بسم الله الرحمن الرحیم

Algorithmic Game Theory

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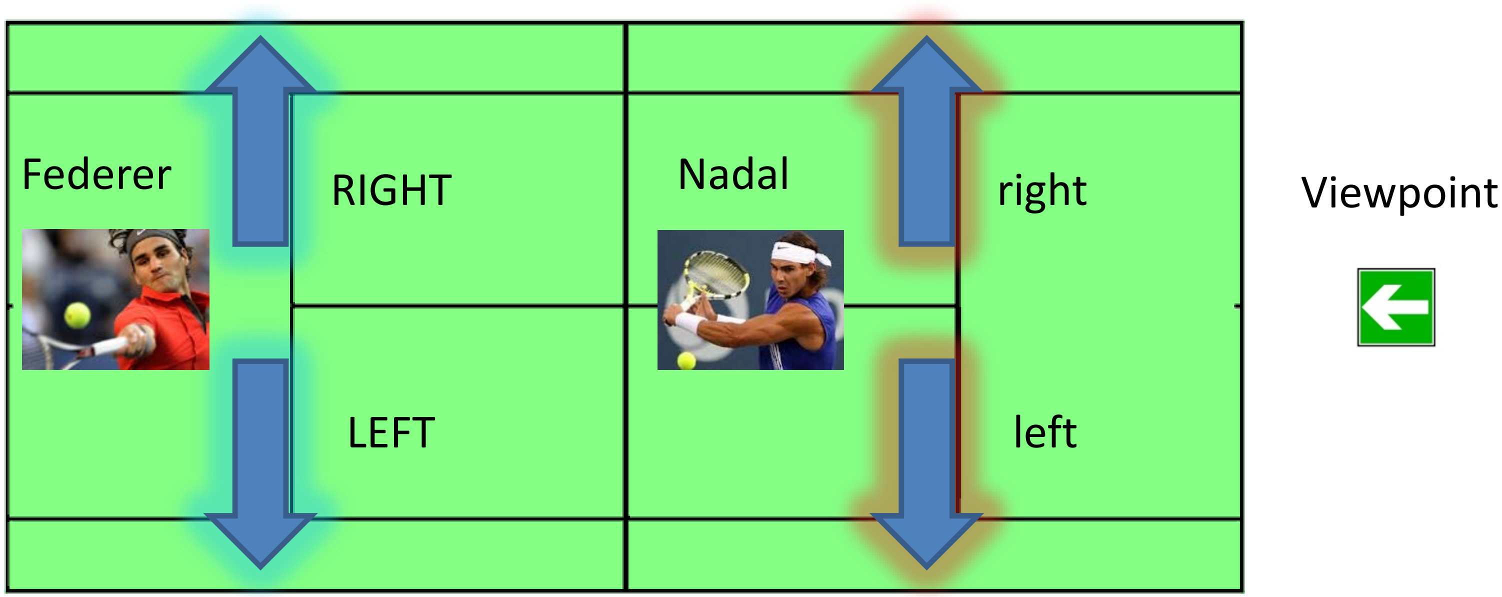
دانشگاه کاشان

MIXED STRATEGY NASH EQUILIBRIUM: AN EXAMPLE

Tennis Game

- We're going to look at a tennis game
- Assume two players (Federer and Nadal)
- Where Nadal is at the net.



Tennis Game

		Nadal *l*	Nadal *r*	
Federer	*L*	50,50	80,20	p
Federer	*R*	90,10	20,80	(1-p)
		q	(1-q)	

- Have a look at the payoffs
  - E.g.: if Federer chooses 'L' and Nadal guesses wrong and jumps to the 'r', Federer wins the point 80% of the time
- Is there any dominated strategy?
- Is there a ***pure strategy*** NE profile?

