MATH 32B Problem Set 6

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Question 5

Compute the integral of the scalar function or vector field over $r(t) = \langle \cos t, \sin t, t\rangle$ for $0 \leq t \leq \pi$

$$f(x, y, z) = x^2 + y^2 + z^2$$

Remember the formula

$$\int_a^b f(r(t))\|r'(t)\|dt$$

$$f(r(t)) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$$

$$r'(t) = \langle -\sin t, \cos t, 1\rangle \quad \|r'(t)\| = \sqrt{(-\sin^t)^2 + (\cos t)^2 + 1^2} = \sqrt{2} \quad \|r'(t)\| = \sqrt{2}$$

$$\int_0^\pi (1 + t^2)\sqrt{2}dt$$

$$\sqrt{2}\Big[t + \frac{t^3}{3}\Big]_0^\pi$$

$$\sqrt{2}\Big(\pi + \frac{\pi}{3}\Big)$$

Question 6

Compute the integral of the scalar function or vector field over $r(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq \pi$

$$f(x, y, z) = xy + z$$

$$f(r(t)) = \cos t \sin t + t$$

$$r'(t) = \langle -\sin t, \cos t, 1 \rangle \quad \|r'(t)\| = \sqrt{(-\sin^t)^2 + (\cos t)^2 + 1^2} = \sqrt{2} \quad \|r'(t)\| = \sqrt{2}$$

$$\int_0^\pi (\cos t \sin t + t)\|\mathbf{r}'(t)\|dt$$

$$\sqrt{2}\int_0^\pi (\cos t \sin t + t)dt$$

$$\sqrt{2}\left(\int_0^\pi \cos t \sin t \, dt + \int_0^\pi t \, dt\right)$$

$$\int_0^\pi \cos t \sin t, dt \quad u = \sin x \, du = \cos x$$

$$\int_0^0 udu = 0$$

$$\int_0^\pi tdt = \left[\frac{t^2}{2}\right]_0^\pi = \frac{\pi^2}{2}$$

$$\frac{\pi^2}{2} \cdot \sqrt{2} = \frac{\sqrt{2}\pi^2}{2}$$

Question 10

Compute $\int_C f\,ds$ for the curve specified

$$f(x,y) = \frac{y^3}{x^7}, \quad y = \frac{x^4}{4} \text{ for } 1 \le x \le 2$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\frac{dy}{dx} = \frac{4x^3}{4} = x^3$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+x^6}\,dx$$

$$f(x,y) = \frac{\left(\frac{x^4}{4}\right)^3}{x^7} = \frac{\frac{x^{12}}{64}}{x^7} = \frac{x^5}{64}$$

$$\int_1^2 \frac{x^5}{64}\sqrt{1+x^6}\,dx$$

$$I = \int_1^2 \frac{x^5}{64}\sqrt{1+x^6}\,dx$$

$$u = 1 + x^6\, du = 6x^5\,dx$$

$$\frac{du}{6} = x^5\,dx$$

$$I = \int_{1+1^6}^{1+2^6} \frac{1}{64}\cdot\frac{du}{6}\cdot\sqrt{u}$$

$$= \frac{1}{384}\int_2^{65} u^{1/2}\,du$$

$$\frac{1}{384}\cdot\frac{2}{3}\left[u^{3/2}\right]_2^{65}$$

$$= \frac{1}{576}\left[65^{3/2} - 2^{3/2}\right]$$

$$\frac{1}{576}\left(65^{3/2} - 2^{3/2}\right)$$

Question 11

Compute $\int_C f\,ds$ for the curve specified

$$f(x,y,z) = z^2, \quad r(t) = \langle 2t, 3t, 4t \rangle \text{ for } 0 \le t \le 2$$

Arc length element

$$ds = \left\| \frac{d\mathbf{r}}{dt} \right\| dt$$

$$\frac{d\mathbf{r}}{dt} = \langle 2, 3, 4 \rangle$$

$$\left\| \frac{d\mathbf{r}}{dt} \right\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$ds = \sqrt{29}\,dt$$

$$f(x,y,z) = z^2 = (4t)^2 = 16t^2$$

$$\int_0^2 16t^2\sqrt{29}\,dt$$

$$\sqrt{29} \cdot 16 \int_0^2 t^2\,dt$$

$$\int t^2\,dt = \frac{t^3}{3}$$

$$\left[\frac{t^3}{3}\right]_0^2 = \frac{8}{3}$$

$$\sqrt{29} \cdot 16 \cdot \frac{8}{3} = \frac{128\sqrt{29}}{3}$$

$$\frac{128\sqrt{29}}{3}$$

Question 18

Calculate $\int_{\mathcal{C}} 1\,ds$, where the curve $\mathcal{C}$ is parameterized by $r(t) = \langle e^t, \sqrt{2}t, e^{-t}$ for $0 \leq t \leq 2$

