ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #4 – ANSWER KEY

Exercise 2 - Chapter 16 Watson

Solving by backward induction:

1. We start from the second stage of the game where both firms compete in prices. Since market demand is $Q = a - p$, then products are homogeneous, and in addition, we are told in the exercise that the firm setting the lowest price gets all the market. Hence, we are in a Bertrand game of price competition, and we know from class that the equilibrium price firms set is $P_1 = P_2 = 0$.

Importantly, note that prices are not functions of the expenditure on advertising that firm 1 makes during the first period.

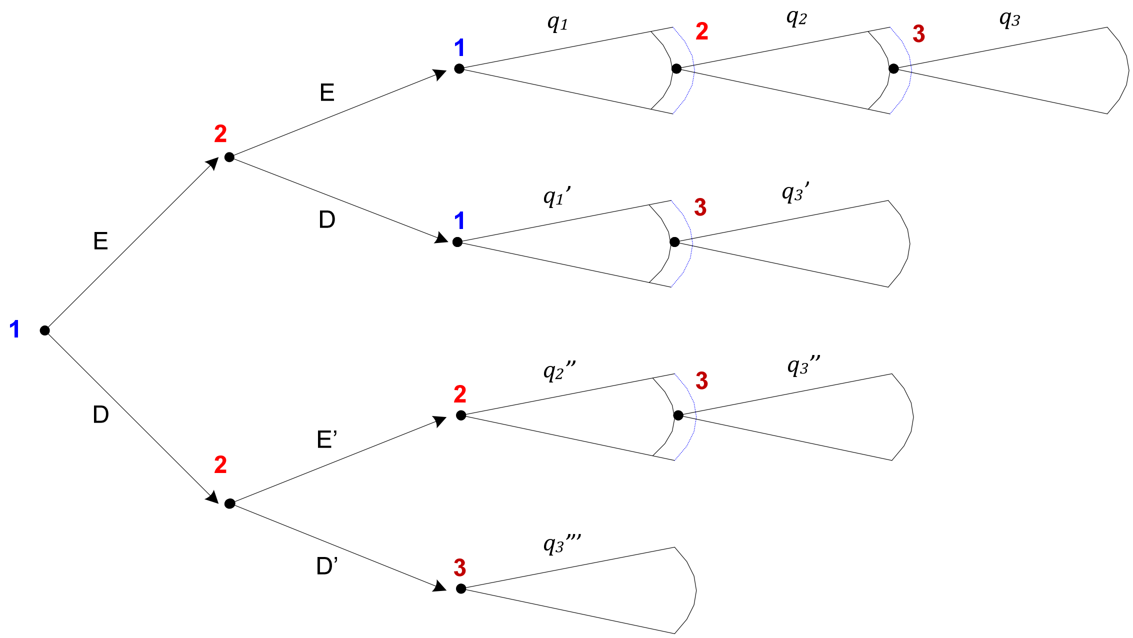
2. Since this is the case, firm 1 knows that by spending more money on advertising it will not increase the profits during the second period. As a consequence, $a = 0$ during the first period.

Therefore, the subgame perfect equilibrium is $a = 0$ during the first stage and $P_1 = P_2 = 0$ during the second stage.

Exercise 8 - Chapter 16 Watson

a.

Without payoffs, the extensive form is as follows [Note that we are using dashed lines to denote that firm 2 chooses q_2 without observing firm 1's output q_1. Similarly, firm 3 chooses q_3 without observing firm 1 and firm 2's output, q_1 and q_2, respectively.]:



Solving by backward induction, we must first find the output level of every possible entry/no entry scenario. By doing so, we will be able to find the profits resulting from every possible entry/no entry scenario, and then we will be ready to compare firms' profits from entering and not entering:

1. We first solve firms' output in the subgame that starts after firm 1 and 2 enter. [In the figure, this subgame is the upper part, where firms are selecting q_1, q2 and q_3] This is just a Cournot game of quantity competition with three firms competing with each other by simultaneously selecting output. Hence, $q_1 = q_2 = q_3 = 3$.

   a. PROFITS: In this case, note that the profits of every firm in this Cournot oligopoly game with three firms are:

$$(12 - Q)q_i = (12 - q_1 - q_2 - q_3)q_i = (12 - 3 - 3 - 3) * 3 = 3 * 3 = 9.$$

b. Note that we must finally subtract 10 (entry costs) in the profits of firm 1 and firm (You don't have to do so for firm 3, since it was already the incumbent in the market). Hence, the payoff vector would be (9-10, 9-10, 9) = (-1, -1, 9)

2. Now we solve the subgame induced after firm 1 enters (E) but firm 2 does not (D). Here we have a Cournot oligopoly game played by firms 1 and 3 (duopoly), where they simultaneously select q_1' and q_3'. Hence, $q_1' = q_3' = 4$.

a. PROFITS: In this case, note that the profits of every active firm in this Cournot oligopoly game with two firms are:

$$(12 - Q)q_i' = (12 - q_1' - q_3')q_i' = (12 - 4 - 4) * 4 = 4 * 4 = 16.$$

b. Note that we must finally subtract 10 (entry costs) from the profits of firm 1 (entrant), which implies that the payoff vector becomes (16-10, 0, 16) = (6, 0, 16).

