University of Illinois **Spring 2013**

ECE 586BH: Problem Set 5: Problems and Solutions
Multistage games, including repeated games, with observed moves

Due: Thursday, April 11 at beginning of class
Reading: Fudendberg and Triole, Sections 3.2 and 5.1. See Chapter 4 for examples.
Ozdaglar lecture notes 12,13,15, & 16 (on extensive form games and repeated games)

1. **[Another subgame perfect equilibria in repeated prisoner's dilemma]**
Consider the repeated game with the stage game being prisoner's dilemma with the payoff functions g_i given by

	c	d
c	1,1	-1,2
d	2,-1	0,0

. Action c means to cooperate. Suppose player 1 has discount factor δ_1 and player 2 has discount factor δ_2, so the payoff for player i is given by $J_i(a) = \sum_{t=0}^{\infty} \delta_i^t g_i(a(t))$, for $a = ((a_1(t), a_2(t)) : t \geq 0)$ with $a_i(t) \in \{c, d\}$. Consider the following pair of scripts, for play beginning at time $t = 0$: player 1: $c, d, d, c, c, c, c, \ldots$ player 2: $d, d, d, d, c, c, c, \ldots$
Both scripts have players playing c for $t \geq 4$. Let μ_i^* be the trigger strategy that follows the script for player i as long as both players have followed the script in the past. If either player has deviated from the script in the past, player i always plays d. Let $\mu^* = (\mu_1^*, \mu_2^*)$.

   (a) Express $J_i(\mu^*)$ in terms of δ_i for $i \in \{1, 2\}$.
   Solution: Under μ^* both players follow their scripts for all time. The payoff sequence for player 1 is $-1, 0, 0, -1, 1, 1, 1, \ldots$ so $J_1(\mu^*) = -1 - \delta_1^3 + \frac{\delta_1^4}{1-\delta_1}$. The payoff sequence for player 2 is $2, 0, 0, 2, 1, 1, 1, \ldots$ so $J_2(\mu^*) = 2 + 2\delta_2^3 + \frac{\delta_2^4}{1-\delta_2}$.

   (b) For what values of (δ_1, δ_2) is μ^* a subgame perfect equilibrium (SPE) of the repeated game? Justify your answer using the one-stage-deviation principle.
   Solution: Consider player 1 assuming player 2 uses μ_2^*. Let $\overline{k} \geq 0$, let $\overline{h}_{\overline{k}}$ denote a history for time $\overline{k}$, and let $\widetilde{\mu}_1$ be the strategy (there is only one because the action space for the stage game has only two actions) that differs from μ_1^* only for the time and history $(\overline{k}, \overline{h}_{\overline{k}})$. If $\overline{h}_{\overline{k}}$ is not equal to the two scripts up to time $\overline{k}-1$ then player 2 will be playing d from time $\overline{k}$ onward. Player 1 under $\widetilde{\mu}_1$ will play c at time $\overline{k}$ and then play d onward. Therefore,
$$J_1^{(\overline{k})}(\widetilde{\mu}_1, \mu_{-1}^* | \overline{h}_{\overline{k}}) - J_1^{(\overline{k})}(\mu_1^*, \mu_{-1}^* | \overline{h}_{\overline{k}}) = -(\delta_1)^{\overline{k}} < 0.$$
   It remains to check the cases when $\overline{h}_{\overline{k}}$ is given by the scripts up to time $\overline{k} - 1$. There is one such case for each value of $\overline{k}$, and the resulting payoff sequences for $\widetilde{\mu}_1$ are listed in the following table.

deviation time	payoff sequence	payoff
none (i.e. μ_1^*)	$-1, 0, 0, -1, 1, 1, 1, \ldots$	$-1 - \delta_1^3 + \frac{\delta_1^4}{1-\delta_1}$
0	$0, 0, 0, 0, 0, 0, 0, \ldots$	0
1	$-1, -1, 0, 0, 0, \ldots$	$-1 - \delta_1 < 0$
2	$-1, 0, -1, 0, 0, 0, \ldots$	$-1 - \delta_1^2 < 0$
3	$-1, 0, 0, 0, \ldots$	$-1 < 0$
$\overline{k} \geq 4$	$-1, 0, 0, -1, 1, \ldots 1, 1, \underbrace{2}_{\text{time } \overline{k}}, 0, 0, 0, \ldots$	$-1 - \delta_1^3 + \frac{\delta_1^4 - \delta_1^{\overline{k}}}{1-\delta_1} + 2\delta_1^{\overline{k}}$

