Econ 1011A: Section 10 Notes[1]

TA: Zhenyu Lai (zlai@fas.harvard.edu), 11/17/2011

- Nash Equilibrium
- Subgame Perfect Nash Equilibrium

1. Finding all Pure and Mixed Nash equilibria

In lecture, we subtly covered a number of concepts. I will describe them in words, followed by math and examples. Mostly, you should treat game theory as a formalization of intuitive concepts.

Principle 1: Do not play a strictly dominated strategy

- Suppose playing strategy 1 yields a strictly lower payoff than playing strategy 2, and that this holds true for each possible action that your opponent can take. Don't play strategy 1.
- **Strict Dominance:** Strategy s_i *strictly dominates* strategy s_i' if playing s_i always yields a superior payoff than playing s_i', no matter what the other players do.

$$\forall s_{-i} \in S_{-i}, \qquad u_i\left(s_i, s_{-i}\right) > u_i\left(s_i', s_{-i}\right)$$

- *Example*: Defect dominates Cooperate. Don't play cooperate.

	$Cooperate$	$Defect$
$Cooperate$	$-1\ ,\ -1$	$-10\ ,\ \underline{0}$
$Defect$	$\underline{0}\ ,\ -10$	$-\underline{5}\ ,\ -\underline{5}$

Principle 2: At Nash equilibrium, no player will have incentive to deviate

- Given what your opponent is doing, you want to play the strategy that maximizes your payoff. When everybody is doing this, nobody can do better for themselves by changing their strategies. Hence, we are at equilibrium.
- **Best Response:** Strategy s_i^* is a *best response* to s_{-i}^* if playing s_i^* always yields at least as good of a payoff as playing any other s_i', given that the opponent plays s_{-i}^*

$$\text{Given } s_{-i}^*,\ \forall s_i' \in S_i, \qquad u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i', s_{-i}^*\right)$$

- **Nash Equilibrium:** Strategy profile $s^* = (s_1^*, s_2^*, ..., s_N^*)$ is a *nash equilibrium* if each player i is playing their best response s_i^* to the strategies s_{-i}^* that all other players are actually playing.

$$\text{For all } i,\ \forall s_i \in S_i, \qquad u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$$

- A pure strategy consists of the agent taking a single action.
- To find a **pure strategy Nash equilibrium**, calculate the player's best response to each of the opponent's possible pure strategies. Any outcome (if it exists) which is a mutual best response is the outcome of a pure strategy Nash equilibrium.
- *Example*: ($Fight$, $Yield$) and ($Yield$, $Fight$) are both pure strategy Nash equilibria.

	$Fight$	$Yield$
$Fight$	$-1\ ,\ -1$	$\underline{2}\ ,\ \underline{0}$
$Yield$	$\underline{0}\ ,\ \underline{2}$	$1\ ,\ 1$

[1] These notes draw heavily on the notes from previous years, and in particular from last year's TF, Matt Fiedler.

Principle 3: In a mixed strategy Nash equilibrium, payoffs need to be equalized.

- In equilibrium, agents are only willing to mix between strategies when the payoff to playing each pure strategy is equal. If one strategy yields a strictly higher payoff, the agent will strictly prefer that strategy, and not want to play a mixed strategy.

General strategy to finding all pure and mixed Nash equilibria of game:

- Please remember to check all the cases. Example: Steps for a 3x3 game are:

  1. First, exclude all strictly dominated strategies from consideration, and find all pure NE.
  2. Look for NE where one player plays a pure strategy, and the other mixes.
     - The pure strategy must lead to the opponent receiving the same payoff no matter what the opponent plays (for him to want to mix).
     - Given that the opponent mixes (with probabilities p), check that the pure strategy yields highest expected payoff. This allows you to solve for mixing probabilities p.
  3. Look for NE where one player mixes between 2 strategies X and Y, and the other mixes.
     - Given that one player mixes between X and Y, there might be a dominated strategy for the other player. Eliminate this from consideration.
     - For each player, express expected payoffs of each pure strategy in terms of opponent's mixing probabilities p. Solve for p that makes players indifferent (and hence mix).
     - Check that mixing between X and Y is not dominated by another pure strategy.
  4. Look for NE where both players mixes between 3 strategies.
     - For this mixed NE to exist, each player has to be *exactly* indifferent between the expected payoffs of **<u>all</u>** his strategies. Solve for mixing probabilities.
     - Note that mixing probabilities may or may not be an exact value. If not exact, remember probabilities sum to 1, hence express mixed strategies as function of p.

Example: Let's do a 2x3 game. Find all NE. B

		L	R
	U	$2, -2$	$0, 0$
A	M	$1, -1$	$1, -1$
	D	$0, 0$	$2, -2$

Step 1: Check for strictly dominated strategies. There are no strictly dominated strategies. In particular, M is not strictly dominated because no single strategy dominates it.

