POS4715-0001. Politics and the Theory of Games

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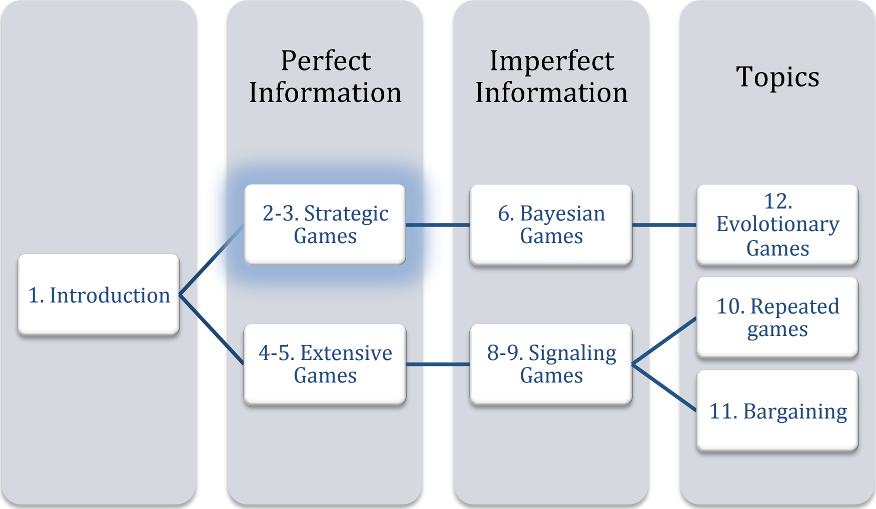
Lecture 1

Introduction

Course outline



Introduction

- Game theory is used widely in social and behavioral sciences.
- This course presents the main concepts of game theory and shows how these can be used to understand political, economic, social, and biological phenomena.
- The only way to appreciate the theory is to see it in action, or better still to put it into action. Hence, the course includes a variety of illustrations and exercises (focusing on political science).

What is game theory?

- A tool to analyze rational players' strategic interactions.
- Examples
  - Political candidates competing for votes;
  - Jury members deciding on a verdict;
  - Legislators' voting behavior under pressure from interest groups;
  - Firms competing for business;
  - The role of threats and punishment in long-term relationships;
  - Animals fighting for prey.
- Conciseness and precision: mathematical symbols

Theory of rational choice

- A component of many models in game theory.
- Brief description
  - A decision-maker (DM) chooses the best action among all the actions available to her according to her preferences.
  - No qualitative restriction is placed on the decision-maker's preferences; her "rationality" lies in the consistency of her decisions when faced with different sets of available actions, not in the nature of her likes and dislikes.
  - Components: 1) actions and 2) preferences and payoff functions.

Actions

- A set **A** consisting of all the actions that are available to DM.
- In any given situation, DM is faced with a subset of **A**, from which she must choose a single element.
- DM knows this subset of available choices and takes it as given.
- Example
  - **A** could be the set of bundles of goods that DM can possibly consume; given her income at any time, she is restricted to choose from the subset of **A** containing the bundles she can afford.

Preferences and payoff functions

- Assumptions
  - DM, when presented with any pair of actions, knows which of the pair she prefers, or knows that she regards both actions as equally desirable.
  - Preferences are consistent in the sense that if DM prefers the action a to the action b, and the action b to the action c, then she prefers the action a to the action c.
- No other restriction is imposed on preferences
  - In particular, we allow a person's preferences to be altruistic in the sense that how much she likes an outcome depends on some other person's welfare.

Description of preferences

- One way is to specify the action DM prefers for each possible pair of actions, or to note that DM is indifferent between the actions.
- Alternatively, we can "represent" the preferences by a payoff function (or "utility function"), which associates a number with each action in such a way that actions with higher numbers are preferred.
- The payoff function u represents a DM's preferences if, for any actions a and b in $\mathbf{A}$,

$$u(a) > u(b) \text{ if and only if the decision-maker prefers } a \text{ to } b.$$

Example 5.2: Payoff function representing preferences

A person is faced with the choice of three vacation packages, to Havana, Paris, and Venice. She prefers the package to Havana to the other two, which she regards as equivalent.

- Action set $\mathbf{A} = \{\text{Havana } (h), \text{Paris } (p), \text{Venice } (v)\}$
- We can compare every possible pair of actions: hPp, hPv, pIv
  - aPb: DM prefers a to b
  - aIb: DM is indifferent between a and b
- Or, we can define a payoff function, u, representing her preferences. For example,
  - $u(h) = 1$ and $u(p) = u(v) = 0$; or
  - $u(h) = 10$ and $u(p) = u(v) = -1$; or
  - $u(h) = -3$ and $u(p) = u(v) = -5$.

Exercise 5.3: Altruistic preferences

Person 1 cares about both her income and person 2's income. Precisely, the value she attaches to each unit of her own income is the same as the value she attaches to any two units of person 2's income. For example, she is indifferent between a situation in which her income is 1 and person 2's is 0, and one in which her income is 0 and person 2's is 2. How do her preferences order the outcomes (1,4), (2,1), and (3,0), where the first component in each case is her income and the second component is person 2's income? Give a payoff function consistent with these preferences.

Ordinal preferences

- A DM's preferences, in the sense used here, convey only ordinal information.
- They may tell us that the DM prefers the action a to the action b to the action c, for example, but they do not tell us "how much" she prefers a to b, or whether she prefers a to b "more" than she prefers b to c.
- It may be tempting to think that the payoff numbers attached to actions by a payoff convey intensity of preferences ? that if, for example, a decision-maker?s preferences are represented by a payoff function u for which $u(a) = 0, u(b) = 1$, and $u(c) = 100$, then the decision-maker likes c a lot more than b but finds little difference between a and b.
- A payoff function contains no such information! It conveys only ordinal information.

Alternative representations of preferences

- A DM's preferences are represented by many different payoff functions.
- If u represents a DM's preferences and v is another payoff function for which

$$v(a) > v(b) \text{ if and only if } u(a) > u(b),$$

then v also represents the DM's preferences.

Exercise 6.1: Alternative representation of preferences

A DM's preferences over the set $A = \{a, b, c\}$ are represented by the payoff function u for which $u(a) = 0, u(b) = 1$, and $u(c) = 4$.

- Are they also represented by the function v for which $v(a) = -1, v(b) = 0$, and $v(c) = 2$?
- How about the function w for which $w(a) = w(b) = 0$ and $w(c) = 8$?

The theory of rational choice

- Allowing for the possibility that there are several equally attractive best actions, the theory of rational choice states:

  the action chosen by a decision-maker is at least as good, according to her preferences, as every other available action.
- For any action, we can design preferences with the property that no other action is preferred.
- Thus, if we have no information about a DM's preferences and make no assumptions about their character, any *single* action is consistent with the theory.

- However, if we assume that a DM who is indifferent between two actions sometimes chooses one action and sometimes the other, not every collection of choices for different sets of available actions is consistent with the theory.
- Suppose, for example, we observe that a DM chooses a whenever she faces the set $\{a, b\}$, but sometimes chooses b when facing the set $\{a, b, c\}$.
- The fact that she chooses a whenever faced with $\{a, b\}$ means that she prefers a to b (if she were indifferent, she would sometimes choose b).
- But then when she faces the set $\{a, b, c\}$, she must choose either a or c, never b.
- Thus, her choices are inconsistent with the theory.

Exercises

- Facing the set $\{a, b, c\}$, a DM always chooses a. Consistent?
- Facing the set $\{a, b, c\}$, a DM always chooses b. Consistent?
- Facing the set $\{a, b, c\}$, a DM sometimes chooses a and sometimes b. Consistent?
- Facing the set $\{a, b, c\}$, a DM sometimes chooses a, sometimes b and sometimes c. Consistent?
- Facing the set $\{a, b, c\}$, a DM always chooses b, and facing the set $\{a, b\}$, the DM always chooses a. Consistent?

Interacting decision-makers: Game theory

- So far, the decision-maker chooses an action from a set $\mathbf{A}$ and cares only about this action.
- A decision-maker in the world often does not have the luxury of controlling all the variables that affect her. If some of the variables that affect her are the actions of other decision-makers, her decision-making problem is altogether more challenging than that of an isolated decision-maker.
- The study of such situations, which we model as games, is the topic of this course.
- Example: Goldenball

Summary

- Game theory is a tool to analyze rational players' strategic interactions.
- The theory of rational choice is a component of many game-theoretic models.
- We assume that the DM knows the set of actions available to her and her preferences over the set of actions are consistent.
- Preferences can be represented by a payoff function (or utility functions).
- A payoff function conveys only ordinal information. So, there are many utility functions representing a DM's preferences.
- Sometimes, other decision-makers' actions affect a decision-maker. We model such situations as games.

Lecture 2

Nash Equilibrium: Theory I

Strategic games

A strategic game is a model of interactive decision-making in which each decision-maker chooses his plan of action once and for all, and these choices are made simultaneously. It is often called a simultaneous game or a normal form game.

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A strategic game is a model of interactive decision-making in which each decision-maker chooses his plan of action once and for all, and these choices are made simultaneously. It is often called a simultaneous game or a normal form game.

A **strategic game** consists of

- a set of players
- a set of actions for each player
- preferences over the set of action profiles for each player.

