Game Theory. Final exam with solutions

February 3, 2012

Problem I (The sunday walk dilemma). Consider the following incomplete information game played between the father and the child during a walk. Half way through, nature decides whether the child is tired (which happens with probability of 1/3) or not. If the child is tired, she asks to be carried. If she is not tired, she can choose whether to admit she is not or to claim she is and request being carried. Father cannot tell whether the child is telling the truth. He can accept or reject the request when he gets one. If the child is not tired and ackonowledges this, both get the utility of zero. Child's utility decreases by one if she is tired. Further, her utility decreases by one if her request is rejected. Her utility increases by one when her request is accepted in case she is not tired and by two when it is accepted while she is indeed tired. Father's utility decreases by one when he takes the inappropriate action (rejects the request of the child that is really tired or accepts the request of the child when she is not).

a) draw the game tree

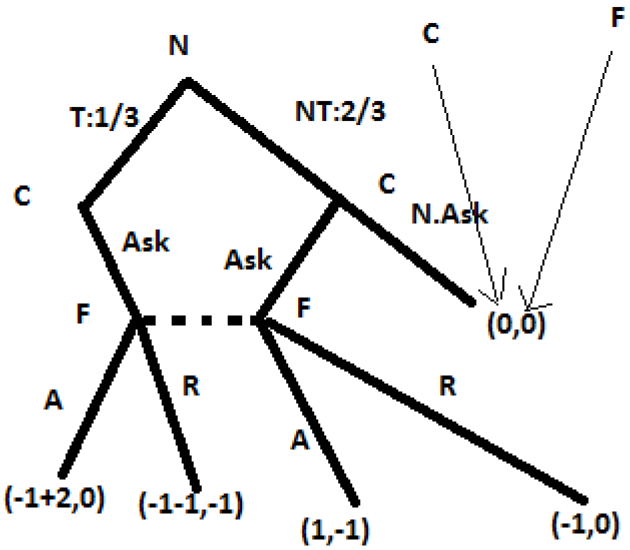
   Solution see figure 1.

b) find sequential equilibrium/equilibria

Solution The easiest way is to represent the game in a matrix form. The child has two strategies: Always Ask (AA) and Ask Only if Tired (AOiT). The father has strategies: Accept and Reject. Thus e.g. when the child has a strategy Always Ask and father playes strategy Reject, then with probability 1/3 she will be tired, will ask and father will reject so the payoffs will be (-2,-1). Else with probability 2/3 she will not be tired, will ask anyway, father will reject and the payffs will be (-1,0) so expected payoffs are $1/3\cdot(-2,-1)+2/3\cdot(-1,0)=(-4/3,-1/3)$. Analogous reasoning for other combinations yields the game matrix.

There is no pure strategy equilibrium. By equating expected payoffs from each strategy, we find that the child should asks if not tired with probability

Figure 1: Sunday Walk Game Tree



	Acc	Rej
AA	1*,-2/3	-4/3,-1/3*
AOiT	1/3,0*	-2/3*,-1/3

Table 1: Sunday Walk Game in normal form

.5, the father thus assign the probability of .5 to the possibility that she is tired when asks and accept also with probability of .5.

Common mistakes In the game tree: many of you forgot about Nature, draw it as if it was child's decision whether she was tired or not. Some made it more difficult by failing to note that the child only had a choice when not tired. Most of you forgot about linking the two nodes that are in the same info set (and some did it incorrectly). When looking for the sequential equilibrium: most of you just did some hand-weaving and came to wrong conclusions :-(.

Problem II. Four buyers compete in an auction. Each buyer i ($i = 1, 2, 3, 4$) observes her private signal s_i only. The singals are drawn independently from a uniform distribution on $[0, 1]$ and the value of the object is the mean of all signals.

a) Consider an ascending (English) auction. What is the maximum amount that a buyer can safely bid as long as nobody has dropped yet? Using arguments discussed in class, show that the buyer with lowest signal will actually drop at this amount in equilibrium.

   Solution The decision when to drop when nobody has dropped yet can only depend on your signal, $b_i(s_i)$. It is reasonable to assume that this is an increasing function. Given that nobody has dropped out yet, in a symmetric equilibrium (so that $b_i()$ functions are identical for all i's) you can infer that when you're about to drop, other people's signals are at least as high as yours. Thus also the value of the object (the mean of all signals) is at least as high as your signal, so you can safely bid up to your signal, s_i. This is also what everyone should do in equilibrium, i.e. $b_i(s_i) = s_i$ for all i. Suppose that you deviate from this behavior slightly. If somebody has a higher signal, you will not win anyway. If everyone has the same signal, then also the value of the object is indeed equal to your signal, so you should not stay any longer.

b) How will the other three bidders update their beliefs about the value of the object after the first bidder has dropped? What is now the maximum amount that any of them can safely bid? Again, show that the buyer with second lowest signal will drop at this amount.

