

# MATH 32B Problem Set 2

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## 1 Question 5

Compute the double integral of  $f(x, y) = x^2y$  over the given shaded domain.

$y$  is bounded by  $y = 0$  and  $y = 2$ ,  $x$  is bounded by  $x = 4 - 2y$  and  $x = 4$

$$\int_0^2 \int_{4-2y}^4 x^2y \, dx \, dy$$

$$\int_0^2 \frac{x^3y}{3} \Big|_{x=4-2y}^{x=4} dy$$

$$\int_0^2 \frac{64y}{3} - \frac{y(4-2y)^3}{3} dy$$

$$\int_0^2 \frac{y}{3} (64 - (4-2y)^3) dy$$

$$\begin{aligned} (4-2y)^3 &= (16 - 16y + 4y^2)(4-2y) = 64 - 64y + 16y^2 - 32y + 32y^2 - 8y^3 \\ &= 64 - 96y + 48y^2 - 8y^3 \end{aligned}$$

$$\int_0^2 \frac{y}{3} (8y^3 - 48y^2 + 96y) dy$$

$$\int_0^2 \frac{8y^4}{3} - 16y^3 + 32y^2 dy$$

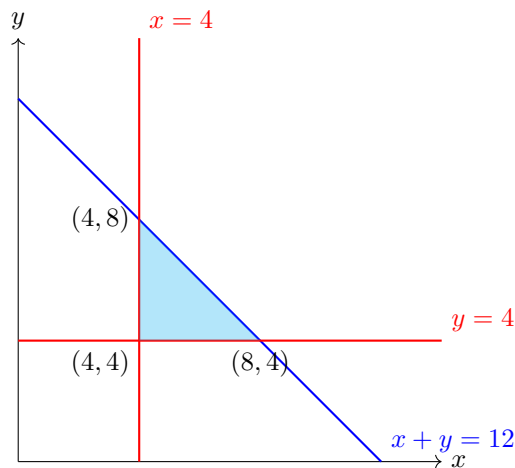
$$\frac{8y^5}{15} - 4y^4 + \frac{32y^3}{3} \Big|_{y=0}^{y=2}$$

$$\frac{256}{15} - 64 + \frac{256}{3} = \frac{256}{15} - \frac{960}{15} + \frac{1280}{15}$$

The double integral computed over the given shaded domain is  $\frac{192}{5}$

## 2 Question 8

Sketch the domain  $\mathcal{D}$  by  $x + y \leq 12, x \geq 4, y \geq 4$  and compute  $\iint_{\mathcal{D}} e^{x+y} dA$



From this sketch, we can see that  $y$  can be bounded by  $y = 4$  and  $y = 12 - x$ .  $x$  is bounded from  $x = 4$  to  $x = 8$

$$\begin{aligned}\iint_{\mathcal{D}} e^{x+y} dA &= \int_4^8 \int_4^{12-x} e^{x+y} dy dx \\ &= \int_4^8 \left( e^{x+y} \Big|_4^{12-x} \right) dx \\ &= \int_4^8 e^{12} - e^{x+4} dx \\ &= e^{12}x - e^{x+4} \Big|_4^8 \\ &= 8e^{12} - e^{12} - (4e^{12} - e^8)\end{aligned}$$

The answer is  $3e^{12} - e^8$

### 3 Question 12

Calculate the double integral of  $f(x, y) = y^2$  over the rhombus  $\mathcal{R}$

Taking into account the symmetry of the rhombus and the symmetry of the  $f(x, y) = y^2$ , we can compute the volume of the curve over  $\frac{1}{4}$  of the rhombus and multiply it by 4 to compute the total volume. We can define the bounds as:

$$\begin{aligned} 0 \leq x \leq 1, \quad 0 \leq y \leq 2 - 2x \\ \int_0^1 \int_0^{2-2x} y^2 dy dx \\ \int_0^1 \frac{y^3}{3} \Big|_{y=0}^{y=2-2x} dx \\ (2 - 2x)^3 = 8 - 24x + 24x^2 - 8x^3 \\ \int_0^1 \frac{8 - 24x + 24x^2 - 8x^3}{3} dx \\ \frac{1}{3} \int_0^1 8 - 24x + 24x^2 - 8x^3 dx \\ \frac{1}{3} \left[ 8x - 12x^2 + 8x^3 - 2x^4 \right]_0^1 \\ \frac{1}{3} (8 - 12 + 8 - 2) = \frac{1}{3} (2) = \frac{2}{3} \end{aligned}$$

