University of Illinois **Spring 2013**

ECE 368BH: Problem Set 2: Problems and Solutions
Analysis of static games (continued), and dynamics involving static games

Due: Thursday, February 14 at beginning of class
Reading: Menache and Ozdaglar, Part I. Also, for more material on fictitious play, see Shamma and Arslan (2004) paper. For material on evolutionary game theory, see Easley and Kleinberg, *Networks, Crowds, and Markets: Reasoning about a highly connected world*, Chapter 7, and Shoham and Leyton-Brown, *Multiagent Systems* pp. 225-230. (Links on course webpage.)

1. **[On quasi-concavity]**
Let $f : C \to \mathbb{R}$, where C is a nonempty convex subset of $\mathbb{R}^n$ for some $n \geq 1$.

   (a) Prove that if f is a concave function, then for any constant t, the *t-upper level set* of f, $L_f(t) \stackrel{\triangle}{=} \{x \in C : f(x) \geq t\}$, is a convex set. (In other words, prove that a concave function is *quasi-concave.*)
   Solution: Fix t. Let $x, y \in L_f(t)$ and $\lambda \in [0, 1]$. It must be shown that $\lambda x + (1-\lambda)y \in L_f(t)$. That is implied by the fact $f(\lambda x+(1-\lambda)y) \geq \lambda f(x)+(1-\lambda)f(y) \geq \lambda t+(1-\lambda)t = t$.

   (b) Show that f is quasi-concave if and only if, for any $x, y \in C$, $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$.
   Solution: (if direction) Suppose that for any $x, y \in C$, $f(\lambda x+(1-\lambda)y) \geq \min\{f(x), f(y)\}$. Let t be an arbitrary constant. Let $x, y \in L_f(t)$ and $\lambda \in [0, 1]$. Then $f(x) \geq t$ and $f(y) \geq t$, so $f(\lambda x+(1-\lambda)y) \geq \min\{f(x), f(y)\} \geq t$, implying that $\lambda x+(1-\lambda)y \in L_f(t)$. Thus, $L_f(t)$ is convex, so that f is quasi-concave.
   (only if direction) Suppose that f is quasi-concave. Let $x, y \in C$ and let $0 \leq \lambda \leq 1$. Let $t = \min\{f(x), f(y)\}$. Then $L_f(t)$ is a convex set (because f is quasi-concave) and $x, y \in L_f(t)$, so $\lambda x+(1-\lambda)y \in L_f(t)$, which yields $f(\lambda x+(1-\lambda)y) \geq t = \min\{f(x), f(y)\}$.

   (c) Suppose f is quasi-concave and g is a nondecreasing function on $\mathbb{R}$. Show that the composition $g \circ f$ is also quasi-concave. (By definition, $g \circ f(x) = g(f(x))$.)
   Solution: We shall use the characterization of quasi-concave functions proved in part (b). Let $x, y \in C$ and $\lambda \in [0, 1]$. Since $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$ and g is nondecreasing, $g \circ f(\lambda x + (1 - \lambda)y) \geq g(\min\{f(x), f(y)\}) = \min\{g \circ f(x), g \circ f(y)\}$. Therefore, $g \circ f$ is quasi-concave.

2. **[On strictly concave functions]**
This straight forward problem is good to have in mind when interpreting the sufficient conditions for existence of pure strategy NE for continuous games. A function f is *strictly concave* if for any distinct x and y and $\lambda \in (0, 1)$, $f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$.

   (a) Let f be a continuously differentiable function with domain $\mathbb{R}^n$. Show that f is strictly concave if and only if for any distinct $x, y \in \mathbb{R}^n$,

$$(\nabla f(x) - \nabla f(y)) \cdot (y - x) > 0.$$

(Hint: Establish the result first for $n = 1$. You may use the fact that a continuously differentiable function on $\mathbb{R}$ is strictly concave if and only if f' is strictly decreasing. Reduce the result for general n to the case for $n = 1$ by considering functions of the form $\theta(t) = f(a + bt)$ for $a, b \in \mathbb{R}^n$ with $b \neq 0$.)

