Mechanism Design and Auctions

Game Theory

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- Reading:
  - Roughgarden’s lecture notes on Mechanism Design
  - Chapter 10 of the MAS book
  - Chapter 11 of the MAS book

Single Item Auctions

- There is some number n of (strategic) bidders who are potentially interested in buying the item.
- Assumptions:
  - Each bidder i has a valuation v_i - its maximum willingness-to-pay for the item being sold. Thus bidder i wants to acquire the item as cheaply as possible, provided the selling price is at most v_i
  - This valuation is private, meaning it is unknown to the seller and to the other bidders.
  - Quasilinear utility model: If a bidder loses an auction, its utility is 0. If the bidder wins at a price p, then its utility is v_i - p.

Sealed-Bid Auctions

- The setting:
  - Each bidder i privately communicates a bid b_i to the auctioneer-in a sealed envelope if you like.
  - The auctioneer decides who gets the good (if anyone).
  - The auctioneer decides on a selling price.
- First Price Auction: ask the winning bidder to pay its bid
- Second Price Auction: ask the winning bidder to pay the highest other bid

Second Price Auctions

- Also known as Vickrey Auctions.
- Theorem: In a second-price auction, every bidder has a dominant strategy: set its bid b_i equal to its private valuation v_i. That is, this strategy maximizes the utility of bidder i, no matter what the other bidders do.
  - Proof: See the blackboard
- Theorem: In a second-price auction, every truthtelling bidder is guaranteed non-negative utility.
  - Proof: See the blackboard

Second Price Auctions

- Theorem: The Vickrey auction is awesome. Meaning, it enjoys three quite different and desirable properties:
  - **[strong incentive guarantees]** It is dominant-strategy incentive-compatible (DSIC), i.e., the previous theorems
  - **[strong performance guarantees]** If bidders report truthfully, then the auction maximizes the social surplus $\sum_{i=1}^{n} x_i v_i$ where x_i is 1 if i wins and 0 if i loses, subject to the obvious feasibility constraint that $\sum_{i=1}^{n} x_i \leq 1$ (i.e., there is only one item).
  - **[computational efficiency]** The auction can be implemented in polynomial (indeed, linear) time

Single Parameter Environments

- Setting:
  - Each bidder i has a private valuation v_i, its value "per unit of stuff" that it gets.
  - There is a feasible set X. Each element of X is an n-vector $(x_1, x_2, \ldots, x_n)$, where x_i denotes the "amount of stuff" given to bidder i.
  - A vector of bids: $\mathbf{b} = (b_1, \ldots, b_n)$
- Allocation Rule: a feasible allocation $\mathbf{x}(\mathbf{b}) \in X \subseteq R^n$ as a function of the bids.
- Payment Rule: payments $\mathbf{p}(\mathbf{b}) \in R^n$ as a function of the bids.
- Quasilinear utility model:

$$u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$$

- Our focus: $p_i(\mathbf{b}) \in [0, b_i \cdot x_i(\mathbf{b})]$
  - $p_i(\mathbf{b}) \geq 0$ is equivalent to prohibiting the seller from paying the bidders.
  - $p_i(\mathbf{b}) \leq b_i \cdot x_i(\mathbf{b})$ ensures that a truthtelling bidder receives nonnegative utility

Example: Sponsored Search Auctions

- The goods for sale are the k slots for sponsored links on a search results page.
- We quantify the difference between different slots using click-through-rates (CTRs). The CTR α_j of a slot j represents the probability that the end user clicks on this slot.
- Ordering the slots from top to bottom, we make the reasonable assumption that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$.
- The bidders are the advertisers who have a standing bid on the keyword that was searched on. The bids : **b**.
- Let **x**(**b**) be the allocation rule that assigns the j^{th} highest bidder to the j^{th} highest slot, for $j = 1, 2, \ldots, k$.
- Is x(b) implementable: can we have a payment rule which yields a DSIC mechanism?