Tennis Game

Let's find the mixed strategy NE

Lesson 1: Each player's randomization is the best response to the other player's randomization

Lesson 2: If players are playing a mixed strategy as part of a NE, then each of the pure strategies involved in the mix must itself be a best response

Tennis Game

- Find a mixture for Nadal and one for Federer that are in equilibrium

- ***TRICK***:
  - To find Nadal's mix (q) I'm going to ***put myself in Federer's shoes*** and look at his payoffs
  - And vice-versa for Federer's mix (p)

Tennis Game

- Federer's expected payoffs:

$$E\left[U_{Federer}\left(L,(q,1-q)\right)\right]=50q+80(1-q)$$

$$E\left[U_{Federer}\left(R,(q,1-q)\right)\right]=90q+20(1-q)$$

- If Federer is mixing in this NE then the payoff to the left and to the right must be equal, they must both be best responses
  - Otherwise Federer would not be mixing

Tennis Game

- Federer's expected payoffs must be equal:

$$E\left[U_{Federer}\left(L,(q,1-q)\right)\right]=50q+80(1-q)$$

$$E\left[U_{Federer}\left(R,(q,1-q)\right)\right]=90q+20(1-q)$$

$$\Rightarrow 50q+80(1-q)=90q+20(1-q)$$

$$\Rightarrow 40q=60(1-q)$$

$$\Rightarrow q=0.6$$

- I was able to derive ***Nadal's mixing probability***
- This is the solution to the equation in one unknown that equates Federer's payoffs in the mix

Tennis Game

- Nadal's expected payoffs:

$$E\left[U_{Nadal}\left((p,1-p),l\right)\right]=50p+10(1-p)$$

$$E\left[U_{Nadal}\left((p,1-p),r\right)\right]=20p+80(1-p)$$

$$\Rightarrow 50p+10(1-p)=20p+80(1-p)$$

$$\Rightarrow 30p=70(1-p)$$

$$\Rightarrow p=0.7$$

	l	*r*	
L	50,50	80,20	p
R	90,10	20,80	(1-p)
	q	(1-q)	

- Similarly, we computed Federer's mixing probability

Tennis Game

- We found the mixed strategy NE:

Federer   Nadal

➔ [(0.7, 0.3) , (0.6, 0.4)]

L   R   l   r

- What would happen if Nadal jumped to the left more often than 0.6?
  - Federer would be better of **playing the pure strategy 'R'!**
- What if he jumped less often than 0.6?
  - Federer would be **shooting to the 'L' all time!**

Tennis Game

			Nadal		
			l	***r***	
Federer	***L***		30,70	80,20	p
	R		90,10	20,80	(1-p)
			q	(1-q)	

- Suppose a new coach teaches Nadal how to forehand, and the payoff would change accordingly
- There is still no pure strategy NE
- What would happen in this game?

Tennis Game

- Let's first let our intuition work
- Basically Nadal is better at his forehand and when Federer shoots there, Nadal scores more often than before

➔ ***Direct effect***: Nadal should increase his q

- But, Federer knows Nadal is better at his forehand, hence he will shoot there less often

➔ ***Indirect effect***: Nadal should decrease his q

Tennis Game

- Let's compute again q:

$$E\left[U_{Federer}\left(L,(q,1-q)\right)\right]=30q+80(1-q)$$

$$E\left[U_{Federer}\left(R,(q,1-q)\right)\right]=90q+20(1-q)$$

$$\Rightarrow 30q+80(1-q)=90q+20(1-q)$$

$$\Rightarrow 60q=60(1-q)$$

$$\Rightarrow q=0.5$$

- We see that in the end Nadal's q went down from 0.6 to 0.5!!
- **The indirect effect was predominant**

Tennis Game

- Nadal's expected payoffs:

$$E\left[U_{Nadal}\left(\left(p,1-p\right),l\right)\right]=70p+10(1-p)$$

$$E\left[U_{Nadal}\left(\left(p,1-p\right),r\right)\right]=20p+80(1-p)$$

$$\Rightarrow 50p+10(1-p)=20p+80(1-p)$$

$$\Rightarrow 50p=70(1-p)$$

$$\Rightarrow p=7/12=0.5833<0.7$$

- **The direct effect was predominant**
- Federer will be shooting to the left with less probability

Tennis Game: Summary

- We just performed a ***comparative statistics*** exercise
  - We looked at a game and found an equilibrium, then we perturbed the original game and found another equilibrium and compared the two NE
- Suppose Nadal's q had not changed
  - Federer would have never shot to the left
  - But this couldn't be a mixed strategy NE
  - There was a force to put back things at equilibrium and that was the force that pulled down Nadal's q

NASH THEOREM

Every Finite Game has
a Mixed Strategy Nash Equilibrium

- Why is this important?
- Without knowing the existence of an equilibrium, it is difficult (perhaps meaningless) to try to understand its properties.
- Armed with this theorem, we also know that every finite game has an equilibrium, and thus we can simply try to locate the equilibria.

