Spring 2011 Final Solutions

1.

a. By the underlining method, there are two pure strategy NE at (B,L) and (T,R).

Player 2

Player 1		**L**	**C**	**R**
	T	**(10 , 10)**	**(0 , 10)**	**(30 , 20)**
	B	**(30 , 20)**	**(0 , 10)**	**(10 , 10)**

b. Player 1 does not have any strictly dominated strategies. However, player 2 can strictly dominate C with a mixture of L and R. Specifically a mixture of L and R with weights a and 1-a where a > 0 and a < 1 will dominate C. I.e., non-zero weight must be placed on L and R to be sure the expected payoff of the mixture is strictly greater than the payoff player 2 gets from playing C alone (10).

c. For the mixed strategy NE, consider player 1 mixing on T and B with probabilities p and 1-p. Player 2 mixes L and R with probabilities q and 1-q.

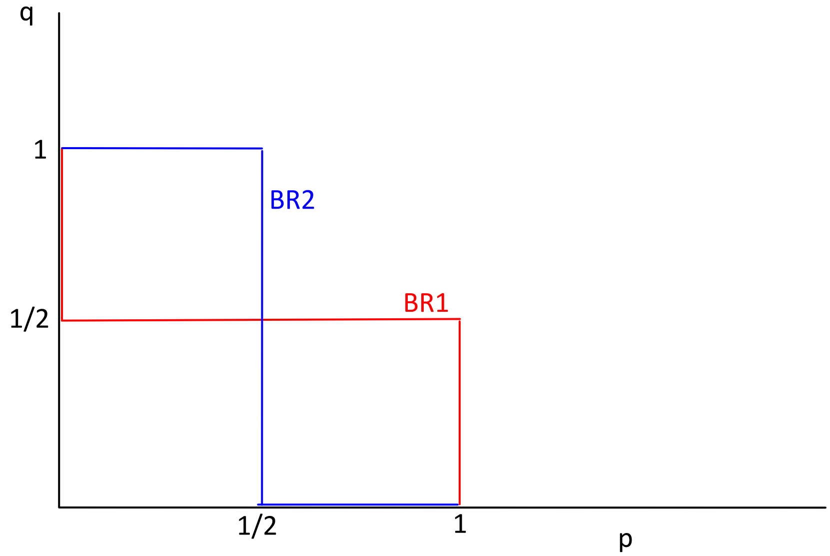
Player 1 solves:

$$\text{Max}(p)\ \{10pq + 30p(1-q) + 30(1-p)q + 10(1-p)(1-q)\ \}$$
$$\text{Max}(p)\ \{p(10q + 30 - 30q - 30q\ \ -10 + 10q) + 30q + 10 - 10q\ \}$$
$$\text{Max}(p)\ \{p(20 - 40q) + 30q + 10 - 10q\ \}$$
$$\rightarrow p=0 \text{ if } 20-40q < 0 \Leftrightarrow q > \tfrac{1}{2}$$
$$\rightarrow p=1 \text{ if } 20-40q > 0 \Leftrightarrow q < \tfrac{1}{2}$$
$$\rightarrow p \in [0,1] \text{ if } 20-40q = 0 \Leftrightarrow q = \tfrac{1}{2}$$

Player 2 solves:

$$\text{Max}(q)\ \{10pq + 20p(1-q) + 20(1-p)q + 10(1-p)(1-q)\ \}$$
$$\text{Max}(q)\ \{q(10p + 20 - 20p - 20p\ \ -10 + 10p) + 20p + 10 - 10p\ \}$$
$$\text{Max}(q)\ \{q(10 - 20p) + 20p + 10 - 10p\ \}$$
$$\rightarrow q=0 \text{ if } 10-20p < 0 \Leftrightarrow p > \tfrac{1}{2}$$
$$\rightarrow q=1 \text{ if } 10-20p > 0 \Leftrightarrow p < \tfrac{1}{2}$$
$$\rightarrow q \in [0,1] \text{ if } 10-20p = 0 \Leftrightarrow p = \tfrac{1}{2}$$

We now plot the best response correspondences in choice space (p,q) which will reveal all Nash Equilibria of the game.



From the intersections of the Best Response curves, we can see all NE are $\{[(0,1),(1,0,0)], [(1,0),(0,0,1)], [(1/2,1/2), (1/2,0,1/2)]\}$.

2.

a. Note that since the receiver's information set following L is "off the equilibrium path," his beliefs, (p,1-p) do not depend on α. However, since both types of sender choose R, q = α. Therefore, we require:

$$E_{Rec}[D|R] >= E_{Rec}[U|R]$$
$$2(1-\alpha) >= \alpha$$
$$3\alpha <= 2$$
$$\alpha <= 2/3$$

b. Given the sender plays (L,L). Then p = 0.2. So following L, the receiver compares $E_{Rec}[U|L]$ = 2*0.2 = 0.4 to $E_{Rec}[D|L]$ = 1(1-0.2) = 0.8. So if he sees L, the receiver plays D.

