MATH 32B Problem Set 1

Allan Zhang

January 6, 2025

1 Question 11

The following table gives the approximate height at quarter-meter intervals of a mound of gravel. Estimate the volume of the mound by computing the average of the two Riemann sums $S_{4,3}$ with lower-left and upper-right vertices of the subrectangles as sample points.

0.75	0.1	0.2	0.2	0.15	0.1
0.5	0.2	0.3	0.5	0.4	0.2
0.25	0.15	0.2	0.4	0.3	0.2
y : **0**	0.1	0.15	0.2	0.15	0.1
	x : **0**	**0.25**	**0.5**	**0.75**	**1**

Computing area using lower-left verticies as sample points: The verticies we will be using are:

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.15 & 0.2 & 0.15 \end{bmatrix}$$

Note that Δx and Δy are both 0.25, meaning the volume of each rectangle will be the height of the rectangles times 0.25^2, which is 0.0625

$$0.0625 \cdot \begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.15 & 0.2 & 0.15 \end{bmatrix} = \begin{bmatrix} 0.0125 & 0.01875 & 0.03125 & 0.025 \\ 0.009375 & 0.0125 & 0.025 & 0.01875 \\ 0.00625 & 0.009375 & 0.0125 & 0.009375 \end{bmatrix}$$

Adding up these areas gives us 0.190625

Now, let's repeat, except we will use the upper-right verticies as sample points

$$0.0625 \cdot \begin{bmatrix} 0.2 & 0.2 & 0.15 & 0.1 \\ 0.3 & 0.5 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.0125 & 0.0125 & 0.009375 & 0.00625 \\ 0.01875 & 0.03125 & 0.025 & 0.0125 \\ 0.0125 & 0.025 & 0.01875 & 0.03125 \end{bmatrix}$$

Adding up these areas gives us 0.196875. The average of these two sums is **0.19375**, which is our estimate for the volume of the mound.

2 Question 15

Use symmetry to evauate the double integral

$$\iint_{\mathcal{R}} x^3\,dA, \quad \mathcal{R} = [-4,4]\times[0,5]$$

Note that that x^3 is odd and lacks horizontal or vertical shifts. This means so for some value a, the area between the curve and $[-a,0]$ will be equal to the area between the curve and $[0,a]$ multiplied by -1.

Also, note that we can split the domain into two sub-domains. Let us split $\mathcal{R}$ into $\mathcal{R}_1 = [-4,0]\times[0,5]$ and $\mathcal{R}_2 = [0,4]\times[0,5]$.

$$\iint_{\mathcal{R}} x^3\,dA, \quad \mathcal{R} = [-4,4]\times[0,5] = \iint_{\mathcal{R}_1} x^3\,dA, \quad \mathcal{R}_1 = [-4,0]\times[0,5] + \iint_{\mathcal{R}_2} x^3\,dA, \quad \mathcal{R}_2 = [0,4]\times[0,5]$$

From the observation we stated before, it can be said that

$$\iint_{\mathcal{R}_1} x^3\,dA, \quad \mathcal{R}_1 = [-4,0]\times[0,5] = -\iint_{\mathcal{R}_2} x^3\,dA, \quad \mathcal{R}_2 = [0,4]\times[0,5]$$

We can rewrite the original expression as

$$-\iint_{\mathcal{R}_2} x^3\,dA + \iint_{\mathcal{R}_2} x^3\,dA = 0$$

This double integral evaluates to 0.

3 Question 17

Use symmetry to evaluate the double integral

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi]$$

Note that $\sin x$ oscillates from -1 to 1. Importantly, without any transformations, $\sin x$ completes one whole revolution from 0 to 2π, and the function from 0 to π is identical to the function from π to 2π reflected over the $x-$axis.

Also remember that we can split the domain into two sub-domains. Let us split $\mathcal{R}$ into $\mathcal{R}_1 = [0, \pi] \times [0, 2\pi]$ and $\mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$.