3. Now we solve the subgame that starts after firm 1 decides not to enter (D), but firm 2 decides to enter (E'). Now we have a Cournot oligopoly game played by firms 2 and 3 (duopoly), where they simultaneously select q_2'' and q_3''. Hence, $q_2'' = q_3'' = 4$.

a. PROFITS: In this case, note that the profits of every active firm in this Cournot oligopoly game with two firms are:

$$(12 - Q)q_i'' = (12 - q_2'' - q_3'')q_i'' = (12 - 4 - 4) * 4 = 4 * 4 = 16.$$

b. Note that we must finally subtract 10 (entry costs) from the profits of firm 2 (entrant), which implies that the payoff vector becomes (0, 16-10, 16) = (0, 6, 16).

4. Now we solve the subgame induced after firm 1 decides not to enter (D) and firm 2 decides not to enter either (D'). Here firm 3 keeps its monopolistic position, and chooses monopoly output, $q_3''' = 6$.

a. PROFITS: In this case, note that the profits of the only monopoly in the market (firm 3), are:

$$(12 - Q)q_3 = (12 - 6)6 = 36$$

b. Note that we don't have to subtract any entry costs from firm 3's profits, given that it was already the incumbent in the market. Hence, the payoff vector in this case is (0, 0, 36).

Plugging all the payoff vectors in the appropriate nodes (see figure at the end of the answer key), and solving by backward induction, we see that:

1. Firm 2 (last mover in this game):
    - After observing that firm 1 entered the market, firm 2 decides to not enter, since its profit from not entering (0) are higher than from entering a "too crowded" market (profits of -1).
    - After observing that firm 1 didn't enter the market, firm 2 chooses to enter, since its profits from doing so (6, now firm 2 would become the only competitor of firm 3) are higher than from not entering (0).
2. Firm 1 (first mover in this game):
    - Firm 1 decides to enter, given that its profits from entering (and inducing firm 2 to stay out afterwards) are 6, while those from not entering (and inducing firm 2 to enter the market afterwards) are only 0. Hence, firm 1 enters.

Hence, at the subgame perfect equilibrium:

1. firm 1 selects Enter,
2. firm 2 chooses not to enter after observing that firm 1 entered, but chooses to enter after observing that firm 1 didn't enter.
3. Equilibrium output levels at every subgame of this game are:

   $q_1 = q_2 = q_3 = 3 \qquad q_1' = q_3' = 4 \qquad q_2'' = q_3'' = 4 \qquad q_3''' = 6$

b.

In the subgame perfect equilibrium only firm 1 enters, inducing firm 2 to stay out of the market.

Exercise 9 - Chapter 16 Watson

a. The government solves:

$$\underset{\dot{p}}{Max}\left\{30 + (\dot{p} - \dot{W}) - \frac{\dot{p}}{2} - 30 = \frac{\dot{p}}{2} - \dot{W}\right\}$$

Taking first order conditions with respect to $\dot{p}$, we obtain $\frac{1}{2}$. Since this result does not depend on $\dot{p}$, it is indicating that the solution to the problem is a corner solution. In particular, given that $\frac{1}{2} > 0$, we can conclude that the solution is the upper corner, i.e., the government sets $\dot{p}$ as high as possible, regardless of the level of $\dot{W}$. So $\dot{p}^* = 10$.

Knowing how the government will behave, the ASE solves:

$$\underset{\dot{W}}{Max}\left\{-(\dot{W} - 10)^2\right\}$$

where we have already replaced $\dot{p}^* = 10$. The first order condition implies:

$$\dot{p}^* = \dot{W}^* = 10$$

So in equilibrium y = 30.

b. If the government could commit ahead of time, it would solve:

$$\underset{\dot{p}}{Max}\left\{30 + (\dot{p} - \dot{W}) - \frac{\dot{p}}{2} - 30\right\}$$

and using the fact that $\dot{p} = \dot{W}$, we can rearrange the above expression to obtain

$$\underset{\dot{W}}{Max}\left\{-\frac{\dot{p}}{2}\right\}$$

Taking first-order conditions with respect to $\dot{p}$ yields $-\frac{1}{2} < 0$, indicating that the solution to this maximization problem is the lower corner, i.e., the government commits $\dot{p} = 0$ and the ASE would set $\dot{W} = 0$.

In (a) u = 0 and v = -5. Now, when commitment is possible, u = 0 and v = 0.

c. One way is to have a separate central bank that does not have a politically elected head that states its goals.

