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Game Theory
Solutions to Problem Set 11

1. A seller owns an object that a buyer wants to buy. The value of the object to the seller is c. The value of the object to the buyer is private information. The buyer's valuation v is a random variable distributed over the interval $[0, V]$ according to the (continuous) c.d.f. F. Assume that $[1 - F(v)]/f(v)$ is a decreasing function of v. The von Neumann-Morgenstern utility of a type v from getting a unit at price p is $v - p$ and the utility of no purchase in 0.

(i) Suppose the seller is constrained to charge just one price. Show that the profit maximizing price satisfies $p = c + [1 - F(p)]/f(p)$.

(ii) Suppose that the seller can commit to a menu of offers $[q(v), p(v)]$, where $q(v)$ is the probability with which a consumer who chooses offer v will get a unit, and $p(v)$ is the price she will pay in the event that she gets a unit. Prove that the menu that maximizes the seller's profit consists of a single price, which is the one found in (i), and that any buyer can get the good at this price with probability 1.

Suggested Answer:
(i) The seller problem is to choose a price $p \in [0, V]$ to maximize:

$$\Pi(p) = [1 - F(p)](p - c) \tag{P}$$

Notice that $\Pi(p)$ is differentiable (thus continuous) and $[0, V]$ is compact. Furthermore, $-c = \Pi(0) < 0 = \Pi(V)$. Thus, the any solution is interior and requires:

$$\Pi'(p^*) = -f(p^*)(p^* - c) + [1 - F(p^*)] = 0$$

Thus:

$$p^* - c = \left(\frac{1 - F(p^*)}{f(p^*)}\right) \tag{1}$$

The LHS of (1) is a strictly increasing function of p^* while the RHS of (1) is decreasing (by assumption), thus the solution is unique.
(ii) The seller problem is to choose a menu $\{q(v), p(v)\}$, (where $p : [0, V] \to \Re$ and $q : [0, V] \to [0, 1]$) to maximize:

$$
\begin{array}{c}
\Pi = \max_{\{q(v),p(v)\}} \int_0^V [p(v) - q(v)c] f(v) dv \\
s.t. : \left\{ \begin{array}{c} U(v) = q(v)v - p(v) \geq q(v')v - p(v') \text{ for all } (v, v') \in [0, V]^2 \\ U(v) \geq 0 \text{ for all } v \in [0, V] \\ q \text{ nondecreasing} \end{array} \right.
\end{array} \tag{P2}
$$

Using the results presented in the lectures we know that $U'(v) = q(v)$. Substituting into the IC and imposing $U(0) = 0$ we have $U(v) = \int_0^v q(\tilde{v}) d\tilde{v}$. Thus:

$$
p(v) = q(v)v - \int_0^v q(\tilde{v}) d\tilde{v} \tag{2}
$$

Substituting 2 into P2 we have the following problem:

$$
\max_{\substack{q(v) \in [0,1] \\ q \text{ nondecreasing}}} \int_0^V [q(v)v - \int_0^v q(\tilde{v}) d\tilde{v} - q(v)c] f(v) dv \tag{3}
$$

Integrating (3) by parts we have:

$$
\begin{aligned}
& \int_0^V [q(v)v - \int_0^v q(\tilde{v}) d\tilde{v} - q(v)c] f(v) dv \\
= & \int_0^V q(v)(v - c) dv + \int_0^V \int_0^v q(\tilde{v}) d\tilde{v} d(1 - F(v)) \\
= & \int_0^V q(v)(v - c) dv - \int_0^V q(v) [1 - F(v)] dv \\
= & \int_0^V q(v) \left(v - c - \left[\frac{1 - F(v)}{f(v)} \right] \right) f(v) dv
\end{aligned}
$$

By assumption $\left[\frac{1-F(v)}{f(v)}\right]$ is decreasing. Thus there is a unique $v^* \in (0, V)$ such that

$$
v - c - \left[\frac{1 - F(v)}{f(v)} \right] \gtreqless 0 \Leftrightarrow v \gtreqless v^*
$$

Thus pointwise maximization implies:

$$
q^*(v) = \left\{ \begin{array}{l} 1 \text{ if } v \geq v^* \\ 0 \text{ if } v < v^* \end{array} \right.
$$

Notice that q^* is obviously nondecreasing, what guarantees the optimality of the presented solution. Finally, (2) implies that the price paid by all types that obtain the object is v^*.

2. Consider the following auction environment. A seller has a single object for sale and can commit to any selling mechanism (the seller's valuation of the object is zero). There are two potential bidders, indexed by $i = 1, 2$. The valuation of the object of bidder $i = 1, 2$ is denoted by v_i and is distributed uniformly over the unit interval. Valuations are independent between the two bidders. Bidder 1 knows her own valuation v_1. However, bidder 2 does *not* know v_2.

The bidders' payoffs are as follows. Suppose bidder $i = 1, 2$ has type v_i and pays the amount t_i to the seller. Her payoff is equal to $v_i - t_i$ if she gets the object, and equal to $-t_i$ otherwise.

