

MATH 32B Problem Set 1

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1 Question 11

The following table gives the approximate height at quarter-meter intervals of a mound of gravel. Estimate the volume of the mound by computing the average of the two Riemann sums $S_{4,3}$ with lower-left and upper-right vertices of the subrectangles as sample points.

0.75	0.1	0.2	0.2	0.15	0.1
0.5	0.2	0.3	0.5	0.4	0.2
0.25	0.15	0.2	0.4	0.3	0.2
$y : \mathbf{0}$	0.1	0.15	0.2	0.15	0.1
	$x : \mathbf{0}$	0.25	0.5	0.75	1

Computing area using lower-left vertices as sample points: The vertices we will be using are:

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.15 & 0.2 & 0.15 \end{bmatrix}$$

Note that Δx and Δy are both 0.25, meaning the volume of each rectangle will be the height of the rectangles times 0.25^2 , which is 0.0625

$$0.0625 \cdot \begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.15 & 0.2 & 0.15 \end{bmatrix} = \begin{bmatrix} 0.0125 & 0.01875 & 0.03125 & 0.025 \\ 0.009375 & 0.0125 & 0.025 & 0.01875 \\ 0.00625 & 0.009375 & 0.0125 & 0.009375 \end{bmatrix}$$

Adding up these areas gives us 0.190625

Now, let's repeat, except we will use the upper-right vertices as sample points

$$0.0625 \cdot \begin{bmatrix} 0.2 & 0.2 & 0.15 & 0.1 \\ 0.3 & 0.5 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.0125 & 0.0125 & 0.009375 & 0.00625 \\ 0.01875 & 0.03125 & 0.025 & 0.0125 \\ 0.0125 & 0.025 & 0.01875 & 0.03125 \end{bmatrix}$$

Adding up these areas gives us 0.196875. The average of these two sums is **0.19375**, which is our estimate for the volume of the mound.

2 Question 15

Use symmetry to evaluate the double integral

$$\iint_{\mathcal{R}} x^3 dA, \quad \mathcal{R} = [-4, 4] \times [0, 5]$$

Note that that x^3 is odd and lacks horizontal or vertical shifts. This means so for some value a , the area between the curve and $[-a, 0]$ will be equal to the area between the curve and $[0, a]$ multiplied by -1 .

Also, note that we can split the domain into two sub-domains. Let us split \mathcal{R} into $\mathcal{R}_1 = [-4, 0] \times [0, 5]$ and $\mathcal{R}_2 = [0, 4] \times [0, 5]$.

$$\iint_{\mathcal{R}} x^3 dA, \quad \mathcal{R} = [-4, 4] \times [0, 5] = \iint_{\mathcal{R}_1} x^3 dA, \quad \mathcal{R}_1 = [-4, 0] \times [0, 5] + \iint_{\mathcal{R}_2} x^3 dA, \quad \mathcal{R}_2 = [0, 4] \times [0, 5]$$

From the observation we stated before, it can be said that

$$\iint_{\mathcal{R}_1} x^3 dA, \quad \mathcal{R}_1 = [-4, 0] \times [0, 5] = - \iint_{\mathcal{R}_2} x^3 dA, \quad \mathcal{R}_2 = [0, 4] \times [0, 5]$$

We can rewrite the original expression as

$$- \iint_{\mathcal{R}_2} x^3 dA + \iint_{\mathcal{R}_2} x^3 dA = 0$$

This double integral evaluates to 0.

3 Question 17

Use symmetry to evaluate the double integral

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi]$$

Note that $\sin x$ oscillates from -1 to 1. Importantly, without any transformations, $\sin x$ completes one whole revolution from 0 to 2π , and the function from 0 to π is identical to the function from π to 2π reflected over the x -axis.

Also remember that we can split the domain into two sub-domains. Let us split \mathcal{R} into $\mathcal{R}_1 = [0, \pi] \times [0, 2\pi]$ and $\mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$.

