Stat 154 Practice Final Exam Spring 2014

Name:

SID:

Attempt all questions and show your working - solutions without explanation will **not** receive full credit. Answer the questions in the space provided. Additional space is available on the final two pages. Two double sided sheets of notes are permitted. Answers can be left in numerical form without simplification except where specified.

Question 1

Find the optimal strategies in the following zero-sum games.

$$(a) \quad \begin{pmatrix} 2 & 4 \\ 7 & 1 \end{pmatrix} \qquad (b) \quad \begin{pmatrix} 3 & 4 & 4 \\ 5 & 1 & 4 \end{pmatrix}$$

Solutions

(a) There are no saddle points so we apply the formula. This gives strategies $x = (\frac{3}{4}, \frac{1}{4})$ and $y = (\frac{3}{8}, \frac{5}{8})$ and value $V = \frac{13}{4}$.

(b) The middle column dominates the rightmost column reducing the problem to

$$\begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix}$$

This again has no saddle points so we apply the formula. This gives strategies $x = (\frac{4}{5}, \frac{1}{5})$ and $y = (\frac{3}{5}, \frac{2}{5})$ and value $V = \frac{17}{5}$.

Question 2

Answer the following:

(a) In the game of Nim if the piles are $17, 21$ and 22 find all winning moves.

(b) Recall that for Misere Nim the aim is to not take the last chip. If the piles are $1, 1, 1, 17, 1$ find all winning moves.

(c) An urn contains 2 yellow, 2 orange balls and two red balls. In a game players take turns to move and in each move may change a red ball into and orange or yellow ball or turn an orange ball into a yellow ball. The game terminates when all the balls are yellow. Is this a P or N position?

Solutions

(a) The nim sum is 18. We can reduce the first pile to 3, the second pile to 7 or the third to 4.

(b) In Misere Nim the winning strategy when there is a single pile with more than 1 chip is to reduce it do that there is an odd number of piles of size 1. In this case it means reducing the pile with 17 to 1.

(c) This is a P position, the second player has a winning strategy. The most straightforward way to solve this is backward induction. A simpler way is that the second player can always copy the move of the first player (i.e. change a red to an orange if the first player just did). This way there will always be an even number of each type after the second player's move and so the game must terminate with 6 yellows after the second players turn.

A final way to solve it is to think of the game as Nim in disguise. If you consider a yellow ball as a pile of 0 chips, an orange as a pile with 1 and a red as a pile with 2 the the allowed moves are exactly the same as Nim and the winner is the one that removes the last chip, i.e. ends with all yellow.

Question 3

Suppose two players share a common resource that needs repairing that will lead to a profit of v to both the players. If both players contribute to the repair it costs c_2 to each. If any of the players repair on their own, it costs c_1 where $c_2 < c_1 < v$. Write down the payoff matrix and find an evolutionary stable strategy in this game.

Solution

The matrix is

$$\begin{pmatrix} (v - c_2, v - c_2) & (v - c_1, v) \\ (v, v - c_1) & (0, 0) \end{pmatrix}$$

Solution

The pure strategies are $x = (1, 0), y = (0, 1)$ $x = (0, 1), y = (1, 0)$ neither of which are symmetric. By equalizing payoffs the symmetric mixed strategy is given by

$$x_1(v - c_2) + (1 - x_1)(v - c_1) = x_1 v$$

so $x = y = (\frac{v - c_1}{v + c_2 - c_1}, \frac{c_2}{v + c_2 - c_1})$ is a symmetric Nash equilibrium.

To check that this is ESS we first note that for both pure strategies z we have that $z^T A x = x^T A x$ since the payoffs for the two strategies are equal (that's how we found x after all). So we need to check that $z^T A z < x^T A z$ for the pure strategies z. For $z = (1, 0)$ we have that $z^T A z = v - c_2$ while $x^T A z = x_1(v - c_2) + (1 - x_1)v > v - c_2$. If $x = (0, 1)$ then $z^T A z = 0$ while if $x^T A z = x_1(v - c_1) > 0$. Hence x is ESS.