Solution: In case $n = 1$, the condition becomes $(f'(x) - f'(y))(y - x) > 0$ if $x \neq y$, or, equivalently, $f'(x) > f'(y)$ if $x < y$. That is, the condition is equivalent to the condition that f' is strictly decreasing. Given that f is continuously differentiable, f' being strictly decreasing is equivalent to f being strictly concave. So the claim is true for $n = 1$.

Now suppose $n \geq 2$. The function f is strictly concave if and only if the restriction of f to any line in $\mathbb{R}^n$ is strictly concave. So it suffices to show that

$$(\nabla f(x) - \nabla f(y)) \cdot (y - x) > 0 \tag{1}$$

holds for all distinct $x, y \in C$ if and only if for any $a, b \in \mathbb{R}^n$ with $b \neq 0$, the function θ defined by $\theta(t) = f(a + bt)$ satisfies

$$(\theta'(s) - \theta'(t))(t - s) > 0 \tag{2}$$

for all $s, t \in \mathbb{R}$ such that $s < t$ and $a + bs, a + bt \in C$.

Suppose $a, b \in \mathbb{R}^n$, $s, t \in \mathbb{R}$ such that $s < t$, and $x, y \in C$ where $x = a + bs$ and $y = a + bt$. By the chain rule of calculus, $\theta'(s) = \nabla f(a + bs) \cdot b$ and $\theta'(t) = \nabla f(a + bt) \cdot b$. So $(\theta'(s) - \theta'(t))(t - s) = (\nabla f(a + bs) - \nabla f(a + bt)) \cdot b(t - s) = (\nabla f(x) - \nabla f(y)) \cdot (y - x)$. Thus, (**??**) holds for all distinct $x, y \in C$ if and only if (**??**) holds for all $a, b \in \mathbb{R}^n$, with $b \neq 0$ and $s, t \in \mathbb{R}$ such that $s < t$ and $a + bs, a + bt \in C$, which, as mentioned above, suffices to complete the proof.

(b) Suppose f is a twice continuously differentiable function on $\mathbb{R}^n$ and suppose its Hessian matrix $H(x)$ defined by

$$H_{ij}(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \qquad i, j \in \{1, \cdots, n\}$$

is (strictly) negative definite for all $x \in \mathbb{R}^n$ (i.e. $b^T H(x) b < 0$ for all nonzero $b \in \mathbb{R}^n$.) Show that f is strictly concave. (Same hint as in previous part applies.)

Solution: If $n = 1$ the condition is just that $f''(x) < 0$, which implies that f is strictly concave. Now suppose $n \geq 2$. Let θ be defined as in part (a). Then by the chain rule of calculus once, $\theta'(t) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a + bt) b_j = \nabla f(a + bt) \cdot b$, and then again,

$\theta''(t) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a + bt) b_i b_j = b^T H(x) b < 0$, where $x = a + bt$. Therefore, θ is strictly concave, so f is strictly concave.

3. **[Existence and uniqueness of NE for some games with quadratic payoff functions]** This problem concerns an n player game with strategy space $S_i = [0, 1]$ for all players i, and space of strategy vectors $S = S_1 \times \cdots \times S_n$.

(a) Given a vector of strategies $x \in S$, let $\overline{x} = \frac{x_1 + \cdots + x_n}{n}$. Consider the payoff functions $u_i(x) = c x_i (1 - x_i) - \frac{1}{2}(x_i - \overline{x})^2$, where $c \geq 0$. For what values of $c \geq 0$ does there exist a pure strategy Nash equilibrium?

Solution: Note that $\frac{\partial^2 u_i(x)}{(\partial x_i)^2} = -2c - \left(\frac{n-1}{n}\right)^2 \leq 0$ for any $c \geq 0$, so $u_i(x)$ is a concave function of x_i for x_{-i} fixed. Also, the payoff functions are continuous and the strategy spaces compact, so there exists a pure strategy NE for any $c \geq 0$ by the Debreu, Glicksberg, and Fan existence theorem. Actually, we could skip using this theorem and just identify the NE, as shown next.