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi] = \iint_{\mathcal{R}_1} \sin x \, dA, \quad \mathcal{R}_1 = [0, \pi] \times [0, 2\pi] + \iint_{\mathcal{R}_2} \sin x \, dA, \quad \mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$$

From what we said before, we can observe that

$$\iint_{\mathcal{R}_1} \sin x \, dA, \quad \mathcal{R}_1 = [0, \pi] \times [0, 2\pi] = -\iint_{\mathcal{R}_2} \sin x \, dA, \quad \mathcal{R}_2 = [\pi, 2\pi] \times [0, 2\pi]$$

Finally, we can state that

$$\iint_{\mathcal{R}} \sin x \, dA, \quad \mathcal{R} = [0, 2\pi] \times [0, 2\pi] = -\iint_{\mathcal{R}_2} \sin x \, dA + \iint_{\mathcal{R}_2} \sin x \, dA = 0$$

The double integral evaluates to 0.

4 Question 19

Evaluate the iterated integral

$$\int_1^3 \int_0^2 x^3 y \, dy \, dx$$

$$\int_1^3 \int_0^2 x^3 y \, dy \, dx = \int_1^3 \left(\int_0^2 x^3 y \, dy \right) dx$$

$$= \int_1^3 \left(x^3 \cdot \frac{y^2}{2} \Big|_{y=0}^{y=2} \right) dx$$

$$= \int_1^3 \left(x^3 \cdot (\frac{4}{2} - \frac{0}{2}) \right) dx$$

$$= \int_1^3 2x^3 \, dx = \frac{x^4}{2} \Big|_{x=1}^{x=3} = \frac{81}{2} - \frac{1}{2} = \frac{80}{2} = 40$$

The iterated integral evaluates to 40.

5 Question 21

Evaluate the iterated integral

$$\int_4^9 \int_{-3}^8 1\,dx\,dy$$

$$\int_4^9 \left(\int_{-3}^8 1\,dx\right) dy =$$

$$\int_4^9 x\Big|_{x=-3}^{x=8} dy =$$

$$\int_4^9 11\,dy = 11x\Big|_{x=4}^{x=9} = 99 - 44 = 55$$

The iterated integral evaluates to 55.

6 Question 23

Evaluate the iterated integral

$$\int_{-1}^{1}\int_{0}^{\pi} x^2 \sin y \, dy \, dx$$

$$\int_{-1}^{1}\left(\int_{0}^{\pi} x^2 \sin y \, dy\right) dx$$

$$\int_{-1}^{1}\left(x^2 \cdot - \cos y\Big|_{y=0}^{y=\pi}\right) dx$$

$$\int_{-1}^{1} x^2(1-(-1))\, dx$$

$$\int_{-1}^{1} 2x^2 \, dx = \frac{2x^3}{3}\Big|_{x=-1}^{x=1} = \frac{2}{3} - \frac{-2}{3} = \frac{4}{3}$$

The iterated integral evaluates to $\frac{4}{3}$.

7 Question 25

Evaluate the iterated integral

$$\int_2^6 \int_1^4 x^2 \, dx \, dy$$

$$\int_2^6 \left(\int_1^4 x^2 \, dx \right) dy$$

$$\int_2^6 \left(\frac{x^3}{3} \bigg|_{x=1}^{x=4} \right) dy$$

$$\int_2^6 \frac{64}{3} - \frac{1}{3} \, dy = \int_2^6 \frac{63}{3} \, dy$$

$$\frac{63x}{3} \bigg|_{x=2}^{x=6} = \frac{378 - 126}{3} = \frac{252}{3} = 84$$

The iterated integral evaluates to 84.

8 Question 37

Evaluate the integral

$$\iint_{\mathcal{R}} \frac{x}{y}\, dA, \quad \mathcal{R} = [-2,4] \times [1,3]$$

$$\iint_{\mathcal{R}} \frac{x}{y}\, dx\, dy, \quad \mathcal{R} = [-2,4] \times [1,3]$$

$$\int_1^3 \int_{-2}^4 \frac{x}{y}\, dx\, dy$$

$$\int_1^3 \left(\frac{x^2}{2y}\bigg|_{x=-2}^{x=4} dx \right) dy$$

$$\int_1^3 \frac{1}{2y}(16-4)\, dy$$

$$\int_1^3 \frac{6}{y}\, dy$$

$$6 \ln y\bigg|_{y=1}^{y=3} = 6\ln 3 - 6\ln 1 = 6\ln 3$$

9 Question 40

Evaluate the integral

$$\iint_{\mathcal{R}} \frac{y}{x+1}\, dA, \quad \mathcal{R} = [0,2] \times [0,4]$$

