MATH 32A Problem Set 8

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1 Question 1

Suppose $h(x, y) = 7y\sin(x) + 19y$ is a function of two variables x and y. Suppose that $g(s,t) = 4t^2 + s$, and $f(s,t) = \frac{e^t}{s}$ are two functions of s and t. Define a composite function $P(s,t) = h(g(s,t), f(s,t))$. What are $\frac{\partial s}{\partial t}$ and $\frac{\partial P}{\partial t}$?

$$P(s,t) = h(4t^2 + s, \frac{e^t}{s}) = 7(\frac{e^t}{s})sin(4t^2 + s) + 19(\frac{e^t}{s})$$

$$\text{In this context, s is not a function of t so } \frac{\partial s}{\partial t} = 0$$

$$\frac{\partial P}{\partial t} = \frac{\partial h}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial h}{\partial y}\frac{\partial f}{\partial t} = 7y\cos(x) \cdot 8t + (7\sin x + 19)\frac{e^t}{s}$$

$$\text{Remember that } x = g(s,t) \text{ and } y = f(s,t)$$

$$\frac{\partial P}{\partial t} = 7(\frac{e^t}{s})\cos(4t^2 + s) + (7\sin(4t^2 + s) + 19)\frac{e^t}{s}$$

2 Question 2

Suppose there exists a magic cylinder whose dimensions change based on the temperature t (in degrees Fahrenheit) and the and the absolute humidity h (in grams per cubic meter): the base of the cylinder has radius $t + \frac{\sin(h)}{t}$, and the cylinder has height ht (both in miles). When the temperature is 2 degrees Fahrenheit and the humidity is 3 grams per cubic meter, what is the rate of change of V the volume of the cylinder with respect to temperature?

$$V = 2(\text{height}) \cdot \pi r^2$$

$$V = 2ht\pi(t + \frac{\sin(h)}{t})^2$$

$$\frac{\partial V}{\partial t} = 2h\pi(t + \frac{\sin(h)}{t})^2 + 2ht\pi(2(t + \frac{\sin(h)}{t})(1 - \frac{\sin(h)}{t^2}))$$

$$6\pi(2 + \frac{\sin(3)}{2})^2 + 24\pi(\frac{\sin(3)}{2})(1 - \frac{\sin(3)}{4}) = 85.94 \text{ miles}^3/\text{Fahrenheit}$$

3 Question 3

Suppose z is defined implicitly in terms of x and y via the equation $z^4 + z^2x^2 - y - 8 = 0$. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(3, 2, 1)$.

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \text{ where } F(x, y, z) = 0$$

In our case,

$$F(x, y, z) = z^4 + z^2x^2 - y - 8$$

$$\frac{\partial F}{\partial x} = 2z^2x \qquad \frac{\partial F}{\partial z} = 4z^3 + 2zx^2 \qquad \frac{\partial F}{\partial y} = -1$$

$$\text{At } (3, 2, 1), \frac{\partial F}{\partial x} = 6 \qquad \frac{\partial F}{\partial z} = 22$$

$$\frac{\partial z}{\partial x} = -\frac{3}{11}$$

$$\frac{\partial z}{\partial y} = \frac{1}{22}$$

4 Question 4

Find the critical points of the function $e^{x^2-y^2+4y}$ and for each, use the second derivative test to determine whether it is a local maximum, local minimum, saddle point, or state that the second derivative test fails