By the concavity of u_i in x_i, we know that $x_i \in B(x_{-i})$ if $\frac{\partial u_i(x)}{\partial x_i} = 0$ for $1 \leq i \leq n$. Equivalently,

$$c(1-2x_i) - \frac{n-1}{n}(x_i - \overline{x}) = 0 \quad \text{or} \quad x_i = \frac{c + \overline{x}\left(\frac{n-1}{n}\right)}{2c + \frac{n-1}{n}}$$

Since x_i is the same for all i, $\overline{x} = x_i$. Replacing x_i by $\overline{x}$ in the above equations yield

$$(2\overline{x} - 1)c = 0$$

If $c = 0$ then for any $\theta \in [0,1]$ it is a Nash equilibrium for $x_i = \theta$ for all i. If $c > 0$ then a NE is given by $x_i = \frac{1}{2}$ for all i. Note that x_i would be a best response to x_{-i} if $x_i = 0$ and $\frac{\partial u_i(0)}{\partial x_i} < 0$, but $\frac{\partial u_i(0)}{\partial x_i} = c + \frac{(n-1)\overline{x}}{n} \geq 0$ so we can't get another solution that way. Similarly, there are no other solutions with $x_i = 1$ and $\frac{\partial u_i(0)}{\partial x_i} > 0$. Thus, the NE already identified are the only pure strategy NE that exist for the given payoff functions.

(b) For the payoff functions of part (a), for what values of c is the pure strategy Nash equilibrium unique?

Solution: As found in part (a), the NE is unique if $c > 0$.

(c) Now consider the payoff functions $u_i(x) = \frac{c}{2}x_i(1-x_i) + x_i\left(\sum_{j=1}^n a_{i,j}x_j\right)$, where $|a_{i,j}| \leq 1$ and $a_{i,i} = 0$ for all i, j. Find a constant c_0 so that for $c \geq c_0$ there exists a pure strategy Nash equilibrium.

Solution: Note that $\frac{\partial^2 u_i(x)}{(\partial x_i)^2} = -c \leq 0$ for all i, so that for any $c \geq 0$ and any i, $u_i(x_i, x_{-i})$ is a concave function of x_i. Also, the strategy sets are compact and the functions u_i are continuous, so there exists a pure strategy NE by the Debreu, Glicksberg, and Fan existence theorem. So we can take $c_0 = 0$.

(d) For the payoff functions of part (c), give a value c_1 so that there is a unique pure strategy NE if $c > c_1$. (Hint: A sufficient condition for a symmetric matrix to be negative definite is that the diagonal elements be strictly negative, and the sum of the absolute values of the off-diagonal elements in any row be strictly smaller than the absolute value of the diagonal element in the row.)

Solution: The Jacobian of ∇u is given by

$$U(x) = \begin{pmatrix} \frac{\partial^2 u_1(x)}{(\partial x_1)^2} & \frac{\partial^2 u_1(x)}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 u_2(x)}{\partial x_2 \partial x_1} & \ddots & \\ \vdots & & \end{pmatrix} = -cI + A,$$

where I is the identify matrix and A is the matrix with entries $a_{i,j}$. A sufficient condition for uniqueness of the NEP is that $U(x) + U(x)^T$ be negative definite (e.g. Rosen's paper, or Proposition 1.44 in Menache and Ozdaglar.) We find

$$U(x) + U(x)^T = -2cI + A + A^T.$$

A sufficient condition for a symmetric matrix to be negative definite is that the diagonal elements are strictly negative, and the sum of the absolute values of the off-diagonal elements in any row be strictly smaller than the absolute value of the diagonal element in the row. Such condition is true for $U(x)+U(x)^T$ if $c > n-1$. Thus, if $c > n-1$, then there is a unique NE for this game. (So we can use $c_1 = n-1$.)

4. **[Guessing within one]**
Let L be a positive integer multiple of three, and consider the following zero sum, two player game. Let $S = \{1, \dots L\}$. Player one selects $i \in S$ and player two selects $j \in S$. Player one wins if $|i-j| \leq 1$ and player two wins otherwise. Each player wishes to maximize her probability of winning.