$$\int_0^4 \int_0^2 \frac{y}{x+1}\, dx\, dy$$

$$\int_0^4 \int_0^2 \frac{y}{x+1}\, dx\, dy$$

$$\int_0^4 \left(y \ln|x+1| \Big|_{x=0}^{x=2} \right) dy$$

$$\int_0^4 y(\ln|3| - \ln|1|)\, dy$$

$$\int_0^4 y \ln|3|\, d$$

$$\frac{\ln|3|\, y^2}{2} \Big|_{y=0}^{y=4}$$

$$8 \ln 3$$

The solution to this evaluated integral is $8 \ln 3$

10 Question 45

Let $f(x, y) = mxy^2$, where m is a constant. Find a value of m such that $\iint_{\mathcal{R}} f(x, y)\, dA = 1$, where $\mathcal{R} = [0, 1] \times [0, 2]$

$$\int_0^2 \int_0^1 mxy^2 \, dx \, dy$$

$$\int_0^2 \left(\frac{mx^2y^2}{2} \bigg|_{x=0}^{x=1} \right) dy$$

$$\int_0^2 \frac{my^2}{2} \, dy$$

$$\frac{my^2}{2} \bigg|_{y=0}^{y=2}$$

$$2m = 1$$

$$m = \frac{1}{2}$$

When $m = \frac{1}{2}$, the expression $\iint_{\mathcal{R}} f(x, y)\, dA, \mathcal{R} = [0, 1] \times [0, 2] = 1$ is true!

11 Question 48

a) Which is easier, antidifferentiating xe^{xy} with respect to x or with respect to y? Explain.
b) Evaluate $\iint_{\mathcal{R}} xe^{xy}\, dA$, where $\mathcal{R} = [0,1] \times [0,1]$

a) It is easier antidifferentiating xe^{xy} with respect to y, as we can simply treat x as a constant. Computing $\int xe^{xy}\, dy$ requires the same steps as solving $\int 4e^{4y}\, dy$, which is relatively simple

b)

$$\iint_{\mathcal{R}} xe^{xy}\, dA, \quad \mathcal{R} = [0,1] \times [0,1]$$

$$\int_0^1 \int_0^1 xe^{yx}\, dy\, dx$$

$$\int_0^1 \left(e^{yx} \Big|_{y=0}^{y=1} \right) dx$$

$$\int_0^1 e^x - 1\, dx$$

$$e^x - x \Big|_{x=0}^{x=1}$$

$$e - 1 - 1 - 0 = e - 2$$

The integral evaluates to $e - 2$

12 Question 49

a) Which is easier, antidifferentiating $\frac{y}{1+xy}$ with respect to x or with respect to y? Explain.
b) Evaluate $\iint_{\mathcal{R}} \frac{y}{1+xy}\,dA$, where $\mathcal{R} = [0,1] \times [0,1]$

a) It is easier to antidifferentiate $\frac{y}{1+xy}$ with respect to x, as we would only need to worry about one variable in the denominator. If we had integrated with respect to y, we would need to consider the y in the numerator and the y in the denominator. Integrating $\int \frac{2}{1+2x} dx$ can be solved with basic integration rules. However, antidifferentiating $\int \frac{y}{1+2y}$ would likely need to be solved using a technique like integration by parts.

b)

$$\iint_{\mathcal{R}} \frac{y}{1+xy}\,dA, \quad \mathcal{R} = [0,1] \times [0,1]$$

$$\int_0^1 \int_0^1 \frac{y}{1+xy}\,dx\,dy$$

$$\int_0^1 \left(\frac{y \ln|1+yx|}{y} \Bigg|_{y=0}^{y=1} \right) dx$$

$$\int_0^1 \ln|1+x|\,dx$$

$$(1+x)\ln(1+x) - (1+x)\Big|_{x=0} x = 1$$

$$2\ln 2 - 2 - (0-1) = \ln 2 - 1$$

This integral evaluates to $2\ln 2 - 1$, or $ln4 - 1$