LET'S SEE PAYOFFS!

Tennis Game

		Nadal		
		l	***r***	
Federer	***L***	50,50	80,20	p
	R	90,10	20,80	(1-p)
		q	(1-q)	

- We identified the ***mixed strategy NE*** for this game

Federer Nadal

➔[(0.7, 0.3) , (0.6, 0.4)]

L R l r

p^* $(1-p^*)$ q^* $(1-q^*)$

Tennis Game

- How do we actually ***check*** that this is indeed an equilibrium?
- Let's verify that in fact p* is BR(q*)
- Federer's payoffs:
  - Pure strategy L ➔ 50*0.6 + 80*0.4 = 62
  - Pure strategy R ➔ 90*0.6 + 20*0.4 = 62
  - Mix p* ➔ 0.7*62 + 0.3*62 = 62
- Nadal's payoffs:
  - Pure Strategy l ➔ 50*0.7 + 10*0.3 = 38
  - Pure Strategy r ➔ 20*0.7 + 80*0.3 = 38
  - Mix q* ➔ 0.6*38 + 0.4*38 = 38

- Federer has no ***strictly profitable*** ***pure-strategy*** deviation

Note (again): You cannot always win by playing NE

Tennis Game

- But is this enough? There are no pure-strategy deviations, but could there be any other mixes?
- Any mixed strategy yields a payoff that is a weighted average of the pure strategy payoffs
  - This already tells us: if you didn't find any pure-strategy deviations then you'll not find any other mixes that will be profitable

> To check if a mixed strategy is a NE
> we only have to check if
> there are any pure-strategy profitable deviations

Discussion

- Since we’re in a mixed strategy equilibrium, it must be the case that the payoffs are equal

- Indeed, if it was not the case, then you shouldn’t be randomizing!!

Applied Example: Security Check at Airport

- After the security problems in the U.S. and worldwide airports due to high risks of attacks, the need for devices capable of inspecting luggage has raised considerably
- The problem is that there are not enough of such machines
- Wrong statements have been promoted by local governments:
  - If we put a check device in NY then all attacks will be shifted to Boston, but if we put a check device in Boston, the attacks will be shifted to yet another city

  ➔The claim was that whatever the security countermeasure, it would only shift the problem

Applied Example: Security Check at Airport

What if you **wouldn't notify** where you would actually put the check devices, which boils down to randomizing?

The hard thing to do in practice is how to mimic randomization!!

MIXED STRATEGIES NE: INTERPRETATIONS

The Battle of the Sexes

		Player 2: M	Player 2: N	
Player 1	M	2,1	0,0	p
	N	0,0	1,2	(1-p)
		q	1-q	

- We already know a lot about this game
- There are two pure-strategy NE: (M,M) and (N,N)
- We know that there is a problem of ***coordination***
- We know that without communication, it is possible (and quite probable) that the two players might fail to coordinate

The Battle of the Sexes

- Player 1 perspective, find NE q:

$$\left.\begin{aligned} E\left[U_1\left(M,(q,1-q)\right)\right] &= 2q+0(1-q) \\ E\left[U_1\left(N,(q,1-q)\right)\right] &= 0q+1(1-q) \end{aligned}\right\} 2q=(1-q) \Rightarrow q=\frac{1}{3}$$

- Player 2 perspective, find NE p:

$$\left.\begin{aligned} E\left[U_2\left((p,1-p),M\right)\right] &= 1p+0(1-p) \\ E\left[U_2\left((p,1-p),N\right)\right] &= 0p+2(1-p) \end{aligned}\right\} 1p=2(1-p) \Rightarrow p=\frac{2}{3}$$

