_{1R}$.

   (b) **Stage Game:**

1\2	A	B
A	2,2	1,3
B	3,1	0,0

   Strategy profile: The play depends on three states. In state S_0, each player plays A; in states S_1 and S_2, each player plays B. The game start at state S_0. In state S_0, if each player plays A or if each player plays B, we stay at S_0, but if a player i plays B while the other is playing A, then we switch to state S_i. At any S_i, if player i plays B, we switch to state S_0; otherwise we state at state S_i.

   Answer: It is not subgame-perfect. At state S_2, player 2 is to play B, and we will switch to state S_0 no matter what 1 plays. In that case, 1 would gain by deviating and playing A (in state S_2).

3. Consider the following first-price, sealed-bid auction where an indivisible good is sold. There are $n \geq 2$ buyers indexed by $i = 1, 2, \ldots, n$. Simultaneously, each buyer i submits a bid $b_i \geq 0$. The agent who submits the highest bid wins. If there are $k > 1$ players submitting the highest bid, then the winner is determined randomly among these players — each has probability $1/k$ of winning. The winner i gets the object and pays his bid b_i, obtaining payoff $v_i - b_i$, while the other buyers get 0, where $v_1, \ldots, v_n$ are independently and identically distributed with probability density function f where

$$f(x) = \begin{cases} 3x^2 & x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the symmetric, linear Bayesian Nash equilibrium.

Answer: We look for an equilibrium of the form

$$b_i = a + cv_i$$

where $c > 0$. Then, the expected payoff from bidding b_i with type v_i is

$$\begin{aligned} U(b_i; v_i) &= (v_i - b_i) \Pr(b_i > a + cv_j \quad \forall j \neq i) \\ &= (v_i - b_i) \prod_{j \neq i} \Pr(b_i > a + cv_j) \\ &= (v_i - b_i) \prod_{j \neq i} \Pr\left(v_j < \frac{b_i - a}{c}\right) \\ &= (v_i - b_i) \prod_{j \neq i} \left(\frac{b_i - a}{c}\right)^3 \\ &= (v_i - b_i) \left(\frac{b_i - a}{c}\right)^{3(n-1)} \end{aligned}$$

for $b_i \in [a, a + c]$. The first order condition is

$$\frac{\partial U(b_i; v_i)}{\partial b_i} = -\left(\frac{b_i - a}{c}\right)^{3(n-1)} + 3(n-1)\frac{1}{c}(v_i - b_i)\left(\frac{b_i - a}{c}\right)^{3(n-1)-1} = 0;$$

i.e.,

$$-\left(\frac{b_i - a}{c}\right) + 3(n-1)\frac{1}{c}(v_i - b_i) = 0;$$

i.e.,

$$b_i = \frac{a + 3(n-1)v_i}{3(n-1)+1}.$$

Since this is an identity, we must have

$$a = \frac{a}{3(n-1)+1} \Longrightarrow a = 0,$$

and

$$c = \frac{3(n-1)}{3(n-1)+1}.$$

(b) What happens as $n \to \infty$?

Answer: As $n \to \infty$,

$$b_i \to v_i.$$

In the limit, each bidder bids his valuation, and the seller extracts all the gains from trade.

[**Hint:** Since $v_1, v_2, \ldots, v_n$ is independently distributed, for any $w_1, w_2, \ldots, w_k$, we have

$$\Pr(v_1 \leq w_1, v_2 \leq w_2, \ldots, v_k \leq w_k) = \Pr(v_1 \leq w_1) \Pr(v_2 \leq w_2) \ldots \Pr(v_k \leq w_k).]$$

4. This question is about a thief and a policeman. The thief has stolen an object. He can either hide the object INSIDE his car on in the TRUNK. The policeman stops the thief. He can either check INSIDE the car or the TRUNK, but not both. (He cannot let the thief go without checking, either.) If the policeman checks the place where the thief hides the object, he catches the thief, in which case the thief gets -1 and the police gets 1; otherwise, he cannot catch the thief, and the thief gets 1, the police gets -1.

(a) Compute all the Nash equilibria.

Answer: This is a matching-pennies game. There is a unique Nash equilibrium, in which Thief hides the object INSIDE or the TRUNK with equal probabilities, and the Policeman checks INSIDE or the TRUNK with equal probabilities.

(b) Now imagine that we have 100 thieves and 100 policemen, indexed by $i = 1, \ldots, 100$, and $j = 1, \ldots, 100$. In addition to their payoffs above, each thief i gets extra payoff b_i form hiding the object in the TRUNK, and each policeman j gets extra payoff d_j from checking the TRUNK. We have

$$\begin{aligned} b_1 &< b_2 < \cdots < b_{50} < 0 < b_{51} < \cdots < b_{100}, \\ d_1 &< d_2 < \cdots < d_{50} < 0 < d_{51} < \cdots < d_{100}. \end{aligned}$$

Policemen cannot distinguish the thieves from each other, nor can the thieves distinguish the policemen from each other. Each thief has stolen an object, hiding it either in the TRUNK or INSIDE the car. Then, each of them is randomly matched to a policeman. Each matching is equally likely. Again, a policeman can either check INSIDE the car or the TRUNK, but not both. Compute a pure-strategy Bayesian Nash equilibrium of this game.

Answer: A Bayesian Nash equilibrium: A thief i hides the object in

$$\begin{array}{ll} \text{INSIDE} & \text{if } b_i < 0 \\ \text{TRUNK} & \text{if } b_i > 0; \end{array}$$

a policeman j checks

$$\begin{array}{ll} \text{INSIDE} & \text{if } d_j < 0 \\ \text{TRUNK} & \text{if } d_j > 0. \end{array}$$

This is a Bayesian Nash equilibrium, because, from the thief's point of view the policeman is equally likely to to check TRUNK or INSIDE the car, hence it is the best response for him to hide in the trunk iff the extra benefit from hiding in the trunk is positive. Similar for the policemen.