Solution. Initially, any bidder i with singal s_i doesn't know others' singals, so he expects them to be .5 on average and his expected value of the object is $\frac{s_i+3\cdot .5}{4}$. Then, somebody drops out at $p_{(n)}$. Thus i infers this bidder's signal was $p_{(n)}$. Now the expected value of the object is $\frac{s_i+2\cdot .5+p_{(n)}}{4}$. And, as before, we can safely stay in the bidding under the assumptions that other people's signals are just as high as ours, except for the bidder that has already dropped out. So we can stay until the price reaches $\frac{s_{(4)}+3\cdot s_i}{4}$. Again, the same arguement goes: this is the only relevant case, because if one of the remaining two biddrs has a higher signal than i, i will not win anyway.

c) Who will drop at which price and what will be the seller's revenue if actual signals are $s_1 = .6, s_2 = .8, s_3 = .4, s_4 = .2$

   Solution Applying the logic of b), we get $s_4 = .2$, $s_3 = \frac{.2+3\cdot .4}{4} = .35$. Then bidder 1 already knows s_3 and s_4 and so he drops at $\frac{.2+.4+2\cdot .6}{4} = .45$ and that will be the seller's revenue. Btw., some of you had bids and values higher than 1, which is nonsensical if v is the *mean* of four signals lower or equal than 1.

d) What bids would the buyers with these signals submit in a second-price sealed-bid auction? What would be the seller's revenue?

   Solution Again, the only relevant case is when bidder i ties with another bidder. Thus if my signal is s_i I should assume that there is another player with s_i and the other two have lower signals, $s_i/2$ on average. Then the expected value is the mean of these four signals, $3/4s_i$. With these particular values bidder 2 will win and pay the seller .45 (this happens to be identical as in the ascending auction, but only due to equidistant signals. You could also remember the formula for the case when v is the sum of signals and simply divide by n.

Common mistakes Remembering models discussed in the class is one thing but most of all you need to carefully read the problems and think. If the value is the MEAN of four numbers ≤ 1, then why would anyone ever want to bid above 1? (but many of you said so). I have also seen many m's, (presumably due to confusion with the model in which the signal is the value +/- m, and the value has diffuse prior, but this is a totally different model. If there is no parameter called m in the problem, then how come it suddenly appears in the solution?) And no, ascending auction is NOT strategically equivalent to second price sealed-bid auctions in general. Because you can condition on actions of other players (when they drop out). The slides clearly say "Note: ascending auctions [...] are strategically equivalent to second price sealed-bid auctions *if* $n = 2$ *only.*" [emphasis added] (this is because in the case of $n = 2$ once you observe something you can condition on (i.e. the other guy quitting), it's too late to do anything anyway.)

Problem III (Football poker). Before the last round of O Rany Ekstraklasa the top of the table is as shown.

Table 2: O Rany Ekstraklasa table before the last round

	team	points	goal difference
1	Legia Włodawa	52	+37
2	Kurnik Zabrze	50	+30
3	Wyszła Krakow	49	+22
4	Jarosław Poznan	45	+15

Other teams cannot make it to the top four any more. In the last round Legia (L) is playinig Kurnik (K) and simultaneously Wyszła(W) is playing Jarosław (J). At the beggining of a match, both teams simultaneously choose a strategy: self-destructive (S), defensive (D) or aggresive (A). Self-destructive manner of playing always costs the team PLN 100 thousand worth of prestige. Similarly, defensive play costs PLN 10 thousand. When both teams play self-destructively or both play defensively, the match will be drawn. When only one team plays self-destructively, it will always lose. In other cases (aggresive vs. aggresive or aggresive vs. defensive), each outcome (home teams wins, draw, home team loses) is equally likely. Victory yields three points, draw one point and defeat zero points. When the dust cleares, the team with the greatest number of points wins the league, which is worth PLN 18 mln, the second gets PLN 6 mln and the third gets PLN 3 mln (in the case of equal number of points, goal difference decides and it is assumed that the ordering based on goal difference will not change. For example, if Legia loses its match and Wyszła wins, they will have 52 points each and it is assumed that Legia will still have better goal difference, so the final raking will be 1. Kurnik (53 points), 2. Legia (52), 3. Wyszła (52), 4. Jarosław (45)). Other than that, winning or losing makes no difference for the teams. Assume risk-neutrality.

Represent this game as a cooperative game: derive the characteristic function reflecting expected payoffs that each coalition can secure (hint: teams may sometimes want to fix a match – i.e. agree what the outcome is going to be, by using defensive or self-destructive strategies). Explain where each value comes from (what strategies will teams within the coalition and outside of coalition take. For example consider a coalition $\{K\}$. The strategies will be Kurnik: A, Legia: A or D, Wyszła: A or D, Jarosław: S. Kurnik will end up first if it wins the game (the chance of 1/3) and third otherwise (because Wyszła will have 52 pts.) and its prestige will not suffer, so $v(K) = 1/3 \cdot 18 + 2/3 \cdot 3 = 8$million).

Solution. The problem was a bit tedious so if someone could identify strategies and compute value for just a few coalitions, would get most points, as I said. Again: read carefully! There were quite a few poeople who lost time searching for the NE of the non-coop. game and even drawing game trees, although you were not asked to do so. Anyway, to find the solution, note that loss of prestige is only of secondary importance – you should think of the payoffs for final standing first. Also note that Jarosaw cannot make it to the top three anyway. Further, note that drawing is generally not a good deal (there is no match between teams that would be happy with just 1 point; further, there is loss of prestige if you play defensively, so teams will typically play Aggressive). Finally note that fixing can only happen within coalition S or within $N \backslash S$.

