Chapter 4.1

Problem 6

The set described is not a subspace of $\mathbb{P}_n$ because it not closed under scalar multiplication. Take an arbitrary element from the set, $p(t) = t^2 + a$ and multiply it by some $b \in \mathbb{R}$. $b \cdot p(t) = b(t^2 + a) = bt^2 + ba$. Because $bt^2 + ba$ is not in the set, this is not a subspace.

Problem 8

This set is a subspace of $\mathbb{P}_n$. Because the zero polynomial satisfies $p(0) = 0$, the zero polynomial is in this set. If $p_1(t) = 0$ and $p_2(t) = 0$, then $p_{1+}p_2 = 0$. Also, if $p_1(t) = 0$, then $p_1(t) \cdot a = 0$ when $a \in \mathbb{R}$.

Because the set contains the zero polynomial and is closed under vector addition and scalar multiplication, it is a subspace of $\mathbb{P}_n$.

Problem 11

Let $W = \left\{ \begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix} : b, c \in \mathbb{R} \right\}$. We can represent the vector that describes the elements of W as the following linear combination:

$$\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix} = b \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

Thus, we can say $W = Span \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}$. Because we can represent W as the span of some vectors, we know that W is a subspace of $\mathbb{R}^3$ by **theorem 1**.

Problem 12

We can the vector that describes the elements of W as the following linear combination:

$$\begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ -3 \\ 5 \end{bmatrix}$$

Because we can represent W as the span of two vectors in $\mathbb{R}^4$, we know that W is a subspace of $\mathbb{R}^4$.

Problem 16

Take two arbitrary vectors of this form and add them together:

$$\begin{bmatrix} 1 \\ 3a_1 - 5b_1 \\ 3b_1 + 2a_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3a_2 - 5b_2 \\ 3b_2 + 2a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3(a_1 + a_2) - 5(b_1 + b_2) \\ 2(a_1 + a_2) + 3(b_1 + b_2) \end{bmatrix}$$

Because the first row is not of the form $\begin{bmatrix} 1 \\ 3a - 5b \\ 3b + 2a \end{bmatrix}$, the vectors of this form are not closed under vector addition and thus do not form a vector subspace.

Problem 18

We can represent the vector that forms the elements of this set as a linear combination: $\begin{bmatrix} 4a + 3b \\ 0 \\ a + 3b + c \\ 3b - 2c \end{bmatrix} = a \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$

This means that we can define the set of vectors of that form to be $Span \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$.

Because we can define vectors of this form as a span, we know that those vectors form a subspace of $\mathbb{R}^4$.

Problem 21

First, we identify that the matrix of all zeroes is in H when $a = 0, b = 0$, and $c = 0$.

Second, we will take two arbitrary elements of H and add them:

$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & c_1 + c_2 \end{bmatrix}$$

Because the result of this addition is in H, the set is closed under addition. Lastly, we will check that H is closed under scalar multiplication:

$$k \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix}$$

This result is in H, meaning that H is closed under scalar multiplication. Because all 3 of these properties hold, H is a vector subspace.

Chapter 4.2

Problem 2

Let $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}$.

If $\vec{w} \in Null(A)$, then $A\vec{w} = \vec{0}$. We will calculate $A\vec{w}$:

$$\begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} + \begin{bmatrix} -6 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because $A\vec{w} = \vec{0}$, we know that $\vec{w} \in Null(A)$.

Problem 4

Let $A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

To find a description of the $Null(A)$, we will solve the homogenous equation, $A\vec{x} = \vec{0}$:

$$[A|\vec{0}] \backsim \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

We can rewrite this like so:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R}$$

This means that we can describe $Null(A)$ as $Span \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 6

Let $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

To find an explicit description of $Null(A)$, we will solve the homogenous equation $A\vec{x} = \vec{0}$:

$$[A|\vec{0}] = \left[\begin{array}{ccccc|c} 1 & 3 & -4 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \backsim \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We an rewrite this like so:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5x_3 + 6x_4 - x_5 \\ 3x_3 - x_4 x_3 x_4 x_5 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This means that we can describe $Null(A)$ as $Span \left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 16

Let $H = \left\{ \begin{bmatrix} b - c \\ 2b + 3d \\ b + 3c - 3d \\ c + d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}$.

We can rewrite the vector describing elements in this set as the following linear combination:

$H = b \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix}$

We can conclude that: $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix}$

Problem 24

Let $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}, w = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$.

To determine if $w \in Null(A)$, we can check to see if $A\vec{w} = \vec{0}$:

$$\begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -16 \\ 4 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Because $A\vec{w} = \vec{0}$, we can conclude that $w \in Null(A)$.

Now, to determine if $\vec{w} \in Coll(A)$, we must show whether or not $[A|\vec{w}]$ is consistent:

$$[A|\vec{w}] \backsim \left[\begin{array}{cccc|c} 10 & -8 & -2 & -2 & 2 \\ 0 & 2 & 2 & -2 & 2 \\ 1 & -1 & 6 & 0 & 0 \\ 1 & 1 & 0 & -2 & 2 \end{array}\right] \backsim \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Because this augmented matrix is consistent, we know that $\vec{w} \in Coll(A)$.