(i) Construct the optimal direct mechanism for the seller (i.e., find the incentive compatible, individually rational mechanism that maximizes the seller's expected revenues). Compute the seller's revenues.

(ii) Can you find a simple *indirect* mechanism that gives to the seller the same expected revenues as the optimal direct mechanism?

Suggested Answer:
i) From the fact that bidders are risk-neutral bidder 2 behaves as if his valuation were $\frac{1}{2}$. The expected revenues of the seller are:

$$\int_0^1 \left[Q_1(v_1) \left(v_1 - \left(\frac{1 - F(v_1)}{f(v_1)} \right) \right) + Q_2(v_1) \frac{1}{2} \right] f(v_1) dv_1. \tag{4}$$

Using the fact that $F(v) = v$ (4) can be rewritten as:

$$\int_0^1 \left[Q_1(v_1) (2v_1 - 1) + Q_2(v_1) \frac{1}{2} \right] f(v_1) dv_1 \tag{5}$$

Maximizing (5) pointwise subject to $Q_i(v_1) \geq 0$, $Q_1(v_1) + Q_2(v_1) \leq 1$ we obtain:

$$\begin{aligned} Q_1(v_1) &= \begin{cases} 1 \text{ if } v_1 > \frac{3}{4} \\ 0 \text{ otherwise} \end{cases} \\ Q_2(v_1) &= 1 - Q_1(v_1) \end{aligned} \tag{6}$$

The solution is clearly nondecreasing. Finally notice that $T_1(v_1) = \frac{3}{4} \mathbf{1}_{\left\{v_1 > \frac{3}{4}\right\}}$. In order to calculate T_2 just notice that bidder 2 will obtain no rent. Thus: $T_2 = \frac{1}{2} \Pr(Q_2 = 1) = \frac{1}{2} \left(\frac{3}{4} \right) = \frac{3}{8}$.
ii) Indirect Mechanism: ask a price of $\frac{3}{4}$ to bidder 1 and $\frac{1}{2}$ to bidder 2. Sell to 2 only if 1 refuses to pay the price.

3. A seller has a unit for sale. Its quality is either high (H) or low (L). The quality is known to the seller but not to the buyer, whose prior probability that the quality is high is $1/2$. Their valuations of the unit are as follows.

	Quality H	Quality L
Buyer	V	2
Seller	7	0

where $V > 7$. Thus, the utility to the buyer of getting the unit at price p is $2 - p$ if it is of the low quality, and $V - p$ if it is of the high quality. Similarly, the utility to the seller is p and $p - 7$, respectively.

(i) Find the ex-post efficient outcomes.

(ii) Identify the range of V (above 7) for which there is, and the range of V for which there is no incentive compatible, individual rational mechanism that will achieve the ex-post efficient outcome.

(iii) Describe the best outcome (in the maximizing of the sum of expected utilities) that can be achieved for each V (above 7) and the mechanism that achieves it.

HINT: A mechanism for this Bayesian bargaining problem consists of a pair of functions $q : \{L, H\} \to [0, 1]$ and $t : \{L, H\} \to \mathbb{R}$, where $q(i)$ is the probability that the object will be sold to the buyer and $t(i)$ is the expected net payment from the buyer to the seller if $i = L, H$ is the type reported by the seller to a mediator.

Suggested Answer:

i) Ex-post efficiency means that in every state the party that values more the good, the buyer, always obtains the object.

ii) Since the object should always go to the buyer, the transfer made from the buyer to the seller, T, should be independent of the state (from IC). From (IR) T has to be as least 7.We now check under what values of V there exists a transfer greater than 7 satisfying the (IR) of the buyer. We need:

$$\frac{1}{2}V + \frac{1}{2}2 \geq 7 \Leftrightarrow V \geq 12$$

Thus we need $V \geq 12$.

iii) We will find the best menu: $\{P(L), T(L); P(H), T(H)\}$ where $P(L)$ $(P(H))$ is the probability that the seller keeps the object when he reports low (high) valuation and $T(L)$ $(T(H))$ is the transfer that the seller receives from the buyer when he reports low (high) valuation. We need to check the following constraints for the seller:

$$\begin{aligned} P(H)7 + T(H) &\geq P(L)7 + T(L) && \text{(ICHS)} \\ T(L) &\geq T(H) && \text{(ICLS)} \\ P(H)7 + T(H) &\geq 7 && \text{(IRHS)} \\ T(L) &\geq 0 && \text{(IRLS)} \end{aligned}$$

and only the rationality from the buyer:

$$\frac{1}{2}(1 - P(H))V + (1 - P(L))\frac{1}{2}2 \geq 0 \qquad \text{(IRB)}$$

Thus, we can set up surplus maximization: problem

$$\max_{\substack{P(H),P(L)\in[0,1]\\T(L),T(H)\geq 0}} (1-P(H))\,(V-7)+(1-P(L))2$$

$$s.t.:\begin{cases} T(L)\geq T(H). \\ P(H)7+T(H)\geq 7 \\ (1-P(H))V+(1-P(L))2-T(L)-T(H)\geq 0 \end{cases}$$

Notice that if $V \geq 12$ we are in (ii), then we assume $V \in (7,12)$.This problem can be easily solved by KT techniques. Rather, we give a somewhat more informal derivation using observations (a),(b), (c) and (d) below:

(a) $P(L) = 0$. This follows because both the objective function and (IRB) are strictly decreasing in $P(L)$. Thus, from (ii) we know that in any solution we need $P(H) > 0$.

