Math 33B Lecture 1

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March 31, 2025

Examples and Direction Fields

Example 1: General Questions

A car accelerates s.t. its velocity at time t is $v(t) = e^t$

How far has the car traveled after t seconds?
Solution:

$$\text{Let } x(t) = \text{distance traveled at time } t$$

$$\text{Then by definition, } \frac{\mathrm{d}x}{\mathrm{d}t} = v(t) \quad v(t) = e^t$$

$$\text{Problems like this have a family of solutions: } e^t + C, \quad C \in \mathbb{R}$$

$$\text{In this case, we must solve for } C \text{ which gives the correct solution}$$

$$\text{We want } x(0) = 0 \quad x(0) = e^0 + C \quad 0 = 1 + C \quad C = -1$$

$$\text{More generally, for questions with the form } \frac{\mathrm{d}x}{\mathrm{d}t} = f(t)\text{, we can solve by integrating}$$

$$x(t) = \int^t f(S)\mathrm{d}s + C$$

Example 2: Population Growth

You go to a field and see P_0 rabbits and want to figure out how many rabbits in the future. To model rabbit population, need 2 ingredients:

α = birth rate = probability given rabbit reproduces per unit time

β = death rate = probability given rabbit die in a unit time interval

We want to find the population at time t, call this $P(t)$

$$P(t + \Delta t) - P(t) = \text{change in population}$$

$$= \#\text{ born - }\#\text{ die}$$

$$= \alpha \Delta t P(t) - \beta \Delta t P(t)$$

$$\text{After dividing by } \Delta t$$

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = \alpha P(t) - \beta P(t)$$

$$\text{As } \Delta t \to 0, \frac{\mathrm{d}P}{\mathrm{d}t} = (\alpha - \beta)P(t)$$

$$\text{When substituting } (\alpha - \beta) \text{ with } r\text{, we get } \frac{\mathrm{d}P}{\mathrm{d}t} = rP(t)$$

Example solutions: $P(t) = e^{rt} \quad P(t) = 2e^{rt}$
Generally, solutions are of the form Ce^{rt}, where $C = P_0$, the initial population size $P(0)$

Example 3: Logistic Growth

A more realistic model for population growth is the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(1 - \frac{P}{K})$$

where K is the carrying capacity, the max population size the environment can hold

$$\text{If } P << K \rightarrow \frac{P}{K} \approx 0 \quad \frac{\mathrm{d}P}{\mathrm{d}t} = rP$$

$$\text{If } P \approx K \rightarrow \frac{P}{K} \approx 1 \quad 1 - \frac{P}{K} \rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} = 0$$

Direction Fields

Some equations can't be solved, but we can still learn about the solution from the direction field.

DEF: A DE is in normal form if it is written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$