Miniproject: Competing populations

Spring semester 2021

Read carefully the general instructions before starting the miniproject.

1 Introduction

In this project, your task is to implement the two-competitive populations model of lecture 9 (Ch. 16.3 of *Neuronal dynamics*) and to understand how different levels of inhibition lead to qualitatively different network dynamics.

The two-competitive populations model is one of the simplest attractor network, but as you will see in this miniproject, this simple model is already very rich! As explained in Chapters 16 and 17 of *Neuronal dynamics*, attractor networks can be used to model memory and perceptual decision making.

Imagine that an experimentalist colleague asks you to propose models for two experiments she has recently realized. The first experiment is a two-item memory experiment: The subjects are presented with two stimuli A and B for a very short time interval. After a delay period of several seconds, the subjects are asked if it was stimulus A, stimulus B or stimulus A *and* B. The second experiment is a perceptual decision making experiment: The subjects are familiarized with two stimuli A and B. Subsequently, an ambiguous stimulus is presented and subjects are forced to choose between A *or* B. Having a pronounced taste for simple theory, you want to explain both experiments with a single model.

Your idea is to use the two-competitive populations model presented in Week 9:

$$\tau \frac{d}{dt} h_1(t) = -h_1(t) + b_1 + (w_{ee} - \alpha) g(h_1(t)) - \alpha g(h_2(t)), \tag{1a}$$

$$\tau \frac{d}{dt} h_2(t) = -h_2(t) + b_2 + (w_{ee} - \alpha) g(h_2(t)) - \alpha g(h_1(t)), \tag{1b}$$

where b_1 and b_2 are the biases for populations 1 and 2, w_{ee} is the self-excitation parameter, α is the inhibition parameter, and g is the gain function. For simplicity, we will put $\tau = 1$. In this project, the function g will be the piecewise linear function:

$$g(h) = \begin{cases} 0, & \text{if } h \leq 0, \\ h, & \text{if } 0 < h < 1, \\ 1, & \text{if } 1 \leq h. \end{cases} \tag{2}$$

2 Theory questions

Let us start by building an intuition for the behaviour of the model by performing a fixed-point analysis.

In the quadrant $\{h_1 \leq 0\} \cap \{h_2 \leq 0\}$, Eq. (1) becomes

$$\frac{d}{dt}h_1(t) = -h_1(t) + b_1, \tag{3a}$$

$$\frac{d}{dt}h_2(t) = -h_2(t) + b_2. \tag{3b}$$

Writing

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

we have that the unique fixed-point of Eq. (3) is the solution $H \in \mathbb{R}^2$ of

$$0 = AH + b. \tag{4}$$

The solution of Eq. (4) is clearly $H = b$.

Question 1: (2 point) Is the fixed-point $H = b$ of Eq. (3) a stable fixed-point? Explain your answer.

Importantly, a fixed point H of Eq. (4) is a fixed-point of Eq. (1) if and only if H is in the negative quadrant $\{h_1 \leq 0\} \cap \{h_2 \leq 0\}$. Make sure that you understand this last statement.

In the quadrant $\{h_1 \leq 0\} \cap \{1 < h_2\}$, Eq. (1) becomes

$$\frac{d}{dt}h_1(t) = -h_1(t) + b_1 - \alpha, \tag{5a}$$

$$\frac{d}{dt}h_2(t) = -h_2(t) + b_2 + (w_{ee} - \alpha). \tag{5b}$$

Question 2: (2 point) Compute the fixed-point of Eq. (5) in terms of w_{ee}, α, b_1 and b_2.

Question 3: (2 point) Do the same as for Question 2 but for the quadrant $\{1 < h_1\} \cap \{1 < h_2\}$.

Question 4: (5 points) What is the maximal value of α for which the fixed-point found in Question 3 is also a fixed point of Eq. (1)? Explain your answer.

Hint: If necessary, you can first do Questions 5-7 to gain some intuition and then get back to Question 4.

3 Varying the inhibition parameter α

For this section, you will use the provided Jupyter notebook. The function 'plot_phase_plane' allows you to plot the vector field of Eq. (1) for different values α. In this section, we fix $w_{ee} = 3$ and $b_1 = b_2 = 0.5$. We use the following numbering and colors for the quadrants:

- Quadrant I: $\{1 < h_1\} \cap \{1 < h_2\}$
- Quadrant II: $\{h_1 \leq 0\} \cap \{1 < h_2\}$
- Quadrant III: $\{h_1 \leq 0\} \cap \{h_2 \leq 0\}$
- Quadrant IV: $\{1 < h_1\} \cap \{h_2 \leq 0\}$

Question 5: (3 point) Find a value of α for which Eq. (1) has exactly 3 stable fixed-points. Plot the vector field. What value of α did you choose and in which quadrants are the fixed-points?

Question 6: (3 point) Find a value of α for which Eq. (1) has exactly 2 stable fixed-points. Plot the vector field. Again, what value of α did you choose and in which quadrants are the fixed-points?

