14.12 Game Theory – Midterm I
ANSWERS

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

1. Find all the Nash equilibria in the following game:

1\2	L	M	R
T	1,0	0,1	5,0
B	0,2	2,1	1,0

Answer: By inspection, there is no pure-strategy equilibrium in this game. There is one mixed strategy equilibrium. Since R is strictly dominated, player 2 will assign 0 probability to R. Let p and q be the equilibrium probabilities for strategies T and L, respectively; the probabilities for B and R are $1-p$ and $1-q$, respectively. If 1 plays T, his expected payoff is $q1 + (1-q)\,0 = q$. If he plays B, his expected payoff is $2\,(1-q)$. Since he assigns positive probabilities to both T and B, he must be indifferent between T and B. Hence, $q = 2\,(1-q)$, i.e., $q = 2/3$. Similarly, for player 2, the expected payoffs from playing L and M are $2\,(1-p)$ and 1, respectively. Hence, $2\,(1-p) = 1$, i.e., $p = 1/2$.

2. Find all the pure strategies that are consistent with the common knowledge of rationality in the following game. (State the rationality/knowledge assumptions corresponding to each operation.)

1\2	L	M	R
T	1,1	0,4	2,2
M	2,4	2,1	1,2
B	1,0	0,1	0,2

Answer:

(a) 1. For player 1, M strictly dominates B. Since **Player 1 is rational**, he will not play B, and we eliminate this strategy:

1\2	L	M	R
T	1,1	0,4	2,2
M	2,4	2,1	1,2

2. Since **Player 2 knows that Player 1 is rational**, he will know that 1 will not play B. Given this, the mixed strategy that assigns probability 1/2 to each of the strategies L and M strictly dominates R. Since **Player 2 is rational**, in that case, he will not play R. We eliminate this strategy:

1\2	L	M
T	1,1	0,4
M	2,4	2,1

3. Since **Player 1 knows that Player 2 is rational and that Player 2 knows that Player 1 is rational**, he will know that 2 will not play R. Given this, M strictly dominates T. Since **Player 1 is rational**, he will not play T, either. We are left with

1\2	L	M
M	2,4	2,1

.

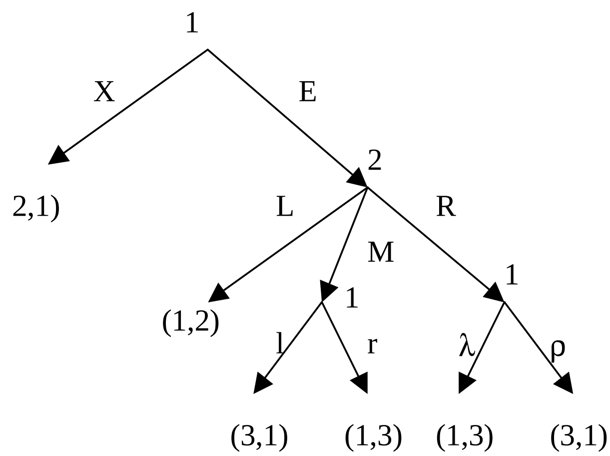
4. Since **Player 2 knows that Player 1 is rational, and that Player 1 knows that Player 2 is rational, and that Player 1 knows that Player 2 knows that Player 1 is rational**, he will know that Player 1 will not play T or B. Given this, L strictly dominates M. Since **Player 2 is rational**, he will not play M, either. He will play L.