The payoff for no deviation is greater than or equal to the payoff for deviation at time $\overline{k}=0$ if $-1-\delta_1^3+\frac{\delta_1^4}{1-\delta_1}\geq 0$, or $\delta_1\in[0.7641922727,1)$. The payoffs for deviation at time $\overline{k}$ with $1\leq\overline{k}\leq 3$ are negative, so are less than the payoff for deviation at time $\overline{k}=0$. Finally, the payoff for deviation at time $\overline{k}\geq 4$ minus the payoff for no deviation is $\delta_1^{\overline{k}}(2-\frac{1}{1-\delta_1})$ which is less than or equal to zero for $\delta_1\in[0.5,1)$. Thus, the conditional payoff for μ_1^* is greater than or equal to the conditional payoff for $\widetilde{\mu}$ for any choice of $(\overline{k},\overline{h}_{\overline{k}})$ if and only if $\delta_1\in[0.7641922727,1)$.

Similarly, consider player 2 assuming player 1 uses μ_1^*. Let $\overline{k}\geq 0$, let $\overline{h}_{\overline{k}}$ denote a history for time $\overline{k}$, and let $\widetilde{\mu}_2$ be the strategy that differs from μ_2^* only for the time and history $(\overline{k},\overline{h}_{\overline{k}})$. By the same reasoning as above, if $\overline{h}_{\overline{k}}$ is not equal to the two scripts up to time $\overline{k}-1$ then

$$J_2^{(\overline{k})}(\widetilde{\mu}_2,\mu_{-2}^*|\overline{h}_{\overline{k}})-J_2^{(\overline{k})}(\mu_2^*,\mu_{-2}^*|\overline{h}_{\overline{k}})=-(\delta_2)^{\overline{k}}<0.$$

It remains to check the cases when $\overline{h}_{\overline{k}}$ is given by the scripts up to time $\overline{k}-1$. There is one such case for each value of $\overline{k}$, and the resulting payoff sequences for $\widetilde{\mu}_2$ are listed in the following table.

deviation time	payoff sequence	payoff
none (i.e. μ_2^*)	$2,0,0,2,1,1,1,\ldots$	$2+2\delta_2^3+\frac{\delta_2^4}{1-\delta_2}$
0	$1,0,0,0,0,0,0,\ldots$	1
1	$2,-1,0,0,0,0,0,\ldots$	$2-\delta_2$
2	$2,0,-1,0,0,0,\ldots$	$2-\delta_2^2$
3	$2,0,0,1,0,\ldots$	$2+\delta_2^3$
$\overline{k}\geq 4$	$2,0,0,2,1,\ldots 1,1,\underbrace{2}_{\text{time } \overline{k}},0,0,0,\ldots$	$2+2\delta_2^3+\frac{\delta_2^4-\delta_2^{\overline{k}}}{1-\delta_2}+2\delta_2^{\overline{k}}$

The payoff for $\overline{k}=4$ is larger than for $0\leq\overline{k}\leq 3$, so there is no need to consider the cases $0\leq\overline{k}\leq 3$ further. The payoff for deviation at time $\overline{k}\geq 4$ minus the payoff for no deviation is $\delta_2^{\overline{k}}(2-\frac{1}{1-\delta_2})$ which is less than or equal to zero for $\delta_2\in[0.5,1)$. Thus, μ_2^* be better than $\widetilde{\mu}_2$ for any choice of $(\overline{k},\overline{h}_{\overline{k}})$ if and only if $\delta_2\in[0.5,1)$.

Combining the results above, we find μ^* is an SPE for the repeated game if and only if $(\delta_1,\delta_2)\in[0.7641922727,1)\times[0.5,1)$.

