14.12 Game Theory
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Solution to Homework 4

1. Note that (A,L) is a Nash equilibrium of the stage game. Thus, for the game where the stage game is repeated five times, to play (A,L) in each period is a strategy profile that is subgame-perfect: neither player has an incentive to deviate in any period. Similarly, since (B,R) is a Nash equilibrium of the stage game, to play (B,R) in each period is a strategy profile that is subgame-perfect.

   Note that (B,L) is not a Nash equilibrium of the stage game. Both players have an incentive to deviate from (B,L). Therefore, to ensure that they play (B,L) in the first period, it is necessary to punish them in future periods if they deviate. Consider the following strategy profile for the 5-period game.

   – play (B,L) in period 1;
   – if (B,L) or (A,R) was played in period 1, play (A,L) in periods 2 and 3 and (B,R) in periods 4 and 5;
   – if (A,L) was played in period 1, play (B,R) in periods 2-5;
   – if (B,R) was played in period 1, play (A,L) in periods 2-5.

   To check if this strategy profile is subgame-perfect, we can apply the single-deviation principle: we check, for each information set, if a player can gain by deviating in that period but following the prescribed strategy in future periods. At all information sets in periods 2 through 5, the players are required to play a Nash equilibrium of the stage game. Therefore, the players do not have an incentive to deviate in these periods. (and such deviation does not lead to future gains). As for period 1, if player 1 deviates he gains 2 in the current period but loses 2 in the future as the strategy profile requires them to play (BR, BR, BR, BR) instead of (AL, AL, BR, BR). Similarly, if player 2 deviates in period 1, she gains 2 in the current period but loses 2 in the future as the strategy profile requires them to play (AL, AL, AL, AL) instead of (AL, AL, BR, BR). Therefore, neither player can profit from a single-period deviation. Therefore, the strategy profile described above is subgame-perfect.

   In the same manner, we can construct a subgame-perfect equilibrum where the players play (A,R) in the first period.

2. In each of the following cases, we apply the single deviation principle to check if the given strategy profile is subgame-perfect:

(a)
- Suppose Goliath Software has updated X every time a startup produced a browser in the past. If the current startup produces a browser instead of a search engine, and Goliath updates according to its strategy, the startup would receive 0 instead of 1. Therefore, the startup loses by deviating.
- Suppose Goliath Software has updated X every time a startup produced a browser in the past. If Goliath chooses not to update after the startup has developed a browser, but all parties play according to the given strategy profile in future periods, then the payoff to Goliath equals

$$\begin{aligned} & 2 + 2\,(0.9) + 2\,(0.9)^2 + ... \\ = \quad & \frac{2}{1-0.9} = 20 \end{aligned}$$

  whereas if Goliath had chosen to update after the startup developed the browser, the payoff to Goliath would equal

$$\begin{aligned} & 1 + 4\,(0.9) + 4\,(0.9)^2 + ... \\ = \quad & 1 + \frac{4\,(0.9)}{1-0.9} = 37 \end{aligned}$$

  Therefore, Goliath Software loses through this single period deviation.
- If Goliath Software has *not* updated X every time a startup produced a browser in the past, the current startup obviously does better by producing a browser rather than a search engine as, according to its strategy, Goliath will not update.
- If Goliath Software has *not* updated X every time a startup produced a browser in the past, and the current startup develops a browser, then a single period deviation in which Goliath chooses to update cannot be profitable: the company would lose 1 in the current period without affecting future payoffs.

Thus, we have established, using the single deviation principle, that the given strategy profile constitutes a subgame-perfect equilibrium. If this strategy profile is used, we observe the outcome that the startup always produces a search engine and Goliath never updates.

(b) If a startup produces a browser, Goliath Software can improve its payoff in the current period by choosing 'No Update' instead of 'Update'; in either case, future startups 'always produce a search engine'. Thus, Goliath Software can profitably deviate from its given strategy. Therefore, the given strategy profile is not subgame perfect. For this strategy profile, we would observe the outcome that the startup always produces a search engine and Goliath never updates.

(c) First consider the *holy years*. For either party, deviating from the given strategy profile results in a loss in the current period; and (if all parties play according to the given strategy profile in the future) does not affect the payoff from future periods. Therefore, single period deviations in holy years are not profitable.

The strategies for the unholy years are the same as those described in part (a). In particular, for startups that appear in unholy years, since they exist for one period only, the holy years do not matter; therefore, the reasoning used in the answer to part (a) for startups applies directly: they cannot profit through single period deviations during these years.

