Spring 2014 Final Solutions

1. Beer/Quiche

a. See lecture notes.

b. If both types of sender have beer for breakfast, then q = 0.2. The receiver will then compare:

$$E_{Rec}[\text{Duel}|\text{Beer}] \text{ vs } E_{Rec}[\text{Not Duel}|\text{Beer}]$$
$$10(0.2)+(-20)(0.8) \text{ vs } 0(0.2)+0(0.8)$$
$$2-16 \text{ vs } 0$$
$$-14 \text{ vs } 0$$

Therefore, the receiver will Not Duel if he see the sender have beer for breakfast. A strong sender would never deviate from this since he gets 20 by having a beer for breakfast and either 0 or 5 if he has Quiche. A wimpy sender though gets 10 by having a beer and either 20 or 5 if he has quiche for breakfast. Therefore, in order for a wimpy sender to not deviate, we need the receiver to duel if he sees the sender have quiche for breakfast. So we require:

$$E_{Rec}[\text{Duel}|\text{Quiche}] >= E_{Rec}[\text{Not Duel}|\text{Quiche}]$$
$$10p-10(1-p) >= 0$$
$$20p >= 10$$
$$p >= 1/2$$

Therefore, we have a PBE as:

$$\{(\text{Beer, Beer}), (\text{Duel, Not Duel}), (p,1-p), (q,1-q) \mid p >= 1/2, q = 0.2\}$$

c. Separating on (Quiche,Beer) implies p = 1 and q = 0 because both of the receiver's information sets are on the equilibrium path. A receiver will therefore play (Duel, Not Duel) following the Quiche and Beer signals respectively. A strong sender gets 20 from having a beer, which is always better than having quiche and getting 0 or 5. A wimpy sender receives 5 from having quiche for breakfast, but could get 10 from having a beer (since the receiver does not duel following Beer). Therefore, a PBE of this type does not exist because a wimpy sender would deviate.

2.
a. Assume X = 5.

Player 2

Player 1

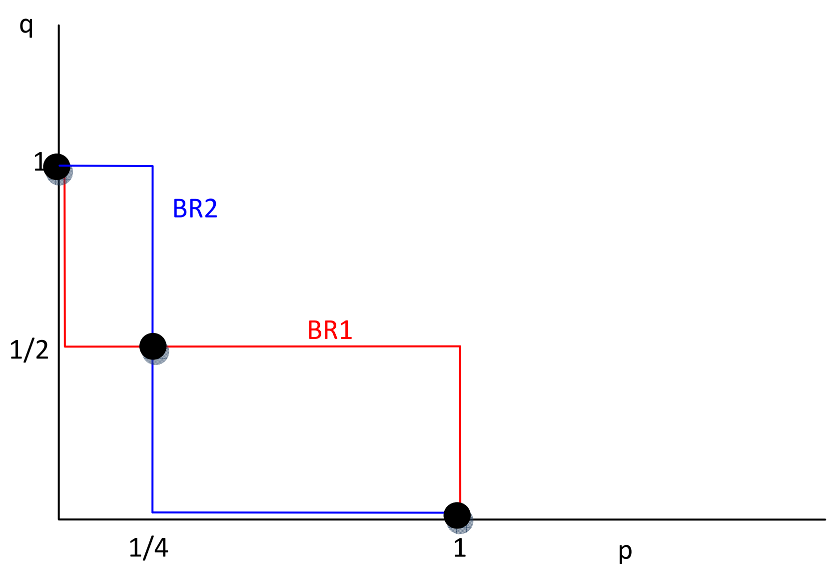
	Left	Right
Up	(3 , 5)	(6 , 8)
Down	(4 , 4)	(5 , 3)

So, by inspection, we have two NE in pure strategies: (D,L) and (U,R). There is also a Mixed Strategy NE.
Player 1 solves:

$$\text{Max(p)} \{3pq + 6p(1\text{-}q) + 4(1\text{-}p)q + 5(1\text{-}p)(1\text{-}q) \}$$
$$\text{Max(p)} \{3pq+6p-6pq+4q-4pq+5-5p-5q+5pq\}$$
$$\text{Max(p)} \{-2pq+p-q+5\}$$
$$\text{Max(p)} \{p(-2q+1)-q+5$$
$$\rightarrow p=0 \text{ if } 1-2q < 0 \Leftrightarrow q > 1/2$$
$$\rightarrow p=1 \text{ if } 1-2q > 0 \Leftrightarrow q < 1/2$$
$$\rightarrow p \in [0,1] \text{ if } 1-2q=0 \Leftrightarrow q = 1/2$$

Player 2 solves:

$$\text{Max(q)} \{5pq + 8p(1\text{-}q) + 4(1\text{-}p)q + 3(1\text{-}p)(1\text{-}q) \}$$
$$\text{Max(q)} \{5pq - 8pq+8p + 4q-4pq + 3-3p-3q+3pq \}$$
$$\text{Max(q)} \{-4pq +q +5p+3 \}$$
$$\text{Max(q)} \{q(1-4p)+5p+3\}$$
$$\rightarrow q=0 \text{ if } 1-4p<0 \Leftrightarrow p > 1/4$$
$$\rightarrow q=1 \text{ if } 1-4p>0 \Leftrightarrow p < 1/4$$
$$\rightarrow q \in [0,1] \text{ if } 1-4p=0 \Leftrightarrow p = 1/4$$



From the intersections of the Best Response curves, we can see all the NE:

{ [(1,0), (0,1)] ; [(0,1),(1,0)] ; [(p , 1-p) , (q , 1-q) | p=1/4, q = 1/2] }

b. Assume X = 10.

