**Midterm Exam # 1 – Answers**
**Graduate Game Theory**

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**Due March 15th, 2010 at 3pm**

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This is an open book and open notes exam. Do not discuss the exam with anyone other than Professor Patty prior to 3pm March 15th. The point value is listed for each question. You may complete as many or as few of the problems as you like. You will receive no more than 100 points total, however.

1. `[10pts]` Find all Nash equilibria of the following 2x2 game.

| | L | R |
|---|---|---|
| U | (2,2) | (0,5) |
| D | (1,4) | (1,3) |

*Answer.* This game has one mixed strategy Nash equilibrium:

$$\begin{aligned} 2x + 4(1-x) &= 5x + 3(1-x) \\ (1-x) &= 3x \\ 1 &= 4x \\ x = Pr[U] &= 0.25 \end{aligned}$$

$$\begin{aligned} 2y &= y + 1 - y \\ 2y &= 1 \\ y = Pr[L] &= 0.5 \end{aligned}$$

The equilibrium is $((0.25, 0.75), (0.5, 0.5))$

2. `[10pts]` Find all Nash equilibria of the following 2x2 game.

| | L | R |
|---|---|---|
| U | (2,2) | (2,1) |
| D | (1,1) | (3,3) |

This game has three Nash equilibria: $(U, L)$, $(D, R)$, and $((2/3, 1/3), (1/2, 1/2))$.

$$\begin{aligned} 2x + (1-x) &= x + 3(1-x) \\ 1 + x &= 3 - 2x \\ 3x &= 2 \\ x = Pr[U] &= 2/3 \end{aligned}$$

$$\begin{aligned} 2y + 2(1-y) &= y + 3(1-y) \\ 2 &= 3 - 2y \\ 2y &= 1 \\ y = Pr[L] &= 0.5 \end{aligned}$$

3. [10pts] Find all Nash equilibria of the following 3x3 game.

| | L | M | R |
|---|---|---|---|
| U | (1,1) | (0,0) | (0,0) |
| M | (0,0) | (2,2) | (0,0) |
| D | (0,0) | (0,0) | (1,1) |

There are three pure strategy Nash equilibria: $(U, L)$, $(M, M)$, and $(D, R)$. There are also 4 mixed strategy equilibria.

These are:

$$((2/3, 1/3, 0), (2/3, 1/3, 0)),$$
$$((0, 1/3, 2/3), (0, 1/3, 2/3)),$$
$$((1/2, 0, 1/2), (1/2, 0, 1/2)),$$
$$((2/5, 1/5, 2/5), (2/5, 1/5, 2/5))$$

4. Consider the 2 player extensive form game displayed in Figure 1.

   (a) [10pts] Find all of the Nash equilibria of the game.
   (b) [5pts] Find all of the subgame perfect Nash equilibria of the game.

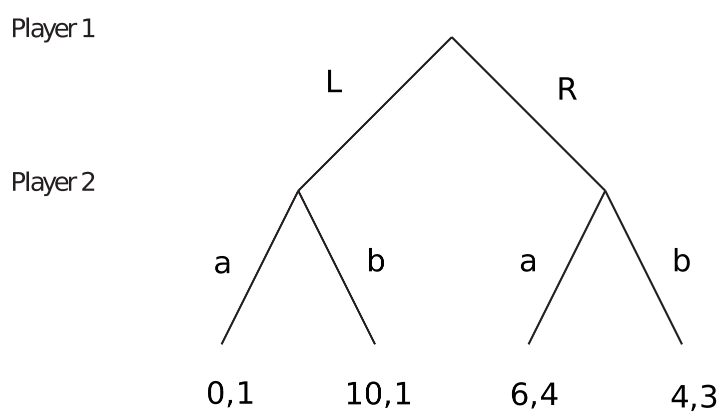


Figure 1:

*Answer.* The pure strategy Nash equilibria are:

$$(L,(b,a))^{\dagger}$$
$$(L,(b,b))$$
$$(R,(a,a))^{\dagger}$$

The mixed strategy Nash equilibria include the following:

$$(L,(b,Pr[a|R]=x)) \qquad \text{for any } x\in(0,1)$$
$$(R,(Pr[b|L]=x,Pr[a|R]=(10x-4)/2)) \qquad \text{for any } x\in(0.4,0.6)$$
$$(R,(Pr[b|L]=0.6x,a))^{\dagger} \qquad \text{for any } x\in(0,1)$$
$$(Pr[L]=x,(Pr[b|L]=0.6,a))^{\dagger}$$

Each Nash equilibrium in which Player 2 plays $a$ with probability one following $R$ is a subgame perfect Nash equilibrium. (These are marked with a † above.)

