1 Fall 2006 Final Solutions

1.1 Question 1

- (a) (T,B,L,R) all survive IESDS.
- (b) 2 Pure: (B,L) and (T,R) and 1 Mixed: $[(0.5, 0.0, 0.5), (0.6, 0.0, 0.4)]$. The mixed strategy is found by equalizing the expected conditional payoffs for each player. For player 1:
$$-10q + 15(1-q) = 0 \Rightarrow 15 = 25q \Rightarrow q = 3/5$$
For player 2 :
$$-10p + 10(1-p) = 0 \Rightarrow 10 = 20p \Rightarrow p = 1/2.$$
- (c) See Osborne: The two conditions are that strategies in which players place a positive probability on must yield the same expected payoff while those that have zero probability placed upon them must NOT yield a higher expected value than those that do.

1.2 Question 2

- (a) Game in strategic form:

	QX	QY	RX	RY
A	$0, 0$	$0, 0$	$0, \underline{1}$	$0, \underline{1}$
B	$0, \underline{1}$	$0, \underline{1}$	$0, 0$	$0, 0$
C	$\underline{5}, \underline{0}$	$\underline{5}, \underline{0}$	$\underline{5}, \underline{0}$	$\underline{5}, \underline{0}$
D	$0, \underline{4}$	$0, 0$	$0, \underline{4}$	$0, 0$
E	$0, 1$	$0, \underline{3}$	$0, 1$	$0, \underline{3}$

Since the game only has ONE subgame,
$$NE = SPNE = \{(C, QX); (C, QY); (C, RX); (C, RY)\}.$$

- (b) Note that player 2 has two information sets, both of which are OFF the equilibrium path given the strategies, (C, RX). So we need beliefs at each. Denote p the probability that player 2 thinks A has been played and denote q the probability that player 2 thinks that D has been played. Requirement 4 of a PBE requires:

$$\begin{aligned} E_2[R] &\geq E_2[Q] \\ 1*p+0*(1-p) &\geq 0*p+1*(1-p) \\ p &\geq \frac{1}{2} \end{aligned}$$

And:

$$\begin{aligned} E_2[X] &\geq E_2[Y] \\ 4*q+1*(1-q) &\geq 0*q+3*(1-q) \\ 6q &\geq 2 \\ q &\geq \frac{1}{3} \end{aligned}$$

1.3 Question 3

- (a) Given that both types of sender chooses L, $\sigma_S = (L, L)$, then $p = 1/4$, nature's probability. Given this belief, a receiver compares:

  $$E_{Rec}[U|L] = 2p + 4(1-p) = 2(1/4) + 4(3/4) = 3.5$$

  to

  $$E_{Rec}[D|L] = 3p + 1(1-p) = 3(1/4) + 1(3/4) = 1.5$$

  So a receiver will play U if he gets the L signal. Thus a weak sender is getting his highest payoff (4) by playing L. By deviating to R, he gets either 1 or 3, so a weak sender will NEVER deviate. However, a strong sender gets 3 by playing L and may get either 4 or 2 by deviating to R. Thus, if we have a PBE, a strong sender requires:

  $$E_{Rec}[D|R] \geq E_{Rec}[U|R]$$

  $$1q + 3(1-q) \geq 4q + 2(1-q)$$

  $$-2q + 3 \geq 2q + 2$$

  $$4q \leq 1$$

  $$q \leq \frac{1}{4}$$

  So our PBE is:

  $$\{(L, L), (U, D), (p, 1-p), (q, 1-q)|p = \frac{1}{4}, q \leq \frac{1}{4}\}.$$

- (b) Separating on $\sigma_S = (L, R)$. Note that since both information sets of the receiver are on the equilibrium path, we have $p = 1$ and $q = 0$. Given these beliefs, the receiver will choose: $\sigma_{Rec} = (D, D)$. The strong and weak sender get 1 and 3 respectively if they play (L,R). By each deviating they would get 2. Thus a STRONG SENDER has an incentive to deviate and play R instead of L.

- (c) See Osborne. Players must have the move at all nodes in the information set and (for information sets involving 2 or more nodes) must not know which node they are operating from. Singleton nodes are information sets. The sender and receiver each have two information sets.

1.4 Question 4

- (a) Each firm solves:

$$Max_{q_i} \ \{q_i(14 - q_i - q_j - 2)\}.$$

Thus,

$$FOC(q_i) : 12 - 2q_i - q_j = 0.$$

This implies a best response function for each firm:

$$q_i(q_j) = 6 - \frac{1}{2}q_j.$$

Plug one BR into the other:

$$q_i = 6 - 0.5(6 - 0.5q_i) = 3 + 0.25q_i$$

$$q_i^c = 4.$$

Thus, $Q = 8$, $P = 14 - 8 = 6$, and:

$$\pi_i^c = 4(6 - 2) = 16.$$

- (b) A monopolist solves:

$$Max_Q \ \{Q(14 - Q - 2)\}.$$

Thus,

$$FOC(Q) : 12 - 2Q = 0.$$

So $Q^m = 6$. Thus $P = 14 - 6 = 8$, and:

$$\pi^m = 6(8 - 2) = 36.$$

- (c) Along the equilibrium path, each firm produces $\frac{1}{2}Q^m = 3$ and earns profits (per period) of $\frac{1}{2}\pi^m = 18$. If one firm wanted to optimally deviate for one period, they would solve:

$$Max_{q_i^d} \ \{q_i^d(14 - q_i^d - \frac{1}{2}Q^m - 2)\}.$$

$$Max_{q_i^d} \ \{q_i^d(14 - q_i^d - 3 - 2)\}.$$

$$Max_{q_i^d} \ \{q_i^d(9 - q_i^d)\}.$$

$$FOC(q_i^d) : 9 - 2q_i^d = 0.$$

So, $q_i^d = 4.5$. Profts in the one period deviation are thus:

$$\pi_i^d = 4.5(14 - 4.5 - 3 - 2) = 4.5(4.5) = 20.25.$$

- (d) If we want to sustain cooperation in all periods, we require the discount factor to satisfy:

$$\begin{aligned}
\frac{1}{2}\pi^m(1 + \delta + \delta^2 + \cdots) &\geq \pi_i^d + \pi_i^c(\delta + \delta^2 + \cdots) \\
18(1 + \delta + \delta^2 + \cdots) &\geq 20.25 + 16(\delta + \delta^2 + \cdots) \\
18\sum_{t=0}^{\infty}\delta^t &\geq 20.25 + 16\sum_{t=1}^{\infty}\delta^t \\
\frac{18}{1-\delta} &\geq 20.25 + \frac{16\delta}{1-\delta} \\
18 &\geq 20.25 - 20.25\delta + 16\delta \\
4.25\delta &\geq 2.25 \\
\delta^* &\geq \frac{2.25}{4.25} = \frac{9}{17} \approx 0.529
\end{aligned}$$