What is v(L)? Legia will end up first unless it loses its game and then it will be second. Strategies L:A, K:A; W,J: doesn't matter. $v(L) = 2/3 \cdot 18 + 1/3 \cdot 6$.

$\{K\}$: covered before

$\{W\}$: Legia and Kurnik can fix the match to make sure that W comes third rather than second: L:S, K:A or D and then Wyszła will be third, so v(W)=6

$\{J\}$: $v(J) = 0$, as said before

$\{L, K\}$: again, L and K can fix the match and take V(L,K)=18+6-.1 (the last term due to Legia's loss of prestige of 100 000 =.1 mio)

$\{L, W\}$: no reason to fix any match. L,W,K:A J:A or D. $v(L, W) = 2/3(18 + 1/3 \cdot 6 + 2/3 \cdot 3) + 1/3(6 + 3)$

$\{L, J\}$: Jarosaw makes no difference, L:A, K:A; W,J: doesn't matter. v(L,J)=v(L)=14

$\{K, W\}$: no fixing. They will have the third place for sure. They will also take second, unless Kurnik wins – then they take first. So $\{K, W\}$=$3 + 1/3 \cdot 18 + 2/3 \cdot 6$

$\{K, J\}$: no fixing. J,K:A, L:A or D, W:A. $v(K, J) = 1/3 \cdot 18 + 1/3(2/3 \cdot 6 + 1/3 \cdot 3) + 1/3(1/3 \cdot 6 + 2/3 \cdot 3)$

$\{W, J\}$: W and J could try to fix their match. But L and K will fix too (K wins), so J would only lose prestige and W would still only be third. $v(W, J) = 3$.

$\{L, K, W\}$: obvious, $v(L, K, w) = 18 + 6 + 3$

$\{L, K, J\}$: the games between L and K is fixed, same situation as $\{L, K\}$

$\{L, W, J\}$: J:S, W,L:A, K:A. Kurnik will end up first only when it wins its game, else it will be third. So $v(L, W, J) = 2/3 \cdot 18 + 1/3 * 3 + 6 - .1$

$\{K, W, J\}$: The same as $\{L, W\}$,

$\{L, K, W, J\}$: All Aggressive, $v = 18 + 6 + 3$

Problem IV. Mr. Akowski has 20 kg of tomatoes, Bekowski 20 kg and Cekowski 60 kg. Total demand for tomatoes is always 60 kg. Each seller can set the price at PLN1 or PLN2 per kg. Unsold tomatoes will be thrown away. Buyers will buy the cheaper tomatoes (if available) first, then possibly turning to the expensive ones. If supply at given price exceeds demand, sales will be reduced proportionally. E.g. when B sets p_B=1 while other players set $p_A = p_C = 2$, B will sell all of his tomatoes (20 kg). Remaining demand for tomatoes at PLN2 per kilogram is 60-20=40, while A and C jointly offer 80, so they will only sell one half of their stock each ($q_A = 20/2 = 10, q_C = 60/2 = 30$). Profits will be ($\pi_A = 10 * 2, \pi_B = 20 * 1, \pi_C = 30 * 2$).

a) Represent this game as a matrix game (two matrices corresponding to two different choices that C can make, A as row player, B as column player)

   Solution

Table 3: $p_C = 1$

	$p_B = 1$	$p_B = 2$
$p_A = 1$	12,12,36	15,0,45
$p_A = 2$	0,15,45	0,0,60

Table 4: $p_C = 2$

	$p_B = 1$	$p_B = 2$
$p_A = 1$	20,20,40	20,20,60
$p_A = 2$	20,20,60	24,24,72

b) Turn this game into a cooperative game

   Solution To turn it into a coalitional game, note that players that are not in the coalition will always choose $p = 1$, because it is generally bad for the coalition.

   $v(A) = 12$ (guaranteed by playing $p_A = 1$) and similarly $v(B) = 12$

   $v(C) = 40$ (by playing $p_C = 2$)

$v(A, B) = 24$ $(p_A = p_B = 1)$

$v(A, C) = 80$ $(p_A = p_C = 2)$ and similarly $v(B, C) = 80$

$v(A, B, C) = 120$ (all $p = 2$)

c) find the core

$120 - 80 \geq x_A \geq 12$
$120 - 80 \geq x_B \geq 12$
$120 - 24 \geq x_C \geq 40$
$x_A + x_B + x_C = 120$

d) compute the Shapley value

Table 5: Shapley Value

	A	B	C
ABC	12	12	96
ACB	12	40	68
BAC	12	12	96
BCA	40	12	68
CAB	40	40	40
CBA	40	40	40
Sh. V.	26	26	66

Solution See the table. Many of you forgot that the sum in each row must be the value of the grand coalition, i.e. 120. And many somehow thought that one-person coaltions must always be worth 0.