Math 33B Problem Set 2

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Problem 1

Obtain the general solution by variation of parameters

Section 2.4 Ex 30

$$y' = -3y + 4$$

First, let's solve for the associated homogenous equation

$$\frac{dy}{dx} = -3y$$

$$y = A\exp\left(\int -3\,dx\right) = e^{-3x}$$

In our case, we'll use $A(x)$

$$y(x) = A(x)e^{-3x}$$

$$\frac{dy}{dx} = A'(x)e^{-3x} - 3A(x)e^{-3x} = -3A(e)^{-3x} + 4$$

$$A'(x) = 4e^{3x}$$

$$A(x) = \int A'(x)\,dx = \int 4e^{3x}\,dx = \frac{4e^{3x}}{3} + C$$

$$y(x) = \left(\frac{4e^{3x}}{3} + C\right)e^{-3x}$$

$$y(x) = \frac{4}{3} + Ce^{-3x}$$

Section 2.4 Ex 35

$$y' + 2xy = 4x$$

Same as above, the homogenous equation would be

$$\frac{dy}{dx} = -2xy$$

$$y = \exp\left(\int -2x\,dx\right) = e^{-x^2}$$

$$y(x) = A(x)e^{-x^2}$$

$$A'(x)e^{-x^2} - 2A(x)xe^{-x^2} = -2xA(x)e^{-x^2} + 4x$$

$$A'(x) = 4xe^{x^2}$$

$$A(x) = \int 4xe^{x^2}\,dx = 2e^{x^2} + C$$

$$y(x) = 2 + Ce^{-x^2}$$

Problem 2

Solve the initial value problems by variation of parameters

Section 2.4 Ex 37

$$y' + \frac{1}{2}y = t \quad y(0) = 1$$

$$\frac{dy}{dt} = -\frac{1}{2}y$$

$$y = A\exp\left(\int -\frac{1}{2}\,dt\right) = Ae^{-t/2}$$

$$y(t) = A(t)e^{-t/2}$$

$$A'(t)e^{-t/2} - \frac{1}{2}A(t)e^{-t/2} = \frac{1}{2}A(t)e^{-t/2} + t$$

$$A'(t) = te^{t/2}$$

$$\int u\,dv = uv - \int v\,du$$

$$A(t) = \int te^{t/2}\,dt = 2te^{t/2} + C - \int 2e^{t/2}\,dt = 2te^{t/2} - 4e^{t/2} + C$$

$$y(t) = 2t - 4 + Ce^{-t/2}$$

$$1 = -4 + C \cdot 1 \quad C = 5$$

$$y(t) = 2t - 4 + 5e^{-t/2}$$

Section 2.4 Ex 40

$$x' + \frac{2}{t^2}x = \frac{1}{t^2} \quad x(1) = 0$$

$$\frac{dx}{dt} = -\frac{2}{t^2}x$$

$$x = A\exp\left(\int -\frac{2}{t^2}\,dt\right) = Ae^{2/t}$$

$$x(t) = A(t)e^{2/t}$$

$$A'(t)e^{2/t} - \frac{2}{t^2}A(t)e^{2/t} = -\frac{2}{t^2}A(t)e^{2/t} + \frac{1}{t^2}$$

$$A'(t) = \frac{1}{t^2}e^{-2/t}$$

$$A(t) = \int \frac{1}{t^2}e^{-2/t}\,dt = \frac{e^{-2/t}}{2} + C$$

$$x(t) = \frac{1}{2} + Ce^{2/t}$$

$$0 = \frac{1}{2} + Ce^2$$

$$-\frac{1}{2} = Ce^2$$

$$-\frac{1}{2e^2} = C$$

$$x(t) = \frac{1}{2} - \frac{1}{2e^2}$$

Problem 3

Suppose that y_1 and y_2 are both solutions to the equation $y' = a(t)y + f(t)$ where $a(t) > 0$ and $f(t) > 0$. For what constants C_1 and C_2 is $y = C_1 y_1 + C_2 y_2$ a solution to the same equation?

$$a(t)y + f(t) = C_1 y_1' + C_2 y_2'$$

$$y_1' = a(t)y_1 + f(t)$$

$$y_2' = a(t)y_2 + f(t)$$

$$y' = C_1(a(t)y_1 + f(t)) + C_2(a(t)y_2 + f(t))$$

$$a(t)(C_1 y_1 + C_2 y_2) + f(t)(C_1 + C_2) = a(t)y + f(t)$$

$$f(t)(C_1 + C_2) = f(t)$$

$$C_1 + C_2 = 1$$

Problem 4

$$T' = -k(T - A\sin\omega t)$$

a) Find a solution T_h of the homogeneous equation $T' + kT = 0$

$$\frac{dT}{dt} = -kT$$

$$T = A\exp\left(\int -k\,dt\right) = Ae^{-kt}$$

b) The equation (4.45) is not autonomous, so finding a particular solution T_p is a bit more difficult. However, it doesn't hurt to guess. [4] As a first guess, substitute $T_p = C\cos\omega t + D\sin\omega t$ into the equation $T' + kT = kA\sin\omega t$ and show that