Myerson's Lemma

- Theorem (Myerson): Fix a single-parameter environment.
  - (a) An allocation rule **x** is implementable if and only if it is monotone.
  - (b) If **x** is monotone, then there is a unique payment rule such that the sealed-bid mechanism (**x**, **p**) is DSIC [assuming the normalization that $b_i = 0$ implies $p_i(\mathbf{b}) = 0$].
  - (c) The payment rule in (b) is given by an explicit formula:

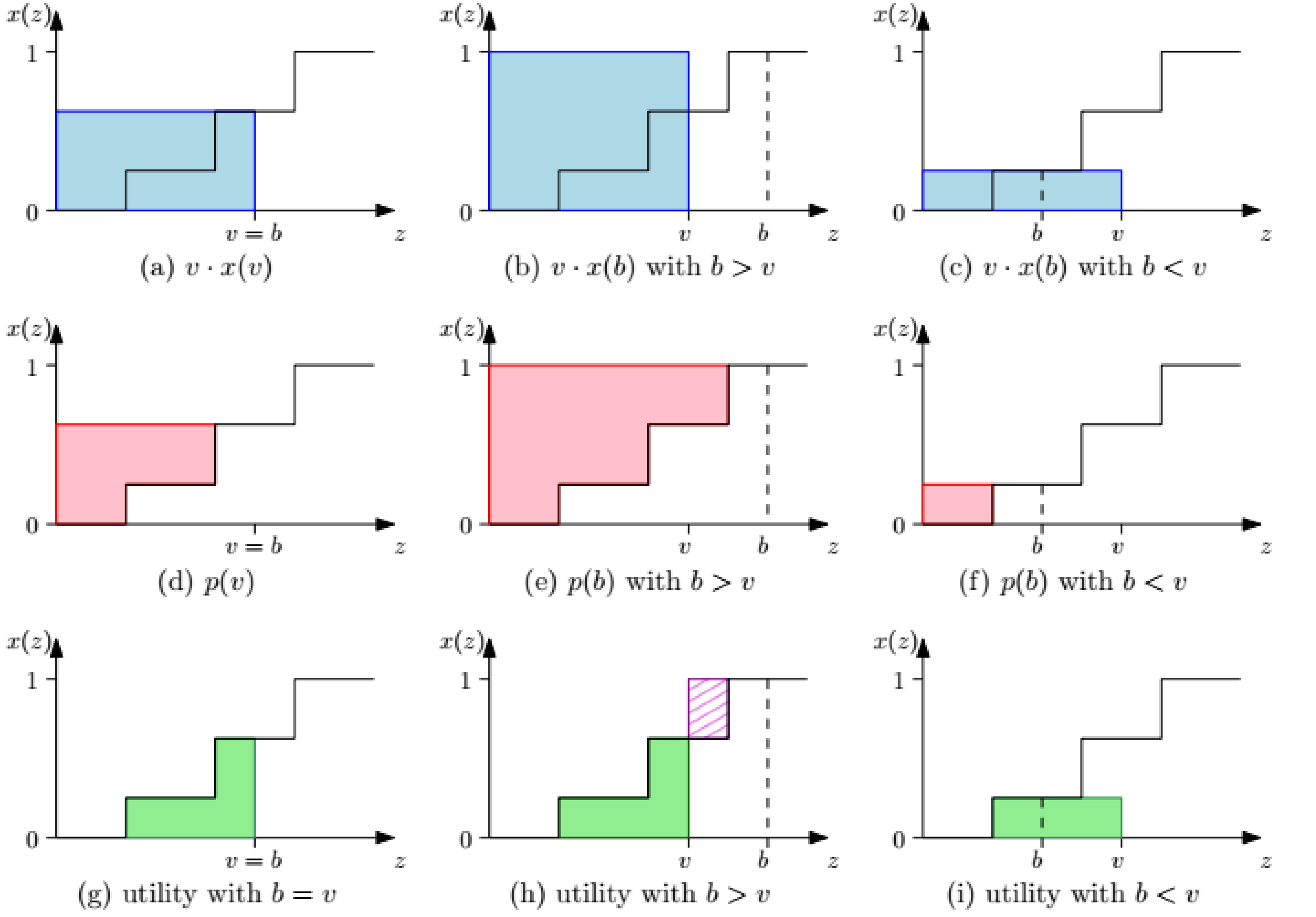
$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j . jump\ in\ x_i(\cdot, b_{-i}) at\ z_j$$

or

$$p_i(b_i, b_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z, b_{-i}) dz$$

