An Introduction to Integer Tiling Modulo p

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Abstract

This paper will present an overview of some of the recent advances in the study of integer tiling and Vuza canons. We will begin with an introduction to the subject and provide an overview of some of the recent results and approaches that have been used to study integer tiling and Vuza canons. Then, we will introduce and investigate integer tiling modulo p as well as present some empirical data and new conjectures about the relationship between different rhythmic patterns and the length of the tilings they form modulo p.

1 Introduction to Rhythmic Canons and Integer Tiling

A rhythmic canon is a musical canon where only the onsets of notes are considered. These canons where studied by the music theorist Dan Tudor Vuza between 1991-1993. Rhythmic canons can also be thought of as tilings of the integers and were also studied by the mathematician Gyorgy Hajos in 1949 while he was working on the factorization of Abelian groups. When working with rhythmic canons, beats are usually represented with the integers $\mathbb{Z}$ and the rhythmic pattern or motif of a canon by a subset of $\mathbb{N}$ that contains zero. We say that a motif or subset forms a rhythmic canon if there exists another set of beats on which the motif can enter such that every beat is played exactly once. This can be formalized as follows:

Definition 1. *The subset $A \subset \mathbb{N}$ tiles $\mathbb{Z}$ if there exists another subset $B \subset \mathbb{N}$ such that $A \oplus B = \mathbb{Z}$ where $\oplus$ is the direct sum of the sets A and B.*

If $A \oplus B = \mathbb{Z}$ we say that A tiles or forms a rhythmic canon with entry set B. If $A \oplus B = \mathbb{Z}$ we also say that (A, B) is a rhythmic canon.

Example 1. *Consider the rhythmic motif or subset $A = \{0, 3, 6\}$. This subset tiles $\mathbb{Z}$ with the set of onsets $B = \{1, 2\}$ as $A \oplus B = \{0, 3, 6\} \oplus \{1, 2\} = \mathbb{Z}_9$ and repeating this canon every 9 beats will tile the integers. A segment of this rhythmic canon is shown below in Figure 1.*

Example 2. *Observe that the rhythmic pattern or subset $A = \{0, 1, 3\}$ does not form a rhythmic canon as there is no set of entries B such that $A \oplus B = \mathbb{Z}$. In other words, given this motif there is no way to tile $\mathbb{Z}$ with translates of itself. The reason for this is that there is no way to cover the integer 2 without covering the integer 3 twice. This is shown in Figure 2 below.*

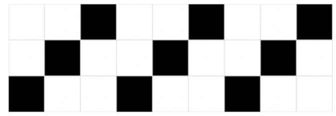

Figure 1: The rhythmic canon with motif $A = \{0, 3, 6\}$ tiles $\mathbb{Z}$

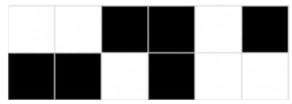

Figure 2: The subset $A = \{0, 1, 3\}$ does not tile $\mathbb{Z}$

An important result in the study of rhythmic canons is that the question of whether a certain subset of integers tiles the integers $\mathbb{Z}$ can always be reduced to the question of whether it tiles or paves a finite subset of the $\mathbb{Z}$ which is denoted by $\mathbb{Z}_n$. In particular, we have the following theorem from the mathematician Nicolaas Govert de Bruijn.

Theorem 1. *Given a rhythmic pattern or subset A and a set of entries B, then $A \oplus B = \mathbb{Z}_n$ if and only if $A \oplus B \oplus n\mathbb{Z} = \mathbb{Z}$*

Example 3. *Consider the rhythmic canon in example 1 with $A = \{0, 3, 6\}$ and $B = \{1, 2\}$. Since $A \oplus B = \{0, 3, 6\} \oplus \{1, 2\} = \mathbb{Z}_9$, then $A \oplus B \oplus 9\mathbb{Z} = \{0, 3, 6\} \oplus \{1, 2\} \oplus \{0, 9, 18, 27...\} = \mathbb{Z}$ and vice versa.*

1.1 Periodic and Non-periodic Canons

For different rhythmic patterns or subsets A, there are two important types of subsets B that occur. Usually, the elements in the set B form a periodic set. Consider the following definition from [Cau].

