EconS 424 – Strategy and Game Theory

Homework #5 – Answer Key

Exercise #1 – *Collusion among N doctors*

Consider an infinitely repeated game, in which there are $n \geq 3$ doctors, who have created a partnership. In each period, each doctor decides how hard to work. Let e_i^t be the effort chosen by doctor *i* in period *t*, and $e_i^t = 1, 2, \ldots, 10$. Doctor i's discount factor is δ_i.

$$\text{Total profit for the partnership: } \quad 2(e_1^t + e_2^t + e_3^t + \cdots + e_n^t)$$

$$\text{A doctor } i\text{'s payoff: } \quad \frac{1}{n} \times 2 \times (e_1^t + e_2^t + e_3^t + \cdots + e_n^t) - e_i^t$$

a. Assume that the history of the game is common knowledge. Derive a subgame perfect NE in which each player chooses effort $e^* > 1$.

To begin, note that doctor's payoff can be rearranged to:

$$\left(\frac{2}{n}\right)(e_1 + e_2 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_n) - \left(\frac{n-2}{n}\right)e_i$$

Since a doctor's payoff is strictly decreasing in her own effort, she wants to minimize it. $e_i = 1$ is then a strictly dominant strategy for doctor *i* and therefore there is a unique stage game Nash equilibrium in which each doctor chooses the minimal effort level of 1.

- Next, note that each doctor's payoff from choosing a common effort level of *e* is:

$$\left(\frac{1}{n}\right) \times 2 \times (e + e + \cdots + e) - e = \left(\frac{1}{n}\right) \times 2 \times ne - e = e$$

- To determine a doctor's best deviation, we must take a partial derivative with respect to e_i of their payoff function when all other (n-1) players select *e*, yielding

$$\frac{\partial u_i}{\partial e_i} = -\left(\frac{n-2}{n}\right)$$

  This suggests a corner solution where doctor *i* wants to minimize effort by playing the lowest e possible, i.e., e_i=1.

- We can now describe a **grim-trigger strategy**. When conditions are met and the strategy is played symmetrically, that will guarantee cooperation at an effort level *e*>1.

Consider the symmetric grim-trigger strategy:

- In period 1: choose $e_i^1 = e^*$
- In period t ≥ 2: choose $e_i^t = e^*$ when $e_j^\tau = e^*$ for all j, for all $\tau \leq t-1$; and

  choose 1 otherwise.

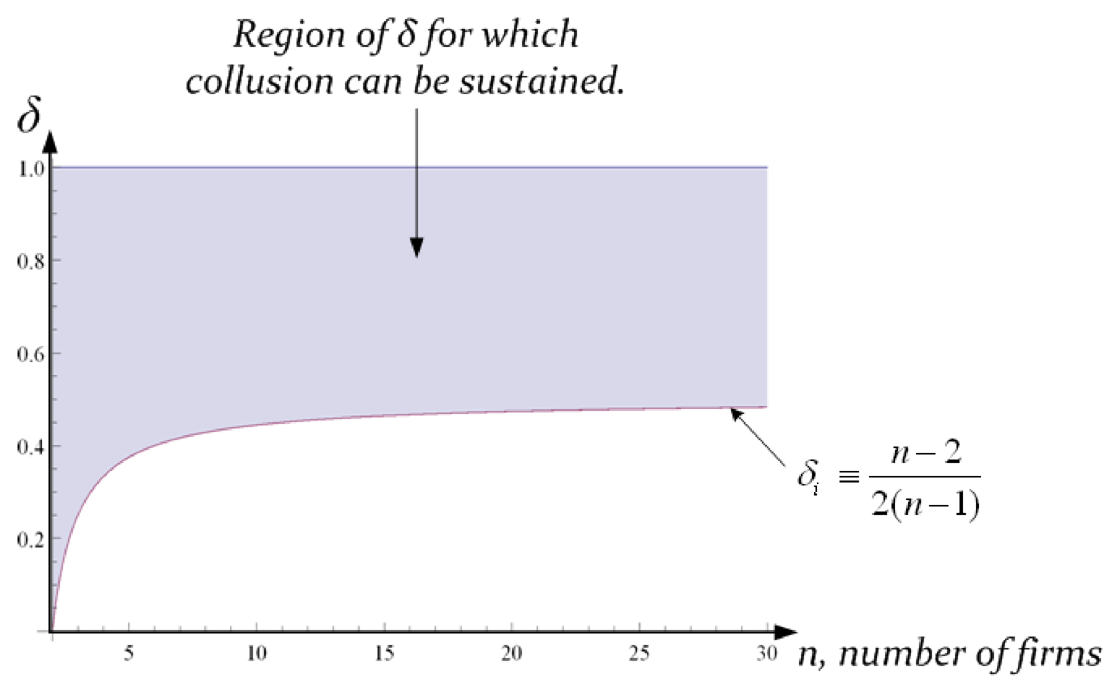
This is a subgame perfect Nash equilibrium if and only if