  - Proof: see the blackboard

Myerson’s Lemma



Ex: Sponsored Search Auctions

- Remind:
  - Ordering the slots from top to bottom, we make the reasonable assumption that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$.
  - The bidders are the advertisers who have a standing bid on the keyword that was searched on. The bids : **b**.
  - Let **x**(**b**) be the allocation rule that assigns the j^{th} highest bidder to the j^{th} highest slot, for $j = 1, 2, \ldots, k$.
- Payment rule:

$$p_i(b) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1})$$

- See the blackboard for the calculations

Surplus Maximizing DSIC Mechanisms

- Defining the allocation rule by

$$x(b) = argmax_x \sum_{i=1}^{n} b_i x_i$$

- If the mechanism is truthful, this allocation rule maximizes the social welfare.
- It is very related to optimization research fields
  - Approximation algorithms
  - Randomized algorithms
  - Complexity theory
  - ….
- Our algorithmic objective: 1) Optimizing the objective 2) while keeping the mechanism DSIC 3) with algorithms running in polynomial time

Ex: Knapsack Auctions

- Each bidder i has a publicly known size w_i and a private valuation v_i
- The seller has a capacity W
- The feasible set X is defined as the 0-1 n-vectors $(x_1, \ldots, x_n)$ such that

$$\sum_{i=1}^{n} w_i x_i \leq W$$

- Our target is to design a surplus maximizing DSIC mechanism for this auction

$$Maximize \sum_{i=1}^{n} b_i x_i$$

Ex: Knapsack Auctions

- Knapsack problem is NP-hard
- There exist approximation algorithms for this problem
- We can not use all of these algorithms (best algorithms) for surplus maximizing mechanism design (At least now), because they are not monotone.
  - Ex: Knapsack has a FPTAS i.e. for each n, ϵ it has a $(1-\epsilon)$ approximation algorithm with polynomial time $Poly(n, \frac{1}{\epsilon})$.
- Our idea: If the proposed allocation rule by the approximation algorithm is monotone, we can use Myerson's lemma.
  - ½ -approximation algorithm has this property

Ex: Knapsack Auctions

- ½ -Approximation Algorithm:
  - Sort and re-index the bidders so that

$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}$$

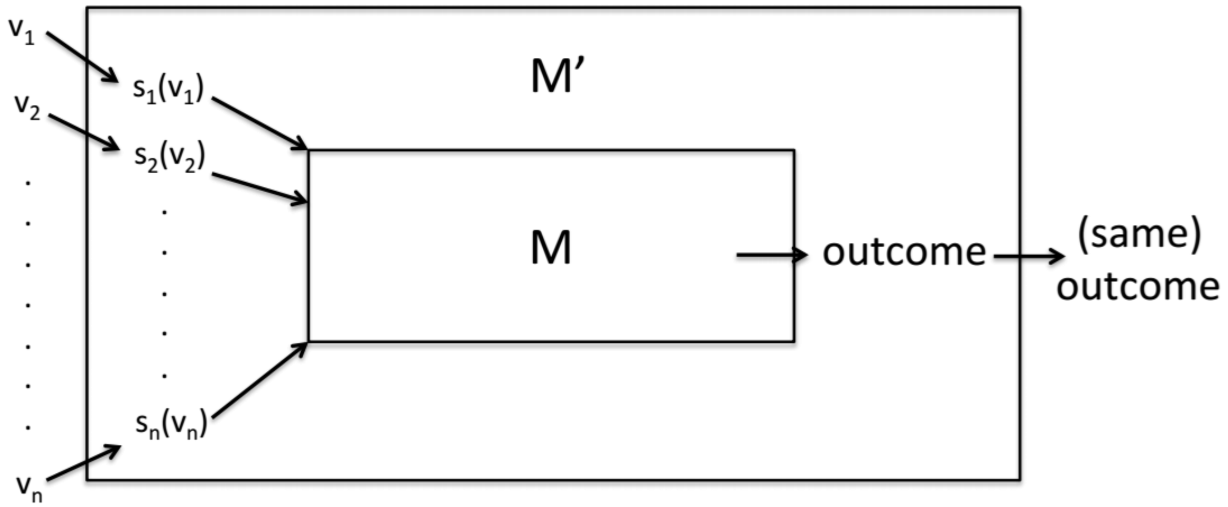
  - Pick winners in this order until one doesn't fit, and then halt.
  - Return either the step-2 solution, or the highest bidder, whichever creates more surplus.
- Theorem: Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum-possible surplus.
  - Proof: see the blackboard

