A Combinatorial Identity for Unicellular Maps

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Outline

- Introduction to Unicellular Maps
- The Gluing and Slicing Operations
- Bijections Between Sets of Unicellular Maps
- Recurrence Relation

Surfaces

- A *surface* is a 2 dimensional object that looks like Euclidean space around each point on it.
- The *genus* of a surface is the number of holes that the surface has.

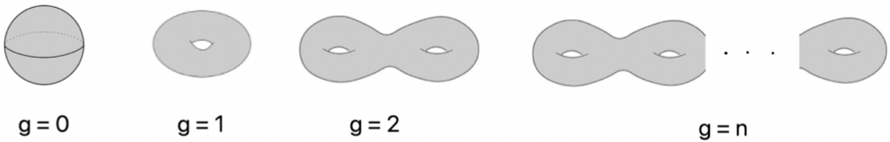
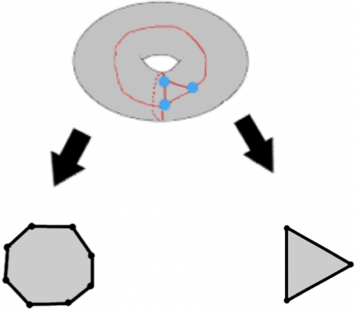


Figure: Surfaces of Genus g

Maps on Surfaces

Definition

A *map* or an *embedding* of a graph on a surface is a drawing of the graph on a surface in which no edges cross. The components of the complement of an embedded graph are called *regions* or *faces*. A map or an embedding of a graph is *cellular* if every region is topologically equivalent to a polygon. [2]



Unicellular Maps

Definition

A *unicellular map* or *one-face map* is a graph embedded on a surface whose complement is topologically equivalent to a polygon. [1]

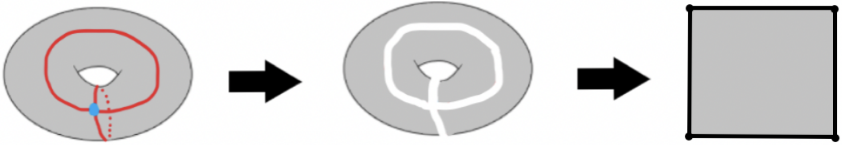


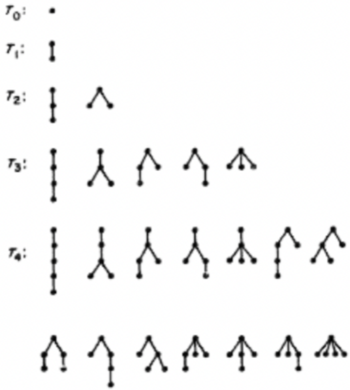
Figure: A Unicellular Map on a Genus 1 Surface

How Many Unicellular Maps are There?

- How many graphs embedded on a surface of genus g are unicellular?
  - A lot.
- How do we count them?
  - Develop a recurrence relation.

Unicellular Maps of Genus 0

- Unicellular maps on a surface of genus 0 are the ordered plane trees.
  - Enumerated by the Catalan numbers
  - $C_n = \frac{1}{n+1}\binom{2n}{n} = 1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$

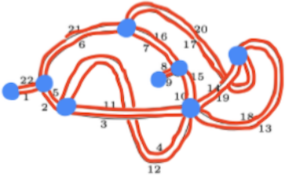


Unicellular Maps of Genus 0

Figure: A Unicellular Map on a Genus 0 Surface

Equivalent Representations for Unicellular Maps

- Three equivalent representations for unicellular maps.
  - A graph embedded on a surface
  - A ribbon graph
  - A triple $\mathbf{m} = (H, \alpha, \sigma)$
    - H is set of cardinality $2n$
    - $\alpha \rightarrow$ pairs of half-edges
    - $\sigma \rightarrow$ information about the vertices.
    - $\gamma \rightarrow$ order of half-edges around the map



$$H = [\![1, 22]\!]$$

$$\alpha = (1,22)(2,5)(3,11)(4,12)(6,21)(7,16)(8,9)(10,15)(13,18)(14,19)(17,20)$$

$$\sigma = (1,5,21)(2,11,4)(3,12,18,14,10)(6,16,20)(7,9,15)(8)(13,19,17)(22)$$

$$\gamma = \alpha\sigma = (1,2,3,\ldots,22)$$

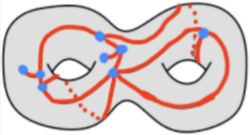


Figure: Three Equivalent Representations of Unicellular Maps [1]