2. **[Feasible payoffs for a trigger strategy in two stage game for a stage game with multiple NEs]**
   Consider a multistage game such that the stage game has the following payoff matrix, where player 1 is the row player and player 2 is the column player:

	1	2	3
1	4,4	5,3	9,3
2	3,5	6,6	9,2
3	3,9	2,9	8,8

   (a) Identify two NE for the stage game.
   Solution: Both (1,1) and (2,2) are NE.

   (b) Consider a two stage game obtained by playing the above stage game twice, with discount factor equal to one, so payoffs for the repeated game are equal to the sum of payoffs for the two stages. Describe a subgame perfect equilibrium for the two stage game so that the payoff vector is $(14, 14)$. Verify that the strategy profile is subgame perfect using the single-deviation principle.

Solution: Player i uses the following strategy μ_i^*: Play 3 in the first stage. If both players play 3 in the first stage, then play 2 in the second stage. Else play 1 in the second stage. Clearly $\mu^* = (\mu_1^*, \mu_2^*)$ has payoff vector (14,14). We show that μ^* is subgame perfect using the single deviation principle. By symmetry, we can focus on the responses of player 1 assuming player 2 uses μ_2^*. For a subgame beginning in stage 2, there are two possibilities. If both players played 3 in the first stage, then player 1 knows player 2 will play 2 in the second stage, and μ_1^* gives the best response to that, namely to play 2 as well. If at least one of the players did not play 3 in the first stage, then player 1 knowns that player 2 will play 1 in the second stage, and μ_1^* gives the best response to that, namely to play 1 as well. So μ_1^* gives the best response for the sub games beginning in the second stage, assuming player two uses μ_2^*. The other subgame to consider for player one is the original game itself, and we need to consider the case player 1 deviates only in the first stage. If player 1 were to play either 1 or 2 in the first stage, and follow μ_1^* in the second stage, then both players will play 1 in the second stage, getting payoffs 4 each in the second stage. Thus, the total payoff of player 1 will be less than or equal to 13 if he deviates from μ_1^* in the first stage only. Therefore, μ^* satisfies the single deviation condition, and is therefore an SPE.

3. **[Repeated play of Cournot game]**
Consider the repeated game with the stage game being two-player Cournot competition with payoffs $g_i(s) = s_i(a - s_1 - s_2 - c)$, where $a > c > 0$. Here s_i represents a quantity produced by firm i, the market price is $a - s_1 - s_2$ and c is the production cost per unit produced. The action space for each player is the interval $[0, \infty)$.

(a) Identify the pure strategy NE of the stage game and the corresponding payoff vector.

Solution: Each g_i is a concave function of s_i and it is maximized at the point $\frac{\partial g_i(s)}{\partial s_i} = 0$ yielding the NE $(s_1, s_2) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right)$ and payoff vector $v^{NE} = \left(\frac{(a-c)^2}{9}, \frac{(a-c)^2}{9}\right)$.

(b) Identify the maxmin value $\underline{v}_i$ for each player for the stage game.

Solution: For each player i, since the other player can produce so much that the price is zero (or even negative),the maxmin strategy of player i is $s_i = 0$, so that $\underline{v}_i = 0$.

(c) Identify the feasible payoff vectors for the stage game.

Solution: The sum of payoffs, $(s_1 + s_2)(a - c - s_1 - s_2)$, ranges over the interval $[0, \frac{(a-c)^2}{4}]$ as $s_1 + s_2$ ranges over the interval $[0, a - c]$. For a given nonzero value of the sum $s_1 + s_2$, the sum payoff can be split arbitrarily between the two players by adjusting the ratio $\frac{s_1}{s_1+s_2}$. Therefore, the set of nonnegative feasible payoff vectors is $\left\{(v_1, v_2) \in \mathbb{R}^+ : v_1 + v_2 \leq \frac{(a-c)^2}{4}\right\}$.

(d) Identify the payoff vectors for the repeated game that are feasible for Nash equilibrium of the repeated game for the discount factor δ sufficiently close to one, guaranteed by Nash's general feasibility theorem.

Solution: Since the vector of maxmin payoffs is the zero vector, Nash's feasible region for the repeated game is $\left\{(v_1, v_2) : v_1 > 0, v_2 > 0, v_1 + v_2 \leq \frac{(a-c)^2}{4}\right\}$.