5. Consider the 2 player extensive form game displayed in Figure 2.

(a) [10pts] Find all of the Nash equilibria of the game.

(b) [5pts] Find all of the subgame perfect Nash equilibria of the game.

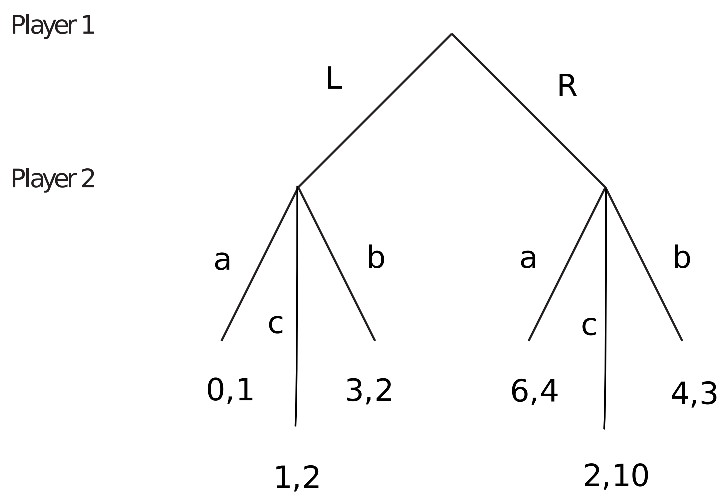


Figure 2:

*Answer.* The pure strategy Nash equilibria of this game are:

$$(L,(b,c))$$
$$(R,(a,c))$$
$$(R,(c,c))$$

The mixed strategy Nash equilibria are (where $(x,y,z|j)$ denotes $(\Pr[a|j],\Pr[c|j],\Pr[b|j])$ (*i.e.*, ordered left-to-right in the figure, rather than alphabetically)

$$(L,(b,(x,y,1-x-y|R))) \text{ such that } (x,y)\in[0,1]^2, x+y\le 1, \text{ and } 2x-2y+1\le 0$$
$$(R,((x,y,1-x-y|L),c) \text{ such that } (x,y)\in[0,1]^2, x+y\le 1, \text{ and } 2x+y\ge 0$$
$$(\alpha,((0,1/2,1/2|L),c) \text{ for any } \alpha\in[0,1]$$

$$\begin{aligned} 6x + 2y + 4(1 - x - y) &\leq 3 \\ 2x - 2y + 4 &\leq 3 \\ 2x - 2y + 1 &\leq 0 \end{aligned}$$

$$\begin{aligned} y + 2(1 - x - y) &\leq 2 \\ 2 - 2x - y &\leq 2 \\ 2x + y &\geq 0 \end{aligned}$$

The subgame perfect Nash equilibria are all Nash equilibria that involve player 2 playing $c$ after $R$ and assigning probability 0 to $a$ after $L$.

6. Consider the following situation. One player, the *agent*, is faced with a choice of how hard to work. This choice is denoted by $e \in [0,1]$, with $e = 1$ denoting full effort and $e = 0$ denoting zero effort. Following this choice, but in ignorance of it, a second player, the *boss*, must choose whether to audit the effort exerted by the agent. This choice is binary and denoted by $a$, with $a = 1$ if the boss audits and $a = 0$ otherwise. Auditing is unpleasant (it costs $c > 0$, but allows the boss to recoup half of the amount of effort that the agent did not exert (*i.e.*, it gains the boss $\frac{1-e}{2}$). The agent incurs a fixed penalty of $P > 0$ if he is caught exerting less than full effort. The players' payoffs are as follows:

$$u_{\text{agent}}(e,a) = \begin{cases} 1-e & \text{if } a = 0 \\ 1 & \text{if } a = 1 \& e = 1 \\ 1-e-P & \text{if } a = 1 \& e \neq 1 \end{cases}$$
$$u_{\text{boss}}(e,a) = e + \left(\frac{1-e}{2} - c\right)a$$

(a) `[5pts]` What are the players' strategy spaces?