Example: Golden ball

- Players
- Each player's set of actions
- The set of action profiles
- Each player's preferences over the set of action profiles

Example: Golden ball

Example: Golden ball

		Abraham (Player 2)	
		Split	Steal
Nick (Player 1)	Split	$(\frac{1}{2}, \frac{1}{2})$	$(0, 1)$
	Steal	$(1, 0)$	$(0, 0)$

Example: the Prisoner's Dilemma

Two suspects in a crime are put into separate cells. Each suspect has two choices: confess or not. If both confess, each will be sentenced to three years in prison. If only one of them confesses, he will be freed and the other will receive a sentence of four years. If neither confesses, they will both be convicted of a minor offense and spend one year in prison.

Prisoner's dilemma

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- Players: two suspects, $N = \{1, 2\}$
- Action set: $A_1 = A_2 = \{\text{Don't Confess, Confess}\}$
- Action profiles (a_1, a_2): (D,D), (D,C), (C,D), (C,C)
- Preferences over the set of action profiles
  – Player 1: (C,D)$\succ$(D,D)$\succ$(C,C)$\succ$(D,C)
  – Player 2: (D,C)$\succ$(D,D)$\succ$(C,C)$\succ$(C,D)

Prisoner's dilemma

- Preferences over the action profile
  – Player 1: (C,D)$\succ$(D,D)$\succ$(C,C)$\succ$(D,C)
  – Player 2: (D,C)$\succ$(D,D)$\succ$(C,C)$\succ$(C,D)

		Player 2	
		Don't Confess	Confess
Player 1	Don't Confess	(-1,-1)	(-4,0)
	Confess	(0,-4)	(-3,-3)

Example: the Prisoner's Dilemma

Player 2

		Don't Confess	Confess
Player 1	Don't Confess	(3,3)	(0,4)
	Confess	(4,0)	(1,1)

The Prisoner's Dilemma

The Prisoner's Dilemma models a situation in which there are gains from cooperation (each player prefers that both players choose *Don't Confess* rather than both choosing *Confess*), but each player has an incentive to free ride (choose *Confess*) whatever the other player does. The game is important not because we are interested in understanding the incentives for prisoners to confess, but because many other situations have similar structures.

Example: the Arms Race

Suppose that two countries, A and B, are involved in a nuclear arms race. Each country can build an arsenal of nuclear bombs, or can refrain from doing so. Assume also that each country's favorite outcome is that it has bombs and the other country does not; the next best outcome is that neither country has any bombs; the next best outcome is that both countries have bombs (what matters is relative strength, and bombs are costly to build); and the worst outcome is that only the other country has bombs.

Bach or Stravinsky

Two people wish to go out together to a concert of music by either Bach or Stravinsky. They want to go together (going to either concert alone is equally bad for both players), but one person prefers Bach and the other person prefers Stravinsky.

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- Players: two people, $N = \{1, 2\}$
- Action set: $A_1 = A_2 = \{\text{Bach, Stravinsky}\}$
- Action profiles (a_1, a_2): (B,B), (B,S), (S,B), (S,S)
- Preferences over the action profile
  – Player 1: (B,B)$\succ$(S,S)$\succ$(B,S)$\sim$(S,B)
  – Player 2: (S,S)$\succ$(B,B)$\succ$(B,S)$\sim$(S,B)

Bach or Stravinsky

- Preferences over the action profile
  – Player 1: (B,B)$\succ$(S,S)$\succ$(B,S)$\sim$(S,B)
  – Player 2: (S,S)$\succ$(B,B)$\succ$(B,S)$\sim$(S,B)

Player 2

		Bach	Stravinsky
Player 1	Bach	(2,1)	(0,0)
	Stravinsky	(0,0)	(1,2)

Lecture 3

Matching Pennies

Each of two people chooses either Head or Tail. If the choices differ, person 1 pays person 2 a dollar; if they are the same, person 2 pays person 1 a dollar. Each person cares only about the money that he receives.

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- Players: two people, $N = \{1, 2\}$
- Action set: $A_1 = A_2 = \{\text{Head, Tail}\}$
- Action profiles (a_1, a_2): (H,H), (H,T), (T,H), (T,T)
- Preferences over the action profile
  – Player 1: (H,H)$\sim$(T,T)$\succ$(H,T)$\sim$(T,H)
  – Player 2: (H,T)$\sim$(T,H)$\succ$(H,H)$\sim$(T,T)

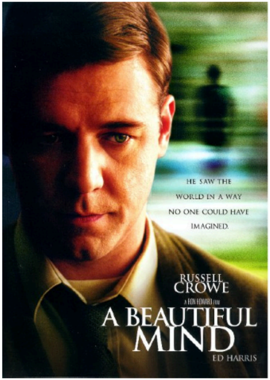
Matching Pennies

- Preferences over the action profile
  - Player 1: (H,H)$\sim$(T,T)$\succ$(H,T)$\sim$(T,H)
  - Player 2: (H,T)$\sim$(T,H)$\succ$(H,H)$\sim$(T,T)

		Player 2	
		Head	Tail
Player 1	Head	(1,-1)	(-1,1)
	Tail	(-1,1)	(1,-1)

Nash equilibrium: Theory II

Nash equilibrium



Nash equilibrium: theory

What actions will be chosen by the players in a strategic game?

- As per rational choice theory, we assume that each player chooses the best available action.
- In a game, the best available action for any given player depends, in general, on the other players' actions.
- Hence, when choosing an action a player must have in mind the actions the other player will choose. That is, she must form a belief about the other players' actions.
- Since no actions have happened already, she must form beliefs about all possible actions of other players.

Nash Equilibrium

The solution concept we study to answer the question what players will choose is Nash equilibrium.

A ***Nash equilibrium*** is an action profile where each player chooses optimally (the best action), given the choices of the other players.

Nash Equilibrium

The action profile a^* in a strategic game with ordinal preferences is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j. Equivalently, for every player i,

$$u_i(a^*) \geq u_i(a_i, a^*_{-i}) \text{ for every action } a_i \text{ of player } i,$$

where u_i is a payoff function that represents player i's preferences.

Interpretation of a Nash equilibrium

Consider a Nash equilibrium as

- a steady state, in which no player wants to change her action because she would not get a higher payoff (given the other players do also not change their actions); or
- a social norm: if everyone else adheres to it, no individual wishes to deviate from it.

Find Nash Equilibrium

- Nash equilibrium is an action profile and not payoffs.
- Check every action profile.
- Examples
  - Prisoner's dilemma
  - Golden ball
  - Matching pennies
  - BoS

Find Nash Equilibrium: an easier way

Use 2 by 2 matrices

Lecture 4

Example: Hawk-Dove, or Chicken Game

Two animals are fighting over some territory. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome in which its opponent is passive to that in which its opponent is aggressive. Formulate the situation as a strategic game and find its Nash equilibria (in pure strategies).

the Stag Hunt

A sentence in *Discourse on the origin and foundations of inequality among men* (1755) by the philosopher Jean-Jacques Rousseau discusses a group of hunters who want to catch a stag:

- They will succeed if they all remain sufficiently attentive, but each is tempted to desert her post and catch a hare.
- Each of a group of hunters has two options: she may remain attentive to the pursuit of the stag, or she may catch a hare.
- If all hunters pursue the stag, they catch it and share it equally.
- If any hunter devotes her energy to catching a hare, the stag escapes, and the hare belongs to the defecting hunter alone (hence, those who made efforts to hunt the stag are empty- handed).
- Each hunter prefers a share of the stag to a hare, and a hare is better than having no food at all.

the Stag Hunt

- Formulate the game and find NE with two hunters
- Formulate the game and find NE with many hunters

A Pure coordination game

You and your friend spend a few hours in New York City. Unfortunately, you were separated and neither of you took his mobile phone. Your plans were to hang out at 'Times Square', 'Union Square', or 'Washington Square', which are about equally far away from where you were separated. But you hadn't made a final decision yet! Both of you prefer to meet each other again to spending time in NYC alone before heading back home. Both of you are indifferent between hanging out on Times Square, Union Square, or Washington Square.

Strict and Weak Nash equilibrium

- Strict Nash equilibria: a deviation from the equilibrium by a player leads to an outcome worse for that player than the equilibrium outcome (PD, BoS, Stag Hunt)
- The definition of Nash equilibrium, however, requires only that the outcome of a deviation be no better for the deviant than the equilibrium outcome.
- And, indeed, some games have Nash equilibria in which a player is indifferent between her equilibrium action and some other action, given the other players' actions. These equilibria are called weak Nash equilibria (Golden ball).

Example of a weak Nash equilibrium

		Player 2		
		L	M	R
Player 1	T	1,1	1,0	0,1
	B	1,0	0,1	1,0

- What is the NE of the game?