   Next, consider the senders payoff from playing L. A strong and weak sender gets 2 and 3 respectively. A weak sender cannot profitably deviate since he is already attaining his highest potential payoff. However, a strong sender could deviate and get either 0 or 3. So a strong sender requires that a receiver play U following the R signal (so the strong sender does NOT want to deviate from L). So we need:

$$E_{Rec}[U|R] >= E_{Rec}[D|R]$$
$$q > = 2(1-q)$$
$$3q >= 2$$
$$q >= 2/3$$

   Thus our PBE is:

   PBE = { (L,L), (D,U), (p,1-p), (q,1-q) | p = 0.2, q >= 2/3 }

c. There is only one subgame (the game itself) because starting from the only two singleton information sets (each of the sender's nodes), the potential subgame would cut one of the receiver's information sets.

3. Consider the game in strategic form:

		Player 2		
		X	Y	Z
Player 1	A	(1 , 1)	(0 , -2)	(10 , -3)
	B	(-2 , 0)	(0 , 0)	(-1 , 0)
	C	(-3 , 10)	(1 , -1)	(5 , 5)

a. The unique pure-strategy Nash Equilibrium (by the underlining/highlighting method) is (A,X).
b. Grim trigger strategies to sustain (5,5) as the average per-period payoff of the infinitely repeated game are as follows:

σ_1 = { Play C in the first periods and in all subsequent periods if (C,Z) has been played by players 1 and 2 respectively in all periods. Play A otherwise. }

σ_2 = { Play Z in the first periods and in all subsequent periods if (C,Z) has been played by players 1 and 2 respectively in all periods. Play X otherwise. }

Payoffs along the equilibrium path are then:

$$\pi_i^e = 5(1 + \delta + \delta^2 + \delta^3 + \ldots) = 5/(1- \delta)$$

Payoffs from deviating (in the first period):

$$\pi_i^d = 10 + 1(\delta + \delta^2 + \delta^3 + \ldots) = 10 + \delta/(1-\delta)$$

The critical discount factor that sustains cooperation (C,Z) in all periods satisfies:

$$\pi_1^e >= \pi_1^d$$
$$5/(1- \delta)>=10 + \delta/(1-\delta)$$
$$5>=10-10\delta + \delta$$
$$9\delta>=5$$
$$\delta >= 5/9 \sim 0.56 = \delta^*$$

c. Assuming δ = 0.6 and T periods of limited punishment, we only need to compare the aggregate discounted profit from T+1 periods of cooperation with the aggregate discounted profit from a deviation in the first period, followed by T periods of the punishment phase. (Note we are comparing an equal number of periods (T+1) in both cases).

Along the equilibrium path:

$$\pi_i^e = 5(1+\delta+\delta^2+\ldots+\mathrm{d}^{\mathrm{T}}) = 5 * \sum_{t=0}^{\mathrm{T}} \delta^{\mathrm{t}}$$
$$= 5\left[\frac{(1-\delta^{T+1})}{1-\delta}\right]$$
$$= 5\left[\frac{(1-0.6^{T+1})}{1-0.6}\right]$$
$$= 12.5(1-0.6^{T+1})$$

And the deviation path payoff:

$$\pi_i^d = 10 + 1(\delta+\delta^2+\ldots+\mathrm{d}^{\mathrm{T}}) = 10 + \delta\sum_{t=0}^{\mathrm{T-1}} \delta^{\mathrm{t}}$$
$$= 10 + \delta\left[\frac{(1-\delta^{T})}{1-\delta}\right]$$
$$= 10 + 0.6\left[\frac{(1-0.6^{T})}{1-0.6}\right]$$
$$= 10 + 1.5*(1-0.6^{T})$$
$$= 11.5 - 1.5(0.6^{T})$$

So cooperation is optimal if:

$$12.5(1-0.6^{T+1}) \geq 11.5 - 1.5(0.6^{T})$$
$$1.5(0.6^{T}) - 12.5*(0.6^{T+1}) \geq -1$$
$$1.5(0.6^{T}) - 12.5*(0.6^{T} * 0.6^{1}) \geq -1$$
$$1.5(0.5^{T}) - 7.5*(0.6^{T}) \geq -1$$
$$0.6^{T}(1.5-7.5) \geq -1$$
$$0.6^{T} \leq 1/6$$
$$T * \log(0.6) \leq \log(1/6)$$
$$T \geq \log(1/6)/\log(0.6)$$
$$T \geq -1.79/-0.51 \sim 3.5$$

So as long as **T ≥ 4 periods**, cooperation is optimal. (Note, log(0.6) < 0 so we need to switch the sign when dividing by a negative.)

d. Assuming δ = 0.5, the simple answer is there does NOT exist a T that will yield cooperation in all periods. Why? Because we know the grim trigger critical discount factor in part (b) was 5/9. Since we are using a less draconian trigger strategy (limited punishment), even a very long punishment phase will not yield cooperation because 0.5

< 5/9. Note that a limited punishment strategy yields the grim trigger critical discount factor as the punishment period approaches infinity. So the critical discount factor required to sustain cooperation will be larger than 5/9 for any finite T.