Step 2: Check for pure strategy NE. Make sure you understand why there are none.

Step 3: Look for NE in which one player plays a pure strategy, opponent mixes.
In the following parts, let q and $1-q$ be the probabilities that B plays L and R respectively.

When B plays a pure strategy, no such equilibrium exists (since A best responses to L by playing U, and A best responses to R by playing D). Notice when A plays M however, B is indifferent between L and R. For A to play M in equilibrium, the following 2 conditions must hold.

$$\begin{aligned} u_A(M, \{q, 1-q\}) &\geq u_A(U, \{q, 1-q\}) \quad \Rightarrow \quad 1 \geq 2q \qquad \Rightarrow \quad q \leq \frac{1}{2} \\ u_A(M, \{q, 1-q\}) &\geq u_A(D, \{q, 1-q\}) \quad \Rightarrow \quad 1 \geq 2-2q \quad \Rightarrow \quad q \geq \frac{1}{2} \end{aligned}$$

Nash Equilibrium: A plays M, and B mixes between $\{L, R\}$ with probabilities $\left\{\frac{1}{2}, \frac{1}{2}\right\}$

Step 4: Look for NE in which A mixes between 2 strategies, and B mixes.

There are 3 cases:

1. **Suppose A mixes between U and M.** Then, B never mixes since L is dominated.
2. **Suppose A mixes between M and D.** Similarly, B never mixes since R is dominated.
3. **Suppose A mixes between U and D with respective probabilities p and $1-p$.**
   Here, B mixes only if he is indifferent between the payoffs from his mixing strategies, or when

$$\begin{aligned} u_B(\{p,0,1-p\},L) &= u_B(\{p,0,1-p\},R) \\ -2p &= -2(1-p) \quad \Rightarrow \quad p=\frac{1}{2} \end{aligned}$$

   Similarly, A mixes between U and D only if

   (a) *A is indifferent between the expected payoffs from U and D*

$$\begin{aligned} u_B(U,\{q,1-q\}) &= u_B(D,\{q,1-q\}) \\ 2q &= 2(1-q) \quad \Rightarrow \quad q=\frac{1}{2} \end{aligned}$$

   (b) *A's payoff from mixing between U and D at $\left(\frac{1}{2},0,\frac{1}{2}\right)$ is undominated.*
   We check that $u_B\left(M,\left\{\frac{1}{2},\frac{1}{2}\right\}\right)=1\leq\frac{1}{2}u_B\left(U,\left\{\frac{1}{2},\frac{1}{2}\right\}\right)+\frac{1}{2}u_B\left(D,\left\{\frac{1}{2},\frac{1}{2}\right\}\right)=1$.

   Nash Equilibrium: A mixes between $\{U,D\}$ with probabilities $\left\{\frac{1}{2},\frac{1}{2}\right\}$
   B mixes between $\{L,R\}$ with probabilities $\left\{\frac{1}{2},\frac{1}{2}\right\}$

Step 5: Look for the NE in which A mixes between 3 strategies and B mixes.

Let p_1, p_2 and $1-p_1-p_2$ be the probabilities A plays U, M and D respectively. B mixes when

$$\begin{aligned} u_B(\{p_1,p_2,1-p_1-p_2\},L) &= u_B(\{p_1,p_2,1-p_1-p_2\},R) \\ -2p_1-p_2 &= -p_2-2(1-p_1-p_2) \\ &\Rightarrow \ p_2=1-2p_1 \end{aligned}$$

For A to mix between 3 strategies, he has to be indifferent between

$$\begin{aligned} u_A(U,\{q,1-q\}) &= u_A(M,\{q,1-q\})=u_A(D,\{q,1-q\}) \\ 2q &= 1=2(1-q) \quad \Rightarrow \quad q=\frac{1}{2} \end{aligned}$$

At equilibrium, we set the probabilities that A plays M as $1-2p_1$, and that A plays D as $1-p_1-p_2=1-p_1-(1-2p_1)=p_1$. Since each of these probabilities have to fall within the range from 0 to 1, we have valid probabilites for the range

$$0\leq 1-2p_1\leq 1 \quad \Rightarrow \quad 0\leq p_1\leq\frac{1}{2}$$

The following strategy profile describes the mixed strategy Nash equilibrium:

$$A \text{ plays } \begin{cases} U & \text{with probability } p_1 \\ M & \text{with probability } 1-2p_1 \\ D & \text{with probability } p_1 \end{cases} \quad \text{where } 0 \le p_1 \le \frac{1}{2}$$

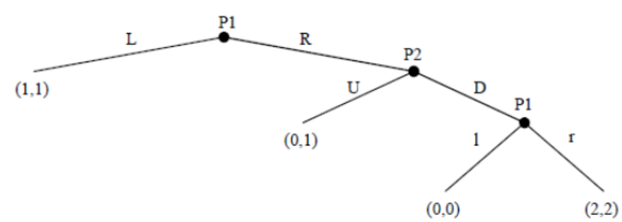
$$B \text{ plays } \begin{cases} L \text{ with probability } \frac{1}{2} \\ R \text{ with probability } \frac{1}{2} \end{cases}$$

We immeidately see that this strategy profile contains all the other equilibrium strategy profiles as special cases. **Please note that this will not always be the case**. In general, you will have to solve for all mixed strategy NE case by case[2].