$$\frac{e^*}{1-\delta_i} \geq \left[\left(\frac{n-1}{n}\right)2e^* - \left(\frac{n-2}{n}\right) * 1\right] + \frac{\delta_i}{1-\delta_i} \quad for\ all\ i.$$

That is to say, the equilibrium will only hold so long as the payoff from remaining in the equilibrium is greater than or equal to the one period payoff from deviating plus the payoff from the 'punishment' equilibrium played every period thereafter. Solving for δ_i yields:

$$\delta_i \geq \frac{n-2}{2(n-1)}$$

The following figure depicts this cutoff of δ_i, shading the region of discount factors above δ_i which would support collusion.



It is now possible to see how the equilibrium responds to changes in *n*. Differentiating the about cutoff of δ_i with respect to *n*, we obtain

$$\frac{\partial \delta_i}{\partial n} = \frac{1}{2(n-1)^2}$$

This partial is positive, indicating that as the group size n increases, δ_i has to increase to maintain the cooperative equilibrium. So it is more difficult to support cooperation as the group size increases.

b. Assume that the history of the game is not common knowledge, i.e., in each period, only the total effort is observed. Find a subgame perfect NE in which each player chooses effort $e^* > 1$.

Consider the strategy profile in part (a), except that it now conditions on total effort. Let e^t denote total effort for period *t.*

- In period 1: choose $e_i^1 = e^*$
- In period t ≥ 2: choose $e_i^t = e^*$ when $e^\tau = ne^*$ for all *j*, for all $\tau \leq t-1$; and
  choose 1 otherwise.

This is a subgame perfect Nash equilibrium under the exact same conditions as in part (a).

Exercise #2 - *Collusion when firms compete in quantities*

Consider two firms competing as Cournot oligopolists in a market with demand:

$$p(q_1, q_2) = a - bq_1 - bq_2$$

Both firms have total costs, $TC(q_i) = cq_i$ where c > 0 is the marginal cost of production.

a. Considering that firms only interact once (playing an unrepeated Cournot game), find the equilibrium output for every firm, the market price, and the equilibrium profits for every firm.

Firm 1 chooses q_1 to solve

$$\max_{q1} \pi_i = p(a - bq_1 - bq_2) - cq_1$$

Taking first order conditions with respect to q_1, we find

$$\frac{\partial \pi_1}{\partial q_1} = a - bq_1 - bq_2 - c = 0$$

and solving for q_1 we obtain firm 1's best response function

$$q_1(q_2) = \frac{a-c}{2b} - \frac{1}{2}q_2$$

and similarly for firm 2, since firms are symmetric,

$$q_2(q_1) = \frac{a-c}{2b} - \frac{1}{2}q_1$$

Plugging $q_2(q_1)$ into $q_1(q_2)$, we find firm 1's equilibrium output:

$$q_1 = \frac{a - b\left(\frac{a - bq_1 - c}{2b}\right) - c}{2b} = \frac{a}{2b} - \left(\frac{a - bq_1 - c}{4b}\right) - \frac{c}{2b} = \frac{2a - 2c - a + c}{3b} = \frac{a-c}{3b}$$