Example of a weak Nash equilibrium

		Player 2		
		L	M	R
Player 1	T	1,1	1,0	0,1
	B	1,0	0,1	1,0

- What is the NE of the game? (T,L)

Example of a weak Nash equilibrium

		Player 2		
		L	M	R
Player 1	T	1,1	1,0	0,1
	B	1,0	0,1	1,0

- What is the NE of the game? (T,L)
- Why is it a weak NE?
  - For player 1
  - For player 2

Best Responses

- A Nash equilibrium is an action profile with the property that no player can do better by changing her action, given the other players' actions. Alternatively, we can define a Nash equilibrium to be an action profile for which every player's action is a best response to the other player's action.
  - We can find the Nash equilibria of a game in which each player has only a few actions by examining each action profile in turn to see if it satisfies the conditions for equilibrium. In more complicated games, it is often better to work with the players' "best response functions".
  - Recall that we determined the Nash equilibria in both ways for the payoff matrix of the Prisoner's Dilemma.

Best response functions

- We denote the set of player i's best actions when the list of the other players' actions is a_{-i} by $B_i(a_{-i})$.
- Precisely, we define the function B_i by,

$$B_i(a_{-i}) = \{a_i \in A_i | u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_i' \in A_i\}.$$

- Any action in $B_i(a_{-i})$ is at least as good for player i as every other action of player i when the other players' actions are given by a_{-i}.
- We call B_i the best response function of player i.

Best response functions

- The function B_i is set-valued: it associates a set of actions with any list of the other players' actions.
- Every member of the set $B_i(a_{-i})$ is a best response of player i to a_{-i}: if each of the other players adheres to a_{-i}, then player i can do no better than choose a member of $B_i(a_{-i})$.
- In some games, like *BoS*, the set of $B_i(a_{-i})$ consists of a single action for every list a_{-i} of actions of the other players: no matter what the other players do, player i has a single optimal action.
- In other games, like the Goldenball, $B_i(a_{-i})$ contains more than one action for some lists a_{-i} of actions of the other players.

Nash equilibrium in terms of best responses

The action profile a^* is a Nash equilibrium of a strategic game with ordinal preferences if and only if every player's action is a best response to the other players' actions:

$$a_i^* \in B_i(a_{-i}^*) \text{ for every player } i.$$

Example: Best responses in BoS

- Draw the payoff matrix.
- Find $B_1(a_2 = \text{Bach}), B_1(a_2 = \text{Stravinsky})$
- Find $B_2(a_1 = \text{Bach}), B_2(a_1 = \text{Stravinsky})$

Example: BR in a game with no strict equilibrium

		Player 2		
		L	M	R
Player 1	T	1,1	1,0	0,1
	B	1,0	0,1	1,0

- $B_1(a_2 = L) = \{T, B\}$, $B_1(a_2 = M) = \{T\}$, and $B_1(a_2 = R) = \{B\}$
- $B_2(a_1 = T) = \{L, R\}$ and $B_2(a_1 = B) = \{M\}$

Example: BR in a game with no strict equilibrium

		Player 2		
		L	M	R
Player 1	T	1,1	1,0	0,1
	B	1,0	0,1	1,0

- $B_1(a_2 = L) = \{T, B\}$, $B_1(a_2 = M) = \{T\}$, and $B_1(a_2 = R) = \{B\}$
- $B_2(a_1 = T) = \{L, R\}$ and $B_2(a_1 = B) = \{M\}$
- $B_i(a_{-i})$ consists of more than a single action for some actions of other players.

Example: BR in a game with no strict equilibrium

		Player 2		
		L	M	R
Player 1	T	1,1	1,0	0,1
	B	1,0	0,1	1,0

- $B_1(a_2 = L) = \{T, B\}$, $B_1(a_2 = M) = \{T\}$, and $B_1(a_2 = R) = \{B\}$
- $B_2(a_1 = T) = \{L, R\}$ and $B_2(a_1 = B) = \{M\}$
- $B_i(a_{-i})$ consists of more than a single action for some actions of other players.
- (T, L) is a NE: $a_1 = T \in B_1(a_2 = L)$ and $a_2 = L \in B_2(a_1 = T)$
- (T, M) is not a NE: $a_1 = T \in B_1(a_2 = M)$ but $a_2 = M \notin B_2(a_1 = T)$

Dominated Actions: Strict domination

- A player's action "strictly dominates" another action if it is superior, *no matter what the other players do*.
- Definition (Strict Domination): In a strategic game with ordinal preferences, player i's action a_i'' strictly dominates her action a_i' if

  $$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

  where u_i is the payoff function that represents player i's preferences. The action a_i' is strictly dominated.
- A strictly dominated action is not a best response to any actions of the other players.
- A strictly dominated action is not used in any Nash equilibrium.

Example: Strict domination

- Is there a strictly dominated action in the prisoner's dilemma?

Example: Strict domination

- Is there a strictly dominated action in the prisoner's dilemma?
- *Confess* strictly dominates *Don't confess*. Equivalently, *Don't Confess* is strictly dominated by *Confess*.
- Regardless of her opponent's action, a player prefers the outcome when she chooses *Confess* to the outcome when she chooses *Don't confess*.
- In the prisoner's dilemma, we can find the unique Nash equilibrium by iterated elimination of strictly dominated actions.

Example: elimination of strictly dominated actions

	L	R
T	1,1	0,0
M	2,1	1,0
B	3,0	2,1

Example: elimination of strictly dominated actions

	L	R
T	1,1	0,0
M	2,1	1,0
B	3,0	2,1

- Player 1's action M strictly dominates T.
- Player 1's action B strictly dominates both T and M.
- Eliminating player 1's actions T and M, player 2 prefers R to L.
- (B,R) is the Nash equilibrium.

Equilibrium Refinements

- Often we have several equilibria. How do we choose which one is most likely?
- An *equilibrium refinement* is a method by which one or a few equilibria are selected from all of the possibilities.
- There are many types of refinements, but the most basic is the iterated removal of weakly dominated strategies (for now, for us, actions).

Dominated Actions: Weak domination

- A player's action "weakly dominates" another action if the first action is at least as good as the second action, no matter what the other players do, and is better than the second action for some actions of the other players.
- Definition (Weak Domination): In a strategic game with ordinal preferences, player i's action a_i'' weakly dominates her action a_i' if

  $$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every } a_{-i} \text{ of the other players' actions,}$$

  and

  $$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \text{ of the other players' actions,}$$

  where u_i is the payoff function that represents player i's preferences. The action a_i' is weakly dominated.

Dominated Actions

- No player's eq. action is weakly dominated in a strict NE.
- An eq. action can be weakly dominated in a nonstrict NE.
- Example

	B	C
B	1,1	0,0
C	0,0	0,0

- Which action is weakly dominated?
- What are the Nash equilibria of the game?
- Which of the NE survives after eliminating weakly dominated actions?

Example: Elimination of weakly dominated actions

The Golden ball

	Split	Steal
Split	1,1	0,2
Steal	2,0	0,0

- Is there a strictly dominated action?
- Is there a weakly dominated action?

Example: Elimination of weakly dominated actions

The Golden ball

	Split	Steal
Split	1,1	0,2
Steal	2,0	0,0

- Is there a strictly dominated action?
- Is there a weakly dominated action?
- What is the Nash equilibrium that survives the iterated elimination of weakly dominated actions?

Equilibrium in a Single Population: Symmetric games and symmetric equilibria

- We sometime model an interaction in which the members of a single homogenous population are involved anonymously and symmetrically.
- A two-player game is "symmetric" if each player has the same set of actions and each player's evaluation of an outcome depends only on her action and that of her opponent, not on whether she is player 1 or player 2.
- Example

	A	B
A	x,x	y,z
B	z,y	w,w

Symmetric Games

- Symmetric two-player strategic game with ordinal preferences: A two-player strategic game with ordinal preferences is symmetric if the players' sets of actions are the same and the players' preferences are represented by payoff functions u_1 and u_2 for which $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair (a_1, a_2).

Symmetric Nash equilibrium

- The solution that corresponds to a steady state of pairwise interactions between the members of a single population is *symmetric Nash equilibrium*: a Nash equilibrium in which both players take the same action.
- An action profile a^* in a strategic game with ordinal preferences in which each player has the same set of actions is a symmetric Nash equilibrium if it is a Nash equilibrium and a_i^* is the same for every player i.
- This is also an equilibrium refinement.
- Example: the Goldenball

Nash equilibrium: Illustrations I

Two-candidate electoral competition

Suppose that there are two candidates, A and B, and ten voters, $1, 2, \ldots, 10$. Before the voting takes place, each candidate makes a tax-rate proposal to implement if elected. Voters care only about the tax rate, and each voter's most preferred tax rate is: voters 1 and 2 (10%), 3 and 4 (20%), 5 and 6 (30%), 7 and 8 (40%), and 9 and 10 (50%). For every voter i, i's preference is represented by a utility function $u_i(x_i, y^*) = -|x_i - y^*|$, where x_i is i's most preferred tax rate and y^* is the winning candidate's declared tax rate. Each candidate prefers to win than to tie (in which case we assume that the winner is determined by a coin toss), and prefers to tie than to lose. No voter is allowed to abstain, and the winner is determined by a simple majority rule.

