

# Collegio Carlo Alberto

## Game Theory Solutions to Problem Set 9

### Question 1

The monopoly output in this case is  $Q = \frac{a}{2}$  with each firm producing  $q_i^M = \frac{a}{2n}$ . The profits for each firm is  $\pi_i^M = \frac{a^2}{4n}$ . If all other firms follow the monopoly share production, the optimal stage-game (one-period) quantity for firm  $i$  is  $q_i^D = \frac{a(n+1)}{4n}$ , which yields firm  $i$  a profit of  $\pi_i^D = \frac{a^2(n+1)^2}{16n^2} = \frac{a^2}{4n} \frac{(n+1)^2}{4n}$ . The unique NE of the stage game, in contrast, is when all players produce  $q_i^C = \frac{a}{n+1}$ , which yields profits of  $\pi_i^C = \frac{a^2}{(n+1)^2}$ .

Together, this implies that we need to look at the following trigger strategy:

$$s_i(h^t) = \begin{cases} q_i^M & \text{,after } h^o \text{ and } h^t = ((q_1^M, \dots, q_n^M), \dots, (q_1^M, \dots, q_n^M)) \text{ } t \text{ times} \\ q_i^C & \text{,otherwise} \end{cases}$$

The above strategy profile is a SPE if and only if:

$$\pi_i^M \geq (1 - \delta) \pi_i^D + \delta \pi_i^C.$$

which implies the following condition for  $\delta$ :

$$\begin{aligned} \delta &\geq \frac{\pi_i^D - \pi_i^M}{\pi_i^D - \pi_i^C} = \frac{\frac{a^2}{4n} \left( \frac{(n+1)^2}{4n} - 1 \right)}{\frac{a^2}{(n+1)^2} \left( \frac{(n+1)^4}{16n^2} - 1 \right)} \\ &= \frac{(n+1)^2}{4n} \frac{1}{\left( \frac{(n+1)^2}{4n} + 1 \right)} = \frac{(n+1)^2}{(n+1)^2 + 4n} \\ &= \frac{1}{1 + \frac{4n}{(n+1)^2}} \equiv \underline{\delta}(n) \end{aligned}$$

For  $n \geq 1$  the function  $\underline{\delta}(n)$  is increasing. Moreover, as  $n \rightarrow \infty$ ,  $\underline{\delta}(n) \rightarrow 1$ . When  $n$  is large both profits of cooperating,  $\pi_i^M$ , and the Cournot profits,  $\pi_i^C$ , are close to zero. However, the profits of deviating are bounded away from zero. Thus, the larger the number of firms the more profitable the deviation. Cooperation can be sustained only if the firms are extremely patient.

### Question 2

- a) The minmax value  $\underline{v}_i$   $i \in \{1, 2\}$  is  $\underline{v}_1 = \underline{v}_2 = 1$ . [To see this, suppose that player 2 mixes between  $\{A_2, B_2\}$  with probability  $\{p, 1 - p\}$ , find the best-response for player 1, and then

find the value of  $p$  that implies the lowest payoff of player 1 when she is playing a best-response.] The game is symmetric and therefore the same minmax value applies for player 2.

**b)** According to the Fudenberg-Maskin Folk Theorem, any feasible pair of payoffs  $(v_1, v_2)$  with  $v_i > \underline{v}_i$ ,  $i = 1, 2$ , can be achieved with a SPE provided that  $\delta$  is sufficiently large.

**c)** Consider the following strategy:

$$s_i(h^t) = \begin{cases} A_i, & \text{if } t = 0 \\ A_i, & \text{if } t > 0 \text{ and } (A_1, A_2) \text{ or } (B_1, B_2) \text{ was played at } t - 1 \\ B_i, & \text{if } t > 0 \text{ and } (B_1, A_2) \text{ or } (A_1, B_2) \text{ was played at } t - 1 \end{cases}$$

For this strategy to be a SPE it has to be that no player has a profitable one-shot deviation after any possible history. Divide the possible histories into four groups  $(h_{AA}, h_{BB}, h_{AB}, h_{BA})$  and define  $w_{AA}, w_{BB}, w_{AB}, w_{BA}$  to be the continuation payoff after each history. Since the game is symmetric, it is sufficient to check one-shot deviations for one player only. Consider player 1. Following her equilibrium strategy, her payoff after each history is:

$$\begin{aligned} w_{AA} &= w_{BB} = \frac{4}{1-\delta} \\ w_{AB} &= w_{BA} = \frac{4\delta}{1-\delta} \end{aligned}$$