2. Normal vs. Extensive-form games

- A **strategy** for player i specifies the contingent action taken *at every decision node* in which player i has to make a decision.
- A **contingent action** is the action that player i takes, *given the history of what has occured up to that point.*
- We can re-express an extensive form game into normal form by specifying the set of contingent actions taken over every possibility that could happen.

Example 1 (Taking Turns) *Player 1 chooses between L and R, where L ends the game. If R, player 2 picks between U and D, where U ends the game. If D, player 1 picks between l and r.*



- In this game, player 2 has 2 strategies: play U at $\{R\}$, play D at $\{R\}$
- Player 1 has 4 contingent strategies:

$$\begin{array}{l} \text{play } L \text{ at } \{\varnothing\}, \text{ play } l \text{ at } \{R, D\} \\ \text{play } L \text{ at } \{\varnothing\}, \text{ play } r \text{ at } \{R, D\} \\ \text{play } R \text{ at } \{\varnothing\}, \text{ play } l \text{ at } \{R, D\} \\ \text{play } R \text{ at } \{\varnothing\}, \text{ play } r \text{ at } \{R, D\} \end{array}$$

- Using these contingent strategies, we can express the strategic form game matrix as:

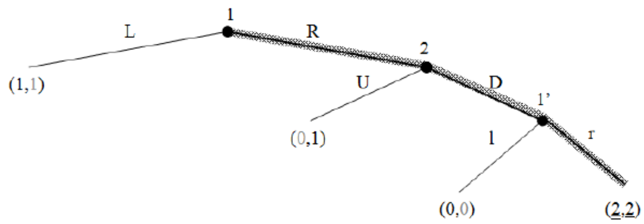
P1 \ P2	U	D
Ll	$(1,1)$	$(1,1)$
Lr	$(1,1)$	$(1,1)$
Rl	$(0,1)$	$(0,0)$
Rr	$(0,1)$	$(2,2)$

- Solving, we see that there are 3 pure strategy Nash equilibria: $\{Ll, U\}$ for a payoff of $(1,1)$, $\{Lr, U\}$ for a payoff of $(1,1)$, and $\{Rr, D\}$ for a payoff of $(2,2)$

[2] This was a rather extended example, and I picked this game as an alternative to some less complicated games in order to illustrate some fundamental concepts in a single example.

- In equilibrium, players play a sequence of strategies defined as being **on-equilibrium path**.
- All other decision nodes never reached in equilibrium are called **off-equilibrium path**.
- In the equilibrium reached by backwards induction (also known as **subgame perfect**), players play strategies that are optimal both **on-equilibrium path** and **off-equilibrium path**.
- In other words, a **subgame perfect Nash equilibrium** has to be a Nash equilibrium in the entire game, and in every subgame of the original game.

Example 2 (Taking Turns Continued) *Find the equilibrium using backwards induction.*



- At node $1'$, player 1 faces a choice between l and r.
  r gives a payoff of 2 and strictly dominates l which gives a payoff of 0.

P1 \ P2	U	D
L	(1,1)	(1,1)
~~Rl~~	~~(0,1)~~	~~(0,0)~~
Rr	(0,1)	(2,2)

- Iterating up the tree, at node 2, player 2 faces a choice between U and D. D gives a payoff of 2 and strictly dominates U which gives a payoff of 1.

P1 \ P2	~~U~~	D
L	~~(1,1)~~	(1,1)
~~Rl~~	~~(0,1)~~	~~(0,0)~~
Rr	~~(0,1)~~	(2,2)

- In the last iteration to the initial node, at node 1, player 1 faces a choice between L and R. R gives a payoff of 2 and strictly dominates L which gives a payoff of 1

P1 \ P2	~~U~~	D
~~L~~	~~(1,1)~~	~~(1,1)~~
~~Rl~~	~~(0,1)~~	~~(0,0)~~
Rr	~~(0,1)~~	$\boxed{(2,2)}$

- This leaves us with $\{Rr, D\}$ as the unique subgame perfect equilibrium, with payoff $(2, 2)$
- Notice $\{Ll, U\}$ and $\{Lr, U\}$ are not subgame perfect because they involve strategies that are not optimal off-equilibrium path.
- *This is the most important sentence to understand*: For either of these outcomes, player 1 and player 2 respectively have to make a **non-credible threat** in order to sustain equilibrium.