   (a) Show that for any Nash equilibrium (in mixed strategies), the probability player one wins is $\frac{3}{L}$.
   Solution: By the theory of zero sum games, all Nash equilibria, also called saddle points, have the same value. The approach we follow is to identify a Nash equilibrium. (An alternative approach would be to identify a min max strategy for one of the players.) Let p and q each be the uniform probability distribution over the set T defined by $T = \left\{3i-1 : 1 \leq i \leq \frac{L}{3}\right\}$. We claim (p,q) is a Nash equilibrium. Indeed, given player two plays q, any pure strategy response i by player one is a best response, so p is also a best response for q. Similarly, given player one plays p, any pure strategy response j by player two is a best response so q is a best response to p. Thus, (p,q) is a Nash equilibrium. The probability player one wins is $\frac{3}{L}$ for this, and thus for any, Nash equilibrium.

   (b) Show that the maxmin strategy for player one is unique, but the maxmin strategy for player two is not unique.
   Solution: A probability distribution $p = (p_1, \dots, p_L)$ is a max min strategy for player one if and only if the minimum payoff of player one is maximized:

$$\min\{p_1+p_2, p_1+p_2+p_3, p_2+p_3+p_4, \dots, p_{L-2}+p_{L-1}+p_L, p_{L-1}+p_L\} = \frac{3}{L}$$

   Adding the L numbers inside the braces and using the fact $p_1 + \cdots + p_L = 1$ yields that $3 - p_1 - p_L \geq 3$, so that $p_1 = p_L = 0$. Since $p_1 = 0$ it follows that $p_2 \geq \frac{3}{L}$. Dropping out the three terms with p_2 in them, we still have

$$\min\{p_3+p_4+p_5, \dots, p_{L-2}+p_{L-1}+p_L, p_{L-1}+p_L\} \geq \frac{3}{L}$$

   Adding the $L-3$ terms inside the braces yields $3(1-p_2) - 2p_3 - p_4 - p_L \geq \frac{3(L-3)}{L}$, from which we can conclude that $p_2 = \frac{3}{L}$, $p_3 = p_4 = 0$. Then $p_5 \geq \frac{3}{L}$. This process can be repeated, so we can show by induction on k that for $1 \leq k \leq \frac{L}{3}$, $(p_1, p_2, \dots, p_{3k-1}) = (0, \frac{3}{L}, 0, 0, \frac{3}{L}, 0, 0, \frac{3}{L}, 0, \cdots, 0, \frac{3}{L}, 0, 0, p_{3k-1})$ where $p_{3k-1} \geq \frac{3}{L}$. This statement for $k = \frac{L}{3}$ shows that p is the probability distribution given in part (a).

   Regarding mixed strategies for player 2, note that, for a given pure strategy i of player one, if $j = 2$ is a winning strategy for player two than $j = 1$ is also a winning strategy for player two. Therefore, if q is a mixed strategy for player 2 and $q_2 > 0$, then, for any ϵ with $0 \leq \epsilon \leq q_2$, $(q_1 + \epsilon, q_2 - \epsilon, q_3, q_4, \dots, q_L)$ weakly dominates q. In particular, if q is the min max optimal strategy of player two described in the solution of part (a), there are other strategies for player two that weakly dominate q, and which therefore must also be maxmin optimal.

5. **[Fictitious play for guessing within one]**
Consider the game of the previous problem for $L = 6$. Player one selects a probability distribution p in an effort to minimize pAq' (which is the probability the first player's number *misses* the second player's number by at least two) and player two selects q to maximize pAq', where

$$A = \begin{pmatrix} 001111 \\ 000111 \\ 100011 \\ 110001 \\ 111000 \\ 111100 \end{pmatrix}$$

Write a computer program for this problem. Starting with some pair of pure strategies (i, j), calculate the evolution of fictitious play, so that for the n^{th} play, each player plays a strategy that is a best response to the empirical average of the previous $n-1$ plays of the other player.