Let $f(x,y) = e^{x^2-y^2+4y}$

$$f_x = 2xe^{x^2-y^2+4y} \qquad f_y = (-2y+4)e^{x^2-y^2+4y}$$

$$2xe^{x^2-y^2+4y} = 0 \quad x = 0$$

$$(-2y+4)e^{x^2-y^2+4y} = 0 \quad y = -2$$

There is a critical point at $(0,-2)$

$$D = f_{xx}(a,b) \cdot f_{yy}(a,b) - (f_{xy}(a,b))^2$$

$$f_{xx} = 2e^{x^2-y^2+4y} + 4x^2e^{x^2-y^2+4y}$$

$$f_{yy} = -2e^{x^2-y^2+4y} + (-2y+4)^2e^{x^2-y^2+4y}$$

$$f_{xy} = 2x(-2y+4)e^{x^2-y^2+4y}$$

$$f_{xx} = e^{-8} \qquad f_{yy} = -2e^{-8} \qquad f_{xy} = 0$$

$$D = e^{-8} \cdot -2e^{-8} + 0 = -2$$

Since D is less than 0, it is a saddle point

5 Question 5

Find the critical points of function $(x+y)\ln(x^2+y^2)$ and for each, use the second derivative test to determine whether it is a local maximum,minimum, saddle point, or state that the second derivative test fails

Let $f(x,y) = (x+y)\ln(x^2+y^2)$

$$f_x = \ln(x^2+y^2) + (x+y)\frac{2x^2}{x^2+y^2} \qquad f_{xx} = \frac{2x^2}{x^2+y^2} + \frac{4x\,(x+y)}{x^2+y^2} + \frac{2x}{x^2+y^2} - \frac{4x^3\,(x+y)}{(x^2+y^2)^2}$$

$$f_y = \ln(x^2+y^2) + (x+y)\frac{2y^2}{x^2+y^2} \qquad f_y = \frac{2y^2}{y^2+x^2} + \frac{4y\,(y+x)}{y^2+x^2} + \frac{2y}{y^2+x^2} - \frac{4y^3\,(y+x)}{(y^2+x^2)^2}$$

$$f_{xy} = \frac{2x}{x^2+y^2} + \frac{2y^2}{x^2+y^2} - \frac{4y^2x\,(x+y)}{(x^2+y^2)^2}$$

$$f_x = 0 \quad \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0$$

$$f_y = 0 \quad \ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0$$

$$f_x = f_y \quad \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = \ln(x^2+y^2) + \frac{2y^2}{x^2+y^2}$$

$$x = y$$

$$\text{Let } f(x,y) = f(x,x)$$

$$f(x,x) = (2x)\ln(2x^2)$$

$$f_x(x,x) = 2\ln(2x^2) + 2x\frac{4x}{2x^2} = 0$$

$$2\ln(2x^2) + 4 = 0$$

$$\ln(2x^2) = -2$$

$$2x^2 = \frac{1}{e^2}$$

$$x^2 = \frac{1}{2e^2}$$

$$x = \pm\frac{1}{\sqrt{2}e}$$

The critical points occur at $(\frac{1}{\sqrt{2}e}, \frac{1}{\sqrt{2}e})$ and $(-\frac{1}{\sqrt{2}e}, -\frac{1}{\sqrt{2}e})$

$$D = f_{xx}(a,b)\cdot f_{yy}(a,b) - (f_{xy}(a,b))^2$$

$$\text{At } (\frac{1}{\sqrt{2}e}, \frac{1}{\sqrt{2}e})$$

$$f_{xx} = 6.84 \qquad f_{yy} = 6.84 \qquad f_{xy} = 2.844$$

$$D > 0 \text{ and } f_{xx} > 0\text{, so point must be local minimum}$$

$$\text{At } (-\frac{1}{\sqrt{2}e}, -\frac{1}{\sqrt{2}e})$$

$$f_{xx} = -0.84 \qquad f_{yy} = -0.84 \qquad f_{xy} = -4.844$$

$$D < 0\text{, so point must be saddle point}$$

6 Question 6

Find the maximum of $f(x, y) = y^2 + xy - x^2$ on the square domain $0 \leq x \leq 2, 0 \leq y \leq 2$.

$$f_x = y - 2x \qquad f_{xx} = -2$$
$$f_y = 2y + x \qquad f_{yy} = 2$$
$$f_{xy} = 1$$
$$f_x = 0 \quad y - 2x = 0$$
$$x = \frac{y}{2}$$
$$y = -\frac{x}{2}$$
$$x = 0 \qquad y = 0$$

