# ECONS 491
## STRATEGY AND GAME THEORY
## HOMEWORK #2 – ANSWER KEY

1. One of the critical moments early on in the *The Lord of the Rings* trilogy is the meeting in Rivendale to decide who should take the ring to Mordor. Gimli the dwarf won't hear of an elf doing it, while Legolas (who is an elf) feels similarly about Gimli. Boromir (who is a man) is opposed to either of them taking charge of the ring. He is also held in contempt, for it was his ancestor who, when given the opportunity to destroy the ring millennia ago, chose to keep it instead. And then there is Frodo the hobbit, who has the weakest desire to take the ring, but knows that someone must throw it into the fires of Mordor. In modeling this scenario as a game, assume there are four players: Boromir, Frodo, Gimli, and Legolas. (There were more, of course, including Aragorn and Elrond, but let's keep it simple.) Each of them has a preference ordering, shown in the following table, as to who should take on the task of carrying the ring.

**PREFERENCE RANKINGS FOR *THE LORD OF THE RINGS***

| Person | First | Second | Third | Fourth | Fifth |
|---|---|---|---|---|---|
| Boromir | Boromir | Frodo | No one | Legolas | Gimli |
| Gimli | Gimli | Frodo | No one | Boromir | Legolas |
| Legolas | Legolas | Frodo | No one | Gimli | Boromir |
| Frodo | Legolas | Gimli | Boromir | Frodo | No one |

Of the three nonhobbits, each prefers to have himself take on the task. Other than themselves and Frodo, each would prefer that no one take the ring. As for Frodo, he doesn't really want to do it and prefers to do so only if no one else will. The game is one in which all players simultaneously make a choice among the four people. Only if they all agree—a unanimity voting rule is put in place—is someone selected; otherwise, no one takes on this epic task. Find all symmetric Nash equilibria.

**ANSWER:** There are four symmetric strategy profiles and thus four candidates for symmetric Nash equilibrium. Note that if a person fails to vote for the person that everyone else votes for, then no one takes on the task. Thus, at a symmetric strategy profile, an individual player's choice is always between the person who the others are voting for and no one. Consider the symmetric strategy profile in which all vote for Boromir. This strategy is optimal for both Boromir and Frodo as each would rather that Boromir take on the task than that no one do so. This is clearly not a Nash equilibrium, however, as both Legolas and Gimli would prefer to vote for someone else, and the result would be that no one takes the ring to Mordor. By a similar argument, it is not a Nash equilibrium for all to vote for Gimli, or for all to vote for Legolas. Now consider all voting for Frodo. Since each person prefers that Frodo do it than that no one do it, each player's strategy is optimal. The unique symmetric Nash equilibrium is then for all to vote for Frodo.

<u>Ex. 1 – Harrington</u>

1) $u_i(Boromir_i, Boromir_{-i}) > u_i(No\ one_i, Boromir_{-i})$ $holds\ for\ i = Boromir\ and\ i = Frodo, but\ doesn't hold\ for\ Gimli\ and\ Legolas.$

2) $u_i(Gimli_i, Gimli_{-i}) > u_i(No\ one_i, Gimli_{-i})$ $holds\ for\ i = Gimli\ and\ i = Frodo, but\ doesn't hold\ for\ Legolas\ and\ Boromir.$

3) $u_i(Legolas_i, Legolas_{-i}) > u_i(No\ one_i, Legolas_{-i})$ $holds\ for\ i = Legolas\ and\ i = Frodo, but\ doesn't\ hold\ for\ Boromir\ and\ Gimli.$

4) $u_i(Frodo_i, Frodo_{-i}) > u_i(No\ one_i, Frodo_{-i})$ $holds\ for\ all\ player\ i.$

For this problem everyone has to agree on who takes on the task. So you have to go down the list of people and find the person were everyone agrees.
To start we look at if Boromir was chosen him and Frodo would agree, but Legolas and Gimili would choose no one.
Next we move to Gimili and him and Frodo would choose Gimili, but Legolas and Boromir would choose no one.
Legolas and Frodo would choose Legolas, but Boromir and Gimili would choose no one.
Finally we get to Frodo and everyone would choose Frodo before they pick no one.