*Answer.* The pure strategy spaces are $S_1 = [0,1]$ and $S_2 = \{0,1\}$. Mixed strategies are simply distributions over $S_1$ and $S_2$, respectively.

(b) `[10pts]` Are there any dominated strategies?

*Answer.* Yes, depending on $c$. If $c > 0.5$, then $a = 1$ is a strictly dominated strategy (as is any mixed strategy) for the boss. If $c = 0.5$, then $a = 1$ is a weakly dominated strategy (as is any mixed strategy) for the boss.

For the agent, any strategy $e \in (0,1)$ is strictly dominated by $e = 0$.

(c) `[10pts]` Find all of the Nash equilibria of the game.

*Answer.* Because $e \in (0,1)$ are all strictly dominated, they will not be played with positive probability in any Nash equilibrium. Accordingly, this is a $2x2$ simultaneous move game. Let $x$ denote the probability of $e = 1$ (with $e = 0$ being played with probability $1-x$) and $\alpha$ denote the probability of $a = 1$. A Nash equilibrium satisfies the following:

$$\begin{aligned} v_{\text{agent}}(e = 0|\alpha) &= 1 - P\alpha \\ v_{\text{agent}}(e = 1|\alpha) &= \alpha \\ v_{\text{agent}}(e = 0|\alpha) = v_{\text{agent}}(e = 1|\alpha) &\Rightarrow 1 - P\alpha = \alpha, \\ &\Rightarrow 1 = (1+P)\alpha, \\ &\Rightarrow \alpha = \frac{1}{1+P}. \end{aligned}$$

$$\begin{aligned} v_{\text{boss}}(a = 0|x) &= x \\ v_{\text{boss}}(a = 1|x) &= 0.5(1+x) - c \\ v_{\text{boss}}(a = 0|x) = v_{\text{boss}}(a = 1|x) &\Rightarrow x = 0.5(1+x) - c, \\ &\Rightarrow 0.5x = 0.5 - c, \\ &\Rightarrow x = 1 - 2c. \end{aligned}$$

Note that for $0 \leq c \leq 0.5$, $x \in [0,1]$. Thus, $(x, \alpha)$ define a mixed strategy Nash equilibrium. It can be verified by inspection that there are no pure strategy Nash equilibria. Thus, this is the unique Nash equilibrium.

7. Consider a simultaneous first price auction between two players. Each player $i$'s type, $t_i$, is privately observed by player $i$ and is uniformly distributed between 0 and 1. The two players' types are independently distributed. The players each simultaneously submit a bid, $b_i$. Player $i$'s payoff, for both $i \in \{1,2\}$ and $j \equiv 3-i$, is

$$u_i(b_i, b_j) = \begin{cases} t_i - b_i & \text{if } b_i > b_j, \\ \frac{t_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_j > b_i. \end{cases}$$

   (a) `[5pts]` What are the dominated strategies in this game? *Answer.* The strategy space is $S_i = \{f : [0,1] \to [0,1]\}$. A weakly dominated strategy is any $s_i \in S_i$ for which there exists a $t_i \in [0,1]$ such that $s_i(t_i) > t_i$.

   (b) `[10pts]` Find a perfect Bayesian equilibrium of this game.