Two-candidate electoral competition

- Policy space: one dimensional
- Voters' preferences: single-peaked
  - Suppose that the winning candidate's proposed tax rate is 20%. What is the utility of voter 1, voter 4, and voter 10?
  - Plot the utility function of voter 3 and voter 7 for winning candidate's proposed tax rates from 10% to 50%, *i.e.*, the graph of $u_3(20, y) = -|20 - y|$ and $u_7(40, y) = -|40 - y|$ for $y \in [10, 50]$.
- Deterministic voting: no uncertainty in the eyes of the candidates
- Two candidates who care only about winning the election
- What is the Nash equilibrium of the electoral competition?
  - Is $(y_A, y_B) = (10, 50)$ a Nash equilibrium?
  - Is $(y_A, y_B) = (10, 30)$ a Nash equilibrium?
  - Is $(y_A, y_B) = (40, 40)$ a Nash equilibrium?
  - Is $(y_A, y_B) = (30, 30)$ a Nash equilibrium?

Two-candidate electoral competition

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  - Is $(y_A, y_B) = (10, 30)$ a Nash equilibrium?
  - Is $(y_A, y_B) = (40, 40)$ a Nash equilibrium?
  - Is $(y_A, y_B) = (30, 30)$ a Nash equilibrium?
- *Median Voter Theorem*

Two-candidate electoral competition: a variation

How does the game change if it is a three-candidate competition? Is $(y_A, y_B, y_C) = (30, 30, 30)$ a Nash equilibrium? Is there a Nash equilibrium?

Two-candidate electoral competition: a variation

Suppose that every voter considers candidate A more competent. Thus, voter i's payoff is $u_i(x_i, y_1) = \alpha - |x_i - y_A|$ if candidate A wins, and $u_i(x_i, y_2) = -|x_i - y_B|$ if candidate B wins, with $\alpha > 0$. Moreover, each candidate's payoff is equal to the number of votes one receives. Is there a Nash equilibrium of this game?

Mixed strategy equilibrium

- A Nash equilibrium of a strategic game is an action profile in which every player's action is optimal given every other players' action.
- In Nash equilibrium, no player has an incentive to deviate from that action profile.
- Such a profile presents a steady state: every player's behavior is the same whenever she plays the game, and no player wishes to change her behavior.
- More general notions of steady state allow the players' choices to vary, as long as the pattern of choices remain constant.
- For example, each individual may, on each occasion she plays the game, choose her action probabilistically according to the same, unchanging distribution.
  - *i.e.*, in each play of the game the individual plays an action a with the same probability p.

Mixed strategy equilibrium

- This notion of a 'stochastic' steady state can be modeled by a mixed strategy Nash equilibrium, a generalization of the notion of Nash equilibrium.
  - Compare the mixed strategy Nash equilibrium with the pure strategy Nash equilibrium that we studied so far, in which the equilibrium actions are chosen with probability 1.
  - Strictly speaking, a pure strategy is a special case of a mixed strategy, where the action is chosen with probability $p = 1$ from all possible $p \in [0, 1]$.

Mixed strategy equilibrium

Why and when does a mixed strategy equilibrium make sense?

- Consider the *rock-paper-scissors* game.
- What would you play if your opponent always plays *rock*?

Mixed strategy equilibrium

Why and when does a mixed strategy equilibrium make sense?

- Consider the *rock-paper-scissors* game.
- What would you play if your opponent always plays *rock*?
- What would your opponent play if she knows that you always play *paper*?

Mixed strategy equilibrium

Why and when does a mixed strategy equilibrium make sense?

- Consider the *rock-paper-scissors* game.
- What would you play if your opponent always plays *rock*?
- What would your opponent play if she knows that you always play *paper*?
- What is the best you can do in this game?

Mixed strategy equilibrium

Why and when does a mixed strategy equilibrium make sense?

- Consider the *rock-paper-scissors* game.
- What would you play if your opponent always plays *rock*?
- What would your opponent play if she knows that you always play *paper*?
- What is the best you can do in this game?
  - To randomize. And this is exactly the meaning of the mixed strategy Nash equilibrium.

Definition: Mixed strategy equilibrium

In the generalization of the notion of the Nash equilibrium that models a stochastic steady state of a strategic game, we allow each player to choose a probability distribution over her set of actions rather than restriction her to choose a deterministic action.

Definition. (Mixed strategy): A mixed strategy of a player in a strategic game is a probability distribution over the players' actions.
- $a_i \in A_i$: player i's action in her set of actions A_i
- α_i: player i's mixed strategy
- $\alpha_i(a_i)$: the probability assigned by i's mixed strategy α_i to her action a_i

Mixed strategy: Example

Matching pennies

- Formulate a strategic game
  - Set of players: $\{player1, player2\}$
  - Each player's set of actions: $A_1 = A_2 = \{H, T\}$
  - Each player's preferences over the set of action profiles
- Player 1's mixed strategy, α_1, assigns a probability to each action of player 1, H and T.
  - $\alpha_1(H)$: the probability assigned by player 1's mixed strategy α_1 to her action H
  - $\alpha_1(T)$: the probability assigned by player 1's mixed strategy α_1 to her action T
- For example, the strategy of player 1 in Matching pennies that assigns probability $\frac{1}{3}$ to H and $\frac{2}{3}$ to T is the mixed strategy α_1 such that $\alpha_1(H) = \frac{1}{3}$ and $\alpha_1(T) = \frac{2}{3}$.
  - A shorthand that is often used: player 1's mixed strategy is $(\frac{1}{3}, \frac{2}{3})$.

Mixed strategy

- If there were three strategies, A, B, and C, and A were the pure strategy Nash equilibrium, we could write this as:
  - $p(A) = 1, p(B) = 0, p(C) = 0$
  - Here each $p()$ stands for the probability of playing the strategy in the parentheses.
- A MSNE allows one to play each strategy with a probability between 0 and 1. So we could have instead:
  - $p(A) = \frac{1}{2}, p(B) = \frac{1}{3}, p(C) = \frac{1}{6}$; or
  - $p(A) = 0, p(B) = \frac{2}{3}, p(C) = \frac{1}{3}$; or
  - Anything else in which all the probabilities add up to one.

Mixed strategy Nash equilibrium: Matching pennies

- **CLAIM**: *There is a mixed strategy Nash equilibrium (MSNE) in which each player chooses each of her two actions with probability* $\frac{1}{2}$.
- That means, when it comes down to decide to play *Head* or *Tail*, you literally flip a coin and play whatever the coin tells you to play.

	H	T
H	1,-1	-1,1
T	1,-1	-1,1

Mixed strategy Nash equilibrium: Matching pennies

- To show that there is a mixed strategy Nash equilibrium in which each player chooses each of her two actions with probability one half, we need to show that neither player wants to deviate from doing this.
- That is to say, if one player flips a coin, it is optimal for the other player to continue flipping a coin as well.
- In other words, both players' strategies should mutually be best responses.

	H	T
H	1,-1	-1,1
T	1,-1	-1,1

Matching pennies: Proving the claim

- Remember: in any equilibrium, no one wants to deviate, so we can check whether something is a MSNE by assuming that both players are playing the equilibrium mixed strategy, and seeing if one player wants to deviate.
- Since both players are the same here, we need to check only one of the players.
- We would need to check both if they were different.

Matching pennies: Proving the claim

- Let's begin by writing out the probability of each outcome's occurring if player 1 player Head with probability p, and player 2 plays Head with probability q.
- Write down the probabilities of (H,H), (H,T), (T,H), (T,T).
- Together with (H,H) at $p \times q$, we know how likely each outcome is to occur, if each player plays a mixed strategy.
- What does this get us?

Expected Value

- To go further, we need to understand the concept of expected value.
- Expected value tells us how much something is worth when it happens with some uncertainty.
- It also relates to our tolerance for risk.

Risk

- The concept of risk in game theory relates to your relative preferences for sure things versus lotteries.
- A sure thing is a payoff that happens with certainty.
  - ex. Sue receives $10.
- A lottery is a probability distribution over payoffs, just like with the common use of the word.
  - ex. John plays a slot machine. He receives $20 with probability $\frac{1}{2}$, and $0 with probability $\frac{1}{2}$.

Expected Value

- The expected value of a lottery is the average payoff one would receive from playing it.
- John's lottery yields \$20 with probability 1/2, and \$0 with probability 1/2.
- The average value of this is:

$$\frac{1}{2}(\$20) + \frac{1}{2}(\$0) = \$10.$$

- More complicated lotteries work the same way: multiply each payoff by the chance it happens.

Risk Neutral

- A risk neutral person is indifferent between playing a lottery, and receiving a sure payoff equal to the expected value of the lottery.
- So, if John is risk neutral, he is indifferent between playing the slot machine and receiving $10.
- Your risk tolerance tell you how to convert an expected value into expected utility

Risk Averse, Risk Accepting

- We will generally assume that people are risk neutral in this class. Thus, their payoffs from lotteries are equal to the expected values of the lotteries.
- However, there are other options:
  - Risk-averse people don't like risk. They value the sure thing higher than the lottery that has the same expected value.
  - Risk-accepting people like risk. They value the sure thing lower than the lottery that has the same expected value.
- We'll cover this in more depth later.

Expected value: Matching pennies

- So, what is the expected value of the matching pennies game, if both people play mixed strategies.
- Multiply the probability of each outcome's occurring by the payoff for that outcome.
- For player 1:

$$\begin{aligned} p \cdot q \cdot 1 + (1-p) \cdot q \cdot (-1) + p \cdot (1-q) \cdot (-1) + (1-p)(1-q) \cdot 1 \\ = 1 - 2q - 2p + 4pq. \end{aligned}$$

- For player 2, the same is true.

