Algorithmic Game Theory

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RANDOMIZATION AND MIXED STRATEGIES

Mixed strategies

- So far, we have been discussing how to achieve NE by players selecting their ***pure strategies***

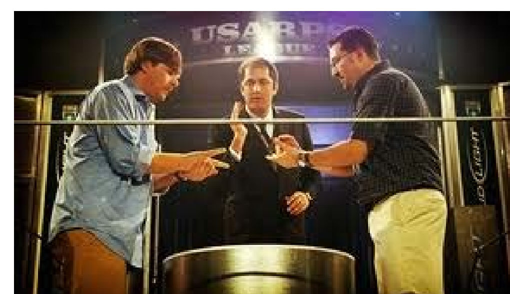
- In principle, players can also randomize over their pure strategies

- Let's see an example before being more formal

Rock, Scissors, Paper Game

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- Is there any dominated strategy?
- What is the NE of this game?
  - Notice the cycle?



- ***Pure strategies*** = {R, S, P}

Rock, Scissors, Paper Game

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- **Claim:** there is a NE if player choose with probability 1/3 each of his pure strategies
- How can we verify this is a NE?

Rock, Scissors, Paper Game

$$E\left[U_1\left(R,\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right]=\frac{1}{3}0+\frac{1}{3}1+\frac{1}{3}(-1)=0$$

$$E\left[U_1\left(S,\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right]=\frac{1}{3}(-1)+\frac{1}{3}0+\frac{1}{3}1=0$$

$$E\left[U_1\left(P,\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right]=\frac{1}{3}1+\frac{1}{3}(-1)+\frac{1}{3}0=0$$

$$\Rightarrow E\left[U_1\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right]=\frac{1}{3}0+\frac{1}{3}0+\frac{1}{3}0=0$$

Rock, Scissors, Paper Game

- In the RSP game, playing each strategy with probability 1/3 against a player **doing the same**, is a Nash Equilibrium

- We'll see in a moment that this is called a ***Mixed Strategies NE***

- Are you convinced it is indeed a BR?

Definition: Mixed strategies

A mixed strategy p_i is a randomization over i's pure strategies

- $p_i(s_i)$ is the probability that p_i assigns to pure strategy s_i
- $p_i(s_i)$ could be zero → in RSP: (1/2, 1/2, 0)
- $p_i(s_i)$ could be one → in RSP: 'P' a pure strategy if $p_i(P)=1$

Mixed Strategies

- The pure strategies are **embedded** in our mixed strategies

- Question: What are the payoffs from playing mixed strategies?
  - In particular, what is the ***expected payoff*** ?

Definition: Expected Payoffs

The expected payoff of the mixed strategy p_i
is the weighted average of
the expected payoffs of each of the
pure strategies in the mix of -i

- Basically, every player is mixing, hence you have to take the **joint probabilities** for a strategy profile to occur

The Battle of the Sexes

		Player 2: M	Player 2: N	
Player 1	M	2,1	0,0	1/5
	N	0,0	1,2	4/5
		½	½	

- Suppose the following mixed strategies:
  - Player 1: p = (1/5, 4/5)
  - Player 2: q = (1/2, 1/2)
- What is the Player 1's expected payoff by using p?

Expected Payoffs

$$E\left[U_1\left(M,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}2+\frac{1}{2}0=1$$

$$E\left[U_1\left(N,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}0+\frac{1}{2}1=\frac{1}{2}$$

$$E\left[U_2\left(\left(\frac{1}{5},\frac{4}{5}\right),M\right)\right]=\frac{1}{5}1+\frac{4}{5}0=\frac{1}{5}$$

$$E\left[U_2\left(\left(\frac{1}{5},\frac{4}{5}\right),N\right)\right]=\frac{1}{5}0+\frac{4}{5}2=\frac{8}{5}$$

Expected Payoffs

$$E\left[U_1\left(\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{5}1+\frac{4}{5}\frac{1}{2}=\frac{3}{5}$$

$$E\left[U_2\left(\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}\frac{1}{5}+\frac{1}{2}\frac{8}{5}=\frac{9}{10}$$

The expected payoffs for both players are computed as the weighted average of the pure strategies expected payoffs against the other player's mix

Important Observation

- Let's focus on player 1's expected payoff 3/5
- Obviously we have:

$$E\left[U_1\left(M,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=1$$

$$E\left[U_1\left(N,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}$$

$$\frac{1}{2}<\frac{3}{5}<1$$

The weighted average must lie between the two pure strategies expected payoffs

Observation

- The expected payoff from mixed strategies must lie between the pure strategies expected payoffs in the mixed
- This simple observation turns out to be the key to compute mixed strategies NE

If a mixed strategy is a best response then each of the pure strategies in the mix must itself be best responses

→ They must yield the same expected payoff

Main Lesson (Formal)

- If player i's mixed strategy p_i is a best response to the (mixed) strategies of the other players p_{-i}, then, for each pure strategy s_i such that $p_i(s_i) > 0$, it must be the case that s_i is itself a best response to p_{-i}

- In particular, $E[u_i(s_i, p_{-i})]$ must be the same for all such strategies

Sketch of Proof

- Suppose it was not true. Then there must be at least one pure strategy s_i that is assigned positive probability by my best-response mix and that yields a lower expected payoff against p_{-i}
- If there is more than one, focus on the one that yields the lowest expected payoff. Suppose I drop that (low-yield) pure strategy from my mix, assigning the weight I used to give it to one of the other (higher-yield) strategies in the mix
- This must raise my expected payoff
- But then the original mixed strategy cannot have been a best response: it does not do as well as the new mixed strategy
- This is a *contradiction*

Example

- Player 1's expected payoff 3/5
- Obviously we have:

$$E\left[U_1\left(M,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=1$$

× 1/5 **× 1**

BUT, what about?

$$E\left[U_1\left(N,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}$$

× 4/5 **× 0**

$$\frac{1}{2}<\frac{3}{5}<1$$

Definition: Mixed Strategies Nash Equilibrium

A mixed strategy profile $\left(p_1^*, p_2^*, \ldots, p_N^*\right)$ is a mixed strategy NE if for each player *i*:
p_i^* is a BR to p_{-i}^*

- This is the same definition of NE we've been using so far, except that we've been looking at pure strategies, and now we'll look at mixed ones

Observation

- Our informal lesson before implies that

$$\text{if} \quad p_i^*(s_i) > 0 \Rightarrow s_i^* \text{ is also a BR to } p_{-i}^*$$

- Let's play a game to fix these ideas