Similarly, firm 2's equilibrium output is

$$q_2 = \frac{a - b\left(\frac{a-c}{3b}\right) - c}{2b} = \frac{a}{2b} - \left(\frac{a-c}{6b}\right) - \frac{c}{2b} = \frac{3a - a + c - 3c}{6b} = \frac{a-c}{3b}$$

Therefore, the market price is

$$p = a - b\left(\frac{a-c}{3b}\right) - b\left(\frac{a-c}{3b}\right)$$
$$= \frac{3a - a + c - a + c}{3} = \frac{a+2c}{3}$$

and the equilibrium profits of each firm are

$$\pi_1^{cournot} = \pi_2^{cournot} = \left(\frac{a+2c}{3}\right)\left(\frac{a-c}{3b}\right) - c\left(\frac{a-c}{3b}\right) = \frac{(a-c)^2}{9b}$$

b. Now assume that they could form a cartel. Which is the output that every firm should produce in order to maximize the profits of the cartel? Find the market price, and profits of every firm. Are their profits higher when they form a cartel than when they compete as Cournot oligopolists?

Since the cartel seeks to maximize their joint profits, they choose output levels q_1 and q_2 that solve

$$\max \ \pi_1 + \pi_2 = (a - bq_1 - bq_2)q_1 - cq_1 + (a - bq_1 - bq_2)q_2 - cq_2$$

which simplifies to

$$\max \quad (a - bq_1 - bq_2)(q_1 + q_2) - c(q_1 + q_2)$$

Notice that this maximization problem can be further reduced to the choice of aggregate output Q=$q_1 + q_2$ that solves

$$\max \quad (a - bQ)Q - cQ$$

Interestingly, this maximization problem coincides with that of a regular monopolist. In other words, the overall production of the cartel of two symmetric firms coincides with that of a standard monopoly. Indeed, taking first order conditions with respect to Q, we obtain

$$a - 2bQ - c \leq 0$$

And solving for Q, we find $Q = \frac{a-c}{2b}$, which is an interior solution given that a>c by definition. Therefore, ach firm's output level in the cartel is

$$q_1 = q_2 = \frac{Q}{2} = \frac{\frac{a-c}{2b}}{2} = \frac{a-c}{4b}$$

And the market price is

$$p = a - bQ = a - b\frac{a-c}{2b} = \frac{a+c}{2}$$

Therefore, each firm's profits in the cartel are

$$\pi_1 = pq_1 - TC(q_1) = \left(\frac{a+c}{2}\right)\left(\frac{a-c}{4b}\right) - c\left(\frac{a-c}{4b}\right) = \frac{(a-c)^2}{8b}$$

$$\pi_1^{cartel} = \pi_2^{cartel} = \frac{(a-c)^2}{8b}$$

Comparing the profits that every firm makes in the cartel, $\frac{(a-c)^2}{8b}$, against those under Cournot competition, $\frac{(a-c)^2}{9b}$, we can conclude that

$$\pi_1^{cartel} = \pi_2^{cartel} > \pi_1^{cournot} = \pi_2^{cournot}$$

c. Can the cartel agreement be supported as the (cooperative) outcome of the infinitely repeated game?

1. First, we find the discounted sum of the infinite stream of profits when firms cooperate in the cartel agreement (they do not deviate).

- Payoff of cartel when they cooperate is $\frac{(a-c)^2}{8b}$
- As a consequence, the discounted sum of the infinite stream of profits from cooperating in the cartel is

$$\frac{(a-c)^2}{8b} + \delta \frac{(a-c)^2}{8b} + \cdots = \frac{1}{1-\delta} \frac{(a-c)^2}{8b}$$

2. Second, we need to find the optimal deviation that, conditional on firm 2 choosing the cartel output, maximizes firm 1's profits. That is, which is the output that maximizes firm 1's profits, and which are its corresponding profits from deviating?

