New York University
Department of Economics

POLSC-AD 112 — C. Wilson

Introduction to Game Theory — September 27, 2012

# MidTerm Exam Answer Key

The number of points for each question is given in the parenthesis.

1. (5) Begin each question on a separate page.

2. (15) Define the following:

   (a) Nash equilibrium.
   **Answer:** A strategy profile $(s_1, \dots, s_K)$ is a NE if the strategy $s_i$ for each player $i$ is a best response to the strategy subprofile $s_{-i}$ of the other players.
   A mixed strategy profile $\sigma = (\sigma_1, \dots, \sigma_K)$ is a NE if $\sigma_i(s_i) > 0$ implies that $s_i$ is a best resonse to $\sigma_{-i}$.

   (b) Dominated strategy.
   **Answer:** A pure strategy $s_i$ is dominated if there is a mixed strategy $\sigma_i$ such that $U_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i}$.

   (c) Mixed strategy.
   **Answer:** A mixed strategy for player $i$ is a probability distribution is pure strategies.

3. (5) Two governments are engaged in a dispute. Each government may choose to take a Hard line or a Soft line. Each prefers to take a Hard line if the other takes a Soft line, and each prefers to take a Soft line if the other takes a Hard line. However, given its own position, each government prefers the other take a Soft line.
   Write down a bimatrix that might represent the strategic form of this game.
   **Answer:** There are two players and each has two strategies, $H$ and $S$. If we denote the payoffs to Rowena by letters in the following matrix,

| | S | H |
|---|---|---|
| S | a | b |
| H | c | d |

then specification above implies:

$$\begin{aligned} c &> a \\ b &> d \\ a &> b \\ c &> d \end{aligned}$$

Combining the first three relations, we have $c > a > b > d$. Therefore, a possible representation of this game is

| | S | H |
|---|---|---|
| S | 2,2 | 1,3 |
| H | 3,1 | 0,0 |

4. (15) For each of the following games:

(i)

| | L | R |
|---|---|---|
| T | 3,3 | 0,2 |
| M | 4,0 | 1,1 |

(ii)

| | L | C | R |
|---|---|---|---|
| T | 5,5 | 0,1 | 2,2 |
| M | 2,4 | 2,5 | 4,3 |
| D | 0,1 | 5,2 | 3,0 |

(iii)

| | L | C | R |
|---|---|---|---|
| T | 1,1 | 2,0 | 0,2 |
| M | 0,2 | 1,1 | 2,0 |
| D | 2,0 | 0,2 | 1,1 |

(a) Find all dominated strategies (against **all** strategies of the other player).
**Answer:** (i) T is dominated by M.
(ii) R is dominated by L.
(iii) no dominated strategies

(b) Find all rationalizable strategies.
**Answer:** (i) M and R
(ii) $\{T, D\}$ and $\{L, C\}$
(iii) $\{T, M, D\}$ and $\{L, C, R\}$

(c) Find all of the pure strategy NE.
**Answer:** (i) $(M, R)$
(ii) $(T, L)$ and $(D, C)$
(iii) none

5. (15) Consider the following game.

(a)

| | L | R |
|---|---|---|
| T | 1,1 | 1,0 |
| M | 0,1 | 2,2 |

(a) Find all of the NE (including mixed strategy NE).
**Answer:** $(T, L)$, $(M, R)$ and $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$

(b) Compare the payoffs of the three NE.
**Answer:** $(T, L)$: Both payoffs are 1
$(M, R)$: Both payoffs are 2.
$\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$: Both payoffs are 1.

(c) Does one of the NE Pareto dominate the other? Which one?
**Answer:** $(M, R)$ Pareto dominates the other two.

(d) Does one of the NE risk dominate the other? Which one?
**Answer:** $(T, L)$ risk dominates $(M, R)$

6. (10) $K$ players must each submit an integer bid between 1 and $N$. The payoff to any player $i$ is 2 times the minimum bid of all the players (including himself) minus his own bid. I.e. If $b_j$ denotes the bid of player $j$, then

$$u_i((b_1, \ldots, b_K) = 2 \min_j b_j - b_i.$$

(a) Write down the bi-matrix that represents this game when $N = K = 2$, i.e, there are only two players and two possible bids.
**Answer:**

| | 1 | 2 |
|---|---|---|
| 1 | 1,1 | 1,0 |
| 2 | 0,1 | 2,2 |

(b) Now suppose that $K$ and $N$ can both be larger than 2.

i. Suppose $b^*_{-i} = \min_{j \neq i} b_j$ is the minimum bid of all players except $i$. What is the best response for player $i$? Explain.
**Answer:** The best response $b_i = b^*_{-i}$. The payoffs for different ranges of $b_i$, given $b^*_{-i}$ are given in the table below.

| | $b^*_{-i}$ |
|---|---|
| $b_i < b^*_{-i}$ | $2b_i - b_i = b_i < b^*_{-i}$ |
| $b_i = b_{-i}$ | $2b^*_{-i} - b^*_{-i} = b^*_{-i}$ |
| $b_i > b^*_{-i}$ | $2b^*_{-i} - b_i < b^*_{-i}$ |

Notice that if $b_i < b^*_{-i}$ every unit increase in $b_i$ increases his payoff by 1. But if $b_i > b^*_{-i}$, then every unit decrease in his bid lowers his payoff by 1.

ii. Find all of the pure strategy NE?
**Answer:** Since it is a best response for each player to bid the minimum bid, the only NE is for all players to bid the same value. Therefore, there are $N$ pure strategy NE in which each player bids $b_i = b$ for some $b \in \{1, \ldots, N\}$.

