Spring 2012 Final Solutions

1.

a. Only considering the game in state 1, we find 2 pure-strategy NE by the underlining/highlighting method at (Up,Left) and (Down,Right). .

Emily

John

	Left	Right
Up	(2 , 2)	(0 , 2)
Down	(2 , 0)	(1 , 1)

In addition, there is the potential for mixed strategy NE.

For the mixed strategy NE, consider player 1 mixing on Up and Down with probabilities p and 1-p. Player 2 mixes Left and Right with probabilities q and 1-q.

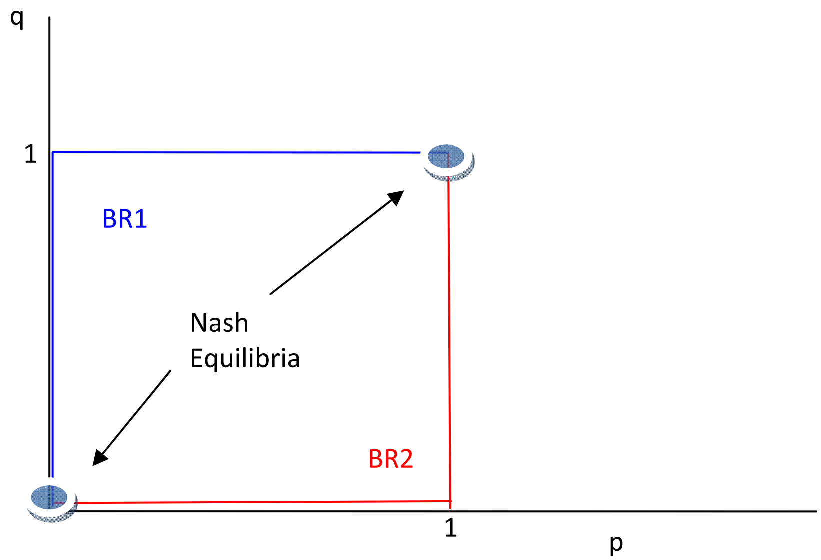
Player 1 solves:

$$\text{Max}(p)\ \{2pq + 0p(1-q) + 2(1-p)q + 1(1-p)(1-q)\ \}$$
$$\text{Max}(p)\ \{2pq + 2q-2pq +1-p -q + pq\ \}$$
$$\text{Max}(p)\ \{p(q - 1) + q +1\ \}$$
$$\rightarrow p=0 \text{ if } q-1<0 \Leftrightarrow q < 1$$
$$\rightarrow p \in [0,1] \text{ if } q-1 = 0 \Leftrightarrow q = 1$$

Player 2 solves:

$$\text{Max}(q)\ \{2pq + 2p(1-q) + 0(1-p)q + 1(1-p)(1-q)\ \}$$
$$\text{Max}(q)\ \{2pq +2p - 2pq +1-p-q+pq\ \}$$
$$\text{Max}(q)\ \{p +1-q+pq\ \}$$
$$\text{Max}(q)\ \{q(p-1) + p +1\ \}$$
$$\rightarrow q=0 \text{ if } p-1<0 \Leftrightarrow p < 1$$
$$\rightarrow q \in [0,1] \text{ if } p-1 = 0 \Leftrightarrow p = 1$$

We now plot the best response correspondences in choice space (p,q) which will reveal all Nash Equilibria of the game.



From the intersections of the Best Response curves, we can see there are no additional NE beyond the pure strategy NE. All NE can be written:

{ [(1,0),(1,0)] , [(0,1),(0,1)] }.

b. Each player is now uncertain about the state of the world (both are equally likely), so they cannot condition their strategy on what state of the world they are in. Therefore both players still have 2 strategies each.

c. Since each state of the world is equally likely, we find the BNE by determining each player's expected payoffs from all combinations of the two players strategies. The game becomes:

Emily

John

	Left	Right
Up	(0 , 0)	(1 , 1)
Down	(1 , 1)	(3/2, 3/2)

Therefore the unique BNE is (Down, Right). Notice it is also strict and dominant for both players.

2. Signaling
    a. Note that since the receiver's information set following L is "off the equilibrium path," his beliefs, (p,1-p) do not depend on α. However, since both types of sender choose R, q = α. Therefore, we require:

$$E_{Rec}[D|R] >= E_{Rec}[U|R]$$
$$5(1-\alpha) >= \alpha$$
$$6\alpha <= 5$$
$$\alpha <= 5/6$$

    b. Given the sender plays (L,L). Then p = 0.5. So following L, the receiver compares $E_{Rec}[U|L] = 2*0.5 = 1$ to $E_{Rec}[D|L] = 1(1-0.5) = 0.5$. So if he sees L, the receiver plays U.