The Battle of the Sexes

- Let's check that p=2/3 is indeed a BR for Player1:

$$\left.\begin{aligned} E\left[U_1\left(M,\left(\frac{1}{3},\frac{2}{3}\right)\right)\right] &= 2\frac{1}{3}+0\frac{2}{3} \\ E\left[U_1\left(N,\left(\frac{1}{3},\frac{2}{3}\right)\right)\right] &= 0\frac{1}{3}+1\frac{2}{3} \end{aligned}\right\} = \frac{2}{3}$$

$$E[U_1] = \frac{2}{3}\frac{2}{3}+\frac{1}{3}\frac{2}{3} = \frac{2}{3}$$

The Battle of the Sexes

- We just found out that there is no strictly profitable pure-strategy deviation

➔ There is no strictly profitable mixed-strategy deviation

- The mixed strategy NE is:

$$\begin{array}{cccc} \multicolumn{2}{c}{\text{Player 1}} & \multicolumn{2}{c}{\text{Player 2}} \\ \multicolumn{4}{c}{\left[\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{2}{3}\right)\right]} \\ p & 1-p & q & 1-q \end{array}$$

The Battle of the Sexes

- What are the payoffs to players when they play such a mixed strategy NE?

$$u_1, u_2 = \left(\frac{2}{3}, \frac{2}{3}\right)$$

- Why are the payoffs so low?
- What is the probability for the two players not to meet?

➔ Prob(meet) = 2/3*1/3+1/3*2/3=4/9

➔ 1- Prob(meet) = 5/9 !!!

The Battle of the Sexes

- This results seems to confirm our intuition that "magically" achieving the pure-strategy NE would be not always possible

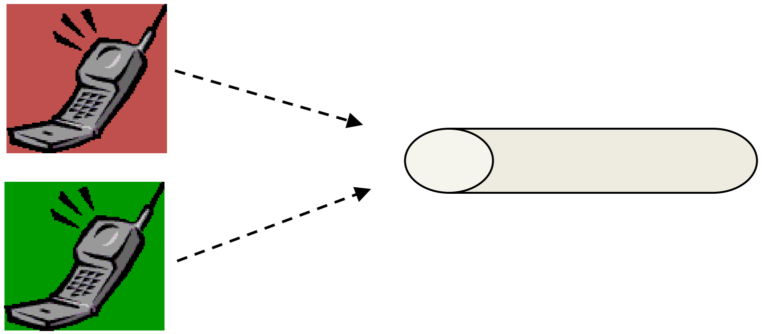
- So the real question is: why are those players randomizing in such a way that it is not profitable?

Mixed Strategies: Interpretation

- Rather than thinking of players actually randomizing over their strategies, we can think of them ***holding beliefs*** of what the other players would play

- What we've done so far is to **find those beliefs** that make players **"indifferent"** over what they play since they're going to obtain the same payoffs

The Multiple Access game



Time-division channel

Reward for successful transmission: 1

Cost of transmission: c
(0 < c << 1)

Blue \ Green	Quiet	Transmit
Quiet	(0, 0)	$\boxed{(0,\ 1\text{-}c)}$
Transmit	$\boxed{(1\text{-}c,\ 0)}$	(-c, -c)

