MATH 32B Problem Set 1

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1 Question 11

The following table gives the approximate height at quarter-meter intervals of a mound of gravel. Estimate the volume of the mound by computing the average of the two Riemann sums $S_{4,3}$ with lower-left and upper-right vertices of the subrectangles as sample points.

0.75	0.1	0.2	0.2	0.15	0.1
0.5	0.2	0.3	0.5	0.4	0.2
0.25	0.15	0.2	0.4	0.3	0.2
y : **0**	0.1	0.15	0.2	0.15	0.1
	x : **0**	**0.25**	**0.5**	**0.75**	**1**

Computing area using lower-left verticies as sample points: The verticies we will be using are:

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.15 & 0.2 & 0.15 \end{bmatrix}$$

Note that Δx and Δy are both 0.25, meaning the volume of each rectangle will be the height of the rectangles times 0.25^2, which is 0.0625

$$0.0625 \cdot \begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.15 & 0.2 & 0.15 \end{bmatrix} = \begin{bmatrix} 0.0125 & 0.01875 & 0.03125 & 0.025 \\ 0.009375 & 0.0125 & 0.025 & 0.01875 \\ 0.00625 & 0.009375 & 0.0125 & 0.009375 \end{bmatrix}$$

Adding up these areas gives us 0.190625

Now, let's repeat, except we will use the upper-right verticies as sample points

$$0.0625 \cdot \begin{bmatrix} 0.2 & 0.2 & 0.15 & 0.1 \\ 0.3 & 0.5 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.0125 & 0.0125 & 0.009375 & 0.00625 \\ 0.01875 & 0.03125 & 0.025 & 0.0125 \\ 0.0125 & 0.025 & 0.01875 & 0.03125 \end{bmatrix}$$

Adding up these areas gives us 0.196875. The average of these two sums is **0.19375**, which is our estimate for the volume of the mound.

2 Question 15

Use symmetry to evauate the double integral

$$\iint_{\mathcal{R}} x^3 \, dA, \quad \mathcal{R} = [-4, 4] \times [0, 5]$$

Note that that x^3 is odd and lacks horizontal or vertical shifts. This means so for some value a, the area between the curve and $[-a, 0]$ will be equal to the area between the curve and $[0, a]$ multiplied by -1.

Also, note that we can split the domain into two sub-domains. Let us split $\mathcal{R}$ into $\mathcal{R}_1 = [-4, 0] \times [0, 5]$ and $\mathcal{R}_2 = [0, 4] \times [0, 5]$.

$$\iint_{\mathcal{R}} x^3 \, dA, \quad \mathcal{R} = [-4, 4] \times [0, 5] = \iint_{\mathcal{R}_1} x^3 \, dA, \quad \mathcal{R}_1 = [-4, 0] \times [0, 5] + \iint_{\mathcal{R}_2} x^3 \, dA, \quad \mathcal{R}_2 = [0, 4] \times [0, 5]$$

From the observation we stated before, it can be said that

$$\iint_{\mathcal{R}_1} x^3 \, dA, \quad \mathcal{R}_1 = [-4, 0] \times [0, 5] = -\iint_{\mathcal{R}_2} x^3 \, dA, \quad \mathcal{R}_2 = [0, 4] \times [0, 5]$$

We can rewrite the original expression as

$$-\iint_{\mathcal{R}_2} x^3 \, dA + \iint_{\mathcal{R}_2} x^3 \, dA = 0$$

This double integral evaluates to 0.

3 Question 17

Use symmetry to evaluate the double integral

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi]$$

Note that $\sin x$ oscillates from -1 to 1. Importantly, without any transformations, $\sin x$ completes one whole revolution from 0 to 2π, and the function from 0 to π is identical to the function from π to 2π reflected over the $x-$axis.

Also remember that we can split the domain into two sub-domains. Let us split $\mathcal{R}$ into $\mathcal{R}_1 = [0, \pi] \times [0, 2\pi]$ and $\mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$.

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi] = \iint_{\mathcal{R}_1} \sin x \, dA, \quad \mathcal{R}_1 = [0, \pi] \times [0, 2\pi] + \iint_{\mathcal{R}_2} \sin x \, dA, \quad \mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$$

From what we said before, we can observe that

$$\iint_{\mathcal{R}_1} \sin x \, dA, \quad \mathcal{R}_1 = [0, \pi] \times [0, 2\pi] = -\iint_{\mathcal{R}_2} \sin x \, dA, \quad \mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$$

Finally, we can state that

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi] = -\iint_{\mathcal{R}_2} \sin x \, dA + \iint_{\mathcal{R}_2} \sin x \, dA = 0$$

The double integral evaluates to 0.

