MATH 32B Problem Set 6

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17.3 Question 17

Find a potential function for $\mathbf{F}$ or determine that $\mathbf{F}$ is not conservative

$$\mathbf{F} = \langle 2xy + 5, x^2 - 4z, -4y \rangle$$

Compute the curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 5 & x^2 - 4z & -4y \end{vmatrix}$$

For the $\hat{\imath}$ component

$$\frac{\partial(-4y)}{\partial y} - \frac{\partial(x^2 - 4z)}{\partial z} = (-4) - (-4) = 0$$

For the $\hat{\jmath}$ component

$$\frac{\partial(2xy + 5)}{\partial z} - \frac{\partial(-4y)}{\partial x} = 0 - 0 = 0$$

For the $\hat{k}$ component

$$\frac{\partial(x^2 - 4z)}{\partial x} - \frac{\partial(2xy + 5)}{\partial y} = (2x) - (2x) = 0$$

Since $\nabla \times \mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative

$$\frac{\partial f}{\partial x} = 2xy + 5$$

Integrate with respect to x

$$f(x, y, z) = x^2 y + 5x + g(y, z)$$

$$\frac{\partial f}{\partial y} = x^2 - 4z$$

$$\frac{\partial}{\partial y}(x^2 y + 5x + g(y, z)) = x^2 + \frac{\partial g}{\partial y} = x^2 - 4z$$

$$\frac{\partial g}{\partial y} = -4z$$

Integrate with respect to y

$$g(y, z) = -4yz + h(z)$$

$$\frac{\partial f}{\partial z} = -4y$$

$$\frac{\partial}{\partial z}(x^2 y + 5x - 4yz + h(z)) = -4y + h'(z) = -4y$$

$$h'(z) = 0 \Rightarrow h(z) = C$$

$$f(x, y, z) = x^2 y + 5x - 4yz + C$$

17.3 Question 18

Find a potential function for $\mathbf{F}$ or determine that $\mathbf{F}$ is not conservative

$$\mathbf{F} = \langle yze^{xy}, xze^{xy}, e^{xy} - y \rangle$$

Compute the curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xy} & xze^{xy} & e^{xy} - y \end{vmatrix}$$

For the $\hat{\imath}$ component

$$\frac{\partial}{\partial y}(e^{xy} - y) - \frac{\partial}{\partial z}(xze^{xy})$$

$$xe^{xy} - 0 = xe^{xy}$$

For the $\hat{\jmath}$ component

$$\frac{\partial}{\partial z}(yze^{xy}) - \frac{\partial}{\partial x}(e^{xy} - y)$$

$$ye^{xy} - (yze^{xy} + xye^{xy}) = ye^{xy} - yze^{xy} - xye^{xy}$$

For the $\hat{k}$ component

$$\frac{\partial}{\partial x}(xze^{xy}) - \frac{\partial}{\partial y}(yze^{xy})$$

$$z(e^{xy} + xye^{xy}) - z(xe^{xy} + yze^{xy}) = ze^{xy} + xzye^{xy} - xze^{xy} - yz^2e^{xy}$$

Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, the field is not conservative

17.3 Question 19

Evaluate

$$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$$

over the path $\mathbf{r}(t) = (t^2, \sin(\pi t/4), e^{t^2-2t})$ for $0 \le t \le 2$

$$\mathbf{F} = (2xyz, x^2z, x^2y)$$

Check if $\mathbf{F}$ is conservative:

$$\frac{\partial f}{\partial x} = 2xyz, \quad \frac{\partial f}{\partial y} = x^2z, \quad \frac{\partial f}{\partial z} = x^2y$$

$$f(x,y,z) = x^2yz + g(y,z)$$

Differentiate with respect to y:

$$\frac{\partial}{\partial y}(x^2yz + g(y,z)) = x^2z + g_y(y,z)$$

$$g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$$

$$\frac{\partial}{\partial z}(x^2yz + h(z)) = x^2y + h'(z)$$

$$h'(z) = 0 \Rightarrow h(z) = C$$

Potential function:

$$f(x,y,z) = x^2yz + C$$

Evaluate at endpoints:

$$\mathbf{r}(2) = (4,1,1), \quad \mathbf{r}(0) = (0,0,1)$$

$$f(4,1,1) = 16, \quad f(0,0,1) = 0$$

Compute the result:

$$f(\mathbf{r}(2)) - f(\mathbf{r}(0)) = 16 - 0 = 16$$

20

Evaluate

$$\oint_{\mathcal{C}} \sin x\, dx + z\cos y\, dy + \sin y\, dz$$

where $\mathcal{C}$ is the ellipse $4x^2 + 9y^2 = 36$, oriented clockwise.