Since this is the volume computed over  $\frac{1}{4}$  of the rhombus, the total volume is  $4(\frac{2}{3}) = \frac{8}{3}$

## 4 Question 21

$f(x, y) = 6xy - x^2$ ; bounded below by  $y = x^2$ , above by  $y = \sqrt{x}$  The bounds are

$$0 \leq x \leq 1, \quad x^2 \leq y \leq \sqrt{x}$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 6xy - x^2 \, dy \, dx$$

$$\int_0^1 3xy^2 - x^2y \Big|_{y=x^2}^{y=\sqrt{x}} \, dx$$

$$\int_0^1 3x(\sqrt{x})^2 - x^2(\sqrt{x}) - 3x(x^2)^2 + x^2(x^2) \, dx$$

$$\int_0^1 3x^2 - x^{5/2} - 3x^5 + x^4 \, dx$$

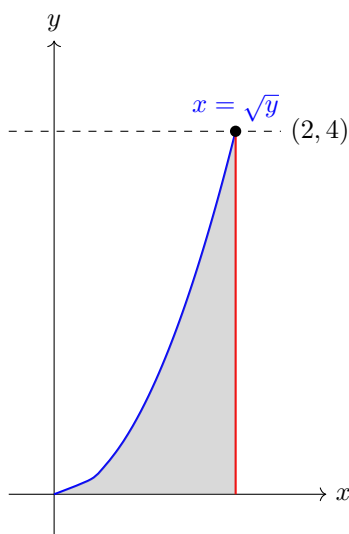
$$x^3 - \frac{2x^{7/2}}{7} - \frac{x^6}{2} + \frac{x^5}{5} \Big|_0^1$$

$$1 - \frac{2}{7} - \frac{1}{2} + \frac{1}{5} = \frac{70}{70} - \frac{20}{70} - \frac{35}{70} + \frac{14}{70} = \frac{29}{70}$$

The answer is  $\frac{29}{70}$

## 5 Question 29

Sketch the domain  $\mathcal{D}$  corresponding to  $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4x^2 + 5y} \, dx \, dy$ . Then change the order of integration and evaluate.



After switching the bounds, we get:

$$\begin{aligned}
 & 0 \leq x \leq 2, \quad 0 \leq y \leq x^2 \\
 & \int_0^2 \int_0^{x^2} \sqrt{4x^2 + 5y} \, dy \, dx \\
 & \int_0^2 \left( \frac{2}{5 \cdot 3} (4x^2 + 5y)^{\frac{3}{2}} \Big|_{y=0}^{y=x^2} \right) \\
 & \frac{2}{15} \int_0^2 (9x^2)^{\frac{3}{2}} - (4x^2)^{\frac{3}{2}} \, dx \\
 & \frac{2}{15} \int_0^2 27x^3 - 8x^3 \, dx \\
 & \frac{2}{15} \int_0^2 19x^3 \, dx \\
 & \frac{2}{15} \left[ \frac{19x^4}{4} \right]_{x=0}^{x=2} \\
 & \frac{2}{15} \cdot 76 = \frac{152}{15}
 \end{aligned}$$