    Next, consider the sender's payoff from playing L. A strong and weak sender gets 1 and 0 respectively. A strong sender could deviate to R and get either 0 or 3 if the Receiver plays U or D following the R signal respectively. So in order for the strong sender to NOT want to deviate, we need the receiver to play U at the right. In order for the weak sender to NOT want to deviate, he also requires that the receiver have beliefs following the R signal such that U is better than D.

$$E_{Rec}[U|R] >= E_{Rec}[D|R]$$
$$q > = 5(1-q)$$
$$6q >= 5$$
$$q >= 5/6$$

    Thus our PBE is:

$$PBE = \{ (L,L), (U,U), (p,1-p), (q,1-q) \mid p = 0.5, q >= 5/6 \}$$

    c. Each player has 2 information sets. The sender's singleton nodes following the two choices of nature, and the receiver's two non-singleton information sets following the sender's L signal and following the sender's R signal.

3. Consider the game in strategic form:

Player 2

Player 1		**Left**	**Right**
	Up	**(Y , B)**	**(0 , A)**
	Down	**(X , 0)**	**(Z , C)**

a. The unique pure-strategy Nash Equilibrium (by the underlining/highlighting method) is (Down Right).

b. Grim trigger strategies to sustain (Y,B) as the average per-period payoff of the infinitely repeated game are as follows:

σ_1 = { Play Up in the first periods and in all subsequent periods if (Up,Left) has been played by players 1 and 2 respectively in all periods. Play Down otherwise. }

σ_2 = { Play Left in the first periods and in all subsequent periods if (Up,Left) has been played by players 1 and 2 respectively in all periods. Play Right otherwise. }

c. Payoffs along the equilibrium path are then:

$$\pi_1^e = Y(1 + \delta_1 + \delta^2{}_1 + \delta^3{}_1 + \ldots) = Y/(1- \delta_1)$$
$$\pi_2^e = B(1 + \delta_2 + \delta^2{}_2 + \delta^3{}_2 + \ldots) = B/(1- \delta_2)$$

Payoffs from deviating (in the first period):

$$\pi_1^d = X + Z(\delta_1 + \delta^2{}_1 + \delta^3{}_1 + \ldots) = X + Z\delta_1/(1-\delta_1)$$
$$\pi_2^d = A + C(\delta_2 + \delta^2{}_2 + \delta^3{}_2 + \ldots) = A + C\delta_2/(1-\delta_2)$$

The critical discount factor that sustains cooperation (Up,Left) in all periods satisfies:

$$\pi_1^e >= \pi_1^d$$
$$Y/(1- \delta_1)>=X + Z\delta_1/(1-\delta_1)$$
$$Y >= X-X\delta_1 + Z\delta_1$$
$$Y-X >= \delta_1(Z-X)$$
$$(Y-X)/(Z-X) <= \delta_1$$
$$\delta_1^* >= (X-Y)/(X-Z)$$

$$\pi_2^e >= \pi_2^d$$
$$B/(1- \delta_2)>=A + C\delta_2/(1-\delta_2)$$
$$B >= A-A\delta_2 + C\delta_2$$
$$B-A >= \delta_2(C-A)$$
$$(B-A)/(C-A) <= \delta_2$$
$$\delta_2^* >= (A-B)/(A-C)$$

*Note (Z-X) and (C-A) are both negative so the sign switches when you divide both sides by a negative number. The last step, I just multiply the top and bottom by -1 so the numerators and denominators are both positive.

d. Using the following values, our critical discount rates become:

A = 30	**X = 40**
B = 25	**Y = 20**
C = 5	**Z = 10**

$$\delta_1^* >= (X-Y)/(X-Z)$$
$$\delta_1^* >= (40-20)/(40-10)$$
$$\delta_1^* >= 2/3$$

$$\delta_2^* >= (A-B)/(A-C)$$
$$\delta_2^* >= (30-25)/(30-5)$$
$$\delta_2^* >= 1/5$$

So player 1 needs to be much more patient than player 2 in order for player 1 not to want to deviate from (Up,Left). The intuition comes from comparing the three key payoffs for the players: the cooperation payoff, the deviation payoff, and the punishment payoff. All of these are stacked against player 1, which means he must value the future a lot in order to not be tempted by the short-term gains from deviating.

Namely,

Player 1's cooperation payoff is lower (Y<B)
Player 1's deviation payoff is higher (X>A)
Player 1's punishment payoff is higher (Z>C)

All of these mean that player 1 doesn't get as much from cooperating, benefits relatively more by deviating, and even in the punishment phase, player 1 does better than player 2.