(a) Attach a copy of the computer program you use for the simulation.
Solution:

```
function v_argmax = v_argmax(X)
%   Given a row vector X, returns a row vector of the same
% length with a one in the position of the largest element of X
% and a zero in the other positions.
k=length(X);
I=1;
max_value=X(1);
for i=2:k
   if X(i) > max_value
        I=i;
        max_value=X(i);
   end
end
v_argmax=zeros(1,k);
v_argmax(I)=1;
```

```
function  fictitious_play
A= [0 0 1 1 1 1
0 0 0 1 1 1
1 0 0 0 1 1
1 1 0 0 0 1
1 1 1 0 0 0
1 1 1 1 0 0 ];
p=[1 0 0 0 0 0];
q=[1 0 0 0 0 0];
for n=1:10
    p=(n/(n+1))*p+v_argmax(-q*A')/(n+1);
    q=(n/(n+1))*q+v_argmax(p*A)/(n+1);
end
gap=max(p*A)-min(A*q')
```

(b) Let p_n and q_n denote the empirical distributions for the two players after the game has been played n times. Define the duality gap for a pair of strategies (p, q) by $\text{gap}(p, q) = \max(p*A) - \min(A*q')$. Compute $\text{gap}(p_n, q_n)$ for your simulation after $n = 10^i$ iterations, for $1 \le i \le 6$. Does your computation indicate that $\text{gap}(p_n, q_n) \to 0$? If so, comment on the rate of convergence.

Solution:

n	10	10^2	10^3	10^4	10^5	10^6
gap	0.45	$4.9e-02$	$3.9e-03$	$8.0e-04$	$7.0e-05$	$1.1e-05$

The above suggests (but doesn't prove!) that the gap is converging to zero at the rather slow rate of $\frac{1}{n}$.

6. **[Evolutionarily stable strategies and states]**

Consider the following symmetric, two-player game:

	1	2
1	0,0	1,2
2	2,1	0,0

That is, each player selects 1 or 2. If they select different numbers, the payoffs are the numbers selected. If they select the same number, the payoffs are zero.

(a) Does either player have a (weaky or strongly) dominant strategy?

Solution: No.

(b) Identify all the pure strategy and mixed strategy Nash equilibria.

Solution: (1,2) and (2,1) are pure strategy NE. As usual, there is no NE in which only one strategy is an NE. If (p, q) is am NE such that both p and q are nondegenerate mixed strategies, either action of player one must be a best response to q, so $2q_1 = q_2$, or $q = \left(\frac{1}{3}, \frac{2}{3}\right)$. Similarly, $p = \left(\frac{1}{3}, \frac{2}{3}\right)$. Thus, $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$ is the unique NE in nondegenerate mixed strategies.

(c) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

Solution: Recall that if p is an ESS, then (p, p) is an NE. The pure strategy NEs found in part (a) are not symmetric, so there are no pure ESSs. It remains to see whether the mixed strategy NE p given by $p = \left(\frac{1}{3}, \frac{2}{3}\right)$ is an ESS. By definition, we need to check whether for any $p' \neq p$, either (i) $u(p', p) < u(p, p)$, or (ii) $(u(p', p) = u(p, p)$ and $u(p, p') > u(p', p'))$. Since $u(p', p) = u(p, p)$ for all choices of p', the question comes down to whether $u(p, p') > u(p', p')$ for all $p' \neq p$. That is, whether $2\left(\frac{2}{3}\right)p'_1 + 1\left(\frac{1}{3}\right)(1 - p'_1) > 2(1 - p'_1)p'_1 + p'_1(1 - p'_1)$ for all $p' \neq p$. Or, equivalently, whether $2\left(p'_1 - \frac{1}{3}\right)^2 > 0$ for $p \neq p'$. This condition is true, so $\left(\frac{1}{3}, \frac{2}{3}\right)$ is a mixed ESS.

(d) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1.