Question 4

In the following symmetric general sum game

$$
\begin{pmatrix}
(2,2) & (0,0) & (0,0) \\
(0,0) & (0,0) & (2,2) \\
(0,0) & (2,2) & (0,0)
\end{pmatrix}
$$

(i) Find all pure Nash equilibria.

(ii) Find all mixed Nash equilibria in which all probabilities are positive.

(iii) Which of these are evolutionary stable strategies?

Solution

(i) The pure strategies are $x = (1,0,0), y = (1,0,0)$ and $x = (0,1,0), y = (0,0,1)$ and $x = (0,0,1), y = (0,1,0)$.

(ii) By equating payoffs we have that

$$2x_1 = 2x_3 = 2x_2$$

and combining this with $x_1+x_2+x_3 = 1$ we have that $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and similarly $y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

(iii) The only two symmetric strategies above are $x = (1,0,0)$ and $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Let us consider $x = (1,0,0)$ first. We have that $x^T A x = 2$. The possible pure strategies are $z = (0,1,0)$ or $(0,0,1)$ both of which give $z^T A x = 0$. Hence $x = (1,0,0)$ is evolutionary stable.

Now consider $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Take $z = (1,0,0)$. We have that $z^T A x = x^T A x = \frac{2}{3}$ and $2 = z^T A Z > x^T A z = \frac{2}{3}$ and hence $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not evolutionary stable.

Question 5 (i) There are 3 men, called A, B, C and 3 women, called X, Y, Z, with the following preference lists:

$$\begin{array}{llll}
\text{For } A: & X > Y > Z & \qquad \text{For } X: & C > A > B \\
\text{For } B: & X > Y > Z & \qquad \text{For } Y: & A > C > B \\
\text{For } C: & Y > X > Z & \qquad \text{For } Z: & A > B > C
\end{array}$$

Find the matchings given by the men proposing and women proposing Gale-Shapley algorithm for these preferences.

(ii) Find a set of preference list for 3 men and women such that in the men-proposing Gale-Shapley algorithm no man gets his first preference.

Solution

(i) The men proposing algorithm give the matching A with X, B with Z and C with Y. The women proposing algorithm gives A with Y B with Z and C with X.

(ii) In the following example no mean gets his first preference in the men-proposing Gale-Shapley algorithm.

$$\begin{array}{llll}
\text{For } A: & X > Y > Z & \qquad \text{For } X: & C > A > B \\
\text{For } B: & X > Y > Z & \qquad \text{For } Y: & B > C > A \\
\text{For } C: & Y > X > Z & \qquad \text{For } Z: & A > B > C
\end{array}$$

The matching is A with Z, B with Y and C with X.

Question 6

Half of the cars a used car salesman gets cars to sell are lemons and the other half of them are good. The salesman knows which are which and always offers a warranty on new cars. He has the choice of offering a warranty on the lemons too. The salesmen sells all the cars for 15000 and unsold a good car is worth 12000 to him and a lemon is worth 9000. The expected cost of a warranty is 1000 for a good car and 5000 for a lemon. To a buyer a good car or a lemon with a warranty is worth 18000 while a lemon without a warranty is worth 12000. The salesman has the option of whether to offer warranties on lemons while the buyer has the option to buy or pass.

Determine the Nash equilibrium for the strategies of the salesman and buyer.

Solution

The seller has two possible strategies, either offering a warranty when it's a lemon or not offering one when it's a lemon (and in both cases offering a warranty when it's a good car). The buyer can either buy when there is no warranty or not buy when there is no warranty. The payoffs matrix is

$$\begin{pmatrix} (\frac{1}{2}(14000 + 10000), 3000) & (\frac{1}{2}(14000 + 10000), 3000) \\ (\frac{1}{2}(14000 + 15000), \frac{1}{2}(3000 - 3000)) & (\frac{1}{2}(14000 + 9000), \frac{1}{2}(3000 + 0)) \end{pmatrix}$$

which simplifies to

$$\begin{pmatrix} (12000, 3000) & (12000, 3000) \\ (14500, 0) & (11500, 1500)) \end{pmatrix}$$

Thus only buying with a warranty dominates for the buyer reducing the matrix to

$$\begin{pmatrix} (12000, 3000) \\ (11500, 1500)) \end{pmatrix}$$

The dominant strategy for player 1 is then to offer a warranty. Thus the Nash equilibrium strategy are for the salesman to offer a warranty and the buyer to only buy with a warranty.