The answer is  $\frac{152}{15}$

## 6 Question 5

$$f(x, y, z) = (x - y)(y - z); \quad [0, 1] \times [0, 3] \times [0, 3].$$

$$\begin{aligned} & \int_0^1 \int_0^3 \int_0^3 (x - y)(y - z) \, dz \, dy \, dx \\ & \int_0^1 \int_0^3 \int_0^3 xy - xz - y^2 + yz \, dz \, dy \, dx \\ & \int_0^1 \int_0^3 \left. xyz - \frac{xz^2}{2} - y^2z + \frac{yz^2}{2} \right|_{z=0}^{z=3} dy \, dx \\ & \int_0^1 \int_0^3 3xy - \frac{9x}{2} - 3y^2 + \frac{9y}{2} \, dx \\ & \int_0^1 \left. \frac{3xy^2}{2} - \frac{9yx}{2} - y^3 + \frac{9y^2}{4} \right|_{y=0}^{y=3} dx \\ & \int_0^1 \frac{27x}{2} - \frac{27x}{2} - 27 + \frac{81}{4} \, dx \\ & \left. \frac{27x^2}{4} - \frac{27x}{2} - 27 + \frac{81}{4} \right|_{x=0}^{x=1} \\ & -27 + \frac{81}{4} = -\frac{27}{4} \end{aligned}$$

The answer is  $-\frac{27}{4}$

## 7 Question 10

Evaluate  $\iiint_{\mathcal{W}} f(x, y, z) dV$  for the function  $f$  and region  $\mathcal{W}$  specified

$$f(x, y, z) = e^{x+y+z}; \quad \mathcal{W} : 0 \leq z \leq 1, 0 \leq y \leq x, 0 \leq x \leq 1$$

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^x e^{x+y+z} dy dx dz \\ & \int_0^1 \int_0^1 \left( e^{x+y+z} \Big|_{y=0}^{y=x} \right) dx dz \\ & \int_0^1 \int_0^1 e^{2x+z} - e^{x+z} dx dz \\ & \int_0^1 \left( \frac{e^{2x+z}}{2} - e^{x+z} \Big|_{x=0}^{x=1} \right) dz \\ & \int_0^1 \left( \frac{e^{z+2}}{2} - e^{z+1} \right) - \left( \frac{e^z}{2} - e^z \right) dz \\ & \left. \frac{e^{z+2}}{2} - e^{z+1} + \frac{e^z}{2} \right|_{z=0}^{z=1} \\ & \frac{e^3}{2} - e^2 + \frac{e}{2} - \left( \frac{e^2}{2} - e + \frac{1}{2} \right) \end{aligned}$$

The answer is  $\frac{e^3}{2} - \frac{3e^2}{2} + \frac{3e}{2} - \frac{1}{2}$

## 8 Question 15

Calculate the integral of  $f(x, y, z) = z$  over the region  $\mathcal{W}$  in Figure 11, below the hemisphere of radius 3 and lying over the triangle  $D$  in the  $xy$ -plane bounded by  $x = 1$ ,  $y = 1$ ,  $x = y$

First, let's calculate the bounds:

$$0 \leq x \leq 1 \quad x \leq y \leq 1 \quad 0 \leq z \leq \sqrt{9 - x^2 - y^2}$$

$$\int_0^1 \int_x^1 \int_0^{\sqrt{9-x^2-y^2}} z \, dz \, dy \, dx$$

$$\int_0^1 \int_x^1 \left( \frac{z^2}{2} \Big|_0^{\sqrt{9-x^2-y^2}} \right) dy \, dx$$

$$\int_0^1 \int_x^1 \frac{9 - x^2 - y^2}{2} dy \, dx$$

$$\frac{1}{2} \int_0^1 \int_x^1 9 - x^2 - y^2 dy \, dx$$

$$\frac{1}{2} \int_0^1 9y - x^2y - \frac{y^3}{3} \Big|_x^1 dx$$

$$\frac{1}{2} \int_0^1 (9 - x^2 - \frac{1}{3}) - (9x - x^3 - \frac{x^3}{3}) dx$$

$$\frac{1}{2} \left[ 9x - \frac{x^3}{3} - \frac{x}{3} - \frac{9x^2}{2} + \frac{x^4}{4} + \frac{x^4}{12} \right]_{x=0}^{x=1}$$

$$\frac{1}{2} (9 - \frac{1}{3} - \frac{1}{3} - \frac{9}{2} + \frac{1}{4} + \frac{1}{12}) = \frac{25}{12}$$