Solution: The replicator dynamics for the population share vector θ_t are given by $\dot{\theta}_t(a) = \theta_t(a)(u(a, \theta_t) - u(\theta_t, \theta_t))$ for $a \in \{1, 2\}$. Although there are two equations, this system is actually one dimensional because $\theta_t(2) = 1 - \theta_t(1)$. Let $x_t = \theta_t(1)$. Then $u(1, \theta_t) = 1 - x_t$, $u(2, \theta_t) = 2x_t$, and $u(\theta_t, \theta_t) = x_t(1 - x_t) + 2(1 - x_t)x_t = 3x_t(1 - x_t)$. So the replicator dynamics become $\dot{x}_t = x_t((1 - x_t) - 3x_t(1 - x_t))$, or, equivalently,

$$\dot{x}_t = x_t(1 - x_t)(1 - 3x_t). \qquad (3)$$

(e) Identify the steady states of the replicator dynamics.

Solution: The right hand side of (**??**) is zero for $x_t \in \left\{0, \frac{1}{3}, 1\right\}$, so there are three steady states for the replicator dynamics: (1,0), $\left(\frac{1}{3}, \frac{2}{3}\right)$, and (1,0).

(f) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.

Solution: Of the three steady states, only $\left(\frac{1}{3}, \frac{2}{3}\right)$ is asymptotically stable. If $x_0 = \epsilon$ for an arbitrarily small but positive ϵ, then $\dot{x}_t > 0$ and x will converge monotonically up to $\frac{1}{3}$, so 0 is not even a stable steady state (so it is not asymptotically stable). Similarly, 1 is not a stable steady state. However, $\frac{1}{3}$ is an asymptotically stable steady state of x because the right hand side of (**??**) has a down crossing of zero at $\frac{1}{3}$. Hence $(\frac{1}{3}, \frac{2}{3})$ is an asymptotically stable state for the replicator dynamics. (Another justification for this problem can be given by applying general facts about ESSs and replicator dynamics. States $(1, 0)$ and $(0, 1)$ can't be asymptotically stable, or even stable, states for the replicator dynamics, because, when played against themselves, they don't give NEs. The mixed state $(\frac{1}{3}, \frac{2}{3})$ is an asymptotically stable state of the replicator dynamics because it is an ESS.)

7. **[Evolutionarily stable strategies and states, II]**

Consider the following symmetric, two-player game:

	1	2	3
1	0,0	1,2	1,3
2	2,1	0,0	2,3
3	3,1	3,2	0,0

That is, each player selects 1,2, or 3. If they select different numbers, the payoffs are the numbers selected. If they select the same number, the payoffs are zero.

(a) Identify all the pure strategy and mixed strategy Nash equilibria.

Solution: The pure strategy NEs are $(2, 3)$ and $(3, 2)$. To reduce the search space for mixed NE, note that strategy 1 is strictly dominated by the mixed strategy $(0, 0.6, 0.4)$. Therefore, no strategy of an NE pair will play strategy 1 with positive probability. Thus, any nondegenerate mixed strategy NE has the form $((0, p_2, p_3), (0, q_2, q_3))$ where $p_2 + p_3 = q_2 + q_3 = 1$ and $p_2, p_3, q_2, q_3 > 0$. Since either strategy 2 or 3 must be a best response to p, we have $3p_2 = 2p_3$, so that $p = (0, 0.4, 0.6)$. Similar, q is the same. So there is a unique nondenerate mixed NE, given by $((0, 0.4, 0.6), (0, 0.4, 0.6))$.

(b) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

Solution: Recall that if p is an ESS, then (p, p) is an NE. The pure strategy NEs found in part (a) are not symmetric, so there are no pure ESSs. It remains to see whether the mixed strategy NE p given by $p = (0, 0.4, 0.6)$ is an ESS. By definition, we need to check whether for any $p' \neq p$, either (i) $u(p', p) < u(p, p)$, or (ii) $(u(p', p) = u(p, p)$ and $u(p, p') > u(p', p'))$. So let p' be a mixed strategy not equal to p. If $p'_1 > 0$ then $u(p', p) < u(p, p)$, so it remains to consider the case $p'_1 = 0$. Then $u(p', p) = u(p, p)$, so the question comes down to whether $u(p, p') > u(p', p')$ for all $p' \neq p$ with $p'_1 = 0$. That is, whether $2(0.6)p'_2 + 1(0.4)p'_3 > 2p'_3p'_2 + p'_2p'_3$ for all $p' \neq p$ with $p'_1 = 0$. Or, equivalently, whether $2\left(p'_2 - 0.4\right)^2 > 0$ for all $p' \neq p$ with $p'_1 = 0$. This condition is true, so the mixed strategy $p = (0, 0.4, 0.6)$ is an ESS.