Definition 2. *A set $A = \{a_0, a_1, a_2, ...\} \subset \mathbb{N}$ is said to be periodic there exists some $k \in \mathbb{N}$ such that $a_n + k = a_{n+1}$ for all $n \in \mathbb{N}$.*

Example 4. *The set $\{0, 3, 6, 9, 12, 15, ...\}$ is periodic with $k = 3$.*

This leads us to the following definition:

Definition 3. *A rhythmic canon (A, B) is periodic if either set A or B are periodic sets.*

Example 5. *The rhythmic canon shown in Example 1 is periodic as the set A is periodic with $k = 3$.*

Definition 4. *A rhythmic canon with motif A is non-periodic or a Vuza canon if its corresponding set of entries B is not a periodic set.*

Vuza canons or non-periodic canons are particularly hard to find. The first one occurs when $n = 72$ and is shown in the example below.

Example 6. *Consider the motif* $A = \{0, 1, 5, 6, 12, 25, 29, 36, 42, 48, 49, 53\}$ *with the set of entries* $B = \{0, 8, 16, 18, 26, 34\}$. *This tiles* $\mathbb{Z}_{72}$ *and the set* B *is also non-periodic so the rhythmic canon* $A \oplus B = \mathbb{Z}_{72}$ *is a non-periodic or Vuza canon. While it appears that the integers* $\{2, 3, 4, 7, 9\}$ *are not covered in Figure* 3*, if the canon were to continue then every integer greater than* 72 *would be covered exactly once. Observe that the motif of the last voice is not shown entirely shown. Also, this canon is said to have length* 72 *because after* 72 *beats the first voice re-enters with the motif.*

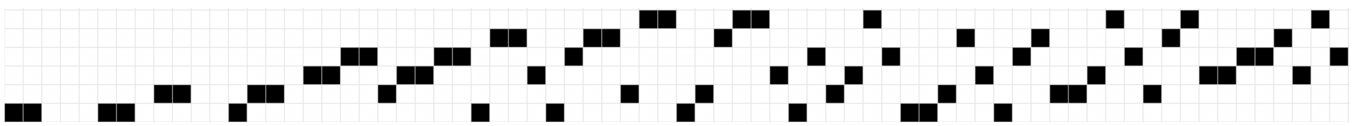

Figure 3: The first non-periodic or Vuza canon of Length 72

1.2 Polynomial Representation

The question of determining if a rhythmic pattern or subset A tiles $\mathbb{Z}$ is difficult. As a result, mathematicians and music theorists have used polynomials to represent rhythmic canons or integer tilings. In particular, we can represent a rhythmic pattern or subset A with the polynomial $A(X)$ as in the following example.

Example 7. *The subset* $A = \{0, 3, 5, 7, 11\}$ *corresponds to the polynomial* $A(X) = 1 + X^3 + X^5 + X^7 + X^{11}$.

Therefore, the question of whether a certain rhythmic pattern or subset A tiles $\mathbb{Z}$ can be rephrased as whether there exists a polynomial $B(X)$ such that $A(X)B(X) = 1 + X + X^2 + X^3 + X^4 + ...$ in $\mathbb{Z}[X]$. Now since theorem 1 tells us that we only need to see if our rhythmic pattern or subset A tiles a finite subset $\mathbb{Z}_n$, the problem now becomes finding a polynomial $B(X)$ and a $n \in \mathbb{N}$ such that $A(X)B(X) = 1 + ... + X^{n-1}$ in $\mathbb{Z}[X]/(X^n - 1)$. Here polynomial multiplication is done in the quotient ring $\mathbb{Z}[X]$ modulo the ideal generated by $(X^n - 1)$ which means that the relation $X^n = 1$ holds when multiplying the polynomials in this ring. Consider the following example:

Example 8. *In example 1, we saw that for* $A = \{0, 3, 6\}$ *and* $B = \{1, 2\}$ *that* $A \oplus B = \{0, 3, 6\} \oplus \{1, 2\} = \mathbb{Z}_9$. *Translating this into polynomials we get* $A(X) = 1 + X^3 + X^6$ *and* $B(X) = X + X^2$ *and that* $A(X)B(X) = (1 + X^3 + X^6)(X + X^2) = 1 + X + X^2 + X^3 + X^4 + X^5 + X^6 + X^7 + X^8$.

1.3 Operations on Rhythmic Canons and Integer Tilings

The importance of non-periodic or Vuza canons becomes clear when one begins to perform different operations on preexisting canons in order to create new canons. Two important operations on rhythmic canons are concatenation and duality. A concatenated rhythmic canon results from appending the motif of the canon onto itself some number of times. The dual canon results from interchanging the rhythmic pattern subset A and the set of entries B. Consider the following formal definitions and propositions from [Cau] and [Ami05].