   *Answer.* I will construct a symmetric PBE in linear strategies: $s_i = \alpha_i t_i$ for some $\alpha_i \in [0,1]$. The interim expected payoff (*i.e.*, conditional on $t_i$) from using $\alpha_i$, given $\alpha_j$, is

$$\begin{aligned} v_i(\alpha_i|\alpha_j, t_i) &= \begin{cases} t_i(1-\alpha_i) & \text{if } t_j\alpha_j < t_i\alpha_i \\ t_i(1-\alpha_i)/2 & \text{if } t_j\alpha_j = t_i\alpha_i \ , \\ 0 & \text{if } t_j\alpha_j > t_i\alpha_i \end{cases} \\ &= \Pr[t_j\alpha_j < t_i\alpha_i](1-\alpha_i)t_i, \\ &= \Pr\left[t_j < t_i\frac{\alpha_i}{\alpha_j}\right](1-\alpha_i)t_i, \\ &= \frac{\alpha_i}{\alpha_j}(1-\alpha_i)t_i^2. \qquad \text{(From } t_j \sim U[0,1]\text{)} \end{aligned}$$

   (Note that the probability of any given type is zero, so I can omit the case of $\alpha_i t_i = \alpha_j t_j$ from the calculation of player $i$'s expected payoff.) Let's maximize $v_i$ with respect to $\alpha_i$, given $\alpha_j$:

$$\begin{aligned} \partial_{\alpha_i} v_i(\alpha_i|\alpha_j, t_i) &= \partial_{\alpha_i}\frac{\alpha_i}{\alpha_j}(1-\alpha_i)t_i^2, \\ &= t_i^2\left(\frac{1-\alpha_i}{\alpha_j} - \frac{\alpha_i}{\alpha_j}\right), \\ &= t_i^2\left(\frac{1-2\alpha_i}{\alpha_j}\right). \end{aligned}$$

   So, supposing that $\alpha_j > 0$ (which will be verified in equilibrium) $\partial_{\alpha_i} v_i(\alpha_i|\alpha_j, t) = 0$ implies that $1 - 2\alpha_i = 0$, implying that $\alpha_i = 0.5$. (Note that the second order sufficient condition for maximization is verified here: $\partial^2_{\alpha_i^2} = \frac{-2t_i}{\alpha_j}$, which is negative for all $t_i > 0$. We don't need to worry about $t_i = 0$, because dominance uniquely pins down $i$'s strategy in this case.) Since this is not a function (thankfully) of $\alpha_j$, we have a PBE by setting $s_i^*(t_i) = 0.5t_i$ and beliefs (here these are over the type of the other player, $t_j$) be given by the uniform distribution.

8. Consider a sequential first price auction between two players. Each player $i$'s type, $t_i$, is privately observed by player $i$ and is uniformly distributed between 0 and 1. The two players' types are independently distributed. Player 1 submits a bid, $b_1$, which is observed

by player 2, who then submits a bid, $b_2$. Player $i$'s payoff, for both $i \in \{1,2\}$ and $j \equiv 3-i$, is

$$u_i(b_i, b_j) = \begin{cases} t_i - b_i & \text{if } b_i > b_j, \\ \frac{t_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_j > b_i. \end{cases}$$

(a) `[5pts]` What are the dominated strategies in this game?

*Answer.* The pure strategy space for player 1 is $S_1 = \{f : [0,1] \to [0,1]\}$. The pure strategy space for player 2 is $S_1 = \{f : [0,1] \times [0,1] \to [0,1]\}$, where $s_2(a,t)$ denotes the bid by player 2 after . A dominated strategy for player 1 is any $s_1$ for which there exists a $t \in [0,1]$ such that $s_1(t) > t$. For player 2, a dominated strategy is any $s_2$ for which

- there exists a $t \in [0,1]$ such that $s_2(a,t) > t$ or
- there exists a pair $(a,t)$ with $a < t$ such that $s_2(a,t) \le a$.

(b) `[10pts]` Find a perfect Bayesian equilibrium of this game.

*Answer.* This is a nasty problem due to an oversight on my part. I should have stated that player 2 wins the auction if the two players' submit identical bids. As currently stated, there is no pure strategy perfect Bayesian equilibrium, as the expected payoff function for player 2 is not a continuous function of his strategy. Another way around this is to presume that there is a smallest bid increment, $\delta$.