The integral evaluates to  $\frac{25}{12}$



## 9 Question 21

Find the volume of the solid in the first octant bounded between the planes  $x + y + z = 1$  and  $x + y + 2z = 1$   
First, let us find where the two planes intersection

$$z = 1 - x - y \quad z = \frac{1 - x - y}{2}$$

$$1 - x - y = \frac{1 - x - y}{2}$$

$$2 - 2x - 2y = 1 - x - y$$

$$x + y = 1$$

Let's compute the bounds

$$\begin{aligned} 0 \leq x \leq 1 \quad 0 \leq y \leq 1 - x \\ \int_0^1 \int_0^{1-x} 1 - x - y - \frac{1 - x - y}{2} dx \\ \int_0^1 \int_0^{1-x} 1 - x - y - \frac{1 - x - y}{2} dx \\ \frac{1}{2} \int_0^1 \int_0^{1-x} 1 - x - y dx \\ \frac{1}{2} \int_0^1 \left( y - xy - \frac{y^2}{2} \Big|_{y=0}^{y=1-x} \right) dx \\ (1 - x) - (1 - x)x - \frac{(1 - x)^2}{2} \\ 1 - 2x + x^2 - \frac{(1 - 2x + x^2)}{2} = \frac{1 - 2x + x^2}{2} \\ \frac{1}{4} \int_0^1 1 - 2x + x^2 \\ \frac{1}{4} \left[ x - x^2 + \frac{x^3}{3} - \right]_{x=0}^{x=1} \\ \frac{1}{4} \left( 1 - 1 + \frac{1}{3} \right) \\ \frac{1}{4} \cdot \frac{1}{3} \end{aligned}$$

The answer is  $\frac{1}{12}$

## 10 Question 22

Evaluate  $\iiint_{\mathcal{W}} y \, dV$  where  $\mathcal{W}$  is the region above  $z = x^2 + y^2$  and below  $z = 5$ , and bounded by  $y = 0$  and  $y = 1$ . First, let us define the bounds:

$$x^2 + y^2 \leq z \leq 5$$

$$0 \leq y \leq 1$$

To solve for the bounds of  $x$ , solve for the intersection of the two surfaces:

$$x^2 + y^2 = 5$$

$$x^2 = 5 - y^2$$

$$x = \pm\sqrt{5 - y^2}$$

$$-\sqrt{5 - y^2} \leq x \leq \sqrt{5 - y^2}$$

Now, we can rewrite the integral as:

$$\begin{aligned} & \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \int_{x^2+y^2}^5 y \, dz \, dx \, dy \\ & \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \left( yz \Big|_{z=x^2+y^2}^{z=5} \right) dx \, dy \\ & \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} 5y - y(x^2 + y^2) dx \, dy \\ & \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} y(5 - x^2 - y^2) dx \, dy \\ & \int_0^1 \frac{y(15x - x^3 - 3y^2x)}{3} \Big|_{x=-\sqrt{5-y^2}}^{x=\sqrt{5-y^2}} dy \\ & \int_0^1 \frac{yx(15 - x^2 - 3y^2)}{3} \Big|_{x=-\sqrt{5-y^2}}^{x=\sqrt{5-y^2}} dy \\ & \frac{(y(\sqrt{5-y^2}))(15 - (5 - y^2) - 3y^2)}{3} - \frac{(y(-\sqrt{5-y^2}))(15 - (5 - y^2) - 3y^2)}{3} \\ & 2\left(\frac{2y\sqrt{5-y^2}(20 - 4y^2)}{3}\right) \\ & \frac{4(y\sqrt{5-y^2}(5 - y^2))}{3} \\ & \frac{4y(5 - y^2)^{\frac{3}{2}}}{3} \\ & \int_0^1 \frac{4y(5 - y^2)^{\frac{3}{2}}}{3} dy \end{aligned}$$

$$\text{Let } u = 5 - y^2 \quad du = -2y \, dy \quad -2 \, du = 4y \, dy$$

$$\int_5^4 \frac{-2u^{\frac{3}{2}}}{3} du$$

$$\frac{2}{3} \left( \frac{2u^{\frac{5}{2}}}{5} \Big|_4^5 \right)$$

$$\frac{4}{5} (5^{\frac{5}{2}} - 4^{\frac{5}{2}})$$