Definition 5. *Let $k \in \mathbb{N}^*$, and A be a rhythmic pattern which tiles $\mathbb{Z}_N$. We note*

$$\overline{A}^k = A \oplus \{0, N, 2N, ..., (k-1)N\}$$

for its k-concatenation.

Example 9. *Consider the following rhythmic pattern or subset $A = \{0,1,2\}$ in $\mathbb{Z}_3$. Then, the first concatenation of the pattern would be $\overline{A}^1 = \{0,1,2,3,4,5\}$ in $\mathbb{Z}_6$ and the second concatenation would be $\overline{A}^2 = \{0,1,2,3,4,5,6,7,8\}$ in $\mathbb{Z}_9$.*

Proposition 1. *(Duality)*
(A,B) is a rhythmic canon of $\mathbb{Z}_N$ if and only if (B,A) is a rhythmic tiling canon of $\mathbb{Z}_N$.

Example 10. *Consider the rhythmic canon with motif set $A = \{0,1,2\}$ and entry set $B = \{0,3,6\}$ that tiles $\mathbb{Z}_9$. The duality proposition tells us that the entry set $A = \{0,3,6\}$ tiles $\mathbb{Z}_9$ with the entry set $B = \{0,1,2\}$ as shown in the figure below.*

Figure 4: A Rhythmic Canon

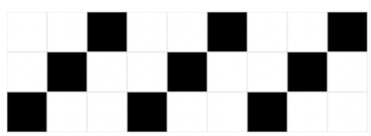

Figure 5: The Dual Canon

Proposition 2. *(Concatenation)*
Let $k \in \mathbb{N}^*$, (A,B) is a rhythmic canon of $\mathbb{Z}_N$ if and only if $(\overline{A}^k, B)$ is a rhythmic canons of $\mathbb{Z}_{kN}$

These propositions lead us to the following theorem:

Theorem 2 (Amiot (2005))**.** *Every rhythmic tiling canon can be deduced by concatenation and duality transformations from a Vuza canon and the trivial canon $(\{0\},\{0\})$.*

An example of how a rhythmic canon can be constructed from by concatenation and duality transformations from a Vuza canon and the trivial canon is given in [Ami05]. This theorem tells us that non-periodic or Vuza canons are similar to prime numbers, prime knots and the finite simple groups in that they are the basic building blocks of rhythmic canons. Therefore, like the search for all the prime numbers, the search for all Vuza or non-periodic canons is important to understanding rhythmic canons and integer tiling. Nonetheless, finding the number of Vuza canons that exist and what they look like has proven to be a hard question is currently an open problem. In spite of this, mathematicians and music theorists have found various algorithms for generating a large number of Vuza canons and have placed some conditions on their existence.

1.4 Existence Conditions for Vuza Canons

A well known result regarding Vuza canons is that they can only exist for certain sizes of N. Consider the following theorem.

Theorem 3. *There exists Vuza canons of $\mathbb{Z}_N$ for, and only for N not of the form:*

$$N = p^\alpha, N = p^\alpha q, N = p^2q^2, N = pqr, N = p^2qr, N = pqrs$$

with distinct primes.

Proof. A full proof is given in [San57] □

Franck Jedrzejewski in "On the Enumeration of Vuza Canons" also provides conditions and ways of generating Vuza canons. For more information on the enumeration of Vuza canons and various algorithms that have been used to generate them see [Jed13] and [KM09]. Mathematicians and music theorists have also found some existence conditions on the more general question of integer tilings.

1.5 Existence Conditions for Integer Tiling

One tool that mathematicians and music theorists have employed to place existence conditions on the question of integer tiling is a special polynomial called the cyclotomic polynomial. The cyclotomic polynomials are closely related to the roots of unity. Consider the following definitions and notations that Amiot introduces in [Ami05].

Definition 6. *Let ξ_n be the n^{th} roots of unity: $\xi_n = \{z \in \mathbb{C}, z^n = 1\}$.*

Definition 7. *The irreducible factors of $1+x+x^2+...+n^{n-1}$ are the cyclotomic polynomials Φ_d with $d > 1$ and $d|n$ (d divides n). These polynomials are in $\mathbb{Z}[x]$, are monic, and in $\mathbb{C}[x]$ we can factor*

$$\Phi_d(x) = \prod_{\xi}(x - \xi)$$

where ξ has exactly the multiplicative order d.