The Revelation Principle

- Can non-DSIC mechanisms accomplish things that DSIC mechanisms cannot?
- Let’s tease apart two separate assumptions that are conflated in our DSIC definition:
    1. Every participant in the mechanism has a dominant strategy, no matter what its private valuation is.
    2. This dominant strategy is *direct revelation*, where the participant truthfully reports all of its private information to the mechanism.
- There are mechanisms that satisfy (1) but not (2). To give a silly example, imagine a single item auction in which the seller, given bids **b**, runs a Vickrey auction on the bids 2**b**.
    - Every bidder’s dominant strategy is then to bid half its value.

The Revelation Principle

- The Revelation Principle states that, given requirement (1), there is no need to relax requirement (2): it comes for free."
- Theorem **(Revelation Principle):** For every mechanism M in which every participant has a dominant strategy (no matter what its private information), there is an equivalent direct-revelation DSIC mechanism M_0.
  - Proof: See the blackboard



The Bayesian Setting

- A single-parameter environment
- The private valuation v_i of participant i is assumed to be drawn from a distribution F_i with density function f_i with support contained in $[0, v_{max}]$. We assume that the distributions $F_1,\ldots, F_n$ are independent (but not necessarily identical). In practice, these distributions are typically derived from data, such as bids in past auctions.
- The distributions $F_1,\ldots, F_n$ are known in advance to the mechanism designer. The realizations $v_1,\ldots,v_n$ of bidders' valuations are private, as usual. Since we focus on DSIC auctions, where bidders have dominant strategies, the bidders do not need to know the distributions $F_1,\ldots,F_n$.

Revenue Maximizing Auctions

- One bidder, one Item: If the seller posts a price of *r*, then its revenue is either *r* (if $v \geq r$) or 0 (if $v < r$)

$$\underbrace{r}_{\text{revenue of a sale}} \cdot \underbrace{(1 - F(r))}_{\text{probability of a sale}}$$

  - If F is the uniform distribution on [0,1], then the best price is 1/2 with revenue 1/4.
- The Vickrey auction on two bidders:
  - If both F_1 and F_2 are uniform distributions on [0,1], then the revenue is the expected value of the smallest bid i.e 1/3 (see the blackboard)
- The Vickrey auction with reserve price on two bidders
  - In a Vickrey auction with reserve *r*, the allocation rule awards the item to the highest bidder, unless all bids are less than *r*, in which case no one gets the item.
  - In previous example, adding a reserve price of 1/2 turns out to be a net gain, raising the expected revenue from 1/3 to 5/12 (see the blackboard)

Revenue Maximizing Auctions

- Our goal is to characterize the optimal (i.e., expected revenue maximizing) DSIC auction for every single-parameter environment and distributions $F_1,\ldots,F_n$.
- We know how to design surplus maximizing (social welfare maximizing) auctions. We show that optimal DSIC auctions are not new.
- Define virtual valuation $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$

Revenue Maximizing Auctions

- Theorem: The expected revenue of a DSIC auction is equal to the expected social welfare of the auction with virtual valuations, that is:

$$\mathbf{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} p_i(\mathbf{v})\right] = \mathbf{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \varphi_i(v_i) \cdot x_i(\mathbf{v})\right]$$

  - Proof: see the blackboard.