Hint for Questions 5 and 6: test values of α in the range $[0, 2]$.

Question 7: (5 points) What do the four colored dots plotted by 'plot_phase_plane' represent? How do they help us to understand the dynamics of Eq. (1)?

4 Adding noise

In the rest of this miniproject, you will study the effect of noise on Eq. (1). We will consider Eq. (1) with additive noise:

$$\frac{d}{dt}h_1(t) = -h_1(t) + b_1 + (w_{ee} - \alpha)g(h_1(t)) - \alpha g(h_2(t)) + \sigma\xi_1(t), \tag{6a}$$

$$\frac{d}{dt}h_2(t) = -h_2(t) + b_2 + (w_{ee} - \alpha)g(h_2(t)) - \alpha g(h_1(t)) + \sigma\xi_2(t), \tag{6b}$$

where $\sigma \geq 0$ is the noise level and ξ_1 and ξ_2 are *independent* Gaussian white noises. To answer the question of the next sections, you need to implement a function that simulate Eq. (6). Here is a suggestion for discretizing Eq. (6a): if $\{t_n\}_{n\in\mathbb{N}}$ is the discretized time with time-step $t_{n+1} - t_n =: \Delta t > 0$,

$$h_1(t_{n+1}) := h_1(t_n) + \Delta t\{-h_1(t_n) + b_1 + (w_{ee} - \alpha)g(h_1(t_n)) - \alpha g(h_2(t_n))\} \\ + \sqrt{\Delta t}\sigma\mathcal{N}_{1,n}(0, 1),$$

where $\{\mathcal{N}_{i,n}(0, 1)\}_{i=1,2;n\in\mathbb{N}}$ are independent standard normal random variables. The reason why there is a $\sqrt{\Delta t}$ scaling is explained in Sec. 8.1 (Eq. (8.8)) of *Neuronal dynamics*.

We recommend using $\Delta t = 0.01$s.

Question 8: (5 points) What happens if, instead of the correct scaling $\sqrt{\Delta t}$ of the noise, you use the the wrong scaling Δt? Plot the trajectories of h_1 and h_2 over time (x-axis time and y-axis h) for $\Delta t = 0.01$ and $\Delta t = 0.001$. Describe what you see.

5 Modeling two-items memory

Let's not make our experimentist colleague wait any longer and let's model her experiments. In this section, you will model the memory experiment. Throughout this section, fix the inhibition parameter α at the value you determined in Question 5. Fix $w_{ee} = 3$ and $b_1 = b_2 = 0.5$

Question 9: (5 points) Pick an initial condition $h(0) = (h_1(0), h_2(0))$ close to a fixed-point. Show that when σ is small, despite the presence of noise, the trajectory stays in the vicinity of the fixed-point (the memory is 'kept'). To show this, plot the trajectory on the phase plane given by the function 'plot_phase_plane'.

Question 10: (5 points) Starting from the parameters of Question 9, increase the noise σ until you see random jumps from one fixed-point to another. Plot the trajectories of $h_1(t)$ and $h_2(t)$ over time (x-axis time and y-axis h).

Hint: you can also play with the duration of the simulation.

The phenomenon of Question 10 is called multi-stability or meta-stability. If you are interested, you can have a look at the talk of Luca Mazzucato (link) or [1].

6 Modeling perceptual decision-making

For this last section, you will use Eq. (6) to model a classical perceptual decision making experiment [2]. Again we fix $w_{ee} = 3$, $b_1 = 0.5$ and you will take the value α you have found in Question 6. We also fix $\sigma = 0.1$ (weak noise). In this section we use the initial condition $h(0) = (-1, -1)$.

Question 12: (3 points) To model perceptual decision making with Eq. (6), you need a decision criterion. Typically, we say that decision A is taken when (h_1, h_2) enters a certain region, and similarly for decision B. What would be reasonable (and practical) regions for decision A and decision B?

Hint: Remember the shape of function g.

Question 13: (10 points) Use the decision criterion you defined in Question 12 to reproduce a plot qualitatively similar to Fig. 16.2.B of *Neuronal dynamics*. To do so, you can keep b_1 fixed and vary b_2. In your plot, the x-axis should be $b_2 - b_1$ and the y-axis should be 'the probability of choosing A'.

Hint: For each value b_2, you will need to simulate several trials to estimate the probability of choosing A.

7 References

[1] L. Mazzucato, A. Fontanini and G. La Camera (2015) *Dynamics of Multistable States during Ongoing and Evoked Cortical Activity.* Journal of Neuroscience:35 (21), pp. 8214–8231. DOI: `https://doi.org/10.1523/JNEUROSCI.4819-14.2015`

[2] Salzman, C. D., Britten, K. H. and Newsome, W. T. (1990) *Cortical microstimulation influences perceptual judgements of motion direction.* Nature, 346(6280), 174-177.