Since firm 2 sticks to cooperation (i.e., produces the cartel output $q_2 = \frac{a-c}{4b}$), if firm 1 seeks to maximize its current profits (optimal deviation), we only need to plug $q_2 = \frac{a-c}{4b}$ into firm 1's best response function, as follows

$$q_1^{dev} \equiv q_1\left(\frac{a-c}{4b}\right) = \frac{a-c}{2b} - \frac{1}{2}\frac{a-c}{4b} = \frac{3(a-c)}{8b}$$

which provides us with firm 1's optimal deviation, given that firm 2 is still respecting the cartel agreement. In this context, firm 1's profit is

$$\pi_1 = \left[a - b\left(\frac{3(a-c)}{8b}\right) - b\left(\frac{a-c}{4b}\right) - c\right]\left(\frac{3(a-c)}{8b}\right)$$

$$= 3\left(\frac{a-c}{8}\right)\left(\frac{3(a-c)}{8b}\right) = \frac{9(a-c)^2}{64b}$$

while that of firm 2 is

$$\pi_2 = \left[a - b\left(\frac{3(a-c)}{8b}\right) - b\left(\frac{a-c}{4b}\right) - c\right]\left(\frac{a-c}{4b}\right)$$

$$= 3\left(\frac{a-c}{8}\right)\left(\frac{a-c}{4b}\right) = \frac{3(a-c)^2}{32b}$$

Hence, firm 1 has incentives to unilaterally deviate since its *current* profits are larger by deviating (while firm 2 respects the cartel agreement) than by cooperating. That is,

$$\pi_1^{deviate} = \frac{9(a-c)^2}{64b} > \pi_1^{cartel} = \frac{(a-c)^2}{8b}$$

3. Finally, we can now compare the profits that firms obtain from cooperating in the cartel agreement (part i) with respect to the profits they obtain from choosing an optimal deviation (part

ii) plus the profits they would obtain from being punished thereafter (discounted profits in the Cournot oligopoly). In particular, for cooperation to be sustained we need

Firm 1:

$$\frac{1}{1-\delta}\frac{(a-c)^2}{8b} > \frac{9(a-c)^2}{64b} + \frac{\delta}{1-\delta}\frac{(a-c)^2}{9b}$$

Solving for discount factor δ, we obtain

$$\frac{1}{8(1-\delta)} > \frac{9}{64} + \frac{\delta}{9(1-\delta)}, \text{ which implies } \delta > \frac{9}{17}$$

Hence, firms need to assign a sufficiently high value to future payoffs, $\delta \in \left[\frac{9}{17}, 1\right]$, for the cartel agreement to be sustained.

Finally, note that firm 2 has incentives to carry out the punishment. Indeed, if it does not revers to the NE of the stage game (producing the Cournot equilibrium output), firm 2 obtains profits of $\frac{3(a-c)^2}{32b}$, since firm 1 keeps producing its optimal deviation of $q_1^{dev} = \frac{3(a-c)}{8b}$ while firm 2 produces the cartel output $q_2^{cartel} = \frac{a-c}{4b}$. If, instead, firm 2 practices the punishment, producing the Cournot output $\frac{a-c}{3b}$, its profits are $\frac{(a-c)^2}{9b}$, which exceed $\frac{3(a-c)^2}{32b}$ for all parameter values. Hence, upon observing that firm 1 deviates, firm 2 prefers to revert to the production of its Cournot output level than being the only firm that produces the cartel output.

Exercise #3 – *Collusion among N firms*

Consider n firms producing homogenous goods and choosing quantities in each period for an infinite number of periods. Demand in the industry is given by $p = 1 - Q$, Q being the sum of individual outputs. All firms in the industry are identical: they have the same constant marginal costs $c < 1$, and the same discount factor δ. Consider the following trigger strategy:

- Each firm sets the output q^m that maximizes joint profits at the beginning of the game, and continues to do so unless one or more firms deviate.
- After a deviation, each firm sets the quantity q^{cn}, which is the Nash equilibrium of the one-shot Cournot game.

(a) Find the condition on the discount factor that allows for collusion to be sustained in this industry.