Expected Value: Matching Pennies

- How do we figure out the equilibrium value of p and q, which tell us how often each player should play head?
- One way: we guess an equilibrium, and check to see if it works.
- Let's try $p = q = \frac{1}{2}$.

Expected Value: Matching Pennies

- How do we figure out the equilibrium value of p and q, which tell us how often each player should play head?
- One way: we guess an equilibrium, and check to see if it works.
- Let's try $p = q = \frac{1}{2}$.
- No player can do better by deviating if the other player plays the strategy (1/2,1/2), because each player receives a payoff of zero no matter what she does.
- Is this unique? Let's try $q = \frac{3}{4}$.

Expected Value: Matching Pennies

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- What about $q = \frac{1}{4}$?

Expected Value: Matching Pennies

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- One way: we guess an equilibrium, and check to see if it works.
- Let's try $p = q = \frac{1}{2}$.
- No player can do better by deviating if the other player plays the strategy (1/2,1/2), because each player receives a payoff of zero no matter what she does.
- Is this unique? Let's try $q = \frac{3}{4}$. This is not an equilibrium.
- What about $q = \frac{1}{4}$? Again, this is not an equilibrium.
- Thus, no q other than $\frac{1}{2}$ works.
- The same logic can be applied to show that no p other than $\frac{1}{2}$ works.
- So, this is the unique MSNE.
- Since this game has no NE in pure strategies, this is also the only equilibrium of Matching Pennies.

Expected Value: Matching Pennies

- Note that in the equilibrium, each player makes the other player indifferent between playing both of the pure strategies that are used in the MSNE.
- The reason for this is that each player must want to play the mixed strategy, which means that each must be willing to randomize. This only happens when neither pure strategy is better on its own.
- If one of the pure strategies were better, the player would just use it.

Best response functions: Matching Pennies

Use best response functions.

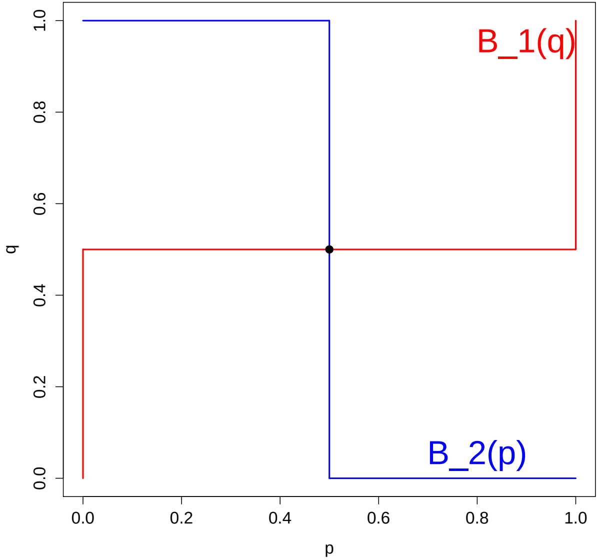
Best response functions: Matching Pennies

Use best response functions.

$$B_1(q) = \begin{cases} \{0\} & \text{if } q < 0.5 \\ \{p : 0 \le p \le 1\} & \text{if } q = 0.5 \\ \{1\} & \text{if } q > 0.5 \end{cases}$$

$$B_2(p) = \begin{cases} \{0\} & \text{if } p < 0.5 \\ \{p : 0 \le p \le 1\} & \text{if } p = 0.5 \\ \{1\} & \text{if } p > 0.5 \end{cases}$$

Best response functions: Matching Pennies



Example: Bach or Stravinski

	Bach	Stravinsky
Bach	(2,1)	(0,0)
Stravinsky	(0,0)	(1,2)

(1) Best response functions
(2) Graphs of best response functions
(3) All Nash equilibria

Example: Coordination game

Find all Nash equilibria of the coordination game with simultaneous choices by using best response functions. Also depict your solution in a figure.

	L	R
T	(4,4)	(0,0)
B	(0,0)	(1,1)

	L	R
T	(2,2)	(0,0)
B	(0,0)	(1,1)

Are they the same game?
- What if we assume ordinal preferences?
- What if the intensity matters?

Extensive Games with Perfect Information: Theory

Extensive games with perfect information

- The model of a strategic game suppresses the sequential structure of decision-making.
  - When applying the model to situations in which players move sequentially, we assume that each player chooses her plan of action once and for all. She is committed to this plan, which she cannot modify as events unfold.
- The model of an extensive game, by contrast, describes the sequential structure of decision-making explicitly, allowing us to study situations in which each player is free to change her mind as events unfold.

Perfection Information

Players are always fulling informed about all previous actions.

- This assumption is used in all following lecture notes that use "perfect information" in the title.
- Later on, we will also study more general cases where players may be only imperfectly informed about previous actions, when choosing an action.

Extensive games with perfect information

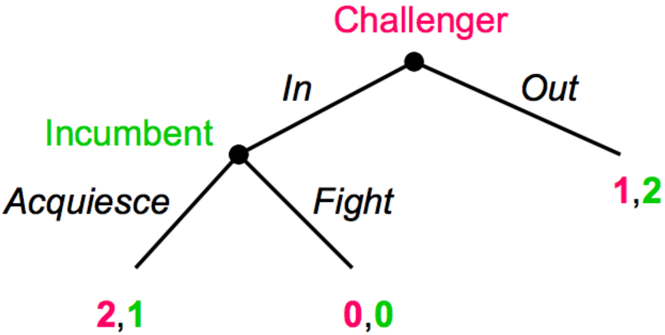
- To describe an extensive game (or extensive form game) with perfect information we need to specify the set of players and their preferences, as for a strategic game.
- In addition, we need to specify the order of players' moves and the actions each player may take at each point (or decision node).
- We do so by specifying the set of all sequences of actions that can possibly occur, together with the player who moves at each point in each sequence.
- We refer to each possible sequence of actions as terminal history and to the function that gives the player who moves at each point in each terminal history as the player function.

Extensive games with perfect information

Thus, an extensive game has four components:

- players
- terminal histories
- player function
- preferences for the players

Example: Entry Game



Extensive games with perfect information

Definition 155.1 (Extensive game with perfect information). An extensive game with perfect information consists of

- a set of players
- Terminal histories: a set of terminal histories with the property that no sequence is a proper subhistory of any other sequence
- Player function: a function that assigns a player to every sequence that is a proper subhistory of some terminal history
- Preferences: for each player, preferences over the set of terminal histories

Example: Entry Game

An incumbent faces the possibility of entry by a challenger. For example, the challenger may be a politician competing for the leadership of a party, or a firm considering entry into an industry currently occupied by a monopolist. The challenger may enter or stay out. If she enters, the incumbent may either acquiesce or fight. Suppose that the challenger's payoff is 2 if the terminal history is (In, Acquiesce), 1 if it is Out, and 0 if it is (In, Fight). And the incumbent's payoff is 2 if the terminal history is Out, 1 if it is (In, Acquiesce), and 0 if it is (In, Fight).

Example: Entry Game

Then the situation may be modeled as the following extensive game with perfect information:

- Players: {Challenger, Incumbent}
- Terminal histories; {(In, Acquiesce), (In, Fight), Out}
- Player function: $P(\emptyset)$=Challenger, and $P(In)$=Incumbent
- Preferences: Payoff functions u_1 for challenger and u_2 for incumbent

$$u_1(In, Acquiesce) = 2, u_1(Out) = 1, \text{ and } u_1(In, Fight) = 0$$
$$u_2(Out) = 2, u_2(In, Acquiesce) = 1, \text{ and } u_2(In, Fight) = 0$$

Extensive Games

Brief Review

- Players move **sequentially** instead of simultaneously.
  $\Rightarrow$ Which player moves when: Player function
  $\Rightarrow$ What actions are available to the player who might act at each point: A set of actions for each history
- Players have preferences over the set of **terminal histories** instead of action profiles.
- We usually use a *game tree* to describe an extensive game.

Extensive game with perfect information

Technical definition.
An extensive game with perfect information consists of

- a set of players,
- a set of consequences (terminal histories) with the property that no sequence is a proper subhistory of any other sequence,
- a function (the player function) that assigns a player to every sequence that is a proper subhistory of some terminal history,
- for each player, preferences over the set of terminal histories.

Players

- A player is an actor who gets to take an action at some point in the game.
- A player function determines which player gets to act at each opportunity for action.
- Players may alternate, or one player may act more than once in a row.
- The opportunity for action appears on a game tree as a *node*.

Actions

- At each node, the player assigned to it may take an action chosen from the set of all available actions at that node.
- Actions are represented on the game tree by lines extending either downward or to the right from the node. Each line represents on action.
- If there are too many actions to draw this way (e.g. a continuous action), we draw an arc.

Preferences and Terminal Histories

- Players have preferences only over the final outcomes of the game, not each action.
- Each final outcome occurs at the end of a terminal history of the game.
- Each possible sequence of action is a *terminal history*.
- Each player has a payoff assigned for each terminal history.
- At the start of a game, and after any sequence of events, a player chooses an action.

