Spring 2013 Final Solutions

1. Beer/Quiche

a. See lecture notes.

b. If both types of sender have beer for breakfast, then q = 0.4. The receiver will than compare:

$$E_{Rec}[\text{Duel}|\text{Beer}] \text{ vs } E_{Rec}[\text{Not Duel}|\text{Beer}]$$
$$10(0.4)+(-20)(0.6) \text{ vs } 0(0.4)+0(0.6)$$
$$4-12 \text{ vs } 0$$
$$-8 \text{ vs } 0$$

Therefore, the receiver will Not Duel if he see the sender have beer for breakfast. A strong sender would never deviate from this since he gets 20 by having a beer for breakfast and either 0 or 5 if he has Quiche. A wimpy sender though gets 10 by having a beer and either 20 or 5 if he has quiche for breakfast. Therefore, in order for a wimpy sender to not deviate, we need the receiver to duel if he sees the sender have quiche for breakfast. So we require:

$$E_{Rec}[\text{Duel}|\text{Quiche}] >= E_{Rec}[\text{Not Duel}|\text{Quiche}]$$
$$20p-10(1-p) >= 0$$
$$30p>= 10$$
$$p>= 1/3$$

Therefore, we have a PBE as:

$$\{(\text{Beer, Beer}), (\text{Duel, Not Duel}), (p,1-p), (q,1-q) \mid p >= 1/3, q = 0.4 \}$$

c. Separating on (Quiche,Beer) implies p = 1 and q = 0 because both of the receiver's information sets are on the equilibrium path. A receiver will therefore play (Duel, Not Duel) following the Quiche and Beer signals respectively. A strong sender gets 20 from having a beer, which is always better than having quiche and getting 0 or 5. A wimpy sender receives 5 from having quiche for breakfast, but could get 10 from having a beer (since the receiver does not duel following Beer). Therefore, a PBE of this type does not exist because a wimpy sender would deviate.

2.

a. Assume X = 20.

Emily

John

	Left	Right
Up	(20 , 5)	(6 , 8)
Down	(20 , 20)	(5 , 12)

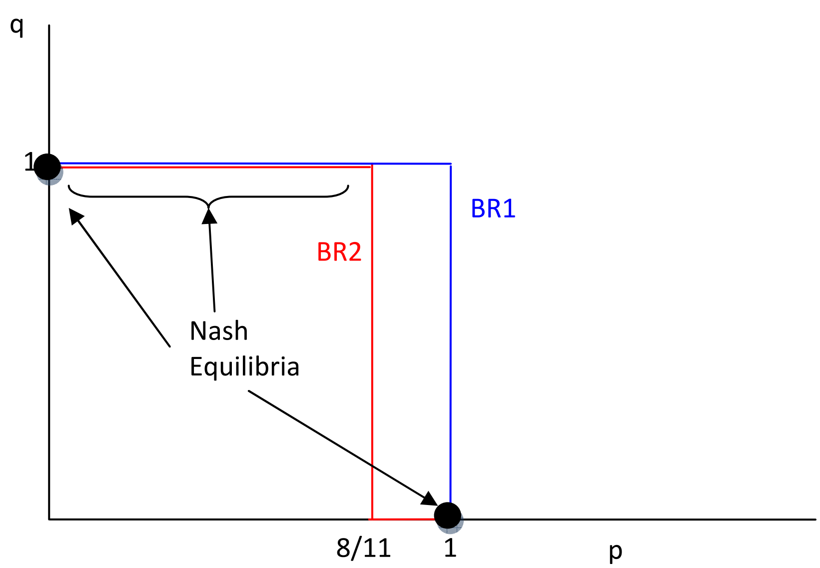
So, by inspection, we have two NE in pure strategies: (D,L) and (U,R). There is also a Mixed Strategy NE.

Player 1 solves:

$$\text{Max}(p)\ \{20pq + 6p(1-q) + 20(1-p)q + 5(1-p)(1-q)\}$$
$$\text{Max}(p)\ \{20pq+6p-6pq+20q-20pq+5-5p-5q+5pq\}$$
$$\text{Max}(p)\ \{p(-q+1)+15q+5$$
$$\rightarrow p=0 \text{ if } 1-q<0 \Leftrightarrow q > 1 \text{ [never]}$$
$$\rightarrow p=1 \text{ if } 1-q>0 \Leftrightarrow q < 1$$
$$\rightarrow p \in [0,1] \text{ if } 1-q=0 \Leftrightarrow q = 1$$

Player 2 solves:

$$\text{Max}(q)\ \{5pq + 8p(1-q) + 20(1-p)q + 12(1-p)(1-q)\}$$
$$\text{Max}(q)\ \{5pq - 8pq+8p + 20q-20pq + 12-12p-12q+12pq\}$$
$$\text{Max}(q)\ \{q(8-11p)-4p+12\}$$
$$\rightarrow q=0 \text{ if } 8-11p<0 \Leftrightarrow p > 8/11$$
$$\rightarrow q=1 \text{ if } 8-11p>0 \Leftrightarrow p < 8/11$$
$$\rightarrow q \in [0,1] \text{ if } 8-11p=0 \Leftrightarrow p = 8/11$$



From the intersections of the Best Response curves, we can see all the NE:

$$\{\quad [(1,0),(0,1)]\ ;\quad [(p,1-p),(1,1-q)\ |\ p<=8/11,\ q = 1]\quad \}$$

Note the first NE is the PSNE (Up,Right) and the second PSNE is found setting p = 0 in the second set of NE.

b. Assume X = 10.