$$r'(t) = \langle e^t, \sqrt{2}, -e^{-t}\rangle$$

$$\|r'(t)\| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$$

$$\int_{\mathcal{C}} 1\,ds = \int_0^2 (e^t + e^{-t})\,dt$$

$$\int_0^2 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^2 = e^2 - e^{-2}$$

$$e^2 - e^{-2}$$

Question 19

Compute $\int_{\mathcal{C}} F \cdot dr$ for the oriented curve specified

$$\mathbf{F}(x,y) = \langle 1+x^2, xy^2 \rangle, \text{ line segment from } (0,0) \text{ to } (1,3)$$

$$F(x,y) = \langle 1+x^2, xy^2 \rangle, \quad r(t) = \langle t, 3t \rangle, \quad 0 \le t \le 1$$

$$F(r(t)) = \langle 1+t^2, 9t^3 \rangle, \quad r'(t) = \langle 1, 3 \rangle$$

$$F(r(t)) \cdot r'(t) = (1+t^2)(1) + (9t^3)(3) = 1 + t^2 + 27t^3$$

$$\int_{\mathcal{C}} F \cdot dr = \int_0^1 (1 + t^2 + 27t^3)\, dt$$

$$\int_0^1 1\, dt = 1, \quad \int_0^1 t^2\, dt = \frac{1}{3}, \quad \int_0^1 27t^3\, dt = \frac{27}{4}$$

$$\int_{\mathcal{C}} F \cdot dr = 1 + \frac{1}{3} + \frac{27}{4} = \frac{12}{12} + \frac{4}{12} + \frac{81}{12} = \frac{97}{12}$$

Question 20

Compute $\int_C F \cdot dr$ for the oriented curve specified

$$F(x,y) = \langle -2, y \rangle, \text{ half circle } x^2 + y^2 = 1 \text{ with } y \geq 0, \text{ oriented clockwise}$$

$$\mathbf{r}(t) = \langle \cos t, -\sin t \rangle, \quad 0 \leq t \leq \pi$$

$$F(r(t)) = \langle -2, -\sin t \rangle$$

$$r'(t) = \langle -\sin t, -\cos t \rangle$$

$$F(r(t)) \cdot \mathbf{r}'(t) = (-2)(-\sin t) + (-\sin t)(-\cos t) = 2\sin t + \sin t \cos t$$

$$\int_C F \cdot dr = \int_0^\pi (2\sin t + \sin t \cos t)\, dt$$

$$\int_0^\pi 2\sin t\, dt = 2\left[-\cos t\right]_0^\pi = 4$$

$$\int_0^\pi \sin t \cos t\, dt = \frac{1}{2}\int_0^\pi \sin 2t\, dt = 0$$

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Question 28

Evaluate the line integral

$$\int_C y dy \text{ over } y = x^3 \text{ for } 0 \leq x \leq 3$$

$$y = x^3, \quad 0 \leq x \leq 3$$

$$dy = 3x^2\, dx$$

$$\int_C y\, dy = \int_0^3 (x^3)(3x^2)\, dx = \int_0^3 3x^5\, dx$$

$$\int_0^3 3x^5\, dx = 3\int_0^3 x^5\, dx = 3\left[\frac{x^6}{6}\right]_0^3$$

$$\int_0^3 x^5\, dx = \left[\frac{x^6}{6}\right]_0^3 = \frac{(3)^6}{6} - \frac{(0)^6}{6} = \frac{729}{6}$$

$$3 \cdot \frac{729}{6} = \frac{729}{2}$$

$$\frac{729}{2}$$

Question 37

Calculate the line integral of $\mathbf{F}(x, y, z) = \langle e^z, e^{x-y}, e^y \rangle$ over the given path

$$\mathbf{r}(t) = \langle 0, 0, t \rangle, \quad 0 \le t \le 1$$

$$\mathbf{r}'(t) = \langle 0, 0, 1 \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \langle e^t, 1, 1 \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 1$$
$$\int_0^1 1 \, dt = 1$$

$$\mathbf{r}(t) = \langle 0, t, 1 \rangle, \quad 0 \le t \le 1$$
$$\mathbf{r}'(t) = \langle 0, 1, 0 \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \langle e, e^{-t}, e^t \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = e^{-t}$$
$$\int_0^1 e^{-t} \, dt = 1 - \frac{1}{e}$$

$$\mathbf{r}(t) = \langle -t, 1, 1 \rangle, \quad 0 \le t \le 1$$
$$\mathbf{r}'(t) = \langle -1, 0, 0 \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \langle e, e^{-t-1}, e \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -e$$
$$\int_0^1 -e \, dt = -e$$

$$1 + \left(1 - \frac{1}{e}\right) - e = \boxed{2 - \frac{1}{e} - e}$$

Question 43

Determine whether the line integrals of the vector fields around the circle are positive, negative, or zero

(A) The vector fields around the circle is zero
(B) The vector fields around the circle is negative
(C) The vector fields around the circle is zero

