Section 4.3

Problem 14

Let $A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

To find $Coll(A)$, we will identify the pivot columns of A. These columns are $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix} \right\}$. Thus, we can say that $Coll(A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix} \right\}$.

To find $Null(A)$, we will determine when $A\vec{x} = \vec{0}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 2x_4 - 5x_5 \\ x_2 \\ 2x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $Null(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 16

Let $V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. To find a basis for the span of the vectors in V, we need to find the linearly independent vectors of V.

Let $A = \begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 2 & -1 & -4 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 1 & -1 & -9 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Taking only the pivot columns, a basis for the space spanned by V is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Problem 24

The matrix whose columns are the vectors of B must have a pivot in every column because the set is linearly independent. Because each vector exists in $\mathbb{R}^n$, the matrix is square. Since the matrix is square and its columns are linearly independent, we know that the vectors span $\mathbb{R}^n$ by the Invertible Matrix Theorem. Since the vectors form a linearly independent spanning set, they form a basis of $\mathbb{R}^n$.

Section 4.4

Problem 2

Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ and let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

$\vec{x} = (-2) \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 5 \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \end{bmatrix} + \begin{bmatrix} -20 \\ 30 \end{bmatrix} = \begin{bmatrix} -26 \\ 40 \end{bmatrix}$

Problem 4

Let $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$ and let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

$\vec{x} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \\ -6 \end{bmatrix}$

Problem 13

Let $\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ be a basis for $\mathbb{P}_2$ and let $p(t) = 1 + 4t + 7t^2$.

First, we will write $p(t)$ in terms of the basis $\mathbb{C} = \{1, t, t^2\}$: $[p(t)]_{\mathbb{C}} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$.

To rewrite this relative to $\mathbb{C}$, we will augment the vectors in $\mathcal{B}$ relative to the standard basis and find a solution to the system:

$[1 + t^2]_{\mathbb{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad [t + t^2]_{\mathbb{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad [1 + 2t + t^2]_{\mathbb{C}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 7 \end{array}\right] \backsim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 8 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 7 \end{array}\right] \backsim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 1 & 7 \end{array}\right] \backsim$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 9 \end{array}\right] \backsim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Thus, $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 9 \\ 8 \\ -2 \end{bmatrix}$.

Problem 28

We will use the basis $\mathcal{B} = \{t^3, t^2, t, 1\}$ for $\mathbb{P}_3$. Rewriting each polynomial as a column vector with respect to $\mathcal{B}$:

$$[1 - 2t^2 - 3t^3]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \qquad [t + 2t^3]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad [1 + t - 2t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

To determine if these polynomials are linearly independent, we can put the column vectors into a matrix and determine if there is a pivot in every column:

$$\begin{bmatrix} -3 & 2 & 0 \\ -2 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \backsim \begin{bmatrix} -3 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \backsim \begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \backsim \begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \backsim$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there is a pivot in every column, we know that the polynomials are linearly independent.

Problem 31

Part a

We will use $\mathcal{B} = \{t^2, t, 1\}$ as a basis for $\mathbb{P}_2$ to rewrite these polynomials as column vectors:

$$[1 - t^2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad [1 - 3t + 5t^2]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

$$[-3+5t-7t^2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix} \qquad [-4+5t-6t^2]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 5 \\ -4 \end{bmatrix}$$

We will now combine these column vectors into a matrix and check for a pivot in each row:

$$\begin{bmatrix} -1 & 5 & -3 & -6 \\ 0 & -3 & 5 & 5 \\ 1 & 1 & -7 & 4 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because there is a pivot in every row, we know that these column vectors span $\mathbb{R}^2$. However, since these column vectors represent polynomials in $\mathbb{P}_2$, we also know that the polynomials span $\mathbb{P}_2$.

Part b

We will use $\mathcal{B} = \{t^2, t, 1\}$ as a basis for $\mathbb{P}_2$ to rewrite these polynomials as column vectors:

$$[5t+t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \qquad [1-8t-2t^2]_{\mathcal{B}} = \begin{bmatrix} -2 \\ -8 \\ 1 \end{bmatrix}$$

$$[-3+4t+2t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} \qquad [1-t^2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$