Emily

John

	Left	Right
Up	(20 , 5)	(6 , 8)
Down	(10 , 10)	(5 , 12)

Only one NE at (Up, Right). Grim Trigger strategies to sustain (10,10) each period:

σ_1 = { Play Down in the first period and in all subsequent periods if (Down,Left) has been played by players 1 and 2 respectively in all periods. Play Up otherwise. }

σ_2 = { Play Left in the first periods and in all subsequent periods if (Down,Left) has been played by players 1 and 2 respectively in all periods. Play Right otherwise. }

Payoffs along the equilibrium path are then:

$$\pi_1^e = 10(1 + \delta_1 + \delta^2_1 + \delta^3_1 + \ldots) = 10/(1- \delta_1)$$
$$\pi_2^e = 10(1 + \delta_2 + \delta^2_2 + \delta^3_2 + \ldots) = 10/(1- \delta_2)$$

Payoffs from deviating (in the first period):

$$\pi_1^d = 20 + 6(\delta_1 + \delta^2_1 + \delta^3_1 + \ldots) = 20 + 6\delta_1/(1-\delta_1)$$
$$\pi_2^d = 12 + 8(\delta_2 + \delta^2_2 + \delta^3_2 + \ldots) = 12 + 8\delta_2/(1-\delta_2)$$

The critical discount factor that sustains cooperation (Down,Left) in all periods satisfies:

$$\pi_1^e >= \pi_1^d$$
$$10/(1- \delta_1)>=20 + 6\delta_1/(1-\delta_1)$$
$$10 >= 20-20\delta_1 + 6\delta_1$$
$$-10 >= -14\delta_1$$
$$10/14 <= \delta_1$$
$$\delta_1^* = 10/14 \text{ (approx 0.714)}$$

$$\pi_2^e >= \pi_2^d$$
$$10/(1- \delta_2)>=12 + 8\delta_2/(1-\delta_2)$$
$$10 >= 12-12\delta_2 + 8\delta_2$$
$$-2 >= -4\delta_2$$
$$1/2 <= \delta_2$$
$$\delta_2^* = 1/2$$

Note that we require player 1 to be more patient than player 2 in order to sustain cooperation. Player 1 has more to gain from deviating than player 2. He does receive less in the punishment phase than player 2 but this is not enough to outweigh the effects of the larger one period deviation.

c. Assume X = 10. Players have a common discount factor, δ=0.9. Given limited punishment with T periods, we have to compare:

$$\pi_1^e = \pi_2^e = 10(1 + \delta + \delta^2 + \delta^3 + \ldots + \delta^T) = 10*(1- 0.9^{T+1})/(1-0.9)$$
$$\pi_1^e = \pi_2^e=100*(1- 0.9^{T+1})$$
$$\pi_1^e = \pi_2^e=100-100* 0.9^{T+1}$$
$$\pi_1^e = \pi_2^e=100-100* (0.9*0.9^T)$$
$$\pi_1^e = \pi_2^e=100-90*0.9^T$$

$$\pi_1^d = 20 + 6(\delta + \delta^2 + \delta^3 +\ldots + \delta^T)$$
$$\pi_1^d = 20+6\delta(1 + \delta + \delta^2 +\ldots + \delta^{T-1})$$
$$\pi_1^d = 20+6*0.9*(1-0.9^T)/(1-0.9)$$
$$\pi_1^d = 20+54*(1-0.9^T)$$
$$\pi_1^d = 74-54*0.9^T$$

$$\pi_2^d = 12 + 8(\delta + \delta^2 + \delta^3 +\ldots + \delta^T)$$
$$\pi_2^d = 12+8\delta(1 + \delta + \delta^2 +\ldots + \delta^{T-1})$$
$$\pi_2^d = 12+8*0.9*(1-0.9^T)/(1-0.9)$$
$$\pi_2^d = 12+72*(1-0.9^T)$$
$$\pi_1^d = 84-72*0.9^T$$

So we require for player 1:

$$100-90*0.9^T >= 74 - 54*0.9^T$$
$$26-90*0.9^T >= - 54*0.9^T$$
$$26 >= 36*0.9^T$$
$$26/36 >= 0.9^T$$
$$\mathrm{Ln}(26/36) >= T*\ln(0.9)$$
$$T>=\ln(26/36)/\ln(0.9)$$

T >= appox 3.08 so T >= 4

So we require for player 2:

$$100-90*0.9^T >= 84 - 72*0.9^T$$
$$16-90*0.9^T >= - 72*0.9^T$$

$16 >= 18*0.9^T$
$16/18 >= 0.9^T$
$Ln(16/18) >= T*ln(0.9)$
$T>=ln(16/18)/ln(0.9)$
T >= appox 1.12 so T >= 2

Again player 1 has a higher critical punishment period for the same reason in part b. Overall, to sustain cooperation for both players, we need the punishment period to be at least 4 periods. This is just enough for player 1 and more than enough for player 2 to never deviate.

d. If we increase the players' discount factor, we are saying they are more patient or they discount the future less. If this is the case, then they care more about getting relatively high payoffs in the future (like those on the equilibrium path) compared with lower payoffs (like those in a punishment period). Therefore, a higher discount rate would drive down the critical punishment period length.

3.

a. Strategies for player 1: $\sigma_1 = \{A, B\}$. Strategies for player 2: $\sigma_2 = \{MP, MQ, NP, NQ\}$
b. The game has 3 subgames. The one following 1's choice of A, the one following 1's choice of B, and the whole game.
c. Working backwards, 2's BR are {MQ, NQ}. Then for each of these, $\sigma_1(\sigma_2=MQ) = \{A,B\}$. And for $\sigma_1(\sigma_2=NQ) = \{B\}$. So the set of SPNE are:
$$\{ \quad (A, MQ); \quad (B,MQ); \quad (B,NQ) \quad \}$$