Hence, no one-shot deviation exists if 1 and 2 both hold:

$$\begin{aligned} w_{AA} &= w_{BB} \geq 6 + \delta w_{BA} & (1) \\ \frac{4}{1-\delta} &\geq 6 + \frac{4\delta^2}{1-\delta} \\ 4 &\geq 6 - 6\delta + 4\delta^2 \\ 0 &\geq 2 - 6\delta + 4\delta^2 \\ &\Downarrow \\ 1 &\geq \delta \geq \frac{1}{2} \end{aligned}$$

$$\begin{aligned} w_{BA} &= w_{AB} \geq 1 + \delta w_{AB} & (2) \\ \frac{4\delta}{1-\delta} &\geq 1 + \frac{4\delta^2}{1-\delta} \\ 4\delta &\geq 1 - \delta + 4\delta^2 \\ 0 &\geq 1 - 5\delta + 4\delta^2 \\ &\Downarrow \\ 1 &\geq \delta \geq \frac{1}{4} \end{aligned}$$

Thus, for values of  $\delta \in [\frac{1}{2}, 0)$ , the above strategy is a SPE.

## Question 3

Notice that the minmax payoff of player 1 is equal to 1. Therefore, there does not exist any SPE in which player 1 plays  $D$  in every period (otherwise her payoff would be zero). This, in turn, implies that in any SPE the payoff of player 2 is strictly greater than one (since player 1 has to play  $A$  in at least one period).

If the players play  $(A, A)$  in every period player's 2 continuation payoff is equal to 3. Suppose that in period  $t$  player 2 deviates and plays  $D$ . Let  $x$  denote the continuation payoff of player 2 in period  $t + 1$  after the deviation. By deviating in period  $t$  player 2 gets:

$$(1 - \delta)5 + \delta x = \frac{5}{2} + \frac{x}{2} > 3,$$

since  $x > 1$ . Thus, the deviation is profitable.

## Question 4

A SPE in which each worker at every period exerts effort could only be if the worker expect to receive a wage of at least  $c$ , to cover her effort costs. Thus, effort has to be conditional on a reputation of the firm to pay wages above  $c$ , where the reputation is some function of the history of wages the company has given in response to previous workers' efforts.

For instance, let  $w \in [c, y)$  and consider the following strategies for the workers:

$$s_w(h^t) = \begin{cases} E & \text{if } t = 0 \text{ or } [t > 0 \text{ and } h^t = \{(w_E^0 \geq w, E), \dots, (w_E^{t-1} \geq w, E)\}] \\ NE & \text{Otherwise} \end{cases}$$

and for the firm:

$$s_F(h^t) = \begin{cases} w & \text{if } t = 0 \text{ and } s_w(h^0) = E \text{ or } [t > 0, h^t = \{(w_E^0 \geq w, E), \dots, (w_E^{t-1} \geq w, E)\}, s_w(h^t) = E] \\ 0 & \text{Otherwise} \end{cases}$$

After a good reputation of paying at least  $w$ , playing the equilibrium strategy, the worker will make an effort and will get  $w - c$  and the firm will have at time  $t$  a profit of  $y - w$ .

By following the equilibrium strategy, the firm will receive  $\frac{y-w}{1-\delta}$  and the worker  $w - c$ .

A deviation of the firm at time  $t$  to pay zero (deviation that gives the greatest possible one-period payoff) would yield a profit of  $y$ . Therefore, if

$$\begin{aligned} y &\leq \frac{y-w}{1-\delta} \\ \Leftrightarrow 1-\delta &\leq \frac{y-w}{y} = 1 - \frac{w}{y} \\ \Leftrightarrow \delta &\geq \frac{w}{y} \end{aligned}$$

the firm has no incentive to deviate. For the worker to have incentive to sustain the above strategy in equilibrium it has to be that  $w \geq c$ . Thus, for any  $1 \geq \delta \geq \frac{c}{y}$  one can find a SPE in which the worker exerts effort in every period and receives a wage of  $\delta y \geq w \geq c$ .