1\2	L
M	2,4

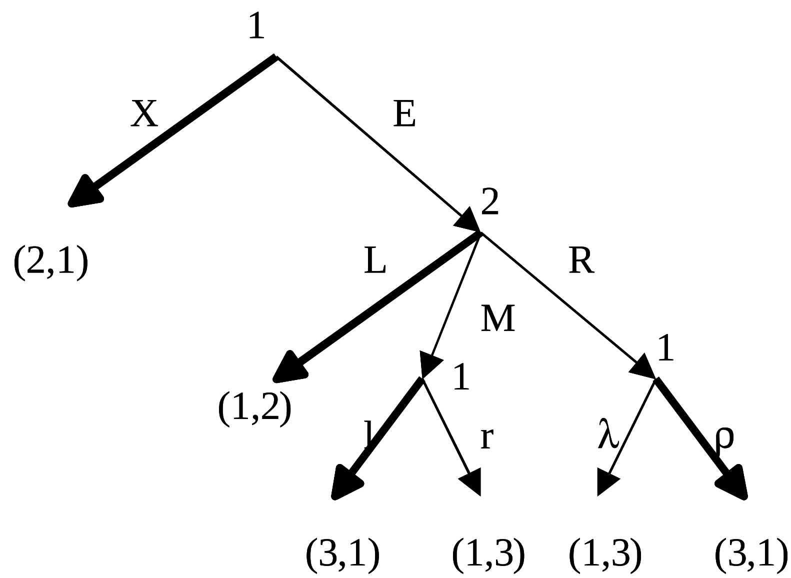
.

Thus, the only strategies that are consistent with the common knowledge of rationality are M for player 1 and L for player 2.

3. Consider the following extensive form game.



(a) Using Backward Induction, compute an equilibrium of this game.



(b) Find the normal form representation of this game.

1\2	L	M	R
X l λ	2,1	2,1	2,1
X l ρ	2,1	2,1	2,1
X r λ	2,1	2,1	2,1
X r ρ	2,1	2,1	2,1
E l λ	1,2	3,1	1,3
E l ρ	1,2	3,1	3,1
E r λ	1,2	1,3	1,3
E r ρ	1,2	1,3	3,1

The points will be taken off from the people who did not distinguish the strategies that start with X from each other.

(c) Find all pure strategy Nash equilibria.

1\2	L	M	R
X l λ	**2,1**	2,1	2,1
X l ρ	**2,1**	2,1	2,1
X r λ	**2,1**	2,1	2,1
X r ρ	**2,1**	2,1	2,1
E l λ	1,2	3,1	1,3
E l ρ	1,2	3,1	3,1
E r λ	1,2	1,3	1,3
E r ρ	1,2	1,3	3,1

The Nash equilibria are (Xlλ,L), (Xlρ,L), (Xrλ,L), (Xrρ,L).

4. In this question you are asked to compute the rationalizable strategies in linear Bertrand-duopoly with discrete prices. We consider a world where the prices must be the positive multiples of cents, i.e.,

$$P = \{0.01, 0.02, \ldots, 0.01n, \ldots\}$$

is the set of all feasible prices. For each price $p \in P$, the demand is

$$Q(p) = \max\{1 - p, 0\}.$$

We have two firms $N = \{1, 2\}$, each with zero marginal cost. Simultaneously, each firm i sets a price $p_i \in P$. Observing the prices p_1 and p_2, consumers buy from the firm with the lowest price; when the prices are equal, they divide their demand equally between the firms. Each firm i maximizes its own profit

$$\pi_i(p_1, p_2) = \begin{cases} p_i Q(p_i) & \text{if } p_i < p_j \\ p_i Q(p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{otherwise,} \end{cases}$$

where $j \neq i$.

(a) Show that any price p greater than the monopoly price $p^{mon} = 0.5$ is strictly dominated by some strategy that assigns some probability $\epsilon > 0$ to the price $p^{\min} = 0.01$ and probability $1 - \epsilon$ to the price $p^{mon} = 0.5$.

Answer: Take any player i and any price $p_i > p^{mon}$. We want to show that the mixed strategy σ^ϵ with $\sigma^\epsilon(p^{mon}) = 1 - \epsilon$ and $\sigma^\epsilon(p^{\min}) = \epsilon$ strictly dominates p_i for some $\epsilon > 0$.