A list of the cyclotomic polynomials can be found on Wikipedia and further information about them can be found in any introductory book on abstract algebra. The following proposition from [Pop17a] illustrates an important fact about cyclotomic polynomials that has proven useful in studying integer tiling.

Proposition 3. *Any polynomial $A(X)$ in $\mathbb{Z}[X]$ can be factorized as $A(X) = \Phi(X)M(X)$ where $\Phi(X)$ is the product of cyclotomic polynomials, and $M(X)$ is an ordinary non-cyclotomic polynomials.*

This proposition illustrates why cyclotomic polynomials are important tools for studying rhythmic canons and integer tiling. In 1999, mathematicians Ethan M. Coven and Aaron Meyerowitz, in their work "Tiling the Integers with Translates of One Finite Set," used cyclotomic polynomials to derive the best series of conditions for integer tiling known to date. Before introducing their conditions, we introduce the following notation and definitions:

Definition 8. *For the polynomial $A(X)$ we define $R_A = \{d \in \mathbb{N}^*$ the d-th cyclotomic polynomial divides $A(X)\}$*

Definition 9. *For the polynomial $A(X)$ we define $S_A = \{p^a \in R_A,\ p$ prime, $a \in \mathbb{N}^*\}$*

Example 11. *Consider the rhythmic motif or subset $A = \{0, 1, 2, 6, 7, 8, 12, 13, 14\}$ from example 10 which corresponds to the polynomials $A(X) = 1 + X + X^2 + X^6 + X^7 + X^8 + X^{12} + X^{13} + X^{14}$. We can factor $A(X)$ as $A(X) = (X^2 + X + 1)(X^6 - X^3 + 1)(X^6 + X^3 + 1) = \Phi_3(X)\Phi_{18}(X)\Phi_9(X)$. Therefore $R_A = \{3, 9, 18\}$ and $S_A = \{3^1, 3^2\} = \{3, 9\}$.*

With this example in mind, we now consider the following conditions and theorem.

Definition 10. *Coven-Myerowitz conditions:*

1. (T_0) : *A tiles*
2. (T_1) : $A(1) = \prod_{p^\alpha \in S_A} p$
3. (T_2) : *If $p_1^\alpha, p_2^\beta, ..., p_r^\gamma \in S_A$, then $p_1^\alpha, p_2^\beta, ..., p_r^\gamma \in R_A$ with p_i distinct primes.*

Theorem 4. *(Coven and Myerowitz (1999))*
Based on the Coven-Myerowitz conditions the following implications hold:

1. $(T_0) \Rightarrow (T_1)$
2. (T_1) *and* $(T_2) \Rightarrow (T_0)$
3. *If #A has at most two prime factors, then $(T_0) \Rightarrow (T_1)$ and (T_2)*

Example 12. *We can apply the Coven-Myerowitz conditions and theorem to the rhythmic pattern or subset A from example 11. Recall that $A(X) = 1 + X + X^2 + X^6 + X^7 + X^8 + X^{12} + X^{13} + X^{14}$, $R_A = \{3, 9, 18\}$ and $S_A = \{3, 9\}$. Condition (T_1) reads as follows: $A(1) = 1 + 1 + 1^2 + 1^6 + 1^7 + 1^8 + 1^{12} + 1^{13} + 1^{14} = 3 * 3$. Observe that the right hand side of (T_1) is the product of the primes p of the $p^\alpha \in S_A$. Condition (T_2) reads as follows: If $3, 9 \in S_A$, then $3, 9 \in R_A$ which is also true. By theorem 4, we have that (T_0) holds or that the subset A tiles.*

Ultimately, the Coven-Myerowtiz theorem allows us to determine if some rhythmic patterns or subsets A tile, but if a rhythmic pattern doesn't satisfy both (T_1) and (T_2) then the Coven-Myerowitz conditions cannot tell us if the pattern tiles. In spite of this, mathematicians and music theorists have continued to explore the Coven and Myerowtiz conditions. In [Ami05], Amiot notes that for all known rhythmic patterns that tile (T_2) holds. The question of whether it is always true that A tiles implies (T_2) holds remains an open problem. Amiot also illustrates that any counterexample for the implication that pattern A that tiles implies that condition (T_2) is false. Recent work in 2021 by Izabella Łaba and Itay Londner have also made some progress on the implications of the Coven-Myerowitz conditions. In particular, they found the following:

Theorem 5. *(Łaba and Londner (2021))*
Let $M = p_i^2 p_j^2 p_k^2$, where p_i, p_j, p_k are distinct odd primes (that is distinct primes not equal to 2). Assume that $A \oplus B = \mathbb{Z}_M$ with $|A| = |B| = p_i p_j p_k$. Then, both A and B satisfy (T_2)

They have also obtained a classification of all tilings with $A \oplus B = \mathbb{Z}_M$ where $M = p_i^2 p_j^2 p_k^2$. For more details, we refer the reader to [LL21].