(c) Identify the steady states of the replicator dynamics.

Solution: As always, the degenerate pure states $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are fixed points of the replicator dynamics. Also, the mixed strategy $p = (0, 0.4, 0.6)$ is a fixed point of the replicator dynamics. There are two more steady states. One is for when there are no individuals of type three present, resulting in the steady state $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$. The other is for when there are no individuals of type two present, resulting in the steady state $\left(\frac{1}{4}, 0, \frac{3}{4}\right)$.

(d) Prove that the strategy (or one of the strategies) identified in the previous part, when used by both players in a two player game, is a trembling hand perfect equilibrium. (Use a proof based directly on the definition of trembling hand perfect equilibrium.)

Solution: We shall show that (p, p) is a trembling hand perfect equilibrium for the strategy $p = (0, 0.4, 0.6)$. By definition, it means that there is a sequence of fully mixed strategies $p^{(n)} \to p$ such that p is a best response to $p^{(n)}$ for all n. We consider $p^{(n)}$ of the form $p^{(n)} = (\frac{a+b}{n}, 0.4 - \frac{a}{n}, 0.6 - \frac{b}{n})$ for some positive constants a and b. The sequence should begin with n so large that $p^{(n)}$ has nonnegative coordinates. Observe that p is a best response for $p^{(n)}$ if and only if strategies 2 and 3 have an equal payoff against $p^{(n)}$ and strategy 1 has a payoff less than or equal to the payoff for either strategies 2 or 3. Equivalently, p is a best response for $p^{(n)}$ if and only if:

$$1 - \frac{a+b}{n} \leq 2\left(1 - \left(0.4 - \frac{a}{n}\right)\right) = 3\left(1 - \left(0.6 - \frac{b}{n}\right)\right)$$

which holds for all large n if $a = 3$ and $b = 2$. We need to take $n \geq 8$ so that $p_2^{(n)} > 0$. Summarizing, $p^{(n)} = (\frac{5}{n}, 0.4 - \frac{3}{n}, 0.6 - \frac{2}{n})$ for $n \geq 8$ is a sequence of fully mixed strategies converging to p, and p is a best response to $p^{(n)}$ for all $n \geq 8$. Hence, by definition, (p, p) is a trembling hand perfect equilibrium for the two-player game.

Note: None of the other stable states identified in the previous part give rise to symmetric trembling hand perfect equilbria, because a trembling hand perfect equilibrium must be an NE.

8. **[Simulation of evolutionary game of doves and hawks]**

The dove hawk game is the two player symmetric static form game given by:

	D	H
D	4,4	1,5
H	5,1	0,0

(a) Find an evolutionarily stable strategy (ESS) and show that it is unique.

Solution: An evolutionarily stable strategy (ESS), when used by both players, must be an NE. The game has two pure strategy NEs, but those aren't symmetric, so there is no pure strategy ESS. There is a unique mixed NE, and it is symmetric, with each player using the strategy $(0.5, 0.5)$. This strategy is the only candidate for being an ESS. To see if $p = (0.5, 0.5)$ is an ESS, since $u(p', p)$ is the same for all p', it comes down to checking the condition: (ii) $u(p, p') > u(p', p')$ for all $p' \neq p$. Equivalently, $(0.5)(4p'_1 + p'_2 + 5p'_1) > p'_1(4p'_1 + p'_2) + p'_2 5p'_1$ or $2(p'_1 - 0.5)^2 > 0$, for $p \neq p'$. Which is true. So $p = (0.5, 0.5)$ is the unique ESS.

