MATH 32B Problem Set 2

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1 Question 5

Compute the double integral of $f(x, y) = x^2y$ over the given shaded domain.

$$y \text{ is bounded by } y = 0 \text{ and } y = 2,\ x \text{ is bounded by } x = 4 - 2y \text{ and } x = 4$$

$$\int_0^2 \int_{4-2y}^4 x^2y\,dx\,dy$$

$$\int_0^2 \left.\frac{x^3y}{3}\right|_{x=4-2y}^{x=4} dy$$

$$\int_0^2 \frac{64y}{3} - \frac{y(4-2y)^3}{3}\,dy$$

$$\int_0^2 \frac{y}{3}(64 - (4-2y)^3)\,dy$$

$$(4-2y)^3 = (16 - 16y + 4y^2)(4-2y) = 64 - 64y + 16y^2 - 32y + 32y^2 - 8y^3$$

$$= 64 - 96y + 48y^2 - 8y^3$$

$$\int_0^2 \frac{y}{3}(8y^3 - 48y^2 + 96y)\,dy$$

$$\int_0^2 \frac{8y^4}{3} - 16y^3 + 32y^2)\,dy$$

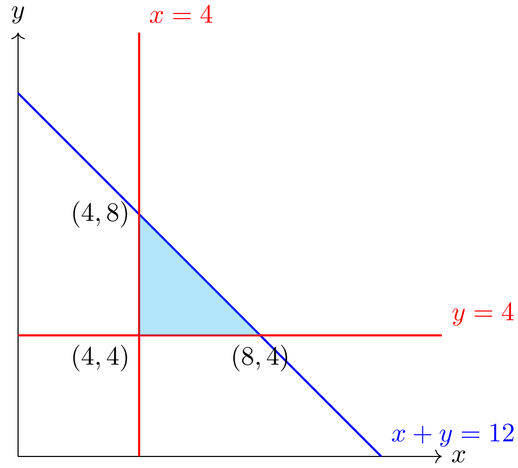
$$\left.\frac{8y^5}{15} - 4y^4 + \frac{32y^3}{3}\right|_{y=0}^{y=2}$$

$$\frac{256}{15} - 64 + \frac{256}{3} = \frac{256}{15} - \frac{960}{15} + \frac{1280}{15}$$

The double integral computed over the given shaded domain is $\frac{192}{5}$

2 Question 8

Sketch the domain $\mathcal{D}$ by $x + y \leq 12, x \geq 4, y \geq 4$ and compute $\iint_{\mathcal{D}} e^{x+y} dA$



From this sketch, we can see that y can be bounded by $y = 4$ and $y = 12 - x$. x is bounded from $x = 4$ to $x = 8$

$$\iint_{\mathcal{D}} e^{x+y} dA = \int_4^8 \int_4^{12-x} e^{x+y} \, dy \, dx$$

$$\int_4^8 \left(e^{x+y} \Big|_4^{12-x} \right) dx$$

$$\int_4^8 e^{12} - e^{x+4} \, dx$$

$$e^{12}x - e^{x+4} \Big|_4^8$$

$$8e^12 - e^12 - (4e^12 - e^8)$$

The answer is $3e^{12} - e^8$

3 Question 12

Calculate the double integral of $f(x, y) = y^2$ over the rhombus $\mathcal{R}$

Taking into account the symmetry of the rhombus and the symmetry of the $f(x, y) = y^2$, we can compute the volume of the curve over $\frac{1}{4}$ of the rhombus and multiply it by 4 to compute the total volume. We can define the bounds as:

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2 - 2x$$

$$\int_0^1 \int_0^{2-2x} y^2 \, dy \, dx$$

$$\int_0^1 \left. \frac{y^3}{3} \right|_{y=0}^{y=2-2x} dx$$

$$(2 - 2x)^3 = 8 - 24x + 24x^2 - 8x^3$$

$$\int_0^1 \frac{8 - 24x + 24x^2 - 8x^3}{3} \, dx$$

$$\frac{1}{3} \int_0^1 8 - 24x + 24x^2 - 8x^3 \, dx$$

$$\frac{1}{3} \Big[8x - 12x^2 + 8x^3 - 2x^4 \Big]_0^1$$

$$\frac{1}{3}(8 - 12 + 8 - 2) = \frac{1}{3}(2) = \frac{2}{3}$$

Since this is the volume computed over $\frac{1}{4}$ of the rhombus, the total volume is $4(\frac{2}{3}) = \frac{8}{3}$

4 Question 21

$f(x, y) = 6xy - x^2$; bounded below by $y = x^2$, above by $y = \sqrt{x}$ The bounds are

$$0 \leq x \leq 1, \quad x^2 \leq y \leq \sqrt{x}$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 6xy - x^2 \, dy \, dx$$

$$\int_0^1 3xy^2 - x^2 y \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$\int_0^1 3x(\sqrt{x})^2 - x^2(\sqrt{x}) - 3x(x^2)^2 + x^2(x^2) \, dx$$

$$\int_0^1 3x^2 - x^{5/2} - 3x^5 + x^4 \, dx$$

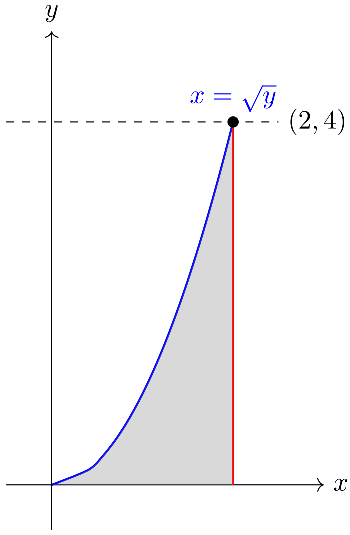
$$x^3 - \frac{2x^{\frac{7}{2}}}{7} - \frac{x^6}{2} + \frac{x^5}{5}\Bigg|_0^1$$

$$1 - \frac{2}{7} - \frac{1}{2} + \frac{1}{5} = \frac{70}{70} - \frac{20}{70} - \frac{35}{70} + \frac{14}{70} = \frac{29}{70}$$

The answer is $\frac{29}{70}$

5 Question 29

Sketch the domain $\mathcal{D}$ corresponding to $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4x^2+5y}\,dx\,dy$. Then change the order of integration and evaluate.



After switching the bounds, we get:

$$0 \le x \le 2, \quad 0 \le y \le x^2$$

$$\int_0^2 \int_0^{x^2} \sqrt{4x^2+5y}\,dy\,dx$$

$$\int_0^2 \left(\frac{2}{5\cdot 3}(4x^2+5y)^{\frac{3}{2}} \Big|_{y=0}^{y=x^2} \right)$$

$$\frac{2}{15}\int_0^2 (9x^2)^{\frac{3}{2}} - (4x^2)^{\frac{3}{2}}\,dx$$

$$\frac{2}{15}\int_0^2 27x^3 - 8x^3\,dx$$

$$\frac{2}{15}\int_0^2 19x^3\,dx$$

$$\frac{2}{15}\left[\frac{19x^4}{4}\right]_{x=0}^{x=2}$$

$$\frac{2}{15}\cdot 76 = \frac{152}{15}$$

The answer is $\frac{152}{15}$

6 Question 5

$f(x,y,z) = (x-y)(y-z); \quad [0,1]\times[0,3]\times[0,3].$

$$\int_0^1\int_0^3\int_0^3 (x-y)(y-z)\,dz\,dy\,dx$$

$$\int_0^1\int_0^3\int_0^3 xy - xz - y^2 + yz\,dz\,dy\,dx$$

$$\int_0^1\int_0^3 xyz - \frac{xz^2}{2} - y^2z + \frac{yz^2}{2}\Big|_{z=0}^{z=3} dy\,dx$$