- Separately for each input **v**, we choose **x**(**v**) to maximize the virtual welfare $\sum_{i=1}^{n} \phi_i(v_i) x_i(v)$ obtained on the input **v** (subject to feasibility of $(x_1,\ldots, x_n) \in X$)

Revenue Maximizing Auctions

- Is this virtual welfare-maximizing rule monotone?
  - If so, then it can be extended to a DSIC auction, and by previous theorem, this auction has the maximum-possible expected revenue
  - The answer depends on F_is, If the corresponding virtual valuation function is increasing, then the virtual welfare-maximizing allocation rule is monotone.
- Definition: A distribution F is regular if the corresponding virtual valuation function $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.
- Ex: Single-item auction with i.i.d. bidders, under the additional assumption that the valuation distribution F is regular:
  - Since all bidders share the same increasing virtual valuation function, the bidder with the highest virtual valuation is also the bidder with the highest valuation.
  - This allocation rule is thus equivalent to the Vickrey auction with a reserve price of $\phi^{-1}(0)$

Near Optimal Auctions

- The optimal auction can get weird, and it does not generally resemble any auctions used in practice
  - Someone other than the highest bidder might win.
  - The payment made by the winner seems impossible to explain to someone who hasn't studied virtual valuations
- We seek for near optimal auctions which are simpler and more practical, but of course approximately optimal
- The tool for this: the Prophet Inequality

The Prophet Inequality

- Consider a game, with has n stages.
  - In stage i, you are offered a nonnegative prize π_i, drawn from a distribution G_i.
  - You are told the distributions $G_1,\ldots, G_n$, and these distributions are independent.
  - You are told the realization π_i only at stage i. After seeing π_i, you can either accept the prize and end the game, or discard the prize and proceed to the next stage.
- Theorem: For every sequence $G_1,\ldots, G_n$ of independent distributions, there is strategy that guarantees expected reward $\frac{1}{2} E_\pi[\max_i \pi_i]$. In fact, there is such a threshold strategy t, which accepts prize i if and only if $\pi_i \geq t$.
  - Proof: see the blackboard

Near Optimal Auctions

- With Prophet inequality we can now consider any allocation rule that has the following form:
  - Choose t such that $pr[\max_i \phi_i(v_i)^+ \geq t] = \frac{1}{2}$ $(z^+ = \max(0, z))$
  - Give the item to a bidder i with $\phi_i(v_i) \geq t$, if any, breaking ties among multiple candidate winners arbitrarily (subject to monotonicity)
  - The Prophet Inequality immediately implies that every auction with an allocation rule of the above type satisfies

$$\mathbf{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \varphi_i(v_i)^+ x_i(\mathbf{v})\right] \geq \frac{1}{2}\mathbf{E}_{\mathbf{v}}\left[\max_i \varphi_i(v_i)^+\right]$$

- Ex:
  - Set a reserve price $r_i = \phi_i^{-1}(t)$ for each bidder i, with t defined as above
  - Give the item to the highest bidder that meets its reserve (if any)

Near Optimal Auctions

- What should the seller do if he does not know, or is not confident about, the valuation distributions?
- The expected revenue of a Vickrey auction can obviously only be less than that of an optimal auction, but it can be shown that this inequality reverses when the Vickrey auction has more players
- Theorem (Bulow and Klemperer): *Let F be a regular distribution and n a positive integer. Then:*

$$\mathbf{E}_{v_1,\ldots,v_{n+1}\sim F}[Rev(VA)\ (n+1\ \ bidders)] \geq \mathbf{E}_{v_1,\ldots,v_n\sim F}[Rev(OPT_F)\ (n\ \ bidders)]$$

  - Proof: see the blackboard
- This theorem implies that in every such environment with $n \geq 2$ bidders, the expected revenue of the Vickrey auction is at least $\frac{n-1}{n}$ times that of an optimal auction. Why?

Multi-Parameter Mechanism Design

- Multi-parameter mechanism design problem:
  - n strategic participants, or agents
  - A finite set Ω of outcomes
  - Each agent i has a private valuation $v_i(\omega)$ for each outcome $\omega \in \Omega$.
  - Each participant i gives a bid $b_i(\omega)$ for each outcome $\omega \in \Omega$.
- In the standard single-parameter model of a single-item auction, we assume that the valuation of an agent is 0 in all of the n-1 outcomes in which it doesn't win, leaving only one unknown parameter per agent.
- Ex: in a bidding war over a hot startup, for example, agent i's highest valuation might be for acquiring the startup, but if it loses it prefers that the startup be bought by a company in a different market, rather than by a direct competitor.

