MATH 33A Problem Set 8

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1 Question 1

Find an eigenbasis for the matrix $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \det(\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 2-\lambda & 1 \\ 6 & 3-\lambda \end{bmatrix})$$

$$(2-\lambda)(3-\lambda) - 6 = 0$$

$$6 - 5\lambda + \lambda^2 - 6 = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0 \qquad \lambda = 5$$

$$E_{\lambda=0} = \ker \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = -1/2y$$

$$\ker(\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}) = \{y \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} : y \in \mathbb{R}\}$$

$$E_{\lambda=5} = \ker \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = 1/3y$$

$$\ker(\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}) = \{y \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} : y \in \mathbb{R}\}$$

$$\text{The eigenbasis is: } \{\{y \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}\}$$

2 Question 2

For each matrix below, find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$$

$$\text{Remember that } \det(A - \lambda I) = 0$$

$$\det(\begin{bmatrix} 2-\lambda & 1 \\ 6 & 3-\lambda \end{bmatrix}) = 0$$

$$(2-\lambda)(3-\lambda) - 6 = 0$$

$$6 - 5\lambda + \lambda^2 - 6 = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0 \qquad \lambda = 5$$

$$\ker(M - \lambda I) = \ker(\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \ker(\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix})$$

$$\begin{bmatrix} -3 & 1 & 0 \\ 6 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}) = \{y\begin{bmatrix} 1 \\ 3 \end{bmatrix} : y \in \mathbb{R}\}$$

$$\ker(M - \lambda I) = \ker(\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} - 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \ker(\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix})$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}) = \{y\begin{bmatrix} 1 \\ -1/2 \end{bmatrix} : y \in \mathbb{R}\}$$

$$S = \begin{bmatrix} 1 & 1 \\ 3 & -1/2 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\det(\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix} - \lambda I) = \det(\begin{bmatrix} 3-\lambda & 6 \\ 7 & 2-\lambda \end{bmatrix})$$

$$(3-\lambda)(2-\lambda) - 42 = 0$$

$$6 - 5\lambda + \lambda^2 - 42 = 0$$

$$\lambda^2 - 5\lambda - 36 = 0$$

$$(\lambda - 9)(\lambda + 4) = 0$$

$$\lambda = 9 \qquad \lambda = -4$$

$$\ker(M - \lambda I) = \ker(\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \ker(\begin{bmatrix} -6 & 6 \\ 7 & -7 \end{bmatrix})$$

$$\begin{bmatrix} -6 & 6 \\ 7 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 7 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\{y\begin{bmatrix} 1 \\ 1 \end{bmatrix} : y \in \mathbb{R}\}$$

$$\ker(M - \lambda I) = \ker(\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix} - -4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \ker(\begin{bmatrix} 7 & 6 \\ 7 & 6 \end{bmatrix})$$

$$\begin{bmatrix} 7 & 6 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 6/7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\{y \begin{bmatrix} 1 \\ -6/7 \end{bmatrix} : y \in \mathbb{R}\}$$

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -6/7 \end{bmatrix} \qquad D = \begin{bmatrix} 9 & 0 \\ 0 & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \ker(\begin{bmatrix} 1-\lambda & 0 & 0 \\ -5 & -\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix})$$

$$= (1-\lambda)((-\lambda)(1-\lambda))$$

$$(1-\lambda)(\lambda^2 - \lambda)$$

$$\lambda(1-\lambda)(\lambda - 1)$$

$$\lambda = -1 \qquad \lambda = 1$$

$$\ker(\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \lambda I)$$

$$\ker(\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} - 1I) = \ker(\begin{bmatrix} 0 & 0 & 0 \\ -5 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix})$$

$$-5x - y + 2z = 0$$

$$y = -5x + 2z$$

$$\text{The kernel is span}(\{\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\})$$

$$\ker(\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} - (-1)I) = \ker(\begin{bmatrix} 2 & 0 & 0 \\ -5 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 Question 3

For what values of a, b, c does the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

have two distinct real eigenvalues

$$\det(A - \lambda I) = \det(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix})$$
$$(a - \lambda)(d - \lambda) - bc = 0$$
$$(ad - a\lambda - d\lambda + \lambda^2) - bc = 0$$

To find out how many roots, in this case λ this equation has, we can use the discriminant. Discriminant $= p^2 - 4q$

$$p = -a - d \qquad q = ad - bc$$

If $(-a - d)^2 - 4(ad - bc) > 0$, then there are two real roots, meaning there are two distinct real eigenvalues

4 Question 4

For what values of a, b, c are the matricies below diagonizable?

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} - \lambda I) = \det(\begin{bmatrix} 1-\lambda & a & b \\ 0 & 1-\lambda & c \\ 0 & 0 & 1-\lambda \end{bmatrix}) = (1-\lambda)^3$$

$$\lambda = 1 \text{ with multiplicity } 3$$

$$\ker(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} - I) = \ker(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix})$$

$$\begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} 0x_0 + ax_1 + bx_2 &= 0 \\ cx_2 &= 0 \end{aligned}$$

As long as $a = 0$ and $c = 0$, we can make 3 linearly independent eigenvectors, meaning the matrix can be diagonal

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} - \lambda I) = \det(\begin{bmatrix} 1-\lambda & a & b \\ 0 & 2-\lambda & c \\ 0 & 0 & 3-\lambda \end{bmatrix}) = (1-\lambda)(2-\lambda)(3-\lambda)$$

$$\lambda = 1 \qquad \lambda = 2 \qquad \lambda = 3$$

$$\ker(\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix} - I) = \ker(\begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix})$$

$$\begin{bmatrix} 0 & a & b & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{aligned} ay + bz &= 0 \\ y + cz &= 0 \\ 2z &= 0 \end{aligned}$$

$$y = 0 \quad z = 0$$

$$\{x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R}\}$$

$$\ker(\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix} - 2I) = \ker(\begin{bmatrix} -1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix})$$

$$\begin{bmatrix} -1 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$
\begin{aligned}
-x + ay + bz &= 0 \\
cz &= 0 \\
1z &= 0
\end{aligned}
$$

$$\{y \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix} : y \in \mathbb{R}\}$$

$$\ker(\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix} - 3I) = \ker(\begin{bmatrix} -2 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix})$$

$$\begin{bmatrix} -2 & a & b & 0 \\ 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
-2x + ay + bz &= 0 \\
-y + cz &= 0
\end{aligned}
$$

$$
\begin{aligned}
y &= cz \\
-2x + acz + bz &= 0 \\
x &= z\frac{ac + b}{-2}
\end{aligned}
$$

$$\{z \begin{bmatrix} \frac{ac+b}{-2} \\ c \\ 1 \end{bmatrix} : z \in \mathbb{R}\}$$

For any value of a, b, c, since there are 3 unique eigenvectors, the matrix will be diagonal