$$\int_0^1\int_0^3 3xy - \frac{9x}{2} - 3y^2 + \frac{9y}{2}\,dx$$

$$\int_0^1 \frac{3xy^2}{2} - \frac{9yx}{2} - y^3 + \frac{9y^2}{4}\Big|_{y=0}^{y=3} dx$$

$$\int_0^1 \frac{27x}{2} - \frac{27x}{2} - 27 + \frac{81}{4}\,dx$$

$$\frac{27x^2}{4} - \frac{27x}{2} - 27 + \frac{81}{4}\Big|_{x=0}^{x=1}$$

$$-27 + \frac{81}{4} = -frac274$$

The answer is $-\frac{27}{4}$

7 Question 10

Evaulate $\iiint_{\mathcal{W}} f(x,y,z)\,dV$ for the function f and region $\mathcal{W}$ specified

$$f(x,y,z) = e^{x+y+z}; \quad \mathcal{W} : 0 \leq z \leq 1,\, 0 \leq y \leq x,\, 0 \leq z \leq 1$$

$$\int_0^1 \int_0^1 \int_0^x e^{x+y+z}\,dy\,dx\,dz$$

$$\int_0^1 \int_0^1 \left(e^{x+y+z}\Big|_{y=0}^{y=x} \right) dx\,dz$$

$$\int_0^1 \int_0^1 e^{2x+z} - e^{x+z}\,dx\,dz$$

$$\int_0^1 \left(\frac{e^{2x+z}}{2} - e^{x+z}\Big|_{x=0}^{x=1} \right) dz$$

$$\int_0^1 \left(\frac{e^{z+2}}{2} - e^{z+1} \right) - \left(\frac{e^z}{2} - e^z \right) dz$$

$$\frac{e^{z+2}}{2} - e^{z+1} + \frac{e^z}{2}\Big|_{z=0}^{z=1}$$

$$\frac{e^3}{2} - e^2 + \frac{e}{2} - (\frac{e^2}{2} - e + \frac{1}{2})$$

The answer is $\frac{e^3}{2} - \frac{3e^2}{2} + \frac{3e}{2} - \frac{1}{2}$

8 Question 15

Calculate the integral of $f(x,y,z) = z$ over the region $\mathcal{W}$ in Figure 11, below the hemisphere of radius 3 and lying over the triangle D in the $xy-$plane bounded by $x = 1$, $y = 1$, $x = y$

First, let's calculate the bounds:

$$0 \leq x \leq 1 \quad x \leq y \leq 1 \quad 0 \leq z \leq \sqrt{9 - x^2 - x^2}$$

$$\int_0^1 \int_x^1 \int_0^{\sqrt{9-x^2-y^2}} z\, dz\, dy\, dx$$

$$\int_0^1 \int_x^1 \left(\frac{z^2}{2} \bigg|_0^{\sqrt{9-x^2-y^2}} \right) dy\, dx$$

$$\int_0^1 \int_x^1 \frac{9 - x^2 - y^2}{2}\, dy\, dx$$

$$\frac{1}{2} \int_0^1 \int_x^1 9 - x^2 - y^2\, dy\, dx$$

$$\frac{1}{2} \int_0^1 9y - x^2 y - \frac{y^3}{3} \bigg|_x^1 dx$$

$$\frac{1}{2} \int_0^1 (9 - x^2 - \frac{1}{3}) - (9x - x^3 - \frac{x^3}{3})\, dx$$

$$\frac{1}{2} \left[9x - \frac{x^3}{3} - \frac{x}{3} - \frac{9x^2}{2} + \frac{x^4}{4} + \frac{x^4}{12} \right]_{x=0}^{x=1}$$

$$\frac{1}{2}(9 - \frac{1}{3} - \frac{1}{3} - \frac{9}{2} + \frac{1}{4} + \frac{1}{12}) = \frac{25}{12}$$

The integral evaluates to $\frac{25}{12}$

9 Question 21

Find the volume of the solid in the first octant bounded between the planes $x+y+z=1$ and $x+y+2z=1$