VCG Mechanism

- Theorem **(The Vickrey-Clarke-Groves (VCG) Mechanism)** In every general mechanism design environment, there is a DSIC welfare-maximizing mechanism.
  - Proof: see the blackboard

$$\mathbf{x}(\mathbf{b}) = \operatorname*{argmax}_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega)$$

$$p_i(\mathbf{b}) = \underbrace{\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)}_{\text{without } i} - \underbrace{\sum_{j \neq i} b_j(\omega^*)}_{\text{with } i}$$

Ex: Combinatorial Auctions

- The model:
  - A combinatorial auction has n bidders for example, Verizon, AT & T, and several regional providers.
  - There is a set M of m items, which are *not* identical for example, a license awarding the right to broadcast on a certain frequency in a given geographic area.
  - The outcome set Ω corresponds to n-vectors $(S_1, \ldots, S_n)$, with S_i denoting the set of items allocated to bidder i (its bundle), and with no item allocated twice. There are $(n + 1)^m$ different outcomes.
  - Each bidder i has a private valuation $v_i(S)$ for each bundle $S \subseteq M$ of items it might get.
  - One generally assumes that $v_i(\emptyset) = 0$ and that $v_i(S) \leq v_i(T)$ whenever $S \subseteq T$

Ex: Combinatorial Auctions

- In principle, the VCG mechanism provides a DSIC solution for maximizing the welfare.
- Challenges:
  - Each bidder has 2^m-1 private parameters, roughly a thousand when m = 10 and a million when m = 20.
  - Even when the first challenge is not an issue, for example, when bidders are single-parameter and direct revelation is practical welfare-maximization can be an intractable problem.
  - The VCG mechanism can have bad revenue and incentive properties, despite being DSIC

Budget Constraints

- The simplest way to incorporate budgets into our existing utility model is to redefine the utility of player i with budget B_i for outcome ω and payment p_i as

$$\begin{array}{rl} v_i(\omega) - p_i & \text{if } p_i \leq B_i \\ -\infty & \text{if } p_i > B_i \end{array}$$

- The Vickrey auction charges the winner the second highest bid, which might well be more its budget. Since the Vickrey auction is the unique DSIC surplus-maximizing auction, in general, surplus-maximization is impossible without violating budgets.

Ex: Clinching Auctions

- There are m identical goods, and each bidder might want many of them
- Each bidder i has a private valuation v_i for each good that it gets, so if it gets k goods, its valuation for them is $k \cdot v_i$.
- Each bidder has a budget B_i that we assume is public, meaning it is known to the seller in advance.
- We define the demand of bidder i at price p as:

$$D_i(p) = \begin{cases} \min\left\{\lfloor \frac{B_i}{p} \rfloor, m\right\} & \text{if } p < v_i \\ 0 & \text{if } p > v_i \end{cases}$$

- If the price is above v_i it doesn't want any (i.e., $D_i(p) = 0$), while if the price is below v_i it wants as many as it can afford. When $v_i = p$ the bidder does not care how many goods it gets, thus $D_i(p^*)$'s for bidders i with $v_i = p^*$ is defined in a way that all m goods are allocated.

Ex: Clinching Auctions

- Using the market-clearing price:
  - Let p^* be the smallest price with $\sum_i D_i(p^*) = m$. Or, more generally, the smallest value such that $\lim_{p \to p^{*-}} \sum_i D_i(p) \geq m \geq \lim_{p \to p^{*+}} \sum_i D_i(p)$
- This auction respects bidders' budgets, but is not DSIC. For example:
  - Suppose there are two goods and two bidders, with $B_1 = +\infty$, $v_1 = 6$, and $B_2 = v_2 = 5$.
  - Truthful bidders: The total demand is at least 3 until the price hits 5, at which point $D_1(5) = 2$ and $D_2(5) = 0$. The auction thus allocates both goods to bidder 1 at a price of 5 each, for a utility of 2.
  - If bidder 1 falsely bids 3, it does better: Bidder 2's demand drops to 1 at the price 2.5, and the auction will terminate at the price 3, at which point $D_1(3)$ will be defined as 1. Bidder 1 only gets one good, but the price is only 3, so its utility is 3, more than with truthful bidding.
- The allocation rule in the market-clearing price auction is monotone, thus we can use Myerson's lemma. What about changing the allocation rule?

