ANSWERS TO PRACTICE PROBLEMS $\boxed{12}$

1. Since B is strictly dominated, it cannot be assigned positive probability at a Nash equilibrium. Let p be the probability of T and q the probability of L. Then p must satisfy the condition:

$$4p + 0(1-p) = 3p + 2(1-p)$$

while q must satisfy the condition:

$$1q + 4(1-q) = 2q + 1(1-q).$$

Thus $p = \frac{2}{3}$ and $q = \frac{3}{4}$. Hence the mixed-strategy equilibrium is given by:

$$\begin{pmatrix} T & C & B & L & R \\ \frac{2}{3} & \frac{1}{3} & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

2. **(a)** The normal form is as follows:

PLAYER 2

PLAYER 1	AC	AD	BC	BD
LEG	2 , 0	2 , 0	0 , 2	0 , 2
LEH	2 , 0	2 , 0	0 , 2	0 , 2
LFG	0 , 6	0 , 6	4 , 1	4 , 1
LFH	0 , 6	0 , 6	4 , 1	4 , 1
REG	1 , 4	4 , 3	1 , 4	4 , 3
REH	2 , 0	1 , 2	2 , 0	1 , 2
RFG	1 , 4	4 , 3	1 , 4	4 , 3
RFH	2 , 0	1 , 2	2 , 0	1 , 2

(b) There are no pure-strategy Nash equilibria.

(c) First let us solve the subgame on the left:

Player 2

Player 1	A	B
E	2 , 0	0 , 2
F	0 , 6	4 , 1

There is no pure-strategy equilibrium. Let us find the mixed-strategy equilibrium. Let p be the probability assigned to E and q the probability assigned to A. Then p must be the solution to $6(1-p) = 2p + (1-p)$ and q must be the solution to $2q = 4(1-q)$. Thus $p = \frac{5}{7}$ and $q = \frac{2}{3}$. The expected payoff of player 1 is

$$2pq + 0p(1-q) + 0(1-p)q + 4(1-p)(1-q) = \frac{4}{3} = 1.33$$

while the expected payoff of player 2 is

$$0pq + 2p(1-q) + 6(1-p)q + 1(1-p)(1-q) = \frac{12}{7} = 1.714$$

Next we solve the subgame on the right:

		Player 2	
		C	D
Player 1	G	1 , 4	4 , 3
	H	2 , 0	1 , 2

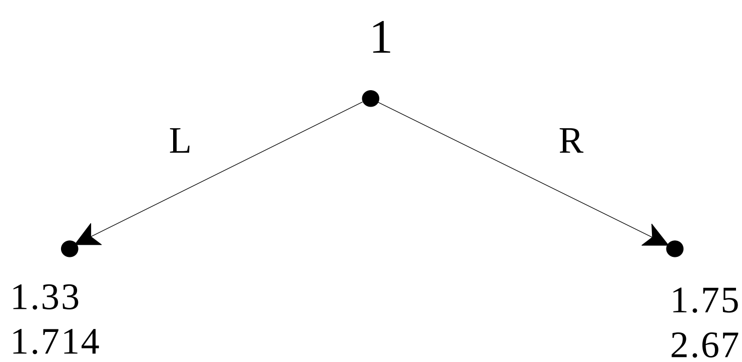
There is no pure-strategy equilibrium. Let us find the mixed-strategy equilibrium. Let p be the probability assigned to G and q the probability assigned to C. Then p must be the solution to $4p = 3p + 2(1-p)$ and q must be the solution to $q + 4(1-q) = 2q + (1-q)$. Thus $p = \frac{2}{3}$ and $q = \frac{3}{4}$. The expected payoff of player 1 is

$$pq + 4p(1-q) + 2(1-p)q + (1-p)(1-q) = \frac{7}{4} = 1.75$$

while the expected payoff of player 2 is

$$4pq + 3p(1-q) + 0(1-p)q + 2(1-p)(1-q) = \frac{8}{3} = 2.67$$

Thus the game reduces to:



Thus the subgame-perfect equilibrium is:

$$\left(\begin{array}{cccccc|cccc} L & R & E & F & G & H & A & B & C & D \\ 0 & 1 & \frac{5}{7} & \frac{2}{7} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{3}{4} & \frac{1}{4} \end{array}\right)$$

3. **(1)** The normal form is:

Player 3 chooses a

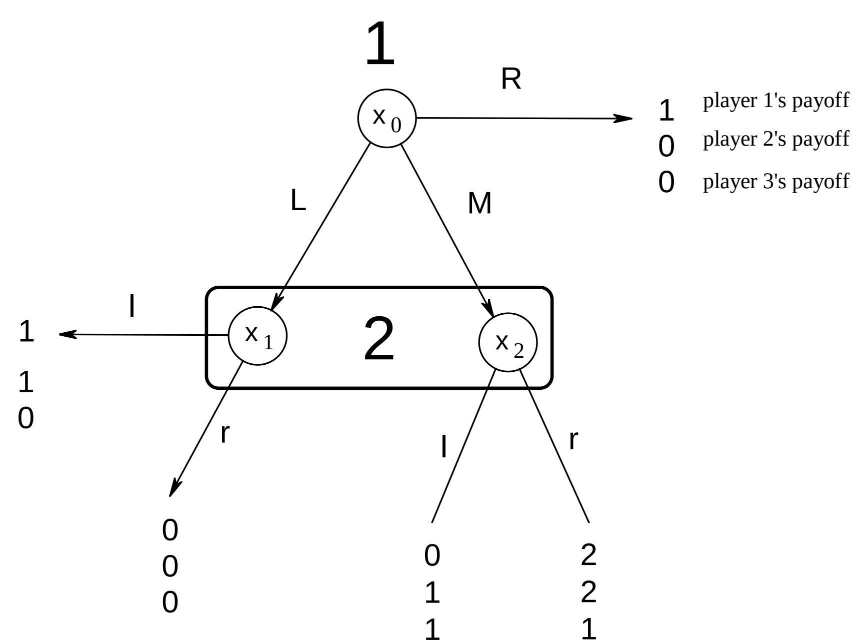
		Player 2: l	Player 2: r
	R	**1 , 0 , 0**	1 , 0 , 0
Player 1	M	0 , 1 , 1	**2 , 2 , 1**
	L	**1 , 1 , 0**	0 , 0 , 0

Player 3 chooses b

		Player 2: l	Player 2: r
	R	**1 , 0 , 0**	**1 , 0 , 0**
Player 1	M	0 , 1 , 0	0 , 0 , 0
	L	**1 , 1 , 0**	0 , 0 , 0

(a) The pure-strategy Nash equilibria are: (R,l,a), (M,r,a), (L,l,a), (R,l,b), (R,r,b) and (L,l,b). They are all subgame perfect because there are no subgames.

(b) Since, for player 3, *a* strictly dominates *b* conditional on his information set being reached, he will have to play *a* if his information set is reached with positive probability. Now, player 3's information set is reached with positive probability if and only if player 1 plays *M* with positive probability. Thus when Pr(*M*) > 0 the game essentially reduces to



Now, in order for player 1 to be willing to assign positive probability to M he must expect a payoff of at least 1 (otherwise R would be better) and the only way he can expect a payoff of at least 1 is if player 2 plays r with probability at least $\frac{1}{2}$. Now if player 2 plays r with probability greater than $\frac{1}{2}$, then M gives player 1 a higher payoff than both L and R and thus he will choose M with probability 1, in which case player 2 will choose r with probability 1 (and player 3 will choose a with probability 1) and so we get a pure strategy equilibrium. If player 2 plays r with probability exactly $\frac{1}{2}$ then player 1 is indifferent between M and R (and can mix between the two), but finds L inferior and must give it probability 0. But then player 2's best reply to a mixed strategy of player 1 that assigns positive probability to M and R and zero probability to L is to play r with probability 1 (if his information set is reached it can only be reached at node x_2). Thus there cannot be a mixed-strategy equilibrium where player 1 assigns to M probability p with $0 < p < 1$. It must be either $\Pr(M) =$ or $\Pr(M) = 1$.

4. There is only one subgame starting from player 2's node. The strategic form is as follows:

		Player 3		
		d	***e***	***f***
	a	**x1**	**x2**	**x3**
Player 2	***b***	**x4**	**x5**	**x6**
	c	**x7**	**x8**	**x9**

For Player 1 strategy c is strictly dominated by strategy b (she prefers x_4 to x_7, and x_5 to x_8 and x_6 to x_9) and for Player 2 strategy f is strictly dominated by strategy d (she prefers x_1 to x_3, and x_4 to x_6 and x_7 to x_9). Thus we can simplify the game to

		Player 3	
		d	*e*
Player 2	*a*	**x1**	**x2**
	b	**x4**	**x5**

Restricted to these outcomes the payers' rankings are: $Player\ 2: \begin{pmatrix} x_1 \\ x_4, x_5 \\ x_2 \end{pmatrix}$, $Player\ 3: \begin{pmatrix} x_2 \\ x_1, x_4 \\ x_5 \end{pmatrix}$.

Let U be 2's von Neumann-Morgenstern utility function. The we can set $U(x_1) = 1$ and $U(x_2) = 0$. Thus, since she is indifferent between x_4 and x_5 and also between x_4 and the lottery $\begin{pmatrix} x_1 & x_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $U(x_4) = U(x_5) = \frac{1}{2}$. Let V be 3's von Neumann-Morgenstern utility function. The we can set $V(x_2) = 1$ and $V(x_5) = 0$. Thus, since she is indifferent between x_1 and x_4 and also between x_1 and the lottery $\begin{pmatrix} x_2 & x_5 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $V(x_1) = V(x_4) = \frac{1}{2}$. Hence the game becomes

Player 3

Player 2		d	e
	a	$\boxed{1, \frac{1}{2}}$	$\boxed{0, 1}$
	b	$\boxed{\frac{1}{2}, \frac{1}{2}}$	$\boxed{\frac{1}{2}, 0}$

There is no pure-strategy Nash equilibrium. Let p be the probability of a and q the probability of d. Then for a Nash equilibrium we need $q = \frac{1}{2}$ and $p = \frac{1}{2}$. Hence in the subgame the outcome will be $\begin{pmatrix} x_1 & x_2 & x_4 & x_5 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. Since all of these outcomes are better than x_{10} for player 1, player 1 will play d. Thus the subgame-perfect equilibrium is $\left(\begin{array}{cc|ccc|ccc} d & u & a & b & c & d & e & f \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right)$.