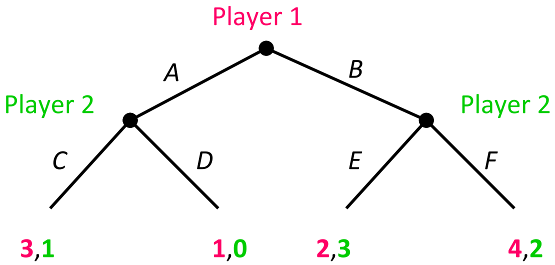
Subgame perfect equilibrium

Or how to solve an extensive game

This notion requires each player's strategy to be optimal, given the other players' strategies, not only at the start of the game but after every possible history h of actions.

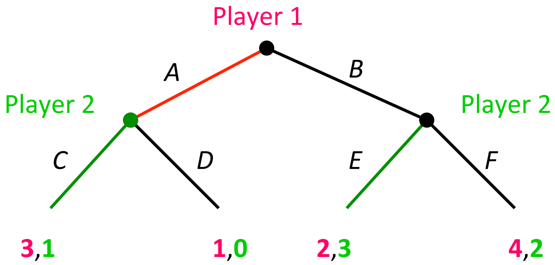
Back to Nash equilibrium

Consider the following game.



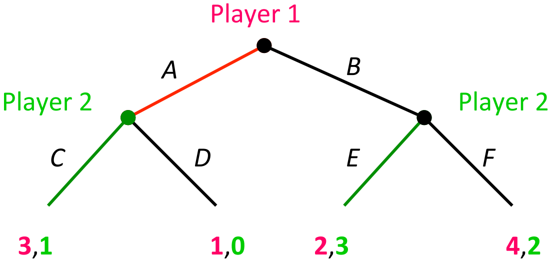
- We refer to the strategies of player 1 in this game simply as A and B, and to the strategies of player 2 simply as CE, CF, DE, and DF.
- What are the Nash equilibria of this game?

Back to Nash equilibrium



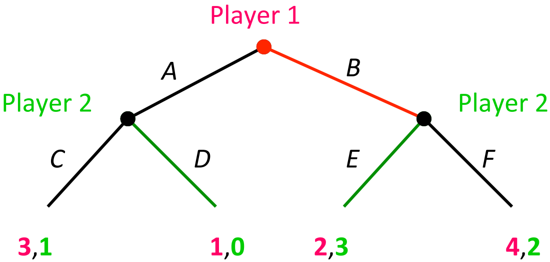
- (A, CE) is a Nash equilibrium. To see this, check whether any player has a profitable unilateral deviation from the strategy profile.

Back to Nash equilibrium



- (A, CE) is a Nash equilibrium. To see this, check whether any player has a profitable unilateral deviation from the strategy profile.
- Is there another Nash equilibrium?

Back to Nash equilibrium



- (B, DE) is a also a Nash equilibrium. To see this, check whether any player has a profitable unilateral deviation from the strategy profile.
- Is this Nash equilibrium plausible?
  $\Rightarrow$ Let's learn *subgame perfect equilibrium*.

Subgame

- For any nonterminal history h, the subgame following h is the part of the game that remains after h has occurred.
- For example the subgame following the history *In* in the entry game is the game in which the incumbent is the only player and there are two terminal histories (of the subgame), *Acquiesce* and *Fight*.
- In the game of the previous slide, the subgame following the history A is the game in which player 2 is the only player and there are two terminal histories (of the subgame), C and D.
- The empty history $\emptyset$ is the state before the game starts (or the state in which no action has taken yet), and the subgame following it is the entire game.
- Every other subgame is called a *proper subgame*.
- The number of subgames is equal to the number of nonterminal histories.

Necessary notation

- Let h be a history and s a strategy profile.
- Suppose that h occurs (even though it is not necessarily consistent with s), and afterward the player adhere to the strategy profile s.
- The resulting terminal history, consisting of h followed by the outcome generated in the subgame following h by the strategy profile induced by s in the subgame, is denoted by $O_h(s)$.
- Note that for any strategy profile s, we have $O_\emptyset(s) = O(s)$.
- In the *Entry Game*, let s be the strategy profile $(Out, Fight)$ and let h be the history *In*. If h occurs, and afterward the players adhere to s, the resulting terminal history is $O_h(s) = (In, Fight)$.

Definition: Subgame perfect equilibrium of extensive game with perfect information

- The strategy profile s^* in an extensive game with perfect information is a subgame perfect equilibrium if, for every player i, every history h after which it is player i's turn to move (*i.e.*, $P(h) = i$), and every strategy r_i of player i, the terminal history $O_h(s^*)$ generated by s^* after the history h is at least as good according to player i's preferences as the terminal history $O_h(r_i, s^*_{-i})$ generated by the strategy profile (r_i, s^*_{-i}) in which player i chooses r_i while every other player j chooses s^*_j.

Definition: Subgame perfect equilibrium of extensive game with perfect information

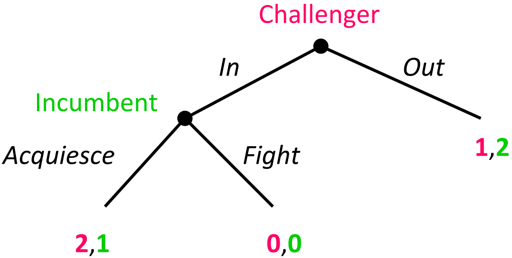
Equivalently, for each player i and for every strategy r_i of player i,

$$u_i(O_h(s^*)) \geq u_i(O_h(r_i, s^*_{-i})),$$

where u_i is a payoff function that represents player i's preferences and $O_h(s)$ is the terminal history consisting of h followed by the sequence of actions generated by s after h.

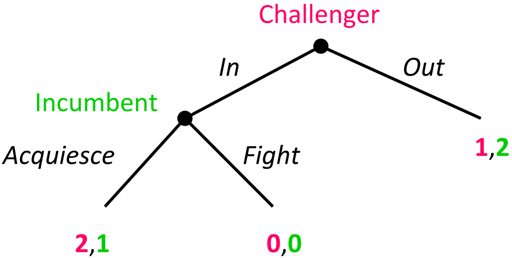
- The important point in this definition is that each player's strategy is required to be optimal for every history after which it is the player's turn to move, not only at the start of the game. Equivalently, the players choose optimal actions in every subgame.
- So we can start from the bottom, the smallest subgame, and work our way up. This is called *backward induction*.

Backward induction: Entry Game



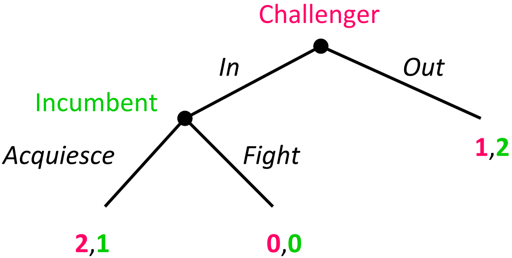
- How many nonterminal histories are there in this game?

Backward induction: Entry Game



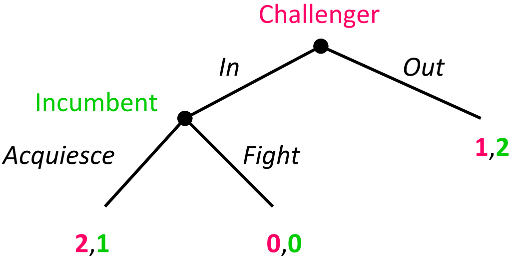
- How many nonterminal histories are there in this game?
  $|\{\emptyset, In\}| = 2$.
- How many subgames are there in this game?

Backward induction: Entry Game



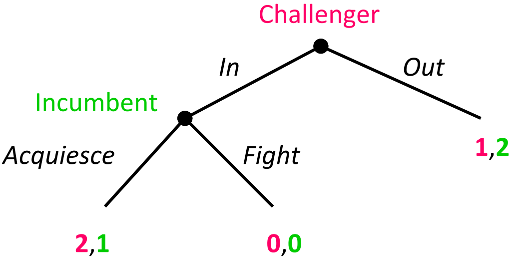
- How many nonterminal histories are there in this game? $|\{\emptyset, In\}| = 2$.
- How many subgames are there in this game? 2

Backward induction: Entry Game



- Find the optimal action(s) for the subgame following teh *Challenger*'s action *In*.

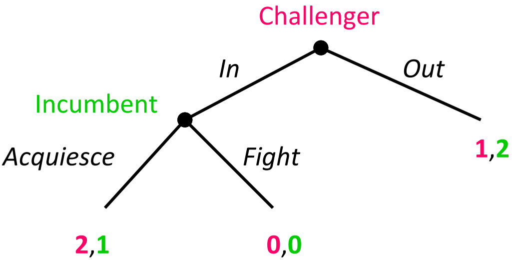
Backward induction: Entry Game



- Find the optimal action(s) for the subgame following teh *Challenger*'s action *In*.
- The incumbent is better off by choosing *Acquiesce*, because

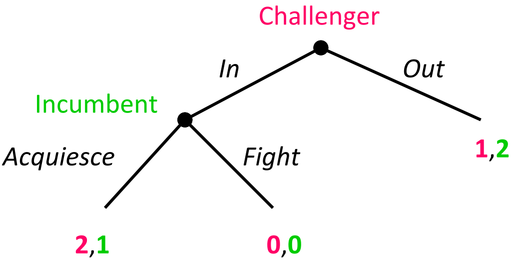
$$u_{Inc}(In, Acquiesce) = 1 > u_{Inc}(In, Fight) = 0.$$

Backward induction: Entry Game



- Continue with larger subgames of the game tree and find the optimal action for this subgame, using the information about what occurs optimally in later subgames.