There is no strictly dominating strategy

There are two Nash equilibria

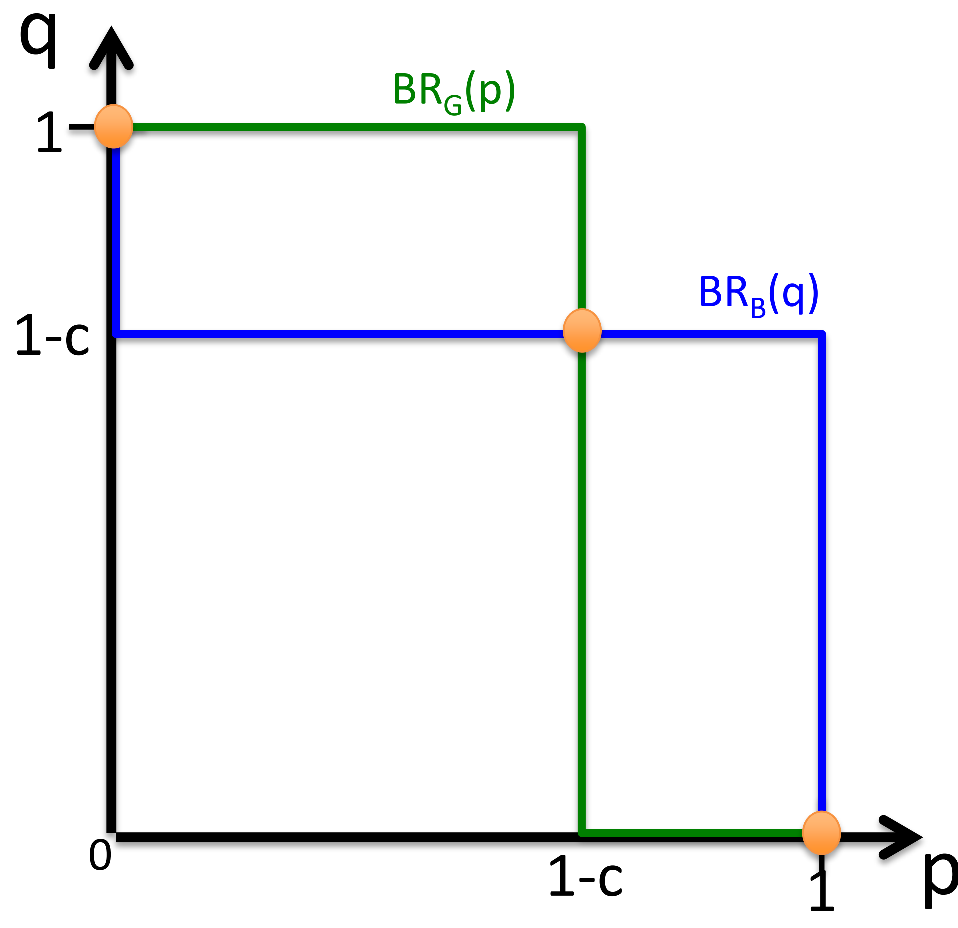
Mixed Strategy Nash equilibrium

p: probability of transmit for Blue

q: probability of transmit for Green

$$u_{blue} = p(1-q)(1-c) - pqc = p(1-c-q)$$

$$u_{green} = q(1-c-p)$$



objectives

- Blue: choose p to maximize u_{blue}
- Green: choose q to maximize u_{green}

$$p^* = 1-c,\ q^* = 1-c$$

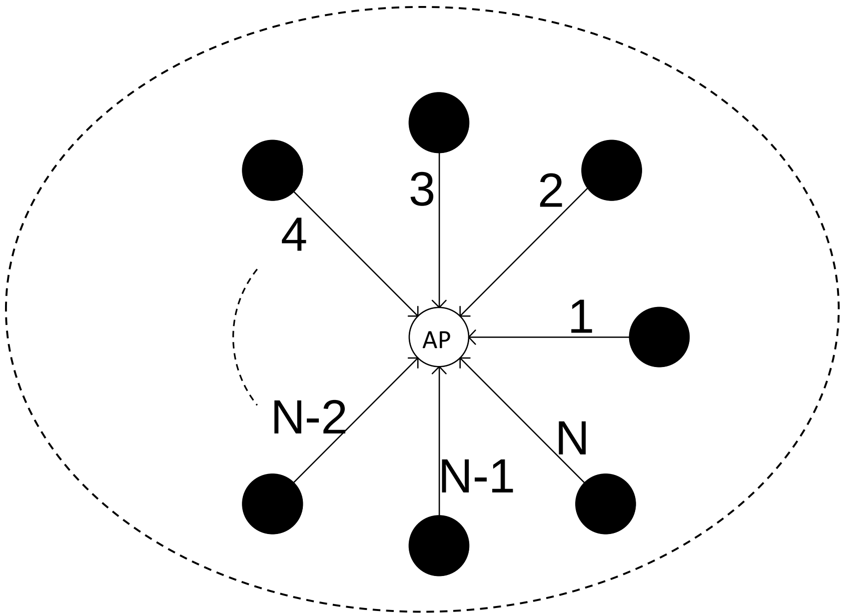
is a Nash equilibrium

802.11 MAC Layer

A Practical Randomization Strategy in Wireless Networks

WiFi Networks

- N links with the same physical condition (single-collision domain):