9. Find all of the Nash equilibria for the three-player game in FIGURE PR4.9.

**FIGURE PR4.9**

Player 3: *A*

Player 2

| Player 1 | | x | y | z |
|---|---|---|---|---|
| | a | 1,1,0 | 2,0,0 | 2,0,0 |
| | b | 3,2,1 | 1,2,3 | 0,1,2 |
| | c | 2,0,0 | 0,2,3 | 3,1,1 |

Player 3: *B*

Player 2

| Player 1 | | x | y | z |
|---|---|---|---|---|
| | a | 2,0,0 | 0,0,1 | 2,1,2 |
| | b | 1,2,0 | 1,2,1 | 1,2,1 |
| | c | 0,1,2 | 2,2,1 | 2,1,0 |

Player 3: *C*

Player 2

| Player 1 | | x | y | z |
|---|---|---|---|---|
| | a | 2,0,0 | 0,1,2 | 0,1,2 |
| | b | 0,1,1 | 1,2,1 | 0,1,2 |
| | c | 3,1,2 | 0,1,2 | 1,1,2 |

**See Review Session #2, and use same methodology.**

**ANSWER:** The Nash equilibria are (*b*, *x*, *A*), (*a* ,*z*, *B*), (*c*, *x*, *C*), (*c*, *z*, *C*).

To find the Nash Equilibrium we'll have to first look at the best response from player 3. When we start to do the underlining we get the following .

Player 3: *A*

Player 2

| | | *x* | *y* | *z* |
|---|---|---|---|---|
| Player 1 | *a* | 1,1,0 | 2,0,0 | 2,0,0 |
| | *b* | 3,2,1 | 1,2,3 | 0,1,2 |
| | *c* | 2,0,0 | 0,2,3 | 3,1,1 |

Player 3: *B*

Player 2

| | | *x* | *y* | *z* |
|---|---|---|---|---|
| Player 1 | *a* | 2,0,0 | 0,0,1 | 2,1,2 |
| | *b* | 1,2,0 | 1,2,1 | 1,2,1 |
| | *c* | 0,1,2 | 2,2,1 | 2,1,0 |

Player 3: *C*

Player 2

| | | *x* | *y* | *z* |
|---|---|---|---|---|
| Player 1 | *a* | 2,0,0 | 0,1,2 | 0,1,2 |
| | *b* | 0,1,1 | 1,2,1 | 0,1,2 |
| | *c* | 3,1,2 | 0,1,2 | 1,1,2 |

Next we do the same for player one and get the following.

Player 3: *A*

Player 2

| | | *x* | *y* | *z* |
|---|---|---|---|---|
| Player 1 | *a* | 1,1,0 | 2,0,0 | 2,0,0 |
| | *b* | 3,2,1 | 1,2,3 | 0,1,2 |
| | *c* | 2,0,0 | 0,2,3 | 3,1,1 |

Player 3: *B*

Player 2

| | | *x* | *y* | *z* |
|---|---|---|---|---|
| Player 1 | *a* | 2,0,0 | 0,0,1 | 2,1,2 |
| | *b* | 1,2,0 | 1,2,1 | 1,2,1 |
| | *c* | 0,1,2 | 2,2,1 | 2,1,0 |

Player 3: *C*

Player 2

| | | *x* | *y* | *z* |
|---|---|---|---|---|
| Player 1 | *a* | 2,0,0 | 0,1,2 | 0,1,2 |
| | *b* | 0,1,1 | 1,2,1 | 0,1,2 |
| | *c* | 3,1,2 | 0,1,2 | 1,1,2 |

Finally we look at the best responses for player 2.

Player 3: *A*

Player 2

| Player 1 | *x* | *y* | *z* |
|---|---|---|---|
| *a* | 1,1,0 | 2,0,0 | 2,0,0 |
| *b* | 3,2,1 | 1,2,3 | 0,1,2 |
| *c* | 2,0,0 | 0,2,3 | 3,1,1 |

Player 3: *B*

Player 2

| Player 1 | *x* | *y* | *z* |
|---|---|---|---|
| *a* | 2,0,0 | 0,0,1 | 2,1,2 |
| *b* | 1,2,0 | 1,2,1 | 1,2,1 |
| *c* | 0,1,2 | 2,2,1 | 2,1,0 |

Player 3: *C*

Player 2

| Player 1 | *x* | *y* | *z* |
|---|---|---|---|
| *a* | 2,0,0 | 0,1,2 | 0,1,2 |
| *b* | 0,1,1 | 1,2,1 | 0,1,2 |
| *c* | 3,1,2 | 0,1,2 | 1,1,2 |

**See Review Session #2, and use same methodology.**

**ANSWER:** The Nash equilibria are $(b, x, A)$, $(a, z, B)$, $(c, x, C)$, $(c, z, C)$.

***Question 5:*** **Harrington, Chapter 5 – Exercise 5**

a) With 2 diners there will be a 3x3 payoff matrix:

| P1\P2 | Pasta | Salmon | Filet |
|---|---|---|---|
| Pasta | | | |
| Salmon | | | |
| Filet | | | |

To find the payoffs for (Pasta, Pasta) for example, we have:
Value(21) - Total Cost(14+14)/2 = Payoff(7)

This is the same for both players as this game is symmetric.