Question 7

Suppose in an election there are 3 candidates A, B and C. The voters have the following preferences. 10 voters have preference $A > C > B$, 8 voters have preference $B > C > A$ and 4 voters have preference $C > B > A$. Who wins the election if the voting rule used is

(i) Plurality.

(ii) Instant run-off.

(iii) Borda count.

(iv) In the instant run-off election do any voters have the an incentive to manipulate the vote individually. What about a collection of voters?

(i) The votes for A, B and C are 10, 8 and 4 so A wins.

(ii) Candidate C has the lowest count of first preferences so would be eliminated. Then between A and B more prefer B by a margin or 12 to 10 so candidate B wins.

(iii) In the Borda count the scores are 42 for A, 42 for B and 48 for C so C wins.

(iv) In instant runoff, since C has 4 votes less than the other candidates no individual can change C being eliminated. Then the resulting election is majority between two candidates A and B and for a majority vote for two candidates no manipulation is possible.

Manipulation by a coalition is possible though. If all the voters who voted $A > C > B$ switched to voting $C > A > B$ then candidate C would win the election which they would prefer to candidate B.

Question 8

A cake is represented by the unit interval $\Omega = [0,1]$ and must be divided between three players. For $i = 1, 2, 3$, player i values the cake according to the measure

$$\mu_i([c,d]) = \int_c^d x^{i-1} dx.$$

Find a cutting of the cake into A_1, A_2 and A_3 such that each person feels like they got a $\frac{1}{3}$ share. That is for each i,

$$\mu_i(A_i) \geq \frac{1}{3}\mu_i[\Omega]$$

Solution

First observe that

$$\mu_i[\Omega] = \int_0^1 x^{i-1} dx = \frac{1}{i}.$$

Suppose voter 1 cuts the piece $A_1 = [0, \frac{1}{3}]$. Then $\mu_1(A_1) = \int_0^{1/3} dx = \frac{1}{3}\mu_1[\Omega]$ while

$$\mu_2(A_1) = \int_0^{1/3} x\, dx = \frac{1}{18} < \frac{1}{3}\mu_2[\Omega], \qquad \mu_3(A_1) = \int_0^{1/3} x^2 dx = \frac{1}{81} < \frac{1}{3}\mu_2[\Omega],$$

So player 1 values the slice to be one third of the cake while players 2 and 3 value it as less than one third of the cake. We will divide the remaining cake between them.

$$\mu_2([\frac{1}{3}, 1]) = \int_{1/3}^1 x\, dx = \frac{4}{9}.$$

Thus we need to find y such that $\mu_2([\frac{1}{3}, y]) = \frac{2}{9}$. Since

$$\mu_2([\frac{1}{3}, y]) = \int_{1/3}^y x\, dx = \frac{1}{2}(y^2 - \frac{1}{9})$$

we take $y = \sqrt{\frac{5}{9}}$. So player 2 divides what is left into $A_2 = [\frac{1}{3}, \sqrt{\frac{5}{9}}]$ and $A_3 = [\sqrt{\frac{5}{9}}, 1]$. We have that

$$\mu_3([\frac{1}{3}, 1]) = \int_{1/3}^1 x^2 dx = \frac{1}{3}(1 - \frac{1}{27}) = \frac{26}{81}, \qquad \mu_3([\frac{1}{3}, \sqrt{\frac{5}{9}}]) = \int_{1/3}^{\sqrt{\frac{5}{9}}} x^2 dx = \frac{1}{3}((\frac{5}{9})^{3/2} - \frac{1}{27}) < \frac{13}{81}$$

and thus player 3 prefers slices A_3. It follows that dividing the cake into $A_1 = [0, \frac{1}{3}]$, $A_2 = [\frac{1}{3}, \sqrt{\frac{5}{9}}]$ and $A_3 = [\sqrt{\frac{5}{9}}, 1]$ provides the required division.

Additional Space

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