(b) For this part you need to write and run a computer simulation using a random number generator (i.e. Monte Carlo simulation). You are to simulate a population of doves and hawks in discrete time. Suppose there are initially $n_D(1)$ dove's and $n_H(1)$ hawks at the initial time, $t = 1$. Given the numbers of each type at time t, $(n_D(t), n_H(t))$, the numbers at time $t+1$ are determined as follows. Two distinct birds are selected from among all $n_D(t) + n_H(t)$ birds present at time t, and the two birds play the above two player game (where the strategy of a bird is the type of the bird). After the game, the two birds are returned to the population. In addition, for each player, more birds of the same type as that player are added to the population as well, with the number added equal to the payoff of the player. For example, if both birds are doves, they each have payoff 4, so the two doves are returned, plus a total of eight more doves (because 8=4+4) are added to the population. Turn in (1) a copy of your computer code and (2) a graph showing the number of doves and the number of hawks versus time t for $1 \leq t \leq 100$, beginning with one dove and ten hawks at time $t = 1$.

Solution: Since ESSs are asymptotically stable points of the deterministic replicator dynamics, and the stochastic replicator dynamics is well approximated by the deterministic replicator dynamics, we expect to observe the share of doves in the population, $\frac{n_D(t)}{n_D(t)+n_H(t)}$, to be close to $p_1 = \frac{1}{2}$. Also, since two individuals play per unit time with mean payoff about 2.5 each, we expect the total population to grow linearly with slope 5. That is indeed observed.

```
function  evolution
%  Simulation of a population of doves and hawks
%  for a stochastic version of replicator dynamics
%  -BH 2/3/13

   %d h        A is the payoff matrix of player 1 for the
A= [4 1  %d    symmetric, two player game.
    5 0] %h
n(1,:)=[1,10]  %start with one dove and ten hawks at time one
              %add a length two row to n at each iteration
for t=1:99,   % sum(n(t,:))is the number of birds at time t
    i=randi(sum(n(t,:)))  % randomly select a bird from population
    j=i;
    while (j==i)  % randomly select birds til get one different from i
      j=randi(sum(n(t,:)))
    end
    if (i<=n(t,1))
        type_one=1  %bird i, the first player, is a dove
    else
        type_one=2  % else bird i is a hawk
    end
    if (j<=n(t,2))
        type_two=1  %bird j, the second player, is a dove
    else
        type_two=2  % else bird j is a hawk
    end
    n(t+1,:)=n(t,:);
    n(t+1,type_one)=n(t+1,type_one)+A(type_one,type_two);
    n(t+1,type_two)=n(t+1,type_two)+A(type_two,type_one);
end
 n
 plot(1:100,n)
```

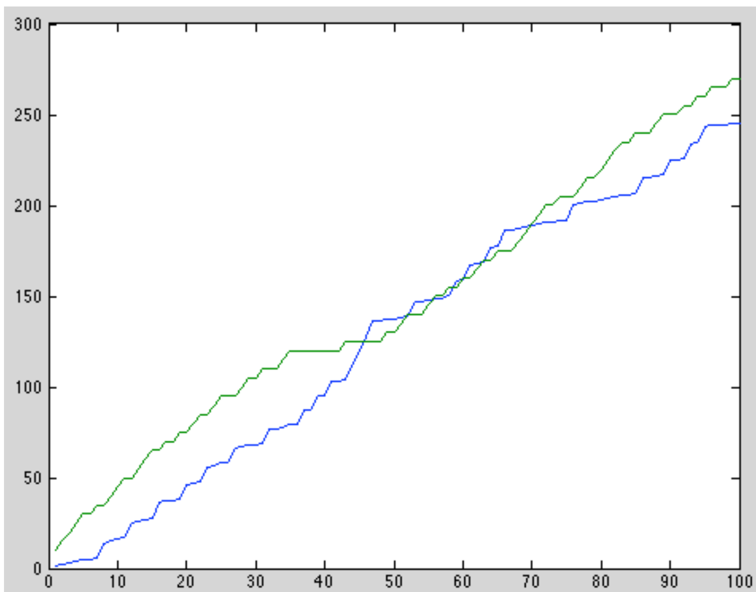

Figure 1: $n_D(t)$ and $n_H(t)$ vs. t for $1 \leq t \leq 100$ for one run of the simulation. As expected, the ratio of Doves to Hawks hovers near one, and the growth of the total population is roughly linear with slope 5.