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi] = \iint_{\mathcal{R}_1} \sin x \, dA, \quad \mathcal{R}_1 = [0, \pi] \times [0, 2\pi] + \iint_{\mathcal{R}_2} \sin x \, dA, \quad \mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$$

From what we said before, we can observe that

$$\iint_{\mathcal{R}_1} \sin x \, dA, \quad \mathcal{R}_1 = [0, \pi] \times [0, 2\pi] = - \iint_{\mathcal{R}_2} \sin x \, dA, \quad \mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$$

Finally, we can state that

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi] = - \iint_{\mathcal{R}_2} \sin x \, dA + \iint_{\mathcal{R}_2} \sin x \, dA = 0$$

The double integral evaluates to 0.

4 Question 19

Evaluate the iterated integral

$$\begin{aligned} & \int_1^3 \int_0^2 x^3 y \, dy \, dx \\ & \int_1^3 \int_0^2 x^3 y \, dy \, dx = \int_1^3 \left(\int_0^2 x^3 y \, dy \right) dx \\ & = \int_1^3 \left(x^3 \cdot \frac{y^2}{2} \Big|_{y=0}^{y=2} \right) dx \\ & = \int_1^3 \left(x^3 \cdot \left(\frac{4}{2} - \frac{0}{2} \right) \right) dx \\ & = \int_1^3 2x^3 \, dx = \frac{x^4}{2} \Big|_{x=1}^{x=3} = \frac{81}{2} - \frac{1}{2} = \frac{80}{2} = 40 \end{aligned}$$

The iterated integral evaluates to 40.

5 Question 21

Evaluate the iterated integral

$$\begin{aligned} & \int_4^9 \int_{-3}^8 1 \, dx \, dy \\ & \int_4^9 \left(\int_{-3}^8 1 \, dx \right) dy = \\ & \int_4^9 x \Big|_{x=-3}^{x=8} dy = \\ & \int_4^9 11 \, dy = 11x \Big|_{x=4}^{x=9} = 99 - 44 = 55 \end{aligned}$$

The iterated integral evaluates to 55.

6 Question 23

Evaluate the iterated integral

$$\begin{aligned} & \int_{-1}^1 \int_0^{\pi} x^2 \sin y \, dy \, dx \\ & \int_{-1}^1 \left(\int_0^{\pi} x^2 \sin y \, dy \right) dx \\ & \int_{-1}^1 \left(x^2 \cdot -\cos y \Big|_{y=0}^{y=\pi} \right) dx \\ & \int_{-1}^1 x^2 (1 - (-1)) \, dx \\ & \int_{-1}^1 2x^2 \, dx = \frac{2x^3}{3} \Big|_{x=-1}^{x=1} = \frac{2}{3} - \frac{-2}{3} = \frac{4}{3} \end{aligned}$$

The iterated integral evaluates to $\frac{4}{3}$.

7 Question 25

Evaluate the iterated integral

$$\begin{aligned}& \int_2^6 \int_1^4 x^2 \, dx \, dy \\& \int_2^6 \left(\int_1^4 x^2 \, dx \right) dy \\& \int_2^6 \left(\frac{x^3}{3} \Big|_{x=1}^{x=4} \right) dy \\& \int_2^6 \frac{64}{3} - \frac{1}{3} \, dy = \int_2^6 \frac{63}{3} \, dy \\& \frac{63x}{3} \Big|_{x=2}^{x=6} = \frac{378 - 126}{3} = \frac{252}{3} = 84\end{aligned}$$

The iterated integral evaluates to 84.

8 Question 37

Evaluate the integral

$$\iint_{\mathcal{R}} \frac{x}{y} dA, \quad \mathcal{R} = [-2, 4] \times [1, 3]$$

$$\iint_{\mathcal{R}} \frac{x}{y} dx dy, \quad \mathcal{R} = [-2, 4] \times [1, 3]$$

$$\int_1^3 \int_{-2}^4 \frac{x}{y} dx dy$$

$$\int_1^3 \left(\frac{x^2}{2y} \Big|_{x=-2}^{x=4} dx \right) dy$$

$$\int_1^3 \frac{1}{2y} (16 - 4) dy$$

$$\int_1^3 \frac{6}{y} dy$$

$$6 \ln y \Big|_{y=1}^{y=3} = 6 \ln 3 - 6 \ln 1 = 6 \ln 3$$

9 Question 40

Evaluate the integral

$$\iint_{\mathcal{R}} \frac{y}{x+1} dA, \quad \mathcal{R} = [0, 2] \times [0, 4]$$