The long answer (if you chose to do the math) is as follows:
Along the equilibrium path:

$$\pi_i^e = 5(1+\delta+\delta^2+\ldots+\mathrm{d}^{\mathrm{T}}) = 5*\sum_{t=0}^{\mathrm{T}}\delta^{\mathrm{t}}$$

$$= 5\left[\frac{(1-\delta^{T+1})}{1-\delta}\right]$$

$$= 5\left[\frac{(1-0.5^{T+1})}{1-0.5}\right]$$

$$= 10(1-0.5^{T+1})$$

And the deviation path payoff:

$$\pi_i^d = 10+1(\delta+\delta^2+\ldots+\mathrm{d}^{\mathrm{T}}) = 10+\delta\sum_{t=0}^{\mathrm{T-1}}\delta^{\mathrm{t}}$$

$$= 10+\delta\left[\frac{(1-\delta^{T})}{1-\delta}\right]$$

$$= 10+0.5\left[\frac{(1-0.5^{T})}{1-0.5}\right]$$

$$= 10+(1-0.5^{T})$$

$$= 11-0.5^{T}$$

So cooperation is optimal if:

$$10(1-0.5^{T+1}) \geq 11-0.5^{T}$$

$$0.5^{T} - 10*(0.5^{T+1}) \geq 1$$

$$0.5^{T} - 10*(0.5^{T}*0.5^{1}) \geq 1$$

$$0.5^{T} - 5*(0.5^{T}) \geq 1$$

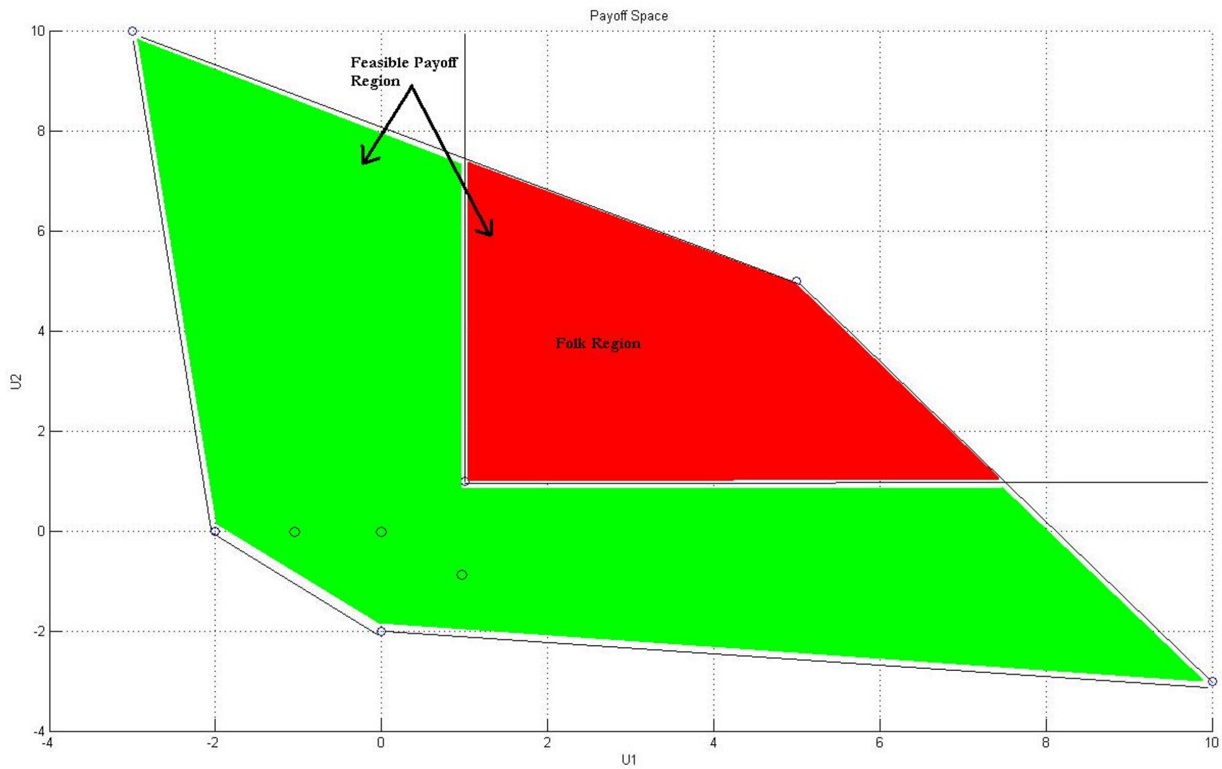
$$0.5^{T}(1-5) \geq 1$$

$$0.5^{T} \leq -1/4$$

$$T \sim Does \sim Not \sim Exist!!$$

So there is NO punish period that will yield cooperating as we expected.

e. The Folk Theorem states that we can obtain any feasible payoffs that are at least the static Nash payoffs for each player, as the average per-period payoff of the infinitely repeated game as long as the discount factor exceeds some critical level. The feasible region and the "folk region" for this game is shown here:



Note the feasible region (green+red) includes all payoff pairs that are convex combinations of all available payoffs in the stage game. The "Folk Region" (red) is the subset of the feasible region such that both players obtain at least their static Nash payoffs of (1,1).