MATH 32B Problem Set ?

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1 Question 5

Sketch the region $\mathcal{D}$ indicated and integrate $f(x, y)$ over $\mathcal{D}$ using polar coordinates

$$f(x,y) = y(x^2+y^2)^{-1}; \quad y \geq \frac{1}{2}, \quad x^2+y^2 \leq 1$$

First, let's solve for the bounds or integration

$$y = r\sin\theta \quad r\sin\theta \geq \frac{1}{2} \quad r \geq \frac{1}{2\sin\theta}$$

$$x^2+y^2 = r \quad r^2 \leq 1 \quad r \leq 1$$

Since we know that $y \geq \frac{1}{2}$

$$\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\frac{1}{2\sin\theta}}^{1} \sin\theta \, dr \, d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 - \frac{1}{2\sin\theta}) \sin\theta \, d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin\theta - \frac{1}{2} \, d\theta$$

$$-\cos\theta - \frac{\theta}{2}\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$\frac{\sqrt{3}}{2} - \frac{5\pi}{6} - (-\frac{\sqrt{3}}{2} - \frac{\pi}{6})$$

$$\sqrt{3} - \frac{\pi}{3}$$

2 Question 24

Evaluate $\iint_{\mathcal{D}} x\sqrt{x^2+y^2}\,dA$, where $\mathcal{D}$ is the shaded region enclosed by the lemniscate curve $r^2 = \sin 2\theta$

First, let us find the bounds of the region. Given the diagram, we can clearly see that

$$0 \le \theta \le \frac{\pi}{2}$$

$$r^2 = \sin 2\theta$$
$$r = \sqrt{\sin 2\theta}$$
$$0 \le r \le \sqrt{\sin 2\theta}$$

Now, let us rewrite the original function in polar coords

$$x\sqrt{x^2+y^2} = r(\cos\theta)r = r^2\cos\theta$$

Now, let us integrate

$$\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\sin 2\theta}} r^2\cos\theta \cdot r\,dr\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\left(\frac{r^4}{4}\cos\theta\Big|_0^{\sqrt{\sin 2\theta}}\right)d\theta$$

$$\int_0^{\frac{\pi}{2}}\frac{(\sin 2\theta)^2}{4}\cos\theta\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\frac{(2\sin\theta\cos\theta)^2}{4}\cos\theta\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\frac{4\sin^2\theta\cos^3\theta}{4}\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^3\theta\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\sin^2\theta\cos\theta(1-\sin^2\theta)\,d\theta$$

$$u = \sin\theta \quad du = \cos\theta\,d\theta$$

$$\int_0^1 u^2(1-u^2)\,du$$

$$\frac{u^3}{3} - \frac{u^5}{5}\Big|_0^1$$

$$\frac{1}{3} - \frac{1}{5} = \frac{1}{2}$$

3 Question 39

Find the volume of the region appearing between the two surfaces

$$z = x^2 + y^2 \qquad z = 8 - x^2 - y^2$$

First, let us find where the surfaces intersect

$$x^2 + y^2 = 8 - x^2 - y^2$$

$$2x^2 + 2y^2 = 8$$

$$x^2 + y^2 = 4$$

$$r^2 = 4 \qquad 0 \leq r \leq 2$$

Just from the shape of the surfaces, we know that

$$0 \leq \theta \leq 2\pi$$

To compute the volume in between the surfaces, we will need to integrate the height over the bounds. The height is the upper bounds minus the lower bound

$$\text{Lower bound: } z = r^2 \quad \text{Upper bound: } z = 8 - r^2$$

$$\text{Height: } 8 - r^2 - r^2 = 8 - 2r^2$$

$$\int_0^{2\pi} \int_0^2 (8 - 2r^2) r \, dr \, d\theta$$

$$4r^2 - \frac{r^4}{2} \Big|_0^2$$

$$16 - 8 = 8$$

$$\int_0^{2\pi} 8 \, d\theta$$

$$16\pi$$