MATH 33A Problem Set 9

Allan Zhang

December 8, 2024

1 Question 1

Find an orthonormal eigenbasis for the following matrices:

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$

$$\det(\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} - \lambda I) = \det(\begin{bmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix})$$

$$(6-\lambda)(3-\lambda) - 4 = 0$$

$$18 - 9\lambda + \lambda^2 - 4 = 0 \qquad 14 - 9\lambda + \lambda^2 = 0$$

$$\lambda = 7 \qquad \lambda = 2$$

$$\text{Let } \lambda = 2\text{: } \ker(\begin{bmatrix} 6-2 & 2 \\ 2 & 3-2 \end{bmatrix})$$

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2x + y = 0 \qquad x = -\frac{1}{2}y \quad \text{The eigenvector is: } \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\text{Let } \lambda = 7\text{: } \ker(\begin{bmatrix} 6-7 & 2 \\ 2 & 3-7 \end{bmatrix})$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x - 2y = 0 \qquad x = 2y \quad \text{The eigenvector is: } \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\| \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \| = \frac{2}{\sqrt{5}} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \qquad \| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \| = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\text{The orthonormal eigenbasis is span}\{\frac{2}{\sqrt{5}} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$$

(b) $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$\det(\begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix}) = (-2-\lambda)((-2-\lambda)^2 - 1) - 1((-2-\lambda) - 1) + 1(1 - (-2-\lambda))$$

$$(2-\lambda)(\lambda^2 - 4\lambda + 3) - 1(-\lambda - 3) + 1(\lambda + 3) = (2-\lambda)(\lambda^2 - 4\lambda + 3) + \lambda + 3 + \lambda + \lambda^2 + 11\lambda - 6 = 0$$

$$-\lambda^3 - 6\lambda^2 - 11\lambda - 6 + 2\lambda + 6 = 0 \qquad \lambda^3 - 6\lambda^2 - 9\lambda = 0$$

$$\lambda(\lambda - 3)^2 = 0 \qquad \lambda = 0 \quad \lambda = 3(\text{multiplicity } 2)$$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$x = z \qquad y = z \quad \text{The eigenvector is } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = -y - z$$

$$\text{The span is: } \{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \}$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{v_2 - \mathrm{proj}_{u_1} v_2}{\|v_2 - \mathrm{proj}_{u_1} v_2\|} = \frac{v_2 - (u_1 \cdot v_2)u_1}{\|v_2 - (u_1 \cdot v_2)u_1\|} = \frac{\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}}{\sqrt{2}}$$

$$\text{The orthonormal eigenbasis is } \{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \}$$

2 Question 2

Find a 2 x 2 matrix A such that $A^3 = \begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix}$

$$P^{-1}AP = D$$

$$\det(\begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix} - \lambda I) = \det(\begin{bmatrix} 13-\lambda & 14 \\ 14 & 13-\lambda \end{bmatrix}) = (13-\lambda)^2 - 14^2 = 0$$

$$169 - 26\lambda + \lambda^2 - 196 = \lambda^2 - 26\lambda - 27 = (\lambda - 27)(\lambda + 1)$$

$$\lambda = 27 \qquad \lambda = -1$$

$$\ker(\begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix} - 27I) = \ker(\begin{bmatrix} -14 & 14 \\ 14 & -14 \end{bmatrix}) = \ker(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix})$$

$$x = y \quad \text{The corresponding eigenvector is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\ker(\begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix} - (-1)I) = \ker(\begin{bmatrix} 14 & 14 \\ 14 & 14 \end{bmatrix}) = \ker(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix})$$

$$x = -y \quad \text{The corresponding eigenvector is } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = D$$

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 27 & 1 \\ 27 & -1 \end{bmatrix} = D$$

$$\begin{bmatrix} 27 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = P{-1}D^{-1/3}P = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

3 Question 3

Find the symmetric matrix of the following quadratic form. Also, determine whether it is positive definite, negative definite, or indefinite.

$$\text{(a)} \quad q(x_1, x_2) = x_1^2 + 4x_1x_2 + 4x_2^2$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\det(\begin{bmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{bmatrix}) = (1-\lambda)(4-\lambda) - 4 = 0$$

$$4 - 5\lambda + \lambda^2 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

$$\lambda = 0 \qquad \lambda = 5$$

Since one of the eigenvalues is positive and the other is negative, the quadratic form is positive semidefinite

$$\text{(b)} \quad q(x_1, x_2) = 2x_1^2 + 11x_1x_2 + 9x_2^2$$

$$\begin{bmatrix} 2 & 11/2 \\ 11/2 & 9 \end{bmatrix}$$

$$\det(\begin{bmatrix} 2-\lambda & 11/2 \\ 11/2 & 9-\lambda \end{bmatrix}) = (2-\lambda)(9-\lambda) - 121/4 = 0$$

$$18 - 11\lambda + \lambda^2 - 121/4 = 72/4 - 121/4 + \lambda^2 - 11\lambda = \lambda^2 - 11\lambda - 49/4 = 0$$

$$\lambda = \frac{11 \pm \sqrt{121 + 49}}{2}$$

From this, we know that one of the eigenvalues is positive and the other is negative. We can conclude that this quadratic form is indefinite.

$$\text{(c)} \quad q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_2$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\det(\begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{bmatrix})$$

$$(2-\lambda)(-1 + (2-\lambda)) - 1(2-\lambda) - 1(2-\lambda)^2 = 0$$

$$(2-\lambda)(\lambda^2 - 4\lambda + 2) = 0$$

$$\lambda = 2 \qquad \lambda = -\sqrt{2} + 2 \qquad \lambda = -\sqrt{2} + 2$$

Since all the eigenvalues are positive, the quadratic form is positive definite.

4 Question 5

A $= \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Find unit vectors $\vec{u}_1$ and $\vec{u_2}$ such that $\|A\vec{u}_1\| = 4$ and $\|A\vec{u}_2\| = 1$

$$A^T = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

$$\det(\begin{bmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}) = (4-\lambda)(13-\lambda) - 36 = 0$$

$$52 - 17\lambda + \lambda^2 - 36 = \lambda^2 = 17\lambda + 16 = (\lambda - 1)(\lambda - 16) = 0$$

$$\lambda = 1 \qquad \lambda = 16$$

$$\ker(\begin{bmatrix} 4-1 & 6 \\ 6 & 13-1 \end{bmatrix})$$

$$\begin{bmatrix} 3 & 6 & 0 \\ 6 & 12 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = -2y \qquad \ker(\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}) = \{\begin{bmatrix} -2y \\ y \end{bmatrix} : y \in \mathbb{R}\}$$

$$\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \| = \sqrt{5}\vec{u_1} = \sqrt{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\ker(\begin{bmatrix} 4-16 & 6 \\ 6 & 13-16 \end{bmatrix})$$

$$\begin{bmatrix} -12 & 6 & 0 \\ 6 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \frac{1}{2}y \qquad \ker(\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}) = \{\begin{bmatrix} \frac{y}{2} \\ y \end{bmatrix} : y \in \mathbb{R}\}$$

$$\| \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \| = \frac{\sqrt{5}}{2}$$

$$\vec{u_2} = \frac{2}{\sqrt{5}} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$