MAC Layer → π = Probability of Transmission

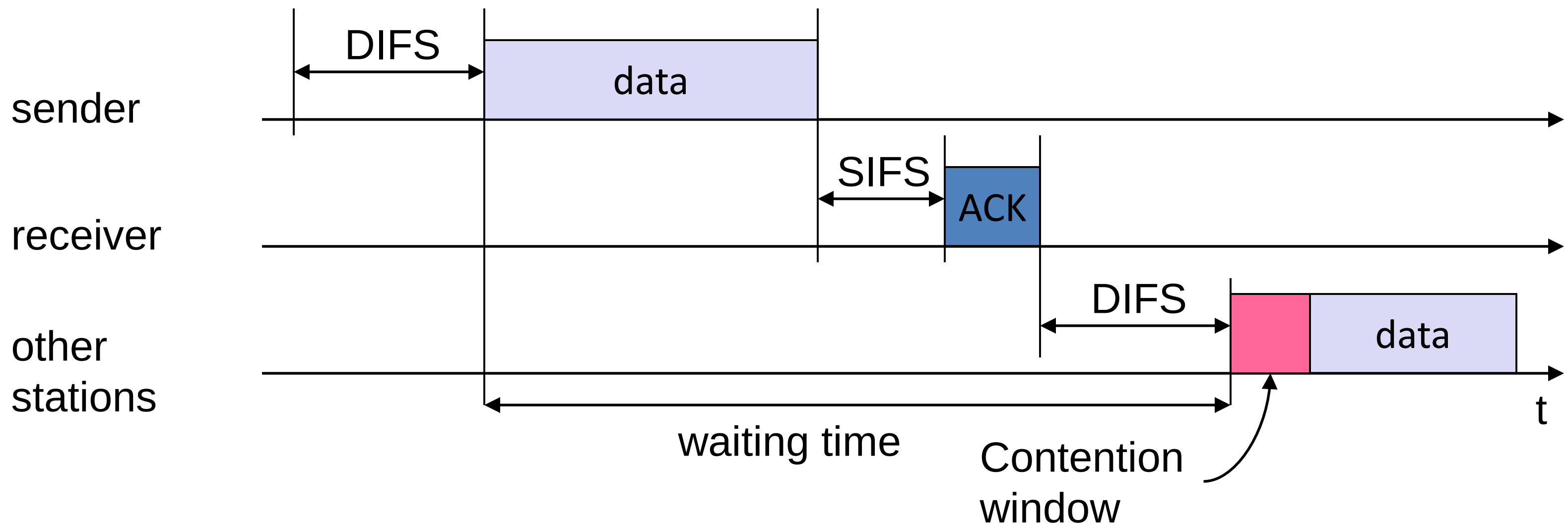
PHY Layer → P = Probability of Collision

= More than one transmission at the same time

$= 1 - (1 - \pi)^{N-1}$

802.11 - CSMA/CA
A MAC Layer for WiFi Networks

- Sending unicast packets
  - station has to wait for DIFS before sending data
  - receiver acknowledges at once (after waiting for SIFS) if the packet was received correctly (CRC)
  - automatic retransmission of data packets in case of transmission errors



The ACK is sent right at the end of SIFS (no contention)

Inter Frame Space and CW Times: Some PHY and MAC Layer Parameters

Parameters	*802.11a*	*802.11b (FH)*	*802.11b (DS)*	*802.11b (IR)*	*802.11b (High Rate)*
Slot Time (μs)	9	50	20	8	20
SIFS (μs)	16	28	10	10	10
DIFS (μs)	34	128	50	26	50
EIFS (μs)	92.6	396	364	205 or 193	268 or 364
$CW_{min}(SlotTime)$	15	15	31	63	31
$CW_{max}(SlotTime)$	1023	1023	1023	1023	1023
Physical Data Rate (Mbps)	6 to 54	1 and 2	1 and 2	1 and 2	1, 2, 5.5, and 11

Bianchi's Model: Solution for *p* and *π*

Basically it is a system of two nonlinear equations with two variables p and π:

$$\begin{cases} p = 1 - (1-\pi)^{N-1} \\ \pi = \dfrac{2}{1 + W_{min} + pW_{min}\sum_{k=0}^{m-1}(2p)^k} \end{cases}$$

A mixing over transmission strategy

In fact π is the mixing probability of pure strategy of transmission for each mobile user

- Try to find all NE of a game between N mobile user?