$$-\omega C + kD = kA \quad \text{and} \quad kC + \omega D = 0.$$

Solution:

$$T_p' = -C\omega\sin\omega t + D\omega\cos\omega t$$

$$-C\omega\sin\omega t + D\omega\cos\omega t + k(C\cos\omega t + D\sin\omega t) = kA\sin\omega t$$

$$(kD - \omega C)\sin\omega t + (\omega D + kC)kA\cos\omega t = kA\sin\omega t$$

Since coefficients need to match,

$$-\omega C + kD = kA \quad \text{and} \quad kC + \omega D = 0.$$

is true

c) Solve the simultaneous equations in part (b) and use Theorem 4.41 to show that the general solution of equation (4.45) is

$$T = T_h + T_p$$

$$= Fe^{-kt} + \frac{kA}{k^2 + \omega^2}[k\sin\omega t - \omega\cos\omega t],$$

where F is an arbitrary constant.

$$C = \frac{-\omega D}{k}$$

$$-\omega\left(-\frac{\omega D}{k}\right) + kD = kA = \frac{\omega^2 D}{k} + kD$$

$$D\frac{w^2 + k^2}{k} = kA \quad D = \frac{k^2 A}{w^2 + k^2}$$

$$C = -\frac{\omega k A}{w^2 + k^2}$$

$$T_p = \frac{k^2 A}{w^2 + k^2}\cos\omega t - \frac{\omega k A}{w^2 + k^2}\sin\omega t = \frac{kA}{k^2 + w^2}(k\sin\omega t - \omega\cos\omega t)$$

$$T = Ae^{-kt} + \frac{kA}{k^2 + w^2}(k\sin\omega t - \omega\cos\omega t)$$

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Problem 5

Section 2.5 Exercise 6

   A tank initially contains 100 gal of a salt-water solution containing 0.05 lb of salt for each gallon of water. At time zero, pure water is poured into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bot-tom of the tank that allows the salt-water solution to leave the tank at a rate of 3 gal per minute. What will be the salt content in the tank when precisely 50 gal of salt solution remain

   Let $V(t)$ represent the volume of salt solution present in the tank

$$V = V(0) + \int^t -1 dV/dt = 100 - t$$

$$\text{For } V = 50, \quad t = 50$$

   Let $S(t)$ represent the amount of salt in the tank

$$\frac{dS}{dt} = 0 - \frac{S(t)}{V(t)} 3 = -\frac{3S}{100 - t}$$

$$\frac{dS}{S} = -3 \frac{dt}{100 - t}$$

$$\ln|S| = 3\ln|100 - t| + C$$

$$S = e^C (100 - t)^3 = C(100 - t)^3$$

$$5 = C(100 - 0)^3 \quad C = \frac{5}{100^3}$$

$$S = \frac{5}{10^6}(100 - 50)^3$$

$$S = 0.625$$

Problem 6

Consider two tanks, labeled tank A and tank B for reference. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/s. There is a drain at the bottom of tank A. The solution leaves tank A via this drain at a rate of 5 gal/s and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 gal/s. What is the salt content in tank B at the precise moment that tank B contains 250 gal of solution?

Let S_a, S_b, V_b represent the salt content of A, salt content of B, and volume of B respectively

$$\frac{dV_b}{dt} = 5 - 2.5 = 2.5$$

$$\frac{dS_a}{dt} = 0 - 5\frac{S}{100} = -\frac{S}{20}$$

$$S = A\exp(\int \frac{-1}{20} dt) = Ae^{-t/20}$$

$$S(0) = 20 \quad A = 20 \quad 20e^{-t/20}$$

So the amount of salt that is flowing into tank B is

$$5\frac{S_a}{100} = e^{-t/20}$$

$$\frac{dS_b}{dt} = e^{-t/20} - 2.5\frac{S_b}{200 + 2.5t} = e^{-t/20} - \frac{Q_b}{80 + t}$$

$$\frac{dS_b}{dt} + \frac{S_b}{80 + t} = e^{-t/20}$$

$$\mu(x) = A\exp\int \frac{1}{80 + t} dt = 80 + t$$

$$(t + 80)\frac{dS_b}{dt} + S_b = (t + 80)e^{-t/20}$$

$$S_b = \int (t + 80)e^{-t/20} dt = (t + 80)e^{-t/20} - \int e^{-t/20} dt = (t + 80)e^{-e/20} + 20e^{-t/20} + C$$

$$40 = 20 + C \quad C = 20$$

$$S_b(t) = (t + 80)e^{-e/20} + 20e^{-t/20} + 40$$

$$S_b(20) = 43.17$$

Question 7

Lake Happy Times contains 100 km^3 of pure water. It is fed by a river at a rate of $50 \text{ km}^3/\text{yr}$. At time zero, there is a factory on one shore of Lake Happy Times that begins introducing a pollutant to the lake at a rate of $2 \text{ km}^3/\text{yr}$. There is another river that is fed by Lake Happy Times at a rate that keeps the volume of Lake Happy Times constant. This means that the rate of flow from Lake Happy Times into the outlet river is $52 \text{ km}^3/\text{yr}$. In turn, the flow from this outlet river goes into another lake, Lake Sad Times, at an equal rate. Finally, there is an outlet river from Lake Sad Times flowing at a rate that keeps the volume of Lake Sad Times at a constant 100 km^3.