Again, the payoff for (Pasta, Salmon) for player 1 is:
21-(14+21)/2 = 21-17.5 = 3.5

For P2 his payoff is 26-(14+21)/2 = 26-17.5 = 8.5

Repeating these calculations we find,

| P1\P2 | Pasta | Salmon | Filet |
|---|---|---|---|
| Pasta | 7,7 | 3.5, <u>8.5</u> | -1, 7 |
| Salmon | <u>8.5</u>, 3.5 | <u>5</u>, <u>5</u> | <u>.5</u>, 3.5 |
| Filet | 7, -1 | 3.5, <u>3.5</u> | -1, -1 |

Using best response for each player, it is simple to show that (salmon, salmon) is the Nash equilibrium.

**b.** Suppose there are four diners (n=4). What will they order (at a Nash equilibrium)?

***Answer***: Note that a diner cannot influence what others order and must pay 25% of the price of the ordered meals. All that a diner an influence is her own order. The key property to note is that whatever she orders, she pays only 25% of the price with the remaining 75% being paid by the other three diners. Once recognizing that this is the actual cost to her, not the price on the menu, a diner should choose the meal that maximizes her surplus. Taking all this into account, Table Sol 5.5.2 shows the costs faced by a diner. For example, a diner who orders the pasta dish pays only 25% of the menu price, which is $3.50. We observe that each diner orders the filet mignon because it really only costs them $7.50, and the surplus is maximized with that order. The unique Nash equilibrium is then that all four diners order the steak. Hence, each gets a meal he or she values at $29, but ends up paying $30!

| | | | |
|---|---|---|---|
| Dish | Value | Actual Cost | Surplus |
| Pasta | $21.00 | $3.50 | $17.50 |
| Salmon | $26.00 | $5.25 | $20.75 |
| Filet | $29.00 | $7.50 | $21.50 |

**Harrington, Chapter 5 – Exercise 6**

To find all of the Nash equilibria we need to find where firms do not wish to enter or leave.

| Number of Firms | Gross Profit Per Firm | Net Profit Per Firm |
|---|---|---|
| 1 | 1000 | 700 |
| 2 | 400 | 100 |
| 3 | 250 | -5 |
| 4 | 150 | -150 |
| 5 | 100 | -200 |

Because each company's "bottom line" is Net Profit, this is the relevant information to analyze. With one firm, it's evident to other firms that they may make money by entry, so an additional firm enters. At two firms though, the next firm entering would lose money, so it's clear that <u>2</u>

firms is the Nash equilibrium. Testing the other possibilities, starting at # of firms=5, there is incentive to exit until only two firms remain. So the only equilibrium is at two firms.

**Harrington, Chapter 5 – Exercise 10**

10. Consider a country with $n$ citizens, and let $v_i$ be the value that citizen $i$ attaches to protesting. Enumerate the citizens so that citizen 1 attaches more value to protesting than citizen 2, who attaches more value than citizen 3, and so forth: $v_1 > v_2 > \cdots > v_n (= 0)$, where citizen $n$ attaches no value to protesting. Assume that the cost of protesting is the same for all citizens and is $\frac{c}{m}$ where $c > 0$ and $m$ is the number of protestors. Then the payoff to citizen $i$ from protesting is $v_i - (\frac{c}{m})$, while the payoff from not protesting is zero. Assume that $v_1 - c < 0$. Find all Nash equilibria.

**ANSWER:** Note that one Nash equilibrium is the strategy profile in which all citizens choose not to protest. The payoff to protesting for citizen $i$ is $v_i - c$, given no one else protests. Since citizen 1 is the most inclined to protest (that is, she

receives the highest benefit) and $v_1 - c < 0$ —so that citizen 1 prefers not to protest on her own—then no other citizen does either. If no one else plans to march against the government, an individual citizen will not want to do so as that would just mean getting thrown into jail. One equilibrium then has no one participating in a protest.