Backward induction: Entry Game



- Continue with larger subgames of the game tree and find the optimal action for this subgame, using the information about what occurs optimally in later subgames.
- Given that the incumbent will choose *Acquiesce* in the later subgame, the challenger is better of by choosing *In*, because

$$u_{Ch}(In, Acquiese) = 2 > u_{ch}(Out, Acquiesce) = 1.$$

- (*In, Acquiese*) is the unique subgame perfect equilibrium.

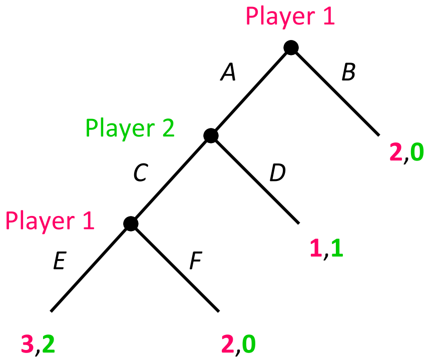
Subgame perfect equilibrium of finite horizon games and backward induction

The set of subgame perfect equilibria of a finite horizon extensive form game with perfect information is equal to the set of strategy profiles isolated by the procedure of backward induction.

Existence of subgame perfect equilibrium

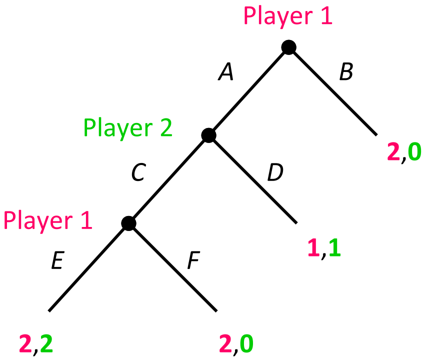
- Every finite extensive game with perfect information has a subgame perfect equilibrium.
- Note that this result does not claim that a finite extensive game has a single subgame perfect equilibrium.

Exercise 1



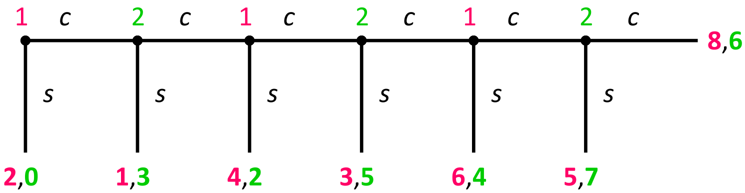
Final all subgame perfect equilibria.

Exercise 2



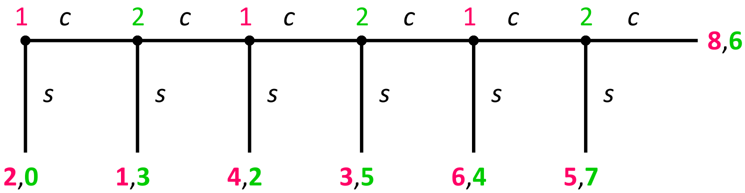
Final all subgame perfect equilibria.

Exercise 3: The Centipede Game



- c is continue, and s is stop.
- How many subgames?
- Do we know whether there exists a subgame perfect equilibrium in this game? Can we find the subgame equilibria of this game by the procedure of backward induction?

Exercise 3: The Centipede Game



- *c* is continue, and *s* is stop.
- How many subgames?
- Do we know whether there exists a subgame perfect equilibrium in this game? Can we find the subgame equilibria of this game by the procedure of backward induction?
  $\Rightarrow$ Yes. This game is a finite extensive form game with perfect information.
- Find all subgame perfect equilibria of this game.

Exercise 4: The Ultimatum Game

- Bargaining over the division of a pie may naturally be modeled as an extensive game.
- Here, we analyze a very simple game that is the basis of many richer models.
- The model is so simple, in fact, that you may not initially think of it as a model of bargaining.

The Ultimatum Game

- Two people use the following procedure to split $\$c$.
  - Player 1 offers player 2 an amount of money up to \$c.
  - If 2 accepts this offer, then 1 receives the remainder of the \$c.
  - If 2 rejects the offer, then neither player receives any payoff.
- Each player cares only about the own amount of money she receives, and prefers to receive as much as possible.
- Player 1 may offer any amount up to \$c, not necessarily an integral number of cents.

The Ultimatum Game

Formulate an extensive game.

- Players
- Terminal histories
- Player function
- Preferences

The Ultimatum Game

Formulate an extensive game.

- **Players**: {player 1, player 2}
- **Terminal histories**: the set of all sequences (x, Z), where $x \in [0, c]$ and $Z \in \{Yes, No\}$
- **Player function**: $P(\emptyset) = 1$ and $P(x) = 2$ for all x
- **Preferences**: For the terminal history (x, Y), player 1 receives $c - x$ and player 2 receives x. For the terminal history (x, N) both players receive 0. Each player prefers more money to less money.

The Ultimatum Game

- The game has a finite horizon, so we can use backward induction to find its subgame perfect equilibria.
- Consider the subgames, in which player 2 either accepts or rejects an offer of player 1. For every possible offer of player 1, there is such a subgame.
- In the subgame that follows an offer x of player 1 for which $x > 0$, player 2's optimal action is to accept (if she rejects, she gets nothing).
- In the subgame that follows the offer $x = 0$, player 2 is indifferent between accepting and rejecting.
- Thus, in a subgame perfect equilibrium player 2's strategy either accepts all offers including 0 or accepts all offers $x > 0$ and rejects the offer $x = 0$.

The Ultimatum Game

- Now consider the entire game. For each possible subgame perfect equilibrium strategy of player, we need to find the optimal strategy of player 1.
- If player 2 accepts all offers including 0, player 1's optimal offer is 0.
- If player 2 accepts all offers except zero, then no offer of player 1 is optimal. Why?

The Ultimatum Game

- Now consider the entire game. For each possible subgame perfect equilibrium strategy of player, we need to find the optimal strategy of player 1.
- If player 2 accepts all offers including 0, player 1's optimal offer is 0.
- If player 2 accepts all offers except zero, then no offer of player 1 is optimal. Why?For any offer $x > 0$, $x/2$ is better, given that player 2 accepts both offers. Offering 0 is not optimal because player 2 rejects it.
- We conclude that the only subgame perfect equilibrium of the game is the strategy pair in which player 1 offers 0 and player 2 accepts all offers.
- In this equilibrium, player 1's equilibrium payoff is \$c and player 2's payoff is 0.

Nash equilibria of the Ultimatum Game

Claim: for every amount x there are Nash equilibria in which player 1 offers x and player 2 accepts any offer $y \geq x$ and rejects others.

Make an argument why it is true.

The holdup game

Before engaging in an ultimatum game in which she may accept or reject an offer of person 1, person 2 takes an action that affects the size c of the pie to be divided. She may exert little effort, resulting in a small pie, or size c_L, or great effort, resulting in a large pie, of size c_H. She dislikes exerting effort. Specifically, assume that her payoff is $x - E$ if her share of the pie is x, where $E = L$ if she exerts little effort and $E = H > L$ if she exerts great effort. The extensive game that models this situation is known as the holdup game.

- Formulate an extensive game and draw a game tree.
- What is the subgame perfect equilibrium of this game?

The holdup game

Before engaging in an ultimatum game in which she may accept or reject an offer of person 1, person 2 takes an action that affects the size c of the pie to be divided. She may exert little effort, resulting in a small pie, or size c_L, or great effort, resulting in a large pie, of size c_H. She dislikes exerting effort. Specifically, assume that her payoff is $x - E$ if her share of the pie is x, where $E = L$ if she exerts little effort and $E = H > L$ if she exerts great effort. The extensive game that models this situation is known as the holdup game.

- Formulate an extensive game and draw a game tree.
- What is the subgame perfect equilibrium of this game?
  - This game has a unique subgame perfect equilibrium, in which person 2 exerts little effort and person 1 obtains all of the resulting small pie.

Agenda control

In some legislatures, proposal for modifications of the law are formulated by committees. Under a “closed rule”, the legislature may either accept or reject a proposed modification, but may not propose an alternative; in the event of rejection, the existing law is unchanged. That is, the committee controls the “agenda”.

Agenda control

Model an outcome as a number y. Assume that the legislature and committee have favorite outcomes that may differ, and that the preferences of each body are represented by a single-peaked payoff function symmetric about its favorite outcome, like the voters' preferences in Hotelling's model of electoral competition. Assign numbers to outcomes so that the legislature's favorite outcome is 0; denote the committee's favorite outcome by $y_c > 0$. Then the following variant of the ultimatum game models the procedure. The players are the committee and the legislature. The committee proposes an outcome y, which the legislature either accepts or rejects. In the event of rejection the outcome is y_0, the "status quo". Note that the main respect in which this game differs from the ultimatum game is that the players' preferences are diametrically opposed only with regard to outcomes between 0 and y_c; if $y' < y'' < 0$ or $y_c < y'' < y'$, then both players prefer y'' to y'.