Now suppose that Goliath Software has *not* updated X every time a startup produced a browser in past unholy years; and that in the present period, also an unholy year, the current startup has developed a browser. Then, a single period deviation in which Goliath chooses to update cannot be profitable: the company would lose 1 in the current period without affecting future payoffs.

Finally, suppose Goliath Software has updated X every time a startup produced a browser in past unholy years. If the startup develops a browser in the current period (an unholy year) and there are τ years between the current period and the next holy year, then Goliath receives the following payoff from its strategy (assuming the startups follow their given strategy):

$$1 + 4\,(0.9) + 4\,(0.9)^2 + ... + 4\,(0.9)^\tau + 2\,(0.9)^{\tau+1} + 4\,(0.9)^{\tau+2} + ...$$

If Goliath deviates in the current period but follows its strategy in future periods, it receives:

$$2 + 2\,(0.9) + 2\,(0.9)^2 + ... + 2\,(0.9)^\tau + 2\,(0.9)^{\tau+1} + 2\,(0.9)^{\tau+2} + ...$$

Therefore, the gain from such a deviation equals

$$1 - 2\,(0.9) - 2\,(0.9)^2 - ... - 2\,(0.9)^\tau - 0\,(0.9)^{\tau+1} - 2\,(0.9)^{\tau+2} - ...$$

$$= 1 - \frac{2\,(0.9)}{1-(0.9)} + \frac{2\,(0.9)^{\tau+1}}{1-(0.9)^{100}}$$

This expression takes its largest value when $\tau = 0$; i.e. when the next year is a holy year. However, plugging $\tau = 0$ into the expression above, we obtain

$$1 - \frac{2\,(0.9)}{1-(0.9)} + \frac{2\,(0.9)}{1-(0.9)^{100}} \approx -15$$

Therefore, Goliath Software cannot profit from a single period deviation at the type of information node considered here.

Thus, we have established, using the single deviation principle that the given strategy profile constitutes a subgame perfect equilibrium. For this strategy profile, we would observe the outcome that the startup produces a browser once every hundred years, and a startup in all other years; Goliath does not update in any year.

(d) As in part (c), if Goliath Software deviates during an unholy year, this has no consequences for its payoff during holy years. However, unlike the strategy profile considered in part (c), in this instance most years are holy. Thus, the incentives for Goliath to follow the given strategy are not as strong as they were in part (c). Consider the case where Goliath Software has updated X every time a startup produced a browser in past unholy years. Suppose that this period, also an unholy year, the startup develops a browser; then Goliath receives the following payoff from its strategy (assuming the startups follow their given strategy):

$$1 + 2\,(0.9) + 2\,(0.9)^2 + ... + 2\,(0.9)^{99} + 4\,(0.9)^{100} + 2\,(0.9)^{101} + ...$$

If Goliath deviates in the current period but follows its strategy in future periods, it receives:

$$2 + 2\,(0.9) + 2\,(0.9)^2 + ... + 2\,(0.9)^{99} + 2\,(0.9)^{100} + 2\,(0.9)^{101} + ...$$

Therefore, the gain from such a deviation equals

$$1 - 0\,(0.9) - 0\,(0.9)^2 - ... - 0\,(0.9)^{99} - 2\,(0.9)^{100} - 0\,(0.9)^{101} - ...$$

$$= 1 - \frac{2\,(0.9)^{100}}{1 - (0.9)^{100}} \approx 1$$

Therefore, Goliath Software can deviate profitably at such an information set. Therefore, the given strategy profile is not subgame perfect. For this strategy profile, we would observe the outcome that the startup produces a search engine once every hundred years, and a browser in all other years; Goliath does not update in any year.

(a) The normal form representation of the stage game is as follows:

	W,W	W,S	S,W	S,S
Pay	$\pi - w, w - c$	$\pi - w, w - c$	$-w, w$	$-w, w$
Not Pay	$\pi, -c$	$0, 0$	$\pi, -c$	$0, 0$

In the table above, the firm is the row-player and the worker is the column-player. For the worker's strategies, the first letter specifies the action if the firm pays, and the second letter specifies the action if the firm does not pay (W for work, S for shirk).By inspection, we see that there is a unique Nash equilibrium of the game, (Not Pay, (S,S)).

i. This strategy profile is not subgame perfect for any value of δ. Given the strategy of the firm, the worker has an incentive to deviate to "shirk" in every period, thus receiving w instead of $w-c$ each time. Given the strategy of the worker, the firm has an incentive to deviate to "not pay" in every period, thus receiving π instead of $\pi - w$ each time.

ii. This strategy profile is not subgame perfect for any value of δ. Given the strategy of the firm, the worker has an incentive to deviate to "shirk" in every period, as the firm will "always pay".

iii. This strategy profile cannot not subgame perfect either. If the firm deviates and chooses to not pay, the worker will still work in every period. So the firm can deviate profitably by choosing to "not pay" in every period.

iv. We consider if a single period deviation can be profitable for either the firm or the worker at each possible type of information set.