2 Integer Tiling Modulo p.

Since understanding integer tiling and finding Vuza canons has proven to be challenging, there has been some work done when the condition that there can only be one onset per integer or beat is relaxed. This approach was first introduced Emmanuel Amiot in his "Rhythmic canons and Galois theory" [Ami05] and was subsequently developed in Helianthe Caurea's dissertation "From covering to tiling modulus p (Modulus p Vuza canons: generalities and resolution of the case $\{0, 1, 2^k\}$ with $p = 2$)".[Cau]

2.1 Introduction to Integer Tiling Modulo p

Integer tiling modulo p is identical to integer tiling with exception that there must exactly 1 modulo p onsets on every beat. A formal definition using polynomials of integer tiling modulo p is given below.

Definition 11. *(A, B) is an integer tiling modulo p if and only if*

$$A(X) \cdot B(X) = 1 + X + ... + X^{N-1} \ mod \ (X^N - 1, p)$$

Here, the condition of *mod* $(X^n - 1, p)$ means that polynomial multiplication is done modulo the ideal generated by $X^n - 1$ and p. Essentially this means that we have the relations $X^n = 1$ and $p = 0$ when multiplying the polynomials. When p is prime, we have $\mathbb{Z}[X]/(X^N - 1, p) = \mathbb{F}_p[X]/(X^N - 1)$. Therefore we will use the notation of $\mathbb{F}_p[X]$ when discussing integer tiling modulo p. Consider the following example:

Example 13. *Consider the rhythmic canon with motif $A = \{0, 1, 3, 5, 6, 7, 9, 11\}$ and length 60 which is shown in figure 6 below. Observe how the number of onsets on each beat is always congruent to 1 modulo 3.*

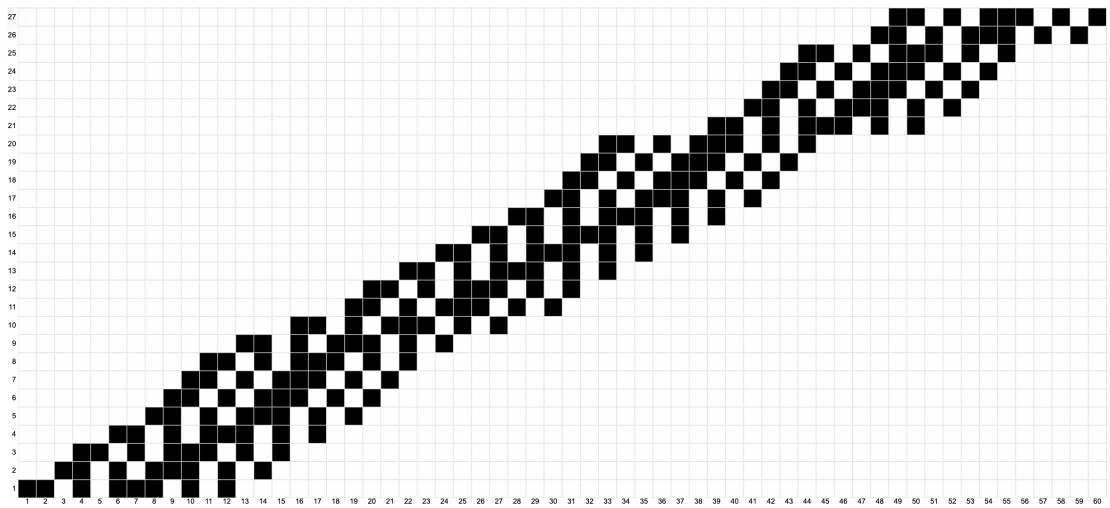

Figure 6: Rhythmic canon modulo 2 with motif $\{0, 1, 3, 5, 6, 7, 9, 11\}$

2.2 Basic Results in Integer Tiling Modulo p

Emmanuel Amiot has provided a few interesting results for the tiling modulo p case in his article [Ami05]. One result follows from a properties of polynomials that was discovered by the mathematician Évariste Galois.