Take any strategy $p_j > p^{mon}$ of the other player j. We have

$$\pi_i(p_i, p_j) \leq p_i Q(p_i) = p_i(1 - p_i) \leq 0.51 \cdot 0.49 = 0.2499,$$

where the first inequality is by definition and the last inequality is due to the fact that $p_i \geq 0.51$. On the other hand,

$$\begin{aligned} \pi_i(\sigma^\epsilon, p_j) &= (1 - \epsilon) p^{mon}(1 - p^{mon}) + \epsilon p^{\min}(1 - p^{\min}) \\ &> (1 - \epsilon) p^{mon}(1 - p^{mon}) \\ &= 0.25(1 - \epsilon). \end{aligned}$$

Thus, $\pi_i(\sigma^\epsilon, p_j) > 0.2499 \geq \pi_i(p_i, p_j)$ whenever $0 < \epsilon \leq 0.0004$. Choose $\epsilon = 0.0004$.

Now, pick any $p_j \leq p^{mon}$. Since $p_i > p^{mon}$, we now have $\pi_i(p_i, p_j) = 0$. But

$$\pi_i(\sigma^\epsilon, p_j) = (1 - \epsilon) p^{mon}(1 - p^{mon}) + \epsilon p^{\min}(1 - p^{\min}) \geq \epsilon p^{\min}(1 - p^{\min}) > 0.$$

That is, $\pi_i(\sigma^\epsilon, p_j) > \pi_i(p_i, p_j)$. Therefore, σ^ϵ strictly dominates p_i.

(b) Iteratively eliminating all the strictly dominated strategies, show that the only rationalizable strategy for a firm is $p^{\min} = 0.01$.

Answer: We have already eliminated the strategies that are larger than p^{mon}. At any iteration t assume that, for each player, the set of all remaining strategies are $P^t = \{0.01, 0.02, \ldots, \bar{p}\}$ where $p^{\min} < \bar{p} \leq p^{mon}$. We want to show that $\bar{p}$ is strictly dominated by the mixed strategy $\sigma_{\bar{p}}^{\epsilon}$ with $\sigma_{\bar{p}}^{\epsilon}(\bar{p} - 0.01) = 1 - \epsilon$ and $\sigma_{\bar{p}}^{\epsilon}\left(p^{\min}\right) = \epsilon$, and eliminate the strategy $\bar{p}$. This process will end when $P^s = \{0.01\}$, completing the proof. Now, for player i,

$$\pi_i(\bar{p}, p_j) = \begin{cases} \bar{p}(1-\bar{p})/2 & \text{if } p_j = \bar{p}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} \pi_i\left(\sigma_{\bar{p}}^{\epsilon}, \bar{p}\right) &= (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) + \epsilon p^{\min}\left(1 - p^{\min}\right) \\ &> (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) \\ &= (1-\epsilon)\left[\bar{p}(1-\bar{p}) - 0.01(1 - 2\bar{p})\right]. \end{aligned}$$

Then, $\pi_i\left(\sigma_{\bar{p}}^{\epsilon}, \bar{p}\right) > \pi_i(\bar{p}, p_j)$ whenever

$$\epsilon \leq 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1-2\bar{p})}.$$

But $\bar{p} \geq 0.02$, hence $0.01(1-2\bar{p}) < \bar{p}(1-\bar{p})/2$, thus the right hand side is greater than 0. Choose

$$\epsilon = 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1-2\bar{p})} > 0$$

so that $\pi_i\left(\sigma_{\bar{p}}^{\epsilon}, \bar{p}\right) > \pi_i(\bar{p}, p_j)$. Moreover, for any $p_j < \bar{p}$,

$$\begin{aligned} \pi_i\left(\sigma_{\bar{p}}^{\epsilon}, p_j\right) &= (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) + \epsilon p^{\min}\left(1 - p^{\min}\right) \\ &\geq \epsilon p^{\min}\left(1 - p^{\min}\right) > 0 = \pi_i(\bar{p}, p_j), \end{aligned}$$

showing that $\sigma_{\bar{p}}^{\epsilon}$ strictly dominates $\bar{p}$, and completing the proof.

(c) What are the Nash equilibria of this game?

Answer: Since any Nash equilibrium is rationalizable, and since the only rationalizable strategy profile is $\left(p^{\min}, p^{\min}\right)$, the only Nash equilibrium is $\left(p^{\min}, p^{\min}\right)$. (Since this is a finite game, there is always a Nash equilibrium — possibly in mixed strategies.)