4 Question 19

Evaluate the iterated integral

$$\int_1^3 \int_0^2 x^3 y \, dy \, dx$$

$$\int_1^3 \int_0^2 x^3 y \, dy \, dx = \int_1^3 \left(\int_0^2 x^3 y \, dy \right) dx$$

$$= \int_1^3 \left(x^3 \cdot \frac{y^2}{2} \Big|_{y=0}^{y=2} \right) dx$$

$$= \int_1^3 \left(x^3 \cdot (\frac{4}{2} - \frac{0}{2}) \right) dx$$

$$= \int_1^3 2x^3 \, dx = \frac{x^4}{2} \Big|_{x=1}^{x=3} = \frac{81}{2} - \frac{1}{2} = \frac{80}{2} = 40$$

The iterated integral evaluates to 40.

5 Question 21

Evaluate the iterated integral

$$\int_4^9 \int_{-3}^8 1 \, dx \, dy$$

$$\int_4^9 \left(\int_{-3}^8 1 \, dx \right) dy =$$

$$\int_4^9 x \Big|_{x=-3}^{x=8} dy =$$

$$\int_4^9 11 \, dy = 11x \Big|_{x=4}^{x=9} = 99 - 44 = 55$$

The iterated integral evaluates to 55.

6 Question 23

Evaluate the iterated integral

$$\int_{-1}^{1}\int_{0}^{\pi} x^2 \sin y \, dy \, dx$$

$$\int_{-1}^{1}\left(\int_{0}^{\pi} x^2 \sin y \, dy\right) dx$$

$$\int_{-1}^{1}\left(x^2 \cdot -\cos y\Big|_{y=0}^{y=\pi}\right) dx$$

$$\int_{-1}^{1} x^2(1-(-1)) \, dx$$

$$\int_{-1}^{1} 2x^2 \, dx = \frac{2x^3}{3}\Bigg|_{x=-1}^{x=1} = \frac{2}{3} - \frac{-2}{3} = \frac{4}{3}$$

The iterated integral evaluates to $\frac{4}{3}$.

7 Question 25

Evaluate the iterated integral

$$\int_2^6 \int_1^4 x^2 \, dx \, dy$$

$$\int_2^6 \left(\int_1^4 x^2 \, dx \right) dy$$

$$\int_2^6 \left(\frac{x^3}{3} \Big|_{x=1}^{x=4} \right) dy$$

$$\int_2^6 \frac{64}{3} - \frac{1}{3} \, dy = \int_2^6 \frac{63}{3} \, dy$$

$$\frac{63x}{3} \Big|_{x=2}^{x=6} = \frac{378 - 126}{3} = \frac{252}{3} = 84$$

The iterated integral evaluates to 84.

8 Question 37

Evaluate the integral

$$\iint_{\mathcal{R}} \frac{x}{y}\, dA, \quad \mathcal{R} = [-2,4] \times [1,3]$$

$$\iint_{\mathcal{R}} \frac{x}{y}\, dx\, dy, \quad \mathcal{R} = [-2,4] \times [1,3]$$

$$\int_1^3 \int_{-2}^4 \frac{x}{y}\, dx\, dy$$

$$\int_1^3 \left(\frac{x^2}{2y}\bigg|_{x=-2}^{x=4} dx \right) dy$$

$$\int_1^3 \frac{1}{2y}(16-4)\, dy$$

$$\int_1^3 \frac{6}{y}\, dy$$

$$6 \ln y \bigg|_{y=1}^{y=3} = 6\ln 3 - 6\ln 1 = 6\ln 3$$

9 Question 40

Evaluate the integral

$$\iint_{\mathcal{R}} \frac{y}{x+1}\, dA, \quad \mathcal{R} = [0,2] \times [0,4]$$

$$\int_0^4 \int_0^2 \frac{y}{x+1}\, dx\, dy$$

$$\int_0^4 \int_0^2 \frac{y}{x+1}\, dx\, dy$$

$$\int_0^4 \left(y \ln|x+1| \Big|_{x=0}^{x=2} \right) dy$$

$$\int_0^4 y(\ln|3| - \ln|1|)\, dy$$

$$\int_0^4 y \ln|3|\, d$$

$$\frac{\ln|3|\, y^2}{2} \Big|_{y=0}^{y=4}$$

$$8 \ln 3$$

The solution to this evaluated integral is $8 \ln 3$

10 Question 45

Let $f(x, y) = mxy^2$, where m is a constant. Find a value of m such that $\iint_{\mathcal{R}} f(x, y)\, dA = 1$, where $\mathcal{R} = [0, 1] \times [0, 2]$

11 Question 48

a. Which is easier, antidifferentiating xe^{xy} with respect to x or with respect to y? Explain.
b. Evaluate $\iint_{\mathcal{R}} xe^{xy}\, dA = 1$, where $\mathcal{R} = [0,1] \times [0,1]$

12 Question 49

a. Which is easier, antidifferentiating $\frac{y}{1+xy}$ with respect to x or with respect to y? Explain.
b. Evaluate $\iint_{\mathcal{R}} \frac{y}{1+xy}\, dA = 1$, where $\mathcal{R} = [0,1] \times [0,1]$