First find the quantities that maximize joint profits $\pi = (1-Q)Q - cQ$. It is easily checked that this output level is $Q = \frac{1-c}{2}$, yielding profits of

$$\pi = \left(1 - \frac{1-c}{2}\right)\frac{1-c}{2} - c\frac{1-c}{2} = \frac{(1-c)^2}{4}$$

for the cartel.

Therefore, at the symmetric equilibrium individual quantities are $q^m = \frac{1}{n}\frac{1-c}{2}$ and individual profits under the collusive strategy are $\pi^m = \frac{1}{n}\frac{(1-c)^2}{4}$.

As for the deviation profits, the optimal deviation by a firm is given by

$$q^d(q^m) = \text{argmax}_q \left[1-(n-1)q^m - q\right]q - cq\,.$$

where note that all other *n*-1 firms are still producing their cartel output $q^m = \frac{1}{n}\frac{1-c}{2}$.

It can be checked that the value of q that maximizes the above expression is $q^d(q^m) = (n+1)\frac{(1-c)}{4n}$, and that the profits that a firm obtains by deviating from the collusive output are, hence,

$$\pi^d = \left[1-(n-1)\left(\frac{1}{n}\frac{1-c}{2}\right) - (n+1)\frac{1-c}{4n}\right](n+1)\frac{1-c}{4n} - c(n+1)\frac{1-c}{4n},$$

which simplifies to

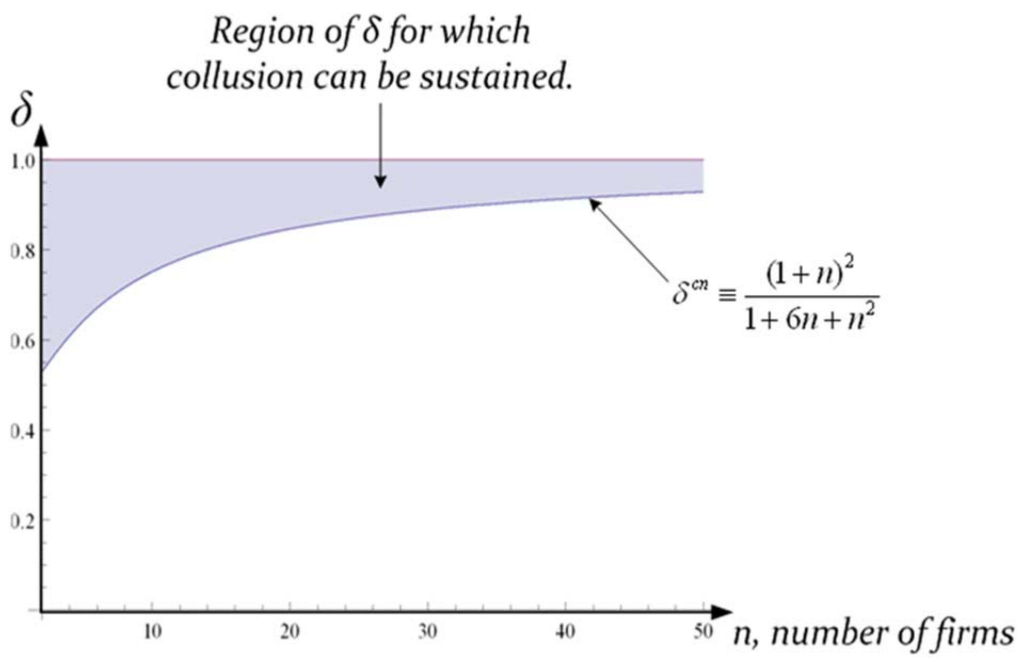
$$\pi^d = \frac{(1-c)^2(n+1)^2}{16n^2}$$

Therefore, collusion can be sustained in equilibrium if

$$\frac{1}{1-\delta}\pi^m \geq \pi^d + \frac{\delta}{1-\delta}\pi^{cn},$$

which after solving for the discount factor, δ, yields $\delta \geq \frac{(1+n)^2}{1+6n+n^2}$. For compactness, we denote this ratio as $\frac{(1+n)^2}{1+6n+n^2} \equiv \delta^{cn}$.