By Stokes' Theorem,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where $\mathbf{F} = (\sin x, z\cos y, \sin y)$

Compute the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & z\cos y & \sin y \end{vmatrix} = \mathbf{0}$$

Since $\nabla \times \mathbf{F} = \mathbf{0}$,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

Answer:

$$\boxed{0}$$

24

The vector field has a uniform curl, which means it might not be conservative. Vector fields are only conservative if their curl is zero.

$$\nabla \times \mathbf{F} = 0$$

From the diagram, it looks like there's a rotational component, there is likely a consistent shear in one direction, meaning the curl is non0. As a result, the vector field is not conservative.

1

a) v
b) iii
c) i
d) iv
e) ii

2

Show that $G(r,\theta) = (r\cos\theta, r\sin\theta, 1 - r^2)$ parameterizes the paraboloid $z = 1 - x^2 - y^2$. Describe the grid curves of this parameterization

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 1 - r^2$$

The equation of the paraboloid is

$$z = 1 - x^2 - y^2$$

Substituting

$$x^2 + y^2 = r^2$$
$$z = 1 - r^2$$

which matches the third component of $G(r,\theta)$

For constant r, the parameterization describes horizontal circles

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 1 - r^2$$

For constant θ, the parameterization describes vertical parabolas

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 1 - r^2$$

7

Calculate $\mathbf{T}_u, \mathbf{T}_v$ and $\mathbf{N}(u,v)$ for the parameterized surface at the given point. Then find the equation of the tangent plane to the surface at that point

$$G(u,v) = (2u+v, u-4v, 3u); \quad u=1, \quad v=4$$

Using the determinant:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & -4 & 0 \end{vmatrix}$$

Expanding:

$$\mathbf{N} = (12, 3, -9).$$

Substituting $u = 1, v = 4$ into $G(u,v)$:

$$G(1,4) = (6, -15, 3).$$

The equation of the tangent plane is:

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

Using $(x_0, y_0, z_0) = (6, -15, 3)$ and $(A, B, C) = (12, 3, -9)$:

$$12(x-6) + 3(y+15) - 9(z-3) = 0.$$

Expanding:

$$12x + 3y - 9z = 0.$$

Dividing by 3:

$$4x + y - 3z = 0.$$

Thus, the equation of the tangent plane is:

$$4x + y - 3z = 0.$$

$$\mathbf{T}_u = \langle 2, 1, 3 \rangle, \mathbf{T}_v = \langle 1, -4, 0 \rangle \mathbf{N}(u,v) = 3\langle 4, 1, -3 \rangle$$

9

Calculate $\mathbf{T}_u, \mathbf{T}_v$ and $\mathbf{N}(u,v)$ for the parameterized surface at the given point. Then find the equation of the tangent plane to the surface at that point

$$G(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi); \quad \theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{4}$$

Compute $\mathbf{T}_\theta$ and $\mathbf{T}_\phi$:

$$\mathbf{T}_\theta = \frac{\partial G}{\partial\theta} = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$

$$\mathbf{T}_\phi = \frac{\partial G}{\partial\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

Evaluate at $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{4}$:

$$\mathbf{T}_\theta = \left(-\frac{\sqrt{2}}{2}, 0, 0\right)$$

$$\mathbf{T}_\phi = \left(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

Compute $\mathbf{N}(\theta,\phi) = \mathbf{T}_\theta \times \mathbf{T}_\phi$:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = \left(0, -\frac{1}{2}, -\frac{1}{2}\right)$$

Find the tangent plane at $G\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$:

$$\mathbf{N} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$0(x-0) - \frac{1}{2}\left(y - \frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(z - \frac{\sqrt{2}}{2}\right) = 0$$

Simplify:

$$y + z - \sqrt{2} = 0$$

$$\mathbf{N} = \mathbf{i}\left(\cos\theta\sin\phi \cdot (-\sin\phi) - 0 \cdot \sin\theta\cos\phi\right)$$

$$-\mathbf{j}\left(-\sin\theta\sin\phi \cdot (-\sin\phi) - 0 \cdot \cos\theta\cos\phi\right)$$

$$+\mathbf{k}\left(-\sin\theta\sin\phi \cdot \sin\theta\cos\phi - \cos\theta\sin\phi \cdot \cos\theta\cos\phi\right)$$

Simplify:

$$\mathbf{N} = \mathbf{i}(-\cos\theta\sin^2\phi) - \mathbf{j}(\sin\theta\sin^2\phi) + \mathbf{k}(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)$$

$$y + z - \sqrt{2} = 0,$$

$\mathbf{T}_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle, \mathbf{T}_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle, \mathbf{N}(u,v) = -\cos\theta\sin^2\phi\mathbf{i} - \sin\theta\sin^2\phi\mathbf{j} - \sin\phi\cos\phi\mathbf{k}$