At $(0, 0)$,

$$D = (2 \cdot -2) - 1 = -5$$

Since $D < 0$, the critical point is a saddle. The global maximum must exist on the boundary. The boundary is a square.

$$B_1 : x = 0 \quad B_2 : x = 2 \quad B_3 : y = 0 \quad B_4 : y = 2$$

$$\text{At } B_1, \quad f(0, y) = y^2$$
$$\frac{d}{dy} y^2 = 2y \quad \text{Critical point} : (0, 0)$$
$$\frac{d^2}{dy^2} y^2 = 2$$

A potential minimum exists at $(0, 0)$

$$\text{At } B_2, \quad f(2, y) = y^2 + 2y - 4$$
$$\frac{d}{dy} y^2 + 2y - 4 = 2y + 2 \quad \text{Critical point} : (2, -1)$$

Since the second derivative is positive, a potential minimum exists at $(2, -1)$

$$\text{At } B_3, \quad f(x, 0) = -x^2$$
$$\frac{d}{dx} - x^2 = -2x \quad \text{Critical point} : (0, 0)$$
$$\frac{d^2}{dx^2} - x^2 = -2$$

A potential max exists at $(0, 0)$

$$\text{At } B_3, \quad f(x, 2) = 4 + 2x - x^2$$
$$\frac{d}{dx} 4 + 2x - x^2 = 2 - 2x \quad \text{Critical point} : (1, 2)$$

A potential max exists on $(1, 2)$

Now, let's test all of the points.
$f(0, 0) = 0 \quad f(2, -1) = -5 \quad f(1, 2) = 5$ Global minimum at $(2, -1)$ at -5, global maximum at $(1, 2)$ at 5

7 Question 7

Find the maximum of $f(x, y) = xy(1 - x - y)$ on the domain D defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Find all critical points of f, and find the global maximum and global minimum for f on D

$$f_x = y(1 - x - y) - xy \qquad f_{xx} = -y - y = -2y$$
$$f_y = x(1 - x - y) - xy \qquad f_{yy} = -x - x = -2x$$
$$f_x = 0 \qquad y(1 - x - y) - xy = 0$$
$$y = 0$$

$$f_y = 0 \qquad x(1 - x - y) - xy = 0$$
$$x = 0$$

A critical point occurs at $(0, 0)$

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - f_{xy}(a, b)$$
$$f_{xy} = (1 - x - y) - y - x$$
$$D = 0 \cdot 0 - 1 = 0$$

$(0, 0)$ is a saddle point, let's test the boundary now

$$B_1 : x = -1 \quad B_2 : x = 1 \quad B_3 : y = -1 \quad B_4 : y = 1$$
$$B_1 : f(-1, y) = -y(2 - y) = -2y + y^2$$
$$\frac{d}{dy} = -2y + y^2 = -2 + 2y \quad \text{Potential min at (-1, 1)}$$
$$B_2 : f(1, y) = y(-y) = -y^2$$
$$\frac{d}{dy} = -y^2 = -2y \quad \text{Potential max at (1, 0)}$$
$$B_3 : f(x, -1) = -x(2 - x) = -2x + x^2$$
$$\frac{d}{dx} = -2x + x^2 = -2 + 2x \quad \text{Potential min at (-1, -1)}$$
$$B_4 : f(x, 1) = x(-x) = -x^2$$
$$\frac{d}{dx} = -x^2 = -2x \quad \text{Potential max at (0, 1)}$$

Now let's test the points

$$f(-1, 1) = -1(1 + 1 - 1) = -1 \quad f(1, 0) = 0 \quad f(-1, -1) = 1(1 + 1 + 1) = 3 \quad f(0, 1) = 0$$

Global maximum occurs at $(-1, -1, 3)$, global mimimum occurs at $(-1, 1, -1)$