Equivalent Representations of Unicellular Maps

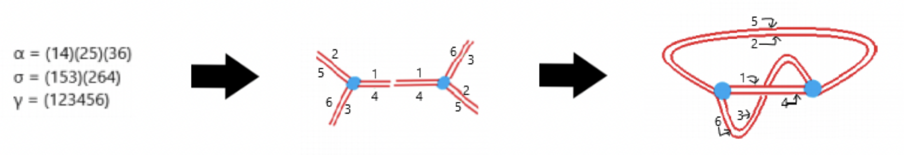


Figure: Constructing the Ribbon Graph from the Permutation Representation

Equivalent Representations of Unicellular Maps

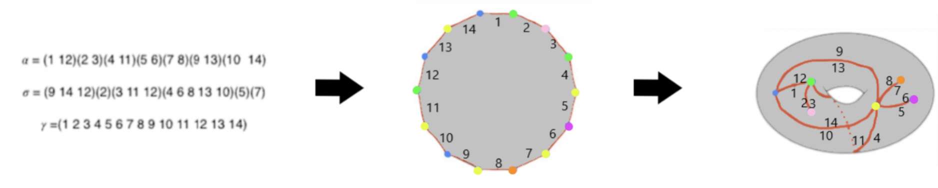
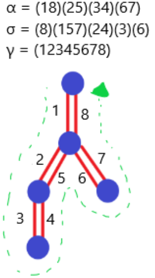


Figure: Constructing the Topological Representation from the Permutation Representation

Properties of Unicellular Maps

- Every unicellular map has a distinguished half-edge called the *root*.
- A *tour* of the face of a unicellular map is given by the permutation γ.
- The *linear order* $<_m$ on H is given by $r <_m \gamma(r) <_m \gamma^2(r) <_m \ldots <_m \gamma^{2n-1}(r)$



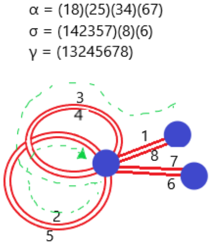


Figure: A tour of the face for two different unicellular maps.

The Gluing Operation

- The gluing operation allows us to create a unicellular map of a larger genus from a unicellular map of smaller genus.
- Requires three distinct half-edges $a_1 <_{\mathbf{m}} a_2 <_{\mathbf{m}} a_3$ belonging to three distinct vertices.

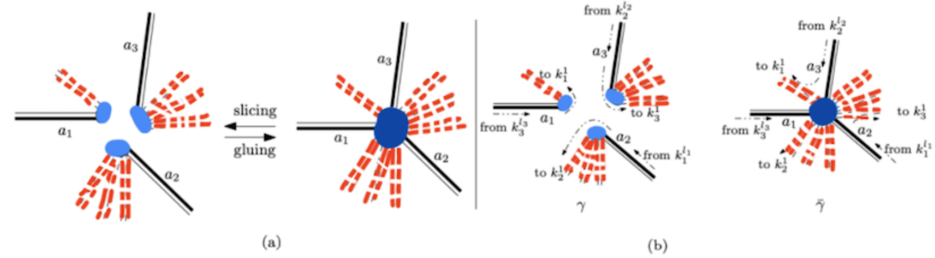


Figure: The Gluing Operation on a Ribbon Graph

The Gluing Operation Example

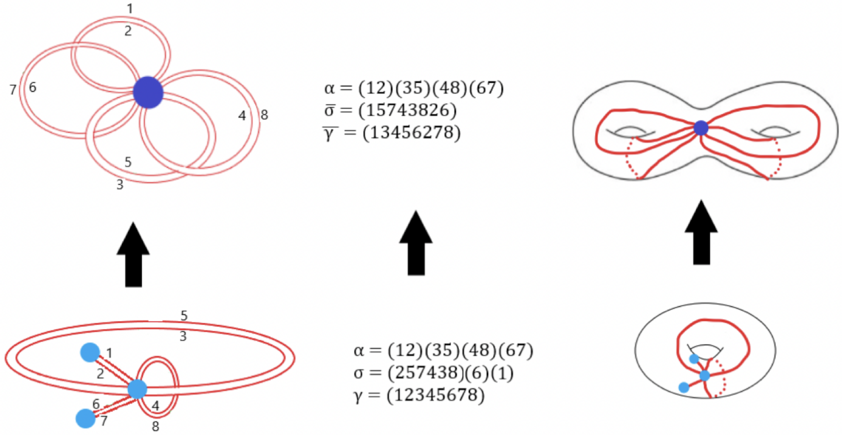


Figure: The Gluing Operation

Intertwined Half-edges

Definition

Let $\overline{\boldsymbol{m}} = (H, \alpha, \overline{\sigma})$ be a map of genus $g+1$, and three half-edges a_1, a_2, a_3 belonging to a same vertex $\overline{v}$ of $\overline{\boldsymbol{m}}$. We say that a_1, a_2, a_3 are *intertwined* if they do not appear in the same order in $\overline{\gamma} = \alpha\overline{\sigma}$ and in $\overline{\sigma}$.[1]

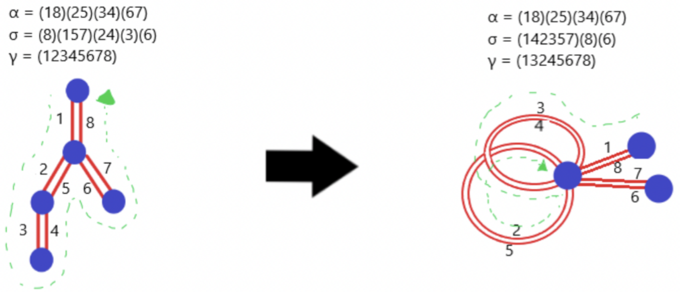


Figure: Intertwining resulting from the gluing operation applied to a plane tree

Trisections

- A trisection is a special half-edge that marks where there is intertwining in the map.
- Every unicellular map of genus g has $2g$ trisections.
- The half-edge 4 has become a trisection.

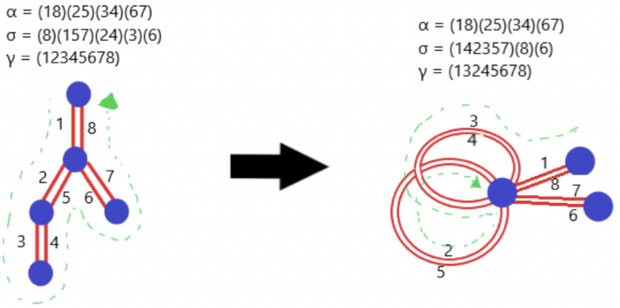


Figure: Intertwining resulting from the gluing operation applied to a plane tree.

The Slicing Operation

- The Slicing operation allows us to create a unicellular map of a smaller genus from a unicellular map of larger genus.
- Requires three intertwined half-edges belonging to the same vertex.

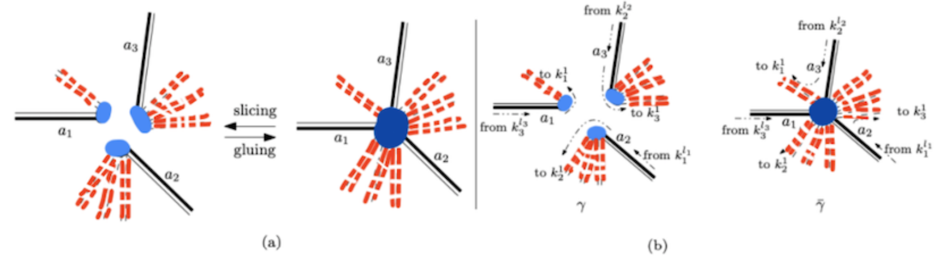


Figure: The Gluing Operation on a Ribbon Graph [1]

Example of the Slicing Operation

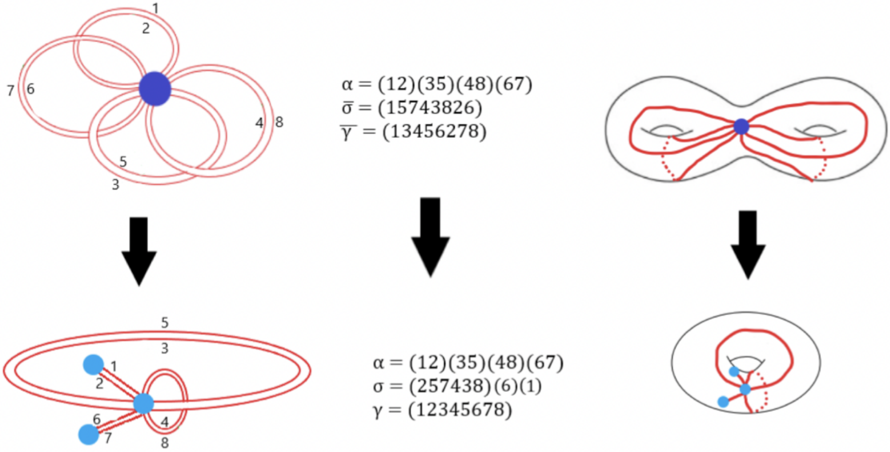


Figure: The Slicing Operation

What have we done so far?

- Unicellular maps
  - There are three equivalent representations.
  - The linear order $<_{\mathbf{m}}$ and the tour of the face.
- The slicing and gluing operations
  - Allowed us to relate unicellular maps of genus g and genus $g + 1$.
  - The gluing operation makes half-edges intertwined.
  - Trisections are special half-edges which mark where there is intertwining.

What's Next?

- Find a recurrence relation that relates unicellular maps of larger genus to unicellular maps of smaller genus.
  - Use the gluing and slicing operations?

Example of Why the Gluing Operation is not Injective

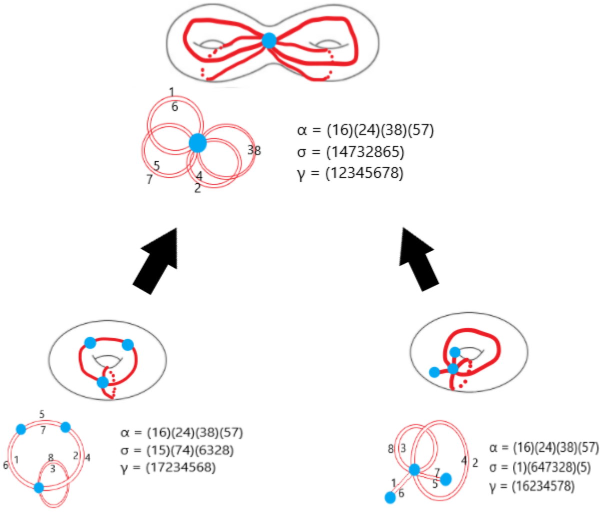


Figure: Two distinct unicellular maps of genus 1 that when glued result in the same unicellular map of genus 2

Making the Gluing Operation Injective

- Specify which trisection to apply the slicing operation to.
- Not all trisections are the same.
  - Use block diagrams to categorize the types of trisections.

Trisections of Type I and II

- What's the difference between I and II?
  - Trisections of type I $\rightarrow$ half-edge with small amount of intertwining
  - Trisections of type II $\rightarrow$ half-edge with large amount of intertwining

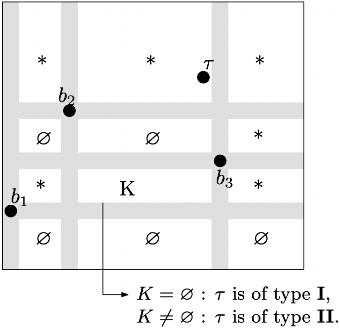


Figure: Distinguishing between Trisections of Type I and II around the vertex $\overline{v}$ [1]

Bijections for Trisections of Type I and II

Theorem (Bijection for Trisections of Type I)

Unicellular maps of genus $g+1$ and a distinguished trisection τ of type I

$$\Updownarrow$$

Unicellular maps of genus g with three distinguished vertices. [1]

Theorem (Bijection for Trisections of Type II)

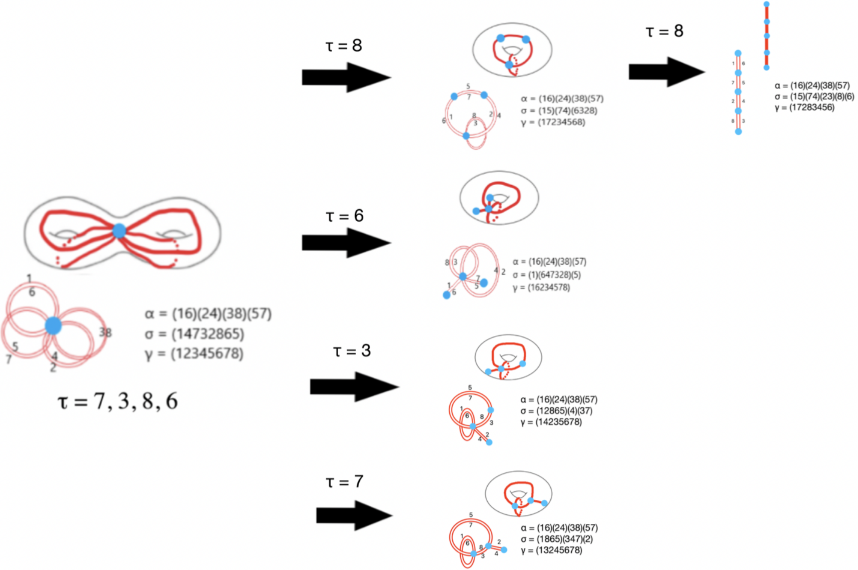
Unicellular maps of genus $g+1$ and a distinguished trisection τ of type II

$$\Updownarrow$$

Unicellular maps of genus g with a distinguished triple (v_1, v_2, τ). [1]

Example of Genus 2 Case with n = 4

- $4 \cdot \epsilon_2(4) = \binom{4-1}{3}\epsilon_1(4) + \binom{4+1}{5} Cat(4)$



A Recurrence Relation for Unicellular Maps

Theorem

The number of $\epsilon_g(n)$ of rooted unicellular maps of genus g with n edges satisfies the following combinatorial identity:

$$2g \cdot \epsilon_g(n) = \binom{n+3-2g}{3}\epsilon_{g-1}(n) + \binom{n+5-2g}{5}\epsilon_{g-2}(n) + \ldots + \binom{n+1}{2g+1}\epsilon_0(n)$$

[1]

Applying the Gluing Operation to a Plane Tree

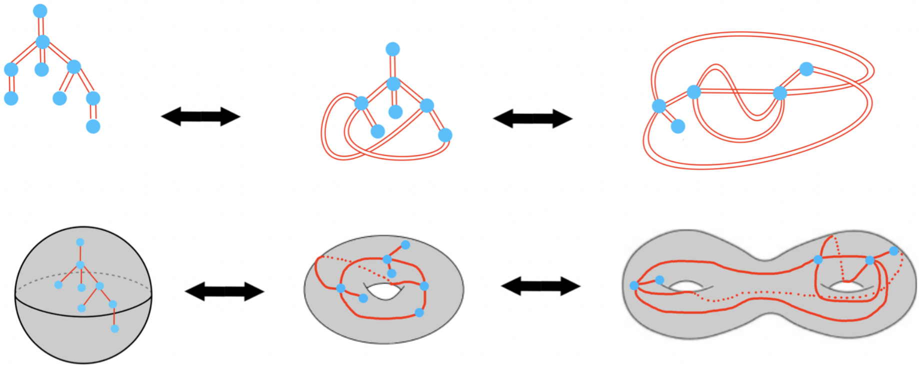


Figure: Applying the Gluing Operation to a Plane Tree

A Formula for the Number of Unicellular Maps

Corollary

The number of $\epsilon_g(n)$ of unicellular maps of genus g with n edges equals:

$$\epsilon_g(n) = R_g Cat(n)$$

where R_g is the polynomial of degree $3g$ defined by the formula:

$$R_g(n) = \sum_{0=g_0<g_1<\ldots<g_r=g} \prod_{i=1}^{r} \frac{1}{2g_i} \binom{n+1-2g_{i-1}}{2(g_i-g_{i-1})+1}$$

[1]

The End

Thank you!

References

Guillaume Chapuy. *A new combinatorial identity for unicellular maps, via a direct bijective approach*. 2018. arXiv: 1006.5053 [math.CO].

Jonathan L. Gross and Thomas W. Tucker. *Topics in Topological Graph Theory*. Ed. by Lowell W. Beineke and Robin J.Editors Wilson. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009. DOI: 10.1017/CBO9781139087223.