Stat 155 Homework # 1 Due February 3

Problems:

Q 1 Let N be uniform on $\{1, 2, \ldots, 10\}$ and let X be a binomial $Bin(N, 1/2)$. Find the mean and variance of X.

Solution Conditional on $N = n$ we have that X is a Bernoulli random variable and so

$$\mathbb{E}[X \mid N] = N/2, \qquad \mathbb{E}[X^2 \mid N] = \mathrm{Var}[X \mid N] + (\mathbb{E}[X \mid N])^2 = N/4 + N^2/4.$$

Since N is uniform on $\{1, 2, \ldots, 10\}$ we have that

$$\mathbb{E}[N] = \frac{1+10}{2} = \frac{11}{2}, \qquad E[N^2] = \mathrm{Var}[N] + (\mathbb{E}[N])^2 = \frac{385}{10} = \frac{77}{2}.$$

Then by the tower property of conditional expectation,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid N]] = \mathbb{E}[N/2] = \frac{11}{4}$$

and

$$\begin{aligned}\mathrm{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[\mathbb{E}[X^2 \mid N]] - (\mathbb{E}[X])^2 = \mathbb{E}[N/4 + N^2/4] - (\frac{11}{4})^2 \\ &= \frac{11}{8} + \frac{77}{8} - \frac{121}{16} = 55/16.\end{aligned}$$

Q 2 Let X_1, X_2, X_3 be independent $Exp(1)$ random variables. Calculate the density and mean of the second largest of the three random variables.

Given three IID random variables with density $f(x)$ and CDF $F(x)$, the second order statistic has density,

$$g(x) = 3f(x)\binom{2}{1}F(x)(1 - F(x)).$$

using the formula from Pitman page 326. Since $f(x) = e^{-x}$ and $F(x) = 1 - e^{-x}$, we have that

$$g(x) = 6e^{-2x}(1 - e^{-x}).$$

Then the mean is given by

$$\int_0^\infty xg(x)dx = \int_0^\infty 6xe^{-2x}(1 - e^{-x})dx = \frac{5}{6}.$$

Q 3 A seller can produce a product for \$1. In each case, what is 1) the optimal price for selling the item and 2) the expected profit per customer if the distribution of the value of the product is (in dollars),

- Distribution Exp(1).
- Has CDF, $F(x) = x/3$ when $0 \le x \le 2$ and $F(x) = \frac{2}{3} + (x-2)/9$ for $2 \le x \le 5$.
- Has CDF, $F(x) = x/9$ when $0 \le x \le 3$ and $F(x) = \frac{1}{3} + (x-3)/3$ for $3 \le x \le 5$.

Let V denote the value of a random customer. As we showed in class, the expected profit per potential customer if the price is x is given by

$$Pr(x) = (x-1)\mathbb{P}[V > x] = (x-1)(1-F(x)).$$

When V has distribution Exp(1), then

$$Pr(x) = (x-1)e^{-x}, \quad Pr'(x) = (2-x)e^{-x}.$$

Then $Pr'(x) = 0$ when $x = 2$ so this gives the optimal price. The expected profit per customer is $Pr(2) = e^{-2}$.

When the distribution function is $F(x) = x/3$ when $0 \le x \le 2$ and $F(x) = \frac{2}{3} + (x-2)/9$ for $2 \le x \le 5$ then

$$Pr(x) = \begin{cases} (x-1)(1-x/3) & x \in [1,2] \\ (x-1)(1/3-(x-2)/9)) & x \in [2,5] \end{cases}, \qquad Pr(x) = \begin{cases} \frac{4}{3} - \frac{2}{3}x & x \in [1,2] \\ \frac{2}{3} - \frac{2}{9}x & x \in [2,5] \end{cases},$$

So on the interval $[1,2]$ the profit is maximized at the endpoint 2 with $Pr(2) = 1/3$. While on the interval $[2,5]$ it is maximized at $x = 3$ with $Pr(3) = \frac{4}{9}$. Hence the optimal price is 3 with profit $\frac{4}{9}$.

When the distribution function is $F(x) = x/9$ when $0 \le x \le 3$ and $F(x) = \frac{1}{3} + (x-3)/3$ for $3 \le x \le 5$ then again we split into cases and have

$$Pr(x) = \begin{cases} (x-1)(1-x/9) & x \in [1,3] \\ (x-1)(2/3-(x-3)/3)) & x \in [3,5] \end{cases}, \qquad Pr(x) = \begin{cases} \frac{10}{9} - \frac{2}{9}x & x \in [1,2] \\ 2 - \frac{2}{3}x & x \in [2,5] \end{cases},$$

The derivative is always positive on $[1,3]$ so it is maximized on that endpoint. On $[3,5]$ we have $Pr'(x) = 0$ solved at $x = 3$. Hence by coincidence the optimal price is again 3 with profit $Pr(3) = \frac{4}{3}$.

Q 4 Suppose that there are three agents in a sealed first price auction with independent values uniform on $[0,1]$.

- Verify from the definition that bidding $\beta(v) = \frac{2}{3}v$ is a Bayes-Nash equilibrium.
- Calculate the expected revenue in this auction.

Let V_1, V_2, V_3 be the values of the 3 agents. We must check that the utility of agent 1 is maximized bidding $\frac{2}{3}v$ if his value is v. We first calculate the probability that a bid b exceeds the other bids,

$$\begin{aligned} \mathbb{P}[b > \max(\frac{2}{3}V_2, \frac{2}{3}V_3)] &= \mathbb{P}[\frac{3}{2}b > V_2, \frac{3}{2}b > V_3)] \\ &= \mathbb{P}[\frac{3}{2}b > V_2]\mathbb{P}[\frac{3}{2}b > V_3)] \\ &= (\frac{3}{2}b)^2. \end{aligned}$$

The profit bidding b is $v - b$ if the agent wins the auction so the expected utility bidding b is,

$$u[b \mid v] = (v - b)(\frac{3}{2}b)^2.$$

Then $\frac{d}{db}u[b \mid v] = \frac{9}{4}b(2v - 3b)$ and so the utility is maximized at $\frac{2}{3}v$ verifying that $\beta(v) = \frac{2}{3}v$ is a Bayes-Nash Equilibrium.

The expected payment of an agent with value v bidding $\frac{2}{3}v$ is $\frac{2}{3}v(\frac{3}{2}\frac{2}{3}v)^2 = \frac{2}{3}v^3$. Hence the expected payment of one agent is

$$E\frac{2}{3}V_1^3 = \int_0^1 \frac{2}{3}v^3 dv = \frac{1}{6}.$$

The total payment is three times this so the expected revenue is 1/2.