Agenda control

Fin the subgame perfect equilibrium of this game as a function of the status quo outcome y_0. Show, in particular, that for a range of values of y_0, an increase in the value of y_0 leads to a decrease in the value of the equilibrium outcome.

Repeated Games: The Prisoner's Dilemma

Repeated Games

- Earlier, we considered one-shot games, in which each player had one chance to take the actions in the game.
- Now we consider repeated games, in which player get to take actions again and again, sometimes forever. We call the base game that is repeated the stage game.

Horizons

- The horizon of a game is the number of times the stage game is repeated.
- A finite-horizon game is one in which the stage game is repeated N times, were N is a finite number.
  - To solve these games, we can use backward induction, as before.
- An infinite-horizon game is one in which the stage game is repeated indefinitely.
  - We have to be more creative to solve these games.

Discounting

- In repeating the stage game, the players and the actions stay the same.
- Payoffs are more tricky.
- One could assume that the payoffs just add each time the stage game is played, but this has two problems.
  (1) It assumes that payoffs that happen after a long time are worth as much as they are now.
  (2) If it is an infinite-horizon game, then all payoffs for the game will also be infinite.
- So instead we allow for discounted payoffs.

Discounting

- Discount factor δ
  - This tells us the rate at which future benefits are discounted compared to today's benefit.
  - Essentially, it tells us how much people care about the future.
  - δ is bounded: i.e. $0 \leq \delta \leq 1$.
- Example: $1000 today or in a month's time
  - If it did not matter to you whether you got the money today or in a month's time, your discount factor would be close to one.
  - If you really wanted the money today, your discount rate would be close to 0.
  - One reason it would be less than 1 is foregone interest.

Present Value

- Say something is worth 5 in every period. It will be worth 5δ in the second period, $5\delta^2$ in the third period, $5\delta^3$ in the fourth period, etc.
- So the present value of the good is:

$$5 + 5\delta + 5\delta^2 + 5\delta^3 + \dots$$

- A useful fact:

$$1 + \delta + \delta^2 + \delta^3 + \dots = \frac{1}{1-\delta}$$

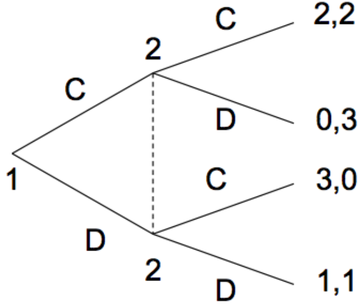
- So, the present value of the good is $\frac{5}{1-\delta}$.

Repeated Prisoner's Dilemma (PD)

- Recall the one-shot PD game but with C (cooperate) and D (defect)
- Each person defects because it is in her best interest to do so.
- Thus cooperation cannot be sustained.
- What happens if this game is repeated?

Finite-Horizon PD

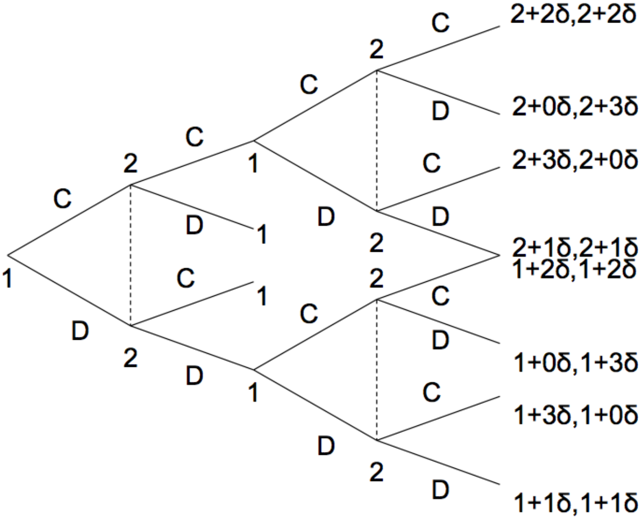
Let's assume that the PD gets repeated twice. To help us think about this, we can represent the PD in the following way:



The dotted line means that player 2 is not sure which node she is at and can only pick one strategy here.

Finite-Horizon PD

Now we add the second stage:



Finite-Horizon PD

- What does backward induction say should happen in the second stage?
- Does it matter what happened in the first stage?
- Given what is going to happen in the second stage, what should happen in the first stage?
- So repeating the game finitely doesn't change the outcome.

Finite-Horizon PD

- Since there is on end to these games, we can't simply work backward.
- Instead we use the *one-shot deviation principle* for sequential games and SPE.
- A one-shot deviation is a deviation from a strategy that differs only in the action taken at one point.

* Recall that a strategy assigns actions for all possible histories.

One-shot Deviation Principle

- The one-shot deviation principle states that a strategy is a SPE if and only if there does not exist a beneficial one-shot deviation.
- That is, a strategy is a SPE if no single change to any action in the equilibrium strategy, by itself, yields a better payoff than the SPE strategy yields.
- Let's apply this to the PD.

Infinite-Horizon PD

- To see if there exists a profitable deviation from a SPE strategy, we first need to postulate a strategy. Let's see if we can sustain cooperation.
- In a cooperative strategy, each person plays C in every period, for all time.
- How can we get this cooperation?
- To answer this, we need to specify strategies.

Infinite-Horizon PD

- First, we dictate that each player always cooperate as long as neither player has ever defected before.
- Second, we dictate that both players always begin the game cooperating in the first period.
- These assumptions dictate behavior *on the equilibrium path*. If no one deviates from these, all we ever see is cooperation.

Infinite-Horizon PD: Grim Trigger

- Since we need to give actions for what to do in all circumstances, our strategy must also dictate what actions to take when the other player does not cooperate.
- To make this easy, we say that, if any player (including you) ever defects, you should defect forever after.
- This is known as a *grim trigger* strategy.
- This describes what happens *off the equilibrium*.

Infinite-Horizon PD: Grim Trigger

- See if this is a SPE, using the one-shot deviation principle.
- Each person's payoff for the game under cooperation is $\frac{2}{1-\delta}$.
- Since payoffs get less and less important over time, the best deviation must happen immediately.
- The best possible deviation is to defect. This gets you a payoff of 3 immediately, instead of 2.
- Now you both defect forever, so you get 1 for all future periods.
- The payoff for the best one-shot deviation is thus $3 + \frac{\delta}{1-\delta}$.
- Whenever this is less than the payoff for cooperation, $\frac{2}{1-\delta}$, no player wants to defect ever, and cooperation can be maintained: that is, $\delta > \frac{1}{2}$.

Infinite-Horizon PD: Grim Trigger

- So, whenever $\delta > \frac{1}{2}$, players do not want to defect.
- We're not done yet, though: we need to check if players want to go through with their grim trigger punishment.
- If you do not punish, you get 0, since the other player continues to defect due to his earlier defection.
- If you punish, you get $0 + \frac{\delta}{1-\delta}$, since you both defect forever.
- So, punishing is always beneficial.

Infinite-Horizon PD: Grim Trigger

- Thus, as long as $\delta > 1/2$, the strategy given provides a SPE in which we observe cooperation all of the time.
- Infinitely repeating the game allows for cooperation, as future gains from cooperation outweigh the one-time benefits from deviation.
- Note that punishments must be credible, in the sense that it must be in the other person's best interests to play the punishment strategy, if the other person deviates from equilibrium play.

Repeated Game

Definition. (Repeated Game) Let G be a strategic game. Denote the set of players by N and the set of actions and payoff function of each player i by A_i and u_i respectively. The T-period repeated game of G for the discount factor δ is the extensive game with perfect information and simultaneous moves in which

- the set of players is N
- the set of terminal histories is the set of sequences $(a^1, a^2, \ldots, a^T)$ of actions profiles in G
- the player function assigns the set of all players to every history $(a^1, a^2, \ldots, a^t)$ (for every value of t)
- the set of actions available to any player i after any history is A_i
- each player i evaluates each terminal history $(a^1, \ldots, a^T)$ according to its discounted sum of payoffs $\sum_{t=1}^{T} \delta^{t-1} u_i(a^t)$.

Infinitely repeated PD: grim trigger strategy

Every player chooses C initially.
Choose C at t if $a^k = (C, C)$ for every period $k < t$.
Choose D otherwise.

Find the smallest value of δ that makes (C, C) in every period as an subgame equilibrium outcome path.

Infinitely repeated PD: limited punishment

Every player chooses C initially.
Choose C at t if t is not in the punishment period; and a^{t-1} is (C, C).
Choose D for k periods from $t+1$ if a^t is not (C, C).
When the punishment ends, both players revert to C.

* Find the smallest value of δ that makes (C, C) in every period as an subgame equilibrium outcome path when $k = 1$.
* Find the smallest value of δ that makes (C, C) in every period as an subgame equilibrium outcome path when $k = 2$.
* Find the smallest value of δ that makes (C, C) in every period as an subgame equilibrium outcome path when $k = 3$.

Infinitely repeated PD: tit-for-tat

Choose (C, C) initially.
Choose D at $t+1$ if the other player chose D at t.
Choose C at $t+1$ if the other player chose C at t.

* Find the all value of δ that makes (C, C) in every period as an subgame equilibrium outcome path when $k = 1$.
This time we need to consider four types of subgame, following histories in which the last outcome is (C, C), (C, D), (D, C) and (D, D).