Hence, under punishment strategies that involve a reversion to Cournot equilibrium forever after a deviation takes place, tacit collusion arises if and only if firms are sufficiently patient. The following figure depicts cutoff δ^{cn}, as a function of the number of firms, *n*, shading the region of δ that exceeds such a cutoff.



(b) Indicate how the number of firms in the industry affects the possibility of reaching the tacit collusive outcome.

By carrying out a simple exercise of comparative statics using the critical threshold for the discount factor, one concludes that

$$\frac{\partial \delta^{cn}}{\partial n} = \frac{4(n^2-1)}{(1+6n+n^2)^2} > 0.$$

(This could be anticipated from our previous figure, where the critical discount factor increases in *n*.)

Intuition*:* Other things being equal, as the number of firms in the agreement increases, the more difficult it is to reach and sustain tacit collusion. Since firms are assumed to be symmetric, an increase in the number of firms is equivalent to a lower degree of market concentration. Therefore, lower levels of market concentration are associated – *ceteris paribus* – with less likely collusion.

BONUS EXERCISES

Bonus Exercise #1 – *Collusion with increasing demand patterns*

Consider a homogenous industry where *n* firms produce at zero cost and play the Bertrand game an infinite number of periods. Assume that:

- When firms choose the same price, they earn a per-period profit $\Pi(p) = p\alpha\frac{D(p)}{n}$, where parameter α represents the state of demand.
- When a firm *i* charges a price of p_i lower than the price of all of the other firms, it earns a profit $\Pi(p_i) = p_i\alpha D(p_i)$, and all of the other firms obtain zero profits.

Imagine that in the current period demand is characterized by $\alpha = 1$, but starting from the following period demand will be characterized by $\alpha = \theta$ in each of the following periods. All the players know exactly the evolution of the demand state at the beginning of the game. Firms have the same common discount factor, δ.

(a) Assume $\theta > 1$ and consider the following trigger strategies. Each firm plays the monopoly price p_m in the first period of the game and continues to charge such a price until a profit equal to zero is observed. When this occurs, each firm charges a price equal to zero forever. Under which conditions does this n-tuple of strategies represent an equilibrium?

- ***[Hint: In particular, show how θ and n affects such a condition, and gives an economic intuition for this result.]***

  Let us denote the collusive price by $p^c \in (c, p_m]$. At time $t = 0$, the demand parameter α satisfies $\alpha = 1$, whereas at time t={1,2,...}, $\alpha = \theta$. The incentive constraint then becomes

  $$\frac{\pi(p^c)}{n} + \delta\theta\frac{\pi(p^c)}{n} + \delta^2\theta\frac{\pi(p^c)}{n} + \cdots \geq \pi(p^c) + \delta 0 + \delta^2 0 + \ldots$$

  where every firm obtains a share $\frac{1}{n}$ of the collusive profits $\pi(p^c)$ in every period in which all firms charge the collusive price $p^c \in (c, p_m]$. If a firm deviates to a price marginally lower than the collusive price p^c, it captures all the demand for the product, obtaining a profit of $\pi(p^c)$ during that period, but triggers an infinite punishment by all other firms, who revert to the Nash equilibrium of the Bertrand game with zero economic profits thereafter. Hence, simplifying the above inequality yields

  $$\frac{\pi(p^c)}{n}(1 + \delta\theta + \delta^2\theta + \delta^3\theta + \cdots) \geq \pi(p^c),$$

or equivalently, $\delta \geq \frac{n-1}{n-1+\theta}$. For compactness, we hereafter denote the previous ratio $\frac{n-1}{n-1+\theta} \equiv \tilde{\delta}(n,\theta)$.