First, let us find where the two planes intersection

$$z = 1-x-y \qquad z = \frac{1-x-y}{2}$$

$$1-x-y = \frac{1-x-y}{2}$$

$$2-2x-2y = 1-x-y$$

$$x+y=1$$

Let's compute the bounds

$$0 \le x \le 1 \qquad 0 \le x \le 1-x$$

$$\int_0^1 \int_0^{1-x} 1-x-y-\frac{1-x-y}{2}\,dx$$

$$\int_0^1 \int_0^{1-x} 1-x-y-\frac{1-x-y}{2}\,dx$$

$$\frac{1}{2}\int_0^1 \int_0^{1-x} 1-x-y\,dx$$

$$\frac{1}{2}\int_0^1 \left(y-xy-\frac{y^2}{2}\Big|_{y=0}^{y=1-x}\right)dx$$

$$(1-x)-(1-x)x-\frac{(1-x)^2}{2}$$

$$1-2x+x^2-\frac{(1-2+x^2}{2} = \frac{1-2x+x^2}{2}$$

$$\frac{1}{4}\int_0^1 1-2x+x^2$$

$$\frac{1}{4}\Big[x-x^2+\frac{x^3}{3}-\Big]_{x=0}^{x=1}$$

$$\frac{1}{4}(1-1+\frac{1}{3})$$

$$\frac{1}{4}\cdot\frac{1}{3}$$

The answer is $\frac{1}{12}$

10 Question 22

Evalulate $\iiint_{\mathcal{W}} y\,dV$ where $\mathcal{W}$ is the region above $z = x^2 + y^2$ and below $z = 5$, and bounded by $y = 0$ and $y = 1$.
First, let us define the bounds:

$$x^2 + y^2 \le z \le 5$$

$$0 \le y \le 1$$

To solve for the bounds of x, solve for the intersection of the two surfaces:

$$x^2 + y^2 = 5$$

$$x^2 = 5 - y^2$$

$$x = \pm\sqrt{5 - y^2}$$

$$-\sqrt{5 - y^2} \le x \le \sqrt{5 - y^2}$$

Now, we can rewrite the integral as:

$$\int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \int_{x^2+y^2}^{5} y\,dz\,dx\,dy$$

$$\int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \left(yz\Big|_{z=x^2+y^2}^{z=5} \right) dx\,dy$$

$$\int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} 5y - y(x^2+y^2) dx\,dy$$

$$\int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} y(5 - x^2 - y^2) dx\,dy$$

$$\int_0^1 \frac{y(15x - x^3 - 3y^2x)}{3}\Big|_{x=-\sqrt{5-y^2}}^{x=\sqrt{5-y^2}} \Big) dy$$

$$\int_0^1 \frac{yx(15 - x^2 - 3y^2)}{3}\Big|_{x=-\sqrt{5-y^2}}^{x=\sqrt{5-y^2}} \Big) dy$$

$$\frac{(y(\sqrt{5-y^2}))(15 - (5-y^2) - 3y^2)}{3} - \frac{(y(-\sqrt{5-y^2}))(15 - (5-y^2) - 3y^2)}{3}$$

$$2(\frac{2y\sqrt{5-y^2}(20 - 4y^2)}{3})$$

$$\frac{4(y\sqrt{5-y^2}(5-y^2))}{3}$$

$$\frac{4y(5-y^2)^{\frac{3}{2}}}{3}$$

$$\int_0^1 \frac{4y(5-y^2)^{\frac{3}{2}}}{3}\,dy$$

Let $u = 5 - y^2 \quad du = -2y\,dy \quad -2\,du = 4y\,dy$

$$\int_5^4 \frac{-2u^{\frac{3}{2}}}{3}\,du$$

$$\frac{2}{3}\left(\frac{2u^{\frac{5}{2}}}{5}\Bigg|_4^5 \right)$$

$$\frac{4}{5}(5^{\frac{5}{2}} - 4^{\frac{5}{2}})$$