(a) Find the amount of pollutant in Lake Sad Times at the end of 3 months.

(b) At the end of 3 months, observers close the factory due to environmental concerns and no further pollutant enters Lake Happy Times. How long will it take for the pollutant in Lake Sad Times (found in part (a)) to be cut in half?

Let P_h, P_s be the amount of pollutant is Happy and Sad lakes

$$\frac{dP_h}{dt} = 2 - 52\frac{P_h}{100}$$

$$\frac{dP_h}{dt} + 52\frac{P_h}{100} = 2$$

$$\mu(t) = A\exp(\int \frac{13t}{25})\,dt = e^{13t/25}$$

$$e^{13t/25}\frac{dP_h}{dt} + \frac{13t}{25}e^{13t/25}P_h = 2e^{13t/25}$$

$$P_h e^{13t/25} = \int 2e^{13t/25}$$

$$P_h e^{13t/25} = \frac{50}{13}e^{13t/25} + C$$

$$P_h = \frac{50}{13} + Ce^{-13t/25}$$

$$P_h(0) = 0 \quad C = -\frac{50}{13}$$

$$P_h = \frac{50}{13} - \frac{50}{13}e^{-13t/25}$$

For Sad lake,

$$\frac{dP_s}{dt} = 52\frac{P_h}{100} - 52\frac{P_s}{100}$$

$$\frac{dP_s}{dt} + \frac{52}{100}P_s = 2(1 - e^{-13t/25})$$

$$\mu = A\exp(\int \frac{52}{100}\,dt) = e^{13t/25}$$

$$e^{13t/25}\frac{dP_s}{dt} + \frac{52}{100}e^{13t/25}P_s = 2(e^{13t/25} - 1)$$

$$P_s = e^{13t/25}\int 2(e^{13t/25} - 1)\,dt$$

$$P_s(t) = 2e^{13t/25}(\frac{25}{13}e^{13t/25} - t) + C$$

$$P_s(0) = 0 \quad C = 0$$

$$P_s(3) = 0.031$$

For part b, we have

$$\frac{dP_s}{dt} = -\frac{52}{100}P_s$$

$$P_s(t) = Ae^{-\frac{52}{100}t}$$

$$0.015 = 0.031e^{-\frac{52}{100}t}$$

$$t = 1.33$$

Problem 8

Determine if the equation is exact or not and solve. Find $y(\pi) = 0$

$$(1 - y\sin x)dx + (\cos x)dy = 0$$

$$\frac{\partial(1 - y\sin x)}{\partial y} = -\sin x$$

$$\frac{\partial(\cos x)}{\partial x} = -\sin x$$

$$\frac{\partial F}{\partial x} = (1 - y\sin x)\,dx$$

$$F = \int (1 - y\sin x)\,dx = x + y\cos x + g(y)$$

$$\frac{\partial F}{\partial y} = \cos x + g'(y) = \cos x$$

$$g'(y) = 0 \quad g(y) = C$$

$$x + y\cos x = K$$

$$\pi = K$$

$$y = \frac{\pi - x}{\cos x}$$

Problem 9

Determine if the equation is exact or not and solve

$$\frac{dy}{dx} = \frac{3x^2 + y}{3y^2 - x}$$

$$(3y^2 - x)dy - (3x^2 + y)dx = 0$$

$$\frac{\partial(3y^2 - x)}{\partial x} = 1$$

$$\frac{\partial(3x^2 + y)}{\partial y} = 1$$

$$F = \int (3y^2 - x)\, dy = y^3 - xy + g(x)$$

$$\frac{\partial F}{\partial x} = -y + g'(x) = -(3x^2 + y)$$

$$g'(x) = -3x^2 \quad g(x) = -x^3 + C$$

$$F = y^3 - xy - x^3 + C$$

Problem 10

Find all solutions to the differential equation

$$(\sin y + x)\left(\frac{dy}{dx}\right)^2 + \left(x^2 + y + x\sin y\right)\frac{dy}{dx} + xy = 0$$

Show that there exist two solution with $y(0) = 0$.

$$(\frac{dy}{dx} + x)((\sin y + x)\frac{dy}{dx} + y) = 0$$

$$\frac{dy}{dx} + x = 0 \quad \frac{dy}{dx} = -x$$

$$dy = -x\,dx$$

$$y = -\frac{x^2}{2} + C$$

$$0 = 0 + C \quad C = 0$$

$$y = -\frac{x^2}{2}$$

The other solution is

$$\frac{dy}{dx} + \frac{y}{\sin y + x} = 0$$

I guess $y(x) = 0$ would fit this solution