Differentiating the minimal discount factor that supports collusion $\tilde{\delta}(n,\theta)$, with respect to θ and n, we obtain

$$\frac{\partial \tilde{\delta}(n,\theta)}{\partial \theta} = \frac{-(n-1)}{(n-1+\theta)^2} < 0, \quad \text{whereas} \quad \frac{\partial \tilde{\delta}(n,\theta)}{\partial n} = \frac{\theta}{(n-1+\theta)^2} > 0.$$

Intuition: the higher the value of θ, the higher the present value of the stream of profits received from $t=1$ onwards. Thus, the opportunity cost of deviation increases with θ. Hence, the higher θ is, the less likely it is that firms will disrupt the collusive agreement today (at t = 0). On the other hand, when n increases, the less likely it is that collusion will be sustained at equilibrium.

(b) *Can other prices be sustained at equilibrium under strategies similar to the ones above? Under which conditions?*

Notice that the incentive constraint derived above is valid for all collusive prices in the interval $p^c \in (c, p_m]$. Hence, under strategies similar to the ones above, a collusive price $p^c \in (c, p_m]$ can be sustained at equilibrium if $\delta \geq \frac{n-1}{n-1+\theta} \equiv \tilde{\delta}(n,\theta)$.

(c) *Assume now $\theta < 1$, and find the conditions under which the collusive strategies found above represent an equilibrium.*

The condition is the same as in (b), i.e., $\delta \geq \tilde{\delta}(n,\theta)$. However, it implies now that an anticipated drop in demand would lead to a more stringent condition for collusion, i.e., cutoff $\tilde{\delta}(n,\theta)$ gets closer to zero.

Bonus Exercise #2 – *Trying to collude while observed by an antitrust authority.*

Consider two perfectly symmetric firms that sell a differentiated good and consider collusion. The fully collusive price in the market is given by p_m, and gives firms a profit π_m each. Firms also have the same discount factor δ. They play the Bertrand game an infinite number of periods.

There also exists an antitrust authority, which investigates the industry in every period. If firms collude, the authority will find them guilty with a probability *p* and will accordingly give them a fine $F > \delta\pi_n$. If they are found colluding, also assume that the authority will prevent them from colluding in the future: they will forever earn market profit $\pi_n > 0$ each, where the index *n* stands for Nash. If firms do not collude, they cannot be fined.

(a) *Focus on simple trigger strategies with Nash reversal forever. Write the incentive constraints for collusion to be sustained at equilibrium, and discuss the effects that p and F have upon collusion.*

If the AA investigates the sector in every period, the present discounted value of collusion is given by

$$V^c = p(\pi_m - F + \frac{\delta}{1-\delta}\pi_n) + (1-p)(\pi_m + \delta V^c),$$

and the incentive constraint each firm faces is given by

$$\frac{p(\pi_m - F + \frac{\delta}{1-\delta}\pi_n) + (1-p)\pi_m}{1-\delta(1-p)} \geq \frac{\pi_d + \delta\pi_n}{1-\delta},$$

where π_d denotes the one-shot deviation profit. Notice that, solving for this profit π_d, the previous condition can be re-written as follows:

$$\frac{\pi_m - p(F - \delta\pi_n) - \delta\pi_n}{1-\delta(1-p)} \geq \pi_d .$$

- ***Intuition:*** collusion is self-enforcing if the long-run expected losses due to the punishment are no smaller than the one-shot expected net gains from deviation. The higher *p* and *F* the less likely for collusion to be sustained at equilibrium, other things being equal.

(b) *Consider a value of the discount factor high enough for collusion to be sustainable. Are prices other than p_m sustainable at equilibrium of this infinite horizon game?*

The described game admits a continuum of solutions. If we consider exactly the same model but assume that firms, along the collusive equilibrium, set the price $\overline{p} \in (c, p_m)$, which gives firms a profit $\overline{\pi}$, then it is straightforward to show that firms' incentive constraint would be similar to the one derived in (a), the only difference being that π_m would be substituted by $\overline{\pi}$.

(c) *Do you know of any other strategies that could allow firms to sustain collusion under a slacker condition?*

Yes, two-phase punishment strategies would increase the interval of discount factors under which collusion might be sustained.