(b) $T(L) = T(H)$. Otherwise one can increase $T(H)$ by ε, decrease $T(L)$ by ε and decrease $P(H)$ by $\frac{\varepsilon}{7}$. For ε small enough this is feasible and increases the objective function.

(c)$P(H)7+T(H) = 7$. Otherwise $P(H)$ can be decreased by some small ε, what increases the value the objective function.

(d)$(1-P(H))V+2-T(L)-T(H) = 0$. Otherwise we can increase both $T(L)$ and $T(H)$ by some small ε and decrease $P(H)$ by $\frac{\varepsilon}{7}$, what increases the value the objective function.

From (b) to (d) we 3 equations and 3 unknowns. Solving the system we have: $T_H = T_H = \frac{14}{14-V}$, $P(H) = \frac{12-V}{14-V}$. From (a) $P(L) = 0$.

4. A seller owns an object that a buyer wants to buy. The quality of the object is a random variable v, with support $[0,1]$ and distribution function $F(v) = v^{\alpha}$, where $\alpha > 0$. The seller knows the quality of the object but the buyer does not. When the quality of the object is v, the value of the object is v to the seller and zv to the buyer, where $z > 1$. Thus, if the object of quality v is traded at price p, the seller gets $p - v$ and the buyer gets $zv - p$. Both players have utility equal to zero if there is no trade.

Consider the function $G : (0,\infty)\times(1,\infty) \to [0,1]$ defined as follows. For each pair (α, z) construct the incentive-compatible individually rational mechanism that maximizes the (ex-ante) probability of trade. Denote this probability by $G(\alpha, z)$. Derive the function G.

(N.B. If the probability of trade is $q(v)$ when the quality is v, then the (ex-ante) probability of trade is equal to $\int_0^1 q(v)\,dF(v)$.

Suggested Answer:

Let $q(v)$ be the probability of trade given v and $t(v)$ the payment from the buyer to the seller given v. The seller (IC), the seller (IR) and the buyer (IR) are respectively:

$$t(v) - vq(v) \geq t(v') - vq(v') \tag{ICS}$$

$$U(v) = t(v) - vq(v) \geq 0 \tag{IRS}$$

$$\int_0^1 \left(zvq(v) - t(v)\right) f_\alpha(v)dv \geq 0 \tag{IRB}$$

where $f_\alpha(v) = \alpha v^{\alpha-1}$.
Notice that (ICS) and (IRS) immediately imply that if $U(1) \geq 0$ then $U(v) \geq 0$ for all v. Furthermore usual analysis implies $U'(v) = q(v)$ and $q(v)$ nonincreasing. From (IRS) we have;

$$U(v) = \int_v^1 q(x)dx + U(1)$$

Thus:

$$t(v) = vq(v) + \int_v^1 q(x)dx + U(1) \tag{7}$$

Substituting (7) into (IRB) we have:

$$\int_0^1 \left(zvq(v) - vq(v) - \int_v^1 q(x)dx - U(1)\right) f_\alpha(v)dv \geq 0 \tag{8}$$

From (8) we can set $U(1) = 0$.
Integrating by parts we have:

$$\begin{aligned} \int_0^1 \int_v^1 q(x)dx f_\alpha(v)dv &= \int_0^1 \int_v^1 q(x)dx dF_\alpha(v) \\ &= \int_0^1 q(v)F_\alpha(v)dv = \int_0^1 q(v)v^\alpha dv \end{aligned} \tag{9}$$

Substituting (9) into (8) we have:

$$\begin{aligned} &\int_0^1 q(v)\left(zv - v - \frac{v}{\alpha}\right) f_\alpha(v)dv \\ = &\int_0^1 q(v)v\left(z - 1 - \frac{1}{\alpha}\right) f_\alpha(v)dv \end{aligned}$$

Thus if $z - 1 - \frac{1}{\alpha} \geq 0$ which holds iff $z \geq \left(\frac{\alpha+1}{\alpha}\right)$ we have $q(v) = 1$ for every v, otherwise $q(v) = 0$ for every v. Therefore the probability of trade is:

$$G(\alpha, z) = \begin{cases} 1 \text{ if } z \geq \left(\frac{\alpha+1}{\alpha}\right) \\ 0 \text{ otherwise} \end{cases}$$