If one presumes that player 2 wins with probability 1 upon submitting $b_2 = b_1$, the following is perfect Bayesian equilibrium. First, set $s_2^*$ as follows:

$$s_2^*(b_1, t_2) = \begin{cases} b_1 & \text{if } b_1 < t_2, \\ 0 & \text{otherwise.} \end{cases}$$

Now, the calculation of player 1's best response is straightforward:

$$\begin{aligned} v_1(b_1 | t_1, s_2^*) =& (t_1 - b_1) \Pr[t_2 \le b_1] \\ =& (t_1 - b_1) b_1 \qquad\qquad \text{(From } t_j \sim U[0,1]\text{)} \end{aligned}$$

As in the previous question, maximization of this leads to the following best response: $s_1^*(t_1) = 0.5 t_1$.

9. Consider the following voting game. There are four alternatives, $a, b, c$, and $d$. Five players, $N = \{1,2,3,4,5\}$, must each cast a vote simultaneously for exactly one of the alternatives. The players preferences are given by the following:

| $i$ | $u_i(a)$ | $u_i(b)$ | $u_i(c)$ | $u_i(d)$ |
|---|---|---|---|---|
| 1 | 5 | 10 | 1 | 20 |
| 2 | 5 | 1 | 25 | 1 |
| 3 | 5 | 1 | 10 | 20 |
| 4 | 5 | 10 | 25 | 1 |
| 5 | 5 | 30 | 1 | 10 |

(a) Suppose that the electoral rule is simple plurality with a fair tie-breaking rule: if exactly two alternatives each get two votes, then a fair coin is flipped to determine which one wins. Players have von Neumann-Morgenstern payoff functions (i.e., their payoff from a lottery is the expected payoff from the lottery).

i. [5pts] What are the dominated strategies in this game?
*Answer.* The pure strategy space for any player $i$ is $S_i = \{a, b, c, d\}$. For any player $i$, voting for the alternative that gives him or her her the lowest payoff (in this example, $x \in S_i$ such that $u_i(x) = 1$) is a weakly dominated strategy (as is any mixed strategy that assigns positive probability to that alternative).

ii. [5pts] Find the pure strategy Nash equilibria of this game.
*Answer.* The Nash equilibria of this game are

- any profile in which four or more individuals are voting for the same alternative (as no one is pivotal),
- any profile in which every individual is voting sincerely between the same pair of alternatives, $(x, y)$ for each pair of alternatives $(x, y)$ in $\{a, b, c, d\}$,
- the profiles in which individuals 1 and 3 vote for $d$, individuals 4 and 5 vote for $b$, and individual 2 votes for any (or mixes between any pair or triple of) alternatives other than $c$.

(b) Suppose that the electoral rule is super-majority (four-fifths rule) with the tie-breaking defined as follows: if any alternative gets four or more votes, that alternative is selected. Otherwise, alternative $a$ is selected.

i. [5pts] What are the dominated strategies in this game?
*Answer.* As above, the pure strategy space for any player $i$ is $S_i = \{a, b, c, d\}$. For any player $i$, voting for the alternative that gives him or her her the lowest payoff (in this example, $x \in S_i$ such that $u_i(x) = 1$) is a weakly dominated strategy (as is any mixed strategy that assigns positive probability to that alternative).

ii. [5pts] Find the pure strategy Nash equilibria of this game.
*Answer.* The Nash equilibria of this game are

- any profile in which all five voters vote for the same alternative,
- any profile in which no alternative receives more than two votes,
- any profile in which any alternative, $x$, receives three votes and every voter who strictly prefers $x$ to $a$ is voting for $x$.
- any profile in which $a$ receives exactly four votes.

Most importantly, given the players' preferences and the nature of the rule, no alternative other than $a$ should ever receive exactly four votes.

(c) Consider a two-player vote-buying game as follows. Three legislators, $X$, $Y$, and $Z$, are considering how to vote on a bill banning sugary sodas from schools. Player 1 represents the Parents Again Yumminess and has \$1000 to contribute to the legislators. Player 2 represents the sugary soda industry trade association, Soda Industry Professionals, and has \$B dollars to contribute. The number $B$ is known by player 2, but is not observed by player 1. Player 1 knows that $B$ is drawn from the following distribution:

$$\Pr[B = 500] = \frac{1}{2}$$
$$\Pr[B = 750] = \frac{1}{2}$$

The game proceeds as follows. Player 2 observes $B$. Player 1 then allocates money between the legislators, $s_1 = (s_1(X), s_1(Y), s_1(Z)) \in [0, 1000]^3$ such that $s_1(X) + s_1(Y) + s_1(Z) \leq 1000$. Player 2 then observes $s_1$ and allocates money between the legislators, $s_2 = (s_2(X), s_2(Y), s_2(Z)) \in [0, B]^3$ such that $s_2(X) + s_2(Y) + s_2(Z) \leq B$. Finally, each legislator votes for or against the bill. Legislator $i \in \{X, Y, Z\}$ votes for the bill if either $\max[s_1(i), s_2(i)] = 0$ or if $s_1(i) > s_2(i)$. Legislator $i \in \{X, Y, Z\}$ votes against the bill if and only if $s_1(i) \leq s_2(i)$ and $s_2(i) > 0$. The bill passes if it gets at least two of the three votes. Otherwise, it fails.

The payoffs are as follows:

$$u_1(s_1, s_2) = \begin{cases} 2000 - s_1(X) - s_1(Y) - s_1(Z) & \text{if bill passes.} \\ 1000 - s_1(X) - s_1(Y) - s_1(Z) & \text{if bill fails.} \end{cases}$$

$$u_2(s_1, s_2, B) = \begin{cases} B - s_2(X) - s_2(Y) - s_2(Z) & \text{if bill passes.} \\ 2B - s_2(X) - s_2(Y) - s_2(Z) & \text{if bill fails.} \end{cases}$$

i. `[15pts]` Find a perfect Bayesian equilibrium of this game.

*Answer.* Again, I screwed up the tie-breaking rule and/or omitted a smallest bid. I will presume that bids have to be integers. First, consider the strategy space of player 2: $S_2\{f : \{0, 1000\}^3 \times \{500, 750\} \to \{0, 750\}^3 | \sum_{i=1}^{3} f^i(s_1, B) \leq B\}$. The set of dominated strategies concists of any strategy in which:

- $\min_{i \in \{X,Y,Z\}}[s_2(i)] > 0$ (there is no reason to buy all legislators' votes)
- $\max_{i \in \{X,Y,Z\}}[(s_2(i) - s_1(i))\, s_1(i)] > 0$ (no reason to pay more than $s_1(i)$ for $i$'s vote (unless $s_1(i) = 0$, in which case you have to pay 1)

There are others, but these are the useful ones to help derive an equilibrium.

Let $s_2$ be the strategy of buy the two least expensive legislators, and randomize between the three if all are equally inexpensive, so long as $B$ can be spent to acquire two votes. Otherwise, give each legislator 0.

Since player 2 will be able to buy two votes if $B = 750$,[1] player 1's expected payoff is given by the following:

$$v_1(s_1 | s_2) = (2000 - \sum s_1)(0.5) + (1000 - \sum s_1)(0.5) \text{ so long as } \left[\sum s_1 - \max_{i \in \{X,Y,Z\}} s_1(i)\right] > 500.$$

This reduces to

$$v_1(s_1 | s_2) = 1500 - \sum s_1 \quad \text{so long as } \left[\sum s_1 - \max_{i \in \{X,Y,Z\}} s_1(i)\right] > 500.$$

[1] Regardless of $B$, some pair of legislators must be given no more than two-thirds of player 1's budget, or 667 dollars. Thus, if $B = 750$, the cheapest pair will cost less than $B$.

Thus, player 1's strategic problem can be expressed as

$$\begin{aligned}&\min \sum s_1 \\ &\text{subject to } s_1 \in \{(a,b,c)\,[a+b+c-\max[a,b,c]] > 500\}\,.\end{aligned} \tag{1}$$

This is satisfied by any permutation of $(250, 251, 251)$. However, this is not a perfect Bayesian equilibrium, because the constrained program in (1) presumes that player 1 must successfully defend the bill. This, of course, is not the case. The expected payoff from this strategy is $0.5(248) + 0.5(1248) = 748$, whereas player 1 can ensure a payoff of 1000 by offering no legislator any payment. Then Player 2's best response along the equilibrium path, of course, is to pay two legislators 1 dollar each.