Question 2

Let $\mathbf{F}(x, y, z) = \langle x^{-1}z, y^{-1}z, \ln(xy)\rangle$

a. Verify that $\mathbf{F} = \nabla f$, where $f(x, y, z) = z\ln(xy)$

$$\mathbf{F}(x, y, z) = \left\langle \frac{z}{x}, \frac{z}{y}, \ln(xy) \right\rangle$$

$$f(x, y, z) = z\ln(xy)$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\frac{\partial f}{\partial x} = z \cdot \frac{1}{x} = \frac{z}{x}$$

$$\frac{\partial f}{\partial y} = z \cdot \frac{1}{y} = \frac{z}{y}$$

$$\frac{\partial f}{\partial z} = \ln(xy)$$

$$\nabla f = \left\langle \frac{z}{x}, \frac{z}{y}, \ln(xy) \right\rangle$$

$$\mathbf{F} = \nabla f$$

$$\mathbf{F} = \nabla f$$

b. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{r}(t) = \langle e^t, e^{2t}, t^2\rangle$ for $1 \le t \le 3$

$$\mathbf{r}(t) = \langle e^t, e^{2t}, t^2\rangle$$

$$\mathbf{r}'(t) = \langle e^t, 2e^{2t}, 2t\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle \frac{t^2}{e^t}, \frac{t^2}{e^{2t}}, \ln(e^t \cdot e^{2t}) \right\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle t^2e^{-t}, t^2e^{-2t}, 3t \right\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (t^2e^{-t} \cdot e^t) + (t^2e^{-2t} \cdot 2e^{2t}) + (3t \cdot 2t)$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^2 + 2t^2 + 6t^2 = 9t^2$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 9t^2\,dt$$

$$\int_1^3 9t^2\,dt = 9\left[\frac{t^3}{3}\right]_1^3 = 9\left(\frac{27}{3} - \frac{1}{3}\right) = 9 \cdot \frac{26}{3} = 78$$

c. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, for any path $\mathcal{C}$ from $P = (\frac{1}{2}, 4, 2)$ to $Q = (2, 2, 3)$ contained in the region $x > 0$, $y > 0$.

$$f(x, y, z) = z \ln(xy)$$

$$P = \left(\frac{1}{2}, 4, 2\right), \quad Q = (2, 2, 3)$$

$$f(P) = 2 \ln\left(\frac{1}{2} \cdot 4\right) = 2 \ln(2)$$

$$f(Q) = 3 \ln(2 \cdot 2) = 3 \ln(4) = 6 \ln(2)$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P) = 6 \ln(2) - 2 \ln(2) = 4 \ln(2)$$

$$4 \ln 2$$

d. In part (c), why is it necessary to specify that the path lies in the region where x and y are positive?

It was necessary to specify that the path lies in the region where x and y are positive due to the constraints of the question. First off, we had the terms x^{-1} and y^{-1}. We can't divide at 0. Also since we have $\ln xy$, we can't go negative.

Question 3

Verify that $\mathbf{F} = \nabla f$ and evaluate the line integral of $\mathbf{F}$ on the interval $1 \leq t \leq 4$

$$\mathbf{F} = \langle 3, 6y \rangle, \quad f(x, y) = 3x + 3y^2; \quad \mathbf{r}(t) = \langle t, 2t^{-1} \rangle, 1 \leq t \leq 4$$

$$f(x, y) = 3x + 3y^2, \quad \nabla f = \langle 3, 6y \rangle$$

$$\mathbf{F} = \langle 3, 6y \rangle, \quad \mathbf{F} = \nabla f$$

$$\mathbf{r}(t) = \langle t, 2t^{-1} \rangle, \quad 1 \leq t \leq 4$$

$$f(\mathbf{r}(t)) = 3t + 3(2t^{-1})^2 = 3t + \frac{12}{t^2}$$

$$f(\mathbf{r}(4)) = 3(4) + \frac{12}{4^2} = 12 + \frac{12}{16} = 12.75$$

$$f(\mathbf{r}(1)) = 3(1) + \frac{12}{1^2} = 3 + 12 = 15$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(4)) - f(\mathbf{r}(1)) = 12.75 - 15 = -2.25$$

Question 4

Verify that $\mathbf{F} = \nabla f$ and evaluate the line integral of $\mathbf{F}$ on the interval $1 \le t \le 4$

$$\mathbf{F} = \langle \cos y, -x \sin y \rangle, \quad f(x,y) = x \cos y;$$

upper half of the unit circle centered at the origin, oriented counterclockwise

$$f(x,y) = x \cos y, \quad \nabla f = \langle \cos y, -x \sin y \rangle$$

$$\mathbf{F} = \langle \cos y, -x \sin y \rangle, \quad \mathbf{F} = \nabla f$$

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le \pi$$

$$f(\mathbf{r}(t)) = (\cos t)(\cos(\sin t))$$

$$f(\mathbf{r}(\pi)) = (-1)(\cos(0)) = -1$$

$$f(\mathbf{r}(0)) = (1)(\cos(0)) = 1$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = -1 - 1 = -2$$