Exercise 5 – Chapter 9 Harrington

A: In this game of "Hunt for Red October" we can only divide the game into two subgames. There is one subgame encompassing the decisions of both Borodin and Melekhin when Ramius sends the letter, and one subgame when Ramius does not send the letter. We can then construct the normal form of these games and determine the best responses, finally analyzing Ramius' decisions from there.

ANSWER: Consider the final subgame associated with the letter having been sent. The strategic form is shown in FIGURE SOL9.5.1. *Defect* is a dominant strategy so there is a unique Nash equilibrium of (*defect*, *defect*).

FIGURE SOL9.5.1

Borodin \ Melekhin	*Defect*	*Renege*
Defect	4,4	2,3
Renege	3,2	1,1

Next, consider the final subgame associated with the letter not having been sent. The strategic form is shown in FIGURE SOL9.5.2. It has two Nash equilibria: (*defect*, *defect*) and (*renege*, *renege*).

FIGURE SOL9.5.2

Borodin \ Melekhin	*Defect*	*Renege*
Defect	8,8	6,5
Renege	5,6	7,7

Having determined the Nash equilibria of the final subgames, we can now look at the last subgame, which is the entire game.

Now turn to the subgame that is the game itself. If the Nash equilibrium is (*defect*, *defect*), at both final subgames the captain prefers to not send the letter—as it results in a payoff of 8—while sending the letter means a lower payoff of 7. This gives us one subgame perfect Nash equilibrium of (*do not send letter, defect/defect*, *defect/defect*). Now suppose the Nash equilibrium at the subgame in which the letter is not sent is (*renege*, *renege*). If the letter is sent the captain again earns a payoff of 7, and if the letter is not sent his payoff is 4, as both officers renege. We then find that (*send letter, defect/renege*, *defect/renege*) is a second subgame perfect Nash equilibrium.

B:

b. Explain why the captain would send the letter.

ANSWER: The payoffs are such that if the letter is sent, then it is a dominant strategy for each of the two officers to defect. However, if the letter is not sent, then there are two Nash equilibria for that subgame: one in which both officers defect and one in which both renege. If the letter is not sent—so there is the opportunity to return to the Soviet Union without their plan to defect having been discovered—then an officer wants to renege unless everyone else is planning to defect. By sending the letter, the captain ensures that the officers will continue to defect; while if the letter is not sent, then induced equilibrium play could mean that they renege and the plan to defect fails.

Or in very simple terms, Ramius <u>must</u> send the letter in order to ensure the defection of his offices with him.

BONUS EXERCISE (*Excessive entry in an industry*)

(a) Since the equilibrium has to be found by backward induction, first solve the last stage of the game, where firms choose quantities given the number of firms, n, that have entered the market in the previous stage. In particular, the n^{th} firm entering produces an output level of $\frac{1-c}{n+1}$, thus obtaining profits of $\pi^c(n) = \frac{(1-c)^2}{(n+1)^2} - F$. Since n is a real number, the equilibrium number of firms in the industry will be given by solving for n in $\pi^c(n) = 0$, given that the n^{th} entrant must be indifferent between entering and staying out. Solving for n we obtain $n^c = \frac{1-c}{\sqrt{F}} - 1$.

(b) The social planner will choose n to maximize total welfare (the sum of consumer and producer surplus). Let us first find consumer surplus, *CS*. Notice that the CS is given by the area of the triangle between the vertical intercept of the demand curve, 1, and the equilibrium price $p = 1 - n\frac{1-c}{n+1} = \frac{1+cn}{n+1}$. Hence, CS is given by $CS = \frac{1}{2}\left(1 - \frac{1+cn}{n+1}\right)n\frac{1-c}{n+1} = \frac{n^2(1-c)^2}{2(n+1)^2}$. Producer surplus is simply given by the aggregate profits all firms make in the industry, i.e., $n\pi^c(n) = n\frac{(1-c)^2}{(n+1)^2} - nF$.

Therefore, the sum of consumer and producer surplus yields a total welfare of

$W = CS + PS = \frac{n(1-c)^2(n+2)}{2(n+1)^2} - nF$. Taking first-order conditions with respect to

n, and solving for n, we obtain $n^* = \sqrt[3]{\frac{(1-c)^2}{F}} - 1$.

- By comparing the optimal number of firms n^* we just found with the number of firms entering at the free entry equilibrium from part (a), n^c, it is clear that there exists excess of entry in the industry.
- The following figure illustrates this result. In particular, the figure depicts n^c and n^* as a function of F in the horizontal axis, and evaluating both of them at a marginal cost of c=0.5. (You can obtain similar figures using another value for firms' marginal costs of production, c). Two important features of the figure are noteworthy. First, note that both n^c and n^* decrease in the entry costs, F. Second, the equilibrium number of firms entering the industry, n^c, lies above the socially optimal number of firms, n^*, for any given level of F; reflecting an excessive entry in the industry when entry is unregulated.