Is there an equilibrium in which a protest emerges? We know there is no equilibrium in which all citizens protest, because the payoff to citizen $n$ from protesting is negative even when everyone else protests; it equals $v_n - \frac{c}{n} = -\frac{c}{n} < 0$. He would prefer to stay home and receive a zero payoff. What about an equilibrium in which some citizens protest? To answer that question, let's derive an important property. If $m'$ citizens protest then it is optimal for citizen $i$ to be one of those protesting citizens when

$$v_i - \frac{c}{m'} \geq 0. \qquad \text{[SOL5.10.1]}$$

Now consider citizen $j$, where $j < i$. Since $v_j > v_i$ (that is, citizen $j$ values protesting more than does citizen $i$), then it follows from (SOL5.10.1) that

$$v_j - \frac{c}{m'} > 0.$$

In other words, if citizen $i$ finds it best to protest than so does citizen $j$. Hence, at a Nash equilibrium, if citizen $i$ protests, then so must citizens $1, 2, \ldots, i-1$. This makes sense because all citizens face the same cost, but citizens $1, 2, \ldots, i-1$ attach greater benefit to protesting than does citizen $i$ With this property, we can proceed. Consider a strategy profile in which $m'$ citizens protest where $2 \leq m' \leq n - 1$. By the argument just made, it must be citizens $1, 2, \ldots, m'$. For this to be an equilibrium, all of those $m'$ citizens must earn a nonnegative payoff from protesting (so that protesting is better than not protesting), and the other $n - m'$ citizens must earn a nonpositive payoff from protesting (so that they prefer not to protest). This involves $n$ conditions. However, note that if citizen $m'$ prefers to protest, then so do citizens $1, 2, \ldots, m' - 1$, since all of them earn a payoff from protesting that is at least as high as that of citizen $m'$. To ensure that protesting is optimal for citizens $1, 2, \ldots, m'$,we just need to make sure it is optimal for citizen $m'$, which is the case when

$$v_{m'} - \frac{c}{m'} \geq 0. \qquad \text{[SOL5.10.2]}$$

Now consider the citizens who are not protesting. Since the benefit to protesting is greatest among them for citizen $m' + 1$, if that citizen finds it optimal to stay home, then so do citizens $m' + 2, m' + 3, \ldots, n$. Thus, we just need to verify that citizen $m' + 1$ finds it optimal not to protest, which is the case when

$$0 \geq v_{m'+1} - \frac{c}{m' + 1}. \qquad \text{[SOL5.10.3]}$$

Note that the cost from protesting is $\frac{c}{m'+1}$, since, if she also protested, there would be $m' + 1$ people at the protest. Putting conditions (SOL5.10.2) and (SOL5.10.3) together, we then require that

$$v_{m'} - \frac{c}{m'} \geq 0 \geq v_{m'+1} - \frac{c}{m' + 1}. \qquad \text{[SOL5.10.4]}$$

$v_{m'} - \frac{c}{m'}$ is the payoff to protesting for citizen $m'$ and $v_{m'+1} - \frac{c}{m+1}$ is the payoff to protesting for citizen $m'$+1. A value for $m'$ that satisfies (SOL5.10.4) means that if $m'$ citizens are expected to protest, then citizens $1, 2, \ldots, m'$ will protest (with citizen $m'$ being the most reluctant protestor in that group) and citizens $m' + 1$, $m'+ 2, \ldots, n$ will not protest (with citizen $m' + 1$ the one in that group most inclined to protest). $m'$ is the equilibrium size of a protest.

**Harrington, Chapter 5 – Exercise 11**

Harrington Chapter 5

Question 11

**ANSWER:** This game has a unique Nash equilibrium, which is for all to take the course: $x_i = 1$ for all $i$ . Suppose all students other than $i$ take the course: $x_j = 1$ for all $j \neq i$. Suppose $i = 1$. Since $a_2 + z > a_1$, then student 1's ranking without having taken the course is no higher than second (it could be lower if $a_3 + z > a_1$.) Thus, her payoff from not taking the course is no higher than $b(n-2)$. If she takes the course, she will be ranked first, since $a_1 + z > a_j + z$ for all $j \neq i$. Hence, her payoff is $b(n-1) - c$. Next, note that $b(n-1) - c > b(n-2)$, which is equivalent to $b > c$, which is true by assumption. It has then been shown that the payoff to student 1 from taking the course exceeds her payoff from not taking the course, given everyone else takes the course. It is then optimal for student 1 to take the course. The gain in score and ranking from taking the prep course exceeds the cost to student 1.

Next, consider student $i$, where $2 \leq i \leq n-2$. The analysis is similar to that for student 1. If student $i$ takes the course—given all other students take the course—her payoff is $b(n-i) - c$. Her payoff from not taking the course is no higher than $b(n-i-1)$. Since $b(n-i) - c > b(n-i-1)$, she prefers to take the course. Finally, consider student $i$, where $i = n-1$ or $i = n$. Her payoff from not taking the course is $b(n-n) = 0$, as she is ranked last. Her payoff from taking the course is $b(n-(n-1)) - c = b - c > 0$, so she prefers to take the course. This completes the proof that all students taking the course is a Nash equilibrium. One can show that this is the unique Nash equilibrium.