10

Calculate $\mathbf{T}_u, \mathbf{T}_v$ and $\mathbf{N}(u,v)$ for the parameterized surface at the given point. Then find the equation of the tangent plane to the surface at that point

$$G(r,\theta) = (r\cos\theta, r\sin\theta, 1-r^2); \quad r = \frac{1}{2}, \quad \theta = \frac{\pi}{4}$$

$$G(r,\theta) = (r\cos\theta, r\sin\theta, 1-r^2); \quad r = \frac{1}{2}, \quad \theta = \frac{\pi}{4}.$$

First, compute the partial derivatives of $G(r,\theta)$ with respect to r and θ:

$$\mathbf{T}_r = \frac{\partial G}{\partial r} = (\cos\theta, \sin\theta, -2r),$$
$$\mathbf{T}_\theta = \frac{\partial G}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0).$$

At $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$:

$$\mathbf{T}_r = \left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}, -2\cdot\frac{1}{2}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right),$$

$$\mathbf{T}_\theta = \left(-\frac{1}{2}\sin\frac{\pi}{4}, \frac{1}{2}\cos\frac{\pi}{4}, 0\right) = \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, 0\right).$$

Next, compute the normal vector $\mathbf{N}(r,\theta)$ as the cross product of $\mathbf{T}_r$ and $\mathbf{T}_\theta$:

$$\mathbf{N}(r,\theta) = \mathbf{T}_r \times \mathbf{T}_\theta.$$

Calculating the cross product:

$$\mathbf{N}(r,\theta) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 \end{vmatrix}.$$

Expanding the determinant:

$$\mathbf{N}(r,\theta) = \mathbf{i}\left(\frac{\sqrt{2}}{2}\cdot 0 - (-1)\cdot\frac{\sqrt{2}}{4}\right) - \mathbf{j}\left(\frac{\sqrt{2}}{2}\cdot 0 - (-1)\cdot\left(-\frac{\sqrt{2}}{4}\right)\right) + \mathbf{k}\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{2}}{4}\right)\right).$$

Simplifying:

$$\mathbf{N}(r,\theta) = \mathbf{i}\left(\frac{\sqrt{2}}{4}\right) - \mathbf{j}\left(-\frac{\sqrt{2}}{4}\right) + \mathbf{k}\left(\frac{1}{4}+\frac{1}{4}\right).$$

Thus:

$$\mathbf{N}(r,\theta) = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{1}{2}\right).$$

$$G\left(\frac{1}{2}, \frac{\pi}{4}\right) = \left(\frac{1}{2}\cos\frac{\pi}{4}, \frac{1}{2}\sin\frac{\pi}{4}, 1 - \left(\frac{1}{2}\right)^2\right) = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{3}{4}\right).$$

The equation of the tangent plane is given by:

$$\mathbf{N}\cdot(\mathbf{r} - \mathbf{r}_0) = 0,$$

$$\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{1}{2}\right)\cdot\left(x - \frac{\sqrt{2}}{4}, y - \frac{\sqrt{2}}{4}, z - \frac{3}{4}\right) = 0.$$

Expanding the dot product:

$$\frac{\sqrt{2}}{4}\left(x-\frac{\sqrt{2}}{4}\right)+\frac{\sqrt{2}}{4}\left(y-\frac{\sqrt{2}}{4}\right)+\frac{1}{2}\left(z-\frac{3}{4}\right)=0.$$

Simplifying:

$$\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}y+\frac{1}{2}z-\frac{\sqrt{2}}{4}\cdot\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{4}\cdot\frac{\sqrt{2}}{4}-\frac{1}{2}\cdot\frac{3}{4}=0.$$

Further simplification:

$$\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}y+\frac{1}{2}z-\frac{2}{16}-\frac{2}{16}-\frac{3}{8}=0.$$

Combine constants:

$$\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}y+\frac{1}{2}z-\frac{1}{8}-\frac{1}{8}-\frac{3}{8}=0.$$

$$\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}y+\frac{1}{2}z-\frac{5}{8}=0.$$

$$2\sqrt{2}x+2\sqrt{2}y+4z-5=0.$$

$$2\sqrt{2}x+2\sqrt{2}y+4z-5=0.$$

$$\mathbf{T}_r=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-1\right),\quad \mathbf{T}_\theta=\left(-\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},0\right),\quad \mathbf{N}(r,\theta)=\left(\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2}\right).$$