$$\int_0^4 \int_0^2 \frac{y}{x+1} dx dy$$

$$\int_0^4 \int_0^2 \frac{y}{x+1} dx dy$$

$$\int_0^4 \left(y \ln |x+1| \Big|_{x=0}^{x=2} \right) dy$$

$$\int_0^4 y(\ln |3| - \ln |1|) dy$$

$$\int_0^4 y \ln |3| dy$$

$$\frac{\ln |3| y^2}{2} \Big|_{y=0}^{y=4}$$

$$8 \ln 3$$

The solution to this evaluated integral is $8 \ln 3$

10 Question 45

Let $f(x, y) = mxy^2$, where m is a constant. Find a value of m such that $\iint_{\mathcal{R}} f(x, y) dA = 1$, where $\mathcal{R} = [0, 1] \times [0, 2]$

$$\begin{aligned} & \int_0^2 \int_0^1 mxy^2 dx dy \\ & \int_0^2 \left(\frac{mx^2y^2}{2} \Big|_{x=0}^{x=1} \right) dy \\ & \int_0^2 \frac{my^2}{2} dy \\ & \frac{my^3}{6} \Big|_{y=0}^{y=2} \\ & \frac{8m}{6} = 1 \\ & m = \frac{3}{4} \end{aligned}$$

When $m = \frac{3}{4}$, the expression $\iint_{\mathcal{R}} f(x, y) dA, \mathcal{R} = [0, 1] \times [0, 2] = 1$ is true!

11 Question 48

a) Which is easier, antidifferentiating xe^{xy} with respect to x or with respect to y ? Explain.

b) Evaluate $\iint_{\mathcal{R}} xe^{xy} dA$, where $\mathcal{R} = [0, 1] \times [0, 1]$

a) It is easier antidifferentiating xe^{xy} with respect to y , as we can simply treat x as a constant. Computing $\int xe^{xy} dy$ requires the same steps as solving $\int 4e^{4y} dy$, which is relatively simple

b)

$$\iint_{\mathcal{R}} xe^{xy} dA, \quad \mathcal{R} = [0, 1] \times [0, 1]$$

$$\int_0^1 \int_0^1 xe^{yx} dy dx$$

$$\int_0^1 \left(e^{yx} \Big|_{y=0}^{y=1} \right) dx$$

$$\int_0^1 e^x - 1 dx$$

$$e^x - x \Big|_{x=0}^{x=1}$$

$$e - 1 - 1 - 0 = e - 2$$

The integral evaluates to $e - 2$

12 Question 49

a) Which is easier, antidifferentiating $\frac{y}{1+xy}$ with respect to x or with respect to y ? Explain.

b) Evaluate $\iint_{\mathcal{R}} \frac{y}{1+xy} dA$, where $\mathcal{R} = [0, 1] \times [0, 1]$

a) It is easier to antidifferentiate $\frac{y}{1+xy}$ with respect to x , as we would only need to worry about one variable in the denominator. If we had integrated with respect to y , we would need to consider the y in the numerator and the y in the denominator. Integrating $\int \frac{2}{1+2x} dx$ can be solved with basic integration rules. However, antidifferentiating $\int \frac{y}{1+2y}$ would likely need to be solved using a technique like integration by parts.

b)

$$\iint_{\mathcal{R}} \frac{y}{1+xy} dA, \quad \mathcal{R} = [0, 1] \times [0, 1]$$

$$\int_0^1 \int_0^1 \frac{y}{1+xy} dx dy$$

$$\int_0^1 \left(\frac{y \ln |1+xy|}{y} \Big|_{y=0}^{y=1} \right) dx$$

$$\int_0^1 \ln |1+x| dx$$

$$(1+x) \ln(1+x) - (1+x) \Big|_{x=0}^{x=1}$$

$$2 \ln 2 - 2 - (0 - 1) = \ln 2 - 1$$

This integral evaluates to $2 \ln 2 - 1$, or $\ln 4 - 1$