Chapter 4.6

Problem 8

The dimension of the null space of A ($dim(Null(A))$), would be 4.

The column space of A would not be equal $\mathbb{R}^4$. The vectors in $Coll(A)$ are in $\mathbb{R}^6$ and could not span $\mathbb{R}^4$.

Problem 10

By Rank-Nullity, dimension of the column space of is 7 (the number of columns in the matrix) minus 5 (the dimension of the null space), which is 2. Thus, $dim(coll(A)) = 2$.

Problem 12

The dimension of the row space of a matrix is equivalent to the rank of the matrix. The rank of A is 4, thus the dimension of the row space of A is also 4.

Chapter 4.7

Problem 8

Let $\vec{b_1} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}, \vec{c_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{c_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find $P_{C \leftarrow B}$, we row reduce this matrix until the left side is equal to the identity matrix:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 1 & 8 & -7 \end{array}\right] \backsim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & -10 & 9 \end{array}\right] \backsim \left[\begin{array}{cc|cc} 1 & 0 & 9 & -9 \\ 0 & 1 & -10 & 9 \end{array}\right]$$

Thus, $P_{C \leftarrow B} = \begin{bmatrix} 9 & -9 \\ -10 & 9 \end{bmatrix}$.

There are two ways we can compute $P_{B \leftarrow C}$. We can either compute it using the same process we used to compute $P_{C \leftarrow B}$ or we can find the inverse of $P_{C \leftarrow B}$. Since $P_{C \leftarrow B}$ is a 2x2 matrix, its inverse has a simple closed form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow \frac{1}{(9)(9)-(-9)(-10)} \begin{bmatrix} 9 & 9 \\ 10 & 9 \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} 9 & 9 \\ 10 & 9 \end{bmatrix}$$

Thus, $P_{B \leftarrow C} = -\frac{1}{9} \begin{bmatrix} 9 & 9 \\ 10 & 9 \end{bmatrix}$.

Problem 10

Let $\vec{b_1} = \begin{bmatrix} 6 \\ -12 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \vec{c_1} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \vec{c_2} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$.

To find $P_{B \leftarrow C}$, we row reduce this matrix until the left side is equal to the identity matrix:

$$\left[\begin{array}{cc|cc} 6 & 4 & 4 & 3 \\ -12 & 2 & 2 & 9 \end{array}\right] \backsim \left[\begin{array}{cc|cc} 6 & 4 & 4 & 3 \\ 0 & 10 & 10 & 15 \end{array}\right] \backsim \left[\begin{array}{cc|cc} 6 & 4 & 4 & 3 \\ 0 & 2 & 2 & 3 \end{array}\right] \backsim$$
$$\left[\begin{array}{cc|cc} 6 & 0 & 0 & -3 \\ 0 & 2 & 2 & 3 \end{array}\right] \backsim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{3}{2} \end{array}\right]$$

Therefore, $P_{B \leftarrow C} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{3}{2} \end{bmatrix}$. Once again, we can find $P_{C \leftarrow B}$ by find the inverse of $P_{B \leftarrow C}$.

$$P_{C \leftarrow B} = P_{B \leftarrow C}^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{3}{2} \end{bmatrix}^{-1} = \frac{1}{\frac{1}{2}} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -1 & 0 \end{bmatrix}^{-1} = 2 \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -1 & 0 \end{bmatrix}^{-1}$$

Chapter 5.1

Problem 2

Let $A = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$ and let $\lambda = -3$.

If λ is an eigenvalue of A then $A\vec{x} = -3\vec{x}$. This means that $(A + 3I)\vec{x} = \vec{0}$, in other words, the columns of $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$ are linearly dependent. Since this is obviously true, $\lambda = 3$ is indeed an eigenvalue of A.

Problem 4

Let $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$ and let $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. If $\vec{u}$ is an eigenvector of A, then $A\vec{u} = c\vec{u}$ where $c \in \mathbb{R}$.

$$A\vec{u} = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

Because there is no $c \in \mathbb{R}$ such that $c\vec{u} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$, we know that $\vec{u}$ is not an eigenvector of A.

Problem 10

Let $A = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}$ and let $\lambda = -5$.

$$(A + 5I) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \backsim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $x_1 = 1$. Then $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to λ.

Problem 12

Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$, $\lambda_1 = 3$, and $\lambda_2 = 7$.

$$(A - 3I) = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $x_2 = 1$. Then $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to λ_1.

$$(A - 3I) = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \backsim \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $x_2 = 1$. Then $\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to λ_2

Problem 14

Let $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}$ and let $\lambda = 3$.

$$(A - 3I) = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

With $x_3 = 1$, $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$ forms a basis for the eigenspace of A corresponding to λ.

Chapter 5.2

Problem 2

Let $A = \begin{bmatrix} -4 & 1 \\ 6 & 1 \end{bmatrix}$. Consider that when $(A - \lambda I)\vec{x} = \vec{0}$, $det(A - \lambda I) = 0$:

$$det\left(\begin{bmatrix} -4 - \lambda & 1 \\ 6 & 1 - \lambda \end{bmatrix} \right) = 0 \Rightarrow (-4 - \lambda)(1 - \lambda) - 6 = 0$$

Now, to solve for the eigenvalues of A, we need to find the solutions to the equation $(-4-\lambda)(1-\lambda)-6=0$:

$$\lambda^2+3\lambda-10=0 \Rightarrow (\lambda+5)(\lambda-2)=0$$

Thus, the eigenvalues of A are $\lambda_1=-5$ and $\lambda_2=2$.

Problem 4

Let $A=\begin{bmatrix}8 & 2\\ 3 & 3\end{bmatrix}$. To find the eigenvalues of A, we find values such that the equation $det(A-\lambda I)=0$ is satisfied:

$$det\left(\begin{bmatrix}8-\lambda & 2\\ 3 & 3-\lambda\end{bmatrix}\right)=0 \Rightarrow (8-\lambda)(3-\lambda)-6=0$$

Now, we must find the solutions to the equation $\lambda^2-11\lambda+18=0$

$$\lambda^2-11\lambda+18=0 \Rightarrow (\lambda-9)(\lambda-2)=0$$

Thus, our eigenvalues for A are $\lambda_1=9$ and $\lambda_2=2$.

Problem 6

Let $A=\begin{bmatrix}9 & -2\\ 2 & 5\end{bmatrix}$. To find the eigenvalues of A, we find values such that the equation $det(A-\lambda I)=0$ is satisfied:

$$det\left(\begin{bmatrix}9-\lambda & -2\\ 2 & 5-\lambda\end{bmatrix}\right)=0 \Rightarrow (9-\lambda)(5-\lambda)+4=0$$

Now we must solve for equation $\lambda^2+49-14\lambda=0$:

$$\lambda^2+49-14\lambda=0 \Rightarrow (\lambda-7)(\lambda-7)=0$$

This means that A only has a single eigenvalue $\lambda=7$