If the worker has worked in all previous periods (or if $t = 0$), the firm cannot improve its payoff by deviating to 'Not Pay'; this will give the firm a payoff of 0 instead of $\pi - w$ in the current period as well as in all future periods.

Also, the firm cannot improve its payoff by deviating to 'Pay' if the worker has not worked in all previous periods; this will give the firm a payoff of $-w$ instead of 0 in the current period, and its future payoff would be unaffected.

If the worker has been paid in the current period and she has worked in all previous periods (or if $t = 0$), then her continuation payoff from her strategy is $\frac{w-c}{1-\delta}$; on the other hand, if she deviates to 'Not Work' in the current period, but follows the given strategy in all future periods, she receives w. Then, she has no incentive to deviate if and only if

$$\frac{w-c}{1-\delta} \geq w \iff \delta \geq \frac{c}{w}$$

If the worker has not been paid in the current period, she receives a continuation payoff of $\delta\frac{w-c}{1-\delta}$ from her strategy; a single period deviation to 'Work' in the current period will simply cost her c in the current period, without affecting her future payoffs. Therefore she has no incentive to deviate at these information sets for any value of δ.

If the worker has not worked in all previous periods, then, once more, a deviation to 'Work' costs her c in the current period without affecting her future payoffs.

Therefore, the given strategy profile is subgame perfect for $\delta \geq \frac{c}{w}$.

v. We need to check the conditions under which neither player has an incentive to deviate in either in the Employment or the Unemployment modes. In the Employment mode, as in (b) (iv), payoffs to the firm and the worker from the given strategy profile equals $\frac{(\pi-w)}{1-\delta}$ and $\frac{(w-c)}{1-\delta}$ respectively.If the worker deviates in the Employment mode (but follows her prescrbed strategy thereafter), she receives w in the current period, and the game goes into the

Unemployment mode for T periods, during which she receives nothing, before returning to the Employment mode. Therefore, for the worker not to deviate in the Employment mode, we need

$$\begin{aligned} \frac{(w-c)}{1-\delta} &\geq w+\delta^{T+1}\frac{(w-c)}{1-\delta} \\ &\Longrightarrow (w-c)\,\delta^{T+1}-w\delta+c\leq 0 \end{aligned} \tag{1}$$

The firm has no incentive to deviate in the Employment mode, because if the firm deviates to "not pay", the worker would "shirk" in the same period, according to her strategy, and the game would go into the Unemployment mode for T periods. Thus, the firm receives 0 instead of $w-c$, in the current as well as the T following periods.Neither player has an incentive to deviate in the Unemployment mode. By deviating, the worker obtains $-c$ instead of 0 and the firm $-w$ instead of 0, in the current period. Moreover, such a deviation delays the return to the Employment mode.

If the firm has deviated in the Employment mode, then the worker receives $\delta^{T+1}\frac{(w-c)}{1-\delta}$ from her strategy (shirk today and shirk during the T periods of unemployment). On the other hand, if the worker chooses instead to work when she has not been paid in the Employment mode, then the game remains in the Employment mode and she receives $-c+\delta\frac{(w-c)}{1-\delta}$. Therefore, for a single period deviation of this kind to be unprofitable, we need

$$\begin{aligned} \delta^{T+1}\frac{(w-c)}{1-\delta} &\geq -c+\delta\frac{(w-c)}{1-\delta} \\ &\Longrightarrow w+\delta^{T+1}\frac{(w-c)}{1-\delta}\geq\frac{(w-c)}{1-\delta} \end{aligned} \tag{2}$$

It is easily verified that neither the firm nor the worker can improve its payoff through a single period deviation in the Unemployment mode. Inequalities (1) and (2) are both satisfied if and only if the discount factor belongs to the set $\left\{\delta\in(0,1):(w-c)\,\delta^{T+1}-w\delta+c=0\right\}$ (which may or may not be empty). Then, using the single deviation principle, the given strategy profile is subgame perfect if and only if the discount factor belongs to this set.