Ex: Clinching Auctions

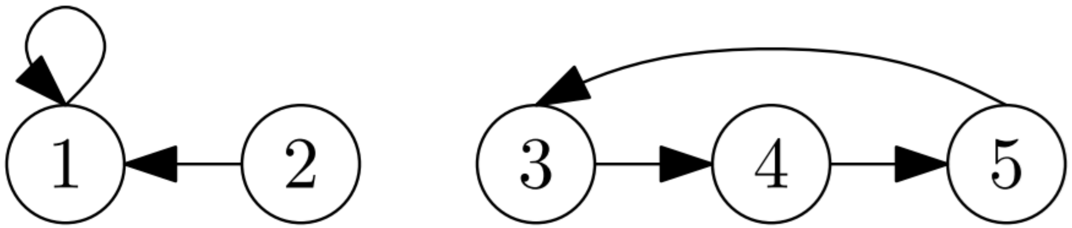
- The Clinching Auction:
  - Initialize p = 0, s = m, $\forall_i \widehat{B}_i = B_i$
  - While s > 0:
    - Increase p until there is a bidder i such that $k = s - \sum_{j \neq i} \widehat{D}_j(p) > 0$, where $\widehat{D}_j(p)$ is $\min\{\left\lfloor \frac{\widehat{B}_j}{p} \right\rfloor, s\}$ for $p < v_i$ and 0 for $p > v_i$.
    - Give k goods to bidder i at price p (theses good are clinched)
    - Decrease s by k
    - Decrease $\widehat{B}_i$ by p.k.
- The last example:
  - $D_2(p)$ drops to 1 at p = 2.5 and bidder 1 clinches one good at a this price.
  - The second good is sold to bidder 1 at price 5, as before.
  - Thus bidder 1 has utility 4.5 when it bids truthfully in the clinching auction.
- Theorem: The clinching auction for bidders with public budgets is DSIC.
  - Proof: see the blackboard

Mechanism Design Without Money

- There are a number of important applications where there are significant incentive issues but where money is infeasible or illegal.
- Mechanism design without money is relevant for designing and understanding methods for voting, organ donation, school choice, and labor markets.
- The designer’s hands are tied without money, even tighter than with budget constraints.
- There is certainly no Vickrey auction! Despite this, and strong impossibility results in general settings, some of mechanism design’s greatest hits are motivated by applications without money.

Ex: House Allocation Problem

- There are n agents, and each initially owns one house. Each agent has a total ordering over the n houses, and need not prefer their own over the others.
- How to sensibly reallocate the houses to make the agents better off?
- Top Trading Cycle Algorithm (TTCA):
  - While agents remain:
    - Each remaining agent points to its favorite remaining house. This induces a directed graph G on the remaining agents in which every vertex has out-degree 1



Ex: House Allocation Problem

- Top Trading Cycle Algorithm (TTCA):
  - While agents remain:
    - The graph G has at least one directed cycle. Self-loops count as directed cycles.
    - Reallocate as suggested by the directed cycles, with each agent on a directed cycle C giving its house to the agent that points to it, that is, to its predecessor on C.
    - Delete the agents and the houses that were reallocated in the previous step.
- Theorem: The TTCA induces a DSIC mechanism.
  - Proof: see the blackboard
- TTCA is in some sense optimal:
  - A core allocation: an allocation such that no coalition of agents can make all of its members better off via internal reallocations.
- Theorem: For every house allocation problem, the allocation computed by the TTCA is the unique core allocation.
  - Proof: see the blackboard