Player 2

Player 1

	Left	Right
Up	(3 , 5)	(6 , 8)
Down	(4 , 4)	(10 , 3)

Only one NE at (Down, Left). Grim Trigger strategies to sustain (6,8) each period:

σ_1 = { Play Up in the first period and in all subsequent periods if (Up,Right) has been played by players 1 and 2 respectively in all periods. Play Down otherwise. }

σ_2 = { Play Right in the first periods and in all subsequent periods if (Up,Right) has been played by players 1 and 2 respectively in all periods. Play Left otherwise. }

Payoffs along the equilibrium path are then:

$$\pi_1^e = 6(1 + \delta_1 + \delta^2{}_1 + \delta^3{}_1 + \ldots) = 6/(1- \delta_1)$$
$$\pi_2^e = 8(1 + \delta_2 + \delta^2{}_2 + \delta^3{}_2 + \ldots) = 8/(1- \delta_2)$$

Payoffs from deviating (in the first period):

$$\pi_1^d = 10 + 4(\delta_1 + \delta^2{}_1 + \delta^3{}_1 + \ldots) = 10 + 4\delta_1/(1-\delta_1)$$

Note that player 2 is already playing a best response strategy to player 1 choosing Up, so she has no deviation. The critical discount factor that sustains cooperation (Up,Right) in all periods satisfies:

$$\pi_1^e >= \pi_1^d$$
$$6/(1- \delta_1)>=10 + 4\delta_1/(1-\delta_1)$$
$$6 >= 10-10\delta_1 + 4\delta_1$$
$$-4 >= -6\delta_1$$
$$2/3 <= \delta_1$$
$$\delta_1^* = 2/3$$
and
$$\delta_2^* = 0$$

Since player 2 never wants to deviate from (Up,Right), we place no requirement on her level of patience (hence the zero required discount factor).

c. Assume X = 10. Players have a common discount factor, δ=0.8. Given limited punishment with T periods, we have to compare:

$$\pi_1^e = 6(1 + \delta + \delta^2 + \delta^3 + \ldots + \delta^T) = 6*(1- 0.8^{T+1})/(1-0.8)$$
$$\pi_1^e = 30*(1- 0.8^{T+1})$$
$$\pi_1^e = 30 - 30 * 0.8^{T+1}$$
$$\pi_1^e = 30 - 30 * (0.8*0.8^T)$$
$$\pi_1^e = 30 - 24 * 0.8^T$$

$$\pi_2^e = 8(1 + \delta + \delta^2 + \delta^3 + \ldots + \delta^T) = 8*(1- 0.8^{T+1})/(1-0.8)$$
$$\pi_2^e = 40*(1- 0.8^{T+1})$$
$$\pi_2^e = 40 - 40 * 0.8^{T+1}$$
$$\pi_2^e = 40-40* (0.8*0.8^T)$$
$$\pi_2^e = 40 - 32 * 0.8^T$$

$$\pi_1^d = 10 + 4(\delta + \delta^2 + \delta^3 + \ldots + \delta^T)$$
$$\pi_1^d = 10+4\delta(1 + \delta + \delta^2 + \ldots + \delta^{T-1})$$
$$\pi_1^d = 10 + 4*0.8*(1-0.8^T)/(1-0.8)$$
$$\pi_1^d = 10 + 16*(1-0.8^T)$$
$$\pi_1^d = 26 - 16 * 0.8^T$$

Again, player 2 has no deviation.

So we require for player 1:

$$30 - 24 * 0.8^T >= 26 - 16 * 0.8^T$$
$$4 >= 8*0.8^T$$
$$1/2 >= 0.8^T$$
$$\mathrm{Ln}(1/2) >= T*\ln(0.8)$$
$$T>=\ln(1/2)/\ln(0.8)$$
$$T >= \text{appox } 3.1 \text{ so } T >= 4$$

Overall, to sustain cooperation for both players, we need the punishment period to be at least 4 periods. This is just enough for player 1 and more than enough for player 2 to never deviate.

3.

a. Strategies for player 1: σ_1 = {A, B, C}. Strategies for player 2: σ_2= {MX, MY, NX, NY}

b. The game has 2 subgames. The one following 1's choice of C, and the whole game.

c. The game in normal form:

Player II

Player I

	MX	MY	NX	NY
A	6 , 4	6 , 4	4 , 3	4 , 3
B	0 , 2	0 , 2	0 , 4	0 , 4
C	6 , 6	9 , 3	6 , 6	9 , 3

Note, the payoffs for (A,MX), for example, are found by calculating ½*(8,4) + ½*(4,4).

By inspection, there are three pure strategy NE at (A,MX), (C,MX), and (C,NX).

d. All NE are subgame perfect because the only subgame besides the game itself starts after player I chooses C. In that case, player II has to choose X, which he does in all three NE.

e. (A,MX) means that player II's information set is on the equilibrium path so if p is the belief of player II following player I's choice of A, then p = 1 is consistent with the strategy according to requirement #3 of a PBE. For the second two NE, player II's information set is off the equilibrium path so we require II's action to be sequentially rational. Therefore, he chooses M if E[M] >= E[N] or 4p+2(1-p) >= 2p+4(1-p) or p >= ½. Similarly he chooses N if p <= ½. Therefore our PBEs are:

{ (A,MX), (p,1-p) | p = 1,
(C,MX), (p,1-p) | p >= ½,
(C,NX), (p,1-p) | p <= ½}