7. (8) Two players, 1 and 2, each have a strategy space $S_i = [0, 3]$ with the following payoff functions:

$$\begin{aligned} u_1(x_1, x_2) &= (3 - x_2)\, x_1 - x_1^2 \\ u_2(x_1, x_2) &= x_1 x_2 - x_2^2 \end{aligned}$$

(a) Compute the best response functions for each player.
**Answer:** For player 1, we set the derivative of his payoff with respect to $x_1$ to 0 to obtain

$$3 - x_2 - 2x_1 = 0 \tag{1}$$

which gives

$$br_1(x_2) = \frac{3 - x_2}{2}$$

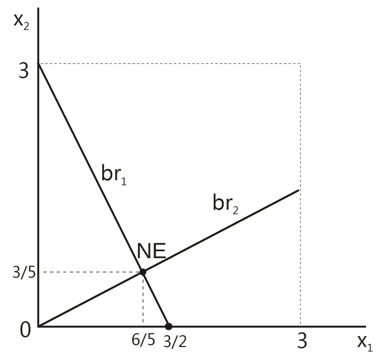
For player 2, we set the derivative of his payoff with respect to $x_2$ to 0 to obtain

$$x_1 - 2x_2 = 0 \tag{2}$$

which gives

$$br_2(x_1) = \frac{x_1}{2}$$

(b) Graph the best response functions in the same graph and identify the NE.




(c) Compute the NE.
**Answer:** Solving (1) into (2) for $x_1$ and $x_2$, we obtain $x_1 = \frac{3}{10}$ and $x_2 = \frac{3}{5}$.

8. (15) True or False. Explain. If true, provide an explanation. If false, give a counterexample.

(a) Any NE profile is Pareto optimal.

**Answer:** False. The prisoner's dilemma game is an example of a NE (D,D) that is Pareto dominated by (C,C).

| | C | D |
|---|---|---|
| C | 3,3 | 0,4 |
| D | 4,0 | 1,1 |

(b) Any NE profile consists of rationalizable strategies.

**Answer:** True. The proof is by contradiction. Suppose $(s_1, \ldots, s_K)$ is a NE, but some component strategy is not rationalizable. Then that strategy must be eliminated by iterated dominance. Now consider the first strategy $s_i$ that is eliminated. By definition of NE, $s_i$ is a best response to the other strategies. But by assumption, as long as $s_i$ remains, none of the other strategies in the NE have been eliminated. Therefore, $s_i$ must be a best response to some of the remaining strategies and cannot be eliminated.

(c) Any profile of rationalizable strategies is a NE.

**Answer:** False. Consider the matching pennies game. All strategies are rationalizable, but there is no NE.

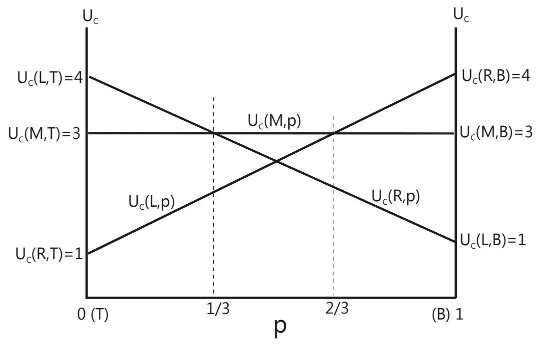
| | H | T |
|---|---|---|
| H | 0,1 | 1,0 |
| T | 1,0 | 0,1 |

9. (12) For the following 3x2 game

| | L | C | R |
|---|---|---|---|
| T | 2,4 | 0,3 | 3,1 |
| B | 1,1 | 2,3 | 1,4 |

(a) Graph the expected payoff to each pure strategy for Colin against the mixed strategies of Rowena.

**Answer:** Let $p$ be the probability that Rowena plays $B$.




(b) Compute the (pure strategy) best response function for Colin.

$$BR_C(p) = \begin{cases} \{L\} & \text{if} \quad p < \frac{1}{3} \\ \{L, M\} & \text{if} \quad p = \frac{1}{3} \\ \{M\} & \text{if} \quad \frac{1}{3} < p < \frac{2}{3} \\ \{M, R\} & \text{if} \quad p = \frac{2}{3} \\ \{R\} & \text{if} \quad p > \frac{2}{3} \end{cases}$$

(c) Compute the pure strategy best responses for Rowena for the following beliefs:

i. $\mu_{-R} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
**Answer:** $T$

ii. $\mu_{-R} = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$
**Answer:** $\{T, B\}$

iii. $\mu_{-R} = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.
**Answer:** $\{T, B\}$

(d) Find all of the NE of the game (including mixed strategy NE).
**Answer:** $(T, L) = ((1,0), (1,0,0))$, $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, 0\right)\right)$, $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)\right)$