(e) Identify the payoff vectors for the repeated game that are feasible for subgame perfect equilibrium of the repeated game for the discount factor δ sufficiently close to one, guaranteed by Friedman's general feasibility theorem.

Solution: $\{(v_1, v_2) : v_i > v_i^{NE}, v_1 + v_2 \leq \frac{(a-c)^2}{4}\}$.

(f) Identify the payoff vectors for the repeated game that are feasible for subgame perfect equilibrium of the repeated game for the discount factor δ sufficiently close to one, guaranteed by the Fudenberg/Maskin general feasibility theorem.

Solution: Same region as in part (d).

4. **[A Cournot game with incomplete information]**
Given $c > a > 0$ and $0 < p < 1$, consider the following version of a Cournot game. Suppose the type θ_1 of player 1 is either zero, in which case his production cost is zero, or one, in which case his production cost is c (per unit produced). Player 2 has only one possible type, and has production cost is c. Player 1 knows his type θ_1. It is common knowledge that player 2 believes player 1 is type one with probability p. Both players sell what they produce at price per unit $a - q_1 - q_2$, where q_i is the amount produced by player i. A strategy for player 1 has the form $(q_{1,0}, q_{1,1})$, where q_{1,θ_1} is the amount produced by player one if player 1 is type θ_1. A strategy for player 2 is q_2, the amount produced by player 2.

(a) Identify the best responses of each player to strategies of the other player.

Solution: The best response functions are given by:

$$\begin{aligned}
q_{1,0} &= \arg\max_{q_1} q_1(a - q_2 - q_1) = \frac{(a-q_2)_+}{2} \\
q_{1,1} &= \arg\max_{q_1} q_1(a - c - q_2 - q_1) = \frac{(a-c-q_2)_+}{2} \\
q_2 &= \arg\max_{q_2'} \left\{(1-p)q_2'(a - c - q_{1,0} - q_2') + pq_2'(a - c - q_{1,1} - q_2)\right\} \\
&= \arg\max_{q_2'} q_2'(a - c - \overline{q}_1 - q_2') = \frac{(a-c-\overline{q}_1)_+}{2}
\end{aligned}$$

where $\overline{q}_1 = (1-p)q_{1,0} + pq_{1,1}$.

(b) Identify the Bayes-Nash equilibrium of the game. *For simplicity, assume $a \geq 2c$.* (Note: After finding the equilibrium, it can be shown that the three corresponding payoffs are given by

$$u_2(q^{NE}) = \frac{(a-(2-p)c)^2}{9}$$

and

$$u_1(q^{NE}|\theta_1 = 0) = \frac{(2a+(2-p)c)^2}{36}, \quad u_1(q^{NE}|\theta_1 = 1) = \frac{(2a-(1+p)c)^2}{36}.$$

As expected, $u_2(q^{NE})$ is increasing in p, and both $u_1(q^{NE}|\theta_1 = 0)$ and $u_1(q^{NE}|\theta_1 = 1)$ are decreasing in p with $u_2(q^{NE}) < u_1(q^{NE}|\theta_1 = 1) < u_1(q^{NE}|\theta_1 = 0)$ for $0 < p < 1$. In the limit $p = 1$, $u_2(q^{NE}) = u_1(q^{NE}|\theta_1 = 1) = \frac{(a-c)^2}{9}$, as in the original Cournot game with complete information and production cost c for both players.)

Solution: Substituting the best response functions for player one into the definition of $\overline{q}_1$ in part (a), yields

$$\overline{q}_1 = \frac{a - pc - q_2}{2}$$

so that we have coupled fixed point equations for $\overline{q}_1$ and q_2. Seeking strictly positive solutions we have

$$q_2 = \frac{(a - c - \overline{q}_1)_+}{2} = \frac{(a - c - \frac{a - pc - q_2}{2})}{2} = \frac{a - (2 - p)c + q_2}{4}$$

which can be solved to get

$$q_2 = \frac{a - (2 - p)c}{3}, \quad q_{1.0} = \quad \frac{2a + (2 - p)c}{6}, \quad q_{1,1} = \frac{2a - (1 + p)c}{6},$$

Furthermore, $\overline{q}_1 = \frac{a + (1 - 2p)c}{3}$.