Theorem 6. *(Galois)*
Any polynomial not vanishing in zero divides $x^n - 1$ in $\mathbb{F}_2[x]$, for some large enough n

Interpreted in the context of rhythmic canons modulo 2, this means there always exists an entry set B for some large enough n which implies the following.

Theorem 7. *(Amiot (2005))*
Any rhythmic pattern or subset A tiles modulo 2.

Amiot also generalizes this result to obtain the following important result.

Theorem 8. *(Amiot (2005))*
Any rhythmic pattern or subset A tiles in $\mathbb{F}_p$, for any prime p.

Amiot notes how theorem 8 follows from a clever lemma involving polynomials. For more details, we refer the reader to [Ami05].

Example 14. *The rhythmic pattern* $\{0, 1, 3, 5, 6, 7, 9, 11\}$ *from example* 13 *tiles modulo* p *for every prime* p.

Unlike the case in $\mathbb{Z}[X]$, there are no conditions similar to the Coven-Myerowitz conditions in $\mathbb{F}_p[X]$. It is important to note that it is not known if there is a connection between tiling modulo p and the general question of integer tiling. Nonetheless, integer tiling modulo p is a fascinating phenomenon which for the most part has been largely unexplored.

2.3 Caurea's Algorithm for Integer Tiling modulo p

Helianthe Caurea in her dissertation "From covering to tiling modulus p (Modulus p Vuza canons: generalities and resolution of the case $\{0, 1, 2^k\}$ with $p = 2$.)" found some interesting results for integer tiling modulo p. [Cau] In particular, Caurea presents an efficient algorithm for constructing rhythmic canons modulo p of minimal length. The essence of the algorithm was summarized by Alexandre Popoff in his blog post "Rhythmic canons modulus p".[Pop16a] The essence of the algorithm is as follows:

1. Place the rhythmic pattern or subset A on beat 0.
2. Locate the first beat that does not have exactly 1 onset modulo p.
3. Place the start of the rhythmic pattern on this beat in a new voice.
4. Review the resulting canon. If every beat has exactly one onset modulo p, then the canon is complete. If not, repeat steps $2 - 4$.

Popoff notes how Caure's algorithm is optimal in that it produces set of entries B with smallest cardinality and therefore produces the rhythmic canon with minimal length for the rhythmic pattern A. In addition to finding this algorithm, Caure's work also resolved the case of tiling modulo 2 for the motif $\{0, 1, 2^k\}$. In particular, she found the following result.[Cau]

Theorem 9. *(Caure (2015))*
For all $k \in \mathbb{N}^*$, *the rhythmic pattern* $(A_k, B_k) = (\{0, 1, 2^k, B_k)$ *is the smallest compact rhythmic canon modulo* 2 *of* $\mathbb{Z}_N$ *and is of size* $N = 4^k - 1$ *with* $B_k = 4^k - 3^k$ *entries, and* $D_{(A_k,D_k),2} = 4^k - \frac{3^{k+1}-1}{2}$ *onsets.*

2.3.1 Length of Rhythmic Canons modulo 2 and Some Open Conjectures

Alexandre Popoff has extended Caurea's work in the case of tiling modulo 2 and has posed the following open conjectures.[Pop17b]

Conjecture 10. *In* $\mathbb{F}_p[X]$ *where* p *is prime (except for* $p = 3$*), (non-compact) canons of motive* $A = 1 + X + X^{p^k}$ *have length* $L = p^{2^k} - 1$.

Conjecture 11. *In $\mathbb{F}_2[X]$, canons of motive $A = 1 + X + X^{2^{2k}+2^k+1}$ have length of $L = \sum_{i=0}^{6} 2^{ki}$.*

Conjecture 12. *In $\mathbb{F}_2[X]$, canons of motif $A = 1 + X + X^{\sum_{j=0}^{3} 2^{jk}}$ have length $L = \sum_{i=0}^{14} 2^{ki}$*

Conjecture 13. *In $\mathbb{F}_2[X]$, canons of motive $A = 1 + X + X^{\sum_{j=0}^{4} 2^{jk}}$ have length $L = \sum_{i=0}^{20} 2^{ki}$.*

Conjecture 14. *In $\mathbb{F}_2[X]$, canons of motive $A = 1 + X + X^{\sum_{j=0}^{5} 2^{jk}}$ have length $L = \sum_{i=0}^{62} 2^{ki}$*

The final sections of this paper will build upon some recent work done by Alexandre Popoff and Helianthe Caurea by providing further empirical results and conjectures for other motifs in the case of tiling modulo p.

3 Further Investigations of Integer Tiling modulo p

Caurea's work investigated the case of tiling modulo 2 with motif $\{0, 1, 2^k\}$ and Alexandre Popoff has presented conjectures for the lengths of rhythmic canons with similar motifs in modulo 2. We will look at tiling modulo p for the cases where $p > 2$ and for different motifs and the tilings they form in different moduli.

3.1 Relationship between Length and Tiling modulus p

In this section we will consider the relationship between the length of a canon the tiling modulus for a particular rhythmic pattern. The same rhythmic pattern can form two very different rhythmic canons modulo different p. Consider the following example.

Example 15. *The rhythmic pattern $A = \{0, 1, 2, 4, 5, 7, 8\}$ and the rhythmic canons it forms modulo 2 and 3 are shown in figures 7 and 8 below. Observe how a larger number of voices enter in the modulo 3 canon than in the modulo 2 canon.*

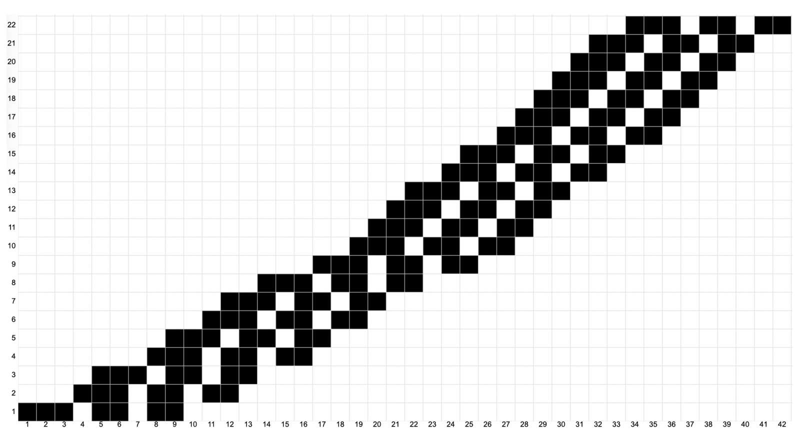

Figure 7: Rhythmic canon (mod 2) with motif $\{0, 1, 2, 4, 5, 7, 8\}$

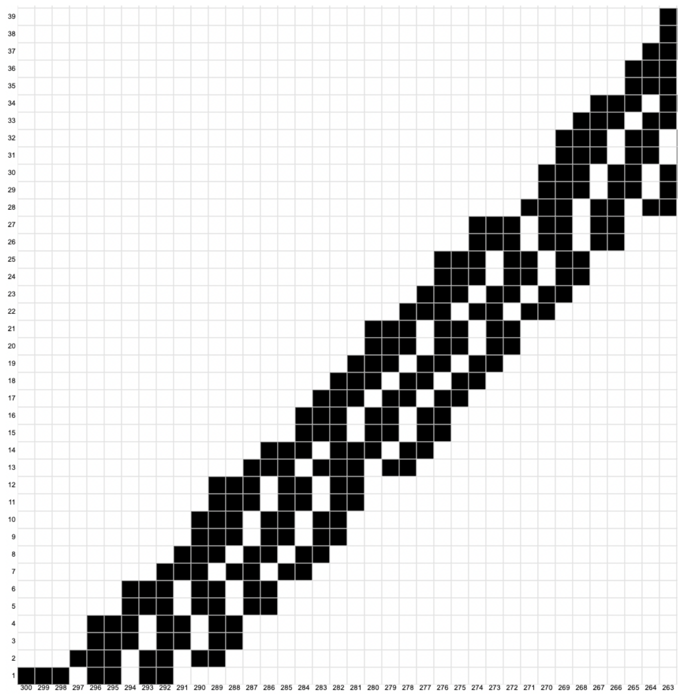

Figure 8: Rhythmic canon (mod 3) with motif $\{0, 1, 2, 4, 5, 7, 8\}$

3.2 Integer tilings modulo 3

In this section, we will investigate the relationship between different motifs and the length of the canons they form modulo 3. The empirical results below were obtained by modifying a program written in C that Alexandre Popoff used to compute the length of rhythmic canons with motif $\{0, 1, N\}$ in modulus 2.[Pop16b] Consider the graphs of the rhythmic canons with motifs $A = \{0, 1, k\}$ and $A = \{0, 1, 2, k\}$ for the relationship between the length of the canon with values of $k \in \mathbb{N}^*$ shown below in figures 9 and 10. From these empirical results, we present the following conjectures:

Conjecture 15. *In $\mathbb{F}_3[X]$, canons of motif $A(X) = 1 + X + X^k$ have length $3n$ for some $n \in \mathbb{N}^*$.*

Conjecture 16. *In $\mathbb{F}_3[X]$, canons of motif $A(X) = 1 + X + X^2 + X^3 + X^4 + X^k$ have length $6n$ for some $n \in \mathbb{N}^*$.*

Conjecture 17. *In $\mathbb{F}_3[X]$, canons of motif $A(X) = 1 + X + ... + X^{N-1} + X^N + X^k$ where N is even have length $(N + 2)n$ for some $n \in \mathbb{N}^*$.*

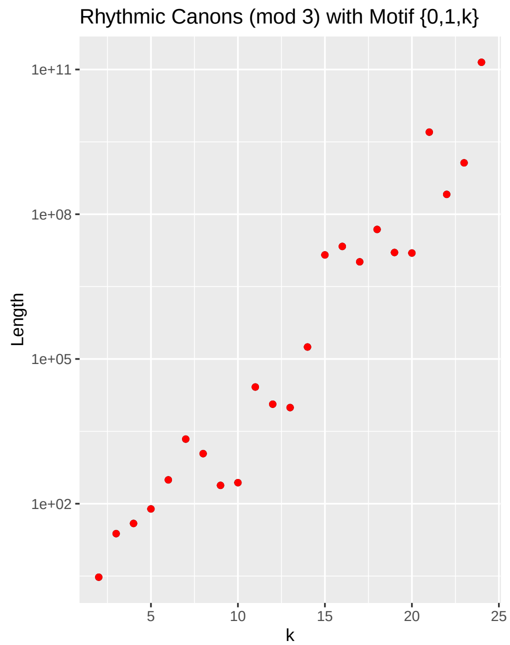

Figure 9: Length of Canons with Motif $\{0, 1, k\}$ modulo 3

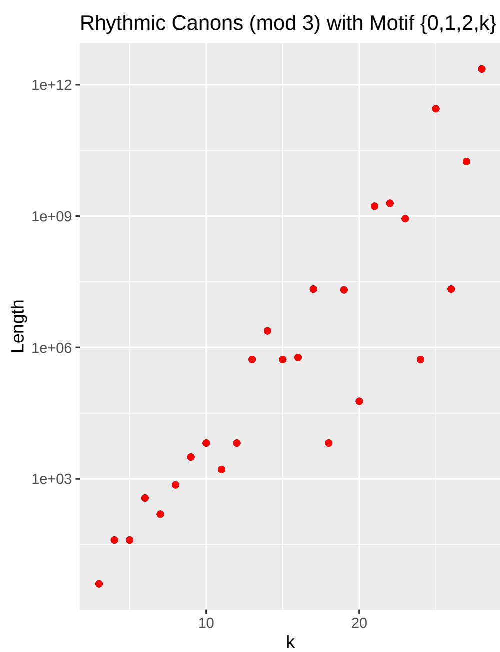

Figure 10: Length of Canons with Motif $\{0, 1, 2, k\}$ modulo 3

4 Future Directions

The relationship between different rhythmic patterns and the lengths of rhythmic canons that they form modulo p is not well understood and it is still unknown if they relate to integer tiling. One future direction of research would be to continue to explore the relationship between different rhythmic patterns and the length of the canons that they form modulo p. Another possible direction of research could investigate rhythmic canons with maximally even motifs in one voice and the lengths of the resulting canons they form modulo p. While there has been some previous work done on maximally even tiling by Jeremiah D. Kastine's dissertation in his dissertation "Maximally Even Tilings: Theory and Algorithms", Kastine looked at maximally even partial tilings which relax the condition that every beat has an onset. For more information see [Kas19]. Further investigation could also look at whether there is a relationship between the length of tilings modulo p and Euler's totient function which gives the number of relatively prime numbers to an natural number or the Carmichael function which gives the smallest positive integer m for which $a^m \equiv 1 \ (mod\ n)$ for every integer a between 1 and n which is coprime to n. It would also be interesting to look at whether certain classes of prime numbers such as the Wieferich primes, which are primes that are solutions to the equation $2^{p-1} \equiv 1 \ (mod\ p^2)$, appear in the length of certain tilings. Overall, integer